MAIK GRÖGER

EXAMPLES OF DYNAMICAL SYSTEMS IN THE INTERFACE BETWEEN ORDER AND CHAOS

EXAMPLES OF DYNAMICAL SYSTEMS IN THE INTERFACE BETWEEN ORDER AND CHAOS

MAIK GRÖGER

DISSERTATION

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)

vorgelegt am
Fachbereich 3 (Mathematik & Informatik)
Universität Bremen

Juni 2015

Dipl.-Phys. Maik Gröger: *Examples of dynamical systems in the interface between order and chaos*, © Juni 2015

GUTACHTER:
Prof. Dr. Bernd O. Stratmann (Universität Bremen)
Prof. Dr. Henk Bruin (Universität Wien)

ÖFFENTLICHE VERTEIDIGUNG:
16. Juli 2015, Bremen

For my grandmother and niece.

ABSTRACT

This thesis deals with the complexity inherent in the long-term behavior of both chaotic and non-chaotic dynamical systems. Thereby, two particular examples form the starting point of our work.

The first example is a simple model for the occurrence of a so-called strange non-chaotic attractor. We study fractal aspects of this attractor by determining several associated dimensional quantities.

Interestingly, the considered model system shows a complex long-term behavior despite having zero topological entropy. It is hence natural to ask whether there exists another topological invariant which is able to detect this inherent complexity. This question is the origin for the investigation launched in the second part of the thesis where we introduce the notion of amorphic complexity. After examining basic properties of this new quantity, we study its applicability to almost sure 1-1 extensions of equicontinuous systems with the particular focus on Sturmian subshifts, Denjoy homeomorphisms on the circle and regular Toeplitz subshifts.

The second motivating example of this thesis is closely related to a parameter family of sets of bounded orbits associated with the classical Farey map. This family of sets was recently studied as a generalization of the sets of bounded continued fraction expansions where several topological and dimensional properties were considered. In particular, it was shown that a natural associated bifurcation set plays a central role in the understanding of this family of sets.

In the last part of the present dissertation, we extend these results to parameter families of sets of bounded orbits associated with more general continuous interval maps and thereby focus on topological aspects.

PUBLICATIONS

Parts of this thesis have already been published as a refereed article or as a preprint on arXiv:

[GJ13] M. Gröger and T. Jäger. Dimensions of attractors in pinched skew products. *Communications in Mathematical Physics*, 320(1):101–119, 2013.

[FGJ15] G. Fuhrmann, M. Gröger, and T. Jäger. Amorphic complexity. *Preprint arXiv:1503.01036*, 2015. Submitted.

Farewell has a sweet sound of reluctance.
Good-bye is short and final, a word with teeth sharp
to bite through the string that ties past to the future.

— John Steinbeck

ACKNOWLEDGMENTS

Since I am writing these lines after my defense in July, I have to start the acknowledgements with a very sad goodbye to my supervisor Prof Dr Bernd O. Stratmann. I am very grateful for having had the chance to have been one of Bernd's doctoral students and for the time we could spend together. Even though we had different opinions from time to time, I was already looking forward to having our first casual discussion after the dissertation and the thought of having no further chance of meeting again is very sorrowful. I want to thank Bernd in particular for pointing me in the direction of the topic of the last section of my thesis and I am sincerely obliged to him for getting me interested in the field of number theory, something that I did not expect at all when I started my PhD.

I want to offer my genuine thanks to my other supervisor Prof Dr Tobias Oertel-Jäger for his constant support and encouragement during the process of writing this thesis. His willingness to engage in a joint endeavor of tackling several mathematical questions that I had during my PhD is something that I really appreciate and I owe to Tobias a lot of the mathematical experience that I have gained in recent years. Furthermore, I want to thank Tobias sincerely for the very open, straightforward and thoughtful relationship that we had during the time of my doctoral studies.

My very special thanks go to Alejandro and Gabriel for their invaluable friendship and for all the unforgettable moments that we enjoyed together in the past years. I am deeply grateful that I was able to meet and get to know both of them during my PhD and I am hoping for many more nice memories that we can share in the future.

Special thanks also go to Anna for sharing the ups and down of a PhD with me as well as for all the discussions we had during our time in Bremen. Furthermore, I have to mention that I owe to Anna almost all of my knowledge about the hyperbolic world.

In the last four years, I was in the very lucky position to be a member of two very nice groups in Dresden and Bremen. In Dresden I could enjoy the company of Alejandro, Gabriel, Tobias and Vasso and in Bremen the company of Anna, Arne, Bernd, Christian, Fabian, Hendrik, Jacob, Johannes, Katie, Kurt, Malte K., Malte S., Marc, Sabrina, Sara, Tanja and Tony. I want to thank all of them for the very pleasant time that we spent together and the nice memories that were created during this period.

I want to thank Dr Sara Munday for her very valuable support in improving the last part of this thesis. Furthermore, many thanks go to Katie Lorenz for proofreading the English of the dissertation.

I would like to give a special thanks to Valeria and Alejandro's family for their very warm welcome in Montevideo as well as to Andrés, Frodo, Martín and Rafael for their hospitality during my stay there. Further, I want to thank Katrin and Lorenzo for taking care of me during my stay in Rio de Janeiro as well as Sofia and Rafael for two very nice weeks that we spent together in Rio. I am very grateful for the opportunity to meet so many nice people in Brazil & Uruguay and for the many enjoyable mathematical discussions I had during my stay there.

My deepest gratitude goes to all of my friends – old and new – and to my family – where the gratitude towards my parents is beyond my words.

CONTENTS

LIST OF FIGURES

BASIC NOMENCLATURE

$\mathbb{R}$	set of reals
$\mathbb{R}_+$	set of positive reals
$\mathbb{Q}$	set of rationals
$\mathbb{Z}$	set of integers
$\mathbb{N}_0$	set of non-negative integers
$\mathbb{N}$	set of positive integers

Part I

GETTING STARTED

1 INTRODUCTION

In the endeavor of gaining new insights in a specific field of mathematical research, the development of the corresponding abstract theory and the understanding of concrete examples go hand in hand. In fact, two motivating examples from the field of dynamical systems form the starting point for the present thesis. In what follows, a dynamical system is a continuous map $f : X \to X$ on a compact metric space (X, d).

One branch of dynamical systems is the study of the complexity inherent in the long-term behavior of a system and one possibility to advance in this task is to investigate dynamical invariants. Arguably, one of the most important topological invariants of a dynamical system is the notion of topological entropy. This notion was first introduced by Adler, Konheim and McAndrew in [AKM65]. Here, we give the definition which is due to Bowen [Bow71] and Dinaburg [Din71]. First, define the *Bowen-Dinaburg metrics* by

$$d_n(x, y) \ := \ \max_{i=0}^{n-1} d(f^i(x), f^i(y))$$

where $x, y \in X$. For $\delta > 0$ and $n \in \mathbb{N}$, a set $S \subseteq X$ is called (f, δ, n)-*separated* if $d_n(x, y) \geqslant \delta$ for all $x \neq y \in S$. Let $\widehat{S}(f, \delta, n)$ denote the maximal cardinality of an (f, δ, n)-separated set. Then, the *topological entropy of f* is defined as

$$h_{\text{top}}(f) \ := \ \sup_{\delta>0} \varlimsup_{n\to\infty} \frac{\log \widehat{S}(f, \delta, n)}{n} \ .$$

One possible interpretation of this notion is that a more involved dynamical behavior apparent in a system is reflected in a larger topological entropy of this system, especially its entropy should be non-zero. However, there are plenty of dynamical systems which have zero topological entropy but at the same time show a certain complex behavior and have interesting dynamical properties.

This brings us to the first of the two motivating examples of this thesis. In [GOPY84], Grebogi and his coworkers introduced the system $F_\kappa : \mathbb{T}^1 \times [0, 1] \to \mathbb{T}^1 \times [0, 1]$, given by

$$F_\kappa(\theta, x) \ = \ (\theta + \rho \bmod 1, \tanh(\kappa x) \cdot \sin(\pi\theta)) \qquad (1)$$

with $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$, $\rho \in \mathbb{R} \setminus \mathbb{Q}$ and real parameter $\kappa > 0$, as a simple model for the existence of a so-called strange non-chaotic attractor (SNA)[1]. The simplicity of (1) is already reflected in its structure: in

1 To be precise, the model studied by Grebogi *et al.* was a four-to-one extension of (1) with a slightly different parametrization.

the base $\mathbb{T}^1$, the dynamics are determined by a rigid rotation by an irrational angle ρ and the fibre maps only consist of strictly monotonically increasing functions. Another reason to consider this system simple is that its topological entropy is zero[2]. Nevertheless, (1) shows a very interesting dynamical behavior: for $\kappa \leqslant 2$ the attractor[3] of the system is just the zero line $\mathbb{T}^1 \times \{0\}$ but for $\kappa > 2$ the attractor is an intricate-looking curve, depicted in Figure 1. We refer to it as the strange non-chaotic attractor of (1).

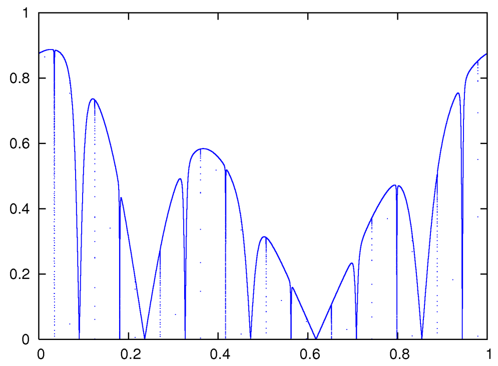


Figure 1: Strange non-chaotic attractor for (1) with $\kappa = 3$ and ρ the golden mean.

To gain further understanding of (1), we will study the structure of the SNA in more detail in Part II. In particular, we are interested in its dimensional properties. Broadly speaking, these measure the size of the SNA from a certain point of view and are able to reveal at least to some extent its complicated structure. Furthermore, since the topological entropy of (1) is zero, it is natural to ask whether there exists another topological invariant which can detect the transition to the strange non-chaotic attractor (for κ changing from $\kappa \leqslant 2$ to $\kappa > 2$) described further above. This will be the starting point for the investigation launched in Part III.

Another possible way of understanding the complex behavior of a dynamical system is to study natural parameter families containing the original system and to analyze how the complexity changes with the parameter when we approach the respective system. The paradigm example for this kind of method is the logistic family

$$f_\mu(x) \;:=\; \mu \cdot x(1-x)$$

on the unit interval $[0,1]$ with $\mu \in [0,4]$. It is well known that the map $\mu \mapsto h_{top}(f_\mu)$ varies continuously and monotonically from 0 to $\log 2$ [Dou95] as μ increases and that before the topological entropy starts

2 This can be deduced for example from [Bow71, Theorem 17].

3 Here, we mean by an attractor a forward invariant set that attracts Lebesgue almost every point in $\mathbb{T}^1 \times [0,1]$.

to grow, $(f_\mu)_{\mu\in[0,4]}$ undergoes so-called period doubling bifurcations. For a good exposition of this bifurcation process, see for example [Ott93, Section 2.2]. From these two properties we gain some insight of how the complexity inherent in $(f_\mu)_{\mu\in[0,4]}$ unfolds while μ is increasing until the full logistic map f_4 is reached.

We would like to apply the same procedure to the classical and well-known Farey map

$$F(x) := \begin{cases} \frac{x}{1-x} & \text{if } x \in [0, 1/2] \\ \frac{1-x}{x} & \text{if } x \in (1/2, 1] \end{cases} . \tag{2}$$

The Farey map has intimate relations to the topic of continued fractions in number theory and is also one of the paradigm examples in the area of infinite ergodic theory, for more information see for example [Iso11] and references therein. Analogous to the logistic family, the obvious approach would seemingly be to consider the parameter family $(F_\lambda)_{\lambda\in[0,1]}$ with $F_\lambda(x) := \lambda \cdot F(x)$. As it turns out, this family behaves quite differently than the classical logistic family: the map $\lambda \mapsto h_{top}(F_\lambda)$ changes drastically (it jumps at the critical parameter $\lambda = 1/2$ directly from 0 to $\log 2$) and there occur no period doubling bifurcations (we give more details in the next chapter).

This leads us to the second motivating example of the present thesis. By restricting F to the surviving sets

$$\mathcal{B}(t) = \bigcap_{n=0}^{\infty} F^{-n}\big([t, 1]\big)$$

for $t \in [0, 1]$, we may interpret the analysis in [CT11] as a study of an alternative parameter family containing the Farey map. Then, one particular result of [CT11] is that $(F|_{\mathcal{B}(t)})_{t\in[0,1]}$ undergoes a process which resembles period doubling bifurcations. Furthermore, from the results of [Rai94], we also have that $t \mapsto h_{top}(F|_{\mathcal{B}(t)})$ varies continuously (and monotonically, which follows directly from the fact that $\mathcal{B}(t) \subseteq \mathcal{B}(t')$ for $t' \leqslant t$). That means the parameter family $(F|_{\mathcal{B}(t)})_{t\in[0,1]}$ behaves much more in accordance with the logistic family. The goal of Part IV of this thesis is to start the process of extending the results of [CT11] and some of the findings in [CT12] to more general interval maps. Thereby, we will focus on generalizing the topological results.

2 MAIN RESULTS

In this chapter, we state the main results of the present thesis. Note that certain preliminary notions, needed for stating some of the assertions, are given in Chapter 4. Readers familiar with those basic dynamical notions can directly proceed and consult Chapter 4 only if needed. Throughout the thesis, we usually state definitions in the normal text and highlight the respective names in a *cursive format*. Only definitions that have a novel character and are specific to this thesis will be highlighted in the same way as propositions, lemmas, theorems, corollaries and special remarks.

2.1 DIMENSIONS OF STRANGE NON-CHAOTIC ATTRACTORS

In this part of the thesis, we study the motivating example (1) from the introduction

$$F_\kappa(\theta, x) = (\theta + \rho \bmod 1, \tanh(\kappa x) \cdot \sin(\pi\theta)) ,$$

where $\rho \in \mathbb{R}\backslash\mathbb{Q}$ and $\kappa > 0$. As already mentioned, this family of maps was introduced by Grebogi and his coworkers in [GOPY84] as a simple model for the existence of a so-called strange non-chaotic attractor (SNA). Later, the term pinched skew products was coined by Glendinning [Gle02] for a general class of systems sharing some essential properties with the motivating example (1). Note that in the following $\mathrm{Leb}_{\mathbb{T}^1}$ and $\mathrm{Leb}_{\mathbb{T}^1\times[0,1]}$ refer to the corresponding Lebesgue measure on $\mathbb{T}^1$ and $\mathbb{T}^1 \times [0, 1]$, respectively.

We call the *upper bounding graph* φ^+ of the *global attractor* $\mathcal{A} := \bigcap_{n\in\mathbb{N}} F_\kappa^n(\mathbb{T}^1 \times [0, 1])$, which is given by

$$\varphi^+(\theta) := \sup\{x \in [0, 1] \mid (\theta, x) \in \mathcal{A}\} ,$$

an SNA without further specifying this notion in this paragraph (cf. Chapter 5 for the precise definition). Due to the monotonicity of the fibre maps $F_{\kappa,\theta} : x \mapsto \tanh(\kappa x) \cdot \sin(\pi\theta)$, one can verify that the function φ^+ satisfies

$$F_{\kappa,\theta}(\varphi^+(\theta)) = \varphi^+(\theta + \rho \bmod 1) .$$

Consequently, the corresponding point set $\Phi^+ := \{(\theta, \varphi^+(\theta)) \mid \theta \in \mathbb{T}^1\}$ is (forward) invariant under F_κ. Slightly abusing terminology, we will call both φ^+ and Φ^+ an *invariant graph*. Keller showed in [Kel96] that for $\kappa > 2$ in (1) the graph φ^+ is $\mathrm{Leb}_{\mathbb{T}^1}$-almost surely strictly positive, its *Lyapunov exponent*

$$\lambda(\varphi^+) := \int \log F'_{\kappa,\theta}(\varphi^+(\theta))\, d\theta$$

is strictly negative and φ^+ attracts $\mathrm{Leb}_{\mathbb{T}^1\times[0,1]}$-a.e. initial condition. Note that Birkhoff's Ergodic Theorem implies that

$$\lim_{n\to\infty}\frac{1}{n}\log\left(F^n_{\kappa,\theta}\right)'(\varphi^+(\theta)) \;=\; \lambda(\varphi^+)$$

for $\mathrm{Leb}_{\mathbb{T}^1}$-a.e. $\theta\in\mathbb{T}^1$ where $F^n_{\kappa,\theta}=F_{\kappa,\theta+(n-1)\rho \bmod 1}\circ\ldots\circ F_{\kappa,\theta}$.

The findings in [GOPY84] attracted substantial interest in the theoretical physics community, and subsequently a large number of numerical studies confirmed the widespread existence of SNA's in quasi-periodically forced systems and explored their behavior and properties (see [PNR01, HP06, Jä09] for an overview and further references). For a long time, however, rigorous results remained rare, and even basic questions are still open nowadays. In particular, this concerns the dimensions and fractal properties of SNA's. A numerical investigation was carried out in [DGO89], and the results indicated that the box-counting dimension of the attractor is two, whereas the information dimension should be one. For sufficiently large κ, the conjecture on the box-counting dimension was verified indirectly in [Jä07], by showing that the topological closure of Φ^+ is equal to the global attractor $\mathcal{A}=\{(\theta,x)\mid 0\leqslant x\leqslant\varphi^+(\theta)\}$ and therefore has positive two-dimensional Lebesgue measure.

Our aim is to determine further dimensions of φ^+ and the associated invariant measure μ_{φ^+} which is obtained by projecting the Lebesgue measure on the base $\mathbb{T}^1$ onto Φ^+. In all of the following assertions, we need that the rotation vector ρ is Diophantine which essentially means that points in the base $\mathbb{T}^1$ do not come back to themselves too quickly under the iteration of the rigid rotation (the precise definition is given in (12) in Section 7.2). For the Hausdorff dimension D_H (see Section 6.1 for the definition) we have

Theorem 2.1. *Suppose ρ is Diophantine and κ is sufficiently large in (1). Then $D_H(\Phi^+)=1$. Furthermore, the one-dimensional Hausdorff measure of Φ^+ is infinite.*

This statement as well as the following ones are special cases of Corollary 7.14, see Section 7.3. Here and in the results below, the largeness condition of κ depends on the constants of the Diophantine condition on ρ.

Remark 2.2. Our results in Section 7.3 also allow us to treat examples with a higher dimensional driving space as given in Example 7.1. In these cases the rotation on $\mathbb{T}^1$ is replaced by a rotation on $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ and we obtain that the Hausdorff dimension of Φ^+ is d. However, at least for sufficiently large d the d-dimensional Hausdorff measure is finite, in contrast to the case $d=1$ (Proposition 7.11). We conjecture that for these examples the d-dimensional Hausdorff measure is infinite only for $d=1$ and finite for all $d\geqslant 2$.

In order to obtain information on the invariant measure μ_{φ^+}, we determine its pointwise dimension

$$d_{\mu_{\varphi^+}}(\theta,x) \;=\; \lim_{\varepsilon\to 0} \frac{\log \mu_{\varphi^+}(B_\varepsilon(\theta,x))}{\log \varepsilon}\;.$$

A priori, it is not clear whether this limit exists such that in general one defines the upper and lower pointwise dimension by taking the limit superior and inferior, respectively (see Section 6.1). Furthermore, even if the limit exists, it may depend on (θ,x). If the pointwise dimension exists and is constant almost surely, the invariant measure is called *exact dimensional*. It turns out that this is the case in the situation considered here. In fact, we obtain the stronger result that μ_{φ^+} is a rectifiabile measure, see Section 6.2 and Theorem 7.13, and this directly implies

Theorem 2.3. *Suppose ρ is Diophantine and κ is sufficiently large in (1). Then for μ_{φ^+}-almost every $(\theta,x) \in \mathbb{T}^1 \times [0,1]$ we have $d_{\mu_{\varphi^+}}(\theta,x) = 1$. In particular, μ_{φ^+} is exact dimensional.*

For an exact dimensional measure μ it is known that the information dimension D_1 (see again Section 6.1 for the definition) coincides with the pointwise dimension. Hence, we obtain

Corollary 2.4. *Suppose ρ is Diophantine and κ is sufficiently large in (1). Then $D_1(\mu_{\varphi^+}) = 1$.*

This confirms the conjecture made in [DGO89]. Since the geometric mechanism for the creation of SNA's in pinched skew products is quite universal and can be found in a similar form in other types of systems, we expect our results to hold in further situations. For instance, this holds true for the SNA found in the Harper map, which describes the projective action of quasiperiodic Schrödinger cocycles, and it should further hold true for SNA's in the quasiperiodically forced version of the Arnold circle map. For more information on these maps see e.g. [HP06] and [Jä09], respectively, and references therein. In Section 3.1, a first outlook in this direction will be given.

Our proof hinges on the fact that the SNA φ^+ can be approximated by the iterates of the upper bounding line $\mathbb{T}^1 \times \{1\}$ of the phase space, whose geometry can be controlled quite accurately. This observation has already been used in [Jä07] and will be further utilized here. An outline of the strategy is given in Section 7.1. In Section 7.2 we derive the required estimates on the approximating curves, which are used to compute the Hausdorff dimension and the pointwise dimension in Section 7.3.

2.2 AMORPHIC COMPLEXITY

As we already pointed out in the introduction, an essential motivation for this part of the dissertation is the question whether there exists a topological invariant which can distinguish pinched skew product systems, like (1), with and without a strange non-chaotic attractor. However, here we will take a broader point of view by studying the very onset of dynamical complexity and the break of equicontinuity in the regime of zero entropy systems. Thereby, we are looking for a dynamically defined positive real-valued quantity which

(a) is an invariant of topological conjugacy (and has other good properties);
(b) gives value zero to isometries and Morse-Smale systems;
(c) is able to detect, as test cases, the complexity inherent in the dynamics of Sturmian subshifts or Denjoy homeomorphisms on the circle, by taking positive values for such systems.

Nevertheless, we want to stress that the original motivating question paves the way for all considerations in this part of the thesis. We refer the reader to Section 3.2 where an outlook for the application of amorphic complexity in the context of pinched skew product systems is given.

There exist several concepts to describe the complexity of systems in the zero entropy regime (see, for example, [Mis81, Smí86, MS88, KS91, Car97, Fer97, KT97, Fer99, BHM00, HK02, FP07, HPY07, HY09, CL10, DHP11, Mar13, KC14]). Some of them have properties that may be considered as shortcomings, although this partly depends on the viewpoint and the particular purpose one has in mind. To be more precise, let us consider one example of a standard approach to measure the complexity of zero entropy systems, namely, the (modified) power entropy (see Section 3.3 and [HK02]). In the context of tiling spaces and minimal symbolic subshifts, power entropy is more commonly known as polynomial word complexity and presents a well-established tool to describe the complexity of aperiodic sequences. However, it turns out that power entropy gives positive values to Morse-Smale systems, whereas modified power entropy is too coarse to distinguish Sturmian subshifts or Denjoy examples from irrational rotations.

We are thus taking an alternative and complementary direction, which leads us to define the notions of asymptotic separation numbers and amorphic complexity. Those are based on an asymptotic notion of separation, which is the main qualitative difference to the previous two concepts, since the latter rely in their definition on the classical Bowen-Dinaburg/Hamming metrics which consider only finite time-scales. As a consequence, ergodic theorems can be applied in a more or less direct way to compute or estimate amorphic complexity in many situations. In order to fix ideas, we concentrate on the dynamics of continuous maps defined on metric spaces.

Definition 2.5. Let (X, d) be a metric space and $f : X \to X$. Given $x, y \in X$, $\delta > 0$, $\nu \in (0, 1]$ and $n \in \mathbb{N}$, we let

$$S_n(f, \delta, x, y) := \#\left\{0 \leqslant k < n \mid d(f^k(x), f^k(y)) \geqslant \delta\right\}.$$

We say that x and y are (f, δ, ν)-*separated* if

$$\varlimsup_{n\to\infty} \frac{S_n(f, \delta, x, y)}{n} \geqslant \nu .$$

A subset $S \subseteq X$ is said to be (f, δ, ν)-*separated* if all $x, y \in S$ with $x \neq y$ are (f, δ, ν)-separated. The *(asymptotic) separation number* $\mathrm{Sep}(f, \delta, \nu)$, for distance $\delta > 0$ and frequency $\nu \in (0, 1]$, is then defined as the largest cardinality of an (f, δ, ν)-separated set in X. If these quantities are finite for all $\delta, \nu > 0$, we say f has *finite separation numbers*, otherwise we say it has *infinite separation numbers*. Further, if $\mathrm{Sep}(f, \delta, \nu)$ is uniformly bounded in ν for all $\delta > 0$, we say that f has *bounded separation numbers*, otherwise we say *separation numbers are unbounded*.

These notions provide a first qualitative indication concerning the complexity of a system. Roughly spoken, finite but unbounded separation numbers correspond to dynamics of intermediate complexity, which are our main focus here. Once a system behaves 'chaotically', in the sense of positive topological entropy or weak mixing, separation numbers become infinite.

Theorem 2.6. *Suppose X is a compact metric space and $f : X \to X$ is continuous. If f has positive topological entropy or is weakly mixing with respect to some invariant probability measure μ with non-trivial support, then it has infinite separation numbers.*

The proof is given in Chapter 8. Obviously, if f is an isometry or, more generally, equicontinuous, then its separation numbers are bounded (see also the short discussion after Proposition 2.9 below). Moving away from equicontinuity one encounters the class of almost automorphic systems, which are central objects of study in topological dynamics and include many examples of both theoretical and practical importance. At least in the minimal case, separation numbers are suited to describe this transition, as the next result shows. Due to Veech's Structure Theorem [Vee65], minimal almost automorphic systems can be defined as minimal almost 1-1 extensions of equicontinuous systems. For their definition, cf. Section 4.4, and for more information on almost automorphic systems, see for example [Vee65, Aus88, AGN14] and references therein.

Theorem 2.7. *Suppose X is a compact metric space and $f : X \to X$ is a homeomorphism.*

(i) If f is minimal and almost automorphic but not equicontinuous, then f has unbounded separation numbers.

(ii) If f *is an almost sure 1-1 extension of an equicontinuous system, then* f *has finite separation numbers.*

Again, the proof is given in Chapter 8. Examples for case (ii) are Sturmian subshifts, Denjoy examples on the circle and regular Toeplitz flows which are discussed further below.

In order to obtain quantitative information, we proceed to study the scaling behavior of separation numbers as the separation frequency ν goes to zero. In principle, one may consider arbitrary growth rates (see Section 9.1). However, as all the examples we discuss indicate, it is polynomial growth which is the most relevant.

Definition 2.8. Given $\delta > 0$, we let

$$\underline{\mathrm{ac}}(f,\delta) := \varliminf_{\nu\to 0} \frac{\log \mathrm{Sep}(f,\delta,\nu)}{-\log \nu}\ , \quad \overline{\mathrm{ac}}(f,\delta) := \varlimsup_{\nu\to 0} \frac{\log \mathrm{Sep}(f,\delta,\nu)}{-\log \nu}$$

and define the *lower* and *upper amorphic complexity of* f as

$$\underline{\mathrm{ac}}(f) := \sup_{\delta>0} \underline{\mathrm{ac}}(f,\delta) \quad \text{and} \quad \overline{\mathrm{ac}}(f) := \sup_{\delta>0} \overline{\mathrm{ac}}(f,\delta)\ ,$$

respectively. If both values coincide, $\mathrm{ac}(f) := \underline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}(f)$ is called the *amorphic complexity of* f.

We note once more that the main difference to the notion of (modified) power entropy is the fact that we use an asymptotic concept of separation, and the scaling behavior that is measured is not the one with respect to time but that with respect to the separation frequency. Somewhat surprisingly, this makes amorphic complexity quite well-accessible to rigorous computations and estimates. The reason is that separation frequencies often correspond to certain ergodic averages or visiting frequencies, which can be determined by the application of ergodic theorems. We have the following basic properties.

Proposition 2.9. *Suppose* X, Y *are compact metric spaces and* $f : X \to X,\ g : Y \to Y$ *are continuous. Then the following statements hold.*

(i) Factor relation: *If* g *is a factor of* f*, then* $\overline{\mathrm{ac}}(f) \geqslant \overline{\mathrm{ac}}(g)$ *and* $\underline{\mathrm{ac}}(f) \geqslant \underline{\mathrm{ac}}(g)$*. In particular, amorphic complexity is an invariant of topological conjugacy.*

(ii) Power invariance: *For all* $m \in \mathbb{N}$ *we have* $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$ *and* $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$.

(iii) Product formula: *If upper and lower amorphic complexity coincide for both* f *and* g*, then the same holds for* $f \times g$ *and we have* $\mathrm{ac}(f \times g) = \mathrm{ac}(f) + \mathrm{ac}(g)$*. Otherwise, we have* $\overline{\mathrm{ac}}(f \times g) \leqslant \overline{\mathrm{ac}}(f) + \overline{\mathrm{ac}}(g)$ *and* $\underline{\mathrm{ac}}(f \times g) \geqslant \underline{\mathrm{ac}}(f) + \underline{\mathrm{ac}}(g)$.

(iv) Commutation invariance: $\overline{\mathrm{ac}}(f \circ g) = \overline{\mathrm{ac}}(g \circ f)$ *and* $\underline{\mathrm{ac}}(f \circ g) = \underline{\mathrm{ac}}(g \circ f)$.

The proofs of the stated assertions can be found in Chapter 9. The last proposition shows that requirement (a) from the beginning is fulfilled by amorphic complexity. With respect to requirement (b) we have the following: amorphic complexity is zero for all isometries $f : X \to X$ because in this case separation numbers $\mathrm{Sep}(f, \delta, \nu)$ do not depend on ν. Similarly, amorphic complexity is zero for Morse-Smale systems. Here, we call a continuous map f on a compact metric space X *Morse-Smale* if its non-wandering set $\Omega(f)$ is finite. This implies that $\Omega(f)$ consists of a finite number of fixed or periodic orbits, and for any $x \in X$ there exists $y \in \Omega(f)$ with $\lim_{n\to\infty} f^{np}(x) = y$ where p is the period of y. Since orbits converging to the same periodic orbit cannot be (f, δ, ν)-separated, we obtain $\mathrm{Sep}(f, \delta, \nu) \leqslant \#\Omega(f)$ for all $\delta, \nu > 0$. Hence, separation numbers are even bounded uniformly in δ and ν. Altogether, this means requirement (b) is also fulfilled by amorphic complexity.

Concerning requirement (c), we have the following statement where the proof is given in Chapter 11.

Proposition 2.10. *Amorphic complexity equals one for Sturmian subshifts and Denjoy examples on the circle.*

The arguments in the proof of Theorem 2.7 (ii) can be quantified, at least to some extent, to obtain an upper bound on amorphic complexity for minimal almost sure 1-1 extensions of isometries. In rough terms, the result reads as follows. Details will be given in Chapter 10. By $\overline{D}_B(A)$ we denote the upper box-counting dimension of a totally bounded subset A of a metric space, see Section 4.6.

Theorem 2.11. *Suppose X and Ξ are compact metric spaces and $f : X \to X$ is an almost sure 1-1 extension of a minimal isometry $g : \Xi \to \Xi$ with factor map h. Further, assume that the upper box-counting dimension of Ξ is finite and strictly positive. Then*

$$\overline{\mathrm{ac}}(f) \leqslant \frac{\gamma(h) \cdot \overline{D}_B(\Xi)}{\overline{D}_B(\Xi) - \sup_{\delta>0} \overline{D}_B(E_\delta)}, \tag{3}$$

where $E_\delta = \{\xi \in \Xi \mid \mathrm{diam}(h^{-1}(\xi)) \geqslant \delta\}$ and $\gamma(h)$ is a scaling factor depending on the local properties of the factor map h.

The proof is given in Chapter 10. It should be mentioned, at least according to our current understanding, that this result is of rather abstract nature. The reason is the fact that the scaling factor $\gamma(h)$, defined in (51), seems to be difficult to determine in concrete examples. However, as Proposition 2.10 and the next theorem demonstrate, for specific families of maps more direct methods can be used to obtain improved explicit estimates.

Finally, we will investigate so-called regular Toeplitz flows in Chapter 12. Given a finite alphabet A, a sequence $\omega = (\omega_k)_{k\in\mathbb{I}} \in A^{\mathbb{I}}$ with $\mathbb{I} = \mathbb{N}_0$ or $\mathbb{Z}$ is called *Toeplitz* if for all $k \in \mathbb{I}$ there exists $p \in \mathbb{N}$

such that $\omega_{k+p\ell} = \omega_k$ for all $\ell \in \mathbb{N}$. In other words, every symbol in a Toeplitz sequence occurs periodically. Thus, if we let $\mathrm{Per}(p,\omega) = \{k \in \mathbb{I} \mid \omega_{k+p\ell} = \omega_k \text{ for all } \ell \in \mathbb{N}\}$, then $\bigcup_{p\in\mathbb{N}} \mathrm{Per}(p,\omega) = \mathbb{I}$. By $D(p) = \#(\mathrm{Per}(p,\omega) \cap [0, p-1])/p$, we denote the density of the p-periodic positions. If $\lim_{p\to\infty} D(p) = 1$, then the Toeplitz sequence is called *regular*. A well-known example of a regular Toeplitz sequence is the paperfolding sequence, also known as the dragon curve sequence [AB92].

We call a sequence $(p_\ell)_{\ell\in\mathbb{N}}$ of integers such that $p_{\ell+1}$ is a multiple of p_ℓ for all $\ell \in \mathbb{N}$ and $\bigcup_{\ell\in\mathbb{N}} \mathrm{Per}(p_\ell,\omega) = \mathbb{I}$ a *weak periodic structure* for ω. More details are given in Chapter 12. We denote the shift orbit closure of ω by Σ_ω such that (Σ_ω, σ) is the subshift generated by ω.

Theorem 2.12. *Suppose ω is a non-periodic regular Toeplitz sequence with weak periodic structure $(p_\ell)_{\ell\in\mathbb{N}}$. Then*

$$\overline{\mathrm{ac}}\left(\sigma|_{\Sigma_\omega}\right) \leqslant \varlimsup_{\ell\to\infty} \frac{\log p_{\ell+1}}{-\log(1 - D(p_\ell))} \, .$$

In Chapter 12, we further demonstrate by means of examples that this estimate is sharp and that a dense set of values in $[1,\infty)$ is attained (Theorem 12.6 and Corollary 12.7).

2.3 BIFURCATIONS OF FAMILIES OF BOUNDED ORBITS

First, let us recall that every irrational number x in $[0,1]$ has a unique continued fraction expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} =: [a_1, a_2, a_3, \dots] \, ,$$

where $a_n \in \mathbb{N}$. Further, each rational number $x \in (0,1)$ has two finite continued fraction expansions, $x = [a_1, \dots, a_k] = [a_1, \dots, a_k - 1, 1]$ where $a_k \geqslant 2$. For a good exposition of continued fractions, see for example [Khi64].

A classical object of interest in the theory of continued fractions is the family of sets $(\mathcal{B}_N)_{N\in\mathbb{N}}$,

$$\mathcal{B}_N := \left\{[a_1, a_2, \dots] \in [0,1] \mid a_n \leqslant N \text{ for all } n \in \mathbb{N}\right\} ,$$

that is, the sets of irrational numbers such that the elements of their continued fraction expansions are uniformly bounded by N, see for example [Hen06, Section 9.1] and references therein. In [CT11], the authors suggest to study a generalization of this family given by

$$\mathcal{B}(t) := \{x \in [0,1] \mid F^n(x) \geqslant t \text{ for all } n \in \mathbb{N}_0\}$$

where $t \in [0,1]$ and F is the Farey map defined in (2) in the introduction[1]. It is almost immediately clear that $\mathcal{B}(1/N) = \mathcal{B}_{N-1}$ for $N > 1$. Furthermore, as already mentioned in Chapter 1, it is not difficult to see that $\mathcal{B}(t)$ equals the *surviving set* of points that never hit the interval $[0,t)$ under the dynamics of F for each $t \in [0,1]$. Accordingly, the sets $\mathcal{B}(t)$ are closed and forward invariant under F.

As it turns out, the map $t \mapsto \mathcal{B}(t)$ is locally constant for a large set of parameters $t \in [0,1]$ and the relevant set where $t \mapsto \mathcal{B}(t)$ changes is the *bifurcation set*

$$\mathcal{E} := \{x \in [0,1] \mid x \in \mathcal{B}(x)\}.$$

In [CT11], several properties of the bifurcation set $\mathcal{E}$ and its relation to the family of sets $(\mathcal{B}(t))_{t\in[0,1]}$ are investigated. Since the definitions of these two are not necessarily restricted to the Farey map, it seems natural to ask whether some of the obtained results can be extended to more general interval maps. Indeed, this will be the case, whereby we focus on generalizing the topological statements. We want to emphasize that all the considerations made here are also guided by the classical results of [Urb86].

In what follows, we study the sets $\mathcal{B}(t)$, $t \in [0,1]$ and $\mathcal{E}$ with respect to general continuous maps $f : [0,1] \to [0,1]$. Let us point out that, from a general perspective, when considering the set $\{x \in [0,1] \mid f^n(x) \geqslant t \text{ for all } n \in \mathbb{N}_0\}$ it is quite natural to think of the analog problem, i.e. to consider the set $\{x \in [0,1] \mid f^n(x) \leqslant t \text{ for all } n \in \mathbb{N}_0\}$ as will be done in Part IV. There are also natural situations for studying this kind of analog problem, see for example [BCIT13] and references therein.

In a first step, we want to generalize the description of the connected components of the complement of $\mathcal{E}$ obtained in the case of the Farey map in [CT11, CT12]. To state their result we need the following notions: for a rational number $r = [a_1, \ldots, a_k]$, $a_k \geqslant 2$ in $(0,1)$, we denote the open interval whose endpoints are the quadratic surds $[\overline{a_1, \ldots, a_k}]$ and $[\overline{a_1, \ldots, a_k - 1, 1}]$ by I_r and call it a *quadratic interval* (further, set $I_1 := ((\sqrt{5}-1)/2, 1])$. Moreover, we say I_r is *maximal* if I_r is not contained in any other quadratic interval. It is shown in [CT12] that two maximal quadratic intervals do not intersect and that every quadratic interval is contained in a unique maximal one. Now, we have that

$$[0,1]\backslash\mathcal{E} = \dot{\bigcup_{\substack{r\in\mathbb{Q}\cap(0,1] \\ I_r \text{ is maximal}}}} I_r \tag{4}$$

and further that $t \mapsto \mathcal{B}(t)$ is constant on quadratic intervals.

1 Originally, in the definition of $\mathcal{B}(t)$ in [CT11], the Gauß map is used instead of the Farey map F, however, it is not difficult to show that the two definitions coincide (see also the proof of [CT11, Lemma 1]).

For a general continuous map f on the unit interval, a natural replacement for the quadratic intervals from a dynamical point of view are the following sets.

Definition 2.13. Suppose $m \in \mathbb{N}$. An open interval $I^m \subset [0,1]$ with $f^m(x) < x$ for all $x \in I^m$ is called a *(lower) m-interval for f* if there exists no open interval $J \subset [0,1]$ such that I^m is strictly contained in J and $f^m(x) < x$ for all $x \in J$. Further, we say I^m is a *(lower) interval of order m* if I^m is a (lower) m-interval and there exists no other (lower) $\tilde{m}$-interval $I^{\tilde{m}}$ with $\tilde{m} \in \mathbb{N}$ such that $I^{\tilde{m}} = I^m$ and $\tilde{m} < m$.

The next statement is a first positive indication that this is a suitable choice. The proof is given in Chapter 13.

Lemma 2.14. *Suppose $I^m \subset [0,1]$ is an m-interval. For $t, s \in I^m$, we have that $\mathcal{B}(t) = \mathcal{B}(s)$.*

Define the same notion of maximality for m-intervals as for quadratic intervals.

Theorem 2.15. *Let $\alpha \in \mathcal{E}$. Suppose that $\beta \in \mathcal{E}$ (or $\beta = 1$) such that $\alpha < \beta$ and no other point in (α, β) belongs to $\mathcal{E}$. Then (α, β) ($(\alpha, 1]$) is a maximal m-interval for some $m \in \mathbb{N}$.*

The theorem is proved in Chapter 14. Using that $\mathcal{E}$ is closed, we can almost immediately conclude that

$$[0,1] \backslash \mathcal{E} = \dot{\bigcup_{\substack{m \in \mathbb{N} \\ I \in \mathcal{I}^m}}} I \tag{5}$$

where $\mathcal{I}^m$ is the collection of all maximal intervals of order m for each $m \in \mathbb{N}$. Relation (5) implies several corollaries. For instance, it yields – analogous to the quadratic intervals – that two maximal intervals of order m and $\tilde{m}$, respectively, are disjoint and that each m-interval is contained in a unique maximal interval of order $\tilde{m}$. Moreover, in the case of the Farey map, we can deduce from (4) and (5) that all the maximal quadratic intervals and all the maximal (lower) intervals of order $m \in \mathbb{N}$ are in one-to-one correspondence.

In the case of the Farey map, it is proven in [CT12] that $\mathcal{E}$ has zero Lebesgue measure but full Hausdorff dimension (for the definition of the latter, see Chapter 6). Here, we focus on the cardinality of $\mathcal{E}$ for general transitive continuous maps. For the notion of transitivity, cf. Section 4.1, and for the definition of piecewise monotone maps on the unit interval, see Chapter 14.

Theorem 2.16. *Suppose $f : [0,1] \to [0,1]$ is a transitive continuous map. We have that the bifurcation set $\mathcal{E}$ is nowhere dense and infinite. Furthermore, if f is piecewise monotone, then $\mathcal{E}$ is uncountable.*

As an immediate consequence, we obtain the following statement, where a *Cantor set* is a perfect (closed, no isolated points) and nowhere dense subset of $[0,1]$.

Corollary 2.17. *If f is a transitive, continuous and piecewise monotone map, then the bifurcation set without its isolated points is a Cantor set.*

The proofs can be found in Chapter 14. In the proof of the second part of Theorem 2.16, we will make use of the very general results of [Rai94]. In particular, [Rai94] implies that the map $t \mapsto h_{top}(f|_{\mathcal{B}(t)})$ is continuous (cf. Chapter 1 for the definition of the top. entropy h_{top}).

In the last chapter, we state some results concerning similarities and relations between the surviving sets and the bifurcation set. In particular, we show that for each $t \in \mathcal{E}$ the connected components of the complement of the surviving set $\mathcal{B}(t)$ can be described in a similar way as in (5) and we prove the following assertion (for the definition of piecewise monotone maps with full branches, see Chapter 15).

Theorem 2.18. *Let f be a transitive, continuous and piecewise monotone map with full branches and suppose $t \in \mathcal{E}$. Then t is isolated in $\mathcal{E}$ if and only if $\mathcal{B}(t)$ contains an isolated point.*

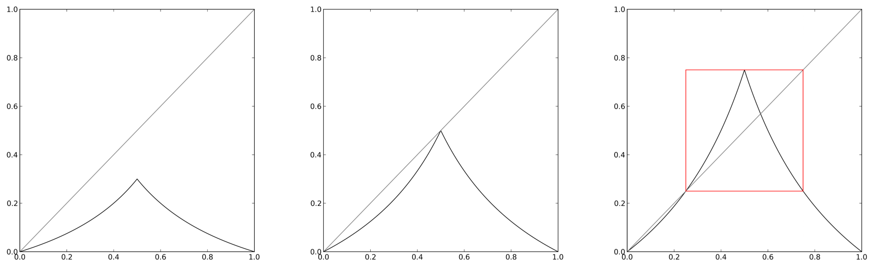
The presence of isolated points is one of the main differences to the corresponding results in [Urb86]. Heuristically speaking, the reason for this deviance is that the maps considered in the last theorem contain orientation-reversing branches, whereas in [Urb86] only orientation-preserving expanding maps on the circle are allowed. The last theorem also has the following interpretation.

Corollary 2.19. *Assume f is a transitive, continuous and piecewise monotone map with full branches and let $t \in \mathcal{E}\backslash\{0, 1\}$. We have that t is a limit point of $\mathcal{E}$ if and only if $\mathcal{B}(t)$ is a Cantor set.*

In Section 3.5, we give a short outlook for further possible directions one can pursue concerning the properties of the surviving and bifurcation sets.

Remark 2.20. As promised in the introduction, we want to explain here in more detail why the parameter family $(F_\lambda)_{\lambda\in[0,1]}$ with $F_\lambda(x) = \lambda \cdot F(x)$ behaves quite differently than the classic logistic family. For $\lambda \in [0, 1/2)$, it is not difficult to see that all points in $[0, 1]$ converge under the dynamics of F_λ to the attracting fixed point 0, and for the critical parameter $\lambda = 1/2$ we get an additional repelling fixed point at $1/2$, cf. Figure 2 (a) and (b). This means that for $\lambda \in [0, 1/2]$ the topological entropy of F_λ is zero[2]. Further, for $\lambda \in (1/2, 1]$ observe that $1-\lambda$ is a fixed point of F_λ and $F_\lambda(\lambda) = 1-\lambda$. Hence, the interval $I_\lambda := [1-\lambda, \lambda]$ is invariant under F_λ (and all points outside of I_λ converge to 0), see Figure 2 (c). We have that the dynamics of F_λ

2 This can be seen by using that $h_{top}(F_\lambda) = h_{top}(F_\lambda|_{\Omega(F_\lambda)})$ where $\Omega(F_\lambda)$ is the non-wandering set of F_λ (which equals $\{0\}$ for $\lambda \in [0, 1/2)$ and $\{0, 1/2\}$ for $\lambda = 1/2$).



(a) (b) (c)

Figure 2: F_λ sketched for (a) $\lambda \in (0, 1/2)$, (b) the critical value $\lambda = 1/2$ and (c) $\lambda \in (1/2, 1)$. With respect to the last case, the relevant subsystem contained in F_λ can be seen in the red-rimmed box.

restricted to I_λ for $\lambda \in (1/2, 1]$ are directly related to the dynamics of the system

$$\hat{F}_\lambda(x) := \begin{cases} \frac{x}{\lambda-(2\lambda-1)x} & \text{if } x \in [0, 1/2] \\ \frac{1-x}{\lambda-(2\lambda-1)(1-x)} & \text{if } x \in (1/2, 1] \end{cases} \tag{6}$$

defined on the unit interval[3]. Observe that $\hat{F}_\lambda$ coincides with the tent map for $\lambda = 1/2$ and the Farey map for $\lambda = 1$. That means (6) interpolates between a uniformly expanding and intermittent map (as the Farey map has an indifferent fixed point at zero). Exactly this kind of scenario was studied in [GI05] and further extended in [EIK07] (in fact, a simple reparametrization of (6) yields the systems studied in these articles). Using the last two references, it follows in particular that $h_{top}(F_\lambda) = \log 2$ for each $\lambda \in (1/2, 1]$. Taken all together, this shows that the map $\lambda \mapsto h_{top}(F_\lambda)$ changes drastically (it jumps at the critical parameter $\lambda = 1/2$ directly from 0 to $\log 2$) and that $(F_\lambda)_{\lambda\in[0,1]}$ undergoes no period doubling bifurcation. Let us emphasize that examples of families of maps where the topological entropy behaves discontinuously are well known, see for example [MS80, Mis89, Mis01]. However, to the best of our knowledge, the particular example $(F_\lambda)_{\lambda\in[0,1]}$ involving the Farey map together with the absence of any periodic doubling bifurcation and the relation to [GI05, EIK07] have not been pointed out in the literature so far. Finally, we want to mention that one can still try to study finer properties of F_λ for $\lambda \in (1/2, 1]$, using for example the techniques from [JKPS09] and [JMS].

3 Namely, $g_\lambda \circ \hat{F}_\lambda = F_\lambda \circ g_\lambda$ on I_λ where $g_\lambda : [0, 1] \to I_\lambda : x \mapsto (2\lambda - 1)x + 1 - \lambda$.

3 OUTLOOK

3.1 Dimensions of strange non-chaotic attractors in non-smooth saddle-node bifurcations

As mentioned at the beginning of Section 2.1, the motivating example (1) from the introduction belongs to the more general class of pinched skew product systems. These systems are characterized by the fact that for some point in the base the fibre over this point is mapped to a single point (see also Chapter 5). This property greatly simplifies their analysis. However, at the same time it gives these systems a certain toy model character (since they are not invertible) and they can therefore not be the time-one maps of flows, which are of particular interest from the applied point of view. A more realistic scenario for the creation of SNA's are so-called *non-smooth saddle-node bifurcations*. There the SNA originates from the collision of two initially continuous invariant curves. In the following, instead of explaining this pattern in general, we will give a concrete example of a family of maps which fits into this general scheme and also state some very recently obtained results from [FGJ14] just for this specific family. These new results generalize some of the assertions obtained in this thesis, where some inspiration for their proof is drawn, at least on a heuristic level, from the strategy applied in our setting. However, we want to stress that on a technical level these new results are much more demanding and heavily rely on the multiscale analysis developed in [Fuh14]. More information can be found in [FGJ14].

The family of maps $(f_\beta)_{\beta\in[0,1]}$ that we want to consider is defined by $f_\beta : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ with

$$f_\beta(\theta, x) := (\theta + \rho \bmod 1, \arctan(\kappa x) - \beta(1 + \cos(2\pi\theta))) ,$$

where $\rho \in \mathbb{R}\backslash\mathbb{Q}$ and $\kappa > 0$. Provided ρ is Diophantine and κ sufficiently large, it can be shown that this family undergoes a non-smooth saddle-node bifurcation. This means there exists a critical parameter $\beta_c \in (0, 1)$ such that

(i) If $\beta < \beta_c$, then f_β has exactly two continuous invariant graphs in $\mathbb{T}^1 \times [0, \infty)$.

(ii) If $\beta > \beta_c$, then f_β has no invariant graphs in $\mathbb{T}^1 \times [0, \infty)$.

(iii) If $\beta = \beta_c$, then f_β has a strange non-chaotic attractor $\varphi^+_{\beta_c}$ (and also a strange non-chaotic repeller) in $\mathbb{T}^1 \times [0, \infty)$.

Now, recall that $\Phi^+_{\beta_c}$ denotes the graph (as the corresponding point set associated with $\varphi^+_{\beta_c}$) and that $\mu_{\varphi^+_{\beta_c}}$ is the invariant measure which is obtained by projecting the Lebesgue measure on $\mathbb{T}^1$ onto $\Phi^+_{\beta_c}$.

Theorem 3.1 ([FGJ14, Theorem 1.4]). *For ρ Diophantine and κ sufficiently large we have that*

(i) The box-counting dimension of $\Phi^+_{\beta_c}$ is 2 and its Hausdorff dimension equals 1.

(ii) The measure $\mu_{\varphi^+_{\beta_c}}$ is exact dimensional with pointwise and information dimension equal to 1.

3.2 PINCHED SKEW PRODUCTS AND AMORPHIC COMPLEXITY

As we already pointed out further above, the question whether there exists a topological invariant which can distinguish pinched skew product systems, like (1), with and without a strange non-chaotic attractor is an essential motivation for the introduction of amorphic complexity. Here, we want to give a first outlook of the applicability of this new concept for pinched skew product systems, where we formulate everything explicitly for the family of maps given by (1). Further information and a more thorough discussion can be found in [FGJ15, Section 6].

Theorem 3.2 ([FGJ15, Theorem 6.1]). *Suppose ρ is Diophantine and κ is sufficiently large in (1). Then there exists an invariant (under the rotation by angle ρ) set $\Omega \subseteq \mathbb{T}^1$ of full Lebesgue measure such that*

$$0 < \underline{ac}\left(F_\kappa|_{\Omega\times[0,1]} \right) \leqslant \overline{ac}\left(F_\kappa|_{\Omega\times[0,1]} \right) < \infty .$$

This approach of considering the dynamics on a restricted subset of full measure in the above statement can be formalized in a more systematic way and leads to the definition of amorphic complexity of a Borel probability measure (again, details can be found in [FGJ15, Remark 6.2]). In fact, this approach seems inevitable. We conjecture, motivated by very recent results [KC14, DG15], that the amorphic complexity of F_κ for $\kappa > 2$ is infinite. This conjecture will be part of future investigation on this topic.

3.3 AMORPHIC COMPLEXITY, POWER ENTROPY AND TRANSIENT BEHAVIOR

In this section, we demonstrate by means of some elementary examples that there is no direct relation – in terms of inequalities – between amorphic complexity and the notions of power entropy and modified power entropy. Furthermore, we give an example which shows that amorphic complexity is sensitive to transient behavior. It should be an interesting task to describe which types of transient behavior have an impact on amorphic complexity and which ones do not, and thus to understand whether this quantity may be used to distinguish qualitatively different types of transient dynamics. More information can be found in [FGJ15] and [GJ15].

Recall the definitions of the Bowen-Dinaburg metrics d_n and of the topological entropy given in Chapter 1. For a continuous map $f : X \to X$ on a compact metric space (X, d), the topological entropy $h_{top}(f)$ measures the exponential growth of the separation numbers $\widehat{S}(f, \delta, n)$ where $\delta > 0$ and $n \in \mathbb{N}$. If topological entropy is zero, then *power entropy* instead simply measures the polynomial growth rate, given by

$$h_{pow}(f) \ := \ \sup_{\delta>0} \varlimsup_{n\to\infty} \frac{\log \widehat{S}(f, \delta, n)}{\log n} \ .$$

We refer to [HK02] and [Mar13] for a more detailed discussion.

Now, note that one wandering point is already enough to ensure that power entropy is at least bigger than one – provided f is a homeomorphism [Lab13, Proposition 2.1]. Given a Morse-Smale homeomorphism on a compact metric space, we hence conclude that the corresponding power entropy is positive, as claimed at the beginning of Section 2.2.

This shows that we may have $h_{pow}(f) > ac(f)$. Conversely, consider the map $f : \mathbb{T}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (x, x + y)$ where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then given $z = (x, y)$ and $z' = (x', y')$, we have that

$$d_n(z, z') \ \leqslant \ n|x - x'| + |y - y'| \ ,$$

which implies that $\widehat{S}(f, \delta, n) \ \leqslant \ \frac{C \cdot n}{\delta^2}$ for some constant $C > 0$. Hence, $h_{pow}(f) \leqslant 1$. However, at the same time we have that if $x \neq x'$, then z and z' rotate in the vertical direction with different speeds, and this makes it easy to show that $\mathbb{T}^1 \times \{0\}$ is an (f, δ, ν)-separated set for suitable $\delta, \nu > 0$, so that $Sep(f, \delta, \nu) = \infty$. Hence, we may also have $ac(f) > h_{pow}(f)$, showing that no inequality holds between the two quantities.

Modified power entropy h^*_{pow} is defined in a similar way as power entropy, with the only difference being that the metrics d_n in the definition are replaced by the *Hamming metrics*

$$d^*_n(x, y) \ := \ \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(x), f^i(y)) \ .$$

Since $d^*_n \leqslant d_n$, modified power entropy is always smaller than power entropy, and it can be shown that for Morse-Smale systems it is always zero. The same is true, however, for Denjoy examples and Sturmian subshifts [HK02], so that modified power entropy does not seem suitable to detect topological complexity on the very fine level we are interested in here. The same example $f(x,y) = (x, x+y)$ as above shows that we may have $ac(f) > h^*_{pow}(f)$. An example for the opposite inequality is more subtle, but can be made such that it demonstrates at the same time the non-existence of a variational principle for the modified power entropy (a question that was left open in [HK02]). It is contained in the forthcoming note [GJ15].

The example of the Morse-Smale systems shows that amorphic complexity is, in some sense, less sensitive to transient behavior than power entropy, since it assigns the value zero to these type of systems. However, amorphic complexity is not entirely insensitive to transient dynamics. An example can be given as follows.

Let $f : [0,1] \times \mathbb{T}^1 \to [0,1] \times \mathbb{T}^1$ be of the form $f(x,y) := (g(x), y + \alpha(x) \bmod 1)$, where $\mathbb{T}^1 = \mathbb{R}^1/\mathbb{Z}^1$, $\alpha : [0,1] \to \mathbb{R}$ is continuous and $g : [0,1] \to [0,1]$ is a Morse-Smale homeomorphism with unique attracting fixed point $x_a = 0$ and unique repelling fixed point $x_r = 1$ so that $\lim_{k\to\infty} g^k(x) = 0$ for all $x \in (0,1)$. Let $x_0 \in (0,1)$ and $x_k := g^k(x_0)$ for $k \in \mathbb{N}$ and $x_0' := (x_0 + x_1)/2$. Suppose α is given by

$$\alpha(x) := \begin{cases} 0 & \text{if } x \in \{0\} \cup (x_0, 1]; \\ 1 - 2\frac{|x_0' - x|}{x_0 - x_1} & \text{if } x \in (x_1, x_0]; \\ \frac{1}{k}\alpha\left(g^{-(k-1)}(x)\right) & \text{if } x \in (x_k, x_{k-1}],\ k \geqslant 2; \end{cases} .$$

Then if $x, x' \in [x_1, x_0']$, we have that

$$\left| \sum_{k=0}^{n-1} \alpha \circ g^k(x) - \sum_{k=0}^{n-1} \alpha \circ g^k(x') \right| = 2\frac{|x - x'|}{x_0 - x_1} \sum_{k=1}^{n} \frac{1}{k} . \tag{7}$$

This means that one of the two points $(x, 0), (x', 0)$ performs infinitely more turns around the annulus $[0,1] \times \mathbb{T}^1$ as $n \to \infty$, and it is not difficult to deduce from (7) that $(x,0), (x',0)$ are (f, δ, ν)-separated for some fixed $\delta, \nu > 0$ independent of x, x'. Hence, $[x_1, x_0'] \times \{0\}$ is an uncountable (f, δ, ν)-separated set, and we obtain $\mathrm{Sep}(f, \delta, \nu) = \infty$.

3.4 SYMBOLIC DYNAMICS AND AMORPHIC COMPLEXITY

In this section we want to briefly explain how amorphic complexity can be interpreted as the box-counting dimension (cf. Section 4.6) of an appropriate metric space in the context of symbolic dynamics. The corresponding statements and more information can be found in [FGJ15, Section 3.8].

Suppose A is a finite set, $\Sigma_A = A^{\mathbb{N}_0}$ and ρ is the Cantor metric on Σ_A, see Section 4.5. For a general continuous map $f : X \to X$ on a compact metric space X and some $\delta > 0$ we cannot expect that $\overline{\lim}_{n\to\infty} S_n(f, \delta, \cdot, \cdot)/n$ (cf. Definition 2.5) is a metric (even not a pseudo-metric since the triangle inequality will usually fail to hold).

However, this changes in the setting of symbolic dynamics. Namely, one can show that $(\tilde{d}_\delta)_{\delta\in(0,1]}$, defined as

$$\tilde{d}_\delta(x,y) := \overline{\lim_{n\to\infty}} \frac{S_n(\sigma,\delta,x,y)}{n} \quad \text{for} \quad x,y\in\Sigma_A ,$$

is a family of equivalent pseudo-metrics, where $\tilde{d}_1$ is usually called the *Besicovitch pseudo-metric*. It turns out that $\tilde{d}_1$ is especially useful for understanding certain dynamical behavior of cellular automata (see, for example, [BFK97] and [CFMM97]). Now, following a standard procedure, we can introduce the equivalence relation

$$x \sim y :\Leftrightarrow \tilde{d}_\delta(x,y) = 0 \quad \text{for} \quad x,y\in\Sigma_A .$$

Due to the previous observation, this relation is well-defined and independent of the chosen δ. We denote the corresponding projection mapping by $[\cdot]$ and equip $[\Sigma_A]$ with the metric $d_\delta([x],[y]) := \tilde{d}_\delta(x,y)$, $[x]$, $[y] \in [\Sigma_A]$ for some $\delta\in(0,1]$. The space $([\Sigma_A],d_\delta)$ is called *Besicovitch space* and given a subshift $\Sigma\subseteq\Sigma_A$, we also call $[\Sigma]$ the *Besicovitch space associated to* Σ.

Now, suppose (Σ,σ) is a subshift of (Σ_A,σ). If $\sigma|_\Sigma$ has finite separation numbers, we observe for each $\delta\in(0,1]$ that

$$\mathrm{Sep}(\sigma|_\Sigma,\delta,\nu) = M_\nu([\Sigma]) \quad \text{in} \quad ([\Sigma_A],d_\delta)$$

and

$$\mathrm{Span}(\sigma|_\Sigma,\delta,\nu) = N_\nu([\Sigma]) \quad \text{in} \quad ([\Sigma_A],d_\delta)$$

for all $\nu\in(0,1]$, where $M_\varepsilon(\cdot)$ and $N_\varepsilon(\cdot)$ with $\varepsilon>0$ are defined in Section 4.6. This immediately implies

Proposition 3.3. *Let Σ be a subshift of Σ_A. Then*

(a) $\sigma|_\Sigma$ has finite separation numbers if and only if $[\Sigma]$ is totally bounded in $[\Sigma_A]$, and

(b) in this setting, $\underline{\mathrm{ac}}(\sigma|_\Sigma) = \underline{D}_B([\Sigma])$ and $\overline{\mathrm{ac}}(\sigma|_\Sigma) = \overline{D}_B([\Sigma])$.

This means, in particular, that all regular Toeplitz subshifts (see Chapter 12) have a totally bounded associated Besicovitch space, using Theorem 8.5, and that we can find regular Toeplitz subshifts with associated Besicovitch spaces of arbitrarily high box-counting dimension, see Theorem 12.6.

3.5 MORE ON BIFURCATIONS OF FAMILIES OF BOUNDED ORBITS

With respect to topological results in the case of the Farey map, there is one more assertion contained in [CT11]. It states that for the map $t\mapsto\mathcal{B}(t)$ the points of discontinuity with respect to the Hausdorff topology are precisely the isolated points contained in the bifurcation set $\mathcal{E}$. We conjecture that the same statement holds true at least in the setting of Theorem 2.18.

Beyond that, a very interesting question is whether the renormalization techniques applied in [CT11] can be extended to more general continuous interval maps. If this is the case, then the dimensional aspects especially of the bifurcation set $\mathcal{E}$ contained in [CT11] can be generalized. Moreover, the period doubling bifurcations described in [CT11] should also be further clarified with respect to other maps on the interval.

Finally, we want to emphasize that a better understanding of how all these results fit into the general theory of dynamical systems with holes should be pursued.

4 SOME PRELIMINARIES

We assume that the reader is familiar with basic notions from topology as well as with essential aspects of measure theory.

4.1 ELEMENTARY DYNAMICAL OBJECTS AND NOTIONS

First, we want to define what we mean by a dynamical system. From a very general point of view, a *dynamical system* is a pair (G, X), where G is a semigroup with unity e and X is a non-empty set, equipped with a mapping $G \times X \to X : (g, x) \mapsto gx$ which is associative ($h(gx) = (hg)x$ for $h, g \in G$ and $x \in X$) and the unity e operates as the identity on X under this map ($ex = x$ for all $x \in X$), see e.g. [Den05] for more information and references therein.

In our case, we will always consider so-called *discrete dynamical systems*. Here, we are given a map $f : X \to X$ on a non-empty set X and the semigroup G consists of elements f^n with $n \in \mathbb{I}$ where $\mathbb{I}$ equals $\mathbb{Z}$ or $\mathbb{N}_0$. Usually, we assume some extra hypothesis on X and f, for example, that X is a topological space and that f acts continuously on X, and if not explicitly stated, the index set $\mathbb{I}$ will be clear from the context. Therefore, we will usually just refer to the map $f : X \to X$ itself as a dynamical system.

For a dynamical system $f : X \to X$, a non-empty subset $A \subseteq X$ is called *invariant (under* f*)* if $f^{-1}(A) = A$ and *forward invariant (under* f*)* if $f(A) \subseteq A$. Furthermore, we call a point $x \in X$ a *periodic point with period* $m \in \mathbb{N}$ or m*-periodic* if $f^m(x) = x$. A periodic point with $m = 1$ is called a *fixed point*.

Suppose $f : X \to X$ is a measurable map with respect to the measurable space $(X, \mathcal{A})$ and let μ be a (probability) measure on $(X, \mathcal{A})$. We call μ *invariant under/with respect to* f or f*-invariant* if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$.

Assume $f : X \to X$ is a continuous map on a topological space X. We call f *(topologically) transitive* if there is a point $x \in X$ whose *(forward) orbit* $\{f^n(x) : n \in \mathbb{N}_0\}$ is dense in X. Furthermore, we call f *minimal* if the orbit of every point in X is dense in X. A point $x \in X$ is *wandering* if there exist an open set $U \ni x$ and an integer $N > 0$ such that for all $n \geqslant N$ we have $f^n(U) \cap U = \emptyset$. If x is not wandering, we call it a *non-wandering point*. The *set of all non-wandering points of* f is denoted by $\Omega(f)$.

4.2 ISOMETRIES AND EQUICONTINUOUS SYSTEMS

One of the simplest class of dynamical systems on a metric space (X, d) are *isometries*, i.e. maps $f : X \to X$ which satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. A particular example are *rigid rotations* on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}$ with angle $\alpha \in \mathbb{R}^d$ which we denote by

$$R_\alpha(x) := x + \alpha \mod 1 .$$

Directly related to isometries are so-called equicontinuous dynamical systems. An invertible dynamical system $f : X \to X$ is called *equicontinuous* if $\{f^n : n \in \mathbb{Z}\}$ forms an equicontinuous family of maps. This means for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$. Clearly, every isometry is equicontinuous and we have the following converse.

Proposition 4.1 ([Aus88, Chapter 2]). *Suppose $f : X \to X$ is an equicontinuous dynamical system on a metric space (X, d). Then there is a metric $\tilde{d}$ on X, inducing the same topology on X as d, such that f is an isometry with respect to $\tilde{d}$.*

4.3 ERGODIC AND WEAK-MIXING MEASURES

Let $(X, \mathcal{A}, \mu)$ be a probability space and let μ be invariant with respect to the measurable map $f : X \to X$. We say μ is *ergodic with respect to* f if all invariant sets $A \in \mathcal{A}$ satisfy $\mu(A) = 0$ or $\mu(A) = 1$. Further, we say μ is *weak-mixing with respect to* f if for all $A, B \in \mathcal{A}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mu(f^{-k}(A) \cap B) - \mu(A)\mu(B)\right| = 0 .$$

Recall that weak-mixing implies ergodicity, see for example [Wal82, Section 1.7].

Theorem 4.2 ([BS02, Theorem 4.10.6]). *The following statements are equivalent*

(i) μ *is weak-mixing with respect to* f.

(ii) $\mu^m = \bigtimes_{k=1}^m \mu$ *is ergodic with respect to* $\bigtimes_{k=1}^m f$ *for all* $m \geqslant 2$.

A continuous map $f : X \to X$ on a compact metric space X is called *uniquely ergodic* if there is only one measure that is invariant with respect to f. Recall that this measure is automatically ergodic, see for example [Wal82, Section 6.5]. Further, it is well known that every continuous map on a compact metric space has at least one f-invariant measure, according to the Krylov–Bogolyubov Theorem, see e.g. [Wal82, Corollary 6.9.1]. One particular class of examples of uniquely ergodic dynamical systems are minimal equicontinuous maps.

Proposition 4.3 ([Pet83, Section 4.2 D]). *Minimal equicontinuous dynamical systems are uniquely ergodic.*

4.4 EXTENSIONS, FACTORS AND CONJUGACY

Suppose we are given two topological spaces X, Y and let $f : X \to X$, $g : Y \to Y$ be continuous. We say f is a *(topological) extension of* g if there exists a continuous onto map $h : X \to Y$ such that $h \circ f = g \circ h$. In this situation, we call h a *factor map* or *semi-conjugacy from* f *to* g and g is called a *(topological) factor of* f. Further, if h is a homeomorphism, then we also say that h is a *conjugacy between* f *and* g.

We call f an *almost 1-1 extension* of g if the set $\{y \in Y \mid \#h^{-1}(y) = 1\}$ is dense in Y. In the case that g is minimal, this condition can be replaced by the weaker assumption that there exists only one $y \in Y$ with $\#h^{-1}(y) = 1$. If further the set $\{y \in Y \mid \#h^{-1}(y) > 1\}$ has measure zero with respect to every g-invariant Borel probability measure μ on Y, we say that f is an *almost sure 1-1 extension*. Note that if g is equicontinuous and minimal, then it is uniquely ergodic, according to Proposition 4.3. Hence, there is only one measure to be considered in this case.

4.5 SYMBOLIC DYNAMICS

Let A be a finite set (*alphabet*). We denote by σ the left shift on

$$\Sigma_A := A^{\mathbb{I}}$$

where $\mathbb{I}$ equals either $\mathbb{N}_0$ or $\mathbb{Z}$. The product topology on Σ_A is induced by the *Cantor metric*

$$\rho(x, y) := 2^{-j} ,$$

where $x = (x_k)_{k\in\mathbb{I}}$, $y = (y_k)_{k\in\mathbb{I}} \in \Sigma_A$ and

$$j := \min\{|k| : x_k \neq y_k \text{ with } k \in \mathbb{I}\} .$$

If $\Sigma \subseteq \Sigma_A$ is closed and σ-invariant, then we call (Σ, σ) a *subshift*. For more information about symbolic dynamics, see for example [BS02].

4.6 BOX-COUNTING DIMENSION

Suppose (X, d) is a metric space and assume A is a *totally bounded* subset of X (meaning that for every $\varepsilon > 0$ there exists a finite cover of A such that each element of the cover has a diameter strictly smaller than ε). Further, denote by $N_\varepsilon(A)$ the smallest number of sets of diameter strictly smaller than $\varepsilon > 0$ needed to cover A. The *lower* and *upper box-counting dimension of* A are defined as

$$\underline{D}_B(A) := \varliminf_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon} ,$$

$$\overline{D}_B(A) := \varlimsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(A)}{-\log \varepsilon} .$$

If $\underline{D}_B(A) = \overline{D}_B(A)$, then we call their common value $D_B(A)$ the *box-counting dimension of* A.

Furthermore, let $M_\varepsilon(A)$ be the maximal cardinality of an ε-separated subset of A, where a set $S \subseteq A$ is called *ε-separated* if $d(x, y) \geqslant \varepsilon$ for all $x \neq y \in S$.

Proposition 4.4 ([Edg98, Proposition 1.4.6]). *One can replace* $N_\varepsilon(A)$ *by* $M_\varepsilon(A)$ *in the definition of the box-counting dimension.*

Part II

STRANGE CHAOTIC ATTRACTORS IN PINCHED SKEW PRODUCT SYSTEMS

5 STRANGE NON-CHAOTIC ATTRACTORS

Recall the motivating example $F_\kappa : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$ from the introduction given by

$$F_\kappa(\theta, x) \ = \ (\theta + \rho \bmod 1, \tanh(\kappa x) \cdot \sin(\pi\theta)) \ ,$$

where $\rho \in \mathbb{R} \backslash \mathbb{Q}$ and $\kappa > 0$, see also (1). In the following we introduce the class of *pinched skew product systems*, which contains the motivating example, and provide some basic definitions in this context.

Let $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ for $d \in \mathbb{N}$. A *quasiperiodically forced interval map* is a skew product map of the form

$$T \ : \ \mathbb{T}^d \times [0,1] \to \mathbb{T}^d \times [0,1] \quad , \quad (\theta, x) \mapsto (R_\rho(\theta), T_\theta(x)) \ ,$$

where $R_\rho : \mathbb{T}^d \to \mathbb{T}^d$ is the rotation with irrational angle $\rho \in \mathbb{R} \backslash \mathbb{Q}$. The maps $T_\theta : [0,1] \to [0,1]$ are called *fibre maps*. We say T is *pinched* if there exists some $\theta_* \in \mathbb{T}^d$ with $\# T_{\theta_*}([0,1]) = 1$.

We denote by $\mathcal{T}$ the class of quasiperiodically forced interval maps T which share the following properties:

($\mathcal{T}1$) the fibre maps T_θ are monotonically increasing;

($\mathcal{T}2$) the fibre maps T_θ are differentiable and $(\theta, x) \mapsto T'_\theta(x)$ is continuous on $\mathbb{T}^d \times [0,1]$;

($\mathcal{T}3$) T is pinched;

($\mathcal{T}4$) $T_\theta(0) = 0$ for all $\theta \in \mathbb{T}^d$.

Note that the last item means that the zero line $\mathbb{T}^d \times \{0\}$ is forward invariant under T. It is straightforward to check that $F_\kappa \in \mathcal{T}$.

An *invariant graph of* T is a Borel measurable map $\varphi : \mathbb{T}^d \to [0,1]$ satisfying

$$T_\theta(\varphi(\theta)) \ = \ \varphi(R_\rho(\theta))$$

for all $\theta \in \mathbb{T}^d$. If all fibre maps are differentiable, then the *Lyapunov exponent of* φ is defined by

$$\lambda(\varphi) \ := \ \int_{\mathbb{T}^d} \log T'_\theta(\varphi(\theta)) \ d\theta \ .$$

The *upper bounding graph* φ^+ is given by

$$\varphi^+(\theta) \ := \ \sup\{x \in [0,1] \mid (\theta, x) \in \mathcal{A}\} \quad \text{for each } \theta \in \mathbb{T}^d \ ,$$

with $\mathcal{A} := \bigcap_{n \in \mathbb{N}} T^n(\mathbb{T}^d \times [0,1]) \supseteq \mathbb{T}^d \times \{0\}$ the *global attractor of* T.

Now, according to [GOPY84], the upper bounding graph φ^+ is called a *strange non-chaotic attractor (SNA)* if it is non-continuous and has a negative Lyapunov exponent. With the help of the next lemma we can sketch the proof of Keller [Kel96] for the existence of an SNA for F_κ with $\kappa > 2$, where we will mainly explain how to obtain the non-continuity of φ^+.

The upper bounding graph is equivalently defined by

$$\varphi^+(\theta) \;=\; \lim_{n\to\infty} T^n_{R_\rho^{-n}(\theta)}(1),$$

where $T^n_\theta = T_{R_\rho^{n-1}(\theta)} \circ \ldots \circ T_\theta$. This means that the *iterated upper bounding lines*

$$\varphi_n(\theta) \;:=\; T^n_{R_\rho^{-n}(\theta)}(1) \tag{8}$$

converge pointwise and, by monotonicity of the fibre maps, in a decreasing way to φ^+. This fact will be crucial for our later analysis. A first consequence of this observation is that, under some mild conditions, the Lyapunov exponent of φ^+ is always non-positive.

Lemma 5.1 ([Jä03, Lemma 3.5])**.** *If* $\theta \mapsto \log\left(\inf_{x\in[0,1]} T'_\theta(x)\right)$ *is integrable, then* $\lambda(\varphi^+) \leqslant 0$*.*

For the maps F_κ the Lyapunov exponent of the zero line is easily computed and one obtains

$$\lambda(0) \;=\; \log\kappa - \log 2.$$

Consequently, when $\kappa > 2$ this exponent is positive and therefore the upper bounding graph cannot be the zero line. However, at the same time, the pinching condition together with the invariance of φ^+ imply that $\varphi^+(\theta) = 0$ for a dense set of $\theta \in \mathbb{T}^1$. Hence, φ^+ cannot be continuous.

Using the concavity of the fibre maps, it is further possible to show that φ^+ is the only invariant graph of the system (1) besides the zero line, that $\lambda(\varphi^+)$ is strictly negative and that φ^+ attracts $\mathrm{Leb}_{\mathbb{T}^1\times[0,1]}$-a.e. initial condition (θ, x), in the sense that

$$\lim_{n\to\infty} F^n_{\kappa,\theta}(x) - \varphi^+(\theta + n\rho \bmod 1) \;=\; 0\,.$$

Finally, we note that to any invariant graph φ of a map T in $\mathcal{T}$ an invariant measure μ_φ can be associated by

$$\mu_\varphi(A) \;:=\; \mathrm{Leb}_{\mathbb{T}^d}(\pi_1(A \cap \Phi))$$

for all Borel measurable sets $A \subseteq \mathbb{T}^d \times [0,1]$ where $\pi_1 : \mathbb{T}^d \times [0,1] \to \mathbb{T}^d$ is the projection to the first coordinate.

For further information on pinched skew product systems and strange non-chaotic attractors we refer to [Jä07, Jä09].

6 MORE ON DIMENSIONS

In this chapter let X be a separable metric space. The diameter of a subset $A \subseteq X$ is denoted by $\operatorname{diam}(A)$. Furthermore, for $\varepsilon > 0$ a finite or countable collection $\{A_i\}$ of subsets of X is called an *ε-cover* of A if $\operatorname{diam}(A_i) \leqslant \varepsilon$ for each i and $A \subseteq \bigcup_i A_i$.

6.1 HAUSDORFF, POINTWISE AND INFORMATION DIMENSION

For $A \subseteq X$, $s \geqslant 0$ and $\varepsilon > 0$ define

$$\mathcal{H}^s_\varepsilon(A) := \inf\left\{ \sum_i (\operatorname{diam}(A_i))^s \;\middle|\; \{A_i\} \text{ is an } \varepsilon\text{-cover of } A \right\}.$$

Then

$$\mathcal{H}^s(A) := \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A)$$

is called the *s-dimensional Hausdorff measure of* A. The *Hausdorff dimension of* A is defined by

$$D_H(A) := \sup\{s \geqslant 0 \mid \mathcal{H}^s(A) = \infty\}.$$

For the definition of box-counting dimension D_B, see Section 4.6. In general, we have $D_H(A) \leqslant D_B(A)$ with A a totally bounded subset of X (see, for example, [Fal03] and [Pes97]). In the following we state some well known properties of the Hausdorff measure and dimension that will be used later on.

Lemma 6.1 ([Pes97]). *Let X, Y be two separable metric spaces and let $g : A \subseteq X \to Y$ be a Lipschitz continuous map with Lipschitz constant K. Then $\mathcal{H}^s(g(A)) \leqslant K^s \mathcal{H}^s(A)$ and $D_H(g(A)) \leqslant D_H(A)$. Further, if g is bi-Lipschitz continuous, then $D_H(g(A)) = D_H(A)$.*

Lemma 6.2 ([Pes97]). *The Hausdorff dimension is countably stable, i.e. $D_H\left(\bigcup_i A_i\right) = \sup_i D_H(A_i)$ for any sequence of subsets $(A_i)_{i \in \mathbb{N}}$ with $A_i \subseteq X$.*

In contrast to the last lemma, we have that the upper box-counting dimension is only finitely stable and that $D_B(A) = D_B\left(\overline{A}\right)$ (see, for example, [Fal03] and [Pes97], again).

Theorem 6.3 ([How96]). *Let X, Y be two separable metric spaces and consider the Cartesian product space $X \times Y$ equipped with the maximum metric. Then for $A \subseteq X$ and $B \subseteq Y$ totally bounded we have*

$$D_H(A \times B) \leqslant D_H(A) + \overline{D}_B(B).$$

Lemma 6.4. *Let* $A \subseteq X$ *be a* $\limsup$ *set meaning that there exists a sequence* $(A_i)_{i\in\mathbb{N}}$ *of subsets of* X *with*

$$A \;=\; \limsup_{i\to\infty} A_i \;=\; \bigcap_{i=0}^{\infty} \bigcup_{k=i+1}^{\infty} A_k \;.$$

If $\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s < \infty$ *for* $s > 0$*, then* $\mathcal{H}^s(A) = 0$ *and* $D_H(A) \leqslant s$.

Proof. Since $\sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s < \infty$, we have $\sum_{i=k}^{\infty} \operatorname{diam}(A_i)^s \to 0$ for $k \to \infty$. That means the diameter of A_i goes to 0 as $i \to \infty$. Therefore, $\{A_i : i \geqslant k\}$ is an ε-cover for k sufficiently large. This implies $\mathcal{H}^s_\varepsilon(A) \leqslant \sum_{i=k}^{\infty} \operatorname{diam}(A_i)^s \to 0$ as $k \to \infty$. Hence, $\mathcal{H}^s(A) = 0$ and $D_H(A) \leqslant s$. ☐

For $x \in X$ and $\varepsilon > 0$ we denote by $B_\varepsilon(x)$ the open ball around x with radius $\varepsilon > 0$.

Let μ be a finite Borel measure in X. For each point x in the support of μ we define the *lower* and *upper pointwise dimension of* μ *at* x as

$$\begin{aligned} \underline{d}_\mu(x) \;&:=\; \liminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} \;, \\ \overline{d}_\mu(x) \;&:=\; \limsup_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log \varepsilon} \;. \end{aligned}$$

If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$, then their common value $d_\mu(x)$ is called the *pointwise dimension of* μ *at* x. We say that the measure μ is *exact dimensional* if the pointwise dimension exists and is constant almost everywhere, meaning that

$$\underline{d}_\mu(x) \;=\; \overline{d}_\mu(x) \;=:\; d_\mu \;,$$

μ-almost everywhere.

The *lower* and *upper information dimension of* μ are defined as

$$\begin{aligned} \underline{D}_1(\mu) \;&:=\; \liminf_{\varepsilon\to 0} \frac{\int \log \mu(B_\varepsilon(x))d\mu(x)}{\log \varepsilon} \;, \\ \overline{D}_1(\mu) \;&:=\; \limsup_{\varepsilon\to 0} \frac{\int \log \mu(B_\varepsilon(x))d\mu(x)}{\log \varepsilon} \;. \end{aligned}$$

If $\underline{D}_1(\mu) = \overline{D}_1(\mu)$, then their common value $D_1(\mu)$ is called the *information dimension of* μ.

Theorem 6.5 ([Cut91, Zin02])**.** *Suppose* $\overline{D}_B(X) < \infty$*. We have*

$$\int \underline{d}_\mu(x)\; d\mu(x) \;\leqslant\; \underline{D}_1(\mu) \;\leqslant\; \overline{D}_1(\mu) \;\leqslant\; \int \overline{d}_\mu(x)\; d\mu(x).$$

In particular, if μ *is exact dimensional, then* $D_1(\mu) = d_\mu$.

Note that also several other dimensions of μ coincide if μ is exact dimensional [You82, Zin02].

6.2 RECTIFIABLE SETS AND MEASURES

Here, we mainly follow [AK00].

For $d \in \mathbb{N}$ a Borel set $A \subseteq X$ is called *countably* d-*rectifiable* if there exists a sequence of Lipschitz continuous functions $(g_i)_{i\in\mathbb{N}}$ with $g_i : A_i \subseteq \mathbb{R}^d \to X$ such that $\mathcal{H}^d(A \setminus \bigcup_i g_i(A_i)) = 0$. A finite Borel measure μ is called d-*rectifiable* if $\mu = \Theta\, \mathcal{H}^d\big|_A$ for some countably d-rectifiable set A and some Borel measurable density $\Theta : A \to [0, \infty)$.

Note that, by the Radon-Nikodym theorem, μ is d-rectifiable if and only if μ is absolutely continuous with respect to $\mathcal{H}^d\big|_A$ with A some countably d-rectifiable set.

Theorem 6.6 ([AK00, Theorem 5.4]). *For a* d-*rectifiable measure* $\mu = \Theta\, \mathcal{H}^d\big|_A$ *we have*

$$\Theta(x) = \lim_{\varepsilon \to 0} \frac{\mu(B_\varepsilon(x))}{V_d\, \varepsilon^d},$$

for $\mathcal{H}^d$*-a.e.* $x \in A$*, where* V_d *is the volume of the* d*-dimensional unit ball. The right hand side of this equation is called* d*-density of* μ*.*

This theorem implies in particular that the d-density exists and is positive μ-almost everywhere for a d-rectifiable measure μ and this gives directly

Corollary 6.7. *A* d*-rectifiable measure* μ *is exact dimensional with* $d_\mu = D_1(\mu) = d$.

7 PROVING THE MAIN RESULT

7.1 OUTLINE OF THE STRATEGY

As we have mentioned in the introduction, our main goal is to analyze the structure of the upper bounding graphs φ^+ when they are different from the zero line. In particular, we want to determine the dimensions of these invariant graphs[1] and of their associated invariant measures. However, the argument for the non-continuity of φ^+ sketched in Chapter 5 is a 'soft' one and does not yield any quantitative information about the structure of the invariant graphs. Hence, it is not clear how such an analysis can be carried out.

However, as mentioned also in Chapter 5, the upper bounding graph φ^+ can be approximated by the iterated upper bounding lines φ_n defined in (8). It turns out that the geometry of the lines φ_n can be controlled well and this is the starting point of our investigation. Figure 3 shows the first six iterates $\varphi_1, \ldots, \varphi_6$ for the map F_κ defined in (1) for some $\kappa > 2$. A clear pattern can be observed. Apparently, when going from φ_{n-1} to φ_n, the only significant change is the appearance of a new 'peak' in a small ball I_n around the n-th iterate $R^n_\rho(\theta_*) = \tau_n$ of the pinching point $\theta_* = 0$. Outside of I_n the graphs seem to remain unchanged. Further, since every new peak is the image of the previous one and due to the expansion around the zero line, the peaks become steeper and sharper in every step. As a consequence, the radius of the balls I_n decreases exponentially.

Of course, this is a very rough picture which can only hold in an approximate sense. Due to the strict monotonicity of the fibre maps for all $\theta \neq \theta_*$, the sequence φ_n is strictly decreasing everywhere except on the countable set $\{\tau_n \mid n \in \mathbb{N}\}$, so the graphs have to change at least a little bit outside of I_n. However, let us assume for the moment that the above description was true and $\varphi_{n-1}(\theta) - \varphi_n(\theta) = 0$ for all $\theta \notin I_n$. In this case the graph φ^+ is already determined on $\mathbb{T}^d \setminus \bigcup_{k=n}^{\infty} I_k = \Lambda_n$ after n steps and equals $\varphi_n|_{\Lambda_n}$ on this set. However, as a finite iterate of $\mathbb{T}^d \times \{1\}$, the function φ_n is Lipschitz continuous and therefore its graph $\Phi_{n|\Lambda_n} = \{(\theta, \varphi_n(\theta)) \mid \theta \in \Lambda_n\}$ has Hausdorff dimension d. Due to the exponential decrease of the radius of the I_n, the set $\Omega_\infty = \mathbb{T}^d \setminus \bigcup_{n\in\mathbb{N}} \Lambda_n$ is a $\limsup$ set and has Hausdorff dimension zero by Lemma 6.4. It follows that Φ^+ is contained in the countable union $\bigcup_{n\in\mathbb{N}} \Phi_n|_{\Lambda_n} \cup (\Omega_\infty \times [0,1])$ of at most d-dimensional sets. By countable stability, this implies that the

1 Recall that we are slightly abusing terminology by calling both φ^+ and its (forward) invariant point set $\Phi^+ = \{(\theta, \varphi^+(\theta)) \mid \theta \in \mathbb{T}^d\}$ an invariant graph.

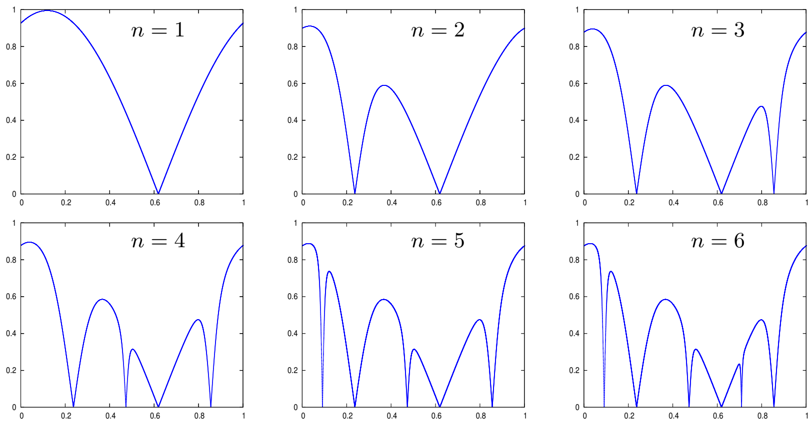


Figure 3: The graphs of the first six iterated upper bounding lines of (1) with $\kappa = 3$ and ρ the golden mean.

Hausdorff dimension of Φ^+ is d. For the pointwise dimension a similar argument could be given, but we will directly conclude from the arguments sketched above that μ_{φ^+} is d-rectifiable.

The remainder of this chapter is devoted to showing that these heuristics can be converted into a rigorous proof, despite the fact that 'nothing changes outside of I_n' has to be replaced by 'almost nothing changes outside of I_n'.

7.2 ESTIMATES ON THE ITERATED UPPER BOUNDING LINES

The purpose of this section is to obtain a good control on the behavior and shape of the iterated upper bounding lines. In order to derive the required estimates, we have to impose a number of assumptions on the geometry of our systems. The hypotheses are formulated in terms of $\mathcal{C}^1$-estimates, and it is straightforward to check that they are fulfilled by (1) whenever κ is large enough (see Lemma 7.2 for the details).

Let $T \in \mathcal{T}$. Suppose there exist $\alpha > 2$, $\gamma > 0$ and $L_0 \in (0, 1)$ such that for all $\theta \in \mathbb{T}^d$

$$|T_\theta(x) - T_\theta(y)| \leqslant \alpha|x - y| \tag{9}$$

for all $x, y \in [0, 1]$ and

$$|T_\theta(x) - T_\theta(y)| \leqslant \alpha^{-\gamma}|x - y| \tag{10}$$

for all $x, y \in [L_0, 1]$. Further, we assume there exists $\beta > 0$ such that for all $x \in [0, 1]$

$$|T_\theta(x) - T_{\theta'}(x)| \leqslant \beta d(\theta, \theta') \, . \tag{11}$$

For example, we may take $\beta = \sup_{(\theta,x)} \|\partial_\theta T_\theta(x)\|$ if T is differentiable in θ. As in the previous section we let $\tau_n := R^n_\rho(\theta_*)$. We suppose that the rotation vector $\rho \in \mathbb{R}^d$ is Diophantine, meaning that there exist constants $c > 0$ and $D > 1$ such that

$$d(\tau_n, \theta_*) \geqslant c \cdot n^{-D} \tag{12}$$

for all $n \in \mathbb{N}$. In addition, we assume there are $m \in \mathbb{N}$, $a > 1$ and $0 < b < 1$ with

$$m > 22\left(1 + \frac{1}{\gamma}\right), \tag{13}$$

$$a \geqslant (m+1)^D, \tag{14}$$

$$b \leqslant c, \tag{15}$$

$$d(\tau_n, \theta_*) > b \text{ for all } n \in \{1, \dots, m-1\} \tag{16}$$

such that

$$T_\theta(x) \geqslant \min\{L_0, ax\} \cdot \min\left\{1, \frac{2}{b} d(\theta, \theta_*)\right\} \tag{17}$$

for all $(\theta, x) \in \mathbb{T}^d \times [0, 1]$. Let

$$\mathcal{T}^* := \{T \in \mathcal{T} \mid T \text{ satisfies (9)–(17)}\}$$

where 'satisfies (9)–(17)' should be understood in the sense of 'there exist constants α, γ, L_0, β, c, D, m, a and b such that (9)–(17) are satisfied'.

Example 7.1. The following map is a simple extension of (1) with a higher-dimensional rotation on the base. We define it as $F_\kappa : \mathbb{T}^d \times [0, 1] \to \mathbb{T}^d \times [0, 1]$ by

$$F_\kappa(\theta, x) = \left(\theta + \rho \bmod 1, \tanh(\kappa x) \cdot \frac{1}{d} \cdot \sum_{i=1}^{d} \sin(\pi\theta_i)\right). \tag{18}$$

Here $\theta = (\theta_1, \dots, \theta_d)$.

As we show now, F_κ satisfies (9) – (17) for all sufficiently large κ.

Lemma 7.2. *Let ρ satisfy the Diophantine condition (12) with constants c, D. Then there exist constants $d_0 = d_0(c, D)$ and $\kappa_0 = \kappa_0(c, D, d)$ such that*

(i) For all $\kappa \geqslant \kappa_0$ the map F_κ belongs to $\mathcal{T}^*$;

(ii) If $d \geqslant d_0$, then the constants α, m and a can be chosen such that

$$d > m^2 \log(\alpha/a). \tag{19}$$

The additional condition (19) will be used to show that for sufficiently large d the d-dimensional Hausdorff measure of the upper bounding graph φ^+ of F_κ is finite, see Proposition 7.11.

Proof. We let $\alpha = \kappa$, $\gamma = \frac{1}{2}$, $L_0 = \frac{\log\kappa}{\kappa}$, $\beta = \pi$, $m = 67$, $b = \frac{1}{2}\min_{n=1}^{m-1} cn^{-D}$ and $a = \frac{2b\kappa}{d(e+1/e)^2}$. Then we choose $d_0 = d_0(c, D)$ such that for all $d \geqslant d_0$

$$d > m^2 \log\left(\frac{d(e+1/e)^2}{2b}\right), \tag{20}$$

and $\kappa_0 = \kappa_0(c, D, d)$ such that for all $\kappa \geqslant \kappa_0$

$$\begin{aligned} \kappa &\geqslant 16\,, \\ \frac{2b\kappa}{d(e+1/e)^2} &\geqslant (m+1)^D\,, && (21) \\ \frac{\log\kappa}{\kappa} &\leqslant \frac{b\tanh(1)}{2d}\,. && (22) \end{aligned}$$

We have

$$[\tanh(\kappa x)]' = \frac{4\kappa}{(e^{\kappa x} + e^{-\kappa x})^2} \leqslant \kappa$$

for all $x \geqslant 0$ and

$$0 \leqslant \frac{1}{d}\sum_{i=1}^{d} \sin(\pi\theta_i) \leqslant 1$$

for all $\theta \in \mathbb{T}^d$. Hence, (9) holds and since

$$F'_{\kappa,\theta}(x) \leqslant F'_{\kappa,\theta}(L_0) \leqslant \frac{4\kappa}{(\kappa + 1/\kappa)^2} \leqslant \frac{4}{\kappa} \leqslant \kappa^{-1/2}$$

for all $x \geqslant L_0$, the same is true for (10). (11) and (13) are easy to check, and (12) holds by assumption. (14) follows from (21), whereas (15) and (16) are obvious from the choice of b and (12). In order to verify (17), note that $[\tanh(\kappa x)]'_{|x=1/\kappa} = \frac{4\kappa}{(e+1/e)^2}$ such that by concavity and monotonicity

$$\tanh(\kappa x) \geqslant \begin{cases} \frac{4\kappa}{(e+1/e)^2} \cdot x & \text{if } x \leqslant 1/\kappa \\ \\ \tanh(1) & \text{if } x > 1/\kappa \end{cases}.$$

Using (22) and the fact that $\sum_{i=1}^{d} \sin(\pi\theta_i) \geqslant d(\theta, \theta_*)$ where $\theta_* = 0$, we obtain

$$\begin{aligned} F_{\kappa,\theta}(x) &\geqslant \min\left\{\tanh(1), \frac{4\kappa}{(e+1/e)^2}x\right\} \cdot \frac{1}{d} \cdot d(\theta, \theta_*) \\ &\geqslant \min\left\{\frac{b\tanh(1)}{2d}, \frac{2b\kappa}{d(e+1/e)^2}x\right\} \cdot \frac{2}{b} \cdot d(\theta, \theta_*) \\ &\geqslant \min\{L_0, ax\} \cdot \min\left\{1, \frac{2}{b}d(\theta, \theta_*)\right\} \end{aligned}$$

as required. Finally, since $\alpha/a = \frac{d(e+1/e)^2}{2b}$, condition (19) follows from (20). Note that, since b and m are constants only depending on c and D, the same is true for the condition (20) on d_0. □

Remark 7.3. Given $T \in \mathcal{T}^*$, note that (17) implies

$$\lambda(0) \;\geqslant\; \log\frac{2a}{b} + \int_{\mathbb{T}^d} \log d(\theta,\theta_*)\, d\theta \;\geqslant\; \log\frac{2a}{b} - \log 2 - 1\,.$$

Since $a \geqslant 23$ by (14), this yields $\lambda(0) > 0$ and hence $\varphi^+(\theta) > 0$ for $\mathrm{Leb}_{\mathbb{T}^d}$-almost every θ.

In order to formulate the main results of this section let $j \in \mathbb{R}$ and

$$r_j \;:=\; \frac{b}{2} a^{-\frac{j-1}{m}}\,.$$

Proposition 7.4. *Let $T \in \mathcal{T}^*$. For $q \in \mathbb{N}$ the following hold:*

(i) $|\varphi_n(\theta) - \varphi_n(\theta')| \leqslant \beta\alpha^n d(\theta,\theta')$ *for all* $n \in \mathbb{N}$ *and* $\theta,\theta' \in \mathbb{T}^d$.

(ii) There exists $\lambda > 0$ *such that if* $n \geqslant mq+1$ *and* $\theta \notin \bigcup_{j=q}^n B_{r_j}(\tau_j)$, *then* $|\varphi_n(\theta) - \varphi_{n-1}(\theta)| \leqslant \alpha^{-\lambda(n-1)}$.

(iii) There exists $K > 0$ *such that if* $\theta,\theta' \notin \bigcup_{j=q}^n B_{r_j}(\tau_j)$, *then* $|\varphi_n(\theta) - \varphi_n(\theta')| \leqslant K\alpha^{mq} d(\theta,\theta')$ *for all* $n \in \mathbb{N}$.

For the proof we need two preliminary statements. The first is a simple observation.

Lemma 7.5. *Suppose (12) holds and let* $n, i \in \mathbb{N}_0$ *and* $n > 0$. *If* $d(\tau_n,\theta_*) \leqslant b \cdot a^{-i}$, *then* $n \geqslant a^{i/D}$.

Proof. (12) implies $c \cdot n^{-D} \leqslant b \cdot a^{-i}$. Using (15) yields $n^{-D} \leqslant a^{-i}$. □

The second statement we need for the proof of Proposition 7.4 is an upper bound on the proportion of time the backwards orbit of a point $(\theta,\varphi_n(\theta)) \in \Phi_n$ spends outside of the contracting region $\mathbb{T}^d \times [L_0, 1]$. Given $\theta \in \mathbb{T}^d$ and $n \in \mathbb{N}$, let $\theta_k := R_\rho^{k-n}(\theta)$ and $x_k := \varphi_k(\theta_k)$ for $0 \leqslant k \leqslant n$. Note that thus $x_k = T^k_{\theta_0}(1)$ and $T^{n-k}_{\theta_k}(x_k) = \varphi_n(\theta)$. Let

$$\begin{aligned} s^n_k(\theta) &:= \#\{k \leqslant j < n \mid x_j < L_0\}\,, \\ s^n_k(\theta,\theta') &:= \#\{k \leqslant j < n \mid \min\{x_j, x'_j\} < L_0\} \end{aligned}$$

and note that $s^n_k(\theta,\theta') \leqslant s^n_k(\theta) + s^n_k(\theta')$. We set $s^n_n(\theta) := 0$ and $s^n_n(\theta,\theta') := 0$.

Lemma 7.6. *Let* $T \in \mathcal{T}^*$ *and* $q, n \in \mathbb{N}$ *with* $n \geqslant mq+1$. *Suppose that* $\theta \notin \bigcup_{j=q}^n B_{r_j}(\tau_j)$. *Then for all* $t \geqslant mq$ *we have*

$$s^n_{n-t}(\theta) \;\leqslant\; \frac{11t}{m}\,.$$

Proof. We divide $A = \{1 \leqslant k < n-q \mid x_k < L_0\}$ into blocks $B = \{l+1,\ldots,p\}$ with $0 \leqslant l < p < n-q$ and the properties

(a) $x_l \geqslant L_0/a$,

(b) $x_k < L_0/a$ for all $k \in \{l+1,\ldots,p-1\}$,

(c) $x_p < L_0$,

(d) either $x_p \geqslant L_0/a$ or $x_{p+1} \geqslant L_0$ or $p+1 = n-q$.

Note that these blocks cover the whole set A and they are uniquely determined by the above requirements. Since we always start a new block when the 'threshold' L_0/a is reached, we may have $p = l'$ for two adjacent blocks $B = \{l+1, \dots, p\}$ and $B' = \{l'+1, \dots, p'\}$.

Now, we first consider a single block $B = \{l+1, \dots, p\}$. We have $\theta_l \in B_{b/2}(\theta_*)$ because otherwise $x_{l+1} \geqslant L_0$ according to (17) and (a). Since $x_{k+1} = T_{\theta_k}(x_k)$, we can use (17) and (b) to obtain that for any $k \in \{l+1, \dots, p-1\}$

$$x_{k+1} \geqslant a x_k \min\left\{1, \frac{2}{b} d(\theta_k, \theta_*)\right\}.$$

Therefore, using (c),(a) and (17) again, we see that

$$1 > \frac{x_p}{L_0} \geqslant a^{p-l-1} \prod_{k=l}^{p-1} \min\left\{1, \frac{2}{b} d(\theta_k, \theta_*)\right\}. \tag{23}$$

Note that

$$\sum_{k=l}^{p-1} \log \min\left\{1, \frac{2}{b} d(\theta_k, \theta_*)\right\}$$
$$\geqslant -(\log a) \cdot \sum_{i=1}^{\infty} i \cdot \#\left\{ l \leqslant k < p \;\middle|\; \frac{b}{2} a^{-i} \leqslant d(\theta_k, \theta_*) < \frac{b}{2} a^{-i+1} \right\}.$$

Therefore, we can deduce from (23) that

$$\begin{aligned} p - l &\leqslant \sum_{i=1}^{\infty} i \cdot \#\left\{ l \leqslant k < p \;\middle|\; \frac{b}{2} a^{-i} \leqslant d(\theta_k, \theta_*) < \frac{b}{2} a^{-i+1} \right\} \quad (24) \\ &= \sum_{i=1}^{\infty} \#\left\{ l \leqslant k < p \;\middle|\; d(\theta_k, \theta_*) < \frac{b}{2} a^{-i+1} \right\}. \end{aligned}$$

We turn to the estimate on $A \cap [n-t, n-q)$ (note that $n-t < n-q$). It may happen that $n-t$ is contained in a middle of a block B. In this case we need two auxiliary statements to estimate the length of this first block intersecting $[n-t, n-q)$.

Let $j \in \mathbb{N}$ be such that $(m-3)(j-1) < t \leqslant (m-3)j$.

Claim 7.7. *If $j' \geqslant 1$ and $d(\theta_k, \theta_*) \geqslant b a^{-j'}/2$ for all $k = l, \dots, p-1$, then $p - l \leqslant \frac{j'}{1-2/m} \leqslant 3j'$.*

Proof. Due to (16), two consecutive visits in $B_{b/2}(\theta_*)$ are at least m times apart, whereas two consecutive visits in $B_{ba^{-i}/2}(\theta_*)$ are at least $a^{i/D}$ times apart by Lemma 7.5. Hence, we obtain from (24) that

$$p - l \leqslant \frac{p-l}{m} + 1 + \sum_{i=2}^{j'} \left(\frac{p-l}{a^{(i-1)/D}} + 1 \right) \overset{(14)}{\leqslant} \frac{2(p-l)}{m} + j'.$$

This finishes the proof of the first claim. ○

Claim 7.8. *Suppose the block* $B = \{l+1, \dots, p\}$ *intersects* $[n-t, n-q)$ *and* $t \leqslant (m-3)j$. *Then* $d(\theta_k, \theta_*) \geqslant ba^{-j+1}/2$ *for all* $k \in B$.

Proof. Suppose for a contradiction that there exist $j' \geqslant j$ and $k' \in B$ with $d(\theta_{k'}, \theta_*) < ba^{-j'+1}/2$. If j' is chosen maximal such that $d(\theta_k, \theta_*) \geqslant ba^{-j'}/2$ for all $k \in B$, then Claim 7.7 implies that $\#B \leqslant 3j'$. However, since $\theta \notin \bigcup_{k=q}^{n} B_{r_k}(\tau_k)$ we have $d(\theta_k, \theta_*) \geqslant r_{n-k}$ for all $k \in \{0, \dots, n-q\}$ and this implies $ba^{-j'}/2 \geqslant r_{n-k'}$, i.e. $k' < n - mj'$. Therefore, $n - t \leqslant \max B \leqslant k' + 3j' < n - (m-3)j'$, contradicting the assumption on t. ○

We can now complete the proof of the lemma. For all blocks B intersecting $[n-t, n-q)$, Claim 7.8 implies $d(\theta_k, \theta_*) \geqslant ba^{-j+1}/2$ for all $k \in B$ such that $\#B \leqslant 3j$, by Claim 7.7. Hence, by the same counting argument as in the proof of Claim 7.7 and summing up over all blocks, we obtain the following estimate from (24)

$$\begin{aligned} s^n_{n-t}(\theta) &\leqslant q + \#(A \cap [n-t, n-q)) \\ &\leqslant q + 3j + \frac{t}{m} + 1 + \sum_{i=2}^{j-1} \frac{t}{a^{(i-1)/d}} + 1 \\ &\leqslant q + 4j + \frac{2t}{m} \overset{(13)}{\leqslant} \frac{11t}{m} \end{aligned}$$

(recall that $t \geqslant mq$). □

This allows us to turn to the

Proof of Proposition 7.4.
(i) For all $\theta, \theta' \in \mathbb{T}^d$ we have

$$\begin{aligned} \left|\varphi_1(\theta) - \varphi_1(\theta')\right| &= \left|T_{R_\rho^{-1}(\theta)}(1) - T_{R_\rho^{-1}(\theta')}(1)\right| \\ &\overset{(11)}{\leqslant} \beta d(R_\rho^{-1}(\theta), R_\rho^{-1}(\theta')) = \beta d(\theta, \theta') \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\left|\varphi_{n+1}(\theta) - \varphi_{n+1}(\theta')\right| \\ &\qquad \leqslant \left|T_{\theta_n}(x_n) - T_{\theta_n}(x_n')\right| + \left|T_{\theta_n}(x_n') - T_{\theta_n'}(x_n')\right|. \end{aligned} \tag{26}$$

We claim that for all $\theta, \theta' \in \mathbb{T}^d$

$$\left|\varphi_n(\theta) - \varphi_n(\theta')\right| \leqslant \beta(\alpha^n - 1) d(\theta, \theta'). \tag{27}$$

For the proof of this assertion we proceed by induction. (27) holds for $n = 1$ because of (25) and the fact that $\alpha > 2$. Moreover,

$$\begin{aligned}
&\left|\varphi_{n+1}(\theta) - \varphi_{n+1}(\theta')\right| \\
\overset{(26)}{\leqslant}\quad & \left|T_{R_\rho^{-1}(\theta)}(\varphi_n(R_\rho^{-1}(\theta))) - T_{R_\rho^{-1}(\theta)}(\varphi_n(R_\rho^{-1}(\theta')))\right| \\
& + \left|T_{R_\rho^{-1}(\theta)}(\varphi_n(R_\rho^{-1}(\theta'))) - T_{R_\rho^{-1}(\theta')}(\varphi_n(R_\rho^{-1}(\theta')))\right| \\
\overset{(9),(11)}{\leqslant}\quad & \alpha|\varphi_n(\theta') - \varphi_n(\theta)| + \beta d(\theta,\theta') \\
\overset{(27)}{\leqslant}\quad & (\alpha\beta(\alpha^n - 1) + \beta)\, d(\theta,\theta') \;\leqslant\; \beta(\alpha^{n+1} - 1)d(\theta,\theta')
\end{aligned}$$

which proves (27) for $n + 1$.

(ii) Fix $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^d$. Let θ_k and x_k be defined as above. If $\varphi_{k-1}(\theta_k) - \varphi_k(\theta_k) = 0$ for some $k \in \{1, \dots, n\}$, then we have $\varphi_{n-1}(\theta_n) - \varphi_n(\theta_n) = 0$. Thus, we may assume that the distance is greater than 0 for all k. In this case we have

$$\begin{aligned}
\varphi_{n-1}(\theta) - \varphi_n(\theta) &= (\varphi_0(\theta_1) - \varphi_1(\theta_1)) \cdot \prod_{k=1}^{n-1} \frac{\varphi_k(\theta_{k+1}) - \varphi_{k+1}(\theta_{k+1})}{\varphi_{k-1}(\theta_k) - \varphi_k(\theta_k)} \\
\leqslant \quad \prod_{k=1}^{n-1} & \frac{T_{\theta_k}(\varphi_{k-1}(\theta_k)) - T_{\theta_k}(\varphi_k(\theta_k))}{\varphi_{k-1}(\theta_k) - \varphi_k(\theta_k)} \;\leqslant\; \alpha^{s_1^n(\theta) - \gamma(n-1-s_1^n(\theta))}\,,
\end{aligned}$$

applying (9) and (10). Since $\theta \notin \bigcup_{j=q}^n B_{r_j}(\tau_j)$, we can use Lemma 7.6 with $t = n - 1$ to obtain $|\varphi_n(\theta) - \varphi_{n-1}(\theta)| \leqslant \alpha^{-\lambda(n-1)}$ where

$$\lambda := \gamma - \frac{11}{m}(1+\gamma) \overset{(13)}{>} 0\,.$$

(iii) We proceed by induction to show that for all $\theta, \theta' \in \mathbb{T}^d$ and $n \in \mathbb{N}$ we have

$$\left|\varphi_n(\theta) - \varphi_n(\theta')\right| \leqslant \beta \left(\sum_{k=0}^{n-1} \alpha^{(1+\gamma)s_{n-k}^n(\theta,\theta') - \gamma k} \right) d(\theta,\theta')\,. \tag{28}$$

For $n = 1$ this is true because of (25). Further, since

$$s_n^{n+1}(\theta,\theta') + s_{n-k}^n(R_\rho^{-1}(\theta), R_\rho^{-1}(\theta')) = s_{n-k}^{n+1}(\theta,\theta')\,, \tag{29}$$

we have

$$\begin{aligned}
&\left|\varphi_{n+1}(\theta) - \varphi_{n+1}(\theta')\right| \\
\overset{(26),(9)-(11)}{\leqslant}\quad & \alpha^{(1+\gamma)s_n^{n+1}(\theta,\theta') - \gamma} \left|\varphi_n(R_\rho^{-1}(\theta)) - \varphi_n(R_\rho^{-1}(\theta'))\right| \\
& + \beta d(R_\rho^{-1}(\theta), R_\rho^{-1}(\theta')) \\
\overset{(28),(29)}{\leqslant}\quad & \beta \left(\sum_{k=0}^{n} \alpha^{(1+\gamma)s_{n+1-k}^{n+1}(\theta,\theta') - \gamma k} \right) d(\theta,\theta')\,.
\end{aligned}$$

This completes the induction step such that (28) holds for all $n \in \mathbb{N}$.

Now, when $\theta, \theta' \notin \bigcup_{j=q}^{n} B_{r_j}(\tau_j)$ and $k \geqslant mq$, then $s_{n-k}^{n}(\theta, \theta') \leqslant \frac{22k}{m}$ by Lemma 7.6. Consequently, (28) yields that

$$\begin{aligned}
&\left|\varphi_n(\theta) - \varphi_n(\theta')\right| \\
&\leqslant \beta \left(\sum_{k=0}^{mq-1} \alpha^k + \sum_{k=mq}^{n-1} \alpha^{(1+\gamma)s_{n-k}^{n}(\theta,\theta') - \gamma k} \right) d(\theta, \theta') \\
&\leqslant \beta \left(\alpha^{mq} + \sum_{k=mq}^{n-1} \alpha^{-\left(\gamma - \frac{22}{m}(1+\gamma)\right)k} \right) d(\theta, \theta') \,.
\end{aligned}$$

Because of (13), we have $\gamma - \frac{22}{m}(1+\gamma) > 0$ and this implies $|\varphi_n(\theta) - \varphi_n(\theta')| \leqslant K\alpha^{mq} d(\theta, \theta')$ with

$$K := \beta \left(1 + \frac{\alpha^{-mq}}{1 - \alpha^{-(\gamma - \frac{22}{m}(1+\gamma))}} \right) . \qquad \square$$

7.3 DIMENSIONS OF φ^+ AND μ_{φ^+}

For $T \in \mathcal{T}^*$ we can now calculate the Hausdorff dimension of the upper bounding graph φ^+, or more precisely of the corresponding point set Φ^+. We will also be able to draw some conclusions regarding the Hausdorff measure of Φ^+. To that end, we will partition φ^+ into countably many subgraphs. First, keeping the notation from the last section, we define a partition of $\mathbb{T}^d$ by subsets $\Omega_j \subset \mathbb{T}^d$ with $j \in \mathbb{N}_0 \cup \{\infty\}$ as

$$\Omega_0 := \mathbb{T}^d \setminus \bigcup_{k=j_0}^{\infty} B_{r_k}(\tau_k) \,, \tag{30}$$

$$\Omega_j := B_{r_{j+j_0-1}}(\tau_{j+j_0-1}) \setminus \bigcup_{k=j+j_0}^{\infty} B_{r_k}(\tau_k) \,, \tag{31}$$

$$\Omega_\infty := \bigcap_{i=0}^{\infty} \bigcup_{k=i+1}^{\infty} B_{r_k}(\tau_k) \,, \tag{32}$$

where we choose $j_0 \in \mathbb{N}$ large enough to ensure $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) > 0$ for all $j \in \mathbb{N}_0$. This works for $j = 0$ because $\sum_{k=1}^{\infty} \mathrm{Leb}_{\mathbb{T}^d}(B_{r_k}(\tau_k)) < \infty$, and for $j \in \mathbb{N}$ because for all $j' > j$ with $B_{r_j}(\tau_j) \cap B_{r_{j'}}(\tau_{j'}) \neq \emptyset$ the Diophantine condition (12) and (15) yields

$$j' > \nu(j) \quad \text{with} \quad \nu(j) := a^{\frac{j-1}{Dm}} + j \,.$$

Hence, we obtain

$$\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) \geqslant \mathrm{Leb}_{\mathbb{T}^D}(B_{r_{j+j_0-1}}(\tau_{j+j_0-1})) - \sum_{j' \geqslant \nu(j+j_0-1)} \mathrm{Leb}_{\mathbb{T}^d}(B_{r_{j'}}(\tau_{j'}))$$

which is strictly positive if $j_0 \in \mathbb{N}$ is sufficiently large. The corresponding subgraphs ψ^j are defined by restricting φ^+ to Ω_j, i.e. $\psi^j := \varphi^+|_{\Omega_j}$.

Proposition 7.9. *Let $T \in \mathcal{T}^*$. Then for all $j \in \mathbb{N}_0$ the graph Ψ^j is the image of a bi-Lipschitz continuous function $g_j : \Omega_j \to \Omega_j \times [0,1]$ and therefore $D_H(\Psi^j) = d$. Further, $D_H(\Psi^\infty) \leqslant 1$.*

Proof. Consider the maps $g_j : \Omega_j \to \Omega_j \times [0,1] : \theta \mapsto (\theta, \psi^j(\theta))$. For all $j \in \mathbb{N}_0 \cup \{\infty\}$ we have $g_j(\Omega_j) = \Psi^j$ and $d_{\mathbb{T}^d \times [0,1]}(g_j(\theta), g_j(\theta')) \geqslant d(\theta, \theta')$ for all $\theta, \theta' \in \Omega_j$. Further, for all $j \in \mathbb{N}_0$ we have

$$d_{\mathbb{T}^d \times [0,1]}(g_j(\theta), g_j(\theta')) \leqslant \left(1 + K\alpha^{(j+j_0)m}\right) d(\theta, \theta')$$

for all $\theta, \theta' \in \Omega_j$. This is true because Proposition 7.4 (iii) implies that $\varphi_n|_{\Omega_j}$ is Lipschitz continuous with Lipschitz constant $K\alpha^{(j+j_0)m}$ independent of n. Since $\psi^j = \lim_{n\to\infty} \varphi_n|_{\Omega_j}$, we also get that ψ^j is Lipschitz continuous with the same constant. This means that g_j is bi-Lipschitz continuous for any $j \in \mathbb{N}_0$. Therefore, $D_H(\Psi^j) = D_H(\Omega_j)$. Hence, $D_H(\Psi^j) = d$ for all $j \in \mathbb{N}_0$ because $0 < \text{Leb}(\Omega_j) < \infty$.

In order to complete the proof, we now show that $D_H(\Psi^\infty) \leqslant 1$. Since Ω_∞ is a lim sup set and $\sum_{k=1}^{\infty} \text{diam}(B_{r_k}(\tau_k))^s < \infty$ for all $s > 0$, we get that $D_H(\Omega_\infty) \leqslant s$ for all $s > 0$, using Lemma 6.4. Hence, $D_H(\Omega_\infty) = 0$. Furthermore, $\Psi^\infty \subset \Omega_\infty \times [0,1]$ and therefore $D_H(\Psi^\infty) \leqslant D_H(\Omega_\infty) + D_B([0,1]) = 1$, applying Theorem 6.3. □

Since the Hausdorff dimension is countably stable, see Lemma 6.2, we immediately obtain

Theorem 7.10. *Let $T \in \mathcal{T}^*$. Then the Hausdorff dimension of the upper bounding graph is d.*

It remains to determine the d-dimensional Hausdorff measure of the upper bounding graph Φ^+.

Proposition 7.11. *Let $T \in \mathcal{T}^*$ and $d > m^2 \log(\alpha/a)$. Then the d-dimensional Hausdorff measure of Φ^+ is finite.*

Proof. Since $D_H(\Psi^\infty) \leqslant 1$, we have $\mathcal{H}^d(\Psi^\infty) = 0$ for $d > 1$. Furthermore, we can consider the maps g_j from the last proposition as Lipschitz continuous maps from $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ and therefore we can use the Area Formula (see, for example, [EG92, Chapter 3]) to deduce

$$\begin{aligned} \mathcal{H}^d(\Psi^j) &\leqslant \sqrt{1 + (K\alpha^{(j+j_0)m+1})^2}\, \text{Leb}_{\mathbb{R}^d}(B_{r_{j+j_0-1}}(\tau_{j+j_0-1})) \\ &= V_d \left(\frac{b}{2}\right)^d \sqrt{1 + (K\alpha^{(j+j_0)m+1})^2}\, a^{-\frac{d}{m}(j+j_0-2)} \end{aligned}$$

where V_d is the volume of the d-dimensional unit ball. When $d > m^2 \log(\alpha/a)$, this implies that $\mathcal{H}^d(\Psi^j)$ is decaying exponentially fast and therefore $\mathcal{H}^d(\Phi^+) = \sum_{j=0}^{\infty} \mathcal{H}^d(\Psi^j) < \infty$. □

Proposition 7.12. *Let* $T \in \mathcal{T}^*$ *and* $d = 1$. *Then the one-dimensional Hausdorff measure of* Φ^+ *is infinite.*

Proof. We show that there exists an increasing sequence of integers $(j_i)_{i\in\mathbb{N}}$ such that $\mathcal{H}^1(\Psi^{j_i}) \geqslant c^+/6$.

Suppose $j_1, \dots, j_N$ are given. Our first goal is to find $j > j_N + j_0 - 1$ such that there exists a point $\tilde{\theta}^+ \in B_{r_j}(\tau_j)$ with $\varphi_j(\tilde{\theta}^+) \geqslant 2c^+/3$. According to Remark 7.3, we can find a $\theta^+ \in \mathbb{T}^1$ with $\theta^+ \notin \Omega'_\infty := \bigcap_{i=0}^\infty \bigcup_{k=i+1}^\infty B_{2r_k}(\tau_k)$ and $c^+ := \varphi^+(\theta^+) > 0$. Since $\theta^+ \notin \Omega'_\infty$, there exists $q \in \mathbb{N}$ such that $\theta^+ \notin \bigcup_{k=q}^\infty B_{2r_k}(\tau_k)$. Now, we can choose $n > \max\{j_N + j_0 - 1, mq\}$ such that for all $j \geqslant n$

$$\frac{1}{6}c^+ \geqslant \frac{1}{1-\alpha^{-\lambda}}\alpha^{-\lambda j}, \tag{33}$$

$$\nu(j) \geqslant m(j+1)+1, \tag{34}$$

$$a^{\frac{\nu(j)-1}{m}} \geqslant \frac{6b}{c^+(1-a^{-1/m})}\left(1+K\alpha^{(j+1)m+1}\right). \tag{35}$$

Note that $B_{r_n}(\theta^+) \cap \bigcup_{k=q}^n B_{r_k}(\tau_k) = \emptyset$, which means that there exists a neighborhood of θ^+ where we can apply Proposition 7.4 (ii) to all points of this neighborhood. Since φ_n is continuous and $\varphi_n(\theta^+) \geqslant \varphi^+(\theta^+) = c^+$, we can find $\delta \leqslant r_n$ such that $\varphi_n(\theta) > 5c^+/6$ for all $\theta \in B_\delta(\theta^+)$. Now, let $j \geqslant n$ be the first time such that $B_\delta(\theta^+) \cap B_{r_j}(\tau_j) \neq \emptyset$. Set $R := B_\delta(\theta^+) \backslash B_{r_j}(\tau_j) \neq \emptyset$. Then for all $\theta \in R$ we have $\theta \notin \bigcup_{k=q}^{n'} B_{r_k}(\tau_k)$ for all $n \leqslant n' \leqslant j$ and therefore

$$\sum_{k=n}^{j-1} \alpha^{-\lambda k} \geqslant \varphi_n(\theta) - \varphi_j(\theta) > \frac{5c^+}{6} - \varphi_j(\theta),$$

using $n \geqslant qm+1$ and Proposition 7.4 (ii). This implies $\varphi_j(\theta) > 2c^+/3$ for all $\theta \in R$, using (33). Since φ_j is continuous, there exists a $\tilde{\theta}^+ \in B_{r_j}(\tau_j)$ such that $\varphi_j(\tilde{\theta}^+) \geqslant 2c^+/3$. Now, using Proposition 7.4 (i), we have that φ_j is Lipschitz continuous with Lipschitz constant $\beta\alpha^j$ and therefore there exists an interval $I \subseteq B_{r_j}(\tau_j)$ such that φ_j is greater than $c^+/2$ on I and

$$\mathrm{Leb}_{\mathbb{T}^1}(I) \geqslant \frac{c^+}{6\beta\alpha^j}.$$

Because of (35), we have that $\mathrm{Leb}_{\mathbb{T}^1}(I \backslash \bigcup_{k=j+1}^\infty B_{r_k}(\tau_k)) > 0$ (note that $\beta < K$). Hence, using (34) plus Proposition 7.4 (ii) and (33) again, there exists $\theta \in I \setminus \bigcup_{k=j+1}^{\infty} B_{r_k}(\tau_k) \subset \Omega_{j_{N+1}}$ such that $\psi^{j_{N+1}}(\theta) \geqslant c^+/3$ where $j_{N+1} := j - j_0 + 1$. Finally, the application of (35) yields

$$\begin{aligned}
&\mathcal{H}^1(\Psi^{j_{N+1}}) \\
&\geqslant \mathcal{H}^1(\psi^{j_{N+1}}(\Omega_{j_{N+1}})) \\
&\geqslant \frac{c^+}{3} - \left(1 + K\alpha^{(j+1)m+1}\right) \mathrm{Leb}_{\mathbb{T}^1}\left(\bigcup_{k=j+1}^{\infty} B_{r_k}(\tau_k)\right) \\
&\geqslant \frac{c^+}{6} .
\end{aligned}$$

□

We turn to the question of rectifiability. Note that by definition μ_{φ^+} is absolutely continuous with respect to $\mathcal{H}^d|_{\Phi^+}$.

Theorem 7.13. *Let $T \in \mathcal{T}^*$. Then μ_{φ^+} is d-rectifiable and we have $d_{\mu_{\varphi^+}} = D_1(\mu_{\varphi^+}) = d$.*

Proof. Observe that $\mu_{\varphi^+}(\Psi^\infty) = 0$. Therefore, μ_{φ^+} is also absolutely continuous with respect to $\mathcal{H}^d|_{\Phi^+ \setminus \Psi^\infty}$ and $\Phi^+ \setminus \Psi^\infty = \bigcup_{j=0}^{\infty} \Psi^j$ is countably d-rectifiable, according to Proposition 7.9. That means μ_{φ^+} is d-rectifiable. Now, use Corollary 6.7 to obtain the dimensional results for μ_{φ^+}. □

Note that for $d \geqslant 2$ we have $\mathcal{H}^d(\Psi^\infty) = 0$ such that Φ^+ is countably d-rectifiable. The question whether Φ^+ is countably 1-rectifiable for $d = 1$ remains open.

We can now apply the above results to the family F_κ defined in Example 7.1 to obtain the following corollary which contains Theorem 2.1, Theorem 2.3 and Corollary 2.4 as a special case.

Corollary 7.14. *Let F_κ be defined by (18). Then there exists a $\kappa_0 = \kappa_0(c, D, d)$ such that for all $\kappa \geqslant \kappa_0$*

(i) the upper bounding graph Φ^+ of F_κ has Hausdorff dimension d,

(ii) the d-dimensional Hausdorff measure of Φ^+ is infinite if $d = 1$ and finite for d sufficiently large,

(iii) μ_{φ^+} is exact dimensional with pointwise dimension d,

(iv) the information dimension of μ_{φ^+} is d and

(v) μ_{φ^+} is d-rectifiable.

Finally, we close by addressing a further obvious question in our context, namely, that of the size of the set of *pinched points* where the upper bounding graph φ^+ equals zero. For $T \in \mathcal{T}$ let

$$\mathcal{P} := \left\{\theta \in \mathbb{T}^d \mid \varphi^+(\theta) = 0\right\} .$$

Then $\mathcal{P}$ is residual in the sense of Baire [Kel96] and therefore its box-counting dimension is d. However, from the point of view of Hausdorff dimension, $\mathcal{P}$ turns out to be small.

Proposition 7.15. *Let* $T \in \mathcal{T}^*$*. Then*

$$\mathcal{P} \subseteq \Omega_\infty \cup \left\{R_\rho^n(\theta_*) \mid n \in \mathbb{N}\right\} ,$$

where Ω_∞ *is the set defined in (32). In particular,* $D_H(\mathcal{P}) = 0$*.*

Proof. Suppose $\theta \notin \Omega_\infty \cup \left\{R_\rho^n(\theta_*) \mid n \in \mathbb{N}\right\}$. Let $q \in \mathbb{N}$ be such that $\theta \notin \bigcup_{j=q}^{\infty} B_{r_j}(\tau_j)$ and fix any $t \geqslant mq$. Let

$$\varepsilon := \min_{k=1}^{t} T^k_{R_\rho^{-k}(\theta)}(L_0) .$$

Note that since $\theta \notin \{R_\rho^n(\theta_*) \mid n \in \mathbb{N}\}$ we have $\varepsilon > 0$. Now, for any $n > t$ Lemma 7.6 implies that $s^n_{n-t}(\theta) \leqslant 11t/m \leqslant t/2$. In particular, there exists $l \in \{n-t, \ldots, n-1\}$ such that $x_l = T^l_{R_\rho^{-n}(\theta)}(1) \geqslant L_0$. Hence,

$$\varphi_n(\theta) = T^{n-l}_{R_\rho^{-(n-l)}(\theta)}(x_l) \geqslant \varepsilon .$$

Since this holds for all $n > t$, we obtain $\varphi^+(\theta) \geqslant \varepsilon$ and thus $\theta \notin \mathcal{P}$ as required. The statement on the Hausdorff dimension then follows from Lemma 6.2 and 6.4. □

Part III

AMORPHIC COMPLEXITY

8 QUALITATIVE BEHAVIOR OF SEPARATION NUMBERS

For the definition of the (asymptotic) separation numbers, see Definition 2.5. In Section 4.3 one can find the definition of ergodic and weak-mixing measures as well as Theorem 4.2 which is used in the proof of the following first theorem.

Theorem 8.1. *Let (X, d) be a metric space. Suppose $f : X \to X$ is Borel measurable and μ is a Borel probability measure invariant under f. Furthermore, assume that μ is weak-mixing with respect to f and its support is not a single point. Then f has infinite separation numbers.*

Note that the assumption that f is only measurable is consistent with the definition of asymptotic separation numbers, since a priori we do not require any regularity of the map f in their definition.

Proof. For each $\delta > 0$ we define the function $h_\delta : X^2 \to \{0, 1\}$ as $h_\delta(z, w) := \Theta(d(z, w) - \delta)$ where $\Theta : \mathbb{R} \to \{0, 1\}$ is the Heaviside step function. Note that

$$\frac{1}{n} S_n(f, \delta, x, y) = \frac{1}{n} \sum_{k=0}^{n-1} h_\delta\big(f^k(x), f^k(y)\big) .$$

Since μ is not supported on a single point, we can find $\delta_0 > 0$ and $\nu_0 > 0$ such that for all $\delta \leqslant \delta_0$ we have

$$\int h_\delta \, d\mu^2 \geqslant \nu_0 . \tag{36}$$

(Note that $\int h_{\delta'} d\mu^2 \geqslant \int h_\delta d\mu^2$ for $\delta' \leqslant \delta$.) In order to see this, assume for a contradiction that there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \searrow 0$ as $n \to \infty$ such that $\int h_{\delta_n} d\mu^2 = 0$ for each δ_n. This implies $d(z, w) < \delta_n$ for μ^2-a.e. point $(z, w) \in X^2$ and moreover $d(z, w) \leqslant \delta_n$ for all $(z, w) \in \operatorname{supp} \mu^2$ because of the continuity of the metric. Since we can choose δ_n arbitrarily small, we get that the support of μ is just a single point, contradicting our assumptions.

Now, fix $\delta \in (0, \delta_0]$, $\nu \in (0, \nu_0]$ and define $\phi_m : X^m \to \mathbb{R}^{m(m-1)/2}$ by

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \mapsto \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \begin{pmatrix} h_\delta(f^k(x_1), f^k(x_2)) \\ h_\delta(f^k(x_1), f^k(x_3)) \\ \vdots \\ h_\delta(f^k(x_{m-1}), f^k(x_m)) \end{pmatrix} \tag{37}$$

for each $m \geqslant 2$. Since h_δ is bounded, observe that the functions $(x_1, \ldots, x_m) \mapsto h_\delta(x_i, x_j)$ with $1 \leqslant i < j \leqslant m$ are in $L^1(\mu^m)$. By ergodicity of μ^m (cf. Theorem 4.2), the limits in (37) exist μ^m-almost everywhere. Further, φ_m is μ^m-almost surely constant and all its entries are different from zero, since we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h_\delta\left(f^k(x_i), f^k(x_j)\right) &= \int_{X^m} h_\delta(x_i, x_j) d\mu^m(x_1, \ldots, x_m) \\ &= \int_{X^2} h_\delta(x_i, x_j) d\mu(x_i) d\mu(x_j) \geqslant \nu_0 > 0\end{aligned}$$

for $1 \leqslant i < j \leqslant m$ by (36). Thus, the above implies that for each $m \in \mathbb{N}$ there exist at least m points that are pairwise (f, δ, ν)-separated, so that

$$\mathrm{Sep}(f, \delta, \nu) \geqslant m\,.$$

Since m was arbitrary and the pair (δ, ν) is fixed, we get $\mathrm{Sep}(f, \delta, \nu)$ is infinite. □

The analogous statement for maps with positive topological entropy is a direct consequence of a result of Downarowicz in [Dow14]. In order to state it, we say that two points x and y in a metric space (X, d) are *DC2-scrambled with respect to* f if the following two conditions are fulfilled

$$\begin{aligned}\forall \delta > 0 \quad &: \quad \overline{\lim_{n\to\infty}} \frac{\#\left\{0 \leqslant k < n \mid d(f^k(x), f^k(y)) < \delta\ \right\}}{n} = 1\,, \\ \exists \delta_0 > 0 \quad &: \quad \underline{\lim_{n\to\infty}} \frac{\#\left\{0 \leqslant k < n \mid d(f^k(x), f^k(y)) < \delta_0\right\}}{n} < 1\,. \end{aligned} \tag{38}$$

Furthermore, we say that a subset $S \subseteq X$ is *DC2-scrambled* if any pair $x, y \in S$ with $x \neq y$ is DC2-scrambled. The set S is called *uniformly DC2-scrambled* if the δ_0's and the lower frequencies in (38) are uniform for all pairs $x, y \in S$ with $x \neq y$. Now by [Dow14, Theorem 1.2], if f has positive topological entropy, then there exists an uncountable DC2-scrambled set S, and as stated in [Dow14, Remark 2] this set can be chosen to be uniformly DC2-scrambled. It follows then directly from (38) that the points in S are pairwise (f, δ, ν)-separated for the respective parameters $\delta, \nu > 0$, i.e. $\mathrm{Sep}(f, \delta, \nu) = \infty$. Thus, we obtain

Theorem 8.2. *Let (X, d) be a compact metric space. Suppose $f : X \to X$ is a continuous map with positive topological entropy. Then f has infinite separation numbers.*

We now turn to the opposite direction and aim to show that almost sure 1-1 extensions of equicontinuous systems have finite separation numbers (for the basic definitions see Sections 4.2 and 4.4). In order to do so, we need to introduce some further notions and preliminary statements. Suppose (X, d) and (Ξ, ρ) are compact metric spaces and $f : X \to X$ is an extension of $g : \Xi \to \Xi$ with factor map $h : X \to \Xi$. For $x \in X$ define the *fibre of* x as $F_x := h^{-1}(h(x))$. Denote the collection of fibres by $\mathcal{F} := \{F_x \mid x \in X\}$. Given $\delta > 0$, let

$$\mathcal{F}_\delta \ := \ \{x \in X \mid \operatorname{diam}(F_x) \geqslant \delta\} \ = \bigcup_{\substack{F \subseteq \mathcal{F} \\ \operatorname{diam}(F) \geqslant \delta}} F \,.$$

Further, let $\mathcal{F}_{>0} := \bigcup_{\delta>0} \mathcal{F}_\delta$, $E_\delta := h(\mathcal{F}_\delta)$ and $E := h(\mathcal{F}_{>0})$. Obviously, both $\mathcal{F}_\delta$ and E_δ are decreasing in δ.

Lemma 8.3. *The set $\mathcal{F}_\delta$ is closed for all $\delta > 0$.*

Proof. Assume $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{F}_\delta$ converging to $x \in X$. For each $k \in \mathbb{N}$ we can find two distinct points x_k^1 and x_k^2 in F_{x_k} such that $d(x_k^1, x_k^2) \geqslant \delta$. We can assume without loss of generality that $(x_k^1)_{k\in\mathbb{N}}$ and $(x_k^2)_{k\in\mathbb{N}}$ converge to x^1 and x^2, respectively, as $k \to \infty$. By continuity of the metric, we have $d(x^1, x^2) \geqslant \delta$. Furthermore,

$$\begin{aligned} &\rho(h(x^i), h(x)) \\ &\qquad \leqslant \ \rho(h(x^i), h(x_k^i)) + \rho(h(x_k^i), h(x_k)) + \rho(h(x_k), h(x)) \\ &\qquad = \ \rho(h(x^i), h(x_k^i)) + \rho(h(x_k), h(x)) \end{aligned}$$

for $i \in \{1, 2\}$, and using the continuity of h we obtain $h(x^i) = h(x)$. In other words $x^1, x^2 \in F_x$, and hence $x \in \mathcal{F}_\delta$. □

Note that as a direct consequence the sets $\mathcal{F}_{>0}$, E_δ and E are Borel measurable. The following basic observation will be crucial in the proof of the next theorem. From now on, we denote by $B_\varepsilon(A)$ for $\varepsilon > 0$ the open ε-neighborhood of a subset A of a metric space.

Lemma 8.4. *For all $\delta > 0$ and $\varepsilon > 0$ there exists $\eta = \eta_\delta(\varepsilon) > 0$ such that if $x, y \in X$ satisfy $d(x, y) \geqslant \delta$ and $\rho(h(x), h(y)) < \eta$, then $h(x)$ and $h(y)$ are contained in $B_\varepsilon(E_\delta)$.*

Proof. Assume for a contradiction that the statement is false. Then there are $\delta, \varepsilon > 0$ and sequences $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ in X such that $h(x_k) \notin B_\varepsilon(E_\delta)$ or $h(y_k) \notin B_\varepsilon(E_\delta)$ and $d(x_k, y_k) \geqslant \delta$ for all $k \in \mathbb{N}$, but $\rho(h(x_k), h(y_k)) \to 0$ as $k \to \infty$. By going over to subsequences if necessary, we may assume that $(h(x_k))_{k\in\mathbb{N}}$ lies in $X \backslash B_\varepsilon(E_\delta)$ and that $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ converge. Let $x := \lim_{k\to\infty} x_k$ and $y := \lim_{k\to\infty} y_k$. Then $d(x, y) \geqslant \delta$ and $h(x) = \lim_{k\to\infty} h(x_k) \notin B_\varepsilon(E_\delta)$. However, $h(x) = h(y)$ and thus $\operatorname{diam}(F_x) = \operatorname{diam}(F_y) \geqslant \delta$ such that $x \in \mathcal{F}_\delta$, which is the required contradiction. □

Note that in the following given two subsets $A, B \subseteq X$, we denote by $d(A, B)$ the minimal distance between A and B.

Theorem 8.5. *Let $f : X \to X$ be a continuous map. Further, assume that f is an almost sure 1-1 extension of an isometry $g : \Xi \to \Xi$. Then f has finite separation numbers.*

Note that this implies Theorem 2.7 (ii), since any equicontinuous system is an isometry with respect to an equivalent metric, see Proposition 4.1.

Proof. Denote by $\mathcal{M}(g)$ the set of all g-invariant Borel probability measures on Ξ. Fix $\delta > 0$ and $\nu > 0$. We claim that since $\mu(E_\delta) \leqslant \mu(E) = 0$ for all $\mu \in \mathcal{M}(g)$, there exists $\varepsilon > 0$ such that

$$\mu\left(\overline{B_\varepsilon(E_\delta)}\right) < \nu \quad \text{for all } \mu \in \mathcal{M}(g) . \tag{39}$$

Otherwise, it would be possible to find a sequence $\mu_n \in \mathcal{M}(g)$ with $\mu_n\big(\overline{B_{1/n}(E_\delta)}\big) \geqslant \nu$, which can be chosen such that it converges to some $\mu \in \mathcal{M}(g)$ in the weak-$*$-topology. If $\varphi_m(\xi) := \max\{1 - m \cdot d(\xi, B_{1/m}(E_\delta)), 0\}$, then we have $\int_\Xi \varphi_m \, d\mu_n \geqslant \nu$ for all $n \geqslant m$ and hence $\int_\Xi \varphi_m \, d\mu \geqslant \nu$ for all $m \in \mathbb{N}$. However, this implies $\mu(E_\delta) \geqslant \nu$ by dominated convergence, contradicting our assumptions. Hence, we may choose $\varepsilon > 0$ as in (39).

This, in turn, implies that

$$\varlimsup_{n\to\infty} \frac{\#\left\{0 \leqslant k < n \mid g^k(\xi) \in \overline{B_\varepsilon(E_\delta)}\right\}}{n} < \nu \tag{40}$$

for all $\xi \in \Xi$. If this were not the case, it would again be possible to construct a g-invariant measure μ contradicting (39), this time as a limit of finite sums $\mu_\ell := \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \delta_{g^k(\xi)}$ of weighted Dirac measures for some $\xi \in \Xi$ that does not satisfy (40). (Note that in this situation we have $\mu_\ell\big(\overline{B_\varepsilon(E_\delta)}\big) \geqslant \nu$ for all $\ell \in \mathbb{N}$, and this inequality carries over to the limit μ by the Portmanteau Theorem.)

Hence, given any pair $x, y \in X$, the frequency by which both of the iterates of $h(x)$ and $h(y)$ visit $\overline{B_\varepsilon(E_\delta)}$ at the same time is smaller than ν. Together with Lemma 8.4, this implies that if $\rho(h(x), h(y)) < \eta_\delta(\varepsilon)$, then the points x and y cannot be (f, δ, ν)-separated. Thus, if $S \subseteq X$ is an (f, δ, ν)-separated set, then the set $h(S)$ must be $\eta_\delta(\varepsilon)$-separated (compare Section 4.6) with respect to the metric ρ. By compactness, the maximal cardinality N of an $\eta_\delta(\varepsilon)$-separated set in Ξ is bounded. We obtain

$$\mathrm{Sep}(f, \delta, \nu) \leqslant N . \tag{41}$$

Since $\delta > 0$ and $\nu > 0$ where arbitrary, this completes the proof. □

As immediate consequences, we obtain

Corollary 8.6. *If for all $\delta > 0$ the set E_δ is finite and contains no periodic point, then f has finite separation numbers.*

Corollary 8.7. *If* $\lim_{n\to\infty} \operatorname{diam}\left(F_{f^n(x)}\right) = 0$ *for all* $x \in X$*, then* f *has finite separation numbers.*

For the second corollary use Poincaré's Recurrence Theorem to get a contradiction.

It remains to prove part (i) of Theorem 2.7, which we restate as

Theorem 8.8. *Let* $f : X \to X$ *be a continuous map. Further, assume that* f *is a minimal almost 1-1 extension of an isometry* $g : \Xi \to \Xi$ *such that the factor map* h *is not injective. Then* f *has unbounded separation numbers.*

For the proof we will again need two preliminary lemmas. Given $x, y \in X$ and $\delta > 0$, we let

$$\nu(f, \delta, x, y) := \varlimsup_{n\to\infty} \frac{1}{n} S_n(f, \delta, x, y) .$$

Lemma 8.9. *Suppose* $V_1, V_2 \subseteq \Xi$ *are two open sets which satisfy*

$$d(h^{-1}(V_1), h^{-1}(V_2)) \geqslant \delta .$$

Then $\nu(f, \delta, x_1, x_2) > 0$ *for all* $x_1 \in h^{-1}(V_1)$ *and* $x_2 \in h^{-1}(V_2)$*.*

Proof. Let $\xi_1 := h(x_1)$ and $\xi_2 := h(x_2)$. By assumption, we have that $d(f^k(x_1), f^k(x_2)) \geqslant \delta$ whenever $g^k(\xi_1) \in V_1$ and $g^k(\xi_2) \in V_2$. Consequently,

$$\nu(f, \delta, x_1, x_2) \geqslant \varlimsup_{n\to\infty} \frac{1}{n} \# \left\{0 \leqslant k < n \mid (g \times g)^k(\xi_1, \xi_2) \in V_1 \times V_2\right\} . \tag{42}$$

However, as g is an isometry, so is $g \times g$. This implies that all points $(\xi_1, \xi_2) \in \Xi \times \Xi$ are almost periodic, and the set of return times to any of their neighborhoods is syndetic [Aus88]. Hence, the right-hand side of (42) is strictly positive. □

Lemma 8.10. *Let* f *be a minimal almost 1-1 extension of* g*. Furthermore, assume* $\operatorname{diam}(h^{-1}(\xi)) > \delta$ *for some* $\xi \in \Xi$*. Then for every neighborhood* U *of* ξ *there exist* $V_1, V_2 \subseteq U$ *such that* $d(h^{-1}(V_1), h^{-1}(V_2)) > \delta$*.*

Proof. Due to minimality, singleton fibres are dense in X. Hence, it is possible to find $x_1, x_2 \in h^{-1}(U)$ such that $F_{x_i} = \{x_i\}$, $i \in \{1, 2\}$ and $d(x_1, x_2) > \delta$. Then, by continuity, any sufficiently small neighborhoods V_i of $h(x_i)$ will satisfy $d(h^{-1}(V_1), h^{-1}(V_2)) > \delta$. □

Proof of Theorem 8.8. Since the factor map h is not injective, there exists $\xi \in \Xi$ with $\operatorname{diam}(h^{-1}(\xi)) > \delta$ for some $\delta > 0$. We will construct, by induction on $k \in \mathbb{N}$ with $k \geqslant 2$, a sequence of finite families of disjoint open sets $V_1^k, \ldots, V_k^k$ with the property that for all $1 \leqslant i < j \leqslant k$ there exists $n_{i,j}^k \in \mathbb{N}_0$ such that

$$d\left(h^{-1}\left(g^{n_{i,j}^k}\left(V_i^k\right)\right), h^{-1}\left(g^{n_{i,j}^k}\left(V_j^k\right)\right)\right) > \delta . \tag{43}$$

For any family of points $x_i^k \in h^{-1}(V_i^k)$, $i \in \{1,\dots,k\}$ and $1 \leqslant i < j \leqslant k$ we will then have

$$\nu\left(f,\delta,x_i^k,x_j^k\right) \;=\; \nu\left(f,\delta,f^{n_{i,j}^k}\left(x_i^k\right),f^{n_{i,j}^k}\left(x_j^k\right)\right) \;>\; 0\,,$$

by Lemma 8.9. Thus, if $\nu_k := \min\left\{\nu\left(f,\delta,x_i^k,x_j^k\right) \mid 1 \leqslant i < j \leqslant k\right\}$, then $\{x_1^k,\dots,x_k^k\}$ is a (f,δ,ν_k)-separated set of cardinality k. This implies that $\sup_{\nu>0} \mathrm{Sep}(f,\delta,\nu)$ is infinite, as required, since k was arbitrary.

It remains to construct the disjoint open sets V_i^k. For $k = 2$ the sets V_1^2 and V_2^2 can be chosen according to Lemma 8.10 with $n_{1,2}^2 = 0$. Suppose that $V_1^k,\dots,V_k^k$ have been constructed as above. By minimality, there exists $n \in \mathbb{N}$ such that $g^n(V_k^k)$ is a neighborhood of ξ. Lemma 8.10 yields the existence of open sets $V, V' \subseteq g^n(V_k^k)$ with $d(h^{-1}(V),h^{-1}(V')) > \delta$. We now set

$$V_i^{k+1} \;:=\; V_i^k \;\text{ for }\; i \in \{1,\dots,k-1\}$$

and

$$V_k^{k+1} \;:=\; g^{-n}(V)\,, \quad V_{k+1}^{k+1} \;:=\; g^{-n}(V')\,,$$

so that $V_k^{k+1} \cup V_{k+1}^{k+1} \subseteq V_k^k$. Choosing $n_{i,j}^{k+1} := n_{i,j}^k$ if $1 \leqslant i < j \leqslant k-1$, $n_{i,j}^{k+1} := n_{i,k}^k$ if $1 \leqslant i \leqslant k-1$ and $j \in \{k,k+1\}$, and $n_{k,k+1}^{k+1} := n$, we obtain that (43) is satisfied for all $1 \leqslant i < j \leqslant k+1$. $\square$

9 PROPERTIES OF AMORPHIC COMPLEXITY

9.1 MORE GENERAL GROWTH RATES

In the definition of amorphic complexity, cf. Definition 2.8, one may consider more general growth rates than just polynomial ones. We call

$$a : \mathbb{R}_+ \times (0,1] \to \mathbb{R}_+$$

a *scale function* if $a(\,\cdot\,,\nu)$ is non-decreasing, $a(s,\,\cdot\,)$ is decreasing and $\lim_{\nu\to 0} a(s,\nu) = \infty$ for all $s \in \mathbb{R}_+$. If the separation numbers of f are finite, then we let

$$\begin{aligned}
\underline{\mathrm{ac}}(f,a,\delta) &:= \sup\left\{ s > 0 \;\middle|\; \varliminf_{\nu\to 0} \frac{\mathrm{Sep}(f,\delta,\nu)}{a(s,\nu)} > 0 \right\} , \\
\overline{\mathrm{ac}}(f,a,\delta) &:= \sup\left\{ s > 0 \;\middle|\; \varlimsup_{\nu\to 0} \frac{\mathrm{Sep}(f,\delta,\nu)}{a(s,\nu)} > 0 \right\}
\end{aligned} \tag{44}$$

and proceed to define the *lower and upper amorphic complexity of* f *with respect to the scale function* a as

$$\begin{aligned}
\underline{\mathrm{ac}}(f,a) &:= \sup_{\delta>0} \underline{\mathrm{ac}}(f,a,\delta) , \\
\overline{\mathrm{ac}}(f,a) &:= \sup_{\delta>0} \overline{\mathrm{ac}}(f,a,\delta) .
\end{aligned} \tag{45}$$

As before, if $\underline{\mathrm{ac}}(f,a) = \overline{\mathrm{ac}}(f,a)$, then their common value is denoted by $\mathrm{ac}(f,a)$. If $a(s,\nu) = \nu^{-s}$, then the above can be reduced to Definition 2.8.

In order to obtain good properties, however, some regularity has to be imposed on the scale function. We say a scale function a is *O-(weakly) regularly varying (at the origin) with respect to* ν if

$$\varlimsup_{\nu\to 0} \frac{a(s,c\nu)}{a(s,\nu)}$$

is finite for each $s, c > 0$.

Under this assumption a part of the theory can be developed in a completely analogous way, until specific properties of polynomial growth start to play a role. For the sake of simplicity, we refrain from stating the results in this chapter in their full generality. However, we provide extra comments in each section to specify the class of scale functions to which the corresponding results extend. For more information on O-regularly varying functions, see for example [AA77, BKS06] and references therein.

9.2 DEFINITION VIA (f, δ, ν)-SPANNING SETS

As in the case of topological entropy, amorphic complexity can be defined in an equivalent way by using spanning sets instead of separating sets. A subset S of a metric space (X, d) is said to be (f, δ, ν)-*spanning* if for all $x \in X$ there exists a $y \in S$ such that

$$\overline{\lim_{n\to\infty}} \frac{S_n(f,\delta,x,y)}{n} < \nu .$$

By $\mathrm{Span}(f,\delta,\nu)$ we denote the smallest cardinality of any (f,δ,ν)-spanning set in X.

Lemma 9.1. *Let* $f : X \to X$ *be a map,* $\delta > 0$ *and* $\nu \in (0,1]$*. We have that*

(i) $\mathrm{Sep}(f,\delta,\nu) \geqslant \mathrm{Span}(f,\delta,\nu)$,

(ii) $\mathrm{Span}(f,\delta,\nu/2) \geqslant \mathrm{Sep}(f,2\delta,\nu)$.

Proof. For the first inequality, assume without loss of generality that $\mathrm{Sep}(f,\delta,\nu) < \infty$. Consider an (f,δ,ν)-separated set $S \subseteq X$ with maximal cardinality $\mathrm{Sep}(f,\delta,\nu)$. Then for any $x \in X$ with $x \notin S$ the set $S \cup \{x\}$ is not (f,δ,ν)-separated. Hence, there exists a $y \in S$ such that

$$\overline{\lim_{n\to\infty}} \frac{S_n(f,\delta,x,y)}{n} < \nu .$$

In other words S is also (f,δ,ν)-spanning. Accordingly, we obtain $\mathrm{Sep}(f,\delta,\nu) \geqslant \mathrm{Span}(f,\delta,\nu)$.

For the second inequality, assume without loss of generality that $\mathrm{Span}(f,\delta,\nu/2) < \infty$. Let $S \subseteq X$ be an $(f,\delta,\nu/2)$-spanning set of cardinality $\mathrm{Span}(f,\delta,\nu/2)$ and assume for a contradiction that $\tilde{S} \subseteq X$ is an $(f,2\delta,\nu)$-separated set with $\#\tilde{S} > \#S$. Then for some $y \in S$ there exist $x_1, x_2 \in \tilde{S}$ such that

$$\overline{\lim_{n\to\infty}} \frac{S_n(f,\delta,x_i,y)}{n} < \frac{\nu}{2}$$

with $i \in \{1,2\}$. However, due to the triangle inequality we have that

$$S_n(f,2\delta,x_1,x_2) \leqslant S_n(f,\delta,x_1,y) + S_n(f,\delta,y,x_2)$$

and consequently

$$\overline{\lim_{n\to\infty}} \frac{S_n(f,2\delta,x_1,x_2)}{n} \leqslant \overline{\lim_{n\to\infty}} \frac{S_n(f,\delta,x_1,y)}{n} + \overline{\lim_{n\to\infty}} \frac{S_n(f,\delta,x_2,y)}{n} < \nu .$$

This contradicts the fact that x_1 and x_2 are $(f,2\delta,\nu)$-separated. □

Corollary 9.2. *Given a metric space* X *and* $f : X \to X$*, we have that*

(i) $\underline{\mathrm{ac}}(f) = \sup_{\delta>0} \underline{\lim}_{\nu\to 0} \frac{\log \mathrm{Span}(f,\delta,\nu)}{-\log \nu}$,

(ii) $\overline{\mathrm{ac}}(f) = \sup_{\delta>0} \overline{\lim}_{\nu\to 0} \frac{\log \mathrm{Span}(f,\delta,\nu)}{-\log \nu}$.

Remarks 9.3.

(a) The above statement remains true if $a(s,\nu) = \nu^{-s}$ is replaced by any O-regularly varying scale function.

(b) In the definition of (f,δ,ν)-separated sets and (f,δ,ν)-spanning sets one could also use lim inf instead of lim sup, and thus define the notions of *strongly* (f,δ,ν)*-separated sets* and *weakly* (f,δ,ν)*-spanning sets*, respectively. However, there is no analog to the second inequality of Lemma 9.1 in this case.

9.3 FACTOR RELATION AND TOPOLOGICAL INVARIANCE

We assume that X and Ξ are arbitrary metric spaces, possibly non-compact. The price to pay for this is that we have to assume the uniform continuity of the factor map. All the assertions of this section remain true for arbitrary scale functions.

Proposition 9.4. *Assume* $g : \Xi \to \Xi$ *is a factor of* $f : X \to X$ *with a uniformly continuous factor map* $h : X \to \Xi$. *Then* $\underline{\mathrm{ac}}(f) \geqslant \underline{\mathrm{ac}}(g)$ *and* $\overline{\mathrm{ac}}(f) \geqslant \overline{\mathrm{ac}}(g)$.

Proof. We denote the metric on X and Ξ with d and ρ, respectively. The uniform continuity of h implies that for every $\delta > 0$ there exists $\tilde{\delta} > 0$ such that $\rho(h(z), h(w)) \geqslant \delta$ implies $d(z,w) \geqslant \tilde{\delta}$. Suppose $\xi, \xi' \in \Xi$ are (g,δ,ν)-separated. Then there exist $x, x' \in X$ such that $h(x) = \xi$ and $h(x') = \xi'$. Since $\rho(g^k(\xi), g^k(\xi')) \geqslant \delta$ implies $d(f^k(x), f^k(x')) \geqslant \tilde{\delta}$, the points x and x' need to be $(f,\tilde{\delta},\nu)$-separated. Given $\nu \in (0,1]$ this means that if $S \subseteq \Xi$ is a (g,δ,ν)-separated set, then there exist $\tilde{S} \subseteq X$ with $h(\tilde{S}) = S$ and $\tilde{\delta} > 0$ such that $\tilde{S}$ is a $(f,\tilde{\delta},\nu)$-separated set. Therefore, for all $\nu \in (0,1]$ we get

$$\mathrm{Sep}(f,\tilde{\delta},\nu) \geqslant \mathrm{Sep}(g,\delta,\nu) .$$

The assertions follow easily. □

Corollary 9.5. *Suppose* X *and* Ξ *are compact and let* $f : X \to X$ *and* $g : \Xi \to \Xi$ *be conjugate. Then* $\underline{\mathrm{ac}}(f) = \underline{\mathrm{ac}}(g)$ *and* $\overline{\mathrm{ac}}(f) = \overline{\mathrm{ac}}(g)$.

For the next corollary, observe that $f \circ g$ is an extension of $g \circ f$ with factor map $h = g$, and conversely $\tilde{h} = f$ is a factor map from $g \circ f$ to $f \circ g$.

Corollary 9.6. *Suppose* $f : X \to X$ *and* $g : X \to X$ *are uniformly continuous. Then* $\underline{\mathrm{ac}}(f \circ g) = \underline{\mathrm{ac}}(g \circ f)$ *and* $\overline{\mathrm{ac}}(f \circ g) = \overline{\mathrm{ac}}(g \circ f)$.

9.4 POWER INVARIANCE AND PRODUCT RULE

We first consider iterates of f. In contrast to topological entropy, taking powers does not affect the amorphic complexity. Throughout this section we assume that X and Y are metric spaces.

Proposition 9.7. *Assume* $f : X \to X$ *is uniformly continuous and let* $m \in \mathbb{N}$. *Then* $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$ *and* $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$.

Proof. Since all iterates of f are uniformly continuous as well, we have that for every $\delta > 0$ there exists $\tilde{\delta} > 0$ such that $d(f^i(z), f^i(w)) \geqslant \delta$ implies $d(z, w) \geqslant \tilde{\delta}$ for all $i \in \{0, \ldots, m-1\}$.

Suppose $x, y \in X$ are (f, δ, ν)-separated. Assume $d(f^k(x), f^k(y)) \geqslant \delta$ with $k = m \cdot \tilde{k} + i$ where $\tilde{k} \in \mathbb{N}_0$ and $i \in \{0, \ldots, m-1\}$. Then by the above we have $d\big(f^{m\tilde{k}}(x), f^{m\tilde{k}}(y)\big) \geqslant \tilde{\delta}$. This means that for $\tilde{n} \in \mathbb{N}$ and $n \in \{m \cdot \tilde{n}, \ldots, m(\tilde{n}+1) - 1\}$ we get

$$\frac{1}{n} S_n(f, \delta, x, y) \;\leqslant\; \frac{1}{n}\big(m \cdot S_{\tilde{n}}(f^m, \tilde{\delta}, x, y) + m\big) \;\leqslant\; \frac{1}{\tilde{n}}(S_{\tilde{n}}(f^m, \tilde{\delta}, x, y) + 1).$$

By taking the lim sup we get that x and y are $(f^m, \tilde{\delta}, \nu)$-separated. Hence,

$$\mathrm{Sep}(f^m, \tilde{\delta}, \nu) \;\geqslant\; \mathrm{Sep}(f, \delta, \nu)\,. \tag{46}$$

Conversely, suppose that x and y are (f^m, δ, ν)-separated. Then for $k \geqslant 1$ it follows from $d(f^{mk}(x), f^{mk}(y)) \geqslant \delta$ that $d\big(f^{\tilde{k}}(x), f^{\tilde{k}}(y)\big) \geqslant \tilde{\delta}$ for all $\tilde{k} \in \{m(k-1)+1, \ldots, mk\}$. Each $\tilde{n} \in \mathbb{N}$ belongs to a block $\{m(n-1)+1, \ldots, m \cdot n\}$ with $n \in \mathbb{N}$ and we have

$$\frac{1}{\tilde{n}} S_{\tilde{n}}(f, \tilde{\delta}, x, y) \;\geqslant\; \frac{1}{\tilde{n}}\big(m \cdot S_n(f^m, \delta, x, y) - m\big) \;\geqslant\; \frac{1}{n}\left(S_n(f^m, \delta, x, y) - 1\right).$$

Again, by taking the lim sup we get that x and y are $(f, \tilde{\delta}, \nu)$-separated. Hence,

$$\mathrm{Sep}(f, \tilde{\delta}, \nu) \;\geqslant\; \mathrm{Sep}(f^m, \delta, \nu)\,. \tag{47}$$

Using (46) and (47), we get $\underline{\mathrm{ac}}(f^m) = \underline{\mathrm{ac}}(f)$ and $\overline{\mathrm{ac}}(f^m) = \overline{\mathrm{ac}}(f)$. □

Remarks 9.8.

(a) The above result remains true for arbitrary scale functions.

(b) In the case that f is not uniformly continuous, we still have $\mathrm{Sep}(f, \delta, \nu/m) \geqslant \mathrm{Sep}(f^m, \delta, \nu)$. This yields $\underline{\mathrm{ac}}(f, a) \geqslant \underline{\mathrm{ac}}(f^m, a)$ and $\overline{\mathrm{ac}}(f, a) \geqslant \overline{\mathrm{ac}}(f^m, a)$ for a O-regularly varying.

In contrast to the above, the product formula is specific to polynomial growth or, more generally, to scale functions satisfying a product rule of the form $a(s+t, \nu) = a(s, \nu) \cdot a(t, \nu)$.

Proposition 9.9. *Let* $f : X \to X$ *and* $g : Y \to Y$. *Then* $\underline{\mathrm{ac}}(f \times g) \geqslant \underline{\mathrm{ac}}(f) + \underline{\mathrm{ac}}(g)$ *and* $\overline{\mathrm{ac}}(f \times g) \leqslant \overline{\mathrm{ac}}(f) + \overline{\mathrm{ac}}(g)$. *Therefore, if the limits* $\mathrm{ac}(f)$ *and* $\mathrm{ac}(g)$ *exist, we get*

$$\mathrm{ac}(f \times g) = \mathrm{ac}(f) + \mathrm{ac}(g) .$$

Proof. We denote the metric on X and Y by d_X and d_Y, respectively. Let d be the maximum metric on the product space $X \times Y$. Using Corollary 9.2, the assertions are direct consequences of the following two inequalities, which we show for all $\delta > 0$ and $\nu \in (0, 1]$,

$$\mathrm{Sep}(f \times g, \delta, \nu) \geqslant \mathrm{Sep}(f, \delta, \nu) \cdot \mathrm{Sep}(g, \delta, \nu) , \tag{48}$$

$$\mathrm{Span}(f \times g, \delta, \nu) \leqslant \mathrm{Span}(f, \delta, \nu/2) \cdot \mathrm{Span}(g, \delta, \nu/2) . \tag{49}$$

For proving (48) assume that $S_X \subseteq X$ and $S_Y \subseteq Y$ are (f, δ, ν)- and (g, δ, ν)-separated sets, respectively, with cardinalities $\mathrm{Sep}(f, \delta, \nu)$ and $\mathrm{Sep}(g, \delta, \nu)$, respectively. Then $S := S_X \times S_Y \subseteq X \times Y$ is an $(f \times g, \delta, \nu)$-separated set. This implies (48).

Now, in order to prove (49), assume that $\tilde{S}_X \subseteq X$ and $\tilde{S}_Y \subseteq Y$ are $(f, \delta, \nu/2)$- and $(g, \delta, \nu/2)$-spanning sets, respectively, with cardinalities $\mathrm{Span}(f, \delta, \nu/2)$ and $\mathrm{Span}(g, \delta, \nu/2)$, respectively. The set $\tilde{S} := \tilde{S}_X \times \tilde{S}_Y \subseteq X \times Y$ is $(f \times g, \delta, \nu)$-spanning, since for arbitrary $(x, y) \in X \times Y$ there are $\tilde{x} \in \tilde{S}_X$ and $\tilde{y} \in \tilde{S}_Y$ such that

$$\begin{aligned} &S_n(f \times g, \delta, (x, y), (\tilde{x}, \tilde{y})) \\ &\quad = \#\left\{0 \leqslant k < n \mid d\big((f \times g)^k(x, y), (f \times g)^k(\tilde{x}, \tilde{y})\big) \geqslant \delta\right\} \\ &\quad \leqslant \#\left\{0 \leqslant k < n \mid d_X(f^k(x), f^k(\tilde{x})) \geqslant \delta\right\} \\ &\qquad\qquad + \#\left\{0 \leqslant k < n \mid d_Y(g^k(y), g^k(\tilde{y})) \geqslant \delta\right\} . \qquad \square \end{aligned}$$

10 QUANTITATIVE ANALYSIS OF ALMOST SURE 1-1 EXTENSIONS OF ISOMETRIES

The aim of this chapter is to give a quantitative version of the argument in the proof of Theorem 8.5 in order to obtain an upper bound for amorphic complexity in this situation. This will involve the usage of the box-counting dimension, see Section 4.6. For the whole chapter let X and Ξ be compact metric spaces and $f : X \to X$ an almost sure 1-1 extension of $g : \Xi \to \Xi$ with factor map h, cf. Section 4.4. Further, assume that g is a minimal isometry with unique invariant probability measure μ (recall that a minimal isometry is necessarily uniquely ergodic, see Proposition 4.3). In this case it is straightforward to check that the measure of an ε-ball $B_\varepsilon(\xi)$ does not depend on $\xi \in \Xi$. For the scaling of this measure as $\varepsilon \to 0$, we have

Lemma 10.1. *In the above situation we get*

$$\varlimsup_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(\xi))}{\log \varepsilon} = \overline{D}_B(\Xi)$$

for all $\xi \in \Xi$ and the analogous equality holds for the limit inferior.

Proof. Recall that we can also use $M_\varepsilon(\Xi)$ in the definition of the box-counting dimension of Ξ (see Proposition 4.4). Let $\hat{\mu}(\varepsilon) := \mu(B_\varepsilon(\xi))$ where $\xi \in \Xi$ is arbitrary and suppose $S \subseteq \Xi$ is an ε-separated subset with cardinality $M_\varepsilon(\Xi)$. Observe that the $\varepsilon/2$-balls $B_{\varepsilon/2}(\xi)$ with $\xi \in S$ are pairwise disjoint. We obtain $1 = \mu(\Xi) \geqslant \sum_{\xi\in S} \hat{\mu}(\varepsilon/2)$ and thus $M_\varepsilon(\Xi) \leqslant 1/\hat{\mu}(\varepsilon/2)$. Hence,

$$\overline{D}_B(\Xi) = \varlimsup_{\varepsilon\to 0} \frac{\log M_\varepsilon(\Xi)}{-\log \varepsilon} \leqslant \varlimsup_{\varepsilon\to 0} \frac{\log \hat{\mu}(\varepsilon/2)}{\log \varepsilon} = \varlimsup_{\varepsilon\to 0} \frac{\log \hat{\mu}(\varepsilon)}{\log \varepsilon} .$$

Conversely, the ε-balls $B_\varepsilon(\xi)$ with centers ξ in S cover Ξ and this directly leads to the reverse inequality. □

By the Minkowski characterization of box-counting dimension, we have for $E \subseteq \Xi$

$$\overline{D}_B(E) = \overline{D}_B(\Xi) - \varliminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(E))}{\log \varepsilon} . \tag{50}$$

The proof of this fact in the setting above is the same as in Euclidean space, see for example [Fal03]. We denote by $\eta_\delta(\varepsilon)$ the constant given by Lemma 8.4 and let

$$\gamma(h) := \varlimsup_{\delta\to 0} \varlimsup_{\varepsilon\to 0} \frac{\log \eta_\delta(\varepsilon)}{\log \varepsilon} . \tag{51}$$

This is the scaling factor from Theorem 2.11 which we restate here as

Theorem 10.2. *Suppose that the upper box-counting dimension of* Ξ *is finite and strictly positive and* $\gamma(h) > 0$*. Then under the above assumptions we have*

$$\overline{ac}(f) \leqslant \frac{\overline{D}_B(\Xi) \cdot \gamma(h)}{\overline{D}_B(\Xi) - \sup_{\delta>0} \overline{D_B}(E_\delta)}\,, \tag{52}$$

where $E_\delta = \{\xi \in \Xi \mid \operatorname{diam}(h^{-1}(\xi)) \geqslant \delta\}$*.*

Proof. Without loss of generality we assume that $\gamma(h)$ is finite and fix $\delta > 0$. Going back to the end of the proof of Theorem 8.5, we find that according to its definition the number N in (41) is equal to $M_{\eta_\delta(\varepsilon)}(\Xi)$. Thus, we have already shown that if $\nu > \mu(B_\varepsilon(E_\delta))$ for some $\varepsilon > 0$, then $\mathrm{Sep}(f, \delta, \nu) \leqslant M_{\eta_\delta(\varepsilon)}(\Xi)$.

Now, note that $\mu(B_\varepsilon(E_\delta))$ is monotonously decreasing to 0 as $\varepsilon \to 0$. For ν small enough choose $k \in \mathbb{N}$ such that

$$\mu(B_{2^{-k-1}}(E_\delta)) < \nu \leqslant \mu(B_{2^{-k}}(E_\delta))\,.$$

We obtain

$$\begin{aligned}
\overline{ac}(f,\delta) &\leqslant \varlimsup_{k\to\infty} \frac{\log M_{\eta_\delta(2^{-k-1})}(\Xi)}{-\log \mu(B_{2^{-k}}(E_\delta))} \\
&= \varlimsup_{k\to\infty} \frac{M_{\eta_\delta(2^{-k-1})}(\Xi)}{-\log \eta_\delta(2^{-k-1})} \cdot \frac{\log \eta_\delta(2^{-k-1})}{\log 2^{-k-1}} \cdot \frac{\log 2^{-k-1}}{\log \mu(B_{2^{-k}}(E_\delta))} \\
&\leqslant \overline{D}_B(\Xi) \cdot \gamma(h) \cdot \left(\varliminf_{k\to\infty} \frac{\log \mu(B_{2^{-k}}(E_\delta))}{\log 2^{-k}} \right)^{-1} \\
&= \frac{\overline{D}_B(\Xi) \cdot \gamma(h)}{\overline{D}_B(\Xi) - \overline{D}_B(E_\delta)}\,,
\end{aligned}$$

where we use (50) for the last equality. Taking the supremum over all $\delta > 0$ yields (52). $\square$

11 STURMIAN SUBSHIFTS AND DENJOY EXAMPLES

We start with some standard notation concerning symbolic dynamics and circle maps. For a finite set A, let σ be the left shift on $\Sigma_A = A^{\mathbb{Z}}$ and ρ the Cantor metric on Σ_A (cf. Section 4.5). Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the circle and denote by d the usual metric on $\mathbb{T}^1$. Further, we denote the open and the closed counter-clockwise interval from a to b in $\mathbb{T}^1$ by (a, b) and $[a, b]$, respectively. The Lebesgue measure on $\mathbb{T}^1$ is denoted by Leb. Moreover, the rigid rotation by an angle $\alpha \in \mathbb{R}$ is denoted by $R_\alpha(x) = x + \alpha \mod 1$.

We first recall some basics about Sturmian subshifts, where we mainly follow [CD05, Section 2.2]. Assume that $\alpha \in (0, 1)$ is irrational. Consider the coding map $\phi_\alpha : \mathbb{T}^1 \to \{0, 1\}$ defined via $\phi_\alpha(x) = 0$ if $x \in I_0 := [0, 1 - \alpha)$ and $\phi_\alpha(x) = 1$ if $x \in I_1 := [1 - \alpha, 1)$. Set

$$\Sigma_\alpha \;:=\; \overline{\{(\phi_\alpha(R_\alpha^k(x)))_{k\in\mathbb{Z}} \mid x \in \mathbb{T}^1\}} \;\subset\; \Sigma_{\{0,1\}}\,.$$

The subshift (Σ_α, σ) is called the *Sturmian subshift generated by* α and its elements are called *Sturmian sequences*. According to [MH40], there exists a map $h : \Sigma_\alpha \to \mathbb{T}^1$ semi-conjugating σ and R_α with the property that $\#h^{-1}(x) = 2$ for $x \in \{k\alpha \mod 1 \mid k \in \mathbb{Z}\}$ and $\#h^{-1}(x) = 1$ otherwise.

Remark 11.1. If $x \in \{k\alpha \mod 1 \mid k \in \mathbb{Z}\}$, then one of the two alternative sequences in $h^{-1}(x)$ corresponds to the coding with respect to the original partition $\{I_0, I_1\}$, whereas the other one corresponds to the coding with respect to the partition $\{(0, 1 - \alpha], (1 - \alpha, 1]\}$. However, the right-hand sides of the two codings coincide after finitely many iterations, i.e. they do not depend on the choice of the partition for high enough iterations. For all other points, the two codings coincide anyway. Further information is given in [BMN00, Section 1.6].

Theorem 11.2. *Let (Σ_α, σ) be a Sturmian subshift. Then* $\mathrm{ac}(\sigma) = 1$.

Proof. First, we show the lower bound. Assume we have two points $x = (x_\ell)_{\ell\in\mathbb{Z}}$, $y = (y_\ell)_{\ell\in\mathbb{Z}} \in \Sigma_\alpha$ with $h(x) \neq h(y)$ and

$$\mathrm{Leb}((h(x), h(y))) \;\leqslant\; \min\{\alpha, 1 - \alpha\} \;<\; 1/2$$

such that

$$R_\alpha^{-k}(0) \in (h(x), h(y))$$

for some $k \in \mathbb{N}_0$. Then $R_\alpha^k(h(x)) \in I_1$ and $R_\alpha^k(h(y)) \in I_0$, i.e. we have $x_k \neq y_k$. Hence, $\rho(\sigma^k(x), \sigma^k(y)) = 1$ and for any $0 < \delta \leqslant 1$

$$\frac{\#\left\{0 \leqslant k < n \mid \rho(\sigma^k(x), \sigma^k(y)) \geqslant \delta\right\}}{n} \geqslant \frac{\#\left\{0 \leqslant k < n \mid R_\alpha^{-k}(0) \in (h(x), h(y))\right\}}{n}$$

and the last term converges to $h(y) - h(x)$ as $n \to \infty$, using Weyl's Equidistribution Theorem [EW11, Example 4.18]. This means for $h(x)$ and $h(y)$ close enough we get

$$\overline{\lim_{n\to\infty}} \frac{S_n(\sigma, \delta, x, y)}{n} \geqslant d(h(x), h(y)) \,. \tag{53}$$

Now, since $\mathbb{T}^1 \backslash \{k\alpha \mod 1 \mid k \in \mathbb{Z}\}$ has full Lebesgue measure, we can find for any $\nu \in (0, 1]$ a set $M \subset \Sigma_\alpha$ with $\lfloor 1/\nu \rfloor$ points such that $h(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lfloor 1/\nu \rfloor \geqslant \nu$ between adjacent vertices. Then for ν small enough, we can use (53) to deduce that M is a (σ, δ, ν)-separated set. Therefore, $\mathrm{Sep}(\sigma, \delta, \nu) \geqslant \lfloor 1/\nu \rfloor$ and this implies $\underline{\mathrm{ac}}(\sigma) \geqslant 1$.

Next, we prove the upper bound. We want to show for $x, y \in \Sigma_\alpha$ with $h(x)$ and $h(y)$ close enough that

$$\overline{\lim_{n\to\infty}} \frac{S_n(\sigma, \delta, x, y)}{n} \leqslant C(\delta) \cdot d(h(x), h(y)) \tag{54}$$

with $C(\delta) > 0$. Suppose for $m \in \mathbb{N}$ that $2^{-m+1} \geqslant \delta > 2^{-m}$. Then

$$\begin{aligned}
&\#\left\{0 \leqslant k < n \mid \rho(\sigma^k(x), \sigma^k(y)) \geqslant \delta\right\} \\
&\quad \leqslant \#\left\{0 \leqslant k < n \mid \rho(\sigma^k(x), \sigma^k(y)) \geqslant 2^{-m}\right\} \\
&\quad \leqslant (2m+1) \cdot \#\left\{0 \leqslant k < n \mid \rho(\sigma^k(x), \sigma^k(y)) = 1\right\} \\
&\quad = (2m+1) \cdot \#\{0 \leqslant k < n \mid x_k \neq y_k\} \,.
\end{aligned}$$

Using a similar argument as in the case of the lower bound, we can conclude (54) with

$$C(\delta) := 2(2(-\log\delta/\log 2 + 1) + 1) \,,$$

taking into account that for k big enough $x_k \neq y_k$ if $R_\alpha^{-k}(0)$ or $R_\alpha^{-k}(R_\alpha^{-1}(0))$ lies in $(h(x), h(y))$ (for $h(x) = h(y)$ we know that the the right-hand sides of x and y coincide after a finite number of iterations, cf. Remark 11.1). Again, since $\mathbb{T}^1 \backslash \{k\alpha \mod 1 \mid k \in \mathbb{Z}\}$ has full Lebesgue measure, we can find for any $\nu \in (0, 1]$ a set $M \subset \Sigma_\alpha$ with $\lceil 2/\nu \rceil$ points such that $h(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lceil 2/\nu \rceil \leqslant \nu/2$ between adjacent vertices. Then for ν small enough, we can use (54) to deduce that M is a (σ, δ, ν)-spanning set. Therefore, $\mathrm{Span}(\sigma, \delta, \nu) \leqslant \lceil 2/\nu \rceil$ and hence $\overline{\mathrm{ac}}(\sigma) \leqslant 1$, using Corollary 9.2. This proves the assertion. □

Now, we want to show the analogous statement for Denjoy examples on the circle. Poincaré's classification of circle homeomorphisms in [Poi85] states that to each orientation-preserving homeomorphism $f : \mathbb{T}^1 \to \mathbb{T}^1$ of the circle we can associate a unique real number $\alpha \in [0,1)$, called the rotation number of f, such that f is semi-conjugate, via an orientation-preserving map, to the rigid rotation R_α, provided α is irrational (see also [dMvS93, HK97]). Another classical result by A. Denjoy [Den32] states that if f is a diffeomorphism such that its derivative is of bounded variation, then f is even conjugate to R_α. In this case, the amorphic complexity is zero. However, Denjoy also constructed examples of C^1 circle diffeomorphism with irrational rotation number that are not conjugate to a rotation and later Herman [Her79] showed that these examples can be made $C^{1+\varepsilon}$ for any $\varepsilon < 1$.

We say a *Denjoy example* or *Denjoy homeomorphism* is an orientation-preserving homeomorphism $f : \mathbb{T}^1 \to \mathbb{T}^1$ such that its rotation number α is irrational and it is not conjugate to a rotation.

Theorem 11.3. *Suppose* $f : \mathbb{T}^1 \to \mathbb{T}^1$ *is a Denjoy example. Then* $\mathrm{ac}(f) = 1$.

The theorem is a direct consequence of the following two lemmas. However, before we proceed, we want to collect some more facts concerning Denjoy examples, following mainly [Mar70, Section 0] and [HOR12, Section 2]. Since f is not conjugate to a rotation, we know from Poincaré's classification result that there is a unique Cantor set $C \subset \mathbb{T}^1$ such that $f|_C$ is minimal. This Cantor set can be described as

$$C = \mathbb{T}^1 \setminus \bigcup_{\ell=1}^{\infty} (a_\ell, b_\ell)$$

where $((a_\ell, b_\ell))_{\ell \in \mathbb{N}}$ is a family of open and pairwise disjoint intervals. The *accessible points* $A \subset \mathbb{T}^1$ *of* C are defined as the union of the endpoints of these intervals and the *inaccessible points of* C are defined as $I := C \setminus A$. A *Cantor function* $p : \mathbb{T}^1 \to \mathbb{T}^1$ *associated to* C is a continuous map satisfying

$$p(x) = p(y) \iff x = y \text{ or } x, y \in [a_\ell, b_\ell] \text{ for some } \ell \geqslant 1 ,$$

that is, p collapses the intervals $[a_\ell, b_\ell]$ to single points and is invertible on I. From this definition one can deduce that p is onto and that $p(A)$ is countable and dense in $\mathbb{T}^1$. Furthermore, we can assume without loss of generality that $p \circ f = R_\alpha \circ p$ where $\alpha \in [0,1) \setminus \mathbb{Q}$ is the rotation number of f, see [Mar70, Section 2].

Lemma 11.4. *Let* $f : \mathbb{T}^1 \to \mathbb{T}^1$ *be a Denjoy homeomorphism. Then there exists* $\delta > 0$ *such that* $\mathrm{Sep}(f, \delta, \nu) \geqslant \lfloor 1/\nu \rfloor$ *for all* $\nu \in (0,1]$.

Note that by definition this implies that $\underline{\mathrm{ac}}(f) \geqslant 1$.

Proof. Suppose $\nu \in (0, 1/2]$. Since $p(A)$ is dense in $\mathbb{T}^1$, we can choose for each $m \in \{1,2,3\}$ a point $\zeta_m \in p(A)$ such that

$$d(\zeta_m, \zeta_n) > 1/4 \quad \text{for } m \neq n . \tag{55}$$

Note that to each ζ_m we can associate an interval $[a_{\ell_m}, b_{\ell_m}]$ with $p\big([a_{\ell_m}, b_{\ell_m}]\big) = \{\zeta_m\}$. Now, choose $\delta > 0$ such that

$$\delta \leqslant \min_{m=1}^{3} d\big(a_{\ell_m}, b_{\ell_m}\big) .$$

Since $p(I)$ has full Lebesgue measure in $\mathbb{T}^1$, we can choose a set of $\lfloor 1/\nu \rfloor$ points

$$M = \{x_1, \ldots, x_{\lfloor 1/\nu \rfloor}\} \subset I$$

s.t. $p(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lfloor 1/\nu \rfloor \geqslant \nu$ between adjacent vertices. Consider distinct points $x_i, x_j \in M$ and assume without loss of generality that $\mathrm{Leb}([p(x_i), p(x_j)]) \leqslant 1/2$. Set $P := [p(x_i), p(x_j)]$. If $\zeta_1 \in R_\alpha^k(P)$ for some $k \geqslant 0$, then due to (55) we have that $\zeta_2 \in \mathbb{T}^1 \backslash R_\alpha^k(P)$ or $\zeta_3 \in \mathbb{T}^1 \backslash R_\alpha^k(P)$ such that both $[f^k(x_i), f^k(x_j)]$ and $[f^k(x_j), f^k(x_i)]$ contain some interval $[a_{\ell_m}, b_{\ell_m}]$ with $m \in \{1, 2, 3\}$. Hence, we have

$$d(f^k(x_i), f^k(x_j)) \geqslant \delta .$$

Consequently, we obtain

$$\frac{S_n(f, \delta, x_i, x_j)}{n} \geqslant \frac{\#\{0 \leqslant k < n \mid \zeta_1 \in R_\alpha^k(P)\}}{n} .$$

By Weyl's Equidistribution Theorem [EW11, Example 4.18], the right-hand side converges to $p(x_j) - p(x_i) \geqslant \nu$ as $n \to \infty$. This means that x_i and x_j are (f, δ, ν)-separated, so that M is an (f, δ, ν)-separated set. $\square$

Lemma 11.5. *Let* $f : \mathbb{T}^1 \to \mathbb{T}^1$ *be a Denjoy homeomorphism. Then for any* $\delta > 0$ *there exists a constant* $\kappa = \kappa(\delta)$ *such that*

$$\mathrm{Span}(f, \delta, \nu) \leqslant \kappa/\nu \quad \textit{for all } \nu \in (0, 1] .$$

Together with Corollary 9.2, this implies that $\overline{\mathrm{ac}}(f) \leqslant 1$, thus completing the proof of Theorem 11.3.

Proof. We show that if $0 < \tilde{\nu} \leqslant 1/(2(\lceil 1/\delta \rceil + 1))$, then

$$\mathrm{Span}(f, \delta, 2\tilde{\nu}(\lceil 1/\delta \rceil + 1)) \leqslant \lceil 1/\tilde{\nu} \rceil .$$

Since $\lceil 1/\tilde{\nu} \rceil \leqslant 2/\tilde{\nu}$, this yields the statement with $\kappa(\delta) := 4(\lceil 1/\delta \rceil + 1)$.

Let $\mu := \mathrm{Leb} \circ p^{-1}$ and define the function $\varphi_{\tilde{\nu}} : \mathbb{T}^1 \to [0, \infty)$ by

$$\varphi_{\tilde{\nu}}(x) := \mu([x, x + \tilde{\nu}]) .$$

Note that $d(x, y) \leqslant \mu([p(x), p(y)])$ and that $\varphi_{\tilde{\nu}}(x) = d(p^{-1}(x), p^{-1}(x + \tilde{\nu}))$ almost everywhere. In particular, $\varphi_{\tilde{\nu}}$ is measurable. Now, consider a subset $\tilde{I} \subseteq I$ such that

$$\frac{\#\{0 \leqslant k < n \mid \varphi_{\tilde{\nu}}(R_\alpha^k(x)) \geqslant \delta\}}{n} \longrightarrow \mathrm{Leb}(\{x \in \mathbb{T}^1 \mid \varphi_{\tilde{\nu}}(x) \geqslant \delta\}) \tag{56}$$

as $n \to \infty$ for all $x \in p(\tilde{I})$. Let $\{\varphi_{\tilde{v}} \geqslant \delta\} := \{x \in \mathbb{T}^1 \mid \varphi_{\tilde{v}}(x) \geqslant \delta\}$. Using Birkhoff's Ergodic Theorem, we know that $\tilde{I}$ can be chosen such that $p(\tilde{I})$ has full Lebesgue measure. Hence, we can choose a set of $\lceil 1/\tilde{v} \rceil$ points

$$M := \{x_1, \dots, x_{\lceil 1/\tilde{v} \rceil}\} \subset \tilde{I}$$

s.t. $p(M)$ is an equidistributed lattice in $\mathbb{T}^1$ with distance $1/\lceil 1/\tilde{v} \rceil \leqslant \tilde{v}$ between adjacent vertices. To show that M is an $(f, \delta, 2\tilde{v}(\lceil 1/\delta \rceil + 1))$-spanning set is our next aim.

For arbitrary $y \in \mathbb{T}^1$, let $x_i, x_j \in M$ be the two adjacent lattice points with $p(y) \in [p(x_i), p(x_j)]$ (that is, $j = i+1$ or $i = \lceil 1/\tilde{v} \rceil$ and $j = 1$). Then

$$R_\alpha^k[p(x_i), p(y)] \subseteq [R_\alpha^k(p(x_i)), R_\alpha^k(p(x_i)) + \tilde{v}]$$

for $k \geqslant 0$, and this implies

$$\begin{aligned} d\left(f^k(x_i), f^k(y)\right) &\leqslant \mu\left([p(f^k(x_i)), p(f^k(y))]\right) \\ &= \mu\left(R_\alpha^k[p(x_i), p(y)]\right) \leqslant \varphi_{\tilde{v}}(R_\alpha^k(p(x_i))) . \end{aligned}$$

We get that

$$\frac{S_n(f, \delta, x_i, y)}{n} \leqslant \frac{\#\left\{0 \leqslant k < n \mid \varphi_{\tilde{v}}(R_\alpha^k(p(x_i))) \geqslant \delta\right\}}{n}$$

and using (56) we know that the right-hand side of this inequality converges to $\mathrm{Leb}(\{\varphi_{\tilde{v}} \geqslant \delta\})$ as $n \to \infty$.

It remains to show that $\mathrm{Leb}(\{\varphi_{\tilde{v}} \geqslant \delta\}) < 2\tilde{v}(\lceil 1/\delta \rceil + 1)$. Suppose by contradiction that this inequality does not hold. Then $\{\varphi_{\tilde{v}} \geqslant \delta\}$ is not contained in a union of fewer than $\lceil 1/\delta \rceil + 1$ intervals of length $2\tilde{v}$. Consequently, there exist at least $\lceil 1/\delta \rceil + 1$ points $\zeta_i \in \mathbb{T}^1$ with $\varphi_{\tilde{v}}(\zeta_i) \geqslant \delta$ and $d(\zeta_i, \zeta_j) \geqslant \tilde{v}$ for $i \neq j$. We thus obtain

$$\mu(\mathbb{T}^1) \geqslant \sum_{i=1}^{\lceil 1/\delta \rceil + 1} \mu([\zeta_i, \zeta_i + \tilde{v}]) = \sum_{i=1}^{\lceil 1/\delta \rceil + 1} \varphi_{\tilde{v}}(\zeta_i) \geqslant 1 + \delta > 1 ,$$

which is a contradiction.

This means $\overline{\lim}_{n \to \infty} S_n(f, \delta, x_i, y)/n \leqslant \mathrm{Leb}(\{\varphi_{\tilde{v}} \geqslant \delta\}) < 2\tilde{v}(\lceil 1/\delta \rceil + 1)$. Since y was arbitrary, this shows that M is an $(f, \delta, 2\tilde{v}(\lceil 1/\delta \rceil + 1))$-spanning set and completes the proof. □

Remark 11.6. There is an intimate connection between Denjoy examples and Sturmian subshifts which allows us to deduce Theorem 11.2 from Theorem 11.3 directly, see [FGJ15, Section 3.6].

12 REGULAR TOEPLITZ SUBSHIFTS

Inspired by earlier constructions of almost periodic functions by Toeplitz, the notions of Toeplitz sequences and Toeplitz subshifts or flows were introduced by Jacobs and Keane in 1969 [JK69]. In the sequel these systems have been used by various authors to provide a series of interesting examples of symbolic dynamics with intriguing dynamical properties, see for example [MP79, Wil84] or [Dow05] and references therein. In what follows, we will study the amorphic complexity for so-called regular Toeplitz subshifts.

Let A be a finite alphabet, $\Sigma_A = A^{\mathbb{I}}$ with $\mathbb{I} = \mathbb{N}_0$ or $\mathbb{Z}$ and ρ the Cantor metric on Σ_A (see Section 4.5). Recall that a sequence $\omega = (\omega_k)_{k\in\mathbb{I}} \in \Sigma_A$ is called Toeplitz if for all $k \in \mathbb{I}$ there exists $p \in \mathbb{N}$ such that $\omega_{k+p\ell} = \omega_k$ for all $\ell \in \mathbb{N}$. Further, we denote the shift orbit closure of ω by Σ_ω such that (Σ_ω, σ) is the subshift generated by ω. Throughout this chapter we assume that ω is non-periodic. Given $p \in \mathbb{N}$ and $x = (x_k)_{k\in\mathbb{I}} \in \Sigma_A$, let

$$\mathrm{Per}(p,x) \ := \ \{k \in \mathbb{I} \mid x_k = x_{k+p\ell} \text{ for all } \ell \in \mathbb{N}\}\,.$$

We call the p-periodic part of ω the p-*skeleton of* ω. To be more precise, define the p-skeleton of ω, denoted by $S(p,\omega)$, as the sequence obtained by replacing ω_k with the new symbol '$*$' for all $k \notin \mathrm{Per}(p,\omega)$. Note that the p-skeletons of two arbitrary points in Σ_ω coincide after shifting one of them by at most $p-1$ positions. We say that p is an *essential period of* ω if $\mathrm{Per}(p,\omega)$ is non-empty and does not coincide with $\mathrm{Per}(\tilde{p},\omega)$ for any $\tilde{p} < p$. A *weak periodic structure of* ω is a sequence $(p_\ell)_{\ell\in\mathbb{N}}$ such that each p_ℓ divides $p_{\ell+1}$ and

$$\bigcup_{\ell\in\mathbb{N}} \mathrm{Per}(p_\ell,\omega) \ = \ \mathbb{I}\,. \tag{57}$$

If, additionally, all the p_l's are essential, we call $(p_\ell)_{\ell\in\mathbb{N}}$ a *periodic structure of* ω. For every (non-periodic) Toeplitz sequence we can find at least one periodic structure [Wil84].

Remark 12.1. Note that from each weak periodic structure we can obtain a periodic structure in the following way. Suppose $(p_\ell)_{\ell\in\mathbb{N}}$ is a weak periodic structure of ω. Without loss of generality we can assume that $\mathrm{Per}(p_\ell,\omega) \neq \emptyset$ and $\mathrm{Per}(p_\ell,\omega) \subsetneq \mathrm{Per}(p_{\ell+1},\omega)$ for all $\ell \in \mathbb{N}$ (recall that ω is non-periodic). For each p_ℓ choose the smallest $\tilde{p}_\ell \in \mathbb{N}$ such that $\mathrm{Per}(\tilde{p}_\ell,\omega)$ coincides with $\mathrm{Per}(p_\ell,\omega)$. Then by definition $\tilde{p}_\ell$ is an essential period. Since p_ℓ divides $p_{\ell+1}$, we have $\mathrm{Per}(\tilde{p}_\ell,\omega) \subsetneq \mathrm{Per}(\tilde{p}_{\ell+1},\omega)$. The next lemma and the minimality of the $\tilde{p}_\ell$'s imply that $\tilde{p}_\ell$ divides $\tilde{p}_{\ell+1}$ for each $\ell \in \mathbb{N}$, so that $(\tilde{p}_\ell)_{\ell\in\mathbb{N}}$ is a periodic structure.

Lemma 12.2. *If* $\mathrm{Per}(p, x) \subseteq \mathrm{Per}(q, x)$*, then* $\mathrm{Per}(\gcd(p, q), x) = \mathrm{Per}(p, x)$ *where* $x \in \Sigma_A$ *and* $p, q \in \mathbb{N}$.

Proof. Since $\gcd(p, q)$ divides p, we have $\mathrm{Per}(\gcd(p, q), x) \subseteq \mathrm{Per}(p, x)$. To prove the other direction we have to show that for each $k \in \mathrm{Per}(p, x)$ and $\ell \in \mathbb{N}$

$$x_k = x_{k+\gcd(p,q)\ell} .$$

Set $\tilde{p} := p/\gcd(p, q)$ and $\tilde{q} := q/\gcd(p, q)$. Note that there exist $\tilde{\ell} \in \mathbb{N}$ and $r \in \{0, \dots, \tilde{q} - 1\}$ such that $\ell = \tilde{q} \cdot \tilde{\ell} + r$. Further, observe that

$$\{\tilde{p} \cdot m \mod \tilde{q} \mid m \in \mathbb{N}\} = \{0, \dots, \tilde{q} - 1\} ,$$

since $\gcd(\tilde{p}, \tilde{q}) = 1$. This means there exist $m, \tilde{m} \in \mathbb{N}$ such that $\tilde{p} \cdot m = \tilde{q} \cdot \tilde{m} + r$, which implies

$$\begin{aligned} \gcd(p, q)\ell &= \gcd(p, q)[\tilde{q} \cdot \tilde{\ell} + r] \\ &= \gcd(p, q)[\tilde{q}(\tilde{\ell} - \tilde{m}) + \tilde{q} \cdot \tilde{m} + r] \\ &= p \cdot m + q(\tilde{\ell} - \tilde{m}) . \end{aligned}$$

Hence,

$$x_{k+\gcd(p,q)\ell} = x_{k+p\cdot m+q(\tilde{\ell}-\tilde{m})} = x_{k+p\cdot m} = x_k ,$$

using that $k + p \cdot m \in \mathrm{Per}(p, x) \subseteq \mathrm{Per}(q, x)$. □

Given $p \in \mathbb{N}$, we define the relative density of the p-skeleton of ω by

$$D(p) := \frac{\#(\mathrm{Per}(p, \omega) \cap [0, p - 1])}{p} .$$

Since ω is non-periodic, we have $D(p) \leqslant 1 - 1/p$. For a (weak) periodic structure $(p_\ell)_{\ell \in \mathbb{N}}$ the densities $D(p_\ell)$ are non-decreasing in ℓ and we say that (Σ_ω, σ) is a *regular Toeplitz subshift* if $\lim_{\ell\to\infty} D(p_\ell) = 1$. Note that regularity of a Toeplitz subshift does not depend on the chosen (weak) periodic structure (use (57) and Lemma 12.2).

It is well known that a regular Toeplitz subshift is an almost sure 1-1 extension of a minimal isometry (an odometer) [Dow05]. Thus, we obtain from Theorem 8.5 that its asymptotic separation numbers are finite. However, as mentioned in the introduction, a quantitative analysis is possible and yields the following.

Theorem 12.3. *Suppose* (Σ_ω, σ) *is a regular Toeplitz subshift. Let* $(p_\ell)_{\ell \in \mathbb{N}}$ *be a (weak) periodic structure of* ω*. For* $\delta, s > 0$ *we have*

$$\varlimsup_{\nu\to 0} \frac{\mathrm{Sep}(\sigma, \delta, \nu)}{\nu^{-s}} \leqslant C \cdot \varlimsup_{\ell\to\infty} \frac{p_{\ell+1}}{(1 - D(p_\ell))^{-s}} ,$$

with $C = C(\delta, s) > 0$.

Note that this directly implies Theorem 2.12.

Proof. Recall that, since ω is a regular Toeplitz sequence, the densities $D(p_\ell)$ are non-decreasing and converge to 1. Choose $m \in \mathbb{N}$ with $2^{-m} < \delta \leqslant 2^{-m+1}$ and $\ell \in \mathbb{N}$ such that

$$(2m+1)2(1-D(p_{\ell+1})) \ < \ \nu \ \leqslant \ (2m+1)2(1-D(p_\ell))\,. \qquad (58)$$

Then we have

$$\mathrm{Sep}(\sigma,\delta,\nu) \ \leqslant \ \mathrm{Sep}(\sigma,2^{-m},(2m+1)2(1-D(p_{\ell+1})))$$

and claim that the second term is bounded from above by p_{l+1}.

Assume for a contradiction that there exists a $(\sigma,2^{-m},(2m+1)2(1-D(p_{\ell+1})))$-separated set $S \subseteq \Sigma_\omega$ with more than $p_{\ell+1}$ elements. Then there are at least two points $x = (x_k)_{k\in\mathbb{I}}$, $y = (y_k)_{k\in\mathbb{I}} \in S$ with the same $p_{\ell+1}$-skeleton. This means that x and y can differ at most at the remaining positions $k \notin \mathrm{Per}(p_{\ell+1},x) = \mathrm{Per}(p_{\ell+1},y)$. Using the fact that $\rho(x,y) \geqslant 2^{-m}$ if and only if $x_k \neq y_k$ for some $k \in \mathbb{I}$ with $|k| \leqslant m$, we obtain

$$\begin{aligned}
\varlimsup_{n\to\infty} \frac{S_n(\sigma,2^{-m},x,y)}{n} \quad &\leqslant \quad (2m+1)\varlimsup_{n\to\infty} \frac{\#\{0 \leqslant k < n \mid x_k \neq y_k\}}{n} \\
&\leqslant \quad (2m+1)\varlimsup_{n\to\infty} \frac{\#([0,n-1] \setminus \mathrm{Per}(p_{\ell+1},\omega))}{n} \\
&= \quad (2m+1)(1-D(p_{\ell+1}))\,.
\end{aligned}$$

However, this contradicts (58). Hence, we obtain

$$\frac{\mathrm{Sep}(\sigma,\delta,\nu)}{\nu^{-s}} \ \leqslant \ C(\delta,s)\cdot\frac{p_{\ell+1}}{(1-D(p_\ell))^{-s}}\,,$$

where $C(\delta,s) := (2m+1)^s$. Note that m only depends on δ. Taking the limit superior yields the desired result. □

For the remainder of this section, our aim is to provide a class of examples demonstrating that the above estimate is sharp and that the amorphic complexity of regular Toeplitz flows takes at least a dense subset of values in $[1,\infty)$. To that end, we first recall an alternative definition of Toeplitz sequences (cf. [JK69]). Consider the extended alphabet $\mathcal{A} := A \cup \{*\}$ where we can think of $*$ as a hole or placeholder like in the definition of the p-skeleton. Then $\omega \in \Sigma_A$ is a Toeplitz sequence if and only if there exists an *approximating sequence* $(\omega^\ell)_{\ell\in\mathbb{N}}$ of periodic points in $(\Sigma_{\mathcal{A}},\sigma)$ such that (i) for all $k \in \mathbb{I}$ we have $\omega_k^{\ell+1} = \omega_k^\ell$ as soon as $\omega_k^\ell \in A$ for some $\ell \in \mathbb{N}$ and (ii) $\omega_k = \lim_{\ell\to\infty}\omega_k^\ell$, see [Ebe71]. Such an approximating sequence of a Toeplitz sequence is not unique. For example, every sequence of p_ℓ-skeletons $(S(p_\ell,\omega))_{\ell\in\mathbb{N}}$ with $(p_\ell)_{\ell\in\mathbb{N}}$ a (weak) periodic structure satisfies these properties.

Let us interpret Theorem 12.3 in this context. For a p-periodic point $x \in \Sigma_{\mathcal{A}}$ we can define the relative density of the holes in x by

$$r(x) := \frac{\#\{0 \leqslant k < p \mid x_k = *\}}{p} .$$

Note that $D(p) = 1 - r(S(p, \omega))$ for every $p \in \mathbb{N}$. Suppose $(\omega^\ell)_{\ell \in \mathbb{N}}$ is an approximating sequence of ω. We say $(p_\ell)_{\ell \in \mathbb{N}}$ is a *sequence of corresponding periods of* $(\omega^\ell)_{\ell \in \mathbb{N}}$ if p_ℓ divides $p_{\ell+1}$ and $\sigma^{p_\ell}(\omega^\ell) = \omega^\ell$ for each $\ell \in \mathbb{N}$. We have that $r(\omega^\ell) \geqslant 1/p_\ell$. Moreover, $r(\omega^\ell) \geqslant 1 - D(p_\ell)$, so that Theorem 12.3 implies

Corollary 12.4. *Assume (Σ_ω, σ) is a regular Toeplitz subshift. Let $(\omega^\ell)_{\ell \in \mathbb{N}}$ be an approximating sequence of ω and let $(p_\ell)_{\ell \in \mathbb{N}}$ be a sequence of corresponding periods of $(\omega^\ell)_{\ell \in \mathbb{N}}$. Furthermore, assume $p_{\ell+1} \leqslant C p_\ell^t$ and $r(\omega^\ell) \leqslant K/p_\ell^u$ for ℓ large enough, where $C, t \geqslant 1$, $u \in (0, 1]$ and $K > 0$. Then*

$$\overline{\mathrm{ac}}(\sigma) \leqslant \frac{t}{u} .$$

For the construction of examples, it will be convenient to use so-called (p, q)-Toeplitz (infinite) words, as introduced in [CK97]. Let $\mathbb{I} = \mathbb{N}_0$. Suppose v is a finite and non-empty word with letters in $\mathcal{A}$ and at least one entry distinct from $*$. Let $|v|$ be its length and $|v|_*$ be the number of holes in v. We use the notation $\overline{v} \in \Sigma_{\mathcal{A}}$ for the one-sided periodic sequence that is created by repeating v infinitely often. Define the sequence $(T_\ell(v))_{\ell \in \mathbb{N}}$ recursively by

$$T_\ell(v) := F_v(T_{\ell-1}(v)) ,$$

where $T_0(v) := \overline{*}$ and $F_v : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ assigns to each $x \in \Sigma_{\mathcal{A}}$ the sequence that is obtained from $\overline{v}$ by replacing the subsequence of all occurrences of $*$ in $\overline{v}$ by x. We get that $(T_\ell(v))_{\ell \in \mathbb{N}}$ is an approximating sequence and denote the corresponding Toeplitz sequence by $T(v)$ [CK97]. Setting $p := |v|$, $q := |v|_*$ and $d := \gcd(p, q)$, we say $T(v)$ is a (p, q)-*Toeplitz word*. One particular nice feature of (p, q)-Toeplitz words is that in order to exclude periodicity, one only has to check a short prefix of the sequence.

Theorem 12.5 ([CK97, Theorem 4])**.** *Let $T(v)$ be a (p, q)-Toeplitz word. Then $T(v)$ is periodic if and only if its prefix of length p is d-periodic.*

Theorem 12.6. *Suppose $m \in \mathbb{N}$ and let $0^m 1$ be the word starting with m zeros and ending with a single one. Furthermore, let v be a word with letters in $\mathcal{A} = \{0, 1, *\}$ such that $1 \leqslant |v|_* \leqslant |v| \leqslant m$. Then $\omega := T(0^m 1 v)$ is a (p, q)-Toeplitz word and the corresponding regular Toeplitz subshift (Σ_ω, σ) has amorphic complexity*

$$\mathrm{ac}(\sigma) = \frac{\log p/d}{\log p/q} .$$

Proof. Define for each $n \in \mathbb{N}$ and $x = (x_k)_{k \in \mathbb{N}_0}, y = (y_k)_{k \in \mathbb{N}_0} \in \Sigma_{\mathcal{A}}$

$$S_n(x, y) := \#\{0 \leqslant k < n \mid x_k, y_k \neq * \text{ and } x_k \neq y_k\} .$$

Observe that

$$\begin{aligned} &S_p\big(T(0^{\,m}1\nu), \sigma^j(T(0^{\,m}1\nu))\big) \\ &\quad \geqslant S_p\big(T_1(0^{\,m}1\nu), \sigma^j(T_1(0^{\,m}1\nu))\big) = S_p\big(\overline{0^{\,m}1\nu}, \sigma^j(\overline{0^{\,m}1\nu})\big) \geqslant 1 \end{aligned}$$

for every $0 < j < p$ due to the special form of the prefix $0^{\,m}1$ and the assumption $|\nu| \leqslant m$. This directly implies that ω is non-periodic, using Theorem 12.5.

To get an upper bound for $\overline{ac}(\sigma)$, note that $(p^\ell/d^{\ell-1})_{\ell \in \mathbb{N}}$ is a sequence of corresponding periods of $(T_\ell(0^{\,m}1\nu))_{\ell \in \mathbb{N}}$ and $r(T_\ell(0^{\,m}1\nu)) = q^\ell/p^\ell$ for each $\ell \in \mathbb{N}$. This is proved easily by induction: The statement is true for $T_1(0^{\,m}1\nu) = \overline{0^{\,m}1\nu}$. When going from ℓ to $\ell+1$, by the induction hypothesis, each of the $p^\ell/d^{\ell-1}$-periodic blocks of $T_\ell(0^{\,m}1\nu)$ has $q^\ell/d^{\ell-1}$ free positions. In order to accommodate q/d such periodic blocks of $T_\ell(0^{\,m}1\nu)$ it needs p^ℓ/d^ℓ of the p-periodic blocks of $\overline{0^{\,m}1\nu}$ with q free positions each. Thus, the resulting periodic block of $T_{\ell+1}(0^{\,m}1\nu)$ has length $p^{\ell+1}/d^\ell$ and $q^{\ell+1}/d^\ell$ free positions. Now, Corollary 12.4 gives the desired upper bound.

In order to prove the lower bound, we show by a similar induction that

$$S_{p^\ell/d^{\ell-1}}\big(T_\ell(0^{\,m}1\nu), \sigma^j(T_\ell(0^{\,m}1\nu))\big) \geqslant q^{\ell-1}/d^{\ell-1} \tag{59}$$

for every $0 < j < p^\ell/d^{\ell-1}$ and $\ell \in \mathbb{N}$. If j is not a multiple of p, then by induction assumption each $p^\ell/d^{\ell-1}$-periodic block of $T_\ell(0^{\,m}1\nu)$ has $p/d \cdot q^{\ell-2}/d^{\ell-2}$ mismatches with $\sigma^j(T_\ell(0^{\,m}1\nu))$ coming from the mismatches of the p/d contained $p^{\ell-1}/d^{\ell-2}$-periodic blocks of the sequence $T_{\ell-1}(0^{\,m}1\nu)$ with its shift $\sigma^j(T_{\ell-1}(0^{\,m}1\nu))$. If j is a multiple of p, then the mismatches result in a similar way from the shift in the sequences that are inserted into $\overline{0^{\,m}1\nu}$, since $\sigma^{ip}(T_\ell(0^{\,m}1\nu)) = F_\nu(\sigma^{iq}(T_{\ell-1}(0^{\,m}1\nu)))$. Note that the fact that $p^\ell/d^{\ell-1}$ is a minimal period comes from the assumption that $d = \gcd(p, q)$.

As a direct consequence from (59), we obtain that for all $\ell \in \mathbb{N}$ and $0 \leqslant i < j < p^\ell/d^{\ell-1}$

$$S_{p^\ell/d^{\ell-1}}\big(\sigma^i(T_\ell(0^{\,m}1\nu)), \sigma^j(T_\ell(0^{\,m}1\nu))\big) \geqslant q^{\ell-1}/d^{\ell-1} .$$

Hence,

$$\{\omega, \sigma(\omega), \ldots, \sigma^{p^\ell/d^{\ell-1}-1}(\omega)\}$$

is a $(\sigma, 1, q^{\ell-1}/p^\ell)$-separated set. For ν small enough choose $\ell \in \mathbb{N}$ such that $q^\ell/p^{\ell+1} < \nu \leqslant q^{\ell-1}/p^\ell$ and observe

$$\frac{\mathrm{Sep}(\sigma, \delta, \nu)}{\nu^{-s}} \geqslant \frac{\mathrm{Sep}(\sigma, 1, q^{\ell-1}/p^\ell)}{\nu^{-s}} > \frac{p^\ell}{d^{\ell-1}} \cdot \frac{q^{ls}}{p^{(l+1)s}}$$

for $\delta, s > 0$. This yields $\underline{ac}(\sigma) \geqslant (\log p/d)/(\log p/q)$. □

Corollary 12.7. *In the class of (p,q)-Toeplitz words, amorphic complexity takes (at least) a dense set of values in $[1,\infty)$.*

Proof. Choose arbitrary $q \in \mathbb{N}$ and $p \in \mathbb{N}$ with $p \geqslant 2q+1$ such that $\gcd(q,p) = 1$. According to the previous theorem, there is a (p,q)-Toeplitz word ω with

$$z(p,q) \;:=\; \mathrm{ac}\left(\sigma|_{\Sigma_\omega}\right) \;=\; \frac{\log p}{\log p/q}.$$

Let $f\colon (1,\infty) \ni x \mapsto \frac{x}{x-1}$ and note that $f(z(p,q)) = \frac{\log p}{\log q}$. Further, for $\varepsilon > 0$ let q_ε be the smallest prime number bigger than $\exp(2/\varepsilon)$ and observe that

$$\{f(z(p,q_\varepsilon))\colon p \in \mathbb{N},\ p \geqslant 2q_\varepsilon + 1 \text{ and } \gcd(p,q_\varepsilon) = 1\}$$

is ε-dense in $(1,\infty)$. Thus, the image of

$$M \;:=\; \{z(p,q)\colon q,\, p \in \mathbb{N},\ p \geqslant 2q+1 \text{ with } \gcd(p,q) = 1\}$$

under f is dense in $f((1,\infty)) = (1,\infty)$. Since f is a monotone function, this shows that M is dense in $(1,\infty)$ which proves the statement. □

Remark 12.8. From the results in [CK97, Theorem 5], one can directly conclude that for all (non-periodic) (p,q)-Toeplitz words the power entropy (see Section 3.3 for the definition) equals $(\log p/d)/(\log p/q)$. Thus, for our examples provided by the last theorem, power entropy and amorphic complexity coincide. It would be interesting to know if this is true for all (p,q)-Toeplitz words, or if not, in which cases this equality holds.

Part IV

BIFURCATIONS OF FAMILIES OF BOUNDED ORBITS ASSOCIATED WITH INTERVAL MAPS

13 SETS OF UNIFORMLY BOUNDED ORBITS AND m-INTERVALS

Suppose $f : [0,1] \to [0,1]$ is a continuous map. In what follows, we want to study the following two families of sets

$$\mathcal{B}_\ell(t) := \{x \in [0,1] \mid f^n(x) \geqslant t \text{ for all } n \in \mathbb{N}_0\}$$

and

$$\mathcal{B}_u(t) := \{x \in [0,1] \mid f^n(x) \leqslant t \text{ for all } n \in \mathbb{N}_0\}$$

with $t \in [0,1]$. In the following we will sometimes use the notation $\mathcal{B}_*(\cdot)$ in the sense of referring to $\mathcal{B}_\ell(\cdot)$ and $\mathcal{B}_u(\cdot)$ at the same time. Observe that

$$\mathcal{B}_\ell(t) = \bigcap_{n=0}^{\infty} f^{-n}\big([t,1]\big) \text{ and } \mathcal{B}_u(t) = \bigcap_{n=0}^{\infty} f^{-n}\big([0,t]\big),$$

meaning $\mathcal{B}_\ell(t)$ and $\mathcal{B}_u(t)$ are also the *surviving sets* of points that never hit the interval $[0,t)$ and $(t,1]$, respectively, under the dynamics of f. From this we directly obtain the next proposition.

Proposition 13.1. *$\mathcal{B}_*(t)$ is closed and forward invariant for each $t \in [0,1]$. Furthermore, $\mathcal{B}_\ell(t) \subseteq \mathcal{B}_\ell(t')$ for $t' \leqslant t$ and $\mathcal{B}_u(s) \subseteq \mathcal{B}_u(s')$ for $s' \geqslant s$, i.e.,*

$$\mathcal{B}_\ell(t) = \bigcap_{t'<t} \mathcal{B}_\ell(t') \quad \text{and} \quad \mathcal{B}_u(s) = \bigcap_{s'>s} \mathcal{B}_u(s').$$

Now, assume that $m \in \mathbb{N}$. If the inequality

$$f^m(x_0) < x_0$$

is fulfilled for some $x_0 \in (0,1]$, then there always exists a non-empty open interval $I \subset [0,1]$ with $x_0 \in I$ such that $f^m(x) < x$ for all $x \in I$, due to the continuity of f^m (note that the same is true for points fulfilling $f^m(x) > x$). We can choose these intervals in a maximal manner:

Definition 13.2. Let $m \in \mathbb{N}$. An open interval $I^m \subset [0,1]$ with $f^m(x) < x$ (respectively, $f^m(x) > x$) for all $x \in I^m$ is called a *lower* (resp., *upper*) *m-interval for f* if there exists no open interval $J \subset [0,1]$ such that I^m is strictly contained in J and $f^m(x) < x$ (respectively, $f^m(x) > x$) for all $x \in J$. Further, we say I^m is a *lower* (resp., *upper*) *interval of order* m if I^m is a lower (upper) m-interval and there exists no other lower (upper) $\tilde{m}$-interval $I^{\tilde{m}}$ with $\tilde{m} \in \mathbb{N}$ such that $I^{\tilde{m}} = I^m$ and $\tilde{m} < m$.

Observe that an m-interval cannot contain a periodic point with period m.

Proposition 13.3. *Let $I^m \subset [0,1]$ be an m-interval and $[\alpha, \beta]$ be its closure. For I^m a lower m-interval α is always m-periodic. If $\beta \neq 1$, then β is also m-periodic. If I^m is an upper m-interval, then the corresponding statements hold as well.*

Proof. We assume that I^m is a lower m-interval (the proof in the other case is analogous). For $\alpha = 0$ we get that $f^m(0) = 0$ due to the continuity of f. For $\alpha > 0$ we have that $f^m(\alpha) \geqslant \alpha$, using the maximality of I^m. Further, by choosing a sequence in I^m converging to α and using the continuity of f, we get $f^m(\alpha) \leqslant \alpha$. Hence, $f^m(\alpha) = \alpha$. For $\beta \neq 1$ we can apply an analogous argument. □

Remark 13.4. With respect to the last proposition if $\beta = 1$, then both cases $f^m(1) = 1$ and $f^m(1) < 1$ can occur.

Recall the basic notion of conjugacy between two continuous maps on the unit interval, see Section 4.4. Note that in our setting the conjugacy map $h : [0,1] \to [0,1]$ is either order-preserving or order-reversing and this directly implies the next proposition.

Proposition 13.5. *Suppose $f, g : [0,1] \to [0,1]$ are continuous maps. If f and g are conjugate, then each m-interval of f is mapped onto an m-interval of g and vice versa.*

Lemma 13.6. *Suppose $I^m \subset [0,1]$ is a lower (upper) m-interval. For $t, s \in I^m$ we have that*

$$\mathcal{B}_\ell(t) = \mathcal{B}_\ell(s) \quad \big(\mathcal{B}_u(t) = \mathcal{B}_u(s)\big).$$

Proof. We give the proof for I^m a lower m-interval. The argument for the upper case is similar. W.l.o.g. assume that $s < t$. Since $f^n(x) \geqslant t > s$ for all $n \geqslant 0$ and $x \in \mathcal{B}_\ell(t)$, we have $\mathcal{B}_\ell(t) \subseteq \mathcal{B}_\ell(s)$. The reverse inclusion will be proved by contradiction. Assume there exists $x \in \mathcal{B}_\ell(s) \backslash \mathcal{B}_\ell(t) \neq \emptyset$. Then by definition $f^n(x) \geqslant s$ for all $n \geqslant 0$ and there is an $n_0 \geqslant 0$ such that $f^{n_0}(x) < t$. Note that in fact $f^n(x) > s$ for all $n \geqslant 0$ because if $f^l(x) = s$ for some $l \in \mathbb{N}_0$, then $f^{m+l}(x) = f^m(f^l(x)) = f^m(s) < s$, since $s \in I^m$. Set $x_0 := f^{n_0}(x)$ and observe that $x_0 \in I^m$ because $s < x_0 < t$. Consider the sequence $(f^{m\cdot k}(x_0))_{k\in\mathbb{N}_0}$. For all $k \geqslant 0$ we have that

$$s \;<\; f^{m\cdot(k+1)}(x_0) \;<\; f^{m\cdot k}(x_0) < t\,,$$

that is, $(f^{m\cdot k}(x_0))_{k\in\mathbb{N}_0}$ is a monotone decreasing and bounded sequence. That means the limit of this sequence exists and we set

$$y \;:=\; \lim_{k\to\infty} f^{m\cdot k}(x_0) \;=\; \inf_{k\in\mathbb{N}_0} \{f^{m\cdot k}(x_0)\} \in [s,t)\,.$$

Furthermore,

$$f^{\mathfrak{m}}(y) = \lim_{k\to\infty} f^{\mathfrak{m}\cdot(k+1)}(x_0) = y\,,$$

in other words, y is an $\mathfrak{m}$-periodic point contained in $I^{\mathfrak{m}}$, which gives the desired contradiction. □

For the next chapter we need the following notion.

Definition 13.7. A lower (upper) $\mathfrak{m}$-interval $I^{\mathfrak{m}}$ with $\mathfrak{m} \in \mathbb{N}$ is called *maximal* if there exists no other lower (upper) $\tilde{\mathfrak{m}}$-interval $I^{\tilde{\mathfrak{m}}}$ with $\tilde{\mathfrak{m}} \in \mathbb{N}$ such that $I^{\mathfrak{m}}$ is strictly contained in $I^{\tilde{\mathfrak{m}}}$.

Similarly to Proposition 13.5, we have

Proposition 13.8. *Let* $f, g : [0,1] \to [0,1]$ *be continuous maps. If* f *and* g *are conjugate, then each maximal* $\mathfrak{m}$*-interval of* f *is mapped onto a maximal* $\mathfrak{m}$*-interval of* g *and vice versa.*

14 LOWER AND UPPER BIFURCATION SETS

First, we define the *lower* and *upper bifurcation set of* f to be

$$\mathcal{E}_\ell := \{x \in [0,1] \mid x \in \mathcal{B}_\ell(x)\} \quad \text{and} \quad \mathcal{E}_u := \{x \in [0,1] \mid x \in \mathcal{B}_u(x)\},$$

respectively. Again, from time to time we will make use of the notation $\mathcal{E}_*$ in the sense of referring to $\mathcal{E}_\ell$ and $\mathcal{E}_u$ at the same time.

Proposition 14.1. *The bifurcation set $\mathcal{E}_*$ is closed.*

Proof. Let $x \in [0,1]$ be a limit point of $\mathcal{E}_\ell$. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{E}_\ell$ such that $x_k \to x$ as $k \to \infty$ and for each $k \in \mathbb{N}$ we have $f^n(x_k) \geqslant x_k$ for all $n \geqslant 0$. Now, fix $n \in \mathbb{N}$ and observe that

$$f^n(x) = \lim_{k\to\infty} f^n(x_k) \geqslant \lim_{k\to\infty} x_k = x.$$

Since n is arbitrary, we get $x \in \mathcal{E}_\ell$. The proof for $\mathcal{E}_u$ is analogous. $\square$

The next result is similar to Propositions 13.5 and 13.8.

Proposition 14.2. *Suppose $f, g : [0,1] \to [0,1]$ are continuous maps. If f and g are conjugate, then the lower (upper) bifurcation set of f is mapped bijectively to the lower or upper bifurcation set of g.*

Theorem 14.3. *Let $\alpha \in \mathcal{E}_\ell$ and suppose that $\beta \in \mathcal{E}_\ell$, or $\beta = 1$, such that $\alpha < \beta$ and no other point in (α, β) belongs to $\mathcal{E}_\ell$. Or, similarly, let $\beta \in \mathcal{E}_u$ and suppose that $\alpha \in \mathcal{E}_u$, or $\alpha = 0$, is such that $\alpha < \beta$ and no other point in (α, β) belongs to $\mathcal{E}_u$. Then in the first case, (α, β) is a maximal lower m-interval for some $m \in \mathbb{N}$, and in the second, (α, β) is a maximal upper m-interval for some $m \in \mathbb{N}$.*

To be precise for $\beta = 1 \notin \mathcal{E}_\ell$ ($\alpha = 0 \notin \mathcal{E}_u$) we actually show that $(\alpha, 1]$ ($[0, \beta)$) is a maximal lower (upper) m-interval.

Proof. We give the proof for maximal lower m-intervals. In the second case the argument works analogously and is thus left to the reader. First, we claim that (α, β) is a lower m-interval for some $m \in \mathbb{N}$ (note that for $\beta = 1 \notin \mathcal{E}_\ell$ we actually have to consider the interval $(\alpha, 1]$). We show this by contradiction. Assume (α, β) is not a lower m-interval for all $m \in \mathbb{N}$ and fix an arbitrary $\gamma_0 \in (\alpha, \beta)$. Since $\gamma_0 \notin \mathcal{E}_\ell$, there exists $q_0 \in \mathbb{N}$ such that $f^{q_0}(\gamma_0) < \gamma_0$. This implies that there is a lower q_0-interval contained in (α, β), where the inclusion follows from the fact that no point in $\mathcal{E}_\ell$ can be contained in any lower m-interval. Furthermore, this inclusion must be strict. Hence, at least one of the endpoints of this q_0-interval, denoted by γ_1, lies in (α, β).

We have that γ_1 is q_0-periodic according to Proposition 13.3 and from this follows that $q_0 > 1$.

Since $\gamma_1 \notin \mathcal{E}_\ell$, there exists $q_1 \in \mathbb{N}$ such that $f^{q_1}(\gamma_1) < \gamma_1$ and w.l.o.g. we can assume that $q_1 < q_0$. With the same argument as above there exists a lower q_1-interval contained in (α, β) and at least one of its endpoints, denoted by γ_2, lies in (α, β). Again, we have that γ_2 is q_1-periodic and $q_1 > 1$.

Now, we can repeat this step indefinitely, since each corresponding $q_i \in \mathbb{N}$ is strictly bigger than 1. However, we also have that $q_0, q_1, \ldots, q_i, \ldots$ are strictly decreasing to 1 which is a contradiction.

The maximality is clear if $\alpha = 0$ and $\beta = 1$, and follows otherwise, again, directly from the fact that no point in $\mathcal{E}_\ell$ can be contained in any lower m-interval. □

Theorem 14.3 together with Proposition 14.1 and the simple observation that each (maximal) m-interval is contained in the complement of $\mathcal{E}_*$ imply immediately that

$$[0,1] \backslash \mathcal{E}_\ell = \dot{\bigcup_{\substack{m \in \mathbb{N} \\ I \in \mathcal{I}_\ell^m}}} I \quad \text{and} \quad [0,1] \backslash \mathcal{E}_u = \dot{\bigcup_{\substack{m \in \mathbb{N} \\ I \in \mathcal{I}_u^m}}} I\,, \tag{60}$$

where $\mathcal{I}_\ell^m$ and $\mathcal{I}_u^m$ is the collection of all maximal lower and upper intervals of order m, respectively, for each $m \in \mathbb{N}$. From this we can directly deduce the following corollaries.

Corollary 14.4. *Two maximal lower (upper) intervals of order m and $\tilde{m}$, respectively, are disjoint and each lower (upper) m-interval is contained in a unique maximal lower (upper) interval of order $\tilde{m}$ where $m, \tilde{m} \in \mathbb{N}$.*

Corollary 14.5. *Assume I^m is a maximal lower m-interval with closure $[\alpha, \beta]$. We have that α is contained in $\mathcal{E}_\ell$ and when $\beta \neq 1$ this is also true for β. If I^m is a maximal upper m-interval, then the corresponding statement holds as well.*

The next corollary also depends upon Proposition 13.3.

Corollary 14.6. *If $x \in \mathcal{E}_*$ is only accumulated from the left or right in $\mathcal{E}_*$, then x is a periodic point. This holds in particular if x is an isolated point in $\mathcal{E}_*$.*

If we assume that $\mathcal{E}_*$ has empty interior (which is equivalent to assuming that $\mathcal{E}_*$ is nowhere dense because of Proposition 14.1), we can use Proposition 13.3 again together with (60) to derive the following assertion.

Corollary 14.7. *If $\mathcal{E}_*$ is nowhere dense, then the periodic points in $\mathcal{E}_*$ form a dense subset.*

Now, we need the notion of transitivity for continuous maps, cf. Section 4.1.

Theorem 14.8. *The bifurcation set $\mathcal{E}_*$ is nowhere dense and infinite if f is transitive.*

As mentioned above, for $\mathcal{E}_*$ to be nowhere dense means that it has empty interior and this is true immediately if f is transitive. The other assertion follows directly from the next lemma.

Lemma 14.9. *If f is transitive, then 0 and 1 are limit points of $\mathcal{E}_\ell$ and $\mathcal{E}_u$, respectively.*

Proof. Note that by definition we always have that 0 belongs to $\mathcal{E}_\ell$ and 1 to $\mathcal{E}_u$. We only show that 0 is a limit point of $\mathcal{E}_\ell$; the other case is analogous and is thus left to the reader.

First, observe that $\mathcal{E}_\ell \cap (0,1)$ is non-empty. Otherwise, every point in $[0,1]$ would converge either to 0 or 1 (since the only possible fixed points of f would be 0 and 1), contradicting our assumption that f is transitive.

We proceed by proving the desired assertion by contradiction. Assume that 0 is an isolated point of $\mathcal{E}_\ell$. Take the smallest value in $\mathcal{E}_\ell \backslash \{0\}$ (note that this set is compact) and denote it by β. We have that $(0,\beta)$ is a lower m-interval for some $m \in \mathbb{N}$, using Theorem 14.3. This implies in particular that 0 is m-periodic, according to Proposition 13.3. For an arbitrary $x \in (0,\beta)$ note that $x > f^m(x) \in [0,\beta)$ and hence $f^{k\cdot m}(x) \to 0$ as $k \to \infty$ (note that 0 must be the limit point using the same argument as in the proof of Lemma 13.6). Further, for each $r \in \{0,\dots,m-1\}$ we have

$$\lim_{k\to\infty} f^{k\cdot m+r}(x) = f^r\left(\lim_{k\to\infty} f^{k\cdot m}(x)\right) = f^r(0) .$$

Since the finite union of the sets $\{k\cdot m + r \mid k \in \mathbb{N}_0\}$ with $r \in \{0,\dots,m-1\}$ equals $\mathbb{N}_0$, we have that for all $x \in (0,\beta)$ the limit points of $(f^n(x))_{n\in\mathbb{N}_0}$ equal $\{0, f(0),\dots,f^{m-1}(0)\}$. However, since $(0,\beta)$ is open and f transitive, there exists a point $x \in (0,\beta)$ such that the closure of $(f^n(x))_{n\in\mathbb{N}_0}$ is $[0,1]$. This contradiction finishes the proof. □

For the last assertions we present here, as well as for the next chapter, we need the following definition. We say that a continuous map $f : [0,1] \to [0,1]$ is *piecewise monotone* if there exist $K \in \mathbb{N}_0$ and $0 = d_0 < d_1 < \dots < d_K < d_{K+1} = 1$ such that f is strictly monotone on each interval $[d_k, d_{k+1}]$ with $k \in \{0,\dots,K\}$. Furthermore, we call the minimal choice of the d_k's, i.e. all the d_k's such that f is not monotone in any neighborhood of d_k for $k \in \{1,\dots,K\}$, the *turning points of* f and denote them by $T(f)$. For more information about piecewise monotonic maps, see for example [Pre88]. The next theorem will be an immediate consequence of the results about the continuity of the topological entropy for the family of systems considered in [Rai94]. For the definition of topological entropy, see Chapter 1.

Theorem 14.10. *Suppose* f *is a continuous piecewise monotone map with topological entropy* $h_{top}(f) > 0$*. We have that the bifurcation set* $\mathcal{E}_*$ *is uncountable.*

Proof. We give the argument for the set $\mathcal{E}_\ell$. For the set $\mathcal{E}_u$, the argument again proceeds in an analogous way. First, note that if $\mathcal{E}_\ell$ has non-empty interior, then the statement is automatically true, thus we assume that $\mathcal{E}_\ell$ is nowhere dense in the following.

As already mentioned, the statement is an immediate consequence of the results shown in [Rai94]. There he considers piecewise monotonic maps $f : [0, 1] \to [0, 1]$, not necessarily continuous on the whole unit interval, with a finite number $K \in \mathbb{N}$ of open holes $(a_1, a_2), \ldots,$ (a_{2K-1}, a_{2K}) and studies the influence of small perturbations of the endpoints of these holes on the dynamical system $f|_{R(a_1,a_2,\ldots,a_{2K})}$ where

$$R(a_1, a_2, \ldots, a_{2K}) := \bigcap_{n=0}^{\infty} \overline{[0, 1] \backslash f^{-n}\big((a_1, a_2) \cup \cdots \cup (a_{2K-1}, a_{2K})\big)} \,.$$

Here, we need only to consider the sets $R(0, t)$ with $t \in [0, 1]$. Since f is continuous, we have for each $t \in [0, 1]$ that

$$R(0, t) = \bigcap_{n=0}^{\infty} f^{-n}(0) \cup f^{-n}\big([t, 1]\big) \supseteq \mathcal{B}_\ell(t) \,.$$

This means that $R(0, t)$ and $\mathcal{B}_\ell(t)$ differ at most by the countable set $\bigcup_{n=0}^{\infty} f^{-n}(0)$, since f is piecewise monotonic. Therefore, the topological entropy of $f|_{R(0,t)}$ and $f|_{\mathcal{B}_\ell(t)}$ agree [Pes97]. Hence, we can also apply the continuity results for the topological entropy of [Rai94] in our setting and we obtain that the map $h : t \mapsto h_{top}\big(f|_{\mathcal{B}_\ell(t)} \big)$ is continuous for all $t \in [0, 1]$. Now, since $h(0) = h_{top}(f) > 0$, we have that h decreases monotonically from $h_{top}(f)$ to 0 in a continuous way. Furthermore, using that $\mathcal{E}_\ell$ is nowhere dense together with the decomposition (60) of the complement of $\mathcal{E}_\ell$ and Lemma 13.6, we know that h must be a devil's staircase map (in the sense that h is locally constant on an open dense subset of $[0, 1]$). Accordingly, the set of points in $[0, 1]$ where h is not locally constant is uncountable and this finishes the proof, since this set is a subset of $\mathcal{E}_\ell$. □

For the next corollary, recall that a transitive interval map always has positive topological entropy [Blo82, BC87]. We call a set $S \subset [0, 1]$ a *Cantor set* if it is perfect (i.e., it is closed and contains no isolated points) and nowhere dense.

Corollary 14.11. *If* f *is a transitive continuous piecewise monotone map, then the bifurcation set without its isolated points is a Cantor set.*

15 SOME RELATIONS BETWEEN BIFURCATION AND SURVIVING SETS

Similarly to the notion of m-intervals, which describe the connected components of the complement of the bifurcation sets as we have seen in the last chapter, we can define for each $t \in \mathcal{E}_*$ intervals that describe the connected components of the complement of $\mathcal{B}_*(t)$. These will be called lower and upper (t, m)-gaps where $m \in \mathbb{N}$.

Throughout this last chapter, we will phrase everything in terms of the lower (t, m)-gaps, but all the definitions and results can be rephrased in terms of upper (t, m)-gaps, as was done in the previous two chapters. For instance, everywhere we have $t \in \mathcal{E}_\ell \backslash \{0\}$, this should be replaced by $t \in \mathcal{E}_u \backslash \{1\}$, and the relevant inequalities should be reversed. Once the definitions and statements of results are altered in this way, the proofs follow immediately in the same way as presented for the lower case.

Definition 15.1. Fix $t \in \mathcal{E}_\ell \backslash \{0\}$ and assume $m \in \mathbb{N}$. An open interval $T^m \subset [0, 1]$ with $f^m(x) < t$ for all $x \in T^m$ is called a *lower* (t, m)*-gap for* f if there exists no open interval $J \subset [0, 1]$ such that T^m is strictly contained in J and $f^m(x) < t$ for all $x \in J$. Further, we call T^m a *lower* t*-gap of order* m if T^m is a lower (t, m)-gap and there exists no other lower $(t, \tilde{m})$-gap $T^{\tilde{m}}$ with $\tilde{m} \in \mathbb{N}$ such that $T^{\tilde{m}} = T^m$ and $\tilde{m} < m$.

Proposition 15.2. *Let* $T^m \subset [0, 1]$ *be a* (t, m)*-gap and* $[\alpha, \beta]$ *be its closure. If* $\alpha \neq 0$*, then* $f^m(\alpha) = t$ *and if* $\beta \neq 1$*, then also* $f^m(\beta) = t$*.*

Proof. Use the maximality of T^m and the continuity of f^m, analogously to the proof of Proposition 13.3. □

Similar to maximal m-intervals, we can also define maximal (t, m)-gaps.

Definition 15.3. A lower (t, m)-gap T^m with $t \in \mathcal{E}_\ell \backslash \{0, 1\}$ and $m \in \mathbb{N}$ is called *maximal* if $T^m \subseteq (t, 1]$ and if there exists no other lower $(t, \tilde{m})$-gap $T^{\tilde{m}}$ with $\tilde{m} \in \mathbb{N}$ such that T^m is strictly contained in $T^{\tilde{m}}$.

Theorem 15.4. *Fix* $t \in \mathcal{E}_\ell \backslash \{0, 1\}$ *and let* $\alpha \in \mathcal{B}_\ell(t)$*. Suppose* $\beta \in \mathcal{B}_\ell(t)$ *or* $\beta = 1$ *such that* $\alpha < \beta$ *and no other point in* (α, β) *belongs to* $\mathcal{B}_\ell(t)$*. Then* (α, β) *is a maximal lower* (t, m)*-gap for some* $m \in \mathbb{N}$*.*

To be precise, similarly to Theorem 14.3, for $\beta = 1 \notin \mathcal{B}_\ell(t)$ ($\alpha = 0 \notin \mathcal{B}_u(t)$) we actually show that $(\alpha, 1]$ (respectively, $[0, \beta)$) is a maximal lower (upper) (t, m)-gap.

Proof. We give the proof for maximal lower (t, m)-gaps and in the other case the argument works analogously. First, we claim that (α, β) is a lower (t, m)-gap for some $m \in \mathbb{N}$ and prove this by contradiction (note that for $\beta = 1 \notin \mathcal{B}_\ell(t)$ we actually have to consider the interval $(\alpha, 1])$. Assume (α, β) is not a lower (t, m)-gap for all $m \in \mathbb{N}$ and fix an arbitrary $\gamma_0 \in (\alpha, \beta)$. Note that $\gamma_0 > \alpha \geqslant t$. Since $\gamma_0 \notin \mathcal{B}_\ell(t)$, there exists $q_0 \in \mathbb{N}$ such that $f^{q_0}(\gamma_0) < t$. From this follows that there exists a lower (t, q_0)-gap included in (α, β) (where again the inclusion uses the fact that no point in $\mathcal{B}_\ell(t)$ can be contained in a lower (t, m)-gap for all $m \in \mathbb{N}$) and this inclusion must be strict. Hence, at least one of the endpoints of this (t, q_0)-gap, denoted by γ_1, lies in (α, β). According to Proposition 15.2, we have that $f^{q_0}(\gamma_1) = t$ and this implies $q_0 > 1$.

Since $\gamma_1 \notin \mathcal{B}_\ell(t)$, there exists $q_1 \in \mathbb{N}$ such that $f^{q_1}(\gamma_1) < t$. We have that $q_1 < q_0$ because if q_1 would be strictly bigger than q_0, then this would imply $f^{q_1}(\gamma_1) = f^{q_1 - q_0}(f^{q_0}(\gamma_1)) = f^{q_1 - q_0}(t) < t$ which is a contradiction (since $t \in \mathcal{E}_\ell$). With the same argument as above there is a lower (t, q_1)-gap contained in (α, β) and at least one of its endpoints, denoted by γ_2, lies in (α, β). Again, it holds that $f^{q_1}(\gamma_2) = t$ and hence $q_1 > 1$.

Now, we can repeat this step indefinitely since each corresponding $q_i \in \mathbb{N}$ is strictly bigger than 1. However, we also have that $q_0, q_1, \ldots, q_i, \ldots$ are strictly decreasing to 1 which is a contradiction.

The maximality is clear if $\alpha = t$ and $\beta = 1$, and follows otherwise, again, directly from the fact that no point in $\mathcal{B}_\ell(t)$ can be contained in any lower (t, m)-gap. □

Similarly to the remark after Theorem 14.3, using the last theorem together with Proposition 13.1, we can describe the complement of $\mathcal{B}_*(t)$ for each $t \in \mathcal{E}_* \backslash \{0, 1\}$ in the following way

$$[t, 1] \backslash \mathcal{B}_\ell(t) = \dot{\bigcup_{\substack{m \in \mathbb{N} \\ T \in \mathcal{T}_\ell^m}}} T \quad \text{and} \quad [0, t] \backslash \mathcal{B}_u(t) = \dot{\bigcup_{\substack{m \in \mathbb{N} \\ T \in \mathcal{T}_u^m}}} T, \tag{61}$$

where $\mathcal{T}_\ell^m$ and $\mathcal{T}_u^m$ is the collection of all maximal lower and upper t-gaps of order m, respectively, for each $m \in \mathbb{N}$. This immediately yields the following three corollaries, which again we state for lower gaps, but the analogous statements (with proofs altered accordingly), also hold for upper gaps.

Corollary 15.5. *Fix $t \in \mathcal{E}_\ell \backslash \{0, 1\}$. Then two maximal lower t-gaps of order m and $\tilde{m}$, respectively, are disjoint and each lower (t, m)-gap which does not lie in $[0, t)$ is contained in a unique maximal lower t-gap of order $\tilde{m}$ where $m, \tilde{m} \in \mathbb{N}$.*

Corollary 15.6. *Assume T^m is a maximal lower (t, m)-gap with $t \in \mathcal{E}_\ell \backslash \{0, 1\}$, $m \in \mathbb{N}$ and let $[\alpha, \beta]$ be its closure. We have that α belongs to $\mathcal{B}_\ell(t)$ and when $\beta \neq 1$ this holds for β, too.*

Corollary 15.7. *Let* $t, s \in \mathcal{E}_\ell$ *with* $0 \leqslant t < s \leqslant 1$. *We have that all* x *in* $\mathcal{B}_\ell(s)$ *are not isolated in* $\mathcal{B}_\ell(t)$ *(in fact, each* $x \in \mathcal{B}_\ell(s) \setminus \{1\}$ *has to be accumulated from the left and right in* $\mathcal{B}_\ell(t)$*).*

Proof. By way of contradiction, assume that $x \in \mathcal{B}_\ell(s)$ is not accumulated from the left in $\mathcal{B}_\ell(t)$. By (61), we can find a lower (t, m)-gap to the left of x with $m \in \mathbb{N}$ and this implies that $f^m(x) \leqslant t < s$ which is a contradiction. Use the corresponding argument if $x \in \mathcal{B}_\ell(s) \setminus \{1\}$ is not accumulated from the right in $\mathcal{B}_\ell(t)$. □

Now, we want to restrict ourselves to a specific class of interval maps. Suppose $f : [0, 1] \to [0, 1]$ is a piecewise monotone map with turning points $T(f)$, see Chapter 14 for the definition. Define $C(f) := T(f) \cup \{0, 1\}$. Then, we say that f *has full branches* if $f(C(f)) = \{0, 1\}$. Observe that $f^{-1}(\{0, 1\}) = C(f) \supseteq \{0, 1\}$ which implies $f^{-n}(C(f)) \supseteq f^{-(n-1)}(C(f))$ and therefore $f^{-n}(C(f)) \supset \bigcup_{k=0}^{n} f^{-k}(T(f))$ for each $n \geqslant 0$. For the next auxiliary statement we need the following notion. We call an interval $S \subseteq [0, 1]$ with non-empty interior an *absorbing set* (or *sink*) *for* f if there exists $p \in \mathbb{N}$ such that f^p is monotone on S and $f^p(S) \subseteq S$.

Proposition 15.8. *Suppose* f *is a continuous piecewise monotone map. If* f *is transitive, then there exist no absorbing sets for* f.

Proof. See [Pre88, Section 4] and the proof of [Rob15, Lemma 13]. □

Theorem 15.9. *Let* f *be a transitive continuous piecewise monotone map with full branches and suppose* $t \in \mathcal{E}_*$. *Then* t *is isolated in* $\mathcal{E}_*$ *if and only if* $\mathcal{B}_*(t)$ *contains an isolated point.*

Proof. As usual, we prove the assertion for the sets $\mathcal{E}_\ell$ and $\mathcal{B}_\ell(t)$. The proof for the upper case is left to the reader.

First, we assume that t is isolated in $\mathcal{E}_\ell$ and we claim that t is also isolated in $\mathcal{B}_\ell(t)$. Observe that by the definition of $\mathcal{B}_\ell(t)$, we only have to show that t is not accumulated from the right in $\mathcal{B}_\ell(t)$. Further, note that 0 cannot be isolated in $\mathcal{E}_\ell$, according to Lemma 14.9, and if $t = 1$, then the statement is automatically true. That means we can assume w.l.o.g. that $t \in (0, 1)$.

Since t is isolated in $\mathcal{E}_\ell$ and by using the decomposition (60) of the complement of $\mathcal{E}_\ell$, we know that there exists a lower m-interval I^m with $m \in \mathbb{N}$ such that the left endpoint of I^m is t. By Proposition 13.3, we have that $f^m(t) = t$. This implies that $t \notin f^{-(m-1)}(C(f))$ because otherwise $f^m(t) \in \{0, 1\}$ which would yield $t \in \{0, 1\}$. Hence, there exists an open interval U containing t such that f^m is strictly monotone in U. We claim that f^m is strictly decreasing in U and prove this by contradiction. Assume f^m is strictly increasing. In particular, this means that $t < f^m(x)$ for all $x \in U \cap I^m$. Furthermore, $f^m(x) < x$ for all $x \in U \cap I^m$, since I^m is a lower m-interval. Hence, $f^m(U \cap I^m) \subseteq U \cap I^m$, i.e. $U \cap I^m$ is an absorbing set for f which contradicts

Proposition 15.8. Now, from the fact that f^m is strictly decreasing in U, it follows that $f^m(x) < t$ for all $x \in U \cap I^m$, that is, there exists a (t, m)-gap with left endpoint equal to t. Accordingly, t cannot be accumulated from the right in $\mathcal{B}_\ell(t)$ and this proves one direction of the stated assertion.

For the other direction, assume that there is an isolated point y in $\mathcal{B}_\ell(t)$. We want to prove that t is isolated in $\mathcal{E}_\ell$. Observe that for $t = 0$ there are no isolated points in $\mathcal{B}_\ell(0)$. If $t = 1$, then $f(1) = 1$ and we claim that $f(x) < x$ for all $x \in (c, 1)$ where $c := \max C(f)\backslash\{1\}$. Otherwise, since f is continuous and strictly increasing in $[c, 1]$, there would exist a fixed point $z \in [c, 1)$ of f such that $[z, 1]$ is an absorbing set with $f([z, 1]) = [z, 1]$, and this would contradict the transitivity of f. Therefore, there is a lower 1-interval such that its right endpoint equals 1 which immediately implies the statement. In the following we always assume $t \in (0, 1)$.

Further, note that if $1 \in \mathcal{B}_\ell(t)$, then it cannot be isolated in $\mathcal{B}_\ell(t)$. To see this, note that 1 must be a fixed point of f in this case and apply Corollary 15.7. Hence, we only have to consider $y \in [t, 1)$.

In the last part of the proof, we will show that there exist $k \in \mathbb{N}$ and two open intervals V, $\tilde{V}$ containing t such that $f^k(t) = t$, f^k strictly decreasing in V and f^{2k} strictly increasing in $\tilde{V}$. From this it follows directly that $f^k(x) < t$ for all $x \in V \cap (t, 1]$ and $f^{2k}(x) < t$ for all $x \in \tilde{V} \cap [0, t) =: W$. Since $t < x$ for $x \in V \cap (t, 1]$, this implies that there is a lower k-interval such that its left endpoint is t. We further claim that $f^{2k}(x) < x$ for all $x \in W$, too. Assuming the contrary we would either get that $x < f^{2k}(x) < t$ for $x \in W$, i.e. $f^{2k}(W) \subseteq W$, or there would exist a $z \in W$ such that $f^{2k}(z) = z$ and $f^{2k}([z, t]) = [z, t]$. This means we would obtain an absorbing set for f, thus contradicting Proposition 15.8. Hence, there also exists a lower $2k$-interval such that its right endpoint is t. This shows that t is isolated in $\mathcal{E}_\ell$ and finishes the proof of the other direction of the statement.

It remains to show the existence of $k \in \mathbb{N}$ and the two open intervals V, $\tilde{V}$ with the desired properties. Since y is isolated in $\mathcal{B}_\ell(t)$ and using (61), we can find a lower (t, m)-gap T^m with $m \in \mathbb{N}$ such that the left endpoint of T^m is y. By Proposition 15.2, we have that $f^m(y) = t$. This implies that $y \notin f^{-(m-1)}(C(f))$, because otherwise $f^m(y) \in \{0, 1\}$, which would mean that $t \in \{0, 1\}$. Therefore, there is an open interval U containing t such that f^m is strictly monotone in U. In fact, f^m is strictly decreasing because assuming otherwise would imply the contradiction $t < f^m(x)$ for all $x \in U \cap T^m$.

If $y = t$, then set $k := m$ and $V := U$. Note that $f^k(t) = t$. Accordingly, $t \notin f^{-(2k-1)}(C(f))$ and hence we can find an open interval $\tilde{V}$ containing t such that f^{2k} is strictly increasing in $\tilde{V}$.

For $y > t$ there exists also a lower $(t, \tilde{m})$-gap with $\tilde{m} \in \mathbb{N}$ such that the right endpoint of $T^{\tilde{m}}$ is y. We have $f^{\tilde{m}}(y) = t$, too. We deduce in an analogous way as above that there is an open interval $\tilde{U}$ containing

y such that $f^{\tilde{m}}$ is strictly increasing in $\tilde{U}$. Note that m and $\tilde{m}$ must be different, since f^m and $f^{\tilde{m}}$ are strictly decreasing and increasing, respectively, in $U \cap \tilde{U} \ni y$. W.l.o.g. assume $m > \tilde{m}$. Accordingly, we have

$$t = f^m(y) = f^{m-\tilde{m}}(f^{\tilde{m}}(y)) = f^{m-\tilde{m}}(t)$$

and set $k := m - \tilde{m}$. Again, from this periodicity of t follows that $t \notin f^{-(k-1)}(C(f))$ and therefore we can find an open interval V containing t such that f^k is strictly monotone in V. In fact, f^k is strictly decreasing in V because in $U \cap \tilde{U} \cap f^{-\tilde{m}}(V) \ni y$ we have that $f^k \circ f^{\tilde{m}}$ is a concatenation of strictly monotone maps, where $f^{\tilde{m}}$ is strictly increasing, and it coincides with f^m which is strictly decreasing. Using the periodicity of t once more, we deduce that there is an open interval $\tilde{V}$ containing t such that f^{2k} is strictly increasing in $\tilde{V}$. $\square$

Corollary 15.10. *Assume* f *is a transitive continuous piecewise monotone map with full branches and let* $t \in \mathcal{E}_* \setminus \{0, 1\}$. *We have that* t *is a limit point of* $\mathcal{E}_*$ *if and only if* $\mathcal{B}_*(t)$ *is a Cantor set.*

BIBLIOGRAPHY

[AA77] S. Aljančić and D. Aranđelović. O-Regularly Varying Functions. *Publications de l'Institut Mathématique. Nouvelle Série*, 22(36):5 – 22, 1977.

[AB92] J.-P. Allouche and R. Bacher. Toeplitz sequences, paper-folding, towers of hanoi and progression-free sequences of integers. *L'Enseignement Mathématique*, 38(3-4):315–327, 1992.

[AGN14] J. Auslander, G. Greschonig, and A. Nagar. Reflections on equicontinuity. *Proceedings of the American Mathematical Society*, 142(9):3129–3137, 2014.

[AK00] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Mathematische Annalen*, 318(3):527–555, 2000.

[AKM65] R.L. Adler, A.G. Konheim, and M.H. McAndrew. Topological entropy. *Transactions of the American Mathematical Society*, 114:309–319, 1965.

[Aus88] J. Auslander. *Minimal Flows and Their Extensions*, volume 153 of *North-Holland Mathematics Studies*. Elsevier Science, 1988.

[BC87] L. Block and E.M. Coven. Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval. *Transactions of the American Mathematical Society*, 300(1):297–306, 1987.

[BCIT13] C. Bonanno, C. Carminati, S. Isola, and G. Tiozzo. Dynamics of continued fractions and kneading sequences of unimodal maps. *Discrete and Continuous Dynamical Systems*, 33(4):1313–1332, 2013.

[BFK97] F. Blanchard, E. Formenti, and P. Kůrka. Cellular automata in the cantor, besicovitch, and weyl topological spaces. *Complex Systems*, 11(2):107–123, 1997.

[BHM00] F. Blanchard, B. Host, and A. Maass. Topological complexity. *Ergodic Theory and Dynamical Systems*, 20:641–662, 2000.

[BKS06] V.V. Buldygin, O.I. Klesov, and J.G. Steinebach. On some extensions of Karamata's theory and their applications. *Publications de l'Institut Mathématique. Nouvelle Série*, 80(94):59–96, 2006.

[Blo82] A.M. Blokh. On sensitive mappings of the interval. *Russian Mathematical Surveys*, 37(2):203–204, 1982. English translation.

[BMN00] F. Blanchard, A. Maass, and A. Nogueira, editors. *Topics in Symbolic Dynamics and Applications*, volume 279 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2000.

[Bow71] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Transactions of the American Mathematical Society*, 153:401–414, 1971.

[BS02] M. Brin and G. Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[Car97] M. Carvalho. Entropy dimension of dynamical systems. *Portugaliae Mathematica*, 54:19–40, 1997.

[CD05] J.-R. Chazottes and F. Durand. Local rates of Poincaré recurrence for rotations and weak mixing. *Discrete and Continuous Dynamical Systems - Series A*, 12(1):175–183, January 2005.

[CFMM97] G. Cattaneo, E. Formenti, L. Margara, and J. Mazoyer. A shift-invariant metric on $S^{\mathbb{Z}}$ inducing a non-trivial topology. In I. Prívara and P. Ružička, editors, *Mathematical Foundations of Computer Science 1997*, volume 1295 of *Lecture Notes in Computer Science*, pages 179–188. Springer Berlin Heidelberg, 1997.

[CK97] J. Cassaigne and J. Karhumäki. Toeplitz Words, Generalized Periodicity and Periodically Iterated Morphisms. *European Journal of Combinatorics*, 18(5):497–510, 1997.

[CL10] W.-C. Cheng and B. Li. Zero entropy systems. *Journal of Statistical Physics*, 140:1006–1021, 2010.

[CT11] C. Carminati and G. Tiozzo. The bifurcation locus for numbers of bounded type. *Preprint arXiv:1109.0516*, 2011.

[CT12] C. Carminati and G. Tiozzo. A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations. *Ergodic Theory and Dynamical Systems*, 32:1249–1269, 2012.

[Cut91] C.D. Cutler. Some results on the behavior and estimation of the fractal dimensions of distributions on attractors. *Journal of Statistical Physics*, 62(3-4):651–708, 1991.

[Den32] A. Denjoy. Sur les courbes définies par les équations différentielles à la surface du tore. *Journal de Mathématiques Pures et Appliquées*, 11:333–376, 1932.

[Den05] M. Denker. *Einführung in die Analysis dynamischer Systeme*. Springer, 2005.

[DG15] T. Downarowicz and E. Glasner. Isomorphic extensions and applications. *Preprint arXiv:1502.06999*, 2015.

[DGO89] M. Ding, C. Grebogi, and E. Ott. Dimensions of strange nonchaotic attractors. *Physics Letters A*, 137(4-5):167–172, 1989.

[DHP11] D. Dou, W. Huang, and K.K. Park. Entropy dimension of topological dynamical systems. *Transactions of the American Mathematical Society*, 363(2):659–680, 2011.

[Din71] E.I. Dinaburg. A connection between various entropy characterizations of dynamical systems. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 35:324–366, 1971.

[dMvS93] W. de Melo and S. van Strien. *One-dimensional dynamics*, volume 25 of *A Series of Modern Surveys in Mathematics*. Springer, 1993.

[Dou95] A. Douady. Topological Entropy of Unimodal Maps. In B. Branner and P. Hjorth, editors, *Real and Complex Dynamical Systems*, volume 464 of *NATO ASI Series*, pages 65–87. Springer Netherlands, 1995.

[Dow05] T. Downarowicz. Survey of odometers and Toeplitz flows. In S.F. Kolyada, Y.I. Manin, and T. Ward, editors, *Algebraic and Topological Dynamics*, volume 385, pages 7–37. American Mathematical Society, 2005.

[Dow14] T. Downarowicz. Positive topological entropy implies chaos DC2. *Proceedings of the American Mathematical Society*, 142(1):137–149, 2014.

[Ebe71] E. Eberlein. Toeplitz-Folgen und Gruppentranslationen. *Archiv der Mathematik*, 22(1):291–301, 1971.

[Edg98] G.A. Edgar. *Integral, probability, and fractal measures*. Springer, 1998.

[EG92] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC-Press, 1992.

[EIK07] M.D. Esposti, S. Isola, and A. Knauf. Generalized Farey Trees, Transfer Operators and Phase Transitions. *Communications in Mathematical Physics*, 275(2):297–329, 2007.

[EW11] M. Einsiedler and T. Ward. *Ergodic Theory: With a View Towards Number Theory*, volume 259 of *Graduate Texts in Mathematics*. Springer, 2011.

[Fal03] K. Falconer. *Fractal Geometry*. 2nd. John Wiley, 2003.

[Fer97] S. Ferenczi. Measure-theoretic complexity of ergodic systems. *Israel Journal of Mathematics*, 100:189–207, 1997.

[Fer99] S. Ferenczi. Complexity of sequences and dynamical systems. *Discrete Mathematics*, 206(1–3):145–154, 1999.

[FGJ14] G. Fuhrmann, M. Gröger, and T. Jäger. Non-smooth saddle-node bifurcations II: dimensions of strange attractors. *Preprint arXiv:1412.6054*, 2014. Submitted.

[FGJ15] G. Fuhrmann, M. Gröger, and T. Jäger. Amorphic complexity. *Preprint arXiv:1503.01036*, 2015. Submitted.

[FP07] S. Ferenczi and K.K. Park. Entropy dimensions and a class of constructive examples. *Discrete and continuous dynamical systems*, 17(1):133–141, 2007.

[Fuh14] G. Fuhrmann. Non-smooth saddle-node bifurcations I: existence of an SNA. *Ergodic Theory and Dynamical Systems*, FirstView:1–26, 2014.

[GI05] M. Giampieri and S. Isola. A one-parameter family of analytic Markov maps with an intermittency transition. *Discrete and Continuous Dynamical Systems*, 12(1):115–136, 2005.

[GJ13] M. Gröger and T. Jäger. Dimensions of attractors in pinched skew products. *Communications in Mathematical Physics*, 320(1):101–119, 2013.

[GJ15] M. Gröger and T. Jäger. Some remarks on modified power entropy. *To appear in Contemporary Mathematics, American Mathematical Society*, 2015.

[Gle02] P. Glendinning. Global attractors of pinched skew products. *Dynamical Systems: An International Journal*, 17(3):287–294, 2002.

[GOPY84] C. Grebogi, E. Ott, S. Pelikan, and J.A. Yorke. Strange attractors that are not chaotic. *Physica D: Nonlinear Phenomena*, 13(1-2):261–268, 1984.

[Hen06] D. Hensley. *Continued fractions*. World Scientific, 2006.

[Her79] M.R. Herman. Sur la Conjugaison Différentiable des Difféomorphismes du Cercle a des Rotations. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 49(1):5–233, 1979.

[HK97] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1997. 8th printing 2006.

[HK02] B. Hasselblatt and A. Katok. Principal structures. In B. Hasselblatt and A. Katok, editors, *Handbook of Dynamical Systems*, volume 1, Part A, pages 1 – 203. Elsevier Science, 2002.

[HOR12] L. Hernández-Corbato, R. Ortega, and F.R. Ruiz del Portal. Attractors with irrational rotation number. *Mathematical Proceedings of the Cambridge Philosophical Society*, 153:59–77, 2012.

[How96] J.D. Howroyd. On Hausdorff and packing dimension of product spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 119(4):715–727, 1996.

[HP06] À. Haro and J. Puig. Strange nonchaotic attractors in Harper maps. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 16(3):033127 (7 pages), 2006.

[HPY07] W. Huang, K.K. Park, and X. Ye. Topological disjointness from entropy zero systems. *Bulletin de la société mathématique de france*, 135(2):259–282, 2007.

[HY09] W. Huang and X. Ye. Combinatorial lemmas and applications to dynamics. *Advances in Mathematics*, 220(6):1689–1716, 2009.

[Iso11] S. Isola. From infinite ergodic theory to number theory (and possibly back). *Chaos, Solitons & Fractals*, 44(7):467–479, 2011.

[JK69] K. Jacobs and M. Keane. 0-1-sequences of Toeplitz type. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 13(2):123–131, 1969.

[JKPS09] T. Jordan, M. Kesseböhmer, M. Pollicott, and B.O. Stratmann. Sets of nondifferentiability for conjugacies between expanding interval maps. *Fundamenta Mathematicae*, 206:161–183, 2009.

[JMS] T. Jordan, S. Munday, and T. Sahlsten. Conjugacies and perturbations of countable Markov maps. In preparation.

[Jä03] T. Jäger. Quasiperiodically forced interval maps with negative Schwarzian derivative. *Nonlinearity*, 16(4):1239–1255, 2003.

[Jä07] T.H. Jäger. On the structure of strange non-chaotic attractors in pinched skew products. *Ergodic Theory and Dynamical Systems*, 27(02):493–510, 2007.

[Jä09] T.H. Jäger. Creation of Strange Non-chaotic Attractors in Non-smooth Saddle-node Bifurcations. *Memoirs of the American Mathematical Society*, 201(945), 2009.

[KC14] D. Kong and E. Chen. Slow Entropy for Noncompact Sets and Variational Principle. *Journal of Dynamics and Differential Equations*, 26(3):477–492, 2014.

[Kel96] G. Keller. A note on strange nonchaotic attractors. *Fundamenta Mathematicae*, 151:139–148, 1996.

[Khi64] A.Ya. Khinchin. *Continued fractions*. University of Chicago Press, 1964.

[KS91] S.F. Kolyada and A.N. Sharkovsky. On topological dynamics of triangular maps of the plane. In *Iteration Theory, Proc. ECIT-89*, pages 177–183. World Scientific Publishing, 1991.

[KT97] A. Katok and J.P. Thouvenot. Slow entropy type invariants and smooth realization of commuting measure-preserving transformations. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 33(3):323–338, 1997.

[Lab13] C. Labrousse. Polynomial entropy for the circle homeomorphisms and for C^1 nonvanishing vector fields on $\mathbb{T}^2$. *Preprint arXiv:1311.0213*, 2013.

[Mar70] N.G. Markley. Homeomorphisms of the circle without periodic points. *Proceedings of the London Mathematical Society*, 3(4):688–698, 1970.

[Mar13] J.-P. Marco. Polynomial entropies and integrable Hamiltonian systems. *Regular and Chaotic Dynamics*, 18(6):623–655, 2013.

[MH40] M. Morse and G.A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *American Journal of Mathematics*, 62(1):1–42, 1940.

[Mis81] M. Misiurewicz. Structure of mappings of an interval with zero entropy. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 53(1):5–16, 1981.

[Mis89] M. Misiurewicz. Jumps of entropy in one dimension. *Fundamenta Mathematicae*, 132(3):215–226, 1989.

[Mis01] M. Misiurewicz. Possible jumps of entropy for interval maps. *Qualitative Theory of Dynamical Systems*, 2(2):289–306, 2001.

[MP79] N.G. Markley and M.E. Paul. Almost automorphic symbolic minimal sets without unique ergodicity. *Israel Journal of Mathematics*, 34(3):259–272, 1979.

[MS80] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. *Studia Mathematica*, 67(1):45–63, 1980.

[MS88] M. Misiurewicz and J. Smítal. Smooth chaotic maps with zero topological entropy. *Ergodic Theory and Dynamical Systems*, 8:421–424, 1988.

[Ott93] E. Ott. *Chaos in dynamical systems*. Cambridge University Press, 1993.

[Pes97] Ya.B. Pesin. *Dimension Theory in Dynamical Systems*. Chicago Lectures in Mathematics. University of Chicago Press, 1997.

[Pet83] K. Petersen. *Ergodic theory*. Cambridge studies in advanced mathematics 2. Cambridge University Press, 1983.

[PNR01] A. Prasad, S.S. Negi, and R. Ramaswamy. Strange nonchaotic attractors. *International Journal of Bifurcation and Chaos*, 11(02):291–309, 2001.

[Poi85] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Sur les courbes définies par les équations différentielles*, 4:167–244, 1885.

[Pre88] C. Preston. *Iterates of Piecewise Monotone Mappings on an Interval*, volume 1347 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1988.

[Rai94] P. Raith. Continuity of the Hausdorff dimension for invariant subsets of interval maps. *Acta Mathematica Universitatis Comenianae*, 63(1):39–53, 1994.

[Rob15] E.A. Robinson, Jr. Parry's topological transitivity and f-expansions. *Proceedings of the American Mathematical Society*, electronically published on August 12, 2015. (to appear in print).

[Smí86] J. Smítal. Chaotic Functions with Zero Topological Entropy. *Transactions of the American Mathematical Society*, 297(1):269–282, 1986.

[Urb86] M. Urbański. On Hausdorff dimension of invariant sets for expanding maps of a circle. *Ergodic Theory and Dynamical Systems*, 6:295–309, 1986.

[Vee65] W. A. Veech. Almost automorphic functions on groups. *American Journal of Mathematics*, 87(3):719–751, 1965.

[Wal82] P. Walters. *An Introduction To Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics*. Spinger, 1982.

[Wil84] S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 67(1):95–107, 1984.

[You82] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory and Dynamical Systems*, 2(1):109–124, 1982.

[Zin02] O. Zindulka. Hentschel-Procaccia spectra in separable metric spaces. *Real Analysis Exchange*, Summer Symposium in Real Analysis XXVI.:115–119, 2002. As well as unpublished notes on http://mat.fsv.cvut.cz/zindulka.

COLOPHON

This document was typeset using the typographical look-and-feel `classicthesis` developed by André Miede. The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*".