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Geometry and dynamics of infinitely generated Kleinian groups - Geometric Schottky groups

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Abstract

We introduce and study a class of Fuchsian groups which we call geometric Schottky groups. Geometric Schottky groups can be constructed in a similar way as classical Schottky groups but their class also contains many infinitely generated groups. We obtain a particularly convenient fundamental domain for such groups and describe the arising tesselation. Using the geometry of the tesselation, we define a coding of the limit set and then characterise in terms of this coding the uniformly radial set, the radial limit set and the set of Jørgensen limit points. We also show that subgroups of geometric Schottky groups are also geometric Schottky groups and then give methods for constructing explicit examples of geometric Schottky groups. Further, we study a certain notion of entropy h_S for the geodesic flow on the quotient manifold, which turns out to be a particular instance of the Bowen-Dinaburg entropy. We show that for finitely generated geometric Schottky groups h_S is equal to the Poincaré exponent of the group and that, in general, for geometric Schottky groups h_S is equal to the convex core entropy of the group. Subsequently, we discuss a subclass of the geometric Schottky groups called the $\mathcal{P}$-class. For groups in this class, the geodesic flow is ergodic with respect to the Liouville-Patterson measure and the Liouville-Patterson measure is finite. We present proofs of these facts. From these facts we then deduce that for groups in the $\mathcal{P}$-class the Patterson measure μ_o of the uniformly radial limit set is zero and that for μ_o-almost-all limit points their coding sequence is saturated. We provide methods of constructing geometric Schottky groups that belong to the $\mathcal{P}$-class. Further, we show that an infinite-index normal subgroup of a finitely generated geometric Schottky group cannot belong to the $\mathcal{P}$-class. Finally, we classify the set of Myrberg limit points in terms of the coding for general geometric Schottky groups. This allows us to obtain a condition for the ergodicity of the geodesic flow with respect to the Liouville-Patterson measure.

Contents

Chapter 1

Introduction

In this work we introduce and study the class of geometric Schottky groups. This class is a subclass of Fuchsian groups, that is of Kleinian groups acting on the two-dimensional hyperbolic space. Geometric Schottky groups bear a resemblance to the classical Schottky groups. But the class of geometric Schottky groups also contains an aboundance of infinitely generated groups. Importantly, the way in which they have been defined allows for straightforward, explicit construction of examples. Moreover, the definition which we use here distinguishes a certain geometric structure which provides a convenient framework for treating geometric Schottky groups.
We develop some tools for working with geometric Schottky groups and study several geometric and dynamical aspects of groups of this type. We also consider a very important subclass of geometric Schottky groups, referred to as the $\mathcal{P}$-class, which is distinguished by the fact that for any group in this class the geodesic flow on the quotiend manifold is ergodic with respect to the Liouville-Patterson measure and the Liouville-Patterson measure is finite.

Through many years of intensive research, a wealth of knowledge about Kleinian groups has been collected. The results obtained for groups acting on the hyperbolic space of dimension two and those acting on the hyperbolic space of dimension three can often be extended to the realm of a general n-dimensional hyperbolic space. But one frequently encounters the requirement that a group is geometrically finite or finitely generated. For a source of examples of easily tractable groups and cases where one can work in an explicit geometric manner, one repeatedly turned to convex-cocompact groups and in particular the classical Schottky groups.
Naturally, there are many results which apply also to geometrically infinite or infinitely generated groups. Yet, this is often because one considers properties which are general enough or algebraic in nature so that the geometric finiteness or finite generatedness is irrelevant. What has received far less attention are the features that are unique to geometrically infinite and infinitely generated groups. In the case of Kleinian groups acting on the three-dimensional hyperbolic space, one has also investigated and classified the

groups which are geometrically infinite but finitely generated, a wonderful exposition of recent results has been given by Series in [Series]. In this work we consider Fuchsian groups, for which the notions of finite generatedness and geometric finiteness coincide, and in this setting we are particularly interested in geometrically infinite groups.
For finitely generated Fuchsian groups a lot is known. For example, the two-dimenisonal manifolds which they uniformise including the nature of their ends, as well as the structure and dimension properties of the limit sets of such groups, are well understood. One also knows that the geodesic flow is ergodic with respect to the Liouville-Patterson measure and that this measure is always finite. There have also been profound developments in the study of Laplace-Beltrami operator and Spectral Theory on surfaces arising from finitely generated Fuchsian groups. For infinitely generated Fuchsian groups these aspects are much less understood and there is still a lot to be investigated.

The class of geometric Schottky groups which we discuss in this work is a way of generalising the class of classical Schottky groups. In particular we obtain a class of Fuchsian groups which are constructed in a similar way, and have similarly friendly geometric properties, but which are infinitely generated. The hope is that geometric Schottky groups can be more easily understood than other infinitely generated groups. As such they could be employed as a field of exploration for investigating infinitely generated groups, in particular their geometric and dynamical aspects. They could also be used as a rich source of explicit examples, where one could observe new phenomena.
The idea of 'infinite Schottky group' has appeared as an example in [Maskit]. But the method of obtaining such groups indicated there does not yield such a natural Fundamental domain as in the case of classical Schottky groups and geometric Schottky groups and does not yield examples as easily as the definition which we use in this work. Particular cases of infinitely generated groups whose construction resembles that of classical Schottky groups are scattered throughout the literature, where they usually appear as examples posessing certain desired properties. For example in [Patterson1983], [StratmannUrbański2007] and [FalkStratmann2004] they appear as examples of groups which have the so called 'dimension gap', and in [Davis2005] as groups uniformising surfaces arising in string perturbation theory.

We start by collecting the necessary background material and setting up the most important notational conventions in Chapter 2. In Chapter 3 we define the geometric Schottky groups. We obtain a particularly convenient fundamental domain for such groups and describe the tesselation which arises when this type of fundamental domain is used. Then in Section 3.2 we define a coding, which encodes the points of the limit set as sequences of integers; a classical idea of symbolic dynamics which goes back to [Morse]. We characterise in terms of the coding the set of uniformly radial limit points and the set of radial limit points by proving the following two theorems, which appear as Theorem 3 and Theorem 4.

Theorem 3.
Let Γ be a geometric Schottky group and let ξ be a point in the limit set $L(\Gamma)$ with coding sequence

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

Then ξ is a uniformly radial limit point if and only if there exists $b \in \mathbb{N}$ such that $|x_i| \leq b$, for all $i \in \mathbb{N}_0$, and $x_i \neq 0$, for all $i \in \mathbb{N}_0$.

Theorem 4.
Let Γ be a geometric Schottky group and let ξ be a point in the limit set $L(\Gamma)$. If ξ is a radial limit point then its coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_1(\xi)$ associated to $\kappa(\xi)$ does not converge to zero. If the coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_2(\xi)$ associated to $\kappa(\xi)$ does not converge to zero then ξ is a radial limit point. Moreover, if the group Γ is regular then ξ is a radial limit point if and only if its coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_2(\xi)$ does not converge to zero.

By clarifying the relationship between the action of the group and the coding in Corollary 2 we also describe the set of Jørgensen limit points. Next we consider subgroups of geometric Schottky groups and show that they are also geometric Schottky groups. We finish Chapter 3 by discussing various ways of constructing geometric Schottky groups. In Chapter 4 we turn to the study of the geodesic flow. Motivated by the work of Sullivan [Sullivan1984] we study a certain notion of entropy for the geodesic flow, which turns out to be a particular instance of the entropy introduced by Bowen [Bowen1973]. We show that for finitely generated geometric Schottky groups this entropy is equal to the Poincaré exponent of the group, which coincides with the Hausdorff dimension of the radial limit set, and for a general geometric Schottky group this entropy is equal to the convex core entropy, which coincides with the upper box-counting dimension of the limit set of the group. Namely, we obtain the following two results which appear as Theorem 5 and Theorem 7.

Theorem 5.
Let Γ be a finitely generated geometric Schottky group. Then the geodesic flow entropy of the group Γ is equal to the Poincaré exponent of the group.

Theorem 7.
Let Γ be a geometric Schottky group. Then the geodesic flow entropy of the group Γ and the convex core entropy of the group coincide.

The first theorem extends the result of Sullivan [Sullivan1984] to the case of geometric Schottky groups. It has been proved by a different method than the one used here in [OtalPeigné2004]. The second theorem is new and gives a new perspective on the earlier results. We relate these theorems to recent research of Handel and Kitchens [HandelKitchens1995], and Otal and Peigné [OtalPeigné2004].

Chapter 5 is devoted to the discussion of the $\mathcal{P}$-class, which is a subclass of the geometric Schottky groups containing many infinitely generated groups. For every group in the $\mathcal{P}$-class, the geodesic flow is ergodic with respect to the Liouville-Patterson measure and the Liouville-Patterson measure is finite. We present proofs of these facts and describe methods of obtaining examples of groups in the $\mathcal{P}$-class. We also address the question of when a group can belong to the $\mathcal{P}$-class. We describe several cases when a group belongs to the $\mathcal{P}$-class and some conditions under which a group belongs to the $\mathcal{P}$-class, and prove the following theorem which appears as Theorem 17.

Theorem 17.
Let Γ be a finitely generated geometric Schottky group and let H be a normal subgroup of Γ. If the index $[\Gamma, H]$ is finite then H belongs to the $\mathcal{P}$-class, and if the index $[\Gamma, H]$ is infinite then H does not belong to the $\mathcal{P}$-class.

As a consequence of the ergodicity of the flow and finiteness of the Liouville-Patterson measure for groups in the $\mathcal{P}$-class we obtain the following result which appears as Theorem 19.

Theorem 19.
Let Γ be an infinitely generated group in the $\mathcal{P}$-class. Then the Patterson measure μ_o of the uniformly radial limit set $L_{ur}(\Gamma)$ is equal to zero.

We deduce another property from the ergodicity of the flow and finiteness of the Liouville-Patterson measure. Namely, we show that almost all limit points with respect to the Patterson measure μ_o are coded by a special type of sequence, a so called saturated sequence, in which every admissible word appears infinitely often; the set of limit points with this type of coding sequence is denoted by $L_s(\Gamma)$. We obtain the following theorem which appears as Theorem 20.

Theorem 20.
Let Γ be a geometric Schottky group for which the geodesic flow on the quotient manifolod is ergodic and the Liouville-Patterson measure is finite. Then the Patterson measure μ_o of the set $L_s(\Gamma)$ is equal to one.

Finally, considering the set of limit points with saturated codes allows us to obtain an if and only if condition for the ergodicity of the geodesic flow with respect to the Liouville-Patterson measure. In particular we obtain the following result which appears as Theorem 22.

Theorem 22.
Let Γ be a geometric Schottky group. Then the geodesic flow on the quotient manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure ν if and only if the Patterson measure μ_o of the set $L_s(\Gamma)$ is equal to one.

This is done by characterising in terms of coding the set of Myrberg limit points, which turns out to coincide with the set of points coded by saturated sequences. This result, which appears as Theorem 21, holds for a general geometric Schottky group, but has been included in Chapter 5 due to the relationship with the concept of saturated codes.

Chapter 2

Preliminaries

In this introductory chapter we will review the well known objects and concepts that constitute the mathematical foundation for of the topics presented in this work. We will also establish some terminological and notational conventions.

2.0.1 Hyperbolic space

Throughout we will denote by $\mathbb{D}$ the Poincaré disc model and by $\mathbb{H}$ the upper half-plane model of the 2-dimensional hyperbolic space. The respective metrics, seen as real valued functions on subsets of $\mathbb{C}^2$, will be denoted by $d_{\mathbb{D}}$ and $d_{\mathbb{H}}$ or simply by d when it is clear which model we are using, and referred to as the **hyperbolic metric**. Occasionally, when referring to higher dimensional hyperbolic spaces, we will denote these by $\mathbb{H}^n$ and if necessary explicitly state what model is used; with this convention $\mathbb{H}^2$ could refer to both $\mathbb{H}$ and $\mathbb{D}$. We will identify the **ideal boundary** of $\mathbb{D}$ with the unit circle $S^1 := \left\{e^{ix} : x \in [0, 2\pi)\right\}$ and the ideal boundary of $\mathbb{H}$ with the extended real line $\mathbb{R} \cup \{\infty\}$.

There is a standard map between the spaces $\mathbb{D}$ and $\mathbb{H}$, which we denote by φ, which has the property that

$$d_{\mathbb{D}}(x, y) = d_{\mathbb{H}}(\varphi(x), \varphi(y))$$

for all $x, y \in \mathbb{D}$. This map is given by:

$$\varphi : \mathbb{D} \to \mathbb{H} \qquad\qquad z \mapsto \frac{-i(z+1)}{(z-1)}$$

The map φ is invertible and its inverse is given by:

$$\varphi^{-1} : \mathbb{H} \to \mathbb{D} \qquad\qquad z \mapsto \frac{z-i}{z+i}$$

We want to distinguish a special point in each of the models. Throughout this work we will denote by o the origin in the Poincaré disc model as well as the corresponding point $i = \varphi(0)$ in the upper half-plane model.

Geodesics in the upper half-plane model are half-circles and half-lines perpendicular at the endpoints to the boundary $\mathbb{R} \cup \{\infty\}$, while in the Poincaré disc model they are circle arcs or line segments perpendicular at the endpoints to the boundary S^1. The points at which a geodesic meets the boundary, which by definition are not part of the geodesic itself, are referred to as the **endpoints at infinity** of the geodesic. The term **geodesic** might be used to describe both a geodesic α as a subset of $\mathbb{H}$ (respectively $\mathbb{D}$) as well as a particular parametrisation of that geodesic, that is a map from $\mathbb{R}$ to $\mathbb{H}$ (resp. $\mathbb{D}$) whose image is equal to α; this map is usually also denoted by α. The points:

$$\xi_-(\alpha) := \lim_{s \to -\infty} \alpha(s) \quad \text{and} \quad \xi_+(\alpha) := \lim_{s \to +\infty} \alpha(s)$$

where the limit is taken with respect to Euclidean distance, are referred to respectively as the **negative endpoint at infinity** and the **positive endpoint at infinity** of the geodesic α. Further, for a restriction of a geodesic α to a set of the form $[a, \infty)$ or an interval $[a, b]$ we will respectively use the terms **geodesic ray** and **geodesic arc**. We will then describe the geodesic ray as a ray from $\alpha(a)$ with endpoint at infinity $\xi_+(\alpha)$, or simply as the geodesic ray between $\alpha(a)$ and $\xi_+(\alpha)$; we will describe the geodesic arc as the arc between $\alpha(a)$ and $\alpha(b)$. Throughout we will always require that the geodesics, geodesic rays and geodesic arcs are all parametrised with respect to their length, by which we mean that any geodesic α has to satisfy $d(\alpha(s), \alpha(s+t)) = t$ for all $t \in \mathbb{R}$.

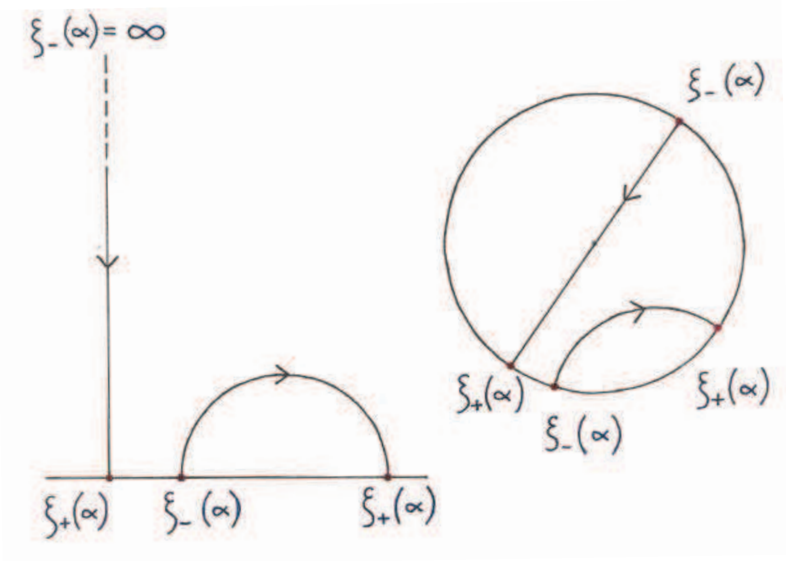


Figure 2.1:

2.0.2 Isometries

We will denote by $\text{Iso}^+(\mathbb{D})$ the group of all orientation preserving isometries of $\mathbb{D}$ and by $\text{Iso}^+(\mathbb{H})$ the group of all orientation preserving isometries of $\mathbb{H}$. The group $\text{Iso}^+(\mathbb{H})$

is equal to the group of maps of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, while the group $\mathrm{Iso}^+(\mathbb{D})$ is equal to the group of maps of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$$

where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. Conjugating by the map φ establishes a one-to-one correspondence between the maps in $\mathrm{Iso}^+(\mathbb{D})$ and $\mathrm{Iso}^+(\mathbb{H})$. The groups $\mathrm{Iso}^+(\mathbb{D})$ and $\mathrm{Iso}^+(\mathbb{H})$ are both isomorphic to the group $PSL(2,\mathbb{R})$. The maps in these groups can also be interpreted as transformations of the complex plane $\mathbb{C}$ preserving respectively the unit disc and the upper half-plane. We will use this point of view when speaking of the action of $\mathrm{Iso}^+(\mathbb{D})$ and $\mathrm{Iso}^+(\mathbb{H})$ on the respective boundaries S^1 and $\mathbb{R} \cup \{\infty\}$.

Each non-identity element in $\mathrm{Iso}^+(\mathbb{H})$, and thus also any element in $\mathrm{Iso}^+(\mathbb{D})$, is conjugate to one of the three maps:

- $z \mapsto az$, where $a \in \mathbb{R}$
- $z \mapsto z+1$
- $z \mapsto \frac{\cos\theta \ z \ + \ \sin\theta}{-\sin\theta \ z \ + \ \cos\theta}$, where $\theta \in \mathbb{R}$

The element is referred to respectively as **hyperbolic**, **parabolic** or **elliptic**.

The action of an element of a particular type retains many geometric features of the action of the map it is conjugate to. In particular:

- A hyperbolic transformation has two fixpoints in the boundary $\mathbb{R} \cup \{\infty\}$ (resp. S^1) and preserves the geodesic whose endpoints at infinity are equal to the two fixpoints. Along this geodesic the hyperbolic transformation acts as a translation, with respect to the hyperbolic distance. We call this geodesic the **axis** of that hyperbolic transformation and we refer to the hyperbolic distance by which it translates the points on the axis as its **translation distance**.

- A parabolic transformation has one fixed point in the boundary and preserves a **horocycle** that is a Euclidean circle tangent to the boundary $\mathbb{R} \cup \{\infty\}$ (resp. S^1) at the fixed point, or if ∞ is the fixed point a horizontal line in the upper half-space. Along this horocycle, the parabolic transformation acts as a translation with respect to the hyperbolic distance.

- An elliptic transformation has one fixed point inside $\mathbb{H}$ (resp. $\mathbb{D}$) and preserves any hyperbolic circle centered at this fixed point.

These geometric properties of various types of transformations have been indicated in Figure 2.2. More details and proofs of the above facts can be found in [Anderson] and [Katok].

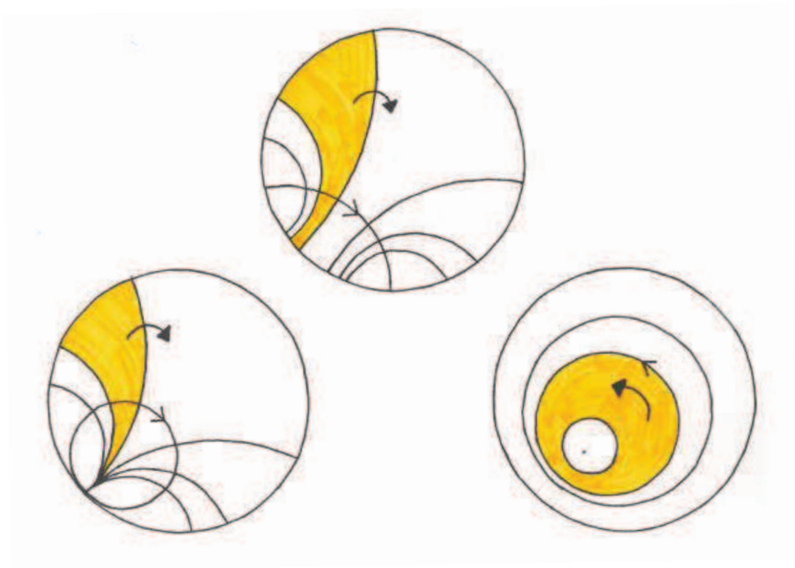


Figure 2.2:

2.0.3 Tangent bundle

The **unit tangent bundle** of the space $\mathbb{D}$ will be denoted by $T^1\mathbb{D}$ while the **unit tangent space** at a particular point $x \in \mathbb{D}$ will be written as $T^1_x\mathbb{D}$; their elements will be referred to as **vectors**. The notation for the unit tangent bundle and unit tangent spaces of $\mathbb{H}$ is analogous. For each $x \in \mathbb{D}$ (resp. $x \in \mathbb{H}$) every element v of $T^1_x\mathbb{D}$ (resp. $T^1_x\mathbb{H}$) defines a unique geodesic α_v satisfying $\alpha_v(0) = x$ and $\alpha'_v(0) = v$.

We define the map $\pi_b : T^1\mathbb{D} \to \mathbb{D}$ by requiring that $\pi_b(T^1_x\mathbb{D}) = x$ and refer to this map as **projection to the basepoint**; the map $\pi_b : T^1\mathbb{H} \to \mathbb{H}$ is defined analogously.

Further, every element of $g \in \mathrm{Iso}^+(\mathbb{D})$ (resp. $\mathrm{Iso}^+(\mathbb{H})$) induces a transformation of the unit tangent bundle $T^1\mathbb{D}$ (resp. $T^1\mathbb{H}$) via its differential, which we denote by Dg. The map Dg then takes each space $T^1_x\mathbb{D}$ (resp. $T^1_x\mathbb{H}$) to the space $T^1_{gx}\mathbb{D}$ (resp. $T^1_{gx}\mathbb{H}$). Sometimes, for a vector v, we will write $g \cdot v$ instead of $Dg(v)$ to denote the action of the element g by the differential Dg. For more details on tangent bundles and differentials, see [doCarmo] and [Lee].

2.0.4 Fuchsian groups and associated manifolds

A subgroup of $\mathrm{Iso}^+(\mathbb{H})$ is called **Fuchsian** if it is discrete with respect to the topology induced on $\mathrm{Iso}^+(\mathbb{H})$ by the norm which is defined for $g \in \mathrm{Iso}^+(\mathbb{H})$ given by $g(z) := \frac{az+b}{cz+d}$ as:

$$||g|| := (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}$$

and a subgroup of $\mathrm{Iso}^+(\mathbb{D})$ is called Fuchsian if the corresponding conjugate with respect to the map φ is a Fuchsian subgroup of $\mathrm{Iso}^+(\mathbb{H})$.

For a subgroup Γ of $\mathrm{Iso}^+(\mathbb{D})$ (resp. $\mathrm{Iso}^+(\mathbb{H})$) one can form the quotient $\mathbb{D}/\Gamma$ (resp. $\mathbb{H}/\Gamma$) by identifying the points in each of the orbits under the action of the group Γ. For a Fuchsian group Γ, this construction yields a hyperbolic manifold of dimension two, that is a hyperbolic surface. We refer to it as the **associated manifold** or the **quotient manifold** of the group Γ.
The **canonical projection** from $\mathbb{D}$ (resp. $\mathbb{H}$) to the manifold $\mathbb{D}/\Gamma$ (resp. $\mathbb{H}/\Gamma$) will be denoted by π throughout this work. Geodesics on $\mathbb{D}/\Gamma$ (resp. $\mathbb{H}/\Gamma$) are defined as the compositions of the form $\pi \circ \alpha$ where α is a geodesic in $\mathbb{D}$ (resp. $\mathbb{H}$); geodesic rays and geodesic arcs on the quotiend manifold are defined analogously. The map π also induces a projection from the the unit tangent bundle $T^1\mathbb{D}$ (resp. $T^1\mathbb{H}$) to the unit tangent bundle of the manifold $T^1(\mathbb{D}/\Gamma)$ (resp. $T^1(\mathbb{H}/\Gamma)$). This projection will also be denoted by π; from the context it will be clear which map is meant. The space $T^1_x\mathbb{D}$ (resp. $T^1_x\mathbb{H}$) is then mapped by π to the space $T^1_{\pi x}(\mathbb{D}/\Gamma)$ (resp. $T^1_{\pi x}(\mathbb{H}/\Gamma)$). On the space $T^1(\mathbb{D}/\Gamma)$ (resp. $T^1(\mathbb{H}/\Gamma)$), the **projection to the basepoint** is defined analogously to the projection to the basepoint on $T^1\mathbb{D}$ (resp. $T^1\mathbb{H}$), and will also be denoted by π_b.

2.0.5 Fundamental domains

For a subgroup Γ of $\mathrm{Iso}^+(\mathbb{D})$, we call an open set $F \subseteq \mathbb{D}$ a **fundamental region** for the action of Γ if the following conditions are satisfied:

1. For $g, h \in \Gamma$ such that $g \neq h$ we have $gF \cap hF = \emptyset$
2. The union $\bigcup_{g\in\Gamma} \overline{gF}$ is equal to $\mathbb{D}$

If the set F is simply connected, then we refer to it as a **fundamental domain** for the action of Γ or simply as a fundamental domain of Γ. When referring to the properties (1) and (2) we will often say that the set F **tesselates** $\mathbb{D}$ and call the collection of its images $\{gF : g \in G\}$ a **tesselation**.

Analogous definitions are made for subgroups of $\mathrm{Iso}^+(\mathbb{H})$. For a group Γ which is Fuchsian, there always exists a fundamental domain. It can be obtained, for example, by the standard Dirichlet construction, but usually there exist many fundamental domains. A fundamental domain of a Fuchsian group Γ together with a part of its boundary, where points in the boundary are added so that this new set contains precisely one point from each orbit under the action of Γ, forms a lift of the quotiend manifold of Γ. To some extent one can then identify this lift with the manifold and thus analyse the manifold inside the hyperbolic plane $\mathbb{D}$ (resp. $\mathbb{H}$).

2.0.6 Limit set

For a Fuchsian group Γ in $\mathrm{Iso}^+(\mathbb{D})$ (resp. $\mathrm{Iso}^+(\mathbb{H})$), the **limit set** of Γ is defined as the set of accumulation points of the orbit Γo, where o denotes the distinguished point in each of the models. We will denote the limit set of the group Γ by $\mathbf{L(\Gamma)}$. In fact, in the definition of $L(\Gamma)$, one could replace the point o by any other point z in $\mathbb{D}$ (resp. $\mathbb{H}$) and would still obtain the same set.

The limit set $L(\Gamma)$ is a subset of the boundary S^1 (resp. $\mathbb{R} \cup \{\infty\}$). It is closed as a subset of S^1 (resp. $\mathbb{R}$) and is invariant under the action of Γ. The limit set can contain 0, 1 or 2 points, in which case we say that the group Γ is **elementary**; or infinitely many points, in which case we say that Γ is **non-elementary**. For a non-elementary group Γ, the limit set is the smallest, with respect to inclusion, closed Γ-invariant subset of the boundary. Moreover, for any normal subgoup G of Γ, we have $L(G) = L(\Gamma)$; proof of this fact has been included in the Appendix. For further information on the limit set, we refer to [Katok].

There exist certain important subsets of the limit set, which we will now define and briefly discuss. Let $\xi \in L(\Gamma)$ and s_ξ the geodesic ray between o and ξ.

- We call ξ a **radial limit point** if there exists $r > 0$ such that the ray s_ξ intersects infinitely many of the open hyperbolic balls $\{B(go, r) : g \in \Gamma\}$. If for some $R > 0$ the ray s_ξ intersects infinitely many of the open hyperbolic balls $\{B(go, R) : g \in \Gamma\}$ we will say that ξ is **radial with respect to the radius** R. We refer to the set of all radial limit points as the **radial limit set** and denote it by $\mathbf{L_r(\Gamma)}$. In terms of dynamics, points $\xi \in L_r(\Gamma)$ describe the direction of those geodesic rays which return infinitely often to a bounded region of the quotiend manifold of the group Γ.

- We call ξ a **uniformly radial limit point** if there exists $r > 0$ such that the ray s_ξ is contained in $\bigcup_{g\in\Gamma} B(go, r)$. If for some $R > 0$ the ray s_ξ is contained in $\bigcup_{g\in\Gamma} B(go, R)$ we will say that ξ is **uniformly radial with respect to the radius** R. We refer to the set of all uniformly radial limit points as the **uniformly radial limit set** and denote it by $\mathbf{L_{ur}(\Gamma)}$. In terms of dynamics, points $\xi \in L_{ur}(\Gamma)$ describe the direction of those geodesic rays which remain within a bounded region of the quotiend manifold of the group Γ.

- We call ξ a **Jørgensen limit point** if there exists a point z on s_ξ, an element $g \in G$ and a fundamental domain of Γ such that the part of the ray s_ξ between z and ξ is contained in gF. We refer to the set of all Jørgensen limit points as the **Jørgensen limit set** and denote it by $\mathbf{L_J(\Gamma)}$.

2.0.7 Poincaré exponent

For a Fuchsian group Γ in $\mathrm{Iso}^+(\mathbb{D})$ we define its **Poincaré series** by:

$$P_\Gamma(x, y, s) := \sum_{g\in\Gamma} e^{-sd(x,gy)} \qquad \text{for} \quad x, y \in \mathbb{D}, \ s \in \mathbb{R}$$

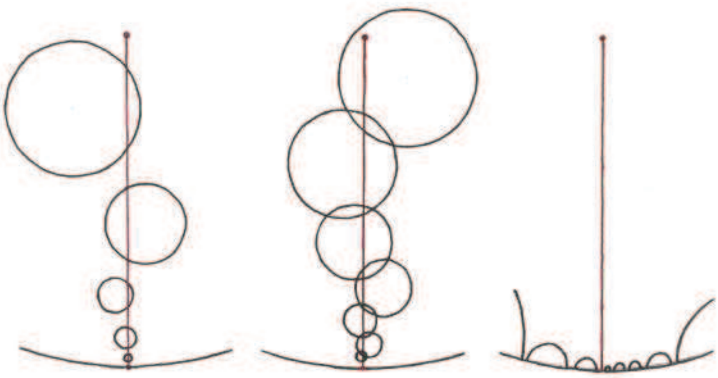


Figure 2.3:

In case $y = o$, we use the simplified notation $P_G(x, s) := P(x, o, s)$, and in case $x = y = o$, we use the simplified notation $P_G(s) := P(o, o, s)$. The abscissa of convergence of the Poincaré series is independent of the points x and y and depends only on the group Γ. This abscissa is of particular interest and we will refer to it as the **Poincaré exponent** of the group Γ. We will denote the Poincaré exponent of Γ by $\delta(\Gamma)$ or simply δ if it is clear which group it corresponds to. That is we have:

$$\delta(\Gamma) := \inf \{s \in \mathbb{R} : \ P_\Gamma(s) \text{ converges}\} = \sup \{s \in \mathbb{R} : \ P_\Gamma(s) \text{ diverges}\}$$

Whether $P_\Gamma(\delta)$ converges or diverges depends on the group Γ. If $P_\Gamma(\delta)$ converges, we say that Γ is of **convergence type**, and if $P_\Gamma(\delta)$ diverges, we say that Γ is of **divergence type**. One can state the above definitions in an analogous form in the setting of the upper half-plane model $\mathbb{H}$. Then the properties of the group Γ correspond to those of its conjugate $\varphi\Gamma\varphi^{-1}$ in $\mathrm{Iso}^+(\mathbb{H})$.

2.0.8 Patterson measures

We will now introduce a family of measures on the boundary S^1 associated to a Fuchsian group Γ in $\mathrm{Iso}^+(\mathbb{D})$. The support of these measures will be contained in the limit set $L(\Gamma)$. An analogous construction can be given in the setting of the upper half-plane model $\mathbb{H}$ and its relationship to the one given here can be easily eastablished with the use of the map $\varphi : \mathbb{D} \to \mathbb{H}$. The construction of these measures is due to Patterson, see [Patterson1976] and [Nicholls], and accordingly we will refer to them as **Patterson measures**.

Let Γ be a Fuchsian group in $\mathrm{Iso}^+(\mathbb{D})$. It has been shown by Patterson that there exists a function $\mathcal{H} : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:

1. $\mathcal{H}$ is continuous
2. $\mathcal{H}$ is non-decreasing
3. The abscissa of convergence of the modified Poincaré series:
$$P'_\Gamma(x,s) := \sum_{g\in\Gamma} \mathcal{H}(d(x,go))\ e^{-sd(x,go)}$$
is equal to the Poincaré exponent δ and $P'_\Gamma(x,\delta)$ diverges
4. For any $q > 0$ there exists $r_q \geq 0$ such that for all $t \geq 0$ and all $r \geq r_q$
$$\mathcal{H}(t+r) \leq e^{qt}\mathcal{H}(r)$$

In the case that the group Γ is of divergence type, we will assume $\mathcal{H} \equiv 1$.

Now for $s > \delta$ we define the following measure on $\overline{\mathbb{D}} = \mathbb{D} \cup S^1$:

$$\mu_{x,s} := \frac{\sum\limits_{g\in\Gamma} \mathcal{H}(d(x,go))\ e^{-sd(x,go)}\ \mathbf{1}_{go}}{P'_\Gamma(o,s)}$$

Here $\mathbf{1}_{go}$ denotes the Dirac measure at the point go. One can show that, for each $x \in \mathbb{D}$, $\tau > 0$ and $s \in (\delta, \delta+\tau)$, the value of $\mu_{x,s}(\overline{\mathbb{D}})$ is bounded independently of s; that is:

$$\mu_{x,s}(\overline{\mathbb{D}}) < c_\tau$$

for some constant $c_\tau \in \mathbb{R}^+$ depending only on τ. This allows us to apply a compactness theorem, namely a version of Helly-Bray's Theorem given by Mattila [Mattila], see Appendix. It tells us that there is a sequence $s_n \searrow \delta$ and a Borel measure μ_x such that:

1. $\mu_{x,s_n} \overset{w}{\to} \mu_x$
2. $\mu_x(\overline{\mathbb{D}}) < c_\tau$

Here $\overset{w}{\to}$ denotes weak convergence, see Appendix, so in our setting the above statement means that:

$$\int_{\overline{\mathbb{D}}} f\ d\mu_{x,s_n} \longrightarrow \int_{\overline{\mathbb{D}}} f\ d\mu_x \quad \text{for all} \quad f \in \mathcal{C}_b(\overline{\mathbb{D}}) \tag{2.1}$$

where $\mathcal{C}_b(\overline{\mathbb{D}})$ denotes the set of bounded continuous functions on $\overline{\mathbb{D}}$ with respect to the topology of $\mathbb{R}^2$. In fact, Helly-Bray's theorem only gives us vague convergence; that is, it tells us that μ_{x,s_n} an μ_x satisfy 2.1 with $\mathcal{C}_b(\overline{\mathbb{D}})$ replaced by the set $\mathcal{C}_K(\overline{\mathbb{D}})$ of continuous functions with compact support. Yet, since $\overline{\mathbb{D}}$ is compact, these two notions of convergence will coincide, see also [Hernandez].

In general, the measure μ_x depends on the choice of the sequence $(s_n)_{n\in\mathbb{N}}$. However, for each $x \in \mathbb{D}$ we can choose μ_x in such a way that for any $x_1, x_2 \in \mathbb{D}$, the measure μ_{x_1} is absolutely continuous with respect to μ_{x_2}, and we will assume that the measures μ_x have this property. One can then show that the Radon-Nikodym derivatives are given by:

$$\frac{d\mu_{x_1}}{d\mu_{x_2}}(\xi) = \left(\frac{p(x_1,\xi)}{p(x_2,\xi)}\right)^{\delta}$$

where $p(x,\xi) := \frac{1-|x|^2}{|x-\xi|^2}$ is the **Poisson kernel**.

The measures μ_x satisfy three further very useful properties:

1. **δ-conformality**: For any $\gamma \in \mathrm{Iso}^+(\mathbb{D})$ and any Borel set $E \subseteq \overline{\mathbb{D}}$

$$\mu_{\gamma^{-1}x} = \int_E \left(\frac{p(\gamma^{-1}x,\xi)}{p(x,\xi)}\right)^{\delta} d\mu_x(\xi)$$

2. **δ-harmonicity**: For any $g \in \Gamma$ and Borel set $E \subseteq \overline{\mathbb{D}}$

$$\mu_x(gE) = \mu_{g^{-1}x}(E)$$

3. For every $x \in \mathbb{D}$

$$\mu_x(\overline{\mathbb{D}}) = \mu_x(S^1)$$

We would also like to point out that by definition $\mu_o(S^1) = 1$; that is, the measure μ_o is a probability measure. Also, by δ-harmonicity, the same is true for any μ_z with $z \in \Gamma o$.

The measure μ_o will play a special role in our work. Whenever we simply speak of **the Patterson measure** we will be referring to the measure μ_o. If one of the other Patterson measures μ_x is mean we will refer to it as the **Patterson measure with respect to the point** x.

2.0.9 Sullivan's Shadow Lemma

On several occasions we will make use of a result usually referred to as Sullivan's Shadow Lemma. It allows us to estimate the Patterson measure $\mu_o(A)$ for certain nice sets $A \subseteq S^1$. To formulate the result we require the notion of a shadow. Let x be a point in $\mathbb{D}$ and D a set in $\mathbb{D}$. Then we define the **shadow** of the set D from the point x, denoted by $\Pi_o(D)$ to be the following set:

$$\Pi_o(D) := \left\{\xi \in S^1 : \; r_\xi \cap D \neq \emptyset\right\}$$

Here r_ξ denotes the ray from the origin o with endpoint at infinity ξ.

Theorem 1. ***Sullivan's Shadow Lemma***
Let $\Gamma \subseteq \mathrm{Iso}^+(\mathbb{D})$ be a non-elementary Fuchsian group and assume that the Patterson-Sullivan measure μ_o has no atoms. There is a constant $\theta > 0$, depending on Γ, such that for any $r \geq \theta$ there exists a constant c, depending on r, such that for all $g \in \Gamma$ we have:

$$\frac{1}{c}e^{-\delta d(o,go)} \leq \mu_o(\Pi_o(B(go,r)) \leq ce^{-\delta d(o,go)}$$

where $\delta = \delta(\Gamma)$.

Proof. For a proof and a more detailed account of this result, which appears in its initial form in [Sullivan1979], we refer the reader to [Nicholls], Chapter (4.3.) or [Borthwick]. □

2.0.10 Parametrisations of $T^1\mathbb{D}$ and $T^1(\mathbb{D}/\Gamma)$

The tangent bundle $T^1\mathbb{D}$ can be parametrised in several ways. We will describe two such parametrisations which are used later in this work. Both of them identify $T^1\mathbb{D}$ with a subset of the space $S^1 \times S^1 \times \mathbb{R}$. Parametrisations in a similar spirit can be given for the tangent bundle $T^1\mathbb{H}$ but we will not need them here.

Let x be a point in $\mathbb{D}$ and v be a vector in $T^1\mathbb{D}$ with $v \in T^1_x\mathbb{D}$. There exists a unique geodesic α satisfying $\alpha'(0) = v$, which we denote by α_v. The positive endpoint at infinity of α_v will be denoted by $\xi_+(v)$ and the negative endpoint at infinity of α_v will be denoted by $\xi_-(v)$. Let t_{top} denote the unique number for which:

$$d(o, \alpha_v(t_{top})) = \min\{d(o, \alpha_v(t)) : \ t \in \mathbb{R}\}$$

and put $s(v) := -t_{top}$. The point $\alpha_v(t_{top})$ is referred to as the **top** of the geodesic α_v and is the 'highest' point on this geodesic. The value $s(v)$ is equal to the distance between the base point x of the vector v and the top of the geodesic α_v if x lies between $\alpha_v(t_{top})$ and $\xi_+(v)$, and is equal to the negative of this distance if x lies between ξ_- and $\alpha_v(t_{top})$. We identify the vector v with the point:

$$(\xi_-(v), \xi_+(v), s(v))$$

We refer to this parametrisation as the **standard parametrisation**. One can easily see that with the above identification the elements of $T^1\mathbb{D}$ are in one to one correspondence with the elements of the set $(S^1 \times S^1 \setminus diag) \times \mathbb{R}$. Here $diag$ stands for the diagonal so that the set $(S^1 \times S^1 \setminus diag) \times \mathbb{R}$ can be written explicitly as

$$S^1 \times S^1 \times \mathbb{R} \ \setminus \ \Big(\bigcup_{x \in S^1} \{x\} \times \{x\} \times \mathbb{R}\Big)$$

To give the other parametrisation of $T^1\mathbb{D}$, we first need to introduce the Busemann function in terms of which this parametrisation is defined. Accordingly, this other parametrisation will be referred to as the **Busemann parametrisation**. For each $\xi \in S^1$, the

Busemann function at ξ, denoted by $B_\xi(\cdot,\cdot)$, is defined by:

$$\begin{aligned} B_\xi : \quad & \mathbb{D}\times\mathbb{D} \to \mathbb{R} \\ (x,y) \;\mapsto\; & \lim_{z\to\xi} d(x,z) - d(y,z) \end{aligned}$$

The Busemann function can be interpreted geometrically. It is the signed hyperbolic distance between the two horocycles based at ξ, one containing x and the other containing y. This distance can be measured along any geodesic with endpoint at ξ. The value of the Busemann function is positive if the horocycle containing y lies inside the horocycle containing x, negative if the horocycle containing x lies inside the horocycle containing y, and is equal to 0 if and only if x and y lie on the same horocycle, see Figure 2.4.

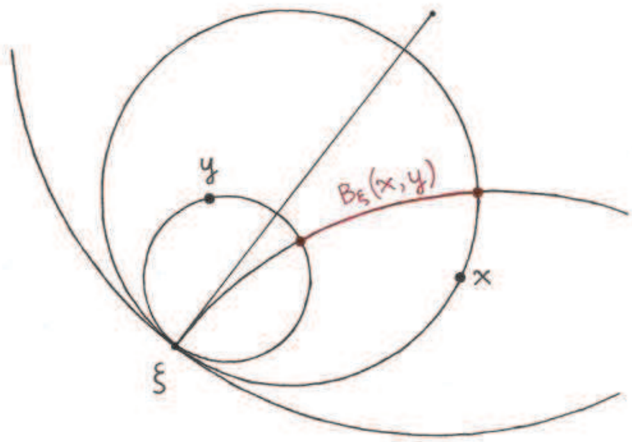


Figure 2.4:

Let x be a point in $\mathbb{D}$ and v be a vector in $T^1\mathbb{D}$ with $v \in T^1_x\mathbb{D}$. Once again we will consider the geodesic α_v satisfying $\alpha_v'(0) = v$ and its endpoints at infinity $\xi_+(v)$ and $\xi_-(v)$. But now we put $\overline{s}(v) := B_{\xi_+(v)}(o,x)$ and identify the vector v with the point:

$$(\xi_-(v), \xi_+(v), \overline{s}(v))$$

In this case one can also show that the Busemann parametrisation gives a one to one correspondence between the elements of the tangent bundle $T^1\mathbb{D}$ and the elements of the set $(S^1 \times S^1 \setminus diag) \times \mathbb{R}$.

The advantage of the Busemann parametrisation lies in the fact that for any $g \in \mathrm{Iso}^+(\mathbb{D})$ one can neatly express the map Dg in terms of this parametrisation, namely:

$$Dg(\xi,\eta,r) = (g\xi, g\eta, r - B_\eta(o, g^{-1}o))$$

For a group $\Gamma \in \mathrm{Iso}^+(\mathbb{D})$, the parametrisations described above will allow us to distinguish an important subset of the tangent bundle of the quotiend manifold. Namely, a central role will be played by the projection under the map $\pi : T^1\mathbb{D} \to T^1(\mathbb{D}/\Gamma)$ of the set $(L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R}$, which we will denote by:

$$((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$$

2.0.11 Geodesic flow

The geodesic flow on both $T^1\mathbb{D}$ and $T^1(\mathbb{D}/\Gamma)$ will play a major role in the investigations which we discuss in this paper. The geodesic flow on $T^1\mathbb{D}$ is a family of maps $\{g^t : t \in \mathbb{R}\}$ where for each $t \in \mathbb{R}$ the map g^t is a certain transformation of the space $T^1\mathbb{D}$. We now explain how the maps g^t are defined. Let x be a point in $\mathbb{D}$, v a vector in $T^1\mathbb{D}$ with $v \in T^1_x\mathbb{D}$ and α_v the unique geodesic with $\alpha'_v(0) = v$. Then g^t maps v to $\alpha'_v(t)$. So, in terms of both the standard parametrisation and the Busemann parametrisation, for any $t \in \mathbb{R}$, the map g^t can be expressed by the following formula:

$$g^t(\xi, \eta, s) = (\xi, \eta, s + t)$$

The geodesic flow on $T^1(\mathbb{D}/\Gamma)$ is defined analogously, and to keep the notation simple we will also write it as $\{g^t : t \in \mathbb{R}\}$. Namely, if v is a vector in $T^1(\mathbb{D}/\Gamma)$ and α_v the unique geodesic on $\mathbb{D}/\Gamma$ with $\alpha'_v(0) = v$, then for any $t \in \mathbb{R}$, we have $g^t(v) := \alpha'_v(t)$.

The relationship between the geodesic flow on $T^1\mathbb{D}$ and the geodesic flow on $T^1(\mathbb{D}/\Gamma)$ can be described by the following commuting diagram:

$$\begin{tikzcd}
T^1\mathbb{D} \arrow[r, "g^t"] \arrow[d, "\pi"'] & T^1\mathbb{D} \arrow[d, "\pi"] \\
T^1(\mathbb{D}/\Gamma) \arrow[r, "g^t"] & T^1(\mathbb{D}/\Gamma)
\end{tikzcd}$$

2.0.12 Liouville-Patterson measure

We will now introduce for a group $\Gamma \in \mathrm{Iso}^+(\mathbb{D})$ a very important measure on the tangent bundle $T^1(\mathbb{D}/\Gamma)$ of the quotiend manifold. This measure is defined via a measure on

$T^1\mathbb{D}$ using the Patterson measure μ_o.

We start by defining a measure $\widetilde{\nu}$ on $T^1\mathbb{D}$, which is given by the infinitesimal formula:

$$d\widetilde{\nu}(\xi, \eta, s) := \frac{d\mu_o(\xi)d\mu_o(\eta)ds}{|\xi - \eta|^{2\delta}}$$

where $|\xi - \eta|$ denotes the chordal distance between ξ and η, that is, the Euclidean distance on S^1 seen as a subset of $\mathbb{R}^2$, and $\delta = \delta(\Gamma)$ is the Poincaré exponent of the group Γ.

The measure $\widetilde{\nu}$ is invariant under the geodesic flow on $T^1\mathbb{D}$ and, due to the factor $\frac{1}{|\xi-\eta|^{2\delta}}$, it can be shown that it is also invariant under the action of Γ; here the action of an element $g \in \Gamma$ on $T^1\mathbb{D}$ is given by the map Dg.

The invariance of $\widetilde{\nu}$ under the action of Γ allows us to consider the projection of $\widetilde{\nu}$ onto $T^1(\mathbb{D}/\Gamma)$ and thus obtain a measure on $T^1(\mathbb{D}/\Gamma)$ which we denote by ν. For a Borel measurable subset V of $T^1(\mathbb{D}/\Gamma)$ its measure $\nu(V)$ is then equal to $\widetilde{\nu}(\widetilde{V})$ where $\widetilde{V}$ is any Borel measurable lift of V with respect to the projection π. Due to the invariance of $\widetilde{\nu}$ under the geodesic flow on $T^1\mathbb{D}$, the new measure ν is invariant under the geodesic flow on $T^1(\mathbb{D}/\Gamma)$.

This measure has been introduced by Patterson [Patterson1976] and further studied by Sullivan [Sullivan1979]. Its construction resembles that of the Liouville measure. The difference is that here instead of the Lebesgue measure which is used for defining Liouville measure we uses the Patterson measure μ_o. Accordingly, we refer to the measure ν as the **Liouville-Patterson measure**. Other constructions exist due to Bowen and Margulis [Bowen1971a] [Bowen1972] [Margulis1970] yielding the same measure; hence this measure often appears in the literature under the name of Bowen-Margulis measure.

Chapter 3

Geometric Schottky groups

In this chapter we describe a special class of Fuchsian groups which we call geometric Schottky groups, and which can be viewed as a generalisation of classical Schottky groups. The class of geometric Schottky groups differs from the classical Schottky groups, since the classical Schottky groups are by definition finitely generated, while the geometric Schottky groups which we consider here can be infinitely generated. Moreover, our way of defining geometric Schottky groups emphasizes their particularly nice geometric structure and allows us to take advantage of this structure.

Geometric Schottky groups are the mathematical objects central to all the investigations described in this work and therefore will be discussed in detail. We will not only give the definition of geometric Schottky groups but also introduce the associated notions which we use in their study and set up certain notational conventions. Then, after establishing some basic properties of geometric Schottky groups, we will examine the geometry of the associated tesselation and explain how the action of a geometric Schottky group can be interpreted geometrically. With this geometric picture we will introduce a coding of the limit set of a geometric Schottky group and then describe in terms of the coding the uniformly radial limit set, the radial limit set and the Jørgensen limit set. Finally, we will show that the property of being a geometric Schottky group is preserved under taking subgroups. We will then give examples of geometric Schottky groups, both finitely generated and infinitely generated. This should aid the understanding of the material presented in this work, and some of these examples will also play a role in our subsequent investigations. We will also provide methods of constructing further examples.

3.1 Definitions and basic properties

3.1.1 Groups of Schottky type and geometric Schottky groups

We will start by introducing structures which we refer to as Schottky descriptions and in terms of which we will later define groups of Schottky type and geometric Schottky groups.

Consider an open interval A in S^1. Let ∂A denote its boundary in S^1, which consists of two points. The two points of ∂A determine a geodesic α_A in $\mathbb{D}$. This geodesic in turn defines two closed half-planes in $\mathbb{D}$. The boundary at infinity of one of these two closed half-planes is equal to the closure $\overline{A} := A \cup \partial A$ and we will denote this closed half-plane by $\widehat{A}$. A demonstration of A, α_A and $\widehat{A}$ has been depicted in Figure 3.1.

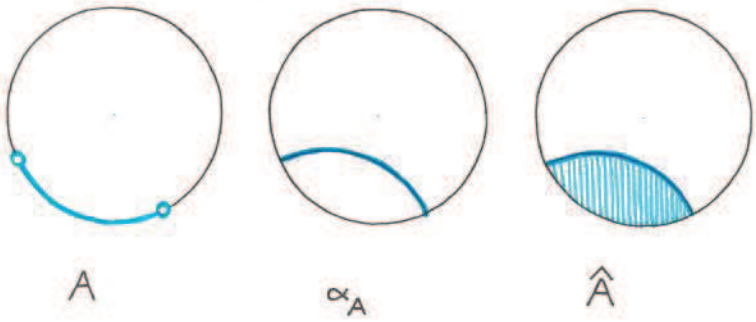


Figure 3.1:

Definition 1. Symmetric subset
Let $I \subseteq \mathbb{Z}$. If $-I := \{-k : k \in I\} = I$ and $0 \notin I$, then we refer to I as a symmmetric subset of $\mathbb{Z}$.

Definition 2. Schottky description
Let $\{A_k : k \in I\}$, where I is a symmetric subset of $\mathbb{Z}$, be a collection of open intervals in S^1, and let $\{g_k : k \in I\}$ be a collection of elements in $\mathrm{Iso}^+(\mathbb{D})$ such that the following conditions are satisfied:

1. The closures $\overline{A}_k$ are mutually disjoint.
2. None of the $\overline{A}_k$ contains a closed half-circle.
3. For each $k \in I$, the element g_k maps the open half-plane $\mathbb{D} \setminus \widehat{A}_{-k}$ to the open-half plane $\widehat{A}_k \setminus \alpha_k$, where $\alpha_k := \alpha_{A_k}$, and $g_{-k} = g_k^{-1}$.
4. For each $k \in I$, the element g_k is hyperbolic.
5. There exists $\epsilon > 0$ such that the closed hyperbolic ϵ-neighborhoods of the geodesics α_k, $k \in I$ are pairwise disjoint.

We call the pair $(\{A_k\}, \{g_k\})_{k \in I}$ a Schottky description. The intervals A_k will be referred to as the **intervals** in the Schottky description while the elements g_k will be referred to as the **generators** in the Schottky description.

Remark 1. Condition (3) in Definition 2 implies that for each $k \in I$ the element g_k also maps the geodesic α_{-k} to the geodesic α_k.

Remark 2. In fact, condition (4) in Definition 2 is superfluous, since it could be deduced from condition (3). By iterating g_k and g_k^{-1} one can exclude the possibility that g_k has a fixpoint inside $\mathbb{D}$ or a single fixpoint in S^1. In particular, one can show that the points $\xi_1 := \lim_{m \to \infty} g_k^m o$ and $\xi_2 := \lim_{m \to \infty} (g_k^{-1})^m o$, where the limit is taken with respect to the Euclidean metric, are the two fixpoints of g_k.

Definition 3. Schottky type
We say that a group $\Gamma < \mathrm{Iso}^+(\mathbb{D})$ is of Schottky type if there exists a Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$ such that:

$$\Gamma = \langle g_k : k \in I \rangle \tag{3.1}$$

We will not be interested in the different ways in which a subgroup of $\mathrm{Iso}^+(\mathbb{D})$ of Schottky type can be described via property 3.1 by a Schottky description. Instead, we will often assume that a group $\Gamma < \mathrm{Iso}^+(\mathbb{D})$ is defined by a Schottky description and keep this Schottky description fixed. This places the focus on the Schottky desription and its geometrical properties, so that we essentially view groups defined in terms of distinct Schottky descriptions as different objects. This motivates the following definition.

Definition 4. Geometric Schottky group
A group $\Gamma < \mathrm{Iso}^+(\mathbb{D})$ together with a Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$ satisfying property 3.1 will be referred to as geometric Schottky group.

We would like to distinguish the following special subset of S^1 determined by a geometric Schottky group as well as two properties of geometric Schottky groups which will all be of importance in our later work.

Definition 5. Set $J(\Gamma)$ and set $\mathcal{J}(\Gamma)$
For a geometric Schottky group Γ with Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$, we denote by $J(\Gamma)$ the set of all accumulation points of the set $\bigcup_{k \in I} \partial A_k$, that is as the set all $\xi \in S^1$ such that there exists a sequence $(\xi_i)_{i \in \mathbb{N}}$ of distinct elements in $\bigcup_{k \in I} \partial A_k$ with $\xi_i \to \xi$. We also define:

$$\mathcal{J}(\Gamma) := \bigcup_{\gamma \in \Gamma} \gamma J(\Gamma)$$

Remark 3. The requirement that the elements of the sequence $(\xi_i)_{i \in \mathbb{N}}$ are distinct is essential since we are trying to capture where the intervals A_k are 'accumulating'.

Proposition 1. *The set $J(\Gamma)$ is a subset of the limit set $L(\Gamma)$.*

Proof. Let ξ be a point in the set $J(\Gamma)$. By definition there exists a sequence $(\xi_i)_{i\in\mathbb{N}}$ of distinct elements in $\bigcup_{k\in I}\partial A_k$ with $\xi_i \to \xi$. For each $i \in \mathbb{N}$ let k_i be such that $\xi_i \in \partial A_{k_i}$ and put $x_i := g_{k_i} o$. Then it is clear that the sequence of points $(x_i)_{i\in\mathbb{N}}$ converges to ξ with respect to the Euclidean metric. This shows that $\xi \in L(\Gamma)$. $\square$

Corollary 1. *The set $\mathcal{J}(\Gamma)$ is a subset of the limit set $L(\Gamma)$.*

Proof. This follows from Proposition 1 and the fact that the limit set $L(\Gamma)$ is invariant under the action of the group Γ. $\square$

Definition 6. Regular
We say that a geometric Schottky group Γ with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ is regular if:

$$J(\Gamma) \cap \bigcup_{k\in I} \partial A_k = \emptyset$$

Definition 7. ϵ-thick
Let $\epsilon > 0$. If, for a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$, the closed ϵ-neighborhoods of the geodesics $\alpha_k := \alpha_{A_k}$, $k \in I$ are pairwise disjoint, we say that Γ is ϵ-thick.

Remark 4. Condition (5) in the definition of Schottky description implies that a geometric Schottky group is ϵ-thick with respect to some $\epsilon > 0$. The set of values of ϵ for which a geometric Schottky group is ϵ-thick will depend on the group.

Observation 1. If a geometric Schottky group Γ is ϵ-thick, then the quotiend manifold $\mathbb{D}/\Gamma$ has bounded injectivity radius.

3.1.2 The geometric picture

For a geometric Schottky group Γ, we can use the fixed Schottky description to define in a canonical way a fundamental domain of Γ. This fundamental domain will allow us to analyse geometrically the action of Γ on $\mathbb{D}$ and this geometrical picture will be the key to our investigations of geometric Schottky groups.

Definition 8. Standard fundamental domain
Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$. Then we call the set:

$$F(\Gamma) := \mathbb{D} \setminus \bigcup_{k\in I} \widehat{A}_k$$

the standard fundamental domain of Γ.

The picture

Of course we still need to prove that the set $F := F(\Gamma)$ is indeed a fundamental domain for Γ. To do this we consider what happens when we apply elements of the group Γ to the region F. We will describe how the images of the region F, that is the sets $\{\gamma F : \gamma \in \Gamma\}$, are arranged in $\mathbb{D}$. Initially this geometric approach will be used to prove that the region F is indeed a fundamental domain for Γ, that a geometric Schottky group is in fact a Fuchsian group and that it is freely generated by the generators in its Schottky description. Later the geometric picture which we describe here will serve to define a coding of the limit set $L(\Gamma)$ of a geometric Schottky group Γ. Eventually our geometric understanding of the action of Γ in $\mathbb{D}$ will also play a major role in several arguments presented in this work.

Let γ be an element of a geometric Schottky group Γ which has Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$. We start by expressing γ in terms of the generators in the Schottky description:

$$\gamma = g_{k_1} \cdots g_{k_m}$$

where $k_{i+1} \neq -k_i$ for $i = 1, ..., m-1$. First we consider the case when $m = 1$, i.e. when $\gamma = g_k$ is a generator in the Schottky description of Γ.

To simplify the notation, we will now simply write F for $F(\Gamma)$. Since the set F satisfies $F \subseteq \mathbb{D} \setminus \widehat{A}_{-k}$ it follows by property (3) in the definition of Schottky description that $g_k F \subseteq \widehat{A}_k \setminus \alpha_k$. We will describe this situation by saying that $g_k F$ **lies under** α_k. Moreover g_k takes α_{-k} to α_k and the images $g_k \alpha_j$, for $j \neq -k$, are geodesics with endpoints in A_k whose arrangement corresponds to that of the geodesics α_j, with $j \neq -k$, on the boundary of F. The geodesics α_k and $g_k \alpha_j$, for $j \neq k$, form the boundary of $g_k F$. Note that in particular $F \cap g_k F = \emptyset$ and $\overline{F} \cap \overline{g_k F} = \alpha_k$.

Now suppose $m = 2$ and $\gamma = g_{k_1} g_{k_2}$. We already know what $g_{k_2} F$ looks like and how it is positioned with respect to F. Namely $\overline{F} \cap \overline{g_{k_2} F} = \alpha_k$ which implies that $\overline{g_{k_1} F} \cap \overline{g_{k_1} g_{k_2} F} = g_{k_1} \alpha_k$. So γF lies in $g_{k_1} \widehat{A}_{k_2} \subset \widehat{A}_{k_1}$, and in particular it lies under α_{k_1} and under $g_{k_1} \alpha_{k_2}$, see Figure 3.2.

To give the general statement we will proceed by induction. Suppose that for some $m \geq 2$ we have already shown that if $m_0 \leq m$ and $\gamma \in \Gamma$ is of the form $g_{k_1} \cdots g_{k_{m_0}}$ then the copy γF of the region F lies under the geodesics α_{k_1}, $g_{k_1} \alpha_{k_2}$, ... and $g_{k_1} ... g_{k_{m_0 - 1}} \alpha_{k_{m_0}}$. Consider an arbitrary element $\gamma' \in \Gamma$ of the form $g_{k'_1} \cdots g_{k'_{m+1}}$. By our assumption the copy $g_{k'_2} \cdots g_{k'_{m+1}} F$ of the region F lies under the geodesics $\alpha_{k'_2}$, $g_{k'_2} \alpha_{k'_3}$, ... and $g_{k'_2} ... g_{k'_m} \alpha_{k'_{m+1}}$. But the fact that $k'_2 \neq -k'_1$ implies that the copy $g_{k'_1} \cdots g_{k'_{m+1}} F$ of the region F lies under the geodesics $g_{k'_1} \alpha_{k'_2}$, $g_{k'_1} g_{k'_2} \alpha_{k'_3}$, ... and $g_{k'_1} g_{k'_2} ... g_{k'_m} \alpha_{k'_{m+1}}$ and as a consequence of that also under the geodesic $\alpha_{k'_1}$. By induction we can now conclude that for each $m > 2$ and an element $\gamma \in \Gamma$ of the form $g_{k_1} \cdots g_{k_m}$ the copy γF of the region F lies under the geodesics α_{k_1}, $g_{k_1} \alpha_{k_2}$, ... and $g_{k_1} ... g_{k_{m-1}} \alpha_{k_m}$, see Figure 3.3.

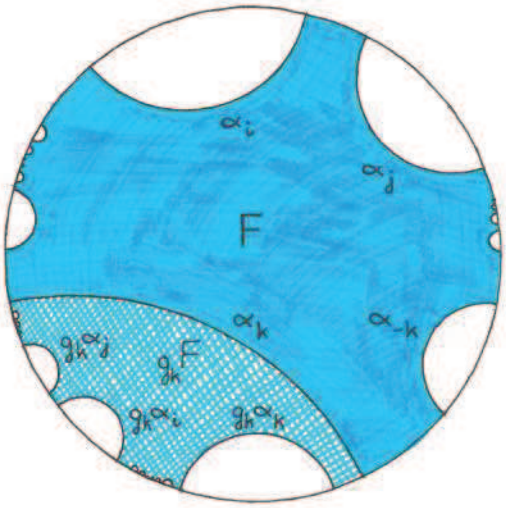


Figure 3.2:

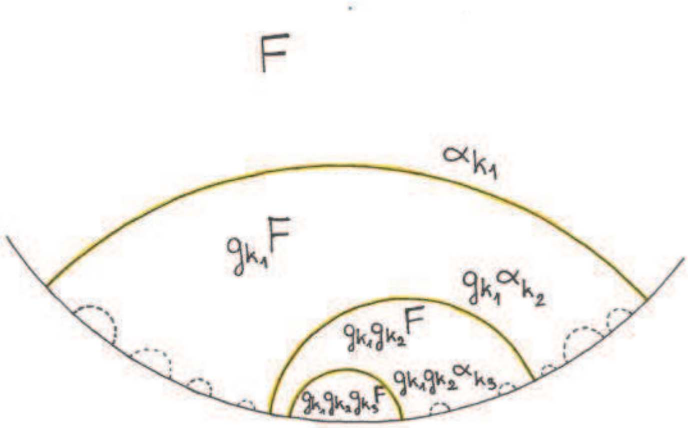


Figure 3.3:

The geometric picture which we discussed above now allows us to show that F is indeed a fundamental domain for Γ. Later we will use it to establish some basic properties of geometric Schottky groups, and it will repeatedly play a major role in our arguments.

Fundamental domain

Proposition 2. *For a geometric Schottky group* Γ *the set* $F(\Gamma)$ *is a fundamental domain.*

Proof. Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$. It is clear that $F := F(\Gamma)$ is open and simply connected. From our discussion of the geometrical picture, that is of the arrangement of the images of F under the action of Γ, it is clear that for all $\gamma \in \Gamma \setminus \{id\}$ we have:

$$\gamma F \cap F = \emptyset$$

So we only need to show that:

$$\bigcup_{\gamma \in \Gamma} \overline{\gamma F} = \mathbb{D}$$

We consider an arbitrary point $x \in \mathbb{D}$ and show that we must have $x \in \overline{\gamma F}$ for some $\gamma \in \Gamma$. If $x \in \overline{F}$ then we are done. If $x \notin \overline{F}$ then we must have $x \in \widehat{A}_{k_1}$ for some $k_1 \in I$. Either $x \in \overline{g_{k_1} F}$ and we are done or $x \in g_{k_1} \widehat{A}_{k_2}$ for some $k_2 \in I, k_2 \neq -k_1$. If the latter holds then either $x \in \overline{g_{k_1} g_{k_2} F}$ and we are done or $x \in g_{k_1} g_{k_2} \widehat{A}_{k_3}$ for some $k_3 \in I, k_3 \neq -k_2$. Now we can repeat this argument and we claim that at some step we must have:

$$x \in \overline{g_{k_1} ... g_{k_m} F}$$

It is clear that the Euclidean diameters of the sets $\widehat{A}_k$ are bounded from above, which implies that there exists $R_0 > 0$ such that $\widehat{A}_{k_1}$ is disjoint from the closed hyperbolic ball $B(o, R_0)$. By property (5) in the definition of Schottky description there exists $\epsilon > 0$ such that the closed ϵ-neighborhoods of the geodesics α_k, $k \in I$ are pairwise disjoint. This, keeping in mind the geometrical picture, implies that $g_{k_1} \widehat{A}_{k_2}$ is disjoint from the closed hyperbolic ball $B(o, R_0 + \epsilon)$. By similar reasoning we see that for each m the closed halfplane $g_{k_1} \dots g_{k_m} \widehat{A}_{k_{m+1}}$ must be disjoint from the closed hyperbolic ball $B(o, R_0 + m\epsilon)$. So as soon as $R_0 + m\epsilon \geq d(o, x)$ we certainly cannot have $x \in g_{k_1} ... g_{k_m} \widehat{A}_k$ for any $k \in I$ and the process must stop. $\square$

Geometric Schottky groups are free products

Proposition 3. *Let* Γ *be a geometric Schottky group and* $(\{A_k\}, \{g_k\})_{k \in I}$ *its Schottky description . Then* Γ *is the free product of the infinite cyclic groups* $\langle g_k \rangle$*, where* k *runs over* $|I| := \{|k| : k \in I\}$*, namely:*

$$\Gamma = \langle g_{k_1} \rangle * \langle g_{k_2} \rangle * \dots * \langle g_{k_m} \rangle * \dots$$

where $k_i \in |I|$ *and each* $k_i \in |I|$ *apears exactly once in the product.*

Proof. It is enough to show that for any choice of $g_{k_i} \in \{g_k : k \in I\}$ satisfying $k_i \neq -k_{i+1}$ for $i = 1, \dots, m-1$ we have:

$$g_{k_1} \cdots g_{k_m} \neq id$$

But if we had $g_{k_1} \cdots g_{k_m} = id$ then, in particular, we would have $g_{k_1} \cdots g_{k_m} o = o$. From our earlier discussion of the geometric picture it follows that this situation cannot occur. $\square$

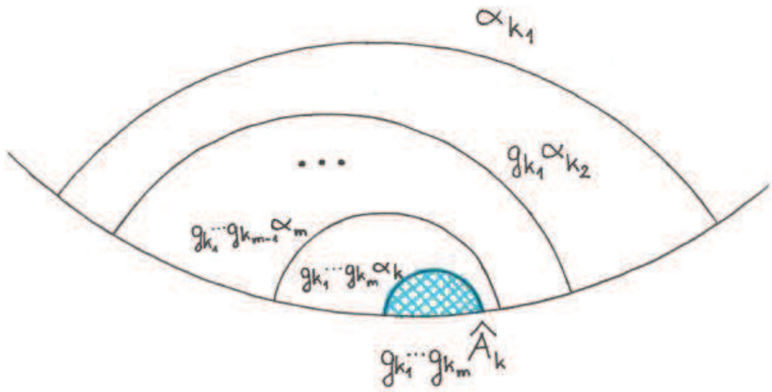


Figure 3.4:

Geometric Schottky groups are Fuchsian

We will prove that every geometrical Schottky group is a Fuchsian group. In the proof we use the following theorem, which gives a convenient standard description of Fuchsian groups.

Theorem 2. *Let G be a subgroup of* $\mathrm{Iso}^+(\mathbb{D})$. *Then G is a Fuchsian group if and only if for all $z \in \mathbb{D}$ the orbit Gz of z is a discrete subset of $\mathbb{D}$.*

Proof. Omitted. We refer the reader to [Katok], in particular to Section 2.2, Theorem 2.2.6. and Corollary 2.2.7. □

Proposition 4. *Every geometric Schottky group is a Fuchsian group.*

Proof. Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$. In order to reach a contradiction, suppose that there exists $z \in \mathbb{D}$ such that Γz is not a discrete subset of $\mathbb{D}$. This implies that there is a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of distinct elements of Γ and some $z_0 \in \mathbb{D}$ with

$$\gamma_i z \to z_0$$

Since $F := F(\Gamma)$ is a fundamental domain for Γ there is some $\gamma \in G$ such that $z \in \overline{\gamma F}$. We now claim that there exists $r > 0$ such that the closed hyperbolic ball $B(z, r)$ does not contain any points of Γz other than z itself. To see this we apply γ^{-1} and consider $\gamma^{-1}z$ in F, and then apply γ. If $\gamma^{-1}z$ lies in the boundary of F then it follows, from properties (3) and (5) in the definition of Schottky description and the geometrical picture, that it is enough to choose r to be equal to the ϵ that appears in property (5). On the other hand, if $\gamma^{-1}z$ lies in F then any point in Γz other than $\gamma^{-1}z$ must lie outside F, so that it is enough to choose r such that the closed ball $B(\gamma^{-1}z, r)$ is contained in F.

But now the existence of $r > 0$ such that the closed hyperbolic ball $B(z, r)$ does not contain any points of Γz other than z itself and the fact that Γ is free together imply that for each $i \in \mathbb{N}$ the hyperbolic distance between $\gamma_i z$ and $\gamma_{i+1} z$ is at least r, which is a contradiction. $\square$

3.2 Coding of the limit set

In this section we will introduce a coding for the limit set of a geometric Schottky group Γ. By this we mean that we will give a way of representing points in the limit set $L(\Gamma)$ by certain sequences of integers. In this way we will be able to single out and identify a point of $L(\Gamma)$ by refering to its associated sequence; and what is more, this sequence will carry precise information about the location of this limit point. We will also explore the existence of various relationships between the features of coding sequences and the properties of the corresponding limit points, which will allow us to describe, both here and in Chapter 5, certain special subsets of $L(\Gamma)$ in terms of the coding.

3.2.1 Defining the coding

Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$. We will define the following map:

$$\kappa : L(\Gamma) \to \mathbb{Z}^{\mathbb{N}_0}$$

We will refer to this map as the **coding map** or simply the **coding** of the limit set $L(\Gamma)$ and for a point $\xi \in L(\Gamma)$ we will refer to $\kappa(\xi)$ as the **coding sequence** or simply the **code** of ξ. The image of of κ will be significantly smaller than the set $\mathbb{Z}^{\mathbb{N}_0}$, in particular the elements of the sequences will belong only to the set $I \cup \{0\}$; the sequences will satisfy other conditions as well.

We will first describe the geometric properties of the limit points which are captured by the definition of the coding κ. Let ξ be a limit point in $L(\Gamma)$ and let $s_\xi : [0, \infty) \to \mathbb{D}$ be the geodesic ray between the origin o and the limit point ξ. The ray s_ξ starts inside the fundamental domain $F := F(\Gamma)$, that is $s_\xi(0) \in F$. As $t \in [0, \infty)$ increases we will examine, which of the copies of F the point $s_\xi(t)$ lies in. Note that by condition (2) in the definition of Schottky description, the geometric picture described in Section 3.1.2 and the fact that F is a fundamental domain of Γ, either $s_\xi([0, \infty))$ intersects infinitely many copies of F or it is contained in the union of closures of finitely many copies of F and if the latter happens then $s_\xi((t, \infty)) \subseteq gF$ for some $t \in [0, \infty)$ and some $g \in \Gamma$. In particular, three scenarios are possible. The first possibility is that there exists a sequence of strictly increasing positive real numbers $(t_i)_{i \in \mathbb{N}_0}$ and a sequence of indices $(k_i)_{i \in \mathbb{N}_0}$, with $k_i \in I$ and $k_i \neq -k_{i+1}$ for all $i \in \mathbb{N}_0$, such that $s_\xi(0, t_0) \subseteq F$, $s_\xi(t_0, t_1) \subseteq g_{k_0} F$ and, for all $i \in \mathbb{N}$, $s_\xi(t_i, t_{i+i}) \subseteq g_{k_i} \cdots g_{k_1} g_{k_0} F$. If this is not the case, then either we have $s_\xi([0, \infty)) \subseteq F$, or for some $m \in \mathbb{N}_0$ there exist positive real numbers:

$$t_0 < t_1 < \ldots < t_m$$

and indices:

$$k_0, \ldots, k_m$$

with $k_i \in I$ and $k_i \neq -k_{i+1}$, such that $s_\xi(0,t_0) \subseteq F$, $s_\xi(t_i,t_{i+1}) \subseteq g_{k_0} \cdots g_{k_i} F$ for all i satisfying $0 \leq i \leq m-1$ (we allow the possibility that no such i exist!), and $s_\xi(t_m,\infty) \subseteq g_{k_0} \cdots g_{k_m} F$. The coding sequence $\kappa(\xi)$ will contain the information about which copies of the region F the ray s_ξ intersects and in which order it intersects them.

Definition 9. Coding κ
Let $\xi \in L(\Gamma)$. If there exists a sequence of strictly increasing positive real numbers $(t_i)_{i\in\mathbb{N}_0}$ and a sequence of indices $(k_i)_{i\in\mathbb{N}_0}$, with $k_i \in I$ and $k_i \neq -k_{i+1}$ for all $i \in \mathbb{N}_0$, such that, $s_\xi(0,t_0) \subseteq F$, $s_\xi(t_0,t_1) \subseteq g_{k_0} F$ and, for all $i \in \mathbb{N}$, $s_\xi(t_i,t_{i+i}) \subseteq g_{k_i} \cdots g_{k_1} g_{k_0} F$, then we define:

$$\kappa(\xi) := [k_0, k_1, \ldots, k_i, \ldots]$$

If $s_\xi([0,\infty)) \subseteq F$, then we define:

$$\kappa(\xi) := [0, 0, \ldots]$$

If for some $m \in \mathbb{N}_0$ there exist positive real numbers $t_0 < t_1 < \ldots < t_m$ and indices $k_0, \ldots, k_m$, with $k_i \in I$ and $k_i \neq -k_{i+1}$, such that $s_\xi(0,t_0) \subseteq F$, $s_\xi(t_i,t_{i+1}) \subseteq g_{k_0} \cdots g_{k_i} F$ for all i satisfying $0 \leq i \leq m-1$, and $s_\xi(t_m,\infty) \subseteq g_{k_0} \cdots g_{k_m} F$, then we define:

$$\kappa(\xi) := [k_0, k_1, \ldots, k_m, 0, 0 \ldots]$$

Observation 2. We make some observations about the relationship between the coding sequence of a point $\xi \in L(\Gamma)$ and its position. The situation that $\kappa(\xi) = [0,0,\ldots]$ occurs when either $\xi \in \partial A_k$, for some $k \in I$, or $\xi \in S^1 - (\bigcup_k \overline{A}_k))$. A limit point ξ has a coding sequence of the form $\kappa(\xi) = [k, \ldots]$, for some $k \in I$, precisely when $\xi \in A_k$. The case $\kappa(\xi) = [k, 0, 0, \ldots]$ occurs when either $\xi \in \partial g_k A_i$, for some $i \in I \setminus \{-k\}$, or $\xi \in A_k \setminus (\bigcup_{i\neq -k} \overline{A}_i)$. A limit point ξ has coding sequence of the form $\kappa(\xi) = [k, j, \ldots]$, for some $k, j \in I$ with $k \neq -i$, precisely when $\xi \in g_k A_j$. In general, a coding sequence of the form $\kappa(\xi) = [k_0, k_1, \ldots, k_m, \ldots]$ corresponds to the fact that $\xi \in g_{k_0} g_{k_1} \cdots g_{k_{m-1}} A_{k_m}$.

3.2.2 Basic properties of the coding

Now we will establish some basic properties of the coding κ.

Proposition 5. *If in the definition of κ we replace the ray s_ξ between o and ξ by the ray s'_ξ between z and ξ, where z is an arbitrary point in F, then we will still obtain the same coding κ.*

Proof. Let z be a point in F. Our proposition follows from the following simple observation. Let ξ be a point in S^1 and z_1 and z_2 two points in $\mathbb{D}$. Suppose that z_1 and z_2 lie in the same open half-plane determined by a geodesic α. Consider the geodesic ray between z_1 and ξ and the geodesic ray between z_2 and ξ. Then either they both eventually intersect α or neither of them does. For any geodesic α which forms part of the boundary of a copy gF of the region F the points o and z lie in the same of the two open half-planes determined by α. It follows immediately that the rays s_ξ and s'_ξ will enter the same copies of the region F in precisely the same order. $\square$

Proposition 6. *If the indexing set I is infinite then the image of κ consists of all the sequences in $I^{\mathbb{N}_0}$ in which no $k \in I$ is followed directly by $-k$ and all sequences of the form $[x_0, \ldots, x_m, 0, 0, \ldots]$ where $[x_0, \ldots, x_m]$ is an element of I^{m+1} in which no $k \in I$ is followed directly by $-k$. If the indexing set I is finite then the image of κ consists only of all the sequences in $I^{\mathbb{N}_0}$ in which $k \in I$ is not followed directly by $-k$.*

Proof. From the definition of the coding κ it is clear that any sequence in the image of κ must be of one of the two types described above. When the indexing set I is finite, it is easy to see that if the ray between o and a point $\xi \in S^1$ intersects only finitely many copies of F, then the orbit Γo cannot accumulate at ξ. Therefore if I is finite then the image of κ will not contain sequences which end in a string of zeros.
For any sequence $[x_0, x_1, \ldots]$ in $I^{\mathbb{N}_0}$ in which $k \in I$ is not followed directly by $-k$, consider:

$$\xi := \widehat{A}_{x_0} \cap g_{x_0} \widehat{A}_{x_1} \cap g_{x_0} g_{x_1} \widehat{A}_{x_2} \cap \ldots$$

The point ξ is a limit point of Γ with coding sequence $\kappa(\xi) = [x_0, x_1, \ldots]$. So all sequences of this type will be contained in the image of κ.
Now assume that I is infinite. For an arbitrary sequence $[x_0, \ldots, x_m, 0, 0, \ldots]$ where $[x_0, \ldots, x_m]$ is an element of I^{m+1} in which $k \in I$ is not followed directly by $-k$, we consider the copy $g_{x_0} \cdots g_{x_m} F$ of the fundamental domain F. Recall that, as explained in Section 3.1.2, the copy $g_{x_0} \cdots g_{x_m} F$ lies 'under' the geodesic $g_{x_0} \cdots g_{x_{m-1}} \alpha_{x_m}$. Since I is infinite, the interval $g_{x_0} \cdots g_{x_{m-1}} A_{x_m}$ contains infinitely many of the intervals $g_{x_0} \cdots g_{x_m} A_k$, $k \in I$, $k \neq -x_m$. The set of endpoints of these intervals must have at least one accumulation point accumulation point, say ξ, which satisfies:

$$\xi \ \in \ \overline{g_{x_0} \cdots g_{x_{m-1}} A_{x_m}} \setminus \bigcup_{k \in I, k \neq -x_m} g_{x_0} \cdots g_{x_m} A_k$$

If $\xi \notin \partial(g_{x_0} \cdots g_{x_{m-1}} A_{x_m})$ then the point ξ is a limit point of Γ with coding sequence $\kappa(\xi) = [x_0, \ldots, x_m, 0, 0, \ldots]$, and if $\xi \in \partial(g_{x_0} \cdots g_{x_{m-1}} A_{x_m})$ then:

$$\kappa(g_{x_0} \cdots g_{x_m} (g_{x_0} \cdots g_{x_{m-1}})^{-1} \xi) = [x_0, \ldots, x_m, 0, 0, \ldots]$$

So when I is infinite, all sequences of this type will also be contained in the image of the coding mapκ. $\square$

The next proposition follows immediately from the discussion in Section 3.1.2 and the definition of the coding map κ. Yet, since it implies the existence of a partial inverse for κ, we shall state it explicitly.

Proposition 7. *Let ξ be a point in the limit set $L(\Gamma)$. If ξ has coding sequence:*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

satisfying $x_i \neq 0$ for all $i \in \mathbb{N}_0$ then the sequence:

$$o,\ g_{x_0}o,\ g_{x_0}g_{x_1}o,\ \ldots\ ,\ g_{x_0}\cdots g_{x_i}o,\ \ldots$$

converges to ξ with respect to the Euclidean metric.

Let $\mathcal{C}_{\kappa,1}$ denote the set of all sequences in $I^{\mathbb{N}_0}$ in which $k \in I$ is not followed directly by $-k$ and let $\mathcal{C}_{\kappa,0}$ denote the set of all sequences of the form $[x_0, \ldots, x_m, 0, 0, \ldots]$ where $[x_0, \ldots, x_m]$ is an element of I^{m+1} in which $k \in I$ is not followed directly by $-k$. As shown in Proposition 6, we have that for I infinite:

$$\kappa(L(\Gamma)) = \mathcal{C}_{\kappa,1}\ \dot{\cup}\ \mathcal{C}_{\kappa,0}$$

while for I finite we have:

$$\kappa(L(\Gamma)) = \mathcal{C}_{\kappa,1}$$

It follows from Proposition 7 that the function:

$$\kappa^* : \mathcal{C}_{\kappa,1} \to L(\Gamma)$$
$$[x_0, x_1, \ldots] \mapsto \lim_{i\to\infty} g_{x_0}\cdots g_{x_i}o$$

is a partial inverse for κ, namely it is the inverse of the restriction of κ to the preimage of $\mathcal{C}_{\kappa,1}$. This implies that κ is injective on the preimage of $\mathcal{C}_{\kappa,1}$.

It is natural to ask what happens on the preimage of $\mathcal{C}_{\kappa,0}$. It is clear that the set $J(\Gamma)$ defined in Section 3.1.1 is precisely the set of limit points which have coding sequence

$$[0, 0, \ldots]$$

One can also use the geometric picture from Section 3.1.2 and property (3) from the definition of Schottky description to argue that the set of points $\xi \in L(\Gamma)$ with $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$ is equal to $\mathcal{J}(\Gamma) = \bigcup_{\gamma\in\Gamma}\gamma J(\Gamma)$, and that, in fact, the set of points with code

$$[x_0, \ldots, x_m, 0, 0, \ldots] \in \mathcal{C}_{\kappa,0}$$

is precisely the set $g_{x_0}\cdots g_{x_m}J$. Instead of proving these statements directly, we will clarify how, in general, the action of the group Γ on the limit set $L(\Gamma)$ manifests itself in the coding sequences of the points in $L(\Gamma)$. The above description of points $\xi \in L(\Gamma)$ with $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$ will then follow immediately from the observation we have made about the set $J(\Gamma)$.

Proposition 8. *Let ξ be a point in the limit set $L(\Gamma)$ with coding sequence*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

and let g_i be a generator in the Schottky description of Γ.
If $i = -x_0$ then the coding sequence of $g_i\xi$ is:

$$\kappa(g_i\xi) = [x_1, x_2, \ldots]$$

If $i \neq -x_0$ then the coding sequence of $g_i\xi$ is:

$$\kappa(\xi) = [i, x_0, x_1, \ldots]$$

Proof. First observe that the ray s_ξ passes through the following copies of F:

$$F,\ g_{x_0}F,\ \ldots\ ,\ g_{x_0}\cdots g_{x_m}F,\ \ldots$$

while the ray $g_i s_\xi$ passes through:

$$g_iF,\ g_ig_{x_0}F,\ \ldots\ ,\ g_ig_{x_0}\cdots g_{x_m}F\ ,\ \ldots$$

If $g_i = g_{-x_0} = g_{x_0}^{-1}$ then there exists a point $z \in F$ such that $g_i s_\xi$ contains the ray between z and $g_i\xi$. By Proposition 5, the coding sequence $[x_1, x_2, ...]$ obtained using the ray between z and $g_i\xi$ is equal to $\kappa(g_i\xi)$.
If $g_i \neq g_{-x_0}$ then $g_i s_\xi$ starts in g_iF, at g_io and does not pass through F. Let z be a point on $s_{g_i\xi}$ such that $z \in g_iF$. By the same argument as in the proof of Proposition 5, the coding sequence obtained using the part of $s_{g_i\xi}$ between z and $g_i\xi$ is the same as the one we would obtain using g_is_ξ, namely $[x_0, x_1, ...]$. Since $\kappa(g_i\xi)$ is by definition equal to the code obtained from $s_{g_i\xi}$ we have $\kappa(g_i\xi) = [i, x_0, x_1, ...]$. □

Corollary 2. *Let ξ be a point in the limit set $L(\Gamma)$ with coding sequence:*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

and let g be an element of Γ with unique representation in terms of the generators in the Schottky description $g = g_{k_0}\cdots g_{k_m}$. If the set $\{0 \le i \le m : k_i \neq -x_{m-i}\}$ is non-empty then the coding sequence of $g\xi$ is:

$$\kappa(g\xi) = [k_0, \ldots, k_l, x_{m-l}, x_{m-l+1}, \ldots]$$

where l is given by:

$$l := \max\{0 \le i \le m : k_i \neq -x_{m-i}\}$$

Otherwise the coding sequence of the point $g\xi$ is:

$$\kappa(g\xi) = [x_{m+1}, x_{m+2}, \ldots]$$

Corollary 3. *Let $\xi \in L(\Gamma)$, then $\xi \in \mathcal{J}(\Gamma)$ if and only if $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$.*

3.2.3 Describing subsets of the limit set

Note that we have already given a description of the set $\mathcal{J}(\Gamma) \subseteq L(\Gamma)$ in terms of the coding. Now we are going to exhibit further connections between the features of coding sequences and the properties of the corresponding limit points. In particular we will characterise the uniformly radial limit set $L_{ur}(\Gamma)$ and the radial limit set $L_r(\Gamma)$ using certain properties of the coding sequences. In case of $L_r(\Gamma)$ full characterisation is given under the additional assumption that Γ is regular, in the sense of Definition 6, and otherwise the characterisation is only partial.

Picture

Before we proceed to discussing the main results of this section we will give a way to interpret geometrically the situation when in the code $\kappa(\xi) = [x_0, x_1, \ldots]$ an integer k_1 is followed directly by an integer k_2; that is when $x_i = k_1$ and $x_{i+1} = k_2$, for some $i \in \mathbb{N}_0$. This interpretation lies at the heart of most propositions which we present in this section.

Fix integers $k_1, k_2 \in I$ with $k_2 \neq -k_1$. Consider a point $\xi \in L(\Gamma)$ with code

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

and suppose that for some $i \in \mathbb{N}_0$ we have:

$$x_i = k_1 \text{ and } x_{i+1} = k_2$$

Because of the way κ was defined, this means that for some $g \in G$ the ray s_ξ leaves a copy $gg_{-k_1}F$ of the fundamental domain $F := F(\Gamma)$ through $gg_{-k_1}\alpha_{k_1}$, enters gF and leaves it through $g\alpha_{k_2}$ to enter $gg_{k_2}F$. We apply the transformation g^{-1} and consider the ray $g^{-1}s_\xi$. The ray $g^{-1}s_\xi$ leaves a copy $g_{-k_1}F$ of the fundamental domain through $g_{-k_1}\alpha_{k_1} = \alpha_{-k_1}$, enters F and leaves it through α_{k_2} to enter $g_{k_2}F$, see Figure 3.5. Note that for each point $z \in s_\xi$ we have:

$$d(z, go) = d(g^{-1}z, o)$$

Let $S(A_{-k_1}, A_{k_2})$ denote the union of all geodesics, seen as subsets of $\mathbb{D}$, such that one of their endpoints at infinity lies in A_{-k_1} and the other endpoint at infinity lies in A_{k_2}. The union $S(A_{-k_1}, A_{k_2})$ is an open subset of $\mathbb{D}$ bounded by two geodesics β_1 and β_2 with endpoints in ∂A_{-k_1} and ∂A_{k_2}, see Figure 3.5. This region looks like a thick stripe and $g^{-1}s_\xi$ lies inside this stripe.

Observe that there exists $R > 0$, depending on k_1 and k_2, such that the hyperbolic ball $B(o, R)$ intersects both β_1 and β_2. If this is the case we will say that the ball $B(o, R)$ **crosses** the stripe $S(A_{-k_1}, A_{k_2})$. In this situation the ray $g^{-1}s_\xi$ passes o within hyperbolic distance R as it passes through F, which implies that s_ξ passes go within hyperbolic distance R as it passes through gF.

Characterising the uniformly radial limit set

We will now prove two propositions, which, combined, yield a full characterisation of the uniformly radial set $L_{ur}(\Gamma)$ in terms of the coding sequences.

Proposition 9. *Let ξ be a point in the limit set $L(\Gamma)$ with coding sequence*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

If $\kappa(\xi) \in \mathcal{C}_{\kappa,1}$ and there exists $b \in \mathbb{N}$ such that $|x_i| \leq b$ for all $i \in \mathbb{N}_0$ then $\xi \in L_{ur}(\Gamma)$.

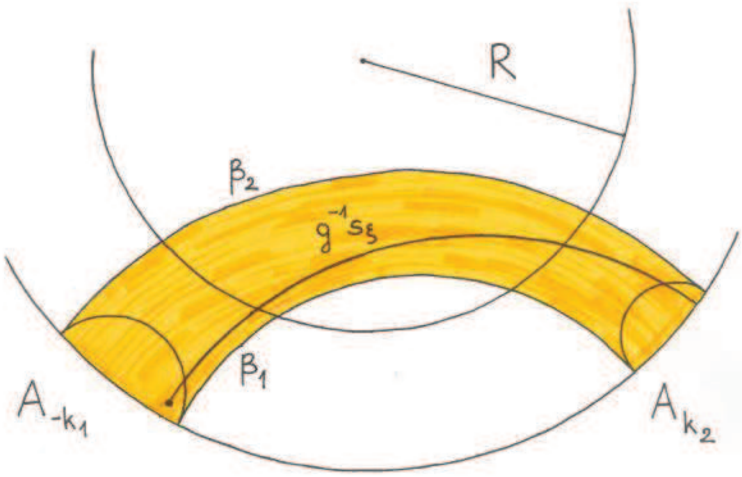


Figure 3.5:

Proof. There are two ways of proving this result. One is to realise that if ξ satisfies the conditions of the proposition, then it is a limit point of the subgroup of Γ generated by $\{g_i : |i| \leq b\}$; this follows, for example, from Proposition 7. Then one can use the well known fact that for a finitely generated Fuchsian group without parabolic elements all limit points are uniformly radial. If ξ is uniformly radial with respect to a subgroup of Γ then it is also uniformly radial with respect to the group Γ itself.

We can also prove this result directly with a geometric argument. Suppose that for some $b \in \mathbb{N}$ we have $|x_1| \leq b$ and $x_i \neq 0$ for all $i \in \mathbb{N}_0$. For $k_1, k_2 \in I$ satisfying $|k_1|, |k_2| \leq b$ and $k_2 \neq -k_1$ consider the stripe $S(A_{-k_1}, A_{k_2})$. The interval A_{-k_1} contains the intervals $g_{-k_1}A_k$ for $k \neq k_1$ while the interval A_{k_2} contains the intervals $g_{k_2}A_l$ for $l \neq -k_2$. Let $\mathcal{S}(k_1, k_2)$ denote the collection of all the stripes of the form $S(g_{-k_1}A_k, g_{k_2}A_k)$ with k and l satisfying $k \neq k_1$, $l \neq -k_2$ and $|k|, |l| \leq b$; this collection is clearly finite. Now consider the following region, a demonstration of which has been depicted in Figure 3.6:

$$Q(k_1, k_2) := \overline{F(\Gamma)} \cap \bigcup_{S \in \mathcal{S}(k_1,k_2)} S \ \subseteq S(A_{-k_1}, A_{k_2})$$

Observe that, since $\mathcal{S}(k_1, k_2)$ contains only finitely many stripes, property (1) from the definition of Schottky description implies that there exists $R(k_1, k_2) > 0$ such that the set $Q(k_1, k_2)$ is contained in the open hyperbolic ball $B(o, R(k_1, k_2))$.
Let $g \in \Gamma \setminus \{id\}$ be arbitrary. The part of s_ξ which lies in $\overline{gF}$ is contained in $gQ(k_1, k_2)$, for some $k_1, k_2 \in I$ with $|k_1|, |k_2| \leq b$, and $k_2 \neq -k_1$ and so it is contained in

$$gB(o, R(k_1, k_2)) = B((go, R(k_1, k_2))$$

Hence the ray $s_\xi \setminus F$ is contained in $\bigcup_{g\in\Gamma} B(go, R_0)$ where

$$R_0 := \max\{R(k_1,k_2) : |k_1|, |k_2| \leq b,\ \ k_2 \neq -k_1\}$$

Further, note that the point ξ must lie in one of the intervals of the form $g_k A_l$ with $k \neq -l$ and $|k|, |l| \leq b$. Since there are only finitely many of them, property (1) from the definition of Schottky description implies that the part of s_ξ which lies in $\overline{F}$ is contained in $B(o, R_1)$, for some $R_1 > 0$, see Figure 3.7. This shows that ξ is uniformly radial with respect to the radius $R_{max} = \max(R_0, R_1)$. $\square$

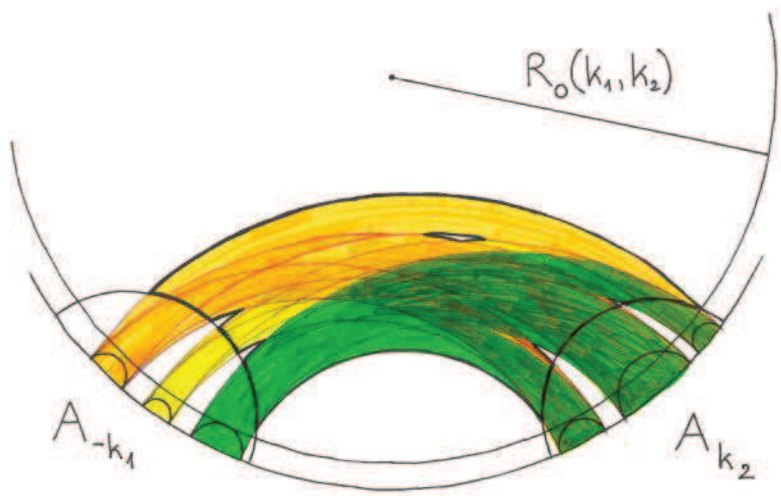


Figure 3.6:

Proposition 10. *Let ξ be a point in the limit set $L(\Gamma)$ with coding sequence*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

If $\xi \in L_{ur}(\Gamma)$ then $\kappa(\xi) \in \mathcal{C}_{\kappa,1}$ and there exists $b \in \mathbb{N}$ such that $|x_i| \leq b$ for all $i \in \mathbb{N}_0$.

Proof. Assume that we have $\xi \in L_{ur}(\Gamma)$ and let $R > 0$ be such that $s_\xi \subseteq \bigcup_{g\in\Gamma} B(go, R)$. Consider the open hyperbolic ball $B(o, R)$. Observe that there are only finitely many $g \in \Gamma$ for which $gF \cap B(o, R) \neq \emptyset$. This implies that there are only finitely many $g \in \Gamma$ for which $F \cap B(go, R) \neq \emptyset$ and thus the set

$$F \ \cap \ \bigcup_{g\in\Gamma} B(go, R)$$

is contained in $B(o, R_0)$, for some $R_0 > 0$. Now note that there exists $b_0 \in \mathbb{N}$ such that all of the geodesics α_k corresponding to the intervals in the Schottky description of Γ

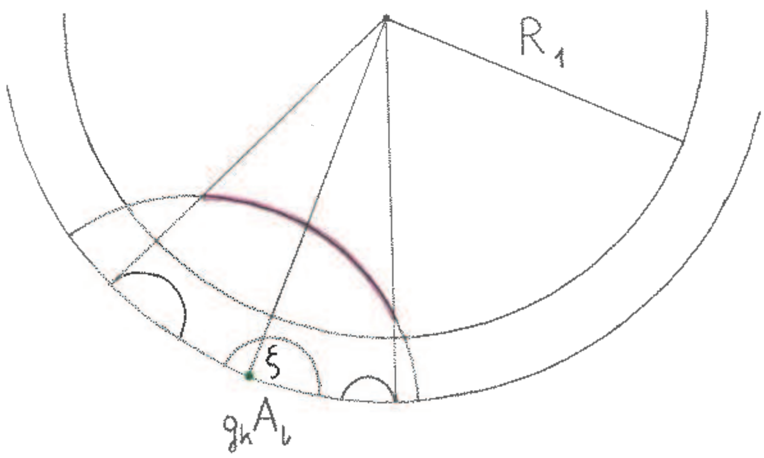


Figure 3.7:

which satisfy $|k| \geq b_0$ do not intersect the ball $B(o, 2R_0)$.
In order to reach a contradiction, suppose that the coding sequence $\kappa(\xi)$ is unbounded i.e. that for all $b \in \mathbb{N}$ there exists $i \in \mathbb{N}_0$ for which $|x_i| > b$. Then, in particular, there is an $i \in \mathbb{N}_0$ such that $|x_i| > b_0$ and without loss of generality we can assume that $i \neq 0$. Consider the part of s_ξ which lies in $g_{x_0} \cdots g_{x_{i-1}} F$ and apply $(g_{x_0} \cdots g_{x_{i-1}})^{-1}$. Note that the ray $(g_{x_0} \cdots g_{x_{i-1}})^{-1} s_\xi$ leaves F through the geodesic α_{x_i}. But then, by the choice of b_0, the part of $(g_{x_0} \cdots g_{x_{i-1}})^{-1} s_\xi$ which lies in F is not contained in $F \cap \bigcup_{g \in \Gamma} B(go, R)$. Since the collection of balls $\{B(go, R) : g \in \Gamma\}$ is invariant under the action of Γ this means that s_ξ is not covered by $\bigcup_{g \in \Gamma} B(go, R)$, which is a contradiction.
Now suppose that we would have $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$. Then there would exist $z \in s_\xi$ such that the part of s_ξ between z and ξ is contained in gF, for some $g \in \Gamma$. Apply g^{-1} and observe that the ray between $g^{-1}z$ and $g^{-1}\xi$ is clearly not contained in $B(o, R_0)$ and therefore, it is not contained in $F \cap \bigcup_{g \in \Gamma} B(go, R)$. Again, since the collection of balls $\{B(go, R) : g \in \Gamma\}$ is invariant under the action of Γ, this means that s_ξ is not covered by $\bigcup_{g \in \Gamma} B(go, R)$, which is a contradiction. □

Combining Propositions 9 and 10 yields the following theorem, which generalises a result of Dal'bo and Starkov [DalboStarkov2000] who worked under the assumption that the set $J(\Gamma)$ is of cardinality one.

Theorem 3. *Let Γ be a geometric Schottky group and let ξ be a point in the limit set $L(\Gamma)$ with coding sequence*

$$\kappa(\xi) = [x_0, x_1, \ldots]$$

Then ξ is a uniformly radial limit point if and only if there exists $b \in \mathbb{N}$ such that $|x_i| \leq b$, for all $i \in \mathbb{N}_0$, and $x_i \neq 0$, for all $i \in \mathbb{N}_0$.

Characterising the radial limit set

Now we will characterise the set of radial limit points $L_r(\Gamma)$ in terms of the coding sequences. For a point $\xi \in L(\Gamma)$ we will consider two ways of associating to its coding sequence $\kappa(\xi)$ another sequence of real numbers. The properties of these two types of associated sequences, in particular whether they converge to zero, will be related to the property of ξ being a radial limit point.

The first step is to consider two functions ρ_1 and ρ_2 whose domain is the set of pairs of subsets of S^1. The function ρ_1 is essentially the Hausdorff distance. For any pair of non-empty sets $A, A' \subseteq S^1$ we define:

$$\rho_1(A, A') := \inf\left\{\epsilon > 0 : \;\; A \subseteq N_\epsilon(A'), \;\; A' \subseteq N_\epsilon(A)\right\}$$

Here $N_\epsilon(\cdot)$ denotes the open ϵ-neighborhood with respect to the chordal distance on S^1. The second function is defined for any pair of non-empty sets $A, A' \subseteq S^1$ by:

$$\rho_2(A, A') := \inf\left\{\epsilon > 0 : \;\; A \cap N_\epsilon(A') \neq \emptyset\right\}$$

We are now going to associate to the codes of limit points two kinds of sequences of real numbers. Let ξ be a point in $L(\Gamma)$ and $\kappa(\xi) = [x_0, x_1, \ldots]$ its code. Moreover, assume that $\kappa(\xi) \in \mathcal{C}_{\kappa,1}$, that is $x_i \neq 0$ for all $i \in \mathbb{N}_0$. We then define:

$$\kappa_1(\xi) := [\rho_1(A_{-x_0}, A_{x_1}), \rho_1(A_{-x_1}, A_{x_2}), \ldots]$$

and

$$\kappa_2(\xi) := [\rho_2(A_{-x_0}, A_{x_1}), \rho_2(A_{-x_1}, A_{x_2}), \ldots]$$

A useful geometrical interpretation of these two sequences can be obtained using the same approach as in the brief discussion at the beginning of this section, where we considered the situation when in the coding sequence $\kappa(\xi)$ an integer k_1 is followed directly by an integer k_2.

Note that we have defined the sequences $\kappa_1(\xi)$ and $\kappa_2(\xi)$ only for those points $\xi \in L(\Gamma)$ which satisfy $\kappa(\xi) \in \mathcal{C}_{\kappa,1}$. But this is all we are going to need since none of the points $\xi \in L(\Gamma)$ with $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$ are radial.

Proposition 11. *Let $\xi \in L(\Gamma)$. If $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$ then ξ is not a radial limit point.*

Proof. Let $\xi \in L(\Gamma)$ and suppose that $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$. Let $R > 0$ be arbitrary and let F be the standard fundamental domain of Γ. Consider the open hyperbolic ball $B(o, R)$. Observe that there are only finitely many $g \in \Gamma$ for which $gF \cap B(o, R) \neq \emptyset$. This implies that there are only finitely many $g \in \Gamma$ for which $F \cap B(go, R) \neq \emptyset$. Similarly, since the collection of balls $\{B(go, R) : g \in \Gamma\}$ is invariant under the action of Γ, for any copy γF of F there are only finitely many $g \in \Gamma$ for which $F \cap B(go, R) \neq \emptyset$. Now, since $\kappa(\xi) \in \mathcal{C}_{\kappa,0}$ there is some point z on the ray s_ξ and an element $\gamma \in \Gamma$ such that the part of s_ξ between z and ξ is contained in γF and, in particular, s_ξ intersects only finitely many copies of F. But since each copy of F is intersected by finitely many balls of the form $B(go, R)$ it follows that only finitely many such balls will intersect s_ξ. This means that ξ is not radial with respect to the radius R but, since R was arbitrary, this shows that ξ is not a radial limit point. $\square$

The following three propositions, when combined, give a characterisation of the set of radial limit points $L(\Gamma)$ in terms of coding sequences. For a general geometric Schottky group, we obtain a necessary and a sufficient condition for a point ξ to be radial; while for a group which is regular, see definition 6, these two conditions reduce to an if and only if statement.

Proposition 12. *Let ξ be a point in $L(\Gamma)$. If ξ is a radial limit point then the sequence $\kappa_1(\xi)$ does not converge to zero.*

Proof. Let ξ be a point in $L(\Gamma)$, $\kappa(\xi) = [x_0, x_1, \ldots]$ its coding sequence and suppose that the sequence $\kappa_1(\xi)$ converges to zero. We will show that ξ is not a radial limit point. Let $R > 0$ be arbitrary. Consider the collection of open hyperbolic balls $\{B(go, R) : g \in \Gamma\}$. As in the proof of Proposition 10, there are only finitely many elements $g \in \Gamma$ for which $F \cap B(go, R) \neq \emptyset$ and thus, the set:

$$F \;\cap\; \bigcup_{g\in\Gamma} B(go, R)$$

is contained in $B(o, R_0)$, for some $R_0 > 0$. The fact that the sequence $\kappa_1(\xi)$ converges to zero implies that there exists some $N \in \mathbb{N}$ such that, for any $n \geq N$, the stripe $S(A_{-x_n}, A_{x_{n+1}})$ is disjoint from $B(o, R_0)$. But this shows that ξ is not radial with respect to R and, since R was chosen arbitrarily, we deduce that ξ is not a radial limit point. $\square$

Proposition 13. *Let ξ be a point in $L(\Gamma)$. If the sequence $\kappa_2(\xi)$ does not converge to zero then ξ is a radial limit point.*

Proof. This is almost immediate. Let ξ be a point in $L(\Gamma)$, $\kappa(\xi) = [x_0, x_1, \ldots]$ its coding sequence and suppose that the sequence $\kappa_2(\xi)$ does not converge to zero. This means that there exists some $\epsilon > 0$ such that for infinitely many $n \in \mathbb{N}$ we have $\rho_2(A_{-x_n}, A_{x_{n+1}}) > \epsilon$. Let $R > 0$ be large enough so that the open hyperbolic ball $B(o, R)$ intersects any geodesic with endpoints at infinity ξ_- and ξ_+ satisfying $|\xi_- - \xi_+| = \epsilon$. Then for every $n \in \mathbb{N}$ for which we have $\rho_2(A_{-x_n}, A_{x_{n+1}}) > \epsilon$ we also have that the ball $B(o, R)$ crosses over the stripe $S(A_{-x_n}, A_{x_{n+1}})$. But this shows that ξ is a radial limit point. $\square$

Proposition 14. *If Γ is regular then for any $\xi \in L(\Gamma)$ the sequence $\kappa_1(\xi)$ converges to zero if and only if the sequence $\kappa_2(\xi)$ converges to zero.*

Proof. Let ξ be a point in $L(\Gamma)$ and $\kappa(\xi) = [x_0, x_1, \ldots]$ its coding sequence. It is clear, by construction, that if the sequence $\kappa_1(\xi)$ converges to zero, then the sequence $\kappa_2(\xi)$ converges to zero as well. We will show that the situation that $\kappa_2(\xi)$ converges to zero but

$\kappa_1(\xi)$ does not converge to zero cannot occur if Γ is regular. Suppose that $\kappa_2(\xi)$ converges to zero but $\kappa_1(\xi)$ does not. This means that there exists some $\epsilon > 0$ and a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$ we have $\rho_1(A_{-x_{n_j}}, A_{x_{n_j+1}}) > \epsilon$. But then either we can find a sequence $(m_j)_{j\in\mathbb{N}}$ such that $diam(A_{x_{m_j}}) > \frac{\epsilon}{3}$, for all $j \in \mathbb{N}$ or we can find a sequence $(m_j)_{j\in\mathbb{N}}$ such that $diam(A_{-x_{m_j}}) > \frac{\epsilon}{3}$, for all $j \in \mathbb{N}$, where *diam* denotes the diameter with respect to the chordal distance. Suppose we can find a sequence $(m_j)_{j\in\mathbb{N}}$ of the first type. Observe that there must exist a $k \in I$, where I is the indexing set of the Schottky description of Γ, such that $x_{m_j} = k$, for infinitely many $j \in \mathbb{N}$. We extract the corresponding subsequence of $(m_j)_{j\in\mathbb{N}}$ but keep the notation unchanged. Now the assumption that the sequence $\kappa_2(\xi)$ converges to zero would imply that the boundary points of the intervals $A_{-x_{m_j-1}}$ accumulate at the boundary of A_k, which is impossible if Γ is regular. If a sequence $(m_j)_{j\in\mathbb{N}}$ of the second type can be found, then we can proceed in an analogous manner as above. Thus for a regular group Γ, if the sequence $\kappa_2(\xi)$ converges to zero then the sequence $\kappa_1(\xi)$ converges to zero as well, which yields the desired equivalence. $\square$

Corollary 4. *If Γ is regular then a point $\xi \in L(\Gamma)$ is a radial limit point if and only if the sequence $\kappa_2(\xi)$ does not converge to zero.*

We will summarise the statements of Propositions 11, 12, 13 and Corollary 4 in the following theorem:

Theorem 4. *Let Γ be a geometric Schottky group and let ξ be a point in the limit set $L(\Gamma)$. If ξ is a radial limit point then its coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_1(\xi)$ associated to $\kappa(\xi)$ does not converge to zero. If the coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_2(\xi)$ associated to $\kappa(\xi)$ does not converge to zero then ξ is a radial limit point. Moreover, if the group Γ is regular then ξ is a radial limit point if and only if its coding sequence $\kappa(\xi)$ does not end in a string of 0's and the sequence $\kappa_2(\xi)$ does not converge to zero.*

3.3 Subgroups

When constructing concrete examples of geometric Schottky groups, and in particular in Chapter 5 where we describe a method of obtaining examples of infinitely generated geometric Schottky groups whose Liouville-Patterson measure is finite and for which the geodesic flow on the quotiend manifold is ergodic with respect to this measure, we will use the fact that a subgroup of a group of Schottky type is also of Schottky type. It is a well known fact that any subgroup of a classical Schottky group is also a classical Schottky group. Yet, we were unable to find a reference with an elementary geometric argument which could be immediately extended to our more general setting. Therefore in this section we provide an explicit construction of a fundamental domain for a subgroup of a geometric Schottky group and explain how from this fundamental domain one can obtain a Schottky description for the subgroup.

3.3.1 Constructing a fundamental domain

Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ and let F denote its standard fundamental domain, as defined in Section 3.1.2. Let H be a subgroup of Γ. Using the fundamental domain F of Γ we will construct a fundamental domain F_H for H.

The construction of the fundamental domain F_H might at first sight seem complicated but, in fact, the idea behind it is very simple. We are going to choose a set S consisting of elements of the group Γ and then define $F(H)$ by:

$$F_H := int\Big(\bigcup_{g\in S} \overline{gF} \Big) \tag{3.2}$$

The idea behind our construction is to choose the set S in such a way that F_H is simply connected and so that S forms a **full set of right coset representatives** for H. By the latter we mean that:

- S contains exacly one representative for each right coset of H; that is, for any coset Hg of H there is some $g^* \in S$ with $Hg^* = Hg$.
- If $g, g^* \in S$ and $g \neq g^*$, then $Hg \neq Hg^*$.

Then we will prove that a region defined by (3.2), where S is a full set of right coset representatives, is a fundamental region.

Every element $g \in \Gamma$ can be expressed uniquely in terms of the generators $\{g_k\}_{k\in I}$ of Γ. Let $l(g)$ denote the number of generators occuring in such an expression of g, where we also count repetitions. That is if $g = g_{k_1} g_{k_2} \cdots g_{k_m}$ then $l(g) = m$ and we refer to $l(g)$ as the **length** of g. For each $m \in \mathbb{N}$ we will now arrange the elements of Γ of length m in a sequence; if the indexing set I is finite these will be finite sequences. Our construction of the fundamental domain F_H for H will depend on how these sequences are chosen. So for each $m \in \mathbb{N}$ we arrange all elements of Γ of length m in a sequence, and denote this sequence by $(\gamma_{m,i})_{i\in\mathbb{N}}$. To be even more explicit, by arranging the elements we mean that every element of Γ of length m has to appear exactly once in the sequence $(\gamma_{m,i})_{i\in\mathbb{N}}$ and for each m the sequence $(\gamma_{m,i})_{i\in\mathbb{N}}$ consists only of elements of length m. Naturally, there are many ways of choosing these sequences but from now on we assume that these sequences are fixed.

Now, in order to defined the set S, we will first construct for $m \in \mathbb{N}_0$ sets S_m consisting of elements of the group Γ; the set S will be defined as their union. We start by setting $S_0 := \{id\}$. Next we define S_1 which will contain id and some of the elements of Γ of length one, that is some of the generators in the Schottky description of Γ. We define it in the following way:

An element $g \in \Gamma$ belongs to S_1 if either $g \in S_0$ or the folowing conditions are satisfied:

1. $l(g) = 1$ i.e. $g = g_j$ for some $j \in I$
2. $g \notin H$ (which is equivalent to $Hg \neq H$)
3. If for an $i_0 \in \mathbb{N}$, $g = \gamma_{1,i_0}$, then for all $i < i_0$ we have that $Hg \neq H\gamma_{1,i}$

Then we define S_2 which will contain all the elements of S_1 and some of the elements of Γ of length two. We define it in the following way:

An element $g \in \Gamma$ belongs to S_2 if either $g \in S_1$ or the following conditions are satisfied:

1. $l(g) = 2$ i.e. $g = g_{j_1}g_{j_2}$ for some $j_1, j_2 \in I$, $j_1 \neq -j_2$
2. $g \notin H$
3. $g_{j_1} \in S_1$
4. For all $g^* \in S_1$ we have $Hg \neq Hg^*$
5. If for an $i_0 \in \mathbb{N}$, $g = \gamma_{2,i_0}$, then for all $i < i_0$ we have that $Hg \neq H\gamma_{2,i}$ or $\gamma_{2,i} = g_{i_1}g_{i_2}$ with $g_{i_1} \notin S_1$

In general, for each $m \in \mathbb{N}$, the set S_m will contain all the elements of S_{m-1} and some of the elements of Γ of length m. We define it in the following way:

An element $g \in \Gamma$ belongs to S_m if either $g \in S_{m-1}$ or the following conditions are satisfied:

1. $l(g) = m$ i.e. $g = g_{j_1}g_{j_2}\cdots g_{j_m}$ for some $j_1, \ldots, j_m \in I$, $j_i \neq -j_{i+1}$
2. $g \notin H$
3. $g_{j_1}g_{j_2}\cdots g_{j_{m-1}} \in S_{m-1}$
4. For all $g^* \in S_{m-1}$ we have $Hg \neq Hg^*$
5. If for an $i_0 \in \mathbb{N}$, $g = \gamma_{k,i_0}$, then for all $i < i_0$ we have that $Hg \neq H\gamma_{m,i}$ or $\gamma_{m,i} = g_{i_1}g_{i_2}\cdots g_{i_m}$ with $g_{i_1}g_{i_2}\cdots g_{i_{m-1}} \notin S_{m-1}$

Finally we define:

$$S := \bigcup_{m \in \mathbb{N}_0} S_m$$

and we define the fundamental domain F_H by:

$$F_H := int\Big(\bigcup_{g \in S} \overline{gF} \Big) \tag{3.3}$$

where *int* denotes the interior and both the interior and the closure are taken in $\mathbb{D}$. The set S is well defined because for every $g \in \Gamma$ we can check in finitely many steps if g

belongs to S or not. Of course it is not immediate that F_H is a fundamental domain for H so we will now prove it.

First note that the third condition in the general step of our construction forces F_H to be connected, and, because of how the copies gF tesselate $\mathbb{D}$, it will also be simply connected. So F_H is indeed a domain, that is an open simply connected set, and we only need to show that it is a fundamental region for the action of H. In order to do so we are going to show that the set S obtained in the above construction is a full set of right coset representatives for H. Then we will prove that a region defined by (3.2), where S is a full set of right coset representatives, is a fundamental region.

Proposition 15. *The set S is a full set of right coset representatives for H.*

Proof. Conditions (4) and (5) in the general step of our construction immediately imply that if $g, g^* \in S$ and $g \neq g^*$ then we have $Hg \neq Hg^*$. The more difficult part is to show that S contains a representative of each right coset of H.
Consider an arbitrary right coset Hg. We have to show that there is some $g^* \in S$ with $Hg = Hg^*$, that is $g = hg^*$ for some $h \in H$; $h = id$ is also possible and means simply that $g \in S$.
We start by expressing g in terms of the generators in the Schottky description of Γ, that is, suppose that g is of the form:

$$g = g_{j_1} g_{j_2} \cdots g_{j_{l(g)}}$$

We will proceed by induction on $l(g)$. By definition, $id \in S$, so there is nothing to check for $l(g) = 0$. Now consider $l(g) = 1$. If $g \in S$, we simply choose $g^* = g$ and $h = id$. If $g \notin S$ and $g \in H$, then we choose $g^* = id$ and $h = g$. If $g \notin S$ and $g \notin H$, then $g = \gamma_{1,i_0}$ and $Hg = H\gamma_{1,i}$, for some $i < i_0$. In this case choose $i_{\min}$ to be smallest such i and put $g^* = \gamma_{1,i_{\min}}$. Then we obviously have $Hg = H\gamma_{1,i_{\min}}$ and by the choice of $i_{\min}$ we must have $g^* \in S$.
Now suppose we have already shown that for an $m \in \mathbb{N}$ and any $g \in \Gamma$ with $l(g) < m$ there is some $g^* \in S$ with $Hg = Hg^*$. Let us consider an arbitrary element $g = g_{j_1} g_{j_2} \cdots g_{j_m} \in G$ with $l(g) = m$. If $g \in S$ then we are done. So suppose $g \notin S$ and consider the possible cases in which our construction could allow that $g \notin S$.

Case A: We have $g \in H$.
Then we simply have $id \in S$ and $Hg = H$.

Case B: There exists $g^* \in S$ with $l(g^*) \leq m$ such that $Hg = Hg^*$.
Then we are done.

Case C: We have $g' := g_{j_1} g_{j_2} \cdots g_{j_{m-1}} \notin S$.
Then, since $l(g') = m - 1 < m$, there exists some $g^* \in S$ with $Hg' = Hg^*$. This implies that $g' = hg^*$, for some $h \in H$, so $g = hg^* g_{j_m}$. If the last letter in the expression of g^* in terms of the generators of Γ is $g_{j_m}^{-1}$ then, by condition (3) in the general step of

our construction, $g^* g_{j_m} \in S$ and we are done. So suppose that the last letter in the expression of g^* in terms of the generators of Γ is not $g_{j_m}^{-1}$. Now if $g^* g_{j_m} \in S$, we are done. If the latter is not the case then, since $g^* \in S$, there are only two options. Either $g^* g_{j_m} \in H$, in which case we are done. Or there exists $g^{\#} \in S$ such that $Hg^* g_{j_m} = Hg^{\#}$, in which case $g^* g_{j_m} = h^{\#} g^{\#}$, for some $h^{\#} \in H$, and we are done as well, since then $g = hh^{\#} g^{\#}$. $\square$

Now that we have established that the set S obtained in our construction is a full set of right coset representatives we will prove that this property implies that F_H is a fundamental region.

Lemma 1. *Let Γ be a geometric Schottky group with its standard fundamental domain F and let H be a subgroup of Γ. Suppose that S is a full set of right coset representatives for H. Then the set:*

$$F_H := int\Big(\bigcup_{g \in S} \overline{gF} \Big)$$

is a fundamental region of H.

Proof. We start by showing that, for $h_1, h_2 \in H$ distinct, we have $h_1 F_H \cap h_2 F_H = \emptyset$. Note first that for any $\gamma \in \Gamma$ we have:

$$\gamma F_H = \gamma \ int\Big(\bigcup_{g \in S} \overline{gF} \Big) = int\Big(\bigcup_{g \in S} \overline{\gamma g F} \Big)$$

Suppose that $h_1, h_2 \in H$ satisfy $h_1 F_H \cap h_2 F_H \neq \emptyset$. Then there exists some point $z \in h_1 F_H \cap h_2 F_H$ and there are two ways in which this can happen.
One possibility is that z lies in the interior of a copy of F. In this case there exist $x_1, x_2 \in S$ such that $z \in h_1 x_1 F$ and $z \in h_2 x_2 F$. Since F is a fundamental domain for Γ this implies that $h_1 x_1 = h_2 x_2$. But then $x_1 = h_1^{-1} h_2 x_2$ so that $Hx_1 = Hx_2$. Since S is a full set of right coset representatives, this implies that $x_1 = x_2$ and hence, $h_1 = h_2$.
The other possiblity is that z lies in the common boundary of two copies of F. In this case there exist $x_1, x_2, x_3, x_4 \in S$ such that $z \in \overline{h_1 x_1 F} \cap \overline{h_1 x_2 F}$ and $z \in \overline{h_2 x_3 F} \cap \overline{h_2 x_4 F}$. But then we must have $\overline{h_1 x_1 F} \cap \overline{h_1 x_2 F} = \overline{h_2 x_3 F} \cap \overline{h_2 x_4 F}$. So, either $h_1 x_1 F = h_2 x_3 F$ or $h_1 x_1 F = h_2 x_4 F$. Since F is a fundamental domain for Γ, this would imply that $h_1 x_1 = h_2 x_3$, or $h_1 x_1 = h_2 x_4$ respectively. In the same way as in the previous case, this allows us to conclude that $h_1 = h_2$.
It now remains to be shown that $\bigcup_{h \in H} \overline{hF_H} = \mathbb{D}$. Consider an arbitrary point $z \in \mathbb{D}$. Since F is a fundamental domain for Γ, there exists a $g \in \Gamma$ such that $z \in \overline{gF}$. Further, since the right cosets of H partition Γ, there exists some $x \in S$ and some $h \in H$ with $g = hx$. Thus $z \in \overline{hxF} \subseteq \overline{hF_H}$. $\square$

3.3.2 Finding a Schottky description

In the previous subsection we have constructed a fundamental domain for a subgroup of a geometric Schottky group. Now we explain how one can obtain a Schottky description for the subgroup from this fundamental domain. This will show that a subgroup of a group of Schottky type is also of Schottky type.

We keep the notation from the previous subsection. The fundamental domain F_H for the subgroup H was constructed from the copies gF of the fundamental domain F of Γ, where the elements g belonged to a suitably chosen set S, a full set of right coset representatives for H. The domain F_H was simply the union of these copies gF with some parts of their boundaries included. It is clear that the complement in $\mathbb{D}$ of the domain F_H which we obtained consists of a possibly infinite number of pairwise disjoint closed half-planes, none of which contain the origin. Let $\widehat{B}$ denote one such closed half-plane and let β denote its boundary in $\mathbb{D}$. Since F_H is a fundamental domain for H there exists a unique element $h \in H$ such that F_H and hF_H 'meet' along the geodesic β, by which we mean that:

$$\overline{F_H} \cap \overline{hF_H} = \beta$$

where the closures are taken in $\mathbb{D}$.

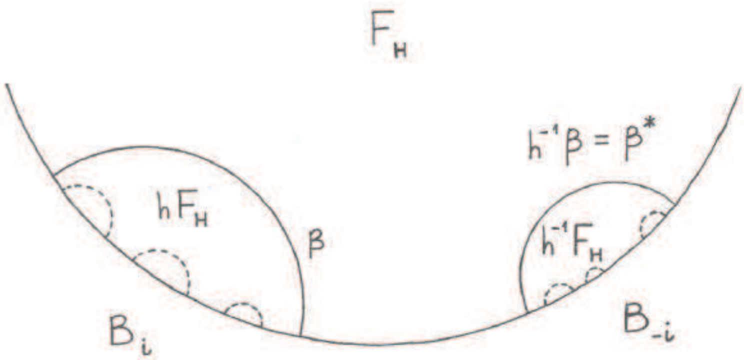


Figure 3.8:

Consider the inverse h^{-1} of h. It maps hF_H to F_H and F_H inside one of the closed half-planes $\widehat{B}^*$, where $\widehat{B}^* \neq \widehat{B}$, for otherwise we would have $h^2F_H = F_H$, which is not possible. Notice that $\mathbb{D} \setminus \widehat{B}^*$ is mapped by h to $\widehat{B}$ and similarly $\mathbb{D} \setminus \widehat{B}$ is mapped by h^{-1} to $\widehat{B}^*$. Similarly, as for $\widehat{B}$, define β^* to be the boundary of $\widehat{B}^*$ in $\mathbb{D}$. Now let B_i and B_{-i} denote the open intervals in S^1 enclosed by β, and β^* respectively, see Figure 3.8, and put $h_i := h$ and $h_{-i} := h^{-1}$.

Now, to obtain a Schottky description for the group H, we do the following:

- Arrange all the closed half-planes which constitute the complement of F_H in a sequence $(\widehat{B}_k)_{k\in\mathbb{N}}$.
- For each $k \in \mathbb{N}$ associate to $\widehat{B}_k$ a pair of intervals B_k, B_{-k} and a pair of transformations h_k, h_{-k}, as described in case of $\widehat{B}$.
- Each of the associated quadruples $(B_k, B_{-k}, h_k, h_{-k})$ will appear twice, if we ignore how the elements of a quadruple are ordered. Eliminate repetitions by deleting the second occurence of the quadruple, thus reducing the set of possible values of the index k from $\mathbb{N}$ to a proper subset I' of it.
- As the Schottky description we take $(\{B_i\}, \{h_i\})_{i\in I_H}$, where $I_H :=\in I' \cup -I'$.

In the above way we clearly get a Schottky description of some Fuchsian group. All we need to check now is that the set of elements $\{h_i : i \in I_H\}$, where I_H is the symmetric subset of $\mathbb{Z}^*$ obtained in the above construction, indeed generate H. But this is not so difficult to see, if we use the geometric picture. Namely, let h be some arbitrary element of $H \setminus \{id\}$. Consider the image hF_H of the fundamental domain F_H. There is some B_j in the constructed Schottky description such that hF_H is contained in the closed half-plane $\widehat{B_j}$. Put $g_1 := h_i^{-1} = h_{-i}$ and consider g_1hF_H. By repeating this process, we eventually obtain $g_m \dots g_1hF_H = F_H$, from which we then deduce that $h = g_1^{-1} \dots g_m^{-1} \in \langle h_i : i \in I_H\rangle$.

3.4 Examples

So far we have not given any concrete examples of groups of geometric Schottky groups and in this section we discuss how such groups can be obtained. In general it is not difficult to construct geometric Schottky groups. We will first describe a direct way of finding both finitely and infinitely generated geometric Schottky groups. Then we will use our earlier discussion about subgroups and give a particular example of an infinitely generated geometric Schottky group which is obtained as a subgroup of a finitely generated geometric Schottky group and discuss how this method can be extended. The latter approach will be used in Chapter 5 to construct geometric Schottky groups whose Liouville-Patterson measure is finite and for which the geodesic flow on the quotiend manifold is ergodic with respect to this measure.

Finitely generated groups

To produce a finitely generated geometric Schottky group we can consider any finite set of even number of intervals satisfying conditions (1) and (2) in the definition of Schottky description, there is plainly an abundance of such collections of intervals, and pair up these intervals by indexing them with a suitable finite and symmetric set $I \subseteq \mathbb{Z}^*$. Then, to associate to a pair of intervals $\{A_i, A_{-i}\}$ a pair of isometries that satisfy conditions (3) and (4), we could consider the unique geodesic β_i which is perpendicular to both of the geodesics α_i and α_{-i}, see Figure 3.9, and choose the isometries g_i and g_{-i} to be the two hyperbolic isometries with axis β_i and translation along the axis equal to $d(\alpha_{-i} \cap \beta, \alpha_i \cap \beta_i)$. Note that for a finite indexing set I condition (5) in the definition of Schottky description is satisfied automatically.

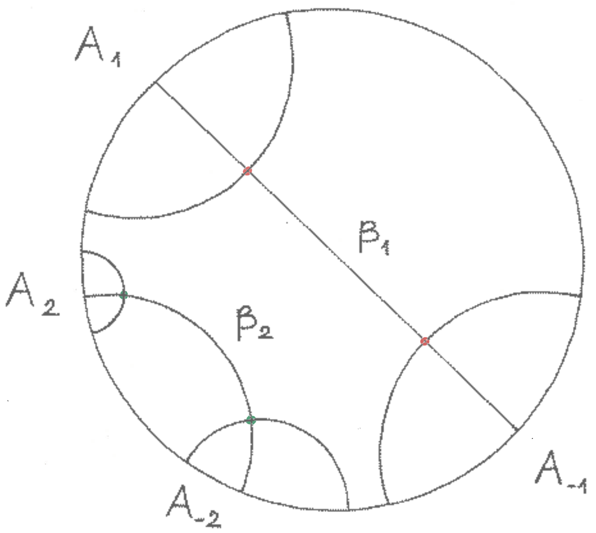


Figure 3.9:

Infinitely generated groups

To produce an infinitely geometric Schottky group we can proceed in the same way as before except that now we need to take care that the intervals we choose are located sparsely enough so that condition (5) in the definition of Schottky description is satisfied. What one can do is to consider an interval A in the boundary of the upper half-plane $\mathbb{H}$ satisfying $A \subseteq \mathbb{R} \setminus \{0\}$ and a map of the form $p : z \mapsto \lambda z$ where $\lambda \in \mathbb{R} \setminus \{0\}$ is chosen so that $A \cap p(A) = \emptyset$. We could then use the intervals $\left\{\varphi^{-1}(p^k A) : k \in \mathbb{N}_0\right\}$, where $\varphi^{-1} : \overline{\mathbb{H}} \to \overline{\mathbb{D}}$ is the extension to the boundary of the standard isometry between the two models of the hyperbolic plane. We could pair them up in any way we wish and continue as for the finitely generated examples, see Figure 3.10.

There are also plenty of ways to extend this construction. We could consider the bigger collection $\left\{\varphi^{-1}(p^k A) : k \in \mathbb{Z}\right\}$ instead. We could also replace any interval by a smaller interval contained in the original one. Another option would be to take several such collections of intervals, rotate them so that the points where the distinct collections accumulate are separated and then delete some of the intervals so that the collections do not intersect each other, see Figure 3.11. This would yield groups Γ for which the set $J(\Gamma)$ is finite. To obtain Γ with $J(\Gamma)$ infinite we could start with one of the above col-

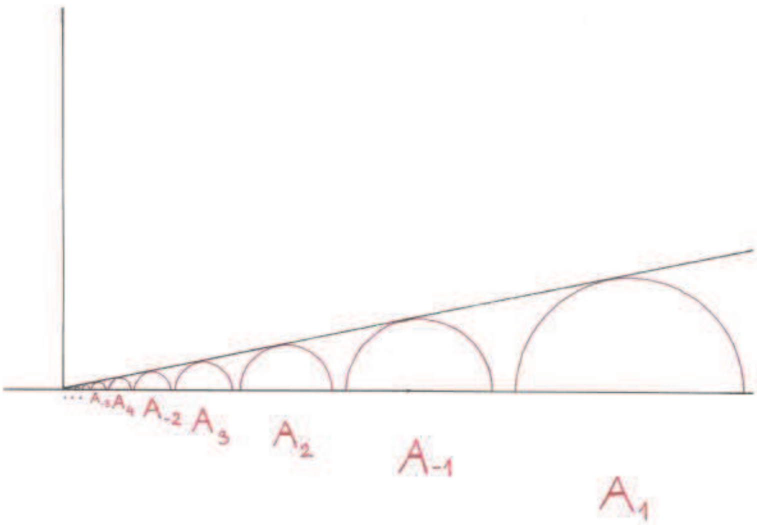


Figure 3.10:

lections and then replace each interval with truncated and rotated version of the initial collection; we truncate by removing a finite number of the intervals.

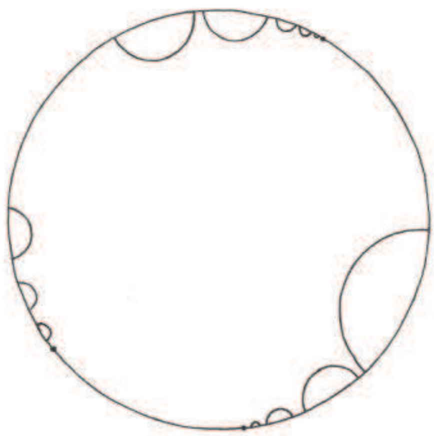


Figure 3.11:

Infinitely generated subgroups of finitely generated groups

Now we give an example of an infinitely generated geometric Schottky group which is obtained as a subgroup of a finitely generated geometric Schottky group.

We start by defining the following four intervals in S^1:

$$A_1 := \left\{ e^{xi} : x \in (\frac{\pi}{3}, \frac{2\pi}{3}) \right\}$$

$$A_{-1} := \left\{ e^{xi} : x \in (-\frac{2\pi}{3}, -\frac{\pi}{3}) \right\}$$

$$A_2 := \left\{ e^{xi} : x \in (-\frac{\pi}{6}, \frac{\pi}{6}) \right\}$$

$$A_{-2} := \left\{ e^{xi} : x \in (\frac{5\pi}{6}, -\frac{5\pi}{6}) \right\}$$

These intervals clearly satisfy the conditions (1) and (2) in the definition of Schottky description. Next, we choose isometries $g_1, g_{-1}, g_2, g_{-2} \in Iso^+(\mathbb{D})$ so that the remaining conditions in the definition of the Schottky description are satisfied. In particular we take the hyperbolic elements with axes equal to $(-i, i)$ and $(-1, 1)$ with translation along the axis chosen in each case in such a way that for $i \in \{1, -1, 2, -2\}$ the element g_i maps the half-plane $\mathbb{D} \setminus \widehat{A}_{-i}$ to the half-plane $\widehat{A}_i \setminus \alpha_i$; this also forces that $g_{-i} = g_i^{-1}$. We define the group Γ to be:

$$\Gamma :=< g_1, g_2 >$$

The standard fundamental domain of this group is the following region, a demonstration of which has been depicted in Figure 3.12:

$$F(\Gamma) = \mathbb{D} \setminus \bigcup_i \widehat{A}_i$$

Now define the following subgroup of Γ:

$$H_0 := \left\langle g_1^k g_2 g_1^{-k} : \quad k \in \mathbb{Z} \right\rangle$$

Note that, since by Proposition 3 each element of Γ has a unique reduced expression in terms of g_1, g_2 and their inverses, it is impossible to find a finite set of generators for H_0, which means that H_0 is infinitely generated.

Now we will find a fundamental domain for H_0, which will allow us to provide a Schottky description for H_0. In the construction of fundamental domain of a subgroup which we have given in the previous section we have used the fact that any full set of right coset representatives of the subgroup yields a fundamental region for this subgroup. We have also shown how one can choose a full set of right coset representatives so that the fundamental region obtained from it is a domain. However, in case of the group H_0

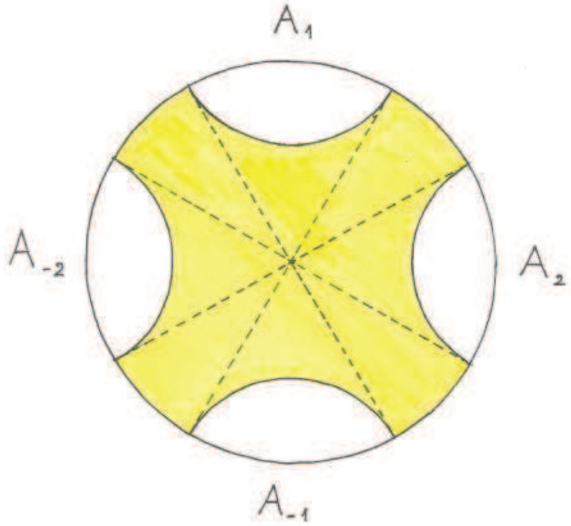


Figure 3.12:

it is straightforward to find a full set of right coset representatives which satisfies this condition. Namely, we consider the following set:

$$S := \left\{ g_1^k : k \in \mathbb{Z} \right\}$$

with the convention that $g_1^0 := id$.

Proposition 16. *The set S is a full set of right coset representatives for H_0.*

Proof. We start by showing that two distinct elements of S represent distinct cosets of H_0. If $H_0 g_1^{k_1} = H_0 g_1^{k_2}$ then $g_1^{k_1} g_1^{-k_2} = g_1^{k_1 - k_2} \in H_0$. From the definition of H_0 it follows that H_0 does not contain g_1 or any of its non-zero powers. So we must have $g_1^{k_1-k_2} = id$ which implies $k_1 = k_2$.
To show that every right coset can be represented by an element of S we consider an arbitrary element γ of Γ. This element has a unique expression in terms of the generators in the Schottky description of of Γ:

$$\gamma = g_1^{k_1} g_2^{j_1} \cdots g_1^{k_m} g_2^{j_m}$$

where $k_i, j_i \in \mathbb{Z}$ and all except possibly k_1 and j_m are non-zero. Suppose first that $k_1 = 0$, that is $\gamma = g_2^{j_1} \cdots g_1^{k_m} g_2^{j_m}$. Then since for:

$$\gamma_1 := g_1^{k_2} g_2^{j_2} \cdots g_1^{k_m} g_2^{j_m}$$

we have $\gamma\gamma_1^{-1} = g_2^{j_1} \in H_0$ it follows that $H\gamma = H\gamma_1$. If $m = 1$ then in this step we simply have $\gamma_1 = id$ and we are done, and if $m = 2$ and $j_2 = 0$ we are also done. Otherwise rewrite γ_1 as:

$$\gamma_1 = g_1^{k_2} g_2^{j_2} g_1^{-k_2} g_1^{k_2} g_1^{k_3} \cdots g_1^{k_m} g_2^{j_m}$$

so that it becomes evident that for:

$$\gamma_2 := g_1^{k_2+k_3} \cdots g_1^{k_m} g_2^{j_m}$$

we have $H\gamma_1 = H\gamma_2$. If $m = 2$ we simply have $\gamma_2 = g_1^{k_2}$ in this step and we are done, and if $m = 3$ and $j_3 = 0$ have $\gamma_2 = g_1^{k_2+k_3}$ and are also done; otherwise, we continue as in the last step. This procedure will finish either after finding γ_{m-1} or after finding γ_m and the so obtained representative will be a power of g_1; this power might be the identity. $\square$

It is clear that the set $F_0 := int\Big(\bigcup_{g\in S} \overline{gF} \Big)$ is a domain, the situation is shown in Figure 3.13. The boundary of this fundamental domain consists of the geodesics:

$$\alpha_2,\ \alpha_{-2},\ g_1\alpha_2,\ g_1\alpha_{-2},\ \dots\ ,\ g_1^k\alpha_2,\ g_1^k\alpha_{-2},\ \dots$$

which accumulate at the point $\xi := \lim_{k\to\infty} g_1^k o$ and of the geodesics

$$g_1^{-1}\alpha_2,\ g_1^{-1}\alpha_{-2},\ \dots\ ,\ g_1^{-k}\alpha_2,\ g_1^{-k}\alpha_{-2},\ \dots$$

which accumulate at the point $\eta := \lim_{k\to\infty} g_1^{-k} o$.

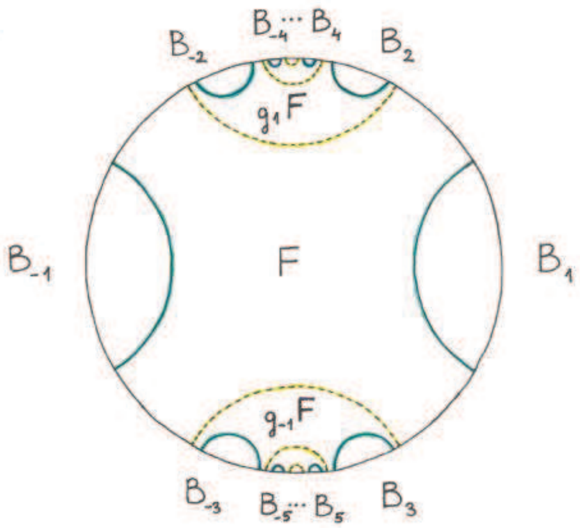


Figure 3.13:

Note that for each $k \in \mathbb{Z}$ the element $g_1^k g_2 g_1^{-k}$ takes the geodesic $g_1^k\alpha_{-2}$ to the geodesic $g_1^k g_2\alpha_{-2}$ and $g_1^k g_2\alpha_{-2} = g_1^k\alpha_2$ because of how we defined g_2. Further, $g_1^k g_2 g_1^{-k}$ takes $g_1^k F$ to $g_1^k g_2 F$, which is the copy of F that lies directly under $g_1^k\alpha_2$. From this we deduce that F_0 and $g_1^k g_2 g_1^{-k} F_0$ are separated by $g_1^k\alpha_2$. Therefore we can use the generators from the definition of H_0 to give a Schottky description for H_0. Namely, we define $B_1 := A_2$,

$B_{-1} := A_{-2}$ and for $k \in \mathbb{N}$ we define $B_{2k} := g_1^k A_2$, $B_{-2k} := g_1^k A_{-2}$, $B_{2k+1} := g_1^{-k} A_2$ and $B_{-(2k+1)} := g_1^{-k} A_{-2}$. Further we define $h_1 := g_2$, $h_j := g_1^k g_2 g_1^{-k}$ for j positive and even, and $h_j := g_1^{-k} g_2 g_1^k$ for j positive and odd, while for j negative we set $h_j := h_{-j}^{-1}$. With this new notation $(\{B_j\}, \{h_j\})_{j \in \mathbb{Z}^*}$ is a Schottky description for H_0.

Chapter 4

Entropy

The work presented in this chapter was inspired by a paper of Sullivan [Sullivan1984] from the 1980's but it is also related to more recent research of Otal and Peigné [OtalPeigné2004]. In his article Sullivan considers a certain notion of entropy for the geodesic flow which we refer to as geodesic flow entropy and denote by h_S; this notion will be introduced in Section 4.1. Sullivan works in the setting of Kleinian groups, acting on the three dimensional hyperbolic space $\mathbb{H}^3$. He shows that, for a geometrically finite Kleinian group Γ, the geodesic flow entropy $h_S(\Gamma)$ of Γ and its Poincaré exponent $\delta(\Gamma)$ coincide.

In this chapter we consider the question of equality of the two invariants for geometric Schottky groups. In case of finitely generated groups we can show that the two invariants agree. Although this is not a new result, see for example [OtalPeigné2004], for the sake of completeness we include a detailed proof of this fact in Section 4.3. The argument presented here has the advantage of being very direct and geometrical in nature.
On the other hand, when the finiteness assumption is dropped, we have found out that for geometric Schottky groups the geodesic flow entropy $h_S(\Gamma)$ is equal to the convex core entropy $h_c(\Gamma)$. We will prove this fact in Section 4.5. The convex core entropy, which in general is different from the Poincaré exponent, will be discussed in Section 4.2.

Bowen defined in [Bowen1973] a notion of entropy $h_{\widetilde{d}}$ for transformations of non-compact spaces in terms of a metric $\widetilde{d}$ on the space; this entropy is usually referred to as Bowen-Dinaburg entropy. So in case of a Fuchsian group Γ the value of $h_{\widetilde{d}}$ calculated for the geodesic flow on $T^1(\mathbb{D}/\Gamma)$, which we denote by $h_{\widetilde{d}}(\Gamma)$ will depend on the choice of a metric $\widetilde{d}$ on the unit tangent bundle $T^1(\mathbb{D}/\Gamma)$. In fact, we will show in Section 4.6 that the geodesic flow entropy $h_S(\Gamma)$ is equal to Bowen's metric entropy $h_{\widetilde{d}}(\Gamma)$ calculated with respect to a certain metric $\widetilde{d}$ on $T^1(\mathbb{D}/\Gamma)$. The equality between the two entropy notions makes our investigation relevant in the context of recent results of Otal and Peigné.
The results of Handel and Kitchens [HandelKitchens1995], and of Otal and Peigné [OtalPeigné2004] yield the following relationship for geodesic flow on $T^1(\mathbb{H}^n/\Gamma)$ for any

non-elementary Kleinian group Γ:

$$\delta(\Gamma) = h_{top}(\Gamma) = \sup_{\mu} h_{\mu}(\Gamma) = \inf_{\widetilde{d}} h_{\widetilde{d}}(\Gamma)$$

where $\delta(\Gamma)$ is the Poincaré exponent of Γ, $h_{top}(\Gamma)$ the topological entropy of the geodesic flow on $T^1(\mathbb{H}^n/\Gamma)$, $h_{\mu}(\Gamma)$ is the measure theoretical entropy with respect to a measure μ, with the supremum taken over all probability measures on $T^1(\mathbb{H}^n/\Gamma)$ invariant under the flow, and $h_{\widetilde{d}}(\Gamma)$ is Bowen's metric entropy with respect to a metric $\widetilde{d}$, with the infimum taken over all metrics on $T^1(\mathbb{H}^n/\Gamma)$ which induce its usual topology. A natural question, already posed by Otal and Peigné, is to ask if there exists a metric $\widetilde{d}$ such that $h_{\widetilde{d}}(\Gamma) = \delta(\Gamma)$. On the other hand it turns out that $h_c(\Gamma) = \overline{\dim}_{\mathrm{B}}(L(\Gamma))$ and Bishop and Jones, see [BishopJones1995] and [Stratmann2004], have shown that $\delta(\Gamma) = \dim_{\mathrm{H}}(L_r(\Gamma))$.
So for Fuchsian geometric Schottky groups either the metric $\widetilde{d}$ for which $h_{\widetilde{d}}(\Gamma) = h_S(\Gamma) = h_c(\Gamma)$ fails the condition $h_{\widetilde{d}}(\Gamma) = \delta(\Gamma)$, or we would obtain a very interesting fact that $\dim_{\mathrm{H}}(L_r(\Gamma)) = \dim_{\mathrm{H}}(L(\Gamma))$. One might be tempted to ask how various metrics on $T^1(\mathbb{D}/\Gamma)$ correspond to dimension properties of various subsets of the limit set $L(\Gamma)$.

4.1 Geodesic flow entropy

In this section we will introduce the notion of geodesic flow entopy of a group Γ which was considered by Sullivan in [Sullivan1984]. Its definition resembles the definition of topological entropy for non-compact spaces given by Bowen, see [Walters] and [Bowen1973]. But while Bowen's definition is derived from the metric of the underlying space, which in our case would be the unit tangent bundle $T^1(\mathbb{D}/\Gamma)$, we use instead the function which takes into account only the distance between the basepoints of vectors in $T^1(\mathbb{D}/\Gamma)$. It is not immediately clear if the two definitions are equivalent. We will clarify this question in Section 4.6. We will introduce geodesic flow entropy in several steps and in the process establish some basic properties.

We start by defining for each $T \geq 0$ the map:

$$d_T : T^1(\mathbb{D}/\Gamma) \times T^1(\mathbb{D}/\Gamma) \to \mathbb{R}$$
$$d_T(u, v) := \sup_{t \in [0,T]} \left\{ d\left(\pi_b(g^t u), \pi_b(g^t v)\right) \right\}$$

where $d(.,.)$ denotes the metric on the manifold $\mathbb{D}/\Gamma$, $\left\{g^t\right\}_{t\in\mathbb{R}}$ is the geodesic flow on $\mathbb{D}/\Gamma$, and $\pi_b : T^1(\mathbb{D}/\Gamma) \to \mathbb{D}/\Gamma$ is the projection onto the basepoint defined in Section 2.0.3. The geometrical meaning of this definition is ilustrated in Figure 4.1

Proposition 17. *For each $T > 0$ the map d_T is a metric on $T^1(\mathbb{D}/\Gamma)$.*

Proof. Fix $T > 0$. It is clear that the function d_T is well defined and real-valued. In fact, since $[0, T]$ is compact, the function:

$$t \mapsto d\left(\pi_b(g^t u), \pi_b(g^t v)\right)$$

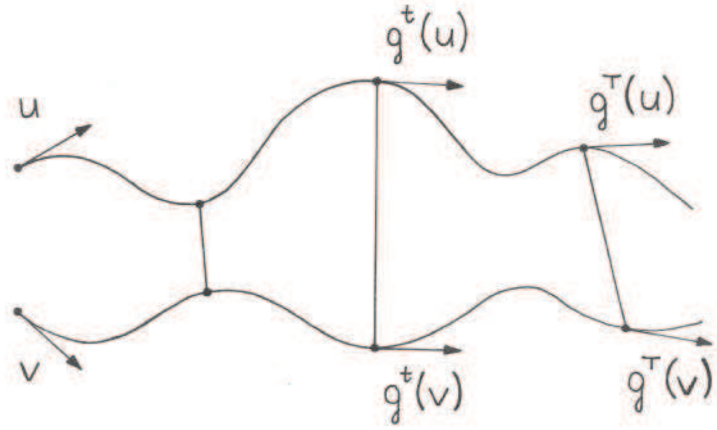


Figure 4.1:

is bounded and in the definition of $d_T(.,.)$ we could replace supremum with maximum. Since $d(.,.)$ is a metric it follows immediately that for all $u, v \in T^1(\mathbb{D}/\Gamma)$ we have:

$$d_T(u,v) = d_T(v,u)$$

and

$$d_T(u,u) = 0$$

If $d_T(u,v) = 0$ then it follows that the geodesics determined by u and v have a common geodesic segment, which can only hapen when the two geodesics are equal. Hence u and v are both tangent to the same geodesic at the point $\pi_b(u) = \pi_b(v)$ and therefore these two vectors must be equal.

Finally, for $u, v, q \in T^1(\mathbb{D}/\Gamma)$, we have:

$$\begin{aligned} d_T(u,v) &= \sup_{t\in[0,T]} \left\{ d\left(\pi_b(g^t u), \pi_b(g^t v)\right) \right\} \\ &\leq \sup_{t\in[0,T]} \left\{ d\left(\pi_b(g^t u), \pi_b(g^t q)\right) + d\left(\pi_b(g^t q), \pi_b(g^t v)\right) \right\} \\ &\leq \sup_{t\in[0,T]} \left\{ d\left(\pi_b(g^t u), \pi_b(g^t q)\right) \right\} + \sup_{t\in[0,T]} \left\{ d\left(\pi_b(g^t q), \pi_b(g^t v)\right) \right\} \\ &= d_T(u,q) + d_T(q,v) \end{aligned} \tag{4.1}$$

□

Definition 10. (T,ϵ)**-separated**
Let $T > 0$, $\epsilon > 0$ and let U be a subset of $T^1(\mathbb{D}/\Gamma)$. We say that U is (T,ϵ)-separated if for all $u, v \in U$ we have:

$$d_T(u,v) > \epsilon$$

Definition 11. sep(T, ϵ, V)
Let $T \geq 0$, $\epsilon > 0$ and let V be a subset of $T^1(\mathbb{D}/\Gamma)$. We define $\mathrm{sep}(T, \epsilon, V)$ to be the supremum over the cardinalities of all (T, ϵ)-separated subsets of V, that is:

$$\mathrm{sep}(T, \epsilon, V) := \sup \{\#U : U \subseteq V,\ \ U \text{ is } (T, \epsilon)\text{-separated}\}$$

In the above definition, $\#U$ denotes the cardinality of the set U. With no restrictions on the set V it is possible that $\mathrm{sep}(T, \epsilon, V)$ is not finite. We will soon provide a condition on V which will guarantee the finiteness of $\mathrm{sep}(T, \epsilon, V)$.

Definition 12. Base-bounded
Let V be a subset of $T^1(\mathbb{D}/\Gamma)$. We say that V is base-bounded if $\pi_b(V)$ is a bounded subset of $\mathbb{D}/\Gamma$.

Proposition 18. *Let V be a base-bounded subset of $T^1(\mathbb{D}/\Gamma)$, $T > 0$ and $\epsilon > 0$. Then $\mathrm{sep}(T, \epsilon, V)$ is finite.*

In the proof of the above proposition we will make use of a little lemma. We present this lemma separately from the rest of the proof since we will use it again later on.

Lemma 2. *Let $T, \epsilon > 0$. If vectors $\widetilde{u}$ and $\widetilde{v}$ in $T^1\mathbb{D}$ satisfy:*

$$d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) \leq \frac{\epsilon}{3}$$

and

$$d(\pi_b(g^{T_1}\widetilde{u}), \pi_b(g^{T_2}\widetilde{v})) \leq \frac{\epsilon}{3}$$

for some $T_1, T_2 \geq T$, then for all $t \in [0, T]$ we have:

$$d(\pi_b(g^t\widetilde{u}), \pi_b(g^t\widetilde{v})) \leq \epsilon$$

Proof. Let $\alpha_{\widetilde{u}}$ and $\alpha_{\widetilde{v}}$ denote the geodesics determined by $\widetilde{u}$ and $\widetilde{v}$ respectively. Since $\pi_b(\widetilde{u})$ and $\pi_b(g^{T_1}\widetilde{u})$ lie in the closed $\frac{\epsilon}{3}$-neighborhood of $\alpha_{\widetilde{v}}$ it follows that any point on $\alpha_{\widetilde{u}}$ between $\pi_b(\widetilde{u})$ and $\pi_b(g^{T_1}\widetilde{u})$ lies in the closed $\frac{\epsilon}{3}$-neighborhood of $\alpha_{\widetilde{v}}$. To see this it is enough to map $\alpha_{\widetilde{v}}$ to the imaginary axis in $\mathbb{H}$, or simply notice that the closed $\frac{\epsilon}{3}$-neighborhood of $\alpha_{\widetilde{v}}$ is convex. Analogous observation holds with the roles of $\widetilde{u}$ and $\widetilde{v}$ interchanged. Now, for $t \in [0, T]$, consider the point $\pi_b(g^t\widetilde{u})$ which lies in $[\pi_b(\widetilde{u}), \pi_b(g^{T_1}\widetilde{u})]$ and the point $\pi_b(g^t\widetilde{v})$ which lies in $[\pi_b(\widetilde{v}), \pi_b(g^{T_2}\widetilde{v})]$. Let $y_{\widetilde{v}}$ be the point on $\alpha_{\widetilde{v}}$ which lies closest to $\pi_b(g^t\widetilde{u})$. Then, since $\pi_b(g^t\widetilde{u})$ lies in the closed $\frac{\epsilon}{3}$-neighborhood of $\alpha_{\widetilde{v}}$, we have:

$$d(\pi_b(g^t\widetilde{u}), y_{\widetilde{v}}) \leq \frac{\epsilon}{3}$$

We also have:

$$d(\pi_b(\widetilde{v}), y_{\widetilde{v}}) \ \leq \ d(\pi_b(\widetilde{v}), \pi_b(\widetilde{u})) + d(\pi_b(\widetilde{u}), \pi_b(g^t\widetilde{u})) + d(\pi_b(g^t\widetilde{u}), y_{\widetilde{v}})$$

$$\leq \frac{\epsilon}{3} + t + \frac{\epsilon}{3} = t + \frac{2\epsilon}{3}$$

and:

$$t = d(\pi_b(\widetilde{u}), \pi_b(g^t\widetilde{u})) \leq d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) + d(\pi_b(\widetilde{v}), y_{\widetilde{v}}) + d(y_{\widetilde{v}}, \pi_b(g^t\widetilde{u}))$$
$$\leq \frac{\epsilon}{3} + d(\pi_b(\widetilde{v}), y_{\widetilde{v}}) + \frac{\epsilon}{3}$$

which by rearranging gives:

$$d(\pi_b(\widetilde{v}), y_{\widetilde{v}}) \geq t - \frac{2\epsilon}{3}$$

Thus:

$$d(y_{\widetilde{v}}, \pi_b(g^t\widetilde{v})) \leq \frac{2\epsilon}{3}$$

Therefore we have:

$$d(\pi_b(g^t\widetilde{u}), \pi_b(g^t\widetilde{v})) \leq d(\pi_b(g^t\widetilde{u}), y_{\widetilde{v}}) + d(y_{\widetilde{v}}, \pi_b(g^t\widetilde{v}))$$
$$\leq \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon$$

□

Proof. Proposition 18
Choose $r > 0$ such that r is less than both $\frac{\epsilon}{6}$ and the injectivity radius of $\mathbb{D}/\Gamma$. Since V is base-bounded there exists a finite collection

$$B_1, \ldots, B_m \subseteq \mathbb{D}/\Gamma$$

of closed balls of radius r such that

$$\pi_b(V)) \subseteq \bigcup_i B_i$$

For each $i \in \{1, \ldots, m\}$ let $\widetilde{B}_i$ be a lift of B_i with respect to the canonical map $\pi : \mathbb{D} \mapsto \mathbb{D}/\Gamma$ chosen in such a way that $\widetilde{B}_i$ is a closed ball in $\mathbb{D}$ whose center lies in $\overline{F}$, the closure of the fundamental domain of Γ. Let x_i denote the center of $\widetilde{B}_i$ and $C_i \in \mathbb{D}$ the circle with center at x_i and of radius $T + r$.

We can clearly partition each C_i into finitely many closed intervals

$$I_{i,1} \ , \ \ldots \ , \ I_{i,n(i)}$$

such that for each $j = 1, \ldots, n(i)$ we have:

$$diam(I_{i,j}) < \frac{\epsilon}{3}$$

Now suppose that $\widetilde{u}$ and $\widetilde{v}$ are two vectors in $T^1(\mathbb{D})$ such that for some $i \in \{1, \ldots, m\}$ both $\pi_b(\widetilde{u})$ and $\pi_b(\widetilde{v})$ lie in $\widetilde{B}_i$. Suppose also that the geodesic rays determined by $\widetilde{u}$ and

$\widetilde{v}$ for some $j \in \{1, \ldots, n(i)\}$ both intersect C_i inside the interval $I_{i,j}$. Let $x_{\widetilde{u}}$ denote the intersection of C_i with the geodesic ray determined by $\widetilde{u}$ and similarly let $x_{\widetilde{v}}$ denote the intersection of C_i with the geodesic ray determined by $\widetilde{v}$.
We then have that:

$$d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) \leq \frac{\epsilon}{3}$$

and

$$d(x_{\widetilde{u}}, x_{\widetilde{v}}) \leq \frac{\epsilon}{3}$$

Moreover, the geodesic segments $[\pi_b(\widetilde{u}), x_{\widetilde{u}}]$ and $[\pi_b(\widetilde{v}), x_{\widetilde{v}}]$ are both of length at least T. Thus, from Lemma 2 it follows that or all $t \in [0, T]$:

$$d(\pi_b(g^t(\widetilde{u})), \pi_b(g^t(\widetilde{u}))) \leq \epsilon$$

This in turn implies that if u and v denote the canonical projection to $T^1(\mathbb{D}/\Gamma)$ of $\widetilde{u}$ and $\widetilde{v}$ respectively then for all $t \in [0, T]$ we have:

$$d(\pi_b(g^t(u)), \pi_b(g^t(u))) \leq \epsilon$$

so that

$$d_T(u, v) \leq \epsilon$$

Now let U be a (T, ϵ)-separated subset of V. For each $u \in U$ choose a lift $\widetilde{u}$ such that $\pi_b(\widetilde{u})$ lies in $\widetilde{B}_i$ for some $i \in \{1, \ldots, m\}$. Our argument has shown that for each pair (i, j) where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n(i)\}$ there can be at most one $u \in U$ for which $\pi_b(\widetilde{u}) \in \widetilde{B}_i$ and the geodesic ray determined by $\widetilde{u}$ intersects C_i inside $I_{i,j}$. Therefore:

$$\#U \leq \sum_{i=1}^{m} n(i)$$

Since U was chosen arbitrarily this shows that:

$$\text{sep}(T, \epsilon, V) \leq \sum_{i=1}^{m} n(i) < \infty$$

$\square$

Definition 13. Geodesic flow entropy with respect to a set
Let V be a base-bounded subset of $T^1(\mathbb{D}/\Gamma)$. We define $h_S(V)$, the geodesic flow entropy with respect to the set V by:

$$h_S(V) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\text{sep}(T, \epsilon, V))}{T}$$

Definition 14. Geodesic flow entropy
We define $h_S(\Gamma)$, the geodesic flow entropy of Γ, by:

$$h_S(\Gamma) := \sup_{V \in \mathcal{V}_1} h_S(V)$$

where:

$$\mathcal{V}_1 := \{V \subseteq T^1(\mathbb{D}/\Gamma) : \ V \subseteq ((L(\Gamma) \times L(\Gamma) - diag) \times \mathbb{R})/\Gamma, \ V \text{ base-bounded}\}$$

We will now prove a proposition which will allow us to give a different definition of $h_S(\Gamma)$. In this new definition the supremum will be taken over a subset $\mathcal{V}_2$ of $\mathcal{V}_1$ which is easier to handle than $\mathcal{V}_1$ itself, yet the two definitions of $h_S(\Gamma)$ will be shown to be equivalent. The subset $\mathcal{V}_2$ which we use in this new definition is given by:

$$\mathcal{V}_2 := \{V \subseteq T^1(\mathbb{D}/\Gamma) : \ V = \pi_b^{-1}(B) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma, \ B \text{ closed ball in } \mathbb{D}/\Gamma\}$$

Proposition 19. *Let V be a base-bounded subset of $T^1(\mathbb{D}/\Gamma)$. Suppose that there are base-bounded sets $K_1, K_2, \dots K_m \subseteq T^1(\mathbb{D}/\Gamma)$ such that:*

$$V \subseteq K_1 \cup \cdots \cup K_m$$

then:

$$h_S(V) \leq \max_i h_S(K_i)$$

Proof. Let $T > 0$ and $\epsilon > 0$. Since V is base-bounded $\text{sep}(T, \epsilon, V)$ is finite. Let U be a (T, ϵ)-separated subset of V with $\#U = \text{sep}(T, \epsilon, V)$. Then for each $i = 1, \cdots, m$ the set $U \cap K_i$ is a (T, ϵ)-separated subset of K_i and therefore:

$$\begin{aligned} \text{sep}(T, \epsilon, V) = \#U &\leq \#(U \cap K_1) + \cdots + \#(U \cap K_m) \\ &\leq \text{sep}(T, \epsilon, K_1) + \cdots + \text{sep}(T, \epsilon, K_m) \end{aligned} \tag{4.2}$$

This implies that:

$$\begin{aligned} h_S(V) :&= \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\text{sep}(T, \epsilon, V))}{T} \\ &\leq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\text{sep}(T, \epsilon, K_1) + \cdots + \text{sep}(T, \epsilon, K_m))}{T} \end{aligned} \tag{4.3}$$

Let us keep ϵ fixed for the moment. Let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence such that $T_k \nearrow \infty$ and:

$$\limsup_{T \to \infty} \frac{\log(\text{sep}(T, \epsilon, K_1) + \cdots + \text{sep}(T, \epsilon, K_m))}{T} = \lim_{k \to \infty} \frac{\log(\text{sep}(T_k, \epsilon, K_1) + \cdots + \text{sep}(T_k, \epsilon, K_m))}{T_k}$$

For each k let $j(k)$ be such that:

$$\text{sep}(T_k, \epsilon, K_{j(k)}) = \max_i \{\text{sep}(T_k, \epsilon, K_i)\}$$

In the sequence $\{j(k)\}_{k\in\mathbb{N}}$ some $q \in \{1, \dots, m\}$ must appear infinitely often. So we choose one such q and extract the subsequence consisting of those T_k where $j(k) = q$. To keep the notation simple, we will still denote this subsequence by $\{T_k\}_{k\in\mathbb{N}}$. For this new sequence we have:

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon, K_1) + \cdots + \text{sep}(T, \epsilon, K_m))}{T} \\
&= \lim_{k\to\infty} \frac{\log(\text{sep}(T_k, \epsilon, K_1) + \cdots + \text{sep}(T_k, \epsilon, K_m))}{T_k} \\
&\le \lim_{k\to\infty} \frac{\log(m \cdot \text{sep}(T_k, \epsilon, K_q))}{T_k} \\
&= \lim_{k\to\infty} \frac{\log m + \log(\text{sep}(T_k, \epsilon, K_q))}{T_k} \\
&= \lim_{k\to\infty} \frac{\log(\text{sep}(T_k, \epsilon, K_q))}{T_k} \\
&\le \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon, K_q))}{T}
\end{aligned} \tag{4.4}$$

The j which we obtained in the above argument depends on ϵ so we will denote it by $j(\epsilon)$. We have:

$$h_S(V) \le \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon, K_{j(\epsilon)}))}{T}$$

Now we choose some sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ such that $\epsilon_k \searrow 0$ and :

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon, K_{j(\epsilon)}))}{T} = \lim_{k\to\infty} \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon_k, K_{j(\epsilon_k)}))}{T}$$

In the sequence $\{j(\epsilon_k)\}_{k\in\mathbb{N}}$ some of the $j \in \{1, \dots, m\}$ must appear infinitely often. So we choose one such j and extract the subsequence consisting of those ϵ_k where $j(\epsilon_k) = j$. Again, to keep the notation simple, we will still denote this subsequence by $\{\epsilon_k\}_{k\in\mathbb{N}}$. For this new sequence we have:

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon, K_{j(\epsilon)}))}{T} \le \lim_{k\to\infty} \limsup_{T\to\infty} \frac{\log(\text{sep}(T, \epsilon_k, K_j))}{T} = h(K_j)$$

(Since we know that the limit in the definition of $h(K_j)$ exists, me might as well consider only a subsequence!) Combining all of the above estimates then gives:

$$h_S(V) \le h(K_j) \le \max_i h_S(K_i)$$

$\square$

Corollary 5. *For $h'_S(\Gamma)$ defined by:*

$$h'_S(\Gamma) := \sup_{V \in \mathcal{V}_2} h_S(V)$$

we have:

$$h_S(\Gamma) = h'_S(\Gamma)$$

Proof. Since $\mathcal{V}_2 \subseteq \mathcal{V}_1$ it is clear that:

$$h'_S(\Gamma) \leq h_S(\Gamma)$$

Let $V \in \mathcal{V}_1$. Since by definition $\pi_b(V)$ is a bounded subset of $\mathbb{D}/\Gamma$ we can cover it with finitely many closed balls in $\mathbb{D}/\Gamma$, say $B_1, \ldots, B_m$. Put

$$K_i := \pi_b^{-1}(B_i) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$$

Then we have:

$$V \subset K_1 \cup \cdots \cup K_m$$

So by Proposition 19 we have:

$$h_S(V) \leq \max_i h_S(K_i) \leq h'_S(\Gamma)$$

Since V was chosen arbitrarily it follows that:

$$h_S(\Gamma) \leq h'_S(\Gamma)$$

which finishes the proof. $\square$

Remark 5. Note that we could also require that the balls B in the definition of $\mathcal{V}_2$ are of diameter less than some constant $\rho > 0$ and the above argument would also work yielding other equivalent definitions of the geodesic flow entropy $h_S(\Gamma)$.

4.2 Convex core entropy

Convex core entropy is a very natural notion, which resembles the notion of volume entropy. For a Kleinian group Γ it approximates the exponential growth of the volume of the convex hull of the limit set $L(\Gamma)$ of Γ. This also explains its name, since by projecting the convex hull of $L(\Gamma)$ to the manifold $\mathbb{D}/\Gamma$ we obtain the convex core of $\mathbb{D}/\Gamma$. In fact it turns out that the convex core entropy of Γ is simply equal to the upper box-counting dimension of the limit set $L(\Gamma)$. This fact does not make the concept of convex core entropy redundant but rather more interesting since this reinterpretation of $\overline{\dim}_{\mathrm{B}}(L(\Gamma))$ allows us to work in the hyperbolic plane rather than on its boundary.

Convex core entropy has been given its name and studied by Falk and Matsuzaki in [FalkMatsuzaki]. Due to the naturality of this concept, it is likely that it has been considered before, but we were unable to locate other references. The definition of convex core entropy, which we give below, is the same as the one that appears in [FalkMatsuzaki] but here we restrict it to the context of Fuchsian groups.

For a Fuchsian group Γ let $Hull(L(\Gamma))$ denote the convex hull of the limit set $L(\Gamma)$. The convex hull of $L(\Gamma)$ is defined as the smallest convex subset of $\mathbb{D}$ containing the union of all the geodesics, seen as subsets of $\mathbb{D}$, whose both endpoints at infinity belong to $L(\Gamma)$. For any $\rho > 0$ let $Hull_\rho(L(\Gamma))$ denote the open hyperbolic ρ-neighborhood of the convex hull $Hull(L(\Gamma))$. Then we define the convex core entropy of Γ, denoted by $h_c(\Gamma)$, as:

$$h_c(\Gamma) := \limsup_{R\to\infty} \frac{\log \mathrm{vol}(\ Hull_\rho(L(\Gamma)) \cap B(z,R)\ )}{R}$$

where $\mathrm{vol}(\cdot)$ denotes the hyperbolic area and $B(z,r)$ is the open ball in $\mathbb{D}$ with center z and radius R. We will usually write h_c instead of $h_c(\Gamma)$ when it is clear which group Γ we are refering to. It can be shown that this definition does not depend the choice of the constant ρ and the point z.

There are well known relations between convex core entropy and other invariants associated to a Fuchsian group Γ. Namely, we have:

$$\delta(\Gamma) = \dim_{\mathrm{H}}(L_r(\Gamma)) \leq \dim_{\mathrm{H}}(L(\Gamma)) \leq h_c(\Gamma) = \overline{\dim}_{\mathrm{B}}(L(\Gamma))$$

The first equality is a consequence, described by Stratmann in [Stratmann2004], of the well known result of Bishop and Jones [BishopJones1995]. The first inequality follows from the monotonicity property of the Hausdorff dimension, while the second one is a standard result from dimension theory. The equality between convex core entropy and upper box-counting dimension of the limit set has been shown in the article of Falk and Matsuzaki [FalkMatsuzaki].

4.3 For finitely generated geometric Schottky groups, geodesic flow entropy is equal to the Poincaré exponent

In this section we will show that for a geometric Schottky group Γ which is finitely generated the geodesic flow entropy $h_S(\Gamma)$ and the Poincaré exponent $\delta(\Gamma)$ coincide. Namely, we will prove the following theorem:

Theorem 5. *Let Γ be a finitely generated geometric Schottky group. Then the geodesic flow entropy of the group Γ is equal to the Poincaré exponent of the group.*

4.3.1 For finitely generated geometric Schottky groups, geodesic flow entropy is bounded from above by the Poincaré exponent

In this section we will show that for a geometric Schottky group Γ which is finitely generated the geodesic flow entropy $h_S(\Gamma)$ is bounded above by the Poincaré exponent $\delta(\Gamma)$. So, we will prove the following proposition:

Proposition 20. *Let Γ be a finitely generated geometric Schottky group. Then the geodesic flow entropy $h_S(\Gamma)$ is bounded from above by the Poincaré exponent $\delta(\Gamma)$.*

Outline of the proof

When arguing that $h_S(\Gamma) \leq \delta(\Gamma)$, we will use the definition of $h_S(\Gamma)$ in terms of the collection $\mathcal{V}_2$. For an arbitrary fixed $V \in \mathcal{V}_2$, we will then show that $h_S(V) \leq \delta(\Gamma)$. To show that $h_S(V) \leq \delta(\Gamma)$, we will find an upper bound for $\text{sep}(T, \epsilon, V)$ of the form:

$$(c_1 \cdot T + c_2) \cdot e^{\ \delta(\Gamma) \cdot (T + c_3)}$$

where c_1, c_2, c_3 are constants independent of T. The above bound is obtained by considerations in the universal cover $\mathbb{D}$. By the definition of $\mathcal{V}_2$ the set $\pi_b(V)$ will be a subset of a closed ball $B \subseteq (\mathbb{D}/\Gamma)$ which by Remark 5 can be assumed to have radius much smaller than the injectivity radius of the manifold $\mathbb{D}/\Gamma$. We lift the ball B to a ball $\widetilde{B}$ in $\mathbb{D}$ and consider circles based at its center whose radii are up to an additive constant equal to T. Once we notice that the lifts of geodesics corresponding to vectors in V must lie inside the convex hull $Hull(L(\Gamma))$ it then seems natural to consider the intersections of these circles with $Hull(L(\Gamma))$. We cover these intersections with collections of 'pearl necklaces' consisting of small balls - 'pearls' of radius equal to a constant multiple of ϵ. The term 'pearl-necklace' is used to draw attention to the special form of these collections in which the balls are allowed to intersect each other in at most one point. We then show that the size of a (T, ϵ)-separated subset of V can be bounded by a constant times the number of 'pearls' in the collection of 'pearl necklaces' corresponding to T. In the course of our argument it will also become clear that the number of distinct 'pearls' which cover a single point is not greater than a constant independent of T; which means that there is no harm if for each T the reader thinks not of a collection of 'pearl-necklaces', but of a single 'pearl-necklace'. The rest of the proof is then devoted to estimating the number of 'pearls' in each collection of 'pearl necklaces'. The crucial step there is an application of Sullivan's Shadow Lemma, which is precisely where the Poincaré exponent $\delta(\Gamma)$ enters our argument.

Proof set up

Throughout let Γ be a geometric Schottky group and $(\{A_k\}, \{g_k\})_{k \in I}$ its Schottky description, where the indexing set I is finite. Let us start by fixing a ball B in $\mathbb{D}/\Gamma$ which is equivalent to fixing a set V in $\mathcal{V}_2$. As already mentioned in Remark 5 it can be assumed that B has radius much smaller than the injectivity radius of the manifold $\mathbb{D}/\Gamma$. Let $\widetilde{B}$ be a lift of B with respect to the map $\pi : \mathbb{D} \mapsto \mathbb{D}/\Gamma$. We choose the lift $\widetilde{B}$ so that it is a ball in $\mathbb{D}$ whose center lies in the closure of the canonical fundamental domain $\overline{F}$ of Γ. We denote the center of $\widetilde{B}$ by x_0 and its radius by r_0. We also fix a real number $T > 0$ and a real number $\epsilon > 0$.

Consider the set $Hull(L(\Gamma)) \cap \overline{F}$ i.e. the intersection of the convex hull of the limit set of Γ with the closure of the canonical fundamental domain of Γ. Since the convex core of $\mathbb{D}/\Gamma$ is defined to be the projection $Hull(L(\Gamma))/\Gamma$, this set is just the closure of a lift of the convex core.

Observation 3. By definition a geodesic α whose endpoints at infinity ξ_- and ξ_+ lie in the limit set $L(\Gamma)$ satisfies:

$$\alpha \subseteq \bigcup_{g\in\Gamma} g\ (Hull(L(\Gamma)) \cap \overline{F}) = Hull(L(\Gamma))$$

Let c_0 be a radius large enough to allow us to use Sullivan's Shadow Lemma for the group Γ. Choose $d_0 > 0$ to be large enough so that the open ball $B(o, d_0) \subset \mathbb{D}$ contains the set $Hull(L(\Gamma)) \cap \overline{F}$ and so that $d_0 \geq c_0$. Note that the constant d_0 depends only on the group Γ.

We denote by $\mathcal{D}$ the collection of all the images of $B(o, d_0)$ under the action of the group Γ; formally:

$$\mathcal{D} := \{gB(o, d_0) : g \in \Gamma\} = \{B(go, d_0) : g \in \Gamma\}$$

Remark 6. In view of Observation 3 the requirement that $B(o, d_0) \subset \mathbb{D}$ contains the set $Hull(L(\Gamma)) \cap \overline{F}$ guarantees that a geodesic α whose endpoints at infinity ξ_- and ξ_+ lie in the limit set $L(\Gamma)$ will satisfy:

$$\alpha \subseteq \bigcup_{D\in\mathcal{D}} D$$

The existance of such d_0 is precisely the point where we use the assumption that Γ is finitely generated.

We define the circle C_T with center at x_0 and of radius $T + r_0$, where x_0 and r_0 are respectively the center and the radius of $\widetilde{B}$, so formally:

$$C_T := \{y \in \mathbb{D} : d(x_0, y) = T + r_0\}$$

Then we denote by $\mathcal{D}_T$ the collection of those balls in $\mathcal{D}$ which intersect C_T; formally:

$$\mathcal{D}_T := \{D \in \mathcal{D} : D \cap C_T \neq \emptyset\}$$

Further to each $D \in \mathcal{D}_T$ we associate a 'pearl-necklace' $\mathcal{P}(D)$. We choose $\mathcal{P}(D)$ so that the following conditions are satisfied:

(1) $\mathcal{P}(D) = \{e_i\}_{i\in J}$ is a collection of 'pearls' - closed balls in $\mathbb{D}$ of radius $\frac{\epsilon}{6}$

(2) $D \cap C_T \subseteq \bigcup_{i\in J} e_i$

(3) For $i \neq j$, $i, j \in J$, the balls e_i and e_j intersect in at most one point

(4) For each $i \in J$ the ball e_i intersects $D \cap C_T$

Remark 7. Since $D \cap C_T$ is a segment of a circle, requiring that $T + r_0 >> d_0$ and $T + r_0 >> \epsilon$ guarantees that it is possible to choose a collection $\mathcal{P}(D)$ which satisfies the requirements (1)-(4) above. In particular, it suffices if $T > 2d_0$ and $T > 2\epsilon$. This restriction will not cause any problems in our final argument.

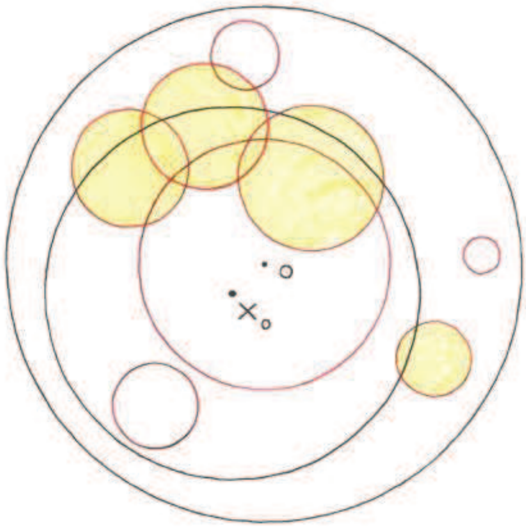


Figure 4.2:

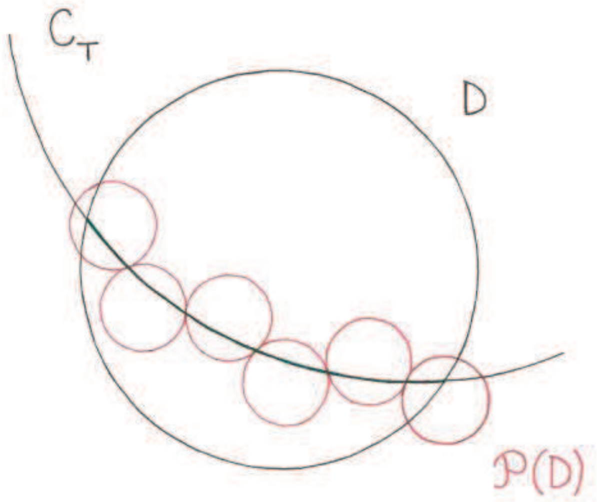


Figure 4.3:

Passing through a common element of $\mathcal{P}(D)$

Let U be a subset of $V = \pi_b^{-1}(B) \cap (((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma)$. Let us also assume that U is (T, ϵ)-separated. Let $\widetilde{U}$ be the unique lift of U with respect to the canonical map $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$ which satisfies $\pi_b(\widetilde{U}) \subseteq \widetilde{B}$, where the $\widetilde{B}$ is the lift of B chosen at the beginning of this section. We want to estimate the number of vectors in U such that the rays determined by their lifts to $\widetilde{U}$ intersect a common element $e \in \mathcal{P}(D)$.

Let $\widetilde{u}$ and $\widetilde{v}$ be two distinct vectors in $\widetilde{U}$. Suppose that the two geodesic rays determined by $\widetilde{u}$ and $\widetilde{v}$ both intersect C_T inside the same 'pearl' $e \in \mathcal{P}(D)$ for some $D \in \mathcal{D}_T$. We claim that then:

$$d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) > \frac{\epsilon}{3}$$

In order to reach a contradiction, suppose we have:

$$d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) \leq \frac{\epsilon}{3}$$

Let $\alpha_{\widetilde{u}}$ and $\alpha_{\widetilde{v}}$ denote the geodesics determined by $\widetilde{u}$ and $\widetilde{v}$ respectively. Let us put $x_{\widetilde{u}} := \alpha_{\widetilde{u}} \cap C_T \cap e$ and $x_{\widetilde{v}} := \alpha_{\widetilde{v}} \cap C_T \cap e$. Observe that the point $\pi_b(\widetilde{u})$ lies in the closed $\frac{\epsilon}{3}$-neighborhood of the geodesic $\alpha_{\widetilde{v}}$ and because of how we chose $\mathcal{P}(D)$ the point $x_{\widetilde{u}}$ also lies in the closed $\frac{\epsilon}{3}$-neighborhood of the geodesic $\alpha_{\widetilde{v}}$. As explained in the proof of Lemma 2 it follows that all the points on $\alpha_{\widetilde{u}}$ which lie between $\pi_b(\widetilde{u})$ and $x_{\widetilde{u}}$ must also lie in the closed $\frac{\epsilon}{3}$-neighborhood of $\alpha_{\widetilde{v}}$. Analogous observations hold with the roles of $\widetilde{u}$ and $\widetilde{v}$ interchanged. Note also that both of the geodesic segments $[\pi_b(\widetilde{u}), x_{\widetilde{u}}]$ and $[\pi_b(\widetilde{v}), x_{\widetilde{v}}]$ are of length at least T. To see this just consider the distances $d(x_0, x_{\widetilde{u}}) = d(x_0, x_{\widetilde{v}}) = T + r_0$. If the segments were of length strictly less than T then using the triangle inequality with $\pi_b(\widetilde{u})$ and $\pi_b(\widetilde{v})$ respectively we would obtain a contradiction. So from Lemma 2 it follows that for $t \in [0, T]$ we have:

$$d(\pi_b(g^t\widetilde{u}), \pi_b(g^t\widetilde{v})) \ \leq \ \epsilon$$

This in turn implies that:

$$d(\pi_b(g^t u), \pi_b(g^t v)) \ \leq \ \epsilon$$

where u and v are the vectors in U corresponding to u and v respectively. But since this holds for any $t \in [0, T]$ we then have:

$$d_T(u, v) \leq \epsilon$$

which contradicts the fact that U is (T, ϵ)-separated.

We also make the following observation. Suppose Y is a subset of $\widetilde{B}$ such that for all $y_1, y_2 \in Y$, $y_1 \neq y_2$ we have $d(y_1, y_2) > \epsilon$. If we place a ball of radius $\frac{\epsilon}{3}$ at each point of Y these balls will be disjoint. So if we consider the area of $\widetilde{B}$ and the areas of these balls we see there cannot be more than:

$$k_1 := \frac{2\pi(\cosh r_0 - 1)}{2\pi(\cosh \frac{\epsilon}{3} - 1)}$$

of these balls, and so $\#Y \leq k_1$. Note that the constant k_1 depends only on ϵ and the radius of the ball $\widetilde{B}$, and so is independent of T.

Now let U_0 be a subset of U and $\widetilde{U}_0 \subseteq \widetilde{U}$ its lift. Suppose that the elements of $\widetilde{U}_0$ determine rays which intersect C_T in a single 'pearl' $e \in \mathcal{P}(D)$ for some $D \in \mathcal{D}_T$. Our discussion in this section has shown that then for all $y_1, y_2 \in \pi_b(\widetilde{U_0})$, $y_1 \neq y_2$ we have $d(y_1, y_2) > \frac{\epsilon}{3}$ and that for distinct $u, v \in \widetilde{U_0}$ we have $\pi_b(u) \neq \pi_b(v)$. So it follows that:

$$\#U_0 = \#\widetilde{U_0} = \#(\pi_b(\widetilde{U_0})) \leq k_1$$

Estimating the size of $\mathcal{P}(D)$

Let $D \in \mathcal{D}_T$ be arbitrary. Now we ask about the maximum size of a 'pearl-necklace' $\mathcal{P}(D)$. By simple comparison of areas we obtain an upper bound which is rather rough but sufficient for our purposes. Observe that the 'pearl-necklace' $\mathcal{P}(D)$ lies in the closed ϵ-neighborhood of D. The area of this neighborhood is equal to $2\pi(\cosh(d_0+\epsilon)-1)$. The 'pearls' $\mathcal{P}(D)$ are closed hyperbolic balls of radius $\frac{\epsilon}{6}$ and thus each has hyperbolic area $2\pi(\cosh(\frac{\epsilon}{6}) - 1)$. So, since any two 'pearls' in $\mathcal{P}(D)$ have at most one point in common there can be at most:

$$k_2 := \frac{2\pi(\cosh(d_0 + \epsilon) - 1)}{2\pi(\cosh(\frac{\epsilon}{6}) - 1)}$$

'pearls' in $\mathcal{P}(D)$. Note that the constant k_2 depends only on ϵ and d_0, and so is independent of T.

Estimating the size of $\mathcal{D}_T$

To estimate the size of $\mathcal{D}_T$ let us start by considering the shadows of the balls in $\mathcal{D}_T$. Namely, for each $D \in \mathcal{D}_T$ let $\Pi_o(D)$ be the shadow from the origin of the ball D, as defined in Section 2.0.9, so formally:

$$\Pi_o(D) := \left\{\xi \in S^1 : r_\xi \cap D \neq \emptyset\right\}$$

where r_ξ denotes the hyperbolic ray between the origin o and the point ξ. To obtain the estimate we will use the following lemma:

Lemma 3. *Let X be a topological space and m a finite Borel measure on X. Let $p > 0$ and $k \in \mathbb{N}$. Suppose $Y_1, \cdots, Y_N$ are Borel subsets of X such that for each $i = 1, \ldots, N$ we have $\mu(Y_i) \geq p$ and that each $x \in X$ belongs to at most k of the subsets $Y_1, \ldots, Y_N$. Suppose further that for each $j = 0, 1, \ldots, N$ the set*

$$A^j := \{x \in X : x \text{ belongs to exactly } j \text{ of the sets } Y_1, ..., Y_N\}$$

is Borel. Then we have:

$$N \leq \frac{m(X)}{p} \cdot k$$

Proof. The proof has been included in the Appendix. □

We want to apply Lemma 3 by taking $X = S^1$, the shadows $\Pi_o(D)$, for $D \in \mathcal{D}_T$, as sets Y_i and $m = \mu_o$, where μ_o is the Patterson measure as defined in Section 2.0.8.

Sullivan's Shadow Lemma gives us the following estimate for the Patterson measure of the shadow from the origin of a ball $D \in \mathcal{D}_T$:

$$\frac{1}{c}e^{-\delta(\Gamma)d(o,x_D)} \leq \mu_o(\Pi_o(D)) \leq ce^{-\delta(\Gamma)d(o,x_D)}$$

where x_D is the center of D and $c > 0$ is a constant which depends only on the radius d_0. From the definition of $\mathcal{D}_T$ it very easily follows that, for $D \in \mathcal{D}_T$, we have:

$$d(o, x_0) - (T + r_0) - d_0 \leq d(o, x_D) \leq d(o, x_0) + (T + r_0) + d_0$$

Thus we have:

$$\mu_o(\Pi_o(D)) \geq \frac{1}{c}e^{-\delta(\Gamma)(T+2r_0+d_0)}$$

Now given a point $\xi_0 \in S^1$ we ask for how many $D \in \mathcal{D}_T$ we can have $\xi_0 \in \Pi_o(D)$. We show that this number can be bounded by a constant which depends on B, where B is the ball corresponding to the set V which we fixed in the proof set up, and T but is independent of the particular choice of ξ_0.

So let ξ_0 be some point in S^1. Suppose that ξ_0 lies in the shadow $\Pi_o(D)$ for some $D \in \mathcal{D}_T$. From the definition of $\mathcal{D}_T$ we know that D is an open ball of radius d_0 which intersects C_T, which implies that D must lie inside the circle with center at x_0 and of radius $T + r_0 + 2d_0$, which we denote here by C_{T+2d_0}. But since $\xi_0 \in \Pi_o(D)$ the ball D must also intersect the ray r_{ξ_0}. The intersection $D \cap r_{\xi_0}$ then also lies inside the circle C_{T+2d_0}. Since r_{ξ_0} is a geodesic ray, the length of the part of r_{ξ_0} which lies inside C_{T+2d_0} is at most the diameter of C_{T+2d_0}. Let r'_{ξ_0} denote the part of r_{ξ_0} which lies inside C_{T+2d_0}. Therefore, since

$$diam(C_{T+2d_0}) \leq 2(T + r_0 + 2d_0)$$

also the length of r'_{ξ_0} is at most $2(T + r_0 + 2d_0)$.

Let us partition r'_{ξ_0} into intervals. We choose them in such a way that all except possibly one are of length one. It is clear that the number of intervals in this partition is less than or equal $2(T + r_0 + 2d_0) + 1$. If some interval in the partition is intersected by two balls $D_1, D_2 \in \mathcal{D}$ then by the triangular inequality the distance between the centers of D_1 and D_2 is at most $2d_0 + 1$. Now suppose $\{D_i\}_{i\in K}$ is a collection of balls in $\mathcal{D}$ such that for all $i, j \in K$ the centers of D_i and D_j are at most $2d_0 + 1$ apart. We apply an element $\gamma \in \Gamma$ such that $\gamma(D_j) = B(o, d_0)$ for some $j \in I$. Since γ is an isometry it then becomes clear that $\#K \leq \# \{\gamma \in \Gamma : \gamma o \leq 2d_0 + 1\}$. Put:

$$k_3 := \# \{\gamma \in \Gamma : \gamma o \leq 2d_0 + 1\}$$

This is a constant that depends only on Γ and the particular choice of d_0 and is independent of T. Thus at most k_3 distinct balls D can intersect the same interval of the partition. Also we have seen that if $\xi_0 \in \Pi_o(D)$ for some $D \in \mathcal{D}_T$ then D must intersect at least one of the intervals in the partition considered above. Therefore there are at most

$$k_3 \cdot (2(T + r_0 + 2d_0) + 1)$$

such balls D.

Now, if in Lemma 3 we take $X = S^1$, $m = \mu_o$ and the shadows $\Pi_o(D)$, for $D \in \mathcal{D}_T$, as our sets Y_i, we deduce that:

$$\#\mathcal{D}_T \leq \frac{k_3 \cdot (2(T + r_0 + 2d_0) + 1)}{\frac{1}{c} e^{-\delta(\Gamma)(T+2r_0+d_0)}}$$

Notice that here the sets A^j from Lemma 3 will be unions of intersections of finitely many shadows $\Pi_o(D)$, hence Borel, so we may indeed apply Lemma 3 in this setting.

Upper bound for $\mathrm{sep}(T, \epsilon, V)$

Now consider again the (T, ϵ)-separated subset U of:

$$V = \pi_b^{-1}(B) \cap (((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma)$$

and its lift $\widetilde{U}$ as described earlier in our discussion. Further, consider the geodesic rays determined by the vectors in $\widetilde{U}$. Denote the set of these rays by $\mathcal{Y}_U$ and note that the elements of $\mathcal{Y}_U$ are in one to one correspondence with the elements of U. From the definition of C_T, it is clear that all the rays in $\mathcal{Y}_U$ have to cross C_T. In view of Observation 3, it is also clear that these rays lie in the union $\bigcup_{D \in \mathcal{D}} D$. In particular, since the intersection

$$C_T \cap \bigcup_{D \in \mathcal{D}} D$$

is contained in

$$\bigcup_{D \in \mathcal{D}_T} \mathcal{P}(D)$$

each ray in $\mathcal{Y}_U$ has to intersect C_T in some 'pearl' $e \in \mathcal{P}(D)$ for some $D \in \mathcal{D}_T$.

Final step

So, now we can combine our earlier estimates to get an upper bound for the size of the set U. We know that at most k_1 rays in $\mathcal{Y}_U$ can intersect C_T inside the same 'pearl' $e \in \bigcup_{D \in \mathcal{D}_T} \mathcal{P}(D)$. For each $D \in \mathcal{D}_T$ we have:

$$\#\mathcal{P}(D) \leq k_2$$

and we also know that:

$$\#\mathcal{D}_T \leq k_3 \cdot (2(T + r_0 + 2d_0) + 1) \cdot c \cdot e^{\delta(\Gamma)(T+2r_0+d_0)}$$

Therefore we conclude that:

$$\#U = \#\mathcal{Y}_U \leq k_1 \cdot k_2 \cdot k_3 \cdot (2(T + r_0 + 2d_0) + 1) \cdot c \cdot e^{\delta(\Gamma)(T+2r_0+d_0)}$$

Since U was an arbitrarily chosen (T, ϵ)-separated subset of V, the expression on the right hand side of the above inequality is also an upper bound for $\operatorname{sep}(T, \epsilon, V)$, and so:

$$h_S(V) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\operatorname{sep}(T, \epsilon, V))}{T}$$

$$\leq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(k_3 k_2 k_1) + \log 2 + \log(T + r_0 + 2d_0 + \frac{1}{2}) + \log c + \delta(\Gamma)(T + 2r_0 + d_0)}{T}$$

Since the constants k_1, k_2, k_3, c, r_0 and d_0 are independent of T and all the terms are positive we have:

$$h_S(V) \leq \lim_{\epsilon \to 0} \delta(\Gamma) = \delta(\Gamma)$$

We should just remark that the restriction that $T > 2d_0$ and $T > 2\epsilon$ clearly plays no role since we are considering only the limits.

Finally, since the closed ball B in our discussion was chosen arbitrarily, V is a general set in $\mathcal{V}_2$, so we conclude that:

$$h_S(\Gamma) = \sup_{V \in \mathcal{V}_2} h_S(V) \leq \delta(\Gamma)$$

This finishes the proof of Proposition 20

4.3.2 For finitely generated geometric Schottky groups, geodesic flow entropy is bounded from below by the Poincaré exponent

In this section we will show that for a geometric Schottky group Γ which is finitely generated, the geodesic flow entropy $h_S(\Gamma)$ is bounded from below by the Poincaré exponent $\delta(\Gamma)$. So, we will prove the following proposition:

Proposition 21. *Let Γ be a finitely generated geometric Schottky group. Then the geodesic flow entropy $h_S(\Gamma)$ is bounded from below by the Poincaré exponent $\delta(\Gamma)$.*

Warning!: The objects and arguments appearing in the proof given here will bear some resemblance to those in the proof of the upper bound. Our notation will reflect these similarities, yet we warn the reader that objects appearing in this section might differ slightly from the ones from the previous section even if the notation we use is identical.

Outline of the proof

When arguing that $h_S(\Gamma) \geq \delta(\Gamma)$ we will consider a particular set $V_0 \in \mathcal{V}_1$, which we will define at the beginning of our argument, and show that $h_S(V_0) \geq \delta(\Gamma)$. To show that $h_S(V_0) \geq \delta(\Gamma)$, we will find a lower bound for $\mathrm{sep}(T, \epsilon, V_0)$ of the form:

$$c_1 \cdot e^{\ \delta(\Gamma)\cdot(T+c_2)} \cdot (T + c_3)^{-1}$$

where c_1, c_2, c_3 are constants independent of T. This lower bound is obtained by explicitly constructing (T, ϵ)-separated subsets of V_0 for all large enough T and small enough ϵ.

Once again we will work in the universal cover $\mathbb{D}$. For each large enough T, we will consider a circle C_T centered at a lift of a point in $\pi_b(V_0)$ whose radius is up to an additive constant equal to T; the precise definition will be given later. In view of the result due to Roblin [Roblin], who has shown that for any $z \in \mathbb{D}$:

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{\log \# \{\gamma \in \Gamma : d(z, \gamma z) \leq R\}}{R}$$

it would be most natural, when constructing a (T, ϵ)-separated subset of V_0, to associate a vector to each orbit point γo which lies inside the circle C_T i.e. to associated vectors to those points of Γo which lie in the ball $\widehat{C_T}$ bounded by the circle C_T. Instead we will consider only the points that are near the boundary of $\widehat{C_T}$ i.e. near the circle C_T itself; what exacly we mean by 'near' will become clear later in the argument. It is not surprising that we can do that, since in hyperbolic space the area and circumference of very large balls are comparable. With the use of Sullivan's Shadow Lemma, we will show that the number of orbit points lying near C_T can be bounded from below by $c_1 \cdot e^{\ \delta(\Gamma)\cdot(T+c_2)}$.

The intuition that guides us in the construction of the (T, ϵ)-separated subsets of V_0 is the following: We are observing the flow of two vectors in $T^1\mathbb{D}$ under the geodesic flow. Suppose they start their journey close to each other, travel together through a sequence of copies of the fundamental domain, and then enter two distinct copies of the fundamental domain. Then due to the distance between the boundaries of these two copies of the fundamental domain for ϵ which is very small compared to this distance we will observe an 'ϵ-separation' of the two paths the vectors are traveling along. If ϵ is also much smaller than the injectivity radius of the manifold $\mathbb{D}/\Gamma$ we expect that this separation will not disappear when we project to the manifold.

The idea described above explains why we concentrate on the orbit points that lie near C_T. Namely, we want to use such events of entering distinct copies of the fundamental domain, as described above, to guarantee the (T, ϵ)-separated property of the subsets of V_0 which we are going to construct. The event of entering distinct copies of the fundamental domain corresponds to different directions in the ideal boundary towards which the two observed vectors are heading, and so we want to eliminate repetitions among directions. We avoid such repetitions among the directions by ignoring the orbit points which lie inside C_T as well as by considering later in our construction only a subset of the orbit points near C_T in which no two points lie 'below one another' in some sense, i.e. in which no two points correspond to essentially the same direction.

Proof set up

Throughout let Γ be a geometric Schottky group and $(\{A_k\}, \{g_k\})_{k\in I}$ its Schottky description, where the indexing set I is finite. In order to detect the orbit points near the circles C_T we will place balls around the orbit points. Namely, we will define collections of closed balls $\mathcal{D}$ and $\mathcal{D}_T$ in a similar way as in the previous section, but here we will choose the radius d_0 differently. For technical reasons, in order to make our work easier, we will choose this radius to be large enough so that several conditions are satisfied. Firstly we will require that all limit points of Γ are uniformly radial with respect to the radius d_0, see Section 2.0.6. The following proposition shows that it is possible to choose d_0 in this way.

Proposition 22. *There exists a real number $R_0 > 0$ such that each $\xi \in L(\Gamma)$ is uniformly radial with respect to R_0.*

Proof. First consider the collection of all geodesic rays from origin o with endpoint at infinity $\xi \in g_i g_j A_k$, for some $i, j, k \in I$ with $i \neq -j$ and $j \neq -k$, where g_i, g_j and A_k are the generators and the intervals in the Schottky description of Γ. Let $\mathcal{R}_1$ denote the union of all these rays seen as subsets of $\mathbb{D}$. We intersect $\mathcal{R}_1$ with the following subset of $\mathbb{D}$:

$$\overline{F} \cup \bigcup_{i\in I} g_i \overline{F}$$

Since Γ is finitely generated and since the intervals A_k do not share boundary points, there exists $R_1 > 0$ such that the intersection of the above set with the set $\mathcal{R}_1$ is contained in the open ball $B(o, R_1)$.
Secondly, consider the collection of all geodesics with endpoints at infinity $\xi_- \in g_i A_j$ and $\xi_+ \in g_k A_q$, for some $i \neq -j$ and $k \neq -q$ such that $g_i A_j \neq g_k A_q$. Let $\mathcal{R}_2$ denote the union of all these geodesics seen as subsets of $\mathbb{D}$. Again since Γ is finitely generated and since the intervals A_k do not share boundary points there exists $R_2 > 0$ such that the intersection of $\mathcal{R}_2$ with $\overline{F}$ is contained in the open ball $B(o, R_2)$.
We now claim that if we put

$$R_0 := \max(R_1, R_2)$$

then for all $\xi \in L(\Gamma)$:

$$r_\xi \subseteq \bigcup_{g\in\Gamma} gB(o, R)$$

where r_ξ is the geodesic ray from origin o with endpoint at infinity ξ.
To see this fix $\xi \in L(\Gamma)$ and partition r_ξ into segments which lie in distinct copies $g\overline{F}$ of the closure of the fundamental domain. Since Γ is finitely generated and since

the intervals A_k do not share boundary points, the ray r_ξ crosses infinitely many such copies. By the choice of R_1, the first two segments, counting from the origin, must be contained in $B(o, R_1)$ and thus also in $B(o, R)$. Let $r_\xi \cap g\overline{F}$ be any other segment. Then after applying the isometry g^{-1} we observe that $g^{-1}r_\xi$ is contained in $\mathcal{R}_2$ while $g^{-1}(r_\xi \cap g\overline{F}) = g^{-1}r_\xi \cap \overline{F}$. So by the choice of R_2, we have that $g^{-1}(r_\xi \cap g\overline{F}) \subseteq B(o, R_2)$ and hence $r_\xi \cap g\overline{F} \subseteq B(go, R_2) \subseteq gB(o, R)$. This proves our claim. $\square$

Let c_0 be a radius large enough to allow us to use Sullivan's Shadow Lemma for group Γ. Choose $d_0 > 0$ to be large enough to guarantee that the open ball $B(o, d_0)$ contains the set $Hull(L(\Gamma)) \cap \overline{F}$, that $d_0 \geq c_0$, and that $d_0 \geq R_0$. Also fix a real number $T > 0$ and a real number $\epsilon > 0$.

With the above definition of d_0 we define the collection of open balls $\mathcal{D}$ to be:

$$\mathcal{D} := \{B(go, d_0) : g \in \Gamma\}$$

Further we define the circle C_T by:

$$C_T := \{y \in \mathbb{D} : d(o, y) = T - 3d_0\}$$

Let $rad(C_T)$ denote the radius of the circle C_T that is:

$$rad(C_T) = T - 3d_0$$

We assume that $T \geq 4d_0$ but this assumption will play no role in our final argument.

Then we define the collection of balls $\mathcal{D}_T$:

$$\mathcal{D}_T := \{D \in \mathcal{D} : D \cap C_T \neq \emptyset\}$$

Finally we define the set V_0. Let B be a closed ball in $\mathbb{D}/\Gamma$ which contains $\pi(B(o, d_0))$. Define $V_0 \subseteq \mathbb{D}/\Gamma$ as:

$$V_0 := \pi_b^{-1}(B) \cap (((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma)$$

It is clear that the set V_0 belongs to the collection $\mathcal{V}_1$.

Estimating size of $\mathcal{D}_T$

The first step in our argument is to give a lower bound for the size of $\mathcal{D}_T$. To do so, we will consider the shadows from the origin of the balls $D \in \mathcal{D}_T$ denoted as before by $\Pi_o(D)$ and then use Sullivan's Shadow Lemma. We start with the following proposition:

Proposition 23. *The limit set $L(\Gamma)$ is contained in:*

$$\bigcup_{D \in \mathcal{D}_T} \Pi_o(D)$$

Proof. Fix $\xi \in L(\Gamma)$. Since all limit points are uniformly radial with respect to the radius R_0, and thus also with respect to the radius d_0, the balls in the collection $\mathcal{D}$ cover the ray r_ξ. In particular the point $r_\xi \cap C_T$ lies in some $D \in \mathcal{D}$. But this implies that $D \in \mathcal{D}_T$ and that $\xi \in \Pi_o(D)$. □

Now Sullivan's Shadow Lemma gives us the following estimate for the Patterson measure of the shadow $\Pi_o(D)$:

$$\frac{1}{c}e^{-\delta(\Gamma)d(o,x_D)} \leq \mu_o(\Pi_o(D)) \leq ce^{-\delta(\Gamma)d(o,x_D)}$$

where x_D is the center of D and $c > 0$ is a constant which depends only on the radius d_0. From the definition of $\mathcal{D}_T$ it follows that for $D \in \mathcal{D}_T$ we have:

$$rad(C_T) - d_0 \leq d(o, x_D) \leq rad(C_T) + d_0$$

Thus we have:

$$\mu_o(\Pi_o(D)) \leq ce^{-\delta(\Gamma)(rad(C_T)-d_0)}$$

Since $\mu_o(L(\Gamma)) = 1$ combining the above inequality with Proposition 23 gives:

$$\#\mathcal{D}_T \geq \frac{1}{c} \cdot e^{\delta(\Gamma)(rad(C_T)-d_0)}$$

Subset $\mathcal{D}'_T$ of $\mathcal{D}_T$

The set $\mathcal{D}_T$ detects the orbit points which lie near the circle C_T. As already indicated we will only be interested in those points which in a sense correspond to distinct directions. This idea is made precise below where we define a subset $\mathcal{D}'_T$ of $\mathcal{D}_T$ which is meant to capture orbit points corresponding to distinct directions.

Each $D \in \mathcal{D}_T$ is of the form $B(ho, d_0)$, where $h \in \Gamma$. We can express h uniquely in terms of the generators $\{g_i : i \in I\}$ in the Schottky description of Γ. It can happen that there are $D, D' \in \mathcal{D}_T$ with $D = B(ho, d_0)$, $D' = B(h'o, d_0)$ such that:

$$h' = hg_{i_1} \dots g_{i_m}$$

In this situation we will say that D' **lies below** D and we will denote it by:

$$D \succ D'$$

It is clear that this relation is transitive. We will also say that the balls $D_0, D_1, \dots, D_m$ with $m \in \mathbb{N}_0$ form a **string** if for each $i = 1, \dots, m$ we have $D_{i-1} \succ D_i$. We will say that this string is of **length** $m + 1$ and the ball D_m will be referred to as the **lowest element** of this string. Further we will say that the string

$$D_0, D_1, \dots, D_m$$

is **contained** in the string

$$D'_0, D'_1, \ldots, D'_n$$

if there exists a strictly increasing function

$$f : \{0, \ldots, m\} \to \{0, \ldots n\}$$

such that $D_i = D'_{f(i)}$. We say that a string is **maximal** if it is not contained in any string other than itself. Consider the collection of all maximal strings in $\mathcal{D}_T$, remembering that we have also allowed for the trivial strings of length one. We let $\mathcal{D}'_T$ be the set of all lowest elements of maximal strings in $\mathcal{D}_T$.

Now we estimate the size of the set $\mathcal{D}'_T$ in relation to the size of the set $\mathcal{D}_T$. Observe that, if for some constant $k_4 \in \mathbb{N}$ the length of any maximal string is at most k_4, then:

$$\#\mathcal{D}'_T \geq \frac{\#\mathcal{D}_T}{k_4}$$

We will now find such a constant k_4. Let $B(ho, d_0)$ be the lowest element of a maximal string in $\mathcal{D}_T$ and $h = g_{i_1} \cdots g_{i_n}$ the unique expression of h in terms of the gnerators $\{g_i : i \in I\}$ in the Schottky description of Γ. We put

$$\epsilon_\Gamma := \min_{i,j \in I, i \neq j} d(\alpha_i, \alpha_j)$$

and

$$\epsilon^o_\Gamma := \min_{i \in I} d(o, \alpha_i)$$

where α_i, for $i \in I$, denote the geodesics corresponding to the intervals A_i from the Schottky description of Γ. Consider the geodesic segment r_h from the origin o to the orbit point ho. The segment r_h intersects

$$\overline{F}, g_{i_1}\overline{F}, g_{i_1}g_{i_2}\overline{F}, \ldots, g_{i_1} \cdots g_{i_n}\overline{F} = h\overline{F}$$

consecutively, which gives a partition of r_h into segments. The segments $r_h \cap \overline{F}$ and $r_h \cap h\overline{F}$ are of length at least ϵ^o_Γ, while the remaining ones are of the form $r_h \cap g_{i_1} \cdots g_{i_k}\overline{F}$, $k \in \{1, \ldots, n-1\}$ and are of length at least ϵ_Γ. Since the length of the segment r_h is at most $rad(C_T) + d_0$ we can have at most:

$$k_4 := \frac{rad(C_T) + d_0}{\min(\epsilon_\Gamma, \epsilon^o_\Gamma)}$$

segments in the partition of r_h. As each element in the maximal string we considered must be of the form $B(g_{i_1} \cdots g_{i_k} o, d_0)$, $k \in \{1, \ldots, n\}$ or be equal to $B(o, d_0)$, this implies that the length of this maximal string is at most k_4.

Defining the subset U of V_0

Now we will construct a subset U of V_0 which we will later show to be (T, ϵ)-separated. We are going to associate a vector $v_D \in V_0$ to each ball D in the collection $\mathcal{D}'_T$. Let

$$D = B(ho, d_0) \in \mathcal{D}'_T$$

and let

$$h = g_{i_1} \cdots g_{i_n}$$

be the unique expression of h in terms of the generators $\{g_i : i \in I\}$ in the Schottky description of Γ. Choose points $\xi_+^D, \xi_-^D \in L(\Gamma)$ such that the coding sequence:

$$\kappa(\xi_+^D) = [x_o, x_1, \ldots]$$

of ξ_+^D satisfies $x_0 = i_1, x_1 = i_2, \ldots, x_{m-1} = i_m$ and the coding sequence:

$$\kappa(\xi_-^D) = [y_o, y_1, \ldots]$$

of ξ_-^D satisfies $y_o \neq x_0$.

It is clear that the geodesic β_D with negative endpoint at infinity ξ_-^D and positive endpoint at infinity ξ_+^D intersects the canonical fundamental domain F of Γ. Choose a vector $\widetilde{v}_D \in T^1\mathbb{D}$ which is tangent to β_D and which satisfies:

$$\pi_b(\widetilde{v}_D) \in F \quad \text{and} \quad d(\pi_b(\widetilde{v}_D), \alpha_i) \geq \frac{2\epsilon_\Gamma}{6}$$

for all $i \in I$. Let v_D denote the projection of $\widetilde{v}_D$ to $T^1(\mathbb{D}/\Gamma)$; from the construction it is clear that $v_D \in V_0$. We now define the set U to be:

$$U := \left\{v_D : D \in \mathcal{D}'_T\right\}$$

Note that because of how we defined $\widetilde{v}_D$, for $D_1, D, 2 \in \mathcal{D}'_T$ distinct, we have $v_{D_1} \neq v_{D_2}$.

The set U is (T, ϵ)-separated

What presently remains to be shown is that the set U is indeed (T, ϵ)-separated. This is what we are going to prove now and we start with some geometrical observations.

Let $D = B(ho, d_0) \in \mathcal{D}'_T$, where $h = g_{i_1} \cdots g_{i_n}$ is the unique expression of h in terms of the generators $\{g_i : i \in I\}$ in the Schottky description of Γ. By the definition of d_0, in particular the fact that $B(o, d_0)$ contains the set $Hull(L(\Gamma)) \cap \overline{F}$, it follows that the intersection $\beta_D \cap h\overline{F}$ lies in D. The geodesic ray β_D^+ between $\pi_b(\widetilde{v}_D)$ and ξ_+^D starts in $\overline{F}$ and intersects

$$g_{i_1}\overline{F}, g_{i_1}g_{i_2}\overline{F}, \ldots, g_{i_1} \cdots g_{i_{n-1}}\overline{F}, h\overline{F}$$

consecutively. Define the following four times:

$$t_1 := \inf\left\{t \in \mathbb{R}^+ : \pi_b(g^{t_1}\widetilde{v}_D) \in D\right\}$$

$$t_2 := \inf\left\{t \in \mathbb{R}^+ : \pi_b(g^{t_2}\widetilde{v}_D) \in h\overline{F}\right\}$$
$$t_3 := \sup\left\{t \in \mathbb{R}^+ : \pi_b(g^{t_2}\widetilde{v}_D) \in h\overline{F}\right\}$$
$$t_4 := \sup\left\{t \in \mathbb{R}^+ : \pi_b(g^{t_2}\widetilde{v}_D) \in D\right\}$$

That t_1 is the time at which γ_D^+ enters D, t_2 the time at which γ_D^+ enters $h\overline{F}$, t_3 the time at which γ_D^+ leaves $h\overline{F}$ and t_4 the time at which γ_D^+ leaves D.

We clearly have $t_1 < t_2 < t_3 < t_4$. Moreover, since $\pi_b(\widetilde{v}_D) \in B(o, d_0)$, by the triangle inequality we have that:

$$t_4 \leq rad(C_T) + 3d_0 = T$$

To see this go from $\pi_b(\widetilde{v}_D)$ to o, to any point in $C_T \cap D$, and then to $\pi_b(g^{t_4}\widetilde{v}_D)$.

Now suppose $D_1 = B(h_1 o, d_0)$ and $D_2 = B(h_2 o, d_0)$ are two distinct elements in $\mathcal{D}'_T$ and let $h_1 = g_{i_1} \cdots g_{i_n}$ and $h_2 = g_{j_1} \cdots g_{j_m}$ be the unique expressions of h_1 and h_2 in terms of the generators in the Schottky description of Γ. The definition of $\mathcal{D}'_T$ implies that there is $k \leq \min(n, m)$ such that $g_{i_k} \neq g_{j_k}$. Since $t_2 < t_4 - \epsilon_\Gamma \leq T - \epsilon_\Gamma$, this means that before time $T - \epsilon_\Gamma$, the rays $\beta_{D_1}^+$ and $\beta_{D_1}^+$ enter distinct copies of $\overline{F}$.

Finally, to prove that U is (T, ϵ)-separated, we show that if this condition failed for some $v_{D_1}, v_{D_2} \in U$, then the corresponding rays $\beta_{D_1}^+$ and $\beta_{D_2}^+$ could not enter distinct copies of $\overline{F}$ before time $T - \epsilon_\Gamma$, in the sense given above.

The failure of the (T, ϵ)-separated condition would mean that for all $t \in [0, T]$ we have:

$$d(\pi_b(g^t v_{D_1}), \pi_b(g^t v_{D_2})) \leq \epsilon$$

For $\widetilde{v}_{D_1}, \widetilde{v}_{D_2}$ this means that for all $t \in [0, T]$ there exists some $h_t \in \Gamma$, depending on t, such that:

$$d(\pi_b(g^t \widetilde{v}_{D_1}), h_t \pi_b(g^t \widetilde{v}_{D_2})) \leq \epsilon$$

To proceed we need to make certain assumptions on ϵ; these assumptions are necessary here, yet they will play no role in our final argument. We assume that:

$$\epsilon \leq \frac{\epsilon_\Gamma}{6} \quad \text{and} \quad \epsilon \leq \frac{\epsilon_\Gamma^*}{4}$$

where ϵ_Γ^* is another constant characterising Γ defined as:

$$\epsilon_\Gamma^* := \min\left\{d(S_{i,j}, \alpha_k) : i, j, k \in I \ : \ i \neq j \neq k \neq i\right\}$$

where $S_{i,j}$ denotes the union of all geodesics with one endpoint at infinity in $L(\Gamma) \cap A_i$ and the other in $L(\Gamma) \cap A_j$.

The rays $\beta_{D_1}^+$ and $\beta_{D_2}^+$ start in $\overline{F}$ at least 2ϵ far away from the boundary of $\overline{F}$. This can be formally expressed by saying that at time $t = 0$ we have $\pi_b(g^t \widetilde{v}_{D_1}), \pi_b(g^t \widetilde{v}_{D_2}) \in \overline{F}$ as well as $d(\pi_b(g^t\widetilde{v}_{D_1}), \alpha_i) \geq 2\epsilon$ and $d(\pi_b(g^t\widetilde{v}_{D_2}), \alpha_i) \geq 2\epsilon$ for all $i \in I$. This implies that for any $\gamma \in \Gamma - \{id\}$ we have:

$$d(\gamma\pi_b(\widetilde{v}_{D_1}), \pi_b(\widetilde{v}_{D_2})) \geq 2\epsilon$$

so that $\pi_b(\widetilde{v}_{D_1})$ must be the closest point in its orbit to $\pi_b(\widetilde{v}_{D_2})$ and we must have:

$$d(\pi_b(\widetilde{v}_{D_1}), \pi_b(\widetilde{v}_{D_2})) \leq \epsilon$$

Without loss of generality, suppose that $\beta^+_{D_1}$ leaves $\overline{F}$ first or at the same time as $\beta^+_{D_2}$ and that $\beta^+_{D_1}$ leaves $\overline{F}$ through the geodesic α_i, for some $i \in I$.

Let us observe the position of $\pi_b(g^t\widetilde{v}_{D_1})$ as t increses from zero to T. Since $\epsilon \leq \frac{\epsilon^*_\Gamma}{4}$ it can only enter the 2ϵ-neighborhood of α_i but of no other geodesic of the boundary of F. Until it enters the ϵ-neighborhood of α_i, all of its images under the action of Γ stay at least ϵ away from $\overline{F}$ and thus $\pi_b(g^t\widetilde{v}_{D_1})$ remains the closest point in its orbit to $\pi_b(g^t\widetilde{v}_{D_1})$. At a time t when $\pi_b(g^t\widetilde{v}_{D_1})$ enters the ϵ-neighborhood of α_i, the point $\pi_b(g^t\widetilde{v}_{D_2})$ must be in the 2ϵ-neighborhood of α_i.

After that, as long as $\pi_b(g^t\widetilde{v}_{D_1})$ remains in the ϵ-neighborhood of α_i, each of its copies under the action of Γ remains in the ϵ neighborhood of the corresponding copy of α_i. Since the 2ϵ-neighborhoods of copies of α_i are at least ϵ away, this implies that $\pi_b(g^t\widetilde{v}_{D_1})$ must stay in the 2ϵ-neighborhood of α_i during this time.

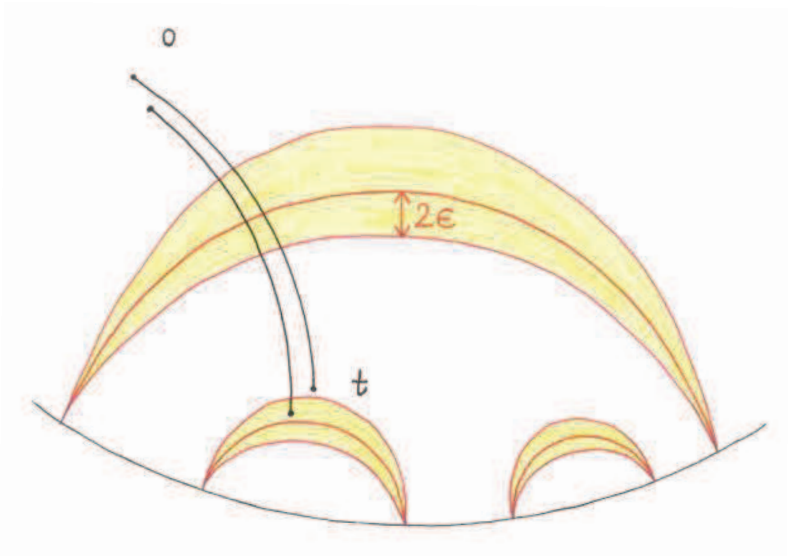


Figure 4.4:

It might happen that by the time $\pi_b(g^t\widetilde{v}_{D_1})$ exits the ϵ-neighborhood of α_i we have already reached the $t = T$, but in this case we already obtain a contradiction, since the two rays would not have entered distinct copies of $\overline{F}$. So suppose that we still have not reached $t = T$. At the time t when $\pi_b(g^t \widetilde{v}_{D_1})$ leaves the ϵ-neighborhood of α_i, since $\pi_b(g^t \widetilde{v}_{D_2})$ must be in the ϵ-ball around $\pi_b(g^t \widetilde{v}_{D_1})$, it either just enters $g_i F$ or has done so already. Without loss of generality, assume that $\pi_b(g^t \widetilde{v}_{D_1})$ is the first one to leave the 2ϵ neighborhood of α_i in $g_i F$. As $\pi_b(g^t \widetilde{v}_{D_1})$ leaves the 2ϵ-neighborhood, any of its images under the action of Γ must lie on the boundary of the 2ϵ neighborhood of the corresponding copy of α_i and therefore we must have $d(\pi_b(g^t \widetilde{v}_{D_1}), \pi_b(g^t \widetilde{v}_{D_2})) \leq \epsilon$. This remains the case all the time until $\pi_b(g^t \widetilde{v}_{D_1})$ enters the 2ϵ-neighborhood of the geodesic $g_i \alpha_j$ which it crosses next. Since $\epsilon \leq \frac{\epsilon_\Gamma}{6}$ this implies that we reach the situation when $\pi_b(g^t \widetilde{v}_{D_1}), \pi_b(g^t \widetilde{v}_{D_2}) \in g_i \overline{F}$ as well as $d(\pi_b(g^t \widetilde{v}_{D_1}), g_i \alpha_j) \geq 2\epsilon$ and $d(\pi_b(g^t \widetilde{v}_{D_2}), g_i \alpha_j) \geq 2\epsilon$, for all $i \in I$.

In $g_i \overline{F}$ we can argue in the same way as we did in $\overline{F}$; and we can argue similarly in further copies of $\overline{F}$. We continue until $t = T$. In this way we show that the rays $\beta^+_{D_1}$ and $\beta^+_{D_2}$ cannot enter distinct copies of $\overline{F}$ before time $T - \epsilon_\Gamma$.

Final step

Now we are finally ready to give the lower bound for the entropy $h_S(\Gamma)$. The set $U \subseteq V_0$ which we defined above depends on T and ϵ. We have shown that provided $T \geq 4d_0$ and $\epsilon \leq \frac{\epsilon_\Gamma}{6}, \frac{\epsilon^*_\Gamma}{4}$ the set U is (T, ϵ)-separated. So for fixed T and ϵ satisfying these assumptions we have:

$$\operatorname{sep}(T, \epsilon, V_0) \geq \#U = \#\mathcal{D}'_T \geq \frac{\#\mathcal{D}_T}{k_4} \geq \frac{\frac{1}{c} e^{\delta(\Gamma)(T - 4d_0)} \cdot \min(\epsilon_\Gamma, \epsilon^o_\Gamma)}{(T - 2d_0)}$$

Therefore:

$$h_S(V_0) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\operatorname{sep}(T, \epsilon, V_0))}{T}$$

$$\geq \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\min(\epsilon_\Gamma, \epsilon^o_\Gamma)) - \log c + \delta(\Gamma)(T - 4d_0) - \log(T - 2d_0)}{T}$$

Hence we obtain:

$$h_S(V_0) \geq \lim_{\epsilon \to 0} \delta(\Gamma) = \delta(\Gamma)$$

And thus:

$$h_S(\Gamma) = \sup_{V \in \mathcal{V}_1} h(V) \geq h_S(V_0) \geq \delta(\Gamma)$$

This finishes the proof of Proposition 21. Combining Proposition 20 and Proposition 21 yields Theorem 5.

4.4 For infinitely generated geometric Schottky groups, geodesic flow entropy is bounded from below by the Poincaré exponent

In this section we drop the assumption that Γ is finitely generated and show that the geodesic flow entropy h_S of Γ is still bounded below by its Poincaré exponent $\delta(\Gamma)$. Namely, we will prove the following proposition:

Proposition 24. *Let Γ be an infinitely generated geometric Schottky group. Then the geodesic flow entropy $h_S(\Gamma)$ is bounded from below by the Poincaré exponent $\delta(\Gamma)$.*

Here Γ will be a geometric Schottky group with Schottky description $(\{g_i\}, \{A_i\})_{i\in I}$, where the indexing set I is infinite. To obtain this result we will look at certain finitely generated subgroups of Γ. The structure of the proof is similar to that of the proof of the lower bound for a finitely generated group in Section 4.3.2. What changes is that some of the objects, constants and properties appearing in the proof will be defined with respect to Γ while others will be defined with respect to some finitely generated subgroup of Γ.

Outline of the proof

Our starting point is a result due to Sullivan which allows to express the Poincaré exponent of Γ in terms of the Poincaré exponents of some of its finitely generated subgroups. The following theorem is a special case of the theorem due to Sullivan [Sullivan1979].

Theorem 6. *Let G ba a Fuchsian group and, for each $N \in \mathbb{N}_0$, let G_N be a subgroup of G such that:*

- $G_N < G_{N+1}$*, for all* $N \in \mathbb{N}_0$
- $G = \bigcup_N G_N$

Then the Poincaré exponent of G is given by:

$$\delta(G) = \sup_{N\in\mathbb{N}_0} \delta(G_N)$$

Proof. The original argument can be found in [Sullivan1979]. For a more recent treatment see [FalkStratmann2004] and [Stratmann2003]. □

For each $N \in \mathbb{N}$ we will define a finitely generated subgroup Γ^N of Γ in such a way that:

$$\Gamma^1 < \Gamma^2 < \ldots < \Gamma^N < \ldots$$

and for which we will have $\Gamma = \bigcup_N \Gamma_N$. Thus Theorem 6 will allow us to deduce that:

$$\delta(\Gamma) = \sup_{N\in\mathbb{N}} \delta(\Gamma^N)$$

We identify the indexing set I with $\mathbb{Z}^*$ and for each $N \in \mathbb{N}$ denote by I_N the subset of I corresponding to $\{i \in \mathbb{Z}^* : |i| \leq N\}$. Then for each $N \in \mathbb{N}$ we define Γ^N as the geometric Schottky group with Schottky description $(\{g_i\}, \{A_i\})_{i \in I_N}$, where the generators g_i and intervals A_i are the same as the generators and intervals in the Schottky description of the original group Γ. We will denote the canonical fundamental domain of Γ^N by F^N. Our plan is to show that for each $N \in \mathbb{N}$, $N > 1$ we have $h_S(\Gamma) \geq \delta(\Gamma^N)$. It will then follow that $h_S(\Gamma) \geq \delta(\Gamma)$.

Proof set up

Let from now on $N \in \mathbb{N}$, $N > 1$ be fixed. In the same way as in Section 4.3.2 we can find a radius R_0 such that all the limit points $L(\Gamma^N)$ are uniformly radial with respect to this radius and the group Γ^N, by which we mean that for all $\xi \in L(\Gamma^N)$:

$$r_\xi \subseteq \bigcup_{g \in \Gamma^N} gB(o, R_0)$$

Further, let c_0 be a radius large enough to allow us to use Sullivan's Shadow Lemma for the group Γ^N. Choose $d_0 > 0$ to be large enough to guarantee that the open ball $B(o, d_0)$ contains the set $Hull(L(\Gamma^N)) \cap \overline{F^N}$, that $d_0 \geq c_0$, and that $d_0 \geq R_0$. Also fix a real number $T > 0$ and a real number $\epsilon > 0$.

Now with this new definition of d_0 we define:

$$\mathcal{D} := \left\{B(go, d_0) : g \in \Gamma^N\right\}$$

$$C_T := \{y \in \mathbb{D} : d(o, y) = T - 3d_0 =: rad(C_T)\}$$

$$\mathcal{D}_T := \{D \in \mathcal{D} : D \cap C_T \neq \emptyset\}$$

Again we assume that $T \geq 4d_0$ and this assumption will play no role in our final argument.

Let B be a closed ball in $\mathbb{D}/\Gamma$ which contains $\pi(B(o, d_0))$, where π denotes the canonical projection from $\mathbb{D}$ to $\mathbb{D}/\Gamma$. Define V_0 as:

$$V_0 := \pi_b^{-1}(B) \;\cap\; (((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma)$$

It is clear that $V_0 \in \mathcal{V}_1$, where $\mathcal{V}_1$ is defined, as in Section 4.1, with respect to the group Γ.

Defining the subset U of V_0

As in Section 4.3.2 we plan to define a subset U of V_0 such that each $u \in U$ satisfies $\pi_b(u) \in \pi(B(o, d_0))$ and which is (T, ϵ)-separated in $\mathbb{D}/\Gamma$.

Note that the collection $\mathcal{D}_T$ has been defined with respect to Γ^N exactly as in Section 4.3.2. Working exclusively with the finitely generated subgroup Γ^N, that is considering its limit subset $L(\Gamma^N)$ and the Patterson measure μ_o associated to Γ^N, we can use Sullivan's Shadow Lemma exacly as in Section 4.3.2 to estimate the size of $\mathcal{D}_T$. Namely, we obtain:

$$\#\mathcal{D}_T \geq \frac{1}{c} \cdot e^{\delta(\Gamma^N)(rad(C_T)-d_0)}$$

Working further exclusively with Γ^N and its canonincal fundamental domain, we define a subcollection $\mathcal{D}'_T$ in exactly the same way as in Section 4.3.2 and estimate its size by:

$$\#\mathcal{D}'_T \geq \frac{\#\mathcal{D}_T}{k_5}$$

where the constant k_5 is given by:

$$k_5 := \frac{rad(C_T)+d_0}{\min(\epsilon_{\Gamma^N}, \epsilon^o_{\Gamma^N})}$$

Similarly as in Section 4.3.2 we associate to each ball $D \in \mathcal{D}'_T$ a vector $v_D \in V_0$. Let $D = B(ho, d_0) \in \mathcal{D}'_T$ and let $h = g_{i_1} \cdots g_{i_n}$ be the unique expression of h in terms of the generators $\{g_i : i \in I_N\}$in the Schottky description of Γ^N. We choose points $\xi^D_+, \xi^D_- \in L(\Gamma)$ such that the coding sequence:

$$\kappa(\xi^D_+) = [x_o, x_1, \ldots]$$

of ξ^D_+ with respect to Γ^N satisfies $x_0 = i_1, x_1 = i_2, \ldots, x_{m-1} = i_m$ and the coding sequence:

$$\kappa(\xi^D_-) = [y_o, y_1, \ldots]$$

of ξ^D_- with respect to Γ^N satisfies $y_o \neq x_0$. It is clear that the geodesic β_D with negative endpoint at infinity ξ^D_- and positive endpoint at infinity ξ^D_+ intersects the canonical fundamental domain F^N of Γ^N. We choose a vector $\widetilde{v}_D \in T^1\mathbb{D}$ which is tangent to β_D and which satisfies $\pi_b(\widetilde{v}_D) \in F^N$ and such that $d(\pi_b(\widetilde{v}_D), \alpha_i) \geq \frac{2\epsilon_{\Gamma^N}}{6}$ for all $i \in I_N$. Let v_D denote the projection of $\widetilde{v}_D$ to $T^1(\mathbb{D}/\Gamma)$; from the construction it is clear that $v_D \in V_0$. We now define the set U to be:

$$U := \left\{ v_D : D \in \mathcal{D}'_T \right\}$$

It is crucial to observe here that, since for each $D \in \mathcal{D}_T$ the basepoint $\pi_b(\widetilde{v}_D)$ not only lies in F^N but also in the smaller fundamental domain F of Γ, we will have $\#U = \#\mathcal{D}'_T$.

The set U is (T, ϵ)-separated

To prove that U is (T, ϵ)-separated in $\mathbb{D}/\Gamma$ we observe, as in Section 4.3.2 but working with Γ^N and its fundamental domain F^N, that if v_{D_1} and v_{D_2} are two distinct vectors in U then the corresponding rays $\beta^+_{D_1}$ and $\beta^+_{D_2}$ enter distinct copies of $\overline{F^N}$ before time

$T - \epsilon_{\Gamma^N}$. We show that if this condition failed for some $v_{D_1}, v_{D_2} \in U$ then the corresponding rays $\beta^+_{D_1}$ and $\beta^+_{D_2}$ could not enter distinct copies of $\overline{F^N}$ before time $T - \epsilon_{\Gamma^N}$.

In this situation the failure of the (T, ϵ)-separated condition means that for all $t \in [0, T]$ we have:

$$d(\pi_b(g^t v_{D_1}), \pi_b(g^t v_{D_2})) \leq \epsilon$$

For $\widetilde{v}_{D_1}, \widetilde{v}_{D_2}$ this means that for all $t \in [0, T]$ there exists some $h_t \in \Gamma$, depending on t, such that

$$d(\pi_b(g^t \widetilde{v}_{D_1}), h_t \pi_b(g^t \widetilde{v}_{D_2})) \leq \epsilon$$

To proceed, we need to make certain assumptions on ϵ; these assumptions are necessary here, yet they will play no role in our final argument. We assume that

$$\epsilon \leq \frac{\epsilon_{\Gamma^N}}{6} \quad \text{and} \quad \epsilon \leq \frac{\epsilon^*_N}{4}$$

where ϵ^*_N is a constant characterising Γ and depending on N. It is defined as:

$$\epsilon^*_N := \min\{d(S_{i,j}, \alpha_k) : i, j \in I_N,\ \ k \in I \ \ : \ \ i \neq j \neq k \neq i\}$$

where $S_{i,j}$ denotes the union of all geodesics with one endpoint at infinity in $L(\Gamma^N) \cap A_i$ and the other in $L(\Gamma^N) \cap A_j$. The sets $L(\Gamma^N) \cap A_i$ and $L(\Gamma^N) \cap A_j$ are properly contained in A_i and A_j respectively, since I_N is finite, and therefore $\epsilon^*_N > 0$.

The rays $\beta^+_{D_1}$ and $\beta^+_{D_2}$ start in $\overline{F^N}$ at least 2ϵ far away from the boundaries of $\overline{F}$. This can be formally expressed by saying that at time $t = 0$ we have $\pi_b(g^t \widetilde{v}_{D_1}), \pi_b(g^t \widetilde{v}_{D_2}) \in \overline{F^N}$ as well as $d(\pi_b(g^t \widetilde{v}_{D_1}), \alpha_i) \geq 2\epsilon$ and $d(\pi_b(g^t \widetilde{v}_{D_2}), \alpha_i) \geq 2\epsilon$ for all $i \in I$. This implies that for any $g \in \Gamma \setminus \{id\}$ we have $d(g\pi_b(\widetilde{v}_{D_1}), \pi_b(\widetilde{v}_{D_2})) \geq 2\epsilon$ so that $\pi_b(\widetilde{v}_{D_1})$ must be the closest point in its orbit under the action of Γ to $\pi_b(\widetilde{v}_{D_2})$ and we must have $d(\pi_b(\widetilde{v}_{D_1}), \pi_b(\widetilde{v}_{D_2})) \leq \epsilon$.

Without loss of generality, suppose that $\beta^+_{D_1}$ leaves $\overline{F^N}$ first or at the same time as $\beta^+_{D_2}$ and that $\beta^+_{D_1}$ leaves $\overline{F}$ through the geodesic α_i, for some $i \in I_N$. Let us observe the movement of $\pi_b(g^t \widetilde{v}_{D_1})$ as t increses from zero to T. Since $\epsilon \leq \frac{\epsilon^*_\Gamma}{4}$, it can only enter the 2ϵ-neighborhood of α_i but of no other geodesic in the boundary of F. Until it enters the ϵ-neighborhood of α_i, all of its images under the action of Γ stay at least ϵ away from $\overline{F}$ and thus $\pi_b(g^t \widetilde{v}_{D_1})$ remains the closest point in its orbit to $\pi_b(g^t \widetilde{v}_{D_1})$.

At a time t when $\pi_b(g^t \widetilde{v}_{D_1})$ enters the ϵ-neighborhood of α_i, the point $\pi_b(g^t \widetilde{v}_{D_2})$ must be in the 2ϵ-neighborhood of α_i. Since $\epsilon \leq \frac{\epsilon^*_\Gamma}{4}$, the point $\pi_b(g^t \widetilde{v}_{D_2})$ cannot enter and stays at least ϵ away from any other 2ϵ-neighborhood of a geodesic of the boundary of F. Therefore it must enter $g_i F$. Again, since $\epsilon \leq \frac{\epsilon^*_\Gamma}{4}$, the first new 2ϵ-neighborhood of a geodesic of the boundary of $g_i F$ which it can enter is the 2ϵ-neighborhood of the geodesic $g_i \alpha_k$ through which it leaves $g_i F$ and it stays at least ϵ away from any other

2ϵ-neighborhood of a geodesic of the boundary of g_iF.

Without loss of generality, suppose that $\pi_b(g^t\widetilde{v}_{D_2})$ is the first one to leave the 2ϵ-neighborhood of α_i in g_iF. As $\pi_b(g^t\widetilde{v}_{D_1})$ leaves the 2ϵ-neighborhood, any of its images under the action of Γ must lie on the boundary of the 2ϵ-neighborhood of the corresponding copy of α_i; so, since $\pi_b(g^t\widetilde{v}_{D_1})$ stays at least ϵ away from it, we must have $d(\pi_b(\widetilde{v}_{D_1}), \pi_b(\widetilde{v}_{D_2})) \leq \epsilon$. This remains the case all the time until $\pi_b(g^t\widetilde{v}_{D_1})$ enters the 2ϵ-neighborhood of the geodesic $g_i\alpha_j$ which it crosses next, because during this time any of its copies outside g_iF is at least ϵ away from the boundary of g_iF. Since $\epsilon \leq \frac{\epsilon_\Gamma}{6}$ this implies that we reach the situation when $\pi_b(g^t\widetilde{v}_{D_1}), \pi_b(g^t\widetilde{v}_{D_2}) \in g_i\overline{F}$ as well as $d(\pi_b(g^t\widetilde{v}_{D_1}), g_i\alpha_j) \geq 2\epsilon$ and $d(\pi_b(g^t\widetilde{v}_{D_2}), g_i\alpha_j) \geq 2\epsilon$ for all $i \in I$.

In $g_i\overline{F}$ we can argue in the same way as we did in $\overline{F}$, and continue similarly in further copies of $\overline{F}$ and we continue until $t = T$. In this way we show that the rays $\beta^+_{D_1}$ and $\beta^+_{D_2}$ cannot enter distinct copies of $\overline{F}$ before time $T - \epsilon_{\Gamma^N}$.

Final step

Now we are finally ready to give the lower bound for the entropy $h_S(\Gamma)$. As in Section 4.3.2, provided $T \geq 4d_0$ and $\epsilon \leq \frac{\epsilon_\Gamma}{6}, \frac{\epsilon^*_\Gamma}{4}$ the set U, which depends on T and ϵ, is (T, ϵ)-separated. For fixed T and ϵ satisfying these assumptions we have:

$$\mathrm{sep}(T, \epsilon, V_0) \geq \#U = \#\mathcal{D}'_T \geq \frac{\#\mathcal{D}_T}{k_5} \geq \frac{\frac{1}{c}e^{\delta(\Gamma^N)(T-4d_0)} \cdot \min(\epsilon_{\Gamma^N}, \epsilon^o_{\Gamma^N})}{(T - 2d_0)}$$

Therefore, taking the limits as in Section 4.3.2, we get:

$$h_S(V_0) \geq \delta(\Gamma^N)$$

And hence:

$$h_S(\Gamma) := \sup_{V \in \mathcal{V}_1} h_S(V) \geq h_S(V_0) \geq \delta(\Gamma^N)$$

As explained at the beginning of this section, this implies that $h_S(\Gamma) \geq \delta(\Gamma)$. This finishes the proof of Proposition 24.

4.5 For geometric Schottky groups, geodesic flow entropy is equal to the convex core entropy

In this section we will relate the two notions of entropy introduced in this chapter. Namely, we will prove that the geodesic flow entropy h_S and the convex core entropy h_c coincide for geometric Schottky groups.

Theorem 7. *Let Γ be a geometric Schottky group. Then the geodesic flow entropy of the group Γ and the convex core entropy of the group coincide.*

Let us recall here the definitions of the two notions of entropy. For a geometric Schottky group Γ, the geodesic flow entropy was defined as:

$$h_S(\Gamma) := \sup_{V \in \mathcal{V}_1} h_S(V) = \sup_{V \in \mathcal{V}_2} h_S(V)$$

where

$$h_S(V) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log \operatorname{sep}(T, \epsilon, V)}{T}$$

While the convex core entopy was defined as:

$$h_c(\Gamma) := \limsup_{R \to \infty} \frac{\log \operatorname{vol}(Hull_\rho(L(\Gamma)) \cap B(z, R))}{R}$$

4.5.1 For geometric Schottky groups, geodesic flow entropy is bounded from above by convex core entropy

Proposition 25. *Let Γ be a geometric Schottky group. Then the geodesic flow entropy $h_S(\Gamma)$ is bounded from above by the convex core entropy $h_c(\Gamma)$.*

The proof of the upper bound is simpler than that of the lower bound. It uses very natural steps and works for any geometric Schottky group. Throughout let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k \in I}$.

In order to prove that the geodesic flow entropy is bounded from above by the convex core entropy, we need to show that for any set V in the collection $\mathcal{V}_2$ we have:

$$h_S(V) \leq h_c(\Gamma)$$

The idea is to use the similarity between the definitions of $h_S(V)$ and $h_c(\Gamma)$ and to show that, for a fixed set V and ϵ small enough, the value of $\operatorname{sep}(T, \epsilon, V)$ can be bounded from above by a constant multiple of $\operatorname{vol}(Hull_\rho(L(\Gamma)) \cap B(z, R))$ for a suitably chosen z and ρ, and for an R, which will depend on T.

Let us start by fixing a set V in the collection $\mathcal{V}_2$, as defined in Section 4.1 which by the definition of $\mathcal{V}_2$ is of the form:

$$V = \pi_b^{-1}(B) \ \cap \ ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$$

where B is a closed ball in $\mathbb{D}/\Gamma$. By Remark 5 we can also assume that the radius of B is much smaller than the injectivity radius of $\mathbb{D}/\Gamma$. So we can lift B to a ball $\widetilde{B}$ in $\mathbb{D}$ with center x_0 and radius r_0. We choose this lift so that x_0 lies in the closure $\overline{F}$ of the standard fundamental domain of Γ. We will denote by C_T the circle in $\mathbb{D}$ centered at x_0 of radius $T+r_0$ and by $\widehat{C}_T$ the closed ball with center x_0 and radius $T+r_0$.

Now let us also fix $\epsilon > 0$ and $T \geq 0$. Suppose that U is a (T, ϵ)-separated subset of V. Let $\widetilde{U}$ be the lift of U satisfying $\pi_b(\widetilde{U}) = \widetilde{B}$. Observe that as the vectors in $\widetilde{U}$ flow under the geodesic flow, their basepoints will eventually leave $\widehat{C}_T$ but will stay all the time in $Hull(L(\Gamma))$. Consider the intersection $Hull(L(\Gamma)) \cap C_T$. We will assume that T is large enough so that $C_T \backslash Hull(L(\Gamma))$ contains an arc. Due to the geometry of $Hull(L(\Gamma))$, for any $T' > T$, the set $C_{T'} \backslash Hull(L(\Gamma))$ will contain an arc which is even longer, in hyperbolic sense. We will assume that ϵ is very small compared to the largest arc in $C_T \backslash Hull(L(\Gamma))$. It is easy to see that with these assumptions the set $Hull(L(\Gamma)) \cap C_T$ can be covered with a collection of closed hyperbolic balls of radius $\frac{\epsilon}{6}$ such that each of them intersects $Hull(L(\Gamma)) \cap C_T$ and such that they are pairwise either disjoint or tangent.

As the vectors in $\widetilde{U}$ flow under the geodesic flow, the basepoint of each vector will leave $\widehat{C}_T$ through one of these $\frac{\epsilon}{6}$-balls. So the size of $\widetilde{U}$ is bounded from above by $c_1 \cdot c_2$, where c_1 is an upper bound for the number of $\frac{\epsilon}{6}$-balls in the cover and c_2 is an upper bound for the number of vectors in $\widetilde{U}$ whose basepoints can leave through the same ball in the cover.

Just as in Section 4.3.1 we can use Lemma 2 to show that if the bases of two distinct vectors $\widetilde{u}$ and $\widetilde{v}$ in $\widetilde{U}$ leave $\widehat{C}_T$ through a common $\frac{\epsilon}{6}$-ball in the cover, then the (T, ϵ)-separability of $\widetilde{U}$ implies that their bases $\pi_b(\widetilde{u})$ and $\pi_b(\widetilde{v})$ satisfy:

$$d(\pi_b(\widetilde{u}), \pi_b(\widetilde{v})) > \frac{\epsilon}{3}$$

So we can take c_2 to be an upper bound for the maximal number of points in $\widetilde{B}$ that are more than $\frac{\epsilon}{3}$ away from each other. In particular we may take:

$$c_2 := \frac{\mathrm{vol}(\widetilde{B})}{\mathrm{vol}(\frac{\epsilon}{6}\text{-ball})} = \frac{2\pi(\cosh(r_0) - 1)}{2\pi(\cosh(\frac{\epsilon}{6}) - 1)}$$

To estimate c_1, notice that all the $\frac{\epsilon}{6}$-balls in our cover are contained in $Hull_\epsilon(L(\Gamma))$. Since the balls in our cover intersect pairwise in at most one point we can take:

$$\begin{aligned} c_1 :&= \frac{\mathrm{vol}(Hull_\epsilon(L(\Gamma)) \cap \widehat{C}_{T+\epsilon})}{\mathrm{vol}(\frac{\epsilon}{6}\text{-ball})} \\ &= \frac{\mathrm{vol}(Hull_\epsilon(L(\Gamma)) \cap B(x_0, T + r_0 + \epsilon))}{\mathrm{vol}(\frac{\epsilon}{6}\text{-ball})} \end{aligned} \tag{4.5}$$

Thus we have shown that for T large enough and ϵ small enough:

$$\mathrm{sep}(T, \epsilon, V) \leq c_1 \cdot c_2 \leq \mathrm{vol}\left(Hull_\epsilon(L(\Gamma)) \cap B(x_0, T + r_0 + \epsilon)\right) \cdot c_3$$

where r_0 was the radius of $B = \pi_b(V)$ and c_3 is a constant depending only on ϵ and r_0.

Therefore we can estimate the value of $h_S(V)$ as follows:

$$\begin{aligned}
h_S(V) :&= \lim_{\epsilon\to 0}\limsup_{T\to\infty} \frac{\log\Big(\operatorname{sep}(T,\epsilon,V)\Big)}{T} \\
&\leq \lim_{\epsilon\to 0}\limsup_{T\to\infty} \frac{\log\Big(\operatorname{vol}\big(Hull_\epsilon(L(\Gamma))\cap B(x_0,T+r_0+\epsilon)\big)\cdot c_3\Big)}{T} \\
&= \lim_{\epsilon\to 0}\limsup_{T\to\infty} \frac{\log\Big(\operatorname{vol}\big(Hull_\epsilon(L(\Gamma))\cap B(x_0,T+r_0+\epsilon)\big)\Big)}{T} \\
&= \lim_{\epsilon\to 0}\limsup_{T\to\infty} \frac{\log\Big(\operatorname{vol}\big(Hull_\epsilon(L(\Gamma))\cap B(x_0,T+r_0+\epsilon)\big)\Big)}{T+r_0+\epsilon}\cdot\frac{T+r_0+\epsilon}{T} \\
&\leq \lim_{\epsilon\to 0} h_c \\
&= h_c
\end{aligned} \tag{4.6}$$

where we obtained h_c from the limit superior since $\frac{T+r_0+\epsilon}{T} \searrow 1$.

This finishes the proof of Proposition 25.

The above result is consistent with the result which we obtained in case of a finitely generated group Γ, since for finitely generated Fuchsian groups, the convex core entropy is equal to the Poincaré exponent. In fact, to prove the result for a finitely generated Γ we could use the above proof together with an argument justifying why $h_c = \delta(\Gamma)$ for a finitely generated geometric Schottky group, and this is not difficult once one notices that for such a group the convex hull $Hull(L(\Gamma))$ is the union of images under the action of Γ of the compact region $Hull(L(\Gamma)) \cap F$. One can then use the result due to Roblin [Roblin], who has shown that for any $z \in \mathbb{D}$:

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{\log \#\{\gamma\in\Gamma : d(z,\gamma z)\leq R\}}{R}$$

4.5.2 For geometric Schottky groups, geodesic flow entropy is bounded from below by convex core entropy

Proposition 26. *Let Γ be a geometric Schottky group. Then the geodesic flow entropy $h_S(\Gamma)$ is bounded from below by the convex core entropy $h_c(\Gamma)$.*

Outline of the proof

Throughout let Γ be a geometric Schottky group and $(\{A_k\},\{g_k\})_{k\in I}$ its Schottky description. The approach in the proof of the lower bound has been inspired by a paper by Falk and Matsuzaki [FalkMatsuzaki]. They have shown that:

$$h_c = \limsup_{k \to \infty} \frac{\log N_k}{k \log 2}$$

where N_k is the number of rectangles in a certain mesh which intersect the convex hull $Hull(L(\Gamma^*))$, where Γ^* is a certain conjugate of the group Γ. The precise definitions of the conjugate Γ^* and the numbers N_k will appear later. In our argument we will show that there exists a set $V \subseteq T^1(\mathbb{D}/\Gamma)$, a sequence $\{T_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{R}$ and a constant $c > 0$ such that for $\epsilon > 0$ sufficiently small we have for all $k \in \mathbb{N}_0$:

$$\operatorname{sep}(T_k, \epsilon, V) \geq c\ N_k$$

The set V and the constant c will be defined later in the argument. The sequence $\{T_k\}_{k\in\mathbb{N}_0}$ will be defined as $T_k := k\log 2$ and the reason for this choice will become apparent in the course of our discussion. To show that the above inequality holds, we will construct for $k \in \mathbb{N}_0$ sets U_k, which will be shown to be (T_k, ϵ)-separated subsets of V for ϵ small enough. Once we have done this, it will follow that:

$$\begin{aligned} h_S &\geq \lim_{\epsilon \to 0} \limsup_{T\to\infty} \frac{\log \operatorname{sep}(T, \epsilon, V)}{T} \\ &\geq \lim_{\epsilon \to 0} \limsup_{k\to\infty} \frac{\log \operatorname{sep}(T_k, \epsilon, V)}{T_k} \\ &\geq \lim_{\epsilon \to 0} \limsup_{k\to\infty} \frac{\log(cN_k)}{T_k} \\ &= \lim_{\epsilon \to 0} \limsup_{k\to\infty} \frac{\log(cN_k)}{k\log 2} \\ &= \lim_{\epsilon \to 0} \limsup_{k\to\infty} \frac{\log(N_k)}{k\log 2} \\ &= \lim_{\epsilon \to 0} h_c \\ &= h_c \end{aligned} \tag{4.7}$$

Passing to the conjugate Γ^*

We have been working predominantly with the Poincaré disc model $\mathbb{D}$, but in this proof we want to benefit from the advantages of the upper half-space model $\mathbb{H}$. Therefore we will consider a suitable conjugate of the group Γ. This conjugate will act on $\mathbb{H}$ and will be chosen in such a way as to assure that its limit set has particularly nice geometric properties.

First we conjugate Γ by a rotation f_r. Since Γ is of the second kind, the complement $S^1 \setminus L(\Gamma)$ of its limit set contains an open interval. So we can choose f_r in such a way that for some $\phi \in (0, \pi)$ the conjugate $f_r \Gamma f_r^{-1}$ has a limit point at $e^{\phi i}$ and $e^{-\phi i}$ while for

$x \in (-\phi, \phi)$ the point e^{xi} is not a limit point. Then in particular e^{xi} is not a limit point for $x = 0$. Having done that we conjugate by the standard map:

$$\varphi : \mathbb{D} \to \mathbb{H}$$

$$\varphi : z \mapsto \frac{-i(z+1)}{(z-1)}$$

Note that φ takes the interval $(e^{-\phi i}, e^{\phi i})$ in S^1 to an interval in $\mathbb{R} \cup \{\infty\}$ containing ∞, i.e. a set of the form $(-\infty, -w) \cup (w, \infty)$ for some $w > 0$. The limit set of the conjugate $\varphi f_r \Gamma f_r^{-1} \varphi^{-1}$ is a subset of the interval $[-w, w]$ and both $-w$ and w belong to this limit set. Next we conjugate by the map:

$$f_w : \mathbb{H} \to \mathbb{H}$$

$$f_w : z \mapsto z + w$$

and finally by the map:

$$f_\lambda : \mathbb{H} \to \mathbb{H}$$

$$f_\lambda : z \mapsto 10 \cdot \frac{z}{2w} = \frac{5}{w} z$$

In this way we obtain the conjugate $\Gamma^* := f \Gamma f^{-1}$, where the map f is given by:

$$f : \mathbb{D} \to \mathbb{H}$$

$$f := f_\lambda \circ f_w \circ \varphi \circ f_r$$

The limit set of the conjugate Γ^*, which is simply equal to $fL(\Gamma)$, lies in the interval $[0, 10]$ and both endpoints of this interval are limit points of Γ^*.

Observation 4. This means, in particular, that $Hull(L(\Gamma^*))$, the convex hull of the limit set of Γ^*, lies in the closed half-plane bounded by the geodesic whose endpoints at infinity are 0 and 10, and this geodesic belongs to $Hull(L(\Gamma^*))$.

Notice that all transformations by which we have conjugated preserve the hyperbolic distances and thus so does f. Moreover, since f_r is a rotation, φ is Lipschitz on $L(f_r \Gamma f_r^{-1})$, f_w is a translation and f_λ is a dilation, f does not change the upper box-counting dimension of the respective limit sets, by which we mean that:

$$\overline{\dim}_{\mathrm{B}}(L(\Gamma^*)) = \overline{\dim}_{\mathrm{B}}(L(\Gamma))$$

Considering the conjugate Γ^* can be thought of as viewing Γ from a different perspective. Nevertheless, we will distinguish between Γ and Γ^* so that it is absolutely clear that this switching of perspective has no influence on our result. The above remark and a result of Falk and Matsuzaki from [FalkMatsuzaki] imply that:

$$h_c(\Gamma) = \overline{\dim}_{\mathrm{B}}(L(\Gamma)) = \overline{\dim}_{\mathrm{B}}(L(\Gamma^*)) = h_c(\Gamma^*)$$

Moreover, Falk and Matsuzaki have shown in [FalkMatsuzaki] that:

$$h_c(\Gamma^*) = \limsup_{k\to\infty} \frac{\log N_k}{k \log 2}$$

Here, for $k \in \mathbb{N}_0$, N_k denotes the number of rectangles of the form $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{(k+1)}}, \frac{1}{2^k)}]$, $m \in \mathbb{Z}$, in the stripe $\mathbb{R} \times [\frac{1}{2^{(k+1)}}, \frac{1}{2^k)}]$ which intersect the convex hull $Hull(L(\Gamma^*))$. Strictly speaking, in the paper of Falk and Matsuzaki, one defines the number $N_k(G)$ for an arbitrary Kleinian group G and there it lies between the number of rectangles which intersect $Hull(L(\Gamma^*))$ and $\frac{1}{n}$ of this number, where n is the dimension of the hyperbolic space on which G acts. Yet, this difference has no influence on the limit superior above, so for our purposes we can use the simplified definition of N_k.

The mesh

Let us take a closer look at the mesh which we use in the definition of N_k. It consists of rectangles of the form $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{(k+1)}}, \frac{1}{2^k)}]$, where $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. This mesh, or rather part of it, is depicted in Figure 4.5.

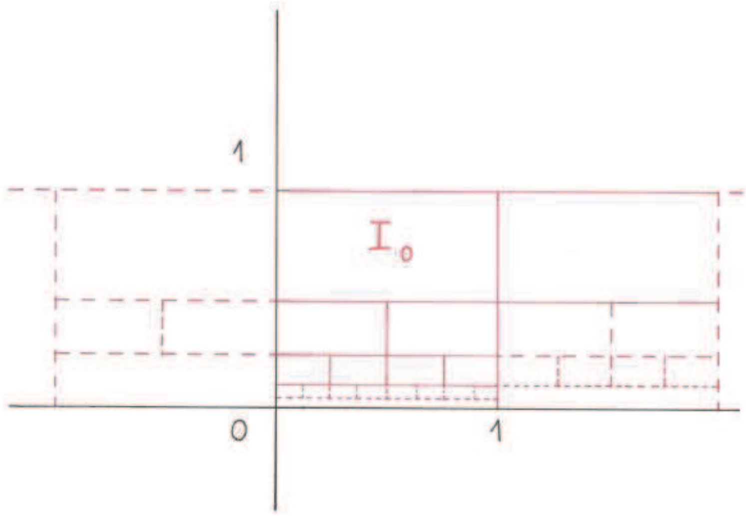


Figure 4.5:

Notice that:

- Every rectangle in this mesh can be mapped to the rectangle $I_0 := [0, 1] \times [\frac{1}{2}, 1]$ by a hyperbolic isometry. (The translation $z \mapsto z - \frac{m}{2^k}$ will take it to a rectangle which intersects the imaginary line with its left vertical side. Then, by applying the dilation $z \mapsto 2^k z$, we will obtain the rectangle I_0.)

- The hyperbolic length of a vertical side of each rectangle is equal to $\log 2$
- The hyperbolic length of the top horizontal side of each rectangle is equal to 1 (This is not to be confused with the hyperbolic distance between the two top vertices!)
- The hyperbolic length of the bottom horizontal side of each rectangle is equal to 2
- The hyperbolic distance between the two top vertices of each rectangle is equal to $d_{\mathbb{H}}(i, 1+i)$

As already mentioned before, the sequence $\{T_k\}_{k\in\mathbb{N}_0}$ will be defined as:

$$T_k := k \log 2$$

In view of the described properties of our mesh, this definition can be interpreted immediately. The value of T_k is the hyperbolic distance between the horizontal line $\mathbb{R} \times \{1\}$ and the horizontal line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$, i.e. the top of the stripe $\mathbb{R} \times [\frac{1}{2^{(k+1)}}, \frac{1}{2^{k)}}]$. So T_k tells us how deep, in hyperbolic sense, this stripe lies in the mesh.

Now, after this preparatory work, we are ready to start constructing the sets U_k. These sets will be constructed in several steps. We will start by defining preliminary sets $\widetilde{U}_k$ in $T^1\mathbb{H}$. Then we will restrict these sets to subsets $\widetilde{U}_k^*$, and finally map to $T^1\mathbb{D}$ and project to $T^1(\mathbb{D}/\Gamma)$ to obtain the desired sets U_k. In the course of this construction, we will also give the definition of the set V. From now on we will work with a fixed $k \in \mathbb{N}_0$.

Constructing the sets $\widetilde{U}_k$

Consider the set of all mesh-boxes of the form $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, $m \in \mathbb{Z}$ in the stripe $\mathbb{R} \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ which intersect the convex hull $Hull(L(\Gamma^*))$. Denote this set by N_k^0. As remarked before, $L(\Gamma^*)$ lies in the interval $[0, 10]$, so $Hull(L(\Gamma^*))$ can only intersect the mesh-boxes between the vertical lines $\{0\} \times \mathbb{R}$ and $\{10\} \times \mathbb{R}$. Thus:

$$|N_k^0| = N_k \leq 10 \cdot 2^k \tag{4.8}$$

We will now choose a subset N_k^1 of N_k^0 in such a way that between any two mesh-boxes in N_k^1 there will be at least five mesh-boxes not in N_k^1. At the same time, we will make this reduction in such a way as to guarantee that the size of N_k^1 has not decreased too much compared to that of N_k^0. In particular we will have that:

$$|N_k^1| \geq \frac{|N_k^0|}{6} \tag{4.9}$$

The reduction from N_k^0 to N_k^1 is very simple. Put first $N_k^1 := N_k^0$. Starting from the left, i.e from the box $[0, \frac{1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, we look for the first box in N_k^0. Say this first box is $[\frac{m_0}{2^k}, \frac{m_0+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$. For each $i \in \{1, 2, 3, 4, 5\}$, check if the box $[\frac{m_0+i}{2^k}, \frac{m_0+i+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ belongs to N_k^0. If it does, redefine N_k^1 by removing this box;

otherwise keep N_k^1 unchanged. Then find the first box $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ in N_k^0 with $m \geq m_0 + 6$ and put $m_1 := m$. Once again, for each $i \in \{1, 2, 3, 4, 5\}$, if the box $[\frac{m_1+i}{2^k}, \frac{m_1+i+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ belongs to N_k^0, redefine N_k^1 by removing this box and then find the $m_2 \geq m_1 + 6$. Continue until all the boxes in N_k^0 have been considered. Because of (4.8) this procedure will certainly stop.

It is clear from the construction that between any two mesh-boxes in N_k^1 there will be at least five mesh-boxes not in N_k^1 and that the inequality (4.9) is satisfied.

Now we can proceed to defining the preliminary sets $\widetilde{U}_k$. To each element of N_k^1 we will associate a vector v in $T^1\mathbb{H}$. This vector v will have the property that the endpoints at infinity of the geodesic β_v determined by v will belong to the limit set $L(\Gamma^*)$.

Let $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ be a mesh-box in N_k^1.

Observation 5. Observe that, since this mesh-box intersects $Hull(L(\Gamma^*))$, the interval $[\frac{m-1}{2^k}, \frac{m+2}{2^k}]$ must contain some point ξ in $L(\Gamma^*)$. The reason why $[\frac{m-1}{2^k}, \frac{m+2}{2^k}]$ must contain a limit point becomes perfectly clear if we look at Figure 4.6. If there were no points of $L(\Gamma^*)$ in the interval $[\frac{m-1}{2^k}, \frac{m+2}{2^k}]$, then the geodesic whose endpoints at infinity are $\frac{m-1}{2^k}$ and $\frac{m+2}{2^k}$ would separate the mesh-box $[\frac{m}{2^k}, \frac{m+1}{2^k}] \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, marked gray in Figure 4.6, from $Hull(L(\Gamma^*))$ since the distance between $\frac{m}{2^k} + \frac{1}{2^{k+1}}$ and $\frac{m}{2^k} + \frac{1}{2^k}i$ is clearly less than $\frac{3}{2^{k+1}}$.

Choose any point ξ satisfying $\xi \in L(\Gamma^*) \cap [\frac{m-1}{2^k}, \frac{m+2}{2^k}]$ and put $\xi_+ := \xi$.

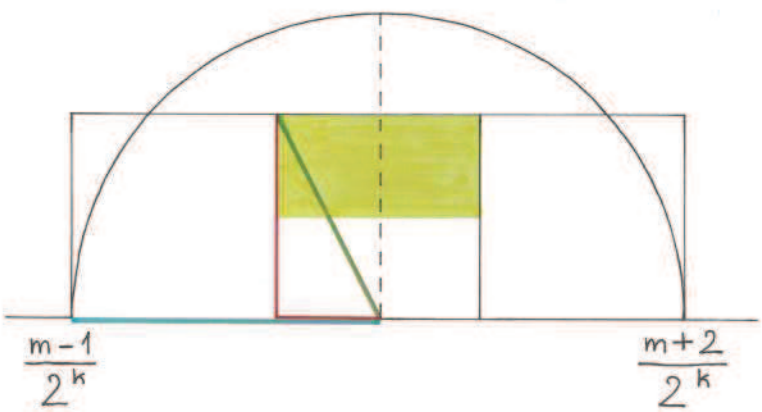


Figure 4.6:

Then we distinguish between two cases:

- If $\xi_+ \in [0,5)$ put $\xi_- := 10$.
- If $\xi_+ \in [5,10]$ put $\xi_- := 0$.

Let β denote the geodesic with negative endpoint at infinity ξ_- and positive endpoint at infinity ξ_+. Notice that β intersects the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$ in two points, one closer to ξ_- and one closer to ξ_+.

Observation 6. Near ξ_+ geodesic β intersects the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$ inside the interval $[\frac{m-2}{2^k}, \frac{m+3}{2^k}] \times \left\{\frac{1}{2^k}\right\}$. To see this, observe that if a geodesic with an endpoint at infinity $\xi \in [\frac{n}{2^k}, \frac{n+1}{2^k}]$ intersects the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$, then near ξ it intersects this line in the interval $[\frac{n-1}{2^k}, \frac{n+2}{2^k}] \times \left\{\frac{1}{2^k}\right\}$. Taking a look at Figure 4.7 will help to appreciate this fact. Actually such a geodesic intersects this line in an even smaller interval, which is marked in Figure 4.7 with a thick line, but for our proof the larger interval suffices.

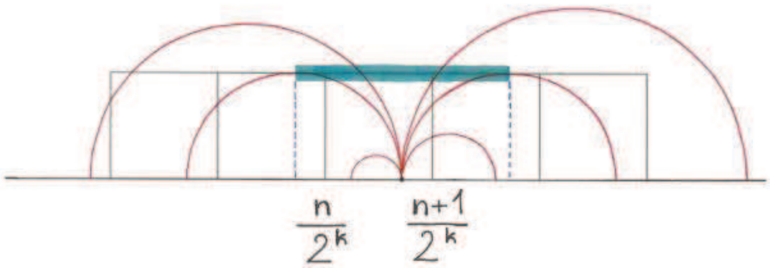


Figure 4.7:

Denote by x_β the point where β intersects the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$ near ξ_+. We want to point out at this moment that, due to the definition of N_k^1, for any two distinct elements of N_k^1, the Euclidean distance between the corresponding points x_{β_1} and x_{β_2} will be at least $\frac{1}{2^k}$.

Observation 7. Consequently the hyperbolic distance betweeen x_{β_1} and x_{β_2} will be at least $d_{\mathbb{H}}(i, 1+i)$.

Now let y_β denote the unique point on β between ξ_- and x_β such that:

$$d(x_\beta, y_\beta) = k \log 2 =: T_k$$

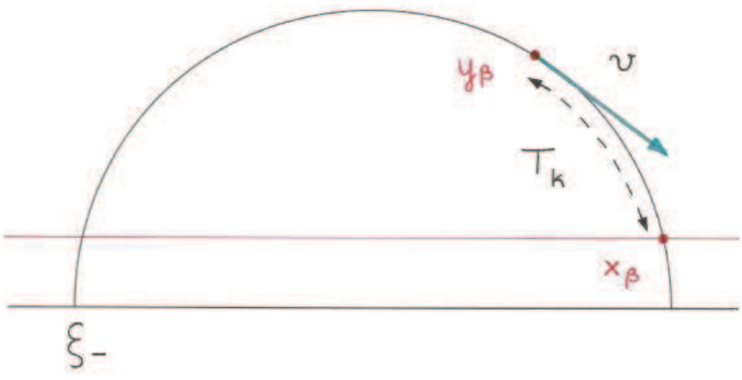


Figure 4.8:

Finally define v to be the vector in $T^1\mathbb{H}$ tangent to β at y_β, see Figure 4.8. So for each mesh-box in N_k^1 we have constructed a vector v. Let $\widetilde{U}_k$ be the set consisting of all these vectors v, one for each mesh-box in N_k^1. In this way we have defined $\widetilde{U}_k \subseteq T^1\mathbb{H}$ which satisfies:

$$\#\widetilde{U}_k = \#N_k^1$$

Location of the sets $\widetilde{U}_k$

The following observation will prepare the way for defining the set V and restricting the sets $\widetilde{U}_k$. We have remarked before that the number T_k measures how deep the stripe $\mathbb{R} \times [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ lies in the mesh and more precisely how far, in hyperbolic sense, the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$ lies from the line $\mathbb{R} \times \{1\}$. This already suggests that for any vector $v \in \widetilde{U}_k$ its base, i.e. the point y_β, should be located, in some sense, near the line $\mathbb{R} \times \{1\}$. We will now show that this is indeed the case. Namely, we will show that there exists a constant d_0, independent of k, such that the base point of every $v \in \widetilde{U}_k$ lies in the hyperbolic d_0-neighborhood of the line $\mathbb{R} \times \{1\}$.

First note that for each v in $\widetilde{U}_k$ the geodesic β corresponding to v intersects the line $\mathbb{R} \times \{1\}$ in two points. One point of intersection lies closer to the negative point at infinity of β and the other point of intersection lies closer to the positive endpoint at infinity. Let ξ_+, ξ_-, x_β and y_β be as in the construction of v. Denote by z_β the point of intersection of β with the line $\mathbb{R} \times \{1\}$ which lies near ξ_+. Let x'_β and z'_β be the points satisfying:

$$\mathrm{Im}(x'_\beta) = \mathrm{Im}(x_\beta)$$

$$\mathrm{Im}(z'_\beta) = \mathrm{Im}(z_\beta)$$

$$\mathrm{Re}(x'_\beta) = \mathrm{Re}(z'_\beta) = \xi_+$$

That is, x'_β and z'_β are points obtained by projecting x_β and z_β respectively, horizontally on the line $\{\xi_+\} \times \mathbb{R}$. It might be helpful to consult Figure 4.9, which illustates the situation just described.

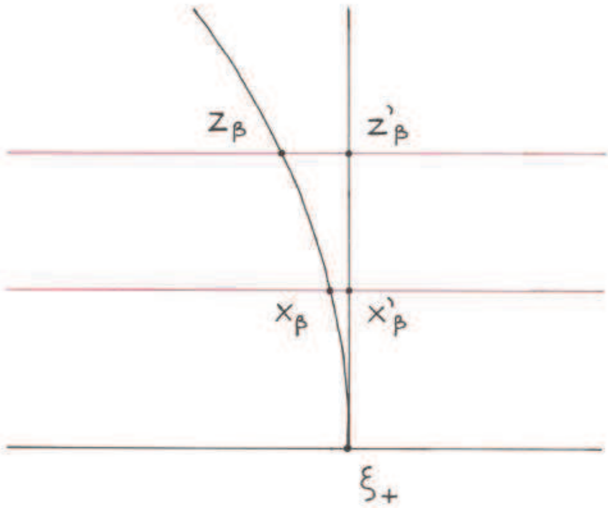


Figure 4.9:

Put $d_1 := d(z_\beta, z'_\beta)$ and $d_2 := d(x_\beta, x'_\beta)$. Also let l denote the length of the subarc of β between z_β and x_β. Since β is a geodesic we have $l = d(z_\beta, x_\beta)$. Moreover, $d(z'_\beta, x'_\beta) = T_k$. Using the triangle inequality we obtain:

$$l \leq d_1 + T_k + d_2$$

$$T_k \leq d_1 + l + d_2$$

Rearranging gives:

$$l - (d_1 + d_2) \leq T_k \leq l + (d_1 + d_2)$$

Recall that y_β was defined as the point on β which lies between ξ_- and x_β with $d(x_\beta, y_\beta) = T_k$. It follows that y_β lies on the subarc of β of length $2(d_1 + d_2)$ with midpoint at z_β. So, y_β lies in the closed ball centered at z_β of radius $(d_1 + d_2)$ and thus it lies in the closed hyperbolic $(d_1 + d_2)$-neighborhood of the line $\mathbb{R} \times \{1\}$. We will now give a bound for $(d_1 + d_2)$.

Proposition 27. *There exists a constant $d_0 \geq 0$ such that for each $k \in \mathbb{N}_0$ and $v \in \widetilde{U}_k$ the base point of v lies in the d_0-neighborhood of the line $\mathbb{R} \times \{1\}$.*

Of course d_1 and d_2 depend on k and the vector $v \in \widetilde{U}_k$ which we are considering. The distance d_1 is determined by the pair (ξ_-, ξ_+), so we will write $d_1(\xi_-, \xi_+)$, while d_2 is determined by the triple (ξ_-, ξ_+, k), and we will write $d_2(\xi_-, \xi_+, k)$. Here we extend the definitions of d_1 and d_2 so that their definitions are not restricted to the values of ξ_-, ξ_+ and the fixed k which appear in the construction of the sets $\widetilde{U}_k$, but $d_1(\xi_-, \xi_+)$ and $d_2(\xi_-, \xi_+, k)$ are defined in the same way as in the construction. Also, whenever it is necessary to distinguish between points x'_β, z_β etc. corrresponding to different geodesics β, we will write $x'(\xi_-, \xi_+, k)$, $z(\xi_-, \xi_+)$, etc. For ξ_- and ξ_+ which appear in the construction of $\widetilde{U}_k$, we will give a bound for $d_1(\xi_-, \xi_+)+d_2(\xi_-, \xi_+, k)$ independent of ξ_-, ξ_+ and k.

To give the bound for $d_1(\xi_-, \xi_+) + d_2(\xi_-, \xi_+, k)$, we need to make two observations:

Observation 8. For $k \in \mathbb{N}_0$ and pairs (ξ_-, ξ_+) for which $z(\xi_-, \xi_+)$ is defined, one has:

$$d_2(\xi_-, \xi_+, k) \leq d_1(\xi_-, \xi_+)$$

The situation is illustrated in Figure 4.10. The easiest way to see that the above inequality holds is to apply the map:

$$h : \mathbb{H} \to \mathbb{H}$$

$$h : z \mapsto \frac{z - \xi_+}{2^k} + \xi_+$$

The map h is just a dilation and it is a hyperbolic isometry. So we have:

$$d(z(\xi_-, \xi_+), z'(\xi_-, \xi_+)) = d(h(z(\xi_-, \xi_+)), h(z'(\xi_-, \xi_+)))$$

But $h(z'(\xi_-, \xi_+)) = x'(\xi_-, \xi_+, k)$, while $h(z(\xi_-, \xi_+))$ is a point on the line $\mathbb{R} \times \left\{\frac{1}{2^k}\right\}$ such that $x(\xi_-, \xi_+, k)$ lies between $h(z(\xi_-, \xi_+))$ and $x'(\xi_-, \xi_+, k)$, which implies:

$$d(h(z(\xi_-, \xi_+)), x'(\xi_-, \xi_+, k)) \geq d(x(\xi_-, \xi_+, k), x'(\xi_-, \xi_+, k))$$

This can be noticed easily once one recalls the fact that hyperbolic balls are Euclidean balls with different centers. A ball of hyperbolic radius $d(x(\xi_-, \xi_+, k), x'(\xi_-, \xi_+, k))$ with center at $x'(\xi_-, \xi_+, k)$, which has been shown in Figure 4.10, clearly cannot contain the point $h(z(\xi_-, \xi_+))$.

Observation 9. For the pairs (ξ_-, ξ_+) from the construction of the sets $\widetilde{U}_k$ one has:

$$d_1(\xi_-, \xi_+) \leq d_1(0, 5)$$

To see this first notice the trivial fact that whenever $|\xi_1 - \xi_2| = 5$ we have:

$$d_1(\xi_1, \xi_2) = d_1(0, 5)$$

Fix a pair (ξ_-, ξ_+) which is admissible in the construction of sets $\widetilde{U}_k$. Let ξ_* be the point between ξ_- and ξ_+ with $|\xi_+ - \xi_*| = 5$. Then, since $|\xi_+ - \xi_-| \geq 5$, the point $z(\xi_-, \xi_+)$

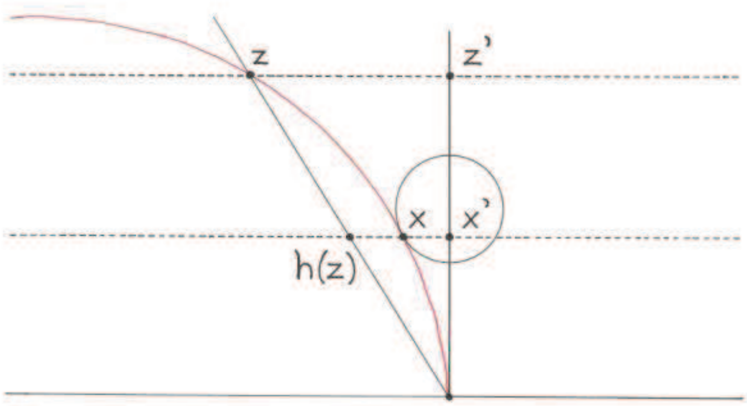


Figure 4.10:

lies between $z'(\xi_-, \xi_+) = z'(\xi_*, \xi_+)$ and $z(\xi_*, \xi_+)$. The possibility that $\xi_* = \xi_-$ and thus $z(\xi_-, \xi_+) = z(\xi_*, \xi_+)$ is also allowed. Therefore:

$$d_1(\xi_-, \xi_+) \leq d_1(\xi_*, \xi_+) = d_1(0.5)$$

Combining our two observations, we deduce that for the pairs (ξ_-, ξ_+) which appear in the construction of the sets $\widetilde{U}_k$, we have:

$$d_1(\xi_-, \xi_+) + d_2(\xi_-, \xi_+, k) \leq 2d_1(\xi_-, \xi_+) \leq 2d_1(0, 5)$$

Put $d_0 := 2d_1(0, 5)$; this is the bound which we were looking for.

Defining the set V and the sets U_k

We are now only one step away from defining the set V and the sets U_k. Before we define them, we introduce a preliminary set $\widetilde{V}$ and reduce the sets $\widetilde{U}_k$ to subsets $\widetilde{U}_k^*$.

Consider the set:

$$\mathcal{K} := N_{d_0}(\mathbb{R} \times \{1\}) \ \cap \ Hull(L(\Gamma^*))$$

where $N_{d_0}(\cdot)$ denotes the closed hyperbolic d_0-neighborhood and the line $\mathbb{R} \times \{1\}$ is naturally interpreted as the set:

$$\{a + bi \in \mathbb{C} : a \in \mathbb{R}, b = 1\}$$

Let $\widetilde{V}$ be the set of all vectors v in $T^1\mathbb{H} \cap ((L(\Gamma^*) \times L(\Gamma^*) \setminus diag) \times \mathbb{R})$ with base in $\mathcal{K}$, so formally:

$$\widetilde{V} := \pi_b^{-1}(\mathcal{K}) \ \cap \ ((L(\Gamma^*) \times L(\Gamma^*) \setminus diag) \times \mathbb{R})$$

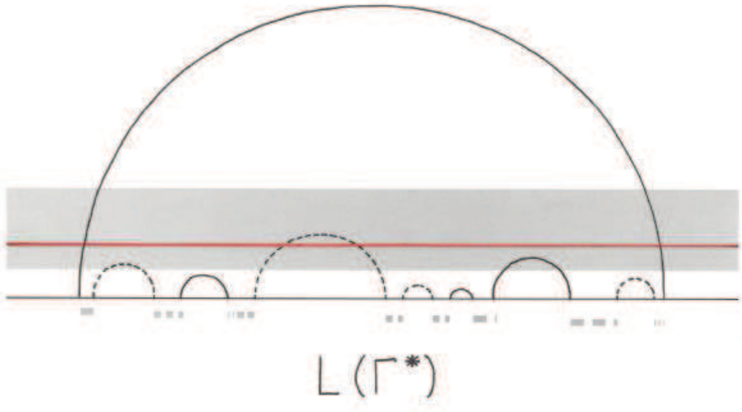


Figure 4.11:

Recall that $\mathbb{D}/\Gamma$ has bounded injectivity radius, see Observation 1. Let $\tau > 0$ be a lower bound for the injectivity radius of $\mathbb{D}/\Gamma$. Fix some finite cover $B_1, \dots, B_{n_0}$ of the set $\mathcal{K}$ by closed hyperbolic balls of radius $\frac{\tau}{3}$. The fact that $\mathcal{K}$ is bounded, which is clear in view of Observation 4, implies that such a cover exists. For each $k \in \mathbb{N}_0$ choose some $i(k)$ satisfying:

$$\#(\widetilde{U}_k \ \cap \ (\pi_b^{-1}(B_{i(k)}))) = \sup_{j \in \{1,..,n_0\}} \left\{ \ \#(\widetilde{U}_k \ \cap \ (\pi_b^{-1}(B_j))) \ \right\}$$

Then for each $k \in \mathbb{N}_0$ define the subset $\widetilde{U}_k^*$ of $\widetilde{U}_k$ by:

$$\widetilde{U}_k^* := \widetilde{U}_k \ \cap \ (\pi_b^{-1}(B_i))$$

From the construction it is clear that:

$$\widetilde{U}_k^* \subseteq \widetilde{U}_k \subseteq \widetilde{V}$$

and that:

$$\#\widetilde{U}_k^* \geq \frac{1}{n_0} \#\widetilde{U}_k$$

where n_0 is the size of the fixed cover of $\mathcal{K}$.

Now we can finally define the sets V and U_k. Recall that we obtained the group Γ^* by conjugating by the map f in the following way:

$$\Gamma^* := f \Gamma f^{-1}$$

where the map f given by:

$$f : \mathbb{D} \to \mathbb{H}$$

$$f := f_\lambda \circ f_w \circ \varphi \circ f_r$$

is the composition of a rotation f_r, the standard map $\varphi : \mathbb{D} \to \mathbb{H}$, a translation f_w and a dilation f_λ. We use the differential:

$$Df^{-1} : T^1\mathbb{H} \to T^1\mathbb{D}$$

of f^{-1} to map the sets $\widetilde{V}$ and $\widetilde{U}_k^*$ into $T^1\mathbb{D}$ and finally by applying the canonical projection:

$$\pi : T^1\mathbb{D} \to T^1(\mathbb{D}/\Gamma)$$

we obtain the set V and for $k \in \mathbb{N}_0$ the sets U_k, so formally:

$$V := \pi(Df^{-1}(\widetilde{V}))$$

$$U_k := \pi(Df^{-1}(\widetilde{U}_k^*))$$

With these definitions we have:

$$U_k \subseteq V$$

Due to the definition of the sets $\widetilde{U}_k^*$, it follows, from the fact that the injectivity radius of $\mathbb{D}/\Gamma$ is bounded from below by τ, that the map π is injective on $Df^{-1}(\widetilde{U}_k^*)$. So using (4.9) we deduce that for each $k \in \mathbb{N}_0$ we have:

$$\#U_k = \#\widetilde{U}_k^* \geq \frac{1}{n_0} N_k$$

Therefore, all that remains to be shown is that for ϵ small enough, for each $k \in \mathbb{N}_0$, the set U_k is (T_k, ϵ)-separated; in particular we will be assuming $\epsilon < \min(\frac{\tau}{3}, d_{\mathbb{H}}(i, i+1))$.

The sets U_k are (T_k, ϵ)-separated

Proposition 28. *If the injectivity radius of the manifold $\mathbb{D}/\Gamma$ is bounded below by τ, then for $k \in \mathbb{N}_0$ and $\epsilon < \min(\frac{\tau}{3}, d_{\mathbb{H}}(i, i+1))$, the set $U_k \subseteq T^1(\mathbb{D}/\Gamma)$ is (T_k, ϵ)-separated.*

Proof. Poposition 28. Fix $k \in \mathbb{N}_0$ and assume that $\epsilon < \min(\frac{\tau}{3}, d_{\mathbb{H}}(i, i+1))$. Let u_1 and u_2 be two arbitrary vectors in U_k. Suppose that u_1 and u_2 are not (T_k, ϵ)-separated. We will show that this leads to a contradiction. Consider the vectors v_1, v_2 and geodesics β_1, β_2 from the construction of the sets U_k which correspond to u_1 and u_2 respectively, so that we have:

$$u_1 = \pi(Df^{-1}v_1) \quad \text{and} \quad u_2 = \pi(Df^{-1}v_2)$$

Also parametrise β_1 and β_2 so that:

$$\beta_1(0) = \pi_b(v_1) \quad \text{and} \quad \beta_2(0) = \pi_b(v_2)$$

The fact that u_1 and u_2 are not (T_k, ϵ)-separated by definition means that:

$$\forall t \in [0, T_k] \qquad d(\pi_b(g^t u_1), \pi_b(g^t u_2)) \leq \epsilon \tag{4.10}$$

where $\{g^t\}_{t\in\mathbb{R}}$ as usual denotes the geodesic flow on $T^1(\mathbb{D}/\Gamma)$. Since f is an isometry between $\mathbb{D}$ and $\mathbb{H}$, for $j \in \{1, 2\}$, the geodesic $f^{-1} \circ \beta_j$ is a lift to $\mathbb{D}$ of the path:

$$t \mapsto \pi_b(g^t u_i)$$

So (4.10) implies that for all $t \in [0, T_k]$ there exists $\gamma_t \in \Gamma$ such that:

$$d(f^{-1} \circ \beta_1(t), \gamma_t(f^{-1} \circ \beta_2(t))) \leq \epsilon$$

But, since f is an isometry, this implies that for all $t \in [0, T_k]$ we have:

$$d(\beta_1(t), (f \circ \gamma_t \circ f^{-1}) \circ \beta_2(t)) \leq \epsilon$$

We now claim that in fact:

$$\forall t \in [0, T_k] \qquad d(\beta_1(t), \beta_2(t)) \leq \epsilon \tag{4.11}$$

First we will establish that this holds for $t = 0$. From the definition of $\widetilde{U}_k^*$ it follows that:

$$d(\beta_1(0), \beta_2(0)) \leq \frac{2\tau}{3}$$

So if some $\gamma \in \Gamma^*$ satisfies:

$$d(\beta_1(0), \gamma \circ \beta_2(0)) \leq \epsilon$$

Then by the triangular inequality:

$$d(\gamma \circ \beta_2(0), \beta_2(0)) \;\leq\; d(\gamma \circ \beta_2(0), \beta_1(0)) \;+\; d(\beta_1(0), \beta_2(0)) \;\leq\; \epsilon + \frac{2\tau}{3} \;\leq\; \tau$$

Since the injectivity radius of $(\mathbb{D}/\Gamma)$ is bounded from below by τ, the distance between two points which lie in the same orbit under the action of Γ must be strictly greater than τ. Because Γ^* is a conjugate of Γ, the same is true for the orbits under the action of Γ^*. Therefore we must have $\gamma = id$, and hence:

$$d(\beta_1(0), \beta_2(0)) \leq \epsilon$$

Let t_* be defined as follows:

$$t_* := \sup\{t_0 \in [0, T_k] : \; \forall t \in [0, t_0] \quad d(\beta_1(t), \beta_2(t)) \leq \epsilon\}$$

We will show that one necessarily has $t_* = T_k$. If we had $t_* < T_k$ it would follow from the definition of t_* that for every $\rho > 0$ there exists a $t' \in [t_*, t_* + \rho] \cap [t_*, T_k]$ satysfying:

$$d(\beta_1(t'), \gamma \circ \beta_2(t')) \leq \epsilon \qquad \text{for some} \quad \gamma \in \Gamma^*, \;\; \gamma \neq id$$

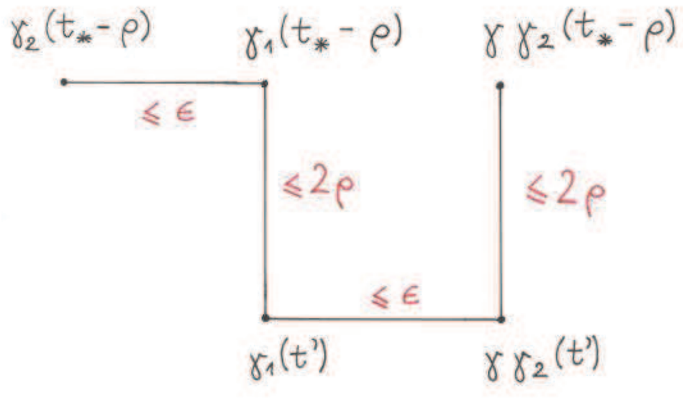


Figure 4.12:

It would hold in particular for some $\rho \leq \min(\frac{\tau}{12}, \frac{t_*}{2})$ and let us assume that ρ satisfies this condition.

Then, by applying the triangle inequality several times, as indicated in Figure 4.12, we would obtain:

$$d(\beta_2(t_* - \rho), \gamma \circ \beta_2(t_* - \rho))$$
$$\leq d(\beta_2(t_*-\rho), \beta_1(t_*-\rho)) + d(\beta_1(t_*-\rho), \beta_1(t')) + d(\beta_1(t'), \gamma\circ\beta_2(t')) + d(\gamma\circ\beta_2(t'), \gamma\circ\beta_2(t_*-\rho))$$
$$\leq \epsilon + 2\rho + \epsilon + 2\rho$$
$$\leq \frac{\tau}{3} + \frac{\tau}{6} + \frac{\tau}{3} + \frac{\tau}{6} = \tau$$

This is a contradiction. So we must indeed have $t_* = T_k$, which means that:

$$\forall t \in [0, T_k) \qquad d(\beta_1(t), \beta_2(t)) \leq \epsilon$$

Now, by applying the above argument with T' replaced by T_k and with $t_* - \rho$ replaced by $T_k - \min(\frac{T_k}{2}, \frac{\tau}{6})$, we again obtain a contradiction, and deduce that:

$$d(\beta_1(T_k), \beta_2(T_k)) \leq \epsilon \tag{4.12}$$

holds as well and therefore the statement (4.11) is true. But then, since β_1 and β_2 are geodesics from the construction of the sets U_k, the inequality (4.12) contradicts the choice of the points $x_{\beta_1} = \beta_1(T_k)$ and $x_{\beta_2} = \beta_2(T_k)$. Recall that the distance between x_{β_1} and x_{β_1} is at least $d_{\mathbb{H}}(i, i+1)$, as noted in Observation 7. This finishes the proof of Proposition 28. □

This finishes the proof of Proposition 26. Combining Proposition 25 and Proposition 26 yields Theorem 7.

4.6 Geodesic flow entopy and Bowen-Dinaburg entropy

We have already remarked in the introduction to this chapter that for a geometric Schottky group Γ the geodesic flow entropy $h_S(\Gamma)$ is in fact equal to the Bowen-Dinaburg entropy of the geodesic flow restricted to the set $((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$ defined relative to a certain metric on the unit tangent bundle $T^1(\mathbb{D}/\Gamma)$. In this section we will show that this is indeed the case. We will then explain how this fact allows us to interpret our result from Section 4.5 in the context of the results of Handel and Kitchens, and Otal and Peigné, and we will discuss its consequences.

Bowen-Dinaburg entropy

We shall start by recalling the definition of the Bowen-Dinaburg entropy for transformations of non-compact metric spaces, see [Bowen1971] and [Bowen1973]. The Bowen-Dinaburg entropy of a transformation ϕ of a non-compact space X relative to a metric $\widetilde{d}$ on X, which we denote here by $h_{\widetilde{d}}$, is defined by:

$$h_{\widetilde{d}} := \sup_{K \in \mathcal{K}} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log(s(n,\epsilon,K))}{n}$$

Here $\mathcal{K}$ denotes the family of all subsets of X which are compact with respect to the metric $\widetilde{d}$, while $s(n,\epsilon,K)$ is defined by:

$$s(n,\epsilon,K) := \sup\left\{\#U : U \subseteq K,\ U \text{ is } (n,\epsilon)^B\text{-separated}\right\}$$

The property of being $(n,\epsilon)^B$-separated, which appears in the above definition, is similar to but not identical with property of being (T,ϵ)-separated from Section 4.1. Namely, we say that a set U is $(n,\epsilon)^B$-separated if for any distinct points $u, v \in U$ we have:

$$\sup_{j \in [0,n-1] \cap \mathbb{Z}} \widetilde{d}(\phi^j(u), \phi^j(v)) > \epsilon$$

For the geodesic flow, or more precisely for the restriction of the geodesic flow to the set $((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, one defines Bowen-Dinaburg entropy by taking

$$X = T^1(\mathbb{D}/\Gamma) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$$

and

$$\phi = g^1$$

We are going to show that geodesic flow entropy h_S is equal to the Bowen-Dinaburg entropy $h_{\widehat{d}}$ of the geodesic flow, restricted to the set $((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, defined relative to the metric $\widehat{d}$ given by:

$$\widehat{d}(u,v) := \sup_{t \in [0,1]} d(\pi_b(g^t u), \pi_b(g^t v))$$

Equivalence of the two notions of entropy

The argument which shows that the two notions of entropy of the geodesic flow agree relies on the following two simple lemmata.

Lemma 4. *For any base-bounded set* $V \subseteq T^1(\mathbb{D}/\Gamma) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, $\epsilon > 0$ *and* $T \geq 0$, *we have:*

$$s(\lfloor T \rfloor, \epsilon, V) \leq \mathrm{sep}(T, \epsilon, V) \leq s(\lfloor T \rfloor + 1, \epsilon, V)$$

Lemma 5. *The families of sets* $\mathcal{V}_1$, $\mathcal{V}_2$ *and* $\mathcal{K}$, *with* $\mathcal{K}$ *the set of all subsets of* $T^1(\mathbb{D}/\Gamma) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$ *which are compact with respect to the metric* $\widehat{d}$, *satisfy the relationship:*

$$\mathcal{V}_2 \subseteq \mathcal{K} \subseteq \mathcal{V}_1$$

With these two lemmata we can argue as follows. From Lemma 4, we immediately obtain that for any base-bounded set $V \subseteq T^1(\mathbb{D}/\Gamma) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, $\epsilon > 0$ and $T \geq 0$ we have:

$$\log(s(\lfloor T \rfloor, \epsilon, V)) \leq \log(\mathrm{sep}(T, \epsilon, V)) \leq \log(s(\lfloor T \rfloor + 1, \epsilon, V))$$

and thus also:

$$\frac{\lfloor T \rfloor}{\lfloor T \rfloor} \cdot \frac{\log(s(\lfloor T \rfloor, \epsilon, V))}{\lfloor T \rfloor + 1} \leq \frac{\log(\mathrm{sep}(T, \epsilon, V))}{T} \leq \frac{\log(s(\lfloor T \rfloor + 1, \epsilon, V))}{\lfloor T \rfloor} \cdot \frac{\lfloor T \rfloor + 1}{\lfloor T \rfloor + 1}$$

By taking the limes superior as T tends to infinity, we obtain that:

$$\limsup_{T \to \infty} \frac{\log(\mathrm{sep}(T, \epsilon, V))}{T} = \limsup_{n \to \infty} \frac{\log(s(n, \epsilon, V))}{n}$$

Thus, we have that:

$$h_{\widehat{d}} = \sup_{K \in \mathcal{K}} \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{\log(\mathrm{sep}(T, \epsilon, K))}{T}$$

Now, since $\mathcal{K} \subseteq \mathcal{V}_1$, it follows from the definition of h_S that:

$$h_{\widehat{d}} \leq h_S$$

and since $\mathcal{V}_2 \subseteq \mathcal{K}$, it follows from the alternative version of the definition of h_S that:

$$h_S \leq h_{\widehat{d}}$$

This yields the desired equivalence of the two notions of entropy.

Proof. Lemma 4.
Let V be a base-bounded subset of $T^1(\mathbb{D}/\Gamma) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, $T \geq 0$, $\epsilon > 0$ and U a subset of V. If U is $(\lfloor T \rfloor, \epsilon)^B$-separated then it follows directly from the definitions that it is (T, ϵ)-separated as well, hence the first inequality in the statement of the lemma holds. On the other hand, if U is (T, ϵ)-separated, then it is automatically $(\lfloor T \rfloor + 1, \epsilon)^B$-separated, which yields the second inequality. $\square$

Proof. Lemma 5.
The inclusion $\mathcal{K} \subseteq \mathcal{V}_1$ is obvious. To obtain the other inclusion we need to show that any set $V \in \mathcal{V}_2$ is compact. So let us fix a set $V \in \mathcal{V}_2$ and consider an arbitrary sequence $(v_i)_{i\in\mathbb{N}}$ of vectors in V. Our aim is to show that this sequence contains a subsequence which converges to some vector $v \in V$. By the definition of $\mathcal{V}_2$, the set V is of the form $\pi_b^{-1}(B) \cap ((L(\Gamma) \times L(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma$, where B is a closed ball in $\mathbb{D}/\Gamma$ and, as remarked in Section 4.1 in Remark 5, without loss of generality we might assume that the diameter of B is significantly smaller than the injectivity radius of $\mathbb{D}/\Gamma$. So we lift B to a closed ball $\widetilde{B}$ in $\mathbb{D}$ and for each $i \in \mathbb{N}$ consider a lift $\widetilde{v}_i$ of the vector v_i satisfying $\pi_b(\widetilde{v}_i) \in \widetilde{B}$. We use the standard parametrisation of $T^1\mathbb{D}$ and for each $i \in \mathbb{N}$ denote by (ξ_i, η_i, r_i) the parameters describing the vector $\widetilde{v}_i$. We first consider the sequence $(\xi_i)_{i\in\mathbb{N}}$. Since $L(\Gamma)$ is a compact set, we can extract a subsequece which converges to some $\xi_0 \in L(\Gamma)$. We extract the corresponding subsequence of $(\widetilde{v}_i)_{i\in\mathbb{N}}$, but for simplicity we still denote these sequences by $(\xi_i)_{i\in\mathbb{N}}$ and $(\widetilde{v}_i)_{i\in\mathbb{N}}$. Notice that, since for each $i \in \mathbb{N}$ the basepoint of the vector $\widetilde{v}_i$ lies in the ball $\widetilde{B}$, there is an interval A in S^1 centered at ξ_0 and $N_0 \in \mathbb{N}$ such that, for all $i > N_0$, the point η_i lies outside the interval A. So we can extract a subsequence of $(\eta_i)_{i\in\mathbb{N}}$ which converges to some $\eta_0 \in L(\Gamma)$ and, by the previous remark, we have $\eta_0 \neq \xi_0$. We extract the corresponding subsequence of $(\widetilde{v}_i)_{i\in\mathbb{N}}$, but again keep the notation unchanged. Finally we consider the sequence of the basepoints $(\pi_b(\widetilde{v}_i))_{i\in\mathbb{N}}$ and extract a convergent subsequence, which can be done since $\widetilde{B}$ is compact. This subsequence converges to a point x in $\widetilde{B}$ lying on the geodesic with endpoints at infinity ξ_0 and η_0. We extract the corresponding subsequence $(\widetilde{v}_i)_{i\in\mathbb{N}}$ and once again keep the notation. Let $\widetilde{v}$ be a vector of the form (ξ_0, η_0, r_0) satisfying $\pi_b(\widetilde{v}) = x$ and put $v := \pi(\widetilde{v})$. Then $(\widetilde{v})_{i\in\mathbb{N}}$ converges to $\widetilde{v}$ and thus the corresponding subsequence of $(v_i)_{i\in\mathbb{N}}$ converges to v, which is a vector that clearly belongs to the set V. □

Consequences

The equivalence of the two notions of entropy allows us to interpret Theorem 7 from Section 4.5 in the context of the results of Handel and Kitchens, and Otal and Peigné metioned in the introduction to this chapter. Their results yield the following relationship:

$$\delta(\Gamma) = \sup_{\mu} h_\mu(\Gamma) = \inf_{\widetilde{d}} h_{\widetilde{d}}(\Gamma)$$

where $\delta(\Gamma)$ is the Poincaré exponent of Γ, $h_\mu(\Gamma)$ is the measure theoretical entropy with respect to a measure μ, with the supremum taken over all probability measures invariant under the flow, and $h_{\widetilde{d}}(\Gamma)$ is Bowen's metric entropy with respect to a metric $\widetilde{d}$, with the infimum taken over all metrics which induce the usual topology on $T^1(\mathbb{D}/\Gamma)$. Theorem 7 addresses the question of Otal and Peigné, who asked if there exists a metric $\widetilde{d}$ such that $h_{\widetilde{d}}(\Gamma) = \delta(\Gamma)$.

For all geometric Schottky groups Γ, it follows from our discussion in Section 4.6 and Theorem 7 that $h_{\widehat{d}}(\Gamma) = h_S(\Gamma) = h_c(\Gamma)$, where $\widehat{d}$ is the metric defined in Section 4.6. Recall that, by a theorem of Bishop and Jones [BishopJones1995], [Stratmann2004], the

Poincaré exponent of Γ is equal to the Hausdorff dimension of the radial limit set $L_r(\Gamma)$, that is:

$$\delta(\Gamma) = \dim_{\mathrm{H}}(L_r(\Gamma))$$

On the other hand, Falk and Matsuzaki show in [FalkMatsuzaki] that the convex-core entropy is equal to the upper box-counting dimension of the entire limit set $L(\Gamma)$, that is:

$$h_c(\Gamma) = \overline{\dim}_{\mathrm{B}}(L(\Gamma))$$

Thus the existence of the so called dimension gap, that is the situation that:

$$\dim_{\mathrm{H}}(L_r(\Gamma)) < \dim_{\mathrm{H}}(L(\Gamma))$$

provides cases where $h_{\widehat{d}}(\Gamma) > \delta(\Gamma)$, so that for the metric $\widehat{d}$ the infimum is not attained. On the other hand, in the case of groups for which $\dim_{\mathrm{H}}(L_r(\Gamma)) = \dim_{\mathrm{B}}(L(\Gamma))$, we have $h_{\widehat{d}}(\Gamma) = \delta(\Gamma)$. Thus in this case the answer to the question of Otal and Peigné is positive, namely the infimum is attained for the metric $\widehat{d}$, which is a very natural metric on $T^1(\mathbb{D}/\Gamma)$.

The class of groups of geometric Schottky groups contains both groups with dimension gap and groups satisfying $\dim_{\mathrm{H}}(L_r(\Gamma)) = \dim_{\mathrm{B}}(L(\Gamma))$. Among geometric Schottky groups interesting examples with dimension gap can be obtained using a theorem of Brooks-Stadlbauer [Stadlbauer2013], namely normal subgroups N of finitely generated geometric Schottky groups G with G/N a free group of order at least two, see [FalkStratmann2004]. Further interesting constructions of groups with dimension gap appear in [StratmannUrbański2007] and [FalkStratmann2004]. On the other hand, any finitely generated geometric Schottky group satisfies $\dim_{\mathrm{H}}(L_r(\Gamma)) = \dim_{\mathrm{B}}(L(\Gamma))$. An interesting direction of further research would be to determine conditions under which infinitely generated geometric Schottky groups have a dimension gap, and conditions under which they satisfy $\dim_{\mathrm{H}}(L_r(\Gamma)) = \dim_{\mathrm{B}}(L(\Gamma))$.

Chapter 5

The $\mathcal{P}$-class of Fuchsian groups

In this chapter we discuss a special class of Fuchsian groups of Schottky type, which we will call the $\mathcal{P}$-class. This class was first introduced by Peigné [Peigné2003], in a slightly more general form in the context of Kleinian groups, which motivates the appearance of the letter $\mathcal{P}$ in the name we have chosen for this class. What makes the $\mathcal{P}$-class interesting is the fact that, for a group Γ in the $\mathcal{P}$-class, the geodesic flow on the quotiend manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure and moreover, in this case, the Liouville-Patterson measure is finite. After defining the $\mathcal{P}$-class, we are going to present the proofs of these two important facts, and then obtain some consequences of these properties. We also investigate the question of when a Fuchsian group of Schottky type belongs to the $\mathcal{P}$-class. Our discussion is complemented by providing explicit methods of constructing examples of Fuchsian groups that belong to the $\mathcal{P}$-class.

5.1 Basic definitions

We start by introducing the notion of the $\mathcal{P}$-property for a geometric Schottky group. The $\mathcal{P}$-class of Fuchsian groups will be the class of those groups of Schottky type which can be expressed as geomeric Schottky groups that have the $\mathcal{P}$-property.

Definition 15. $\mathcal{P}$-property
Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$. We say that Γ has the $\mathcal{P}$-property if there exist disjoint sets $I_1, I_2 \subseteq \mathbb{Z}$ such that $I = I_1 \dot{\cup} I_2$, I_1 and I_2 are symmetric subsets of $\mathbb{Z}$ and the following conditions are satisfied:

1 . $\delta(\Gamma) > \max(\delta(\Gamma_1), \delta(\Gamma_2))$, where $\Gamma_n := \langle g_k : k \in I_n \rangle$, for $n \in \{1, 2\}$

2 . $\exists\ \omega > 0$ such that $|\xi - \eta| > \omega$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{k\in I_2} A_k}$

2'. $\exists\ \theta > 0$ such that $\angle(r_\xi, r_\eta) > \theta$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{k\in I_2} A_k}$

In the above definition, $|\xi - \eta|$ denotes the chordal distance between the points ξ and η, r_ξ the geodesic ray between the origin o and the point ξ, and $\angle(r_\xi, r_\eta)$ the internal angle between the rays r_ξ and r_η; the same notation is also used in the next definition.

Definition 16. $\mathcal{P}$**-class**
Let Γ be a subgroup of $\mathrm{Iso}^+(\mathbb{D})$. We say that Γ belongs to the $\mathcal{P}$-class if there exists a Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ and disjoint symmetric sets $I_1, I_2 \subseteq \mathbb{Z}$ such that $I = I_1 \dot\cup I_2$, $\Gamma = \langle g_k : k \in I\rangle$ and the following conditions are satisfied:

1 . $\delta(\Gamma) > \max(\delta(\Gamma_1), \delta(\Gamma_2))$, where $\Gamma_n := \langle g_k : k \in I_n\rangle$, for $n \in \{1, 2\}$

2 . $\exists\ \omega > 0$ such that $|\xi - \eta| > \omega$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{k\in I_2} A_k}$

2'. $\exists\ \theta > 0$ such that $\angle(r_\xi, r_\eta) > \theta$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{k\in I_2} A_k}$

So a group Γ in $\mathrm{Iso}^+(\mathbb{D})$ belongs to the $\mathcal{P}$-class if it is a group of Schottky type which can be expressed as a geometric Schottky group which has the $\mathcal{P}$-property.

Remark 8. The equivalence of conditions (2) and (2') in the definition of the $\mathcal{P}$-property and in the definition of the $\mathcal{P}$-class is evident.

Observation 10. Note that condition (2') implies that if the expressions of $\gamma, \gamma' \in \Gamma$ in terms of the generators in the Schottky description of Γ are of the form $\gamma = g_{k_1} \cdots g_{k_n}$ and $\gamma' = g_{j_1} \cdots g_{j_m}$ with $k_1 \in I_1$ and $j_1 \in I_2$, then the internal angle between the hyperbolic arc between o and γo and the hyperbolic arc between o and $\gamma' o$ is greater than θ.

Observation 11. In fact condition (2) is equivalent to the requirement that:

$$\overline{\bigcup_{k\in I_1} A_k} \ \cap\ \overline{\bigcup_{k\in I_2} A_k} = \emptyset$$

Condition (2) clearly implies that this intersection is empty. So it is enough to show that if condition (2) fails then the intersection must be non-empty. To see this, first observe that $\overline{\bigcup_{k\in I_1} A_k}$ has countably many boundary points. Consider a sequence $(y_n)_n$ of points in $\overline{\bigcup_{k\in I_2} A_k}$ such that for each $n \in \mathbb{N}$ there exists a boundary point x_n of $\overline{\bigcup_{k\in I_1} A_k}$ for which $|y_n - x_n| < \frac{1}{n}$. If we form a sequence $(x_n)_{n\in\mathbb{N}}$ of such points, then either some x_n appears infinitely often in this sequence, or there is a subsequence of the x_n's which converges to a point $x \in \overline{\bigcup_{k\in I_1} A_k}$. But then the corresponding subsequence of y_n's would also converge to the point x and we would then have $x \in \overline{\bigcup_{k\in I_2} A_k}$. It is also easy to see that condition $\overline{\bigcup_{k\in I_1} A_k} \ \cap\ \overline{\bigcup_{k\in I_1} A_k} = \emptyset$ is equivalent to the condition:

$$J(\Gamma_1) \cap J(\Gamma_2) = \emptyset$$

Here, $J(\Gamma)$ is the set defined in Section 3.1.1.

5.2 Constructing examples

In this section we will show how one can find examples of groups of Schottky type in the $\mathcal{P}$-class. We will give a basic construction of a geometric Schottky group which has the $\mathcal{P}$-property and then describe how by iterating this method one can obtain more interesting examples of groups in the $\mathcal{P}$-class. Importantly, all of these examples will be infinitely generated. Later we will show that all groups Γ in the $\mathcal{P}$-class the geodesic flow on their quotiend manifold is ergodic with respect to the Liouville-Patterson measure, and we will also show that their Liouville-Patterson measure is finite. So our construction will provide an interesting class of examples of infinitely generated groups with these good properties.

When looking for examples of groups in the $\mathcal{P}$-class the diffculty lies in assuring that condition (1) in the definition of the $\mathcal{P}$-class is satisfied. The main strategy for our constructions is to apply the following theorem due to Furusawa.

Theorem 8. ***Furusawa***
Let G_1 and G_2 be geometric Schottky groups with Schottky descriptions $(\{A_k\}, \{g_k\})_{k\in I_1}$ and $(\{A_k\}, \{g_k\})_{k\in I_2}$ respectively, with $I_1 \cap I_2 = \emptyset$, such that:

1. *G_1 is of divergence type*
2. *$\delta(G_1) \geq \delta(G_2)$*
3. *$\exists\ \omega > 0$ such that $|\xi - \eta| > \omega$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{k\in I_2} A_k}$*

Then $\delta(G_1 * G_2) > \delta(G_1)$. Here $G_1 * G_2$ is the geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$, where $I := I_1 \cup I_2$.

Proof. See [Furusawa1991] for the proof of the original more general result, which appears there as Theorem 1. □

Remark 9. It is easily seen that, provided condition (3) of Theorem 8 is satisfied, the geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ is a free product of the groups G_1 and G_2. This also justifies the notation $G_1 * G_2$.

Observation 12. Since finitely generated groups of Schottky type are of divergence type, Theorem 8 implies that any non-elementary finitely generated group of Schottky type belongs to the $\mathcal{P}$-class.

5.2.1 Basic construction

Let us consider two geometric Schottky groups G_1 and G_2 with Schottky descriptions $(\{A_k\}, \{g_k\})_{k\in I_1}$ and $(\{A_k\}, \{g_k\})_{k\in I_2}$ respectively, where $I_1 = \{\pm 1, \pm 2\}$ and $I_2 = \{\pm 3, \pm 4\}$, such that $\overline{\bigcup_{k\in I_1} A_k} \subseteq E_1$ and $\overline{\bigcup_{k\in I_2} A_k} \subseteq E_2$ for some disjoint open intervals $E_1, E_2 \subseteq S^1$. Assume that we have $\delta(G_1) \geq \delta(G_2)$; if this is not the case, we can simply interchange the roles of G_1 and G_2. Further, consider the subgroup $G_0 := \left\langle g_3^k g_4 g_3^{-k} : k \in \mathbb{Z} \right\rangle$ of the group G_2. Note that the group G_0 is defined analogously to the group H_0 which appeared in Section 3.4 as an example of infinitely generated geometric Schottky group. In the same way as for the group H_0, we choose for G_0 a Schottky description $(\{B_j\}, \{h_j\})_{j\in J}$ but this time we choose the indices so that $J = \mathbb{Z} - \{0, \pm1, \pm2\}$. Now we claim that the group $G := G_1 * G_0$ with Schottky description $\big(\{A_k\} \cup \{B_j\}, \{g_k\} \cup \{h_j\}\big)_{k\in I_1, j\in J}$ has the $\mathcal{P}$-property.

To justify this claim we make the following observations:

- Since G_1 is finitely generated it is of divergence type
- Since $G_0 < G_2$ we have $\delta(G_1) \geq \delta(G_2) \geq \delta(G_0)$
- Since for every $j \in J$ the interval B_j is a subset of an interval A_k for some $k \in I_2$ we have that $\overline{\bigcup_{j\in J} B_j} \subseteq E_2$ and thus there exists $\omega > 0$ such that $|\xi - \eta| > \omega$ for any $\xi \in \overline{\bigcup_{k\in I_1} A_k}$ and $\eta \in \overline{\bigcup_{j\in J} B_j}$

So we can apply Theorem 8 and thus deduce that:

$$\delta(G) > \delta(G_1) = \max(\delta(G_1)\delta(G_0))$$

But now it is clear that all the conditions in the definition of the $\mathcal{P}$-property are fulfilled and so G is an example of an infinitely generated geometric Schottky group with the $\mathcal{P}$-property.

Note that in the above construction we could replace G_0 by any other infinitely generated subgroup. Also, instead of requiring that $\overline{\bigcup_{k\in I_1} A_k} \subseteq E_1$ and $\overline{\bigcup_{k\in I_2} A_k} \subseteq E_2$, for some disjoint open intervals $E_1, E_2 \subseteq S^1$, we could simply require that condition (2) in the definition of the $\mathcal{P}$-property is satisfied, which would be a weaker assumption. The groups G_1 and G_2 could also be chosen to have more than two generators and they need not have the same number of generators.

5.2.2 Iterated construction

The construction which we described above is relatively simple. Now we will explain how one can iterate this method to obtain potentially more interesting examples of infinitely generated groups which belong to the $\mathcal{P}$-class.

We start by partitioning the circle S^1 into n ($n \in \mathbb{N}$, $n \geq 3$) open intervals of equal length which we denote $E_1, \dots, E_n$. We choose two finitely generated geometric Schottky groups G_1 and G_2 such that the intervals of their Schottky descriptions are contained in E_1 and E_2 respectively. By relabelling the intervals E_j if necessary, we can assume without loss of generality that $\delta(G_1) \geq \delta(G_2)$. Now we replace the group G_2 by an infinitely generated subgroup, its exponent of convergence will be less or equal to that of the original group, and conjugate this new group G_2 using rotations by multiples of $\frac{2\pi}{n}$ to obtain groups $G_3, \dots, G_n$ with intervals of their Schottky descriptions contained in $E_3, \dots, E_n$ respectively. Note that we have $\delta(G_2) \geq \delta(G_j)$ for $j = 3, \dots, n$. Now in the same way as in the basic construction we consider $G_1 * G_2$, which by Theorem 8 satisfies

$\delta(G_1 * G_2) > \delta(G_1)$, and deduce that this group belongs to the $\mathcal{P}$-class. We will show in Section 5.3 that all groups in the $\mathcal{P}$-class are of divergence type. With this knowledge at hand we could again appeal to Theorem 8 in the case of $(G_1 * G_2) * G_3 = G_1 * G_2 * G_3$ to obtain

$$\delta(G_1 * G_2 * G_3) > \delta(G_1 * G_2) > \delta(G_1) \geq \delta(G_2) = \delta(G_3)$$

so that the group $G_1 * G_2 * G_3$ also belongs to the $\mathcal{P}$-class and hence is of divergence type. We could then repeat this argument to finally obtain that $G_1 * \cdots * G_n$ belongs to the $\mathcal{P}$-class and we also have:

$$\delta(G_1 * \cdots * G_k) > \delta(G_1 * \cdots * G_{n-1}) > \cdots \delta(G_1 * G_2 * G_3) > \delta(G_1 * G_2) > \delta(G_1)$$

In the next section we will see that all groups in the $\mathcal{P}$-class are of divergence type.

Remark 10. Note that the above construction could be extended even further. We could choose a collection $E_1, \ldots, E_n$ which partitions an open interval in the circle S^1 instead of partitioning the entire circle. Then after choosing such collections for several pairwise disjoint open intervals in S^1, we could carry out our iterative construction for each of the collections independently and then combine the so obtained groups. By considering them from the one with largest Poincaré exponent to the one with the least Poincaré exponent we could simply apply Theorem 8 repeatedly to show that their free product belongs to the $\mathcal{P}$-class.

Remark 11. In the above construction we could also choose the groups $G_2, \ldots, G_n$ to be all of convergence type. Therefore, since we are going to show that all groups in the $\mathcal{P}$-class are of divergence type, this means that we can combine one group of divergence type, which is very small in terms of the diameter of its limit set, with many groups of convergence type and obtain a group of divergence type. That it is possible to choose the groups $G_2, \ldots, G_n$ to be all of convergence type can be justified by combining the two following theorems.

Theorem 9. ***Brooks-Stadlbauer***
Let N be a normal subgroup of a finitely generated geometric Schottky group Γ, then we have that:

$$\delta(\Gamma) = \delta(N) \Leftrightarrow \Gamma/N \text{ amenable}$$

Proof. For proof we refer the reader to the original paper [Stadlbauer2013], where a more general version of this result has been proved. □

Remark 12. Theorem 9 is a special case of the result which appears in [Stadlbauer2013]. Namely, Stadlbauer has proved this result for Γ an essentially free Kleinian group. The class of essentially free Kleinian groups is much larger than that of finitely generated geometric Schottky groups but any geometric Schottky group is essentially free. For further details and the definition of essetially free Kleinian groups we refer the reader to [Stadlbauer2013].

Theorem 10. ***Matsuzaki-Yabuki***
Let N be a Fuchsian group of divergence type and suppose that there exists a Fuchsian group Γ which contains N as a normal subgroup. Then $\delta(\Gamma) = \delta(N)$ and Γ is of divergence type.

Proof. For poof we refer the reader to the original paper [MatsuzakiYabuki2009], where a more general version of this result appears. $\square$

So, in particular, any normal subgroup N of a finitely generated geometric Schottky group Γ for which Γ/N is non-amenable must be of convergence type. Recall, that a Fuchsian group Γ is called **amenable** if it contains a **Følner sequence**, that is if there exist finite subsets $F_1, F_2, F_3, \ldots$ of Γ satisfying:

- $\bigcup_{i\in\mathbb{N}} F_i = \Gamma$
- $\lim_{i\to\infty} \frac{\#(gF_i \Delta F_i)}{\#F_i} = 0$, for all $g \in \Gamma$

where $\#(\cdot)$ denotes the cardinality and Δ the symmetric difference. Yet, we will not work directly with the definition of amenability but merely appeal to the standard fact that a free group with at least two generators is non-amenable; this result can be found in [delaHarpe].

With these facts at hand it is easy to see how one can construct a geometric Schottky group of convergence type whose Schottky description intervals are contained in a given small interval E. Namely, we choose a geometric Schottky group Γ with at least three generators such that the intervals of its Schottky description are contained in the interval E. Then we consider a subset W of the set of generators of Γ such that at least two of the generators of Γ are not contained in W and then choose N to be the smallest normal subgroup of Γ which contains all the generators in W. The group N is then of convergence type, since Γ/N contains a free group with two generators, and in a standard way, see Section 3.3, one then finds a Schottky description for it with intervals contained in E.

5.3 Ergodicity of the geodesic flow

We will now prove the first of the properties that make the $\mathcal{P}$-class of particular interest. Namely, we are going to show that for any group Γ in the $\mathcal{P}$-class the geodesic flow on the quotiend manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure.

Theorem 11. *Let Γ be a group in the $\mathcal{P}$-class. Then the geodesic flow on the manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure.*

The construction which we use in the proof and the basic structure of the argument agrees with the one provided by Peigné in [Peigné2003]. Our exposition is slightly different and more detailed so as to make it more accessible to the reader.

At the heart of the proof of the ergodicity of the geodesic flow, which we are going to present here, lies the following theorem due to Sullivan [Sullivan1979]:

Theorem 12. ***Sullivan***
Let G be a non-elementary Fuchsian group, then the following statements are equivalent:

- *The Patterson measure μ_o of the radial limit set of G is equal to one.*
- *The godesic flow on the manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure ν.*

Moreover, each of the above statements implies that G is of divergence type and if the group G satisfies $\delta(G) > \frac{1}{2}$ then each of the above statements is equivalent to:

- *The group G is of divergence type.*

Remark 13. Sullivan's theorem holds for any Kleinian group G acting on $\mathbb{H}^n$ after replacing the condition $\delta(G) > \frac{1}{2}$ by $\delta(G) > \frac{n-1}{2}$.

Our goal is thus to show that for a group Γ in the $\mathcal{P}$-class we have $\mu_o(L_r(\Gamma)) = 1$. In our argument we will make use of certain subsets of the limit set. We have not introduced these sets in Section 2.0.6 along with the radial, uniformly radial and Jørgensen limit sets since their definitions require the notions appearing in Section 5.1. Namely, they are only defined for geometric Schottky groups with the $\mathcal{P}$-property. Moreover, their definitions depend on the symmetric partition of the indexing set with respect to which the group has the $\mathcal{P}$-property and in that respect they differ significantly from $L_r(\Gamma)$, $L_{ur}(\Gamma)$ and $L_J(\Gamma)$ which have been defined in a canonical way.

Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ which has the $\mathcal{P}$-property with respect to a symmetric partition $I_1 \dot\cup I_2$ of I. We define:

$$G := \langle g_k : k \in I_1 \rangle \quad \text{and} \quad H := \langle g_k : k \in I_2 \rangle$$

At this point we also adjust our notation by putting:

$$I_G := I_1 \quad \text{and} \quad I_H := I_2$$

We then make the following definitions.

Definition 17. Let $\xi \in L(\Gamma)$ be a limit point with coding sequence $\kappa(\xi) = [x_0, x_1, ...]$. We define $L_0(\Gamma)$ by requiring that $\xi \in L_0(\Gamma)$ if and only if $\sup\{i \in \mathbb{N}_0 : x_i \in I_G\} = \infty$ and $\sup\{i \in \mathbb{N}_0 : x_i \in I_H\} = \infty$.

Definition 18. We define $L_1(\Gamma)$ as the complement of $L_0(\Gamma)$ in $L(\Gamma)$, that is for $\xi \in L(\Gamma)$ with code $\kappa(\xi) = [x_0, x_1, ...]$ we have $\xi \in L_1(\Gamma)$ if and only if there is some $N \in \mathbb{N}_0$ such that either $x_i \in I_G$ for all $i > N$, or $x_i \in I_H$ for all $i > N$, or $x_i = 0$ for all $i > N$.

Now we shall explain how considering the sets $L_0(\Gamma)$ and $L_1(\Gamma)$ can help us in proving that $\mu_o(L_r(\Gamma)) = 1$. Firstly, we are going to show that $L_0(\Gamma) \subseteq L_r(\Gamma)$, which means that it is enough to show that $\mu_o(L_0(\Gamma)) = 1$. By the definition of $L_1(\Gamma)$ this is equivalent to $\mu_o(L_1(\Gamma)) = 0$. The second step is to express the set $L_1(\Gamma)$ in terms of the limit sets $L(G)$ and $L(H)$. Applying standard properties of the measure μ_o will then allow us to reduce the original problem to showing that $\mu_o(L(G)) = \mu_o(L(H)) = 0$.

Proposition 29. *The set $L_0(\Gamma)$ is a subset of the radial limit set $L_r(\Gamma)$.*

Proof. Let ξ be an element of $L_0(\Gamma)$ with coding sequence $\kappa(\xi) = [x_0, x_1, \ldots]$. As explained in Section 3.2.3, each appearance in the coding sequence of a word of the form x_k, x_{k+1} with $x_k \in I_H$ and $x_{k+1} \in I_G$ corresponds to the fact that the ray s_ξ passes through a copy gF of the standard fundamental domain of Γ, and in particular s_ξ passes through gF within a copy of the stripe $S(A_{-x_k}, A_{x_{k+1}})$, namely $gS(A_{-x_k}, A_{x_{k+1}})$. Because of condition (2') in the definition of the $\mathcal{P}$-property there exists an $R > 0$ such that any stripe $S(A_{-x_k}, A_{x_{k+1}})$ with $x_k \in I_H$ and $x_{k+1} \in I_G$ is crossed, see again Section 3.2.3, by the ball $B(o, R)$. For example, one could take any R big enough so that the ball $B(o, R)$ intersects the geodesic with endpoints at infinity $\xi_- = 1$ and $\xi_+ = e^{i\theta}$, where θ is a constant for which Γ satisfies condition (2') in the definition of the $\mathcal{P}$-property. This implies that as s_ξ passes through gF it intersects $B(go, R)$. Since the copies gF corresponding to each appearance of x_k, x_{k+1} with $x_k \in I_H$ and $x_{k+1} \in I_G$ are distinct, this shows that ξ is a radial limit point. $\square$

Now the following propositions will allow us to express the set $L_1(\Gamma)$ in terms of the limit sets $L(G)$ and $L(H)$.

Proposition 30. *Let $\xi \in L(\Gamma)$ be a limit point with coding sequence $\kappa(\xi) = [x_0, x_1, ...]$. If for all $i \in \mathbb{N}_0$ we have $x_i \in I_G$, then $\xi \in L(G)$. Likewise, if for all $i \in \mathbb{N}_0$ we have $x_i \in I_H$, then $\xi \in L(H)$.*

Proof. Suppose that we have $x_i \in I_G$ for all $i \in \mathbb{N}_0$. Then by Proposition 7 we have $\xi = \lim_{n\to\infty} g_{x_0} \cdots g_{x_n} o$, where the limit is taken with respect to the Euclidean distance. Since $g_{x_0} \cdots g_{x_n} \in G$ for each $n \in \mathbb{N}_0$ we have $\xi \in L(G)$. The proof in the case that $x_i \in I_H$ for all $i \in \mathbb{N}_0$ is analogous. $\square$

Remark 14. The fact that for any $\xi \in L(G)$ we have $x_i \in I_G \cup \{0\}$, for all $i \in \mathbb{N}_0$, and for any $\xi \in L(H)$ we have $x_i \in I_H$, for all $i \in \mathbb{N}_0$, follows from the definition of the coding κ and the geometric picture described in Section 3.1.2.

Proposition 31. *Let $\xi \in L(\Gamma)$ be a limit point with coding sequence $\kappa(\xi) = [x_0, x_1, \ldots]$. If there exists an $N \in \mathbb{N}_0$ such that for all $i > N$ we have $x_i \in I_G$ then $\xi \in g_{x_0} \cdots g_{x_N} L(G)$, and similarly if for all $i > N$ we have $x_i \in I_H$ then $\xi \in g_{x_0} \cdots g_{x_N} L(H)$.*

Proof. Suppose that $N \in \mathbb{N}_0$ has the property that we have $x_i \in I_G$ for all $i > N$. We start by applying Proposition 8. We apply it N times to obtain that:

$$\kappa(g_{x_N}^{-1} \cdots g_{x_0}^{-1} \xi) = [x_{N+1}, x_{N+2}, \ldots]$$

From Proposition 30 it then follows that $g_{x_N}^{-1} \cdots g_{x_0}^{-1}\xi \in L(G)$ and thus $\xi \in g_{x_0} \cdots g_{x_N} L(G)$. The proof in the case that $x_i \in I_H$ for all $i > N$ is analogous. $\square$

Corollary 6. *We can express the set $L_1(\Gamma)$ in the following way:*

$$L_1(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(L(G) \cup L(H))$$

Proof. Using Proposition 31 together with Corollary 3 and the fact that any element of the set $J(\Gamma)$ is either an element of $L(G)$ or an element of $L(H)$ we obtain that $L_1(\Gamma) \subseteq \bigcup_{\gamma\in\Gamma} \gamma(L(G) \cup L(H))$. The other direction follows from Proposition 30 together with Remark 14 and Corollary 2. $\square$

Now the above result allows us to make the following estimate:

$$\mu_o(L_1(\Gamma)) = \mu_o\left(\bigcup_{\gamma\in\Gamma} \gamma(L(G) \cup L(H))\right) \leq \sum_{\gamma\in\Gamma} (\mu_o(\gamma(L(G) \cup L(H))))$$

By the δ-harmonicity property, see Section 2.0.8, for each $\gamma \in \Gamma$ we have that:

$$\mu_o\left(\gamma(L(G) \cup L(H))\right) = \mu_{\gamma^{-1}o}(L(G) \cup L(H))$$

Since for each $\gamma \in \Gamma$ the measure $\mu_{\gamma^{-1}x}$ is absolutely continuous with respect to μ_o it is enough to show that:

$$\mu_o(L(G) \cup L(H)) = 0$$

But, since $\mu_o(L(G) \cup L(H)) \leq \mu_o(L(G)) + \mu_o(L(H))$, this reduces our initial problem to showing that $\mu_o(L(G)) = \mu_o(L(H)) = 0$. We will only show that $\mu_o(L(G)) = 0$. The proof that $\mu_o(L(H)) = 0$ is analogous.

We are going to choose a certain suitable $x \in \mathbb{D}$, show that $\mu_x(L(G)) = 0$, and then use the absolute continuity of the measures μ_x and μ_o to conclude that $\mu_o(L(G)) = 0$. Our strategy of showing that $\mu_x(L(G)) = 0$ will be as follows.

- Define a sequence $\{O_k\}_{k\in\mathbb{N}_0}$ of open sets in $\overline{\mathbb{D}}$ such that:

$$L(G) \subseteq O_k \quad \text{for each} \quad k \in \mathbb{N}_0$$

- For a certain sequence of measures $(\mu_{x,n})_{n\in\mathbb{N}_0}$ with $\mu_{x,n} \xrightarrow{w} \mu_x$ we will then have for each $k \in \mathbb{N}$:

$$\mu_x(L(G)) \leq \mu_x(O_k) \leq \liminf_{n\to\infty} \; \mu_{x,n}(O_k)$$

- Find a sequence of bounds $\{c_k\}_{k\in\mathbb{N}_0}$ such that:

$$\liminf_{n} \; \mu_{x,n}(O_k) \leq c_k$$

- Show that:

$$c_k \to 0 \quad \text{as} \quad k \to \infty$$

The first step is to order all the elements of the group G by attaching to each element of G an index from the set $\mathbb{N}_0$, namely:

$$G = \left\{\gamma_j^G : j \in \mathbb{N}_0\right\}$$

We have added the superscript G to stress the fact that the sequence $(\gamma_j^G)_{j\in\mathbb{N}_0}$ only contains elements of G and not of the entire group Γ. Additionally we will require that the indices are chosen in such a way that $d(\gamma_j^G o, o) \leq d(\gamma_{j+1}^G o, o)$, which forces $\gamma_0^G = id$. We also define the following special subset of the group Γ:

$$\Gamma^H := \{g_{k_0} \cdots g_{k_m} \in \Gamma : k_0 \in I_H\}$$

In the above definition, $g_{k_0} \cdots g_{k_m}$ is the unique expression of an element in terms of the Schottky description of Γ.

Now we can define the sequence $\{O_k\}_{k\in\mathbb{N}_0}$. Let U_H be a closed neighborhood of the union $\overline{\bigcup_{k\in I_H} A_k}$, chosen in such a way that it does not intersect $\overline{\bigcup_{k\in I_G} A_k}$, which can be done due to the condition (2) in the definition of the $\mathcal{P}$-property. Then we define:

$$\begin{aligned}
O_0 &:= \overline{\mathbb{D}} \setminus (\{o\} \cup \Gamma^H o \cup U_H) \\
O_1 &:= O_0 \cap \gamma_1^G O_0 \\
&\cdots \\
O_k &:= O_0 \cap \gamma_1^G O_0 \cap \gamma_2^G O_0 \cap \ldots \cap \gamma_k^G O_0
\end{aligned} \tag{5.1}$$

It is easy to convince oneself that for every element $\gamma \in \Gamma$ we either have $\gamma = \gamma_j^G \in G$, for some $j \in \mathbb{N}_0$, or it can be written in a unique way in the form $\gamma = \gamma_j^G \gamma^*$, with $j \in \mathbb{N}_0$ and $\gamma^* \in \Gamma^H$. We observe that for any fixed k the set O_k has the following important properties:

- The set O_k contains the points in the orbit of o of the form $\gamma_j^G o$ and $\gamma_j^G \gamma^* o$ with $j > k$ and $\gamma^* \in \Gamma^H$.
- The set O_k does not contain the points in the orbit of o of the form $\gamma_j^G o$ and $\gamma_j^G \gamma^* o$ with $j \leq k$ and $\gamma^* \in \Gamma^H$.

As mentioned ealier, we will not treat $\mu_o(L(G))$ directly but instead consider $\mu_x(L(G))$ for a suitable $x \in \mathbb{D}$. We now turn our attention to the measure μ_x and explain how it is chosen.

Put $q := \frac{\delta(\Gamma)-\delta(G)}{2}$ and observe that since $\delta(\Gamma) > \delta(G)$ we have $q > 0$. By property (4) of the function $\mathcal{H}$, which appears in Section 2.0.8 where we discussed Patterson measures, there exists $r_q \geq 0$ such that for all $t \geq 0$ and all $r \geq r_q$ we have:

$$\mathcal{H}(t+r) \leq e^{qt}\mathcal{H}(r)$$

The above inequality motivates our choice of the point $x \in \mathbb{D}$, namely we require that:

$$d(x, \gamma o) \geq r_q \quad \text{for all} \quad \gamma \in \Gamma^H \cup id$$

A point with this property can be easily found by considering the midpoint of some interval in $S^1 \setminus \bigcup_{k \in I} A_k$, which we denote by ξ, and then choosing x to be a point on the ray s_ξ lying close enough to ξ.

Further, we fix a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \searrow \delta(\Gamma)$ and $\mu_{x,s_n} \xrightarrow{w} \mu_x$, where the measures μ_{x,s_n} are defined as in Section 2.0.8. To keep the notation simple, we will denote the measures μ_{x,s_n} by $\mu_{x,n}$. We also observe that for n large enough, so that s_n is close enough to $\delta(\Gamma)$, say for $n \geq N_0$, we have $s_n - q \in (\delta(G), \delta(\Gamma))$.

From the definition of the measures $\mu_{x,n}$ and the properties of the sets O_k it follows that for each $k \in \mathbb{N}_0$, we have:

$$\mu_{x,n}(O_k) = \underbrace{\frac{\sum\limits_{l>k} \sum\limits_{\gamma\in\Gamma^H} \mathcal{H}(d(x,\gamma_l^G\gamma o))e^{-s_n d(x,\gamma_l^G\gamma o)}}{\sum\limits_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o))e^{-s_n d(o,\gamma o)}}}_{K(n)} + \underbrace{\frac{\sum\limits_{l>k} \mathcal{H}(d(x,\gamma_l^G o))e^{-s_n d(x,\gamma_l^G o)}}{\sum\limits_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o))e^{-s_n d(o,\gamma o)}}}_{Q(n)}$$

We will first consider the term $K(n)$. Our choice of x allows us to make, for all $l \in \mathbb{N}_0$ and $\gamma \in \Gamma^H$, the following estimate:

$$\begin{aligned} \mathcal{H}(d(x,\gamma_l^G\gamma o)) &= \mathcal{H}(d((\gamma_l^G)^{-1}x,\gamma o)) \leq \mathcal{H}(d((\gamma_l^G)^{-1}x,x) + d(x,\gamma o)) \\ &= \mathcal{H}(d(x,\gamma_l^G x) + d(x,\gamma o)) \leq e^{q\cdot d(x,\gamma_l^G x)}\mathcal{H}(d(x,\gamma o)) \end{aligned} \tag{5.2}$$

Where the first inequality follows, since the function $\mathcal{H}$ is nondecreasing, and the second inequality follows as well, since $d(x,\gamma o) > r_q$ by our choice of the point x.

Using the reverse triangle inequality for hyperbolic space, see Appendix, we can make an estimate of $d(x,\gamma_l^G\gamma o)$. From condition (2) in the definition of the $\mathcal{P}$-property it follows that there exists $\theta_0 > 0$ such that for all $l \in \mathbb{N}$ and $\gamma \in \Gamma^H$ the angle between the geodesic segment joining $(\gamma_l^G)^{-1}x$ to x and the geodesic segment joining x to γo is at least θ_0. Therefore there is a constant $C \in \mathbb{R}_+$ such that for all $l \in \mathbb{N}$ and $\gamma \in \Gamma^H$, we have:

$$\begin{aligned} d(x,\gamma_l^G\gamma x) &= d((\gamma_l^G)^{-1}x,\gamma o) \geq d((\gamma_l^G)^{-1}x,x) + d(x,\gamma o) - C \\ &= d(x,\gamma_l^G x) + d(x,\gamma o) - C \end{aligned} \tag{5.3}$$

Therefore we can estimate $K(n)$ as follows:

$$
\begin{aligned}
K(n) &:= \frac{\sum\limits_{l>k} \sum\limits_{\gamma \in \Gamma^H} \mathcal{H}(d(x, \gamma_l^G \gamma o)) e^{-s_n d(x, \gamma_l^G \gamma o)}}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}} \\
&\le \frac{\sum\limits_{l>k} \sum\limits_{\gamma \in \Gamma^H} e^{q \cdot d(x, \gamma_l^G x)} \mathcal{H}(d(x, \gamma o)) e^{-s_n d(x, \gamma_l^G x)} e^{-s_n d(x, \gamma o)} e^{s_n C}}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}} \\
&= \frac{e^{s_n C} \sum\limits_{l>k} \left(e^{(q - s_n) d(x, \gamma_l^G x)} \sum\limits_{\gamma \in \Gamma^H} \mathcal{H}(d(x, \gamma o)) e^{-s_n d(x, \gamma o)} \right)}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}} \qquad (5.4) \\
&= \frac{e^{s_n C} \left(\sum\limits_{\gamma \in \Gamma^H} \mathcal{H}(d(x, \gamma o)) e^{-s_n d(x, \gamma o)} \right) \left(\sum\limits_{l>k} e^{(q - s_n) d(x, \gamma_l^G x)} \right)}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}} \\
&\le \frac{e^{s_n C} \left(\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(x, \gamma o)) e^{-s_n d(x, \gamma o)} \right) \left(\sum\limits_{l>k} e^{(q - s_n) d(x, \gamma_l^G x)} \right)}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}}
\end{aligned}
$$

We observe that as s_n decreases, $e^{s_n C}$ decreases, the term $\sum\limits_{l>k} e^{(q-s_n) d(x, \gamma_l^G x)}$ increases, and we have:

$$
\lim_{n \to \infty} \frac{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(x, \gamma o)) e^{-s_n d(x, \gamma o)}}{\sum\limits_{\gamma \in \Gamma} \mathcal{H}(d(o, \gamma o)) e^{-s_n d(o, \gamma o)}} = \mu_x(\overline{\mathbb{D}})
$$

Thus we have:

$$
\liminf_{n \to \infty}(K(n)) \le e^{s_1 C} \mu_x(\overline{\mathbb{D}}) \sum_{l>k} e^{(q - \delta(\Gamma)) d(x, \gamma_l^G x)} =: c_k
$$

Now we will consider the term $Q(n)$. Our choice of x allows us to make for all $l \in \mathbb{N}_0$

the following estimate:

$$\begin{aligned}\mathcal{H}(d(x,\gamma_l^G o)) &\leq \mathcal{H}(d(x,\gamma_l^G x) + d(\gamma_l^G x, \gamma_l^G o)) \\ &\leq e^{q\cdot d(x,\gamma_l^G x)}\mathcal{H}(d(\gamma_l^G x, \gamma_l^G o))\end{aligned} \tag{5.5}$$

Where the first inequality follows since the function $\mathcal{H}$ is nondecreasing and the second inequality since $d(\gamma_l^G x, \gamma_l^G o) = d(x,o) > r_q$ by our choice of the point x.

For each $l \in \mathbb{N}$ we can also estimate $d(x,\gamma_l^G o)$ in the following way:

$$\begin{aligned} & d((\gamma_l^G)^{-1}x, x) \leq d((\gamma_l^G)^{-1}x, o) + d(o,x) \\ \Rightarrow\ & d((\gamma_l^G)^{-1}x, o) \geq d((\gamma_l^G)^{-1}x, x) - d(o,x) \\ \Rightarrow\ & d(x,\gamma_l^G o) \geq d(x,\gamma_l^G x) - d(x,o)\end{aligned} \tag{5.6}$$

Therefore, we can estimate $Q(n)$ as follows:

$$\begin{aligned} Q(n) := &\quad \frac{\sum_{l>k} \mathcal{H}(d(x,\gamma_l^G o)) e^{-s_n d(x,\gamma_l^G o)}}{\sum_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o)) e^{-s_n d(o,\gamma o)}} \\ \leq &\quad \frac{\sum_{l>k} e^{q\cdot d(x,\gamma_l^G x)} \mathcal{H}(d(\gamma_l^G x, \gamma_l^G o)) e^{-s_n d(x,\gamma_l^G x)} e^{s_n d(x,o)}}{\sum_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o)) e^{-s_n d(o,\gamma o)}} \\ = &\quad \frac{\mathcal{H}(d(x,o)) e^{s_n d(x,o)} \sum_{l>k} e^{(q-s_n) d(x,\gamma_l^G x)}}{\sum_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o)) e^{-s_n d(o,\gamma o)}}\end{aligned} \tag{5.7}$$

We observe that as s_n decreases, $\mathcal{H}(d(x,o))e^{s_n d(x,o)}$ decreases, the term $e^{(q-s_n)d(x,\gamma_l^G x)}$ increases, and we have:

$$\lim_{n\to\infty} \sum_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o)) e^{-s_n d(o,\gamma o)} \quad = \infty$$

Thus we have, for all $n \in \mathbb{N}$:

$$Q(n) \leq \quad \frac{\mathcal{H}(d(x,o)) e^{s_1 d(x,o)} \sum_{l>k} e^{(q-\delta(\Gamma)) d(x,\gamma_l^G x)}}{\sum_{\gamma\in\Gamma} \mathcal{H}(d(o,\gamma o)) e^{-s_n d(o,\gamma o)}}$$

Since the numerator is now a constant, we deduce:

$$\lim_{n\to\infty} Q(n) = 0$$

Therefore, for each $k \in \mathbb{N}_0$ we have:

$$\liminf_{n\to\infty} \mu_{x,n}(O_k) \leq c_k$$

To finish the proof we now need to show that:

$$\lim_{k\to\infty} c_k = 0$$

To show this, note that, since $q - \delta(\Gamma) > \delta(G)$, we have:

$$\sum_{l\in\mathbb{N}_0} e^{(q-\delta(\Gamma))d(x,\gamma_l^G x)} \ < \ \infty$$

This implies:

$$\lim_{k\to\infty} \sum_{l>k} e^{(q-\delta(\Gamma))d(x,\gamma_l^G x)} = 0$$

From this it now follows that:

$$\lim_{k\to\infty} c_k = 0$$

This finishes the proof of Theorem 11.

5.4 Finiteness of the Liouville-Patterson measure

One of the reasons why the $\mathcal{P}$-class is interesting is the fact that, for any group Γ which belongs to the $\mathcal{P}$-class, not only is the geodesic flow on $\mathbb{D}/\Gamma$ ergodic with respect to the Liouville-Patterson measure, as has been shown in the previous section, but the Liouville-Patterson measure also turns out to be finite. We have presented in Section 5.2 methods of constructing geometric Schottky groups in the $\mathcal{P}$-class, in particular Schottky groups which are infinitely generated. Thus the $\mathcal{P}$-class is valuable as a source of examples of infinitely generated groups for which the geodesic flow on $\mathbb{D}/\Gamma$ is ergodic with respect to a finite flow invariant measure.

This section is devoted to presenting the proof of the fact that the Liouville-Patterson measure is finite for groups Γ in the $\mathcal{P}$-class. Namely, we are going to prove the following theorem:

Theorem 13. *Let Γ be a group in the $\mathcal{P}$-class. Then the Liouville-Patterson measure is finite.*

The main steps of the argument are similar to those in the proof given by Peigné in [Peigné2003]. Yet here, so as to make the exposition more tractable, we provide more of the details which were not included in [Peigné2003]. In particular, we carry out in detail the necessary calculations and give additional arguments to justify the properties used in the argument. Also, the way in which we reduce the question of finiteness of the Liouville-Patterson measure to the estimation of a particular integral is different from the way it has been done in [Peigné2003].

Outline of the proof

Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$ which has the $\mathcal{P}$-property with respect to a symmetric partition $I_1 \dot\cup I_2$ of I. We define:

$$G := \langle g_k : k \in I_1 \rangle \quad \text{and} \quad H := \langle g_k : k \in I_2 \rangle$$

At this point we also adjust our notation by putting:

$$I_G := I_1 \quad \text{and} \quad I_H := I_2$$

Recall the Busemann parametrisation of $T^1\mathbb{D}$ defined in Section 2.0.10 and the fact that for an element $g \in \mathrm{Iso}^+(\mathbb{D})$ and $(\xi, \eta, s) \in T^1\mathbb{D}$ one can express the map:

$$Dg : T^1\mathbb{D} \to T^1\mathbb{D}$$

in terms of this parametrisation as:

$$Dg(\xi, \eta, s) = (g\xi, g\eta, s - B_\eta(o, g^{-1}o))$$

Also recall that the Liouville-Patterson measure ν on $T^1(\mathbb{D}/\Gamma)$ is defined as the projection of the measure $\widetilde{\nu}$ on $T^1\mathbb{D}$, which is given by the infinitesimal formula:

$$d\widetilde{\nu}(\xi, \eta, s) := \frac{d\mu_o(\xi)d\mu_o(\eta)ds}{|\xi - \eta|^{2\delta}}$$

For a Borel measurable subset V of $T^1(\mathbb{D}/\Gamma)$, its measure $\nu(V)$ is equal to $\widetilde{\nu}(\widetilde{V})$, where $\widetilde{V}$ can be taken to be any Borel measurable lift of V with respect to the canonical projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$.

In Section 5.3 we have considered the set $L_0(\Gamma)$ and have shown that $\mu_o(L_0(\Gamma)) = 1$. From this fact it follows that:

$$\nu(T^1(\mathbb{D}/\Gamma)) = \nu(((L_0(\Gamma) \times L_0(\Gamma) \setminus diag) \times \mathbb{R})/\Gamma)$$

Our intention is to define a set $\mathcal{F} \subseteq T^1\mathbb{D}$ with the property that for any vector:

$$\widetilde{v} \in (L_0(\Gamma) \times L_0(\Gamma) \setminus diag) \times \mathbb{R}$$

there exists an element $g \in \Gamma$ such that:

$$Dg(\widetilde{v}) \in \mathcal{F}$$

In other words, the set $\mathcal{F}$ will contain a lift of $((L_0(\Gamma)\times L_0(\Gamma)\backslash diag)\times\mathbb{R})/\Gamma$, and therefore we will be able to use the following estimate for the Liouville-Patterson measure:

$$\nu(T^1(\mathbb{D}/\Gamma)) \leq \widetilde{\nu}(\mathcal{F})$$

We will then bound $\widetilde{\nu}(\mathcal{F})$ by a certain integral and show that this integral is finite.

Defining the set $\mathcal{F}$

Defining the set $\mathcal{F}$ with the properties just described requires some preliminary definitions. Let L_0^G be the set af all those limit points in $L_0(\Gamma)$ whose coding sequences are of the form $[x_0, x_1, \ldots, x_k, x_{k+1}, \ldots]$, where $k \geq 0$ and $x_i \in I_G$ for $i \leq k$, and $x_i \in I_H$ for $i = k+1$. Similarly, let L_0^H be the set af all those limit points in $L_0(\Gamma)$ whose coding sequences are of the form $[x_0, x_1, \ldots, x_k, x_{k+1}, \ldots]$, where $k \geq 0$, $x_i \in I_H$ for $i \leq k$, and $x_i \in I_G$ for $i = k+1$. It is easy to see that:

$$L_0^G = L_0(\Gamma) \cap \bigcup_{k \in I_G} A_k$$

and

$$L_0^H = L_0(\Gamma) \cap \bigcup_{k \in I_H} A_k$$

and thus, we also have $L_0(\Gamma) = L_0^G \,\dot\cup\, L_0^H$. We define the set:

$$K := (L_0^H \times L_0^G) \,\dot\cup\, (L_0^G \times L_0^H)$$

Further, we define the function $h : L_0(\Gamma) \to \mathbb{R}$ in the following way. For a point $\xi \in L_0^G$ with coding sequence $[x_0, x_1, \ldots, x_k, x_{k+1}, \ldots]$, as in the definition of L_0^G, put:

$$\gamma_\xi := g_{x_0} g_{x_1} \cdots g_{x_k}$$

where g_i are the generators in the Schottky description of the group Γ and define:

$$h(\xi) := B_\xi(o, \gamma_\xi o)$$

For points in L_0^H, we define the value of h analogously. Finally, we define the set $\mathcal{F}$ as follows:

$$\mathcal{F} := \left\{(\xi, \eta, r) \in T^1\mathbb{D} : \ (\xi, \eta) \in K, \ r \in [0, |h(\eta)|]\right\}$$

Now we are going to show that the set $\mathcal{F}$ indeed has the property that for any vector $\widetilde{v} \in (L_0(\Gamma) \times L_0(\Gamma) \setminus diag) \times \mathbb{R}$ there exists an element $g \in \Gamma$ such that $Dg(\widetilde{v}) \in \mathcal{F}$.

Lemma 6. *Let $\widetilde{v}$ be a vector in $(L_0(\Gamma)\times L_0(\Gamma)\setminus diag)\times\mathbb{R}$. Then there exists an element $g \in \Gamma$ such that $g \cdot \widetilde{v} \in \mathcal{F}$.*

Proof. Let $\widetilde{v} = (\xi, \eta, B_\eta(o, \pi_b(\widetilde{v})))$ be an arbitrary element of $(L_0(\Gamma) \times L_0(\Gamma) \setminus diag) \times \mathbb{R}$. Suppose that the coding sequences of ξ and η are given by:

$$\begin{aligned} \kappa(\xi) &= [x_0, \ldots, x_{k_1}, x_{k_1+1}, \ldots, x_{k_2}, \ldots] \\ \kappa(\eta) &= [y_0, \ldots, y_{l_1}, y_{l_1+1}, \ldots, y_{l_2}, \ldots] \end{aligned} \tag{5.8}$$

where $(k_i)_{i\in\mathbb{N}}$ and $(l_i)_{i\in\mathbb{N}}$ are strictly increasing sequences in $\mathbb{N}_0$, and we have:

$$\begin{aligned} &x_0, \ldots, x_{k_1} \in I_G, \quad x_{k_1+1}, \ldots, x_{k_2} \in I_H, \quad x_{k_2+1}, \ldots, x_{k_3} \in I_G \ldots \\ &y_0, \ldots, y_{l_1} \in I_G, \quad y_{l_1+1}, \ldots, y_{l_2} \in I_H, \quad y_{l_2+1}, \ldots, y_{l_3} \in I_G \ldots \end{aligned} \tag{5.9}$$

where we allow the possiblity that k_1, l_1 or both are equal to zero.

Now consider the following points in the orbit of the origin under the action of Γ:

$$\begin{aligned} &\ldots \\ &g_{x_0} \cdots g_{x_{k_2}} \cdot o \\ &g_{x_0} \cdots g_{x_{k_1}} \cdot o \\ &id \cdot o \\ &g_{y_0} \cdots g_{y_{l_1}} \cdot o \\ &g_{y_0} \cdots g_{y_{l_2}} \cdot o \\ &\ldots \end{aligned} \tag{5.10}$$

For $j \in \mathbb{Z}$, define the points z_j in the following way. For $j < 0$, z_j denotes the point on the geodesic α with endpoints ξ and η which lies on the same horocycle based at η as the point $g_{x_0} \cdots g_{x_{k_{-j}}} o$. We will say that the point z_j, for $j < 0$, **corresponds** to the orbit point $g_{x_0} \cdots g_{x_{k_{-j}}} o$. For $j = 0$, z_j denotes the point on the geodesic α which lies on the same horocycle based at η as the point o. We will say that the point z_0 corresponds to the orbit point o. For $j > 0$, z_j denotes the point on the geodesic α which lies on the same horocycle based at η as the point $g_{y_0} \cdots g_{x_{l_j}} o$. We will say that the point z_j, for $j > 0$, corresponds to the orbit point $g_{y_0} \cdots g_{x_{l_j}} o$.
We choose an integer j_0 such that:

$$B_\eta(o, z_{j_0}) < B_\eta(o, \pi_b(\widetilde{v}) \leq B_\eta(o, z_{j_0+1})$$

Here $\pi_b : T^1\mathbb{D} \to \mathbb{D}$ is the projection defined in Section 2.0.3. In order to see why there exists an integer j_0 which satisfies the above conditions, we argue as follows. First, observe that there exists some $j < 0$ such that $B_\eta(o, z_j) < B_\eta(o, \pi_b(\widetilde{v}))$. Otherwise, all the points $g_{x_0} \cdots g_{x_{k_1}} o, g_{x_0} \cdots g_{x_{k_2}} o, \ldots$ would lie inside or on the horocycle based at η

and containing $\pi_b(\widetilde{v})$, which is impossible since these points accumulate at ξ. Secondly, there must exist some j such that $B_\eta(o, z_j) \geq B_\eta(o, \pi_b(\widetilde{v}))$. The justification that this is indeed the case is delivered by examining the proof of Proposition 29. There, we have shown that the point η, being an element of $L_0(\Gamma)$, is a radial limit. But in fact, when proving this, we have also shown that the point η is radial with respect to the orbit points $g_{y_0} \cdots g_{y_{l_1}} o, g_{y_0} \cdots g_{y_{l_2}} o, g_{y_0} \cdots g_{y_{l_3}} o, \ldots$, by which we mean that there exists a radius $R > 0$ such that all these points lie in the R-neighborhood of the ray s_η. This implies that any horocycle based at η must contain infinitely many of these points, and, in particular, at least one must lie inside the horocycle based at η and containing the point $\pi_b(\widetilde{v})$. The two observations combined imply the existence of j_0, since at some point 'we need to jump to the other side of $\pi_b(\widetilde{v})$'.

Now z_{j_0} corresponds to an orbit point $\gamma_0 o$, where $\gamma_0 = g_{x_0} \cdot g_{x_{k_{-j_0}}}$ or $\gamma_0 = g_{y_0} \cdot g_{y_{l_{j_0}}}$ or $\gamma_0 = id$. If $\gamma_0 \neq id$ we consider $\gamma_0^{-1} \cdot \widetilde{v}$:

$$\gamma_0^{-1} \cdot \Big(\xi, \eta, B_\eta(o, \pi_b(\widetilde{v}))\Big) = \Big(\gamma_0^{-1}\xi, \gamma_0^{-1}\eta, B_\eta(o, \pi_b(\widetilde{v})) - B_\eta(o, \gamma_0 o)\Big)$$

By the choice of γ_0 we have that $(\gamma_0^{-1}\xi, \gamma_0^{-1}\eta) \in K$. Further, by the choice of j_0 we have:

$$B_\eta(o, \pi_b(\widetilde{v})) - B_\eta(o, \gamma_0 o) = B_\eta(o, \pi_b(\widetilde{v})) - B_\eta(o, z_{j_0}) > 0$$

Also, by the choice of j_0 we have:

$$\begin{aligned} B_\eta(o, \pi_b(\widetilde{v})) - B_\eta(o, \gamma_0 o) &= B_\eta(o, \pi_b(\widetilde{v})) - B_\eta(o, z_{j_0}) \\ &\leq B_\eta(o, z_{j_0+1}) - B_\eta(o, z_{j_0}) \\ &= B_\eta(z_{j_0}, z_{j_0+1}) \qquad (5.11) \\ &= B_\eta(\gamma_{j_0} o, \gamma_{j_0+1} o) \\ &= B_{\gamma_0^{-1}\eta}(o, \gamma_0^{-1}\gamma_1 o) \end{aligned}$$

where γ_1 is such that $\gamma_1 o$ is the point corresponding to z_{j_0+1}. Since $\gamma_0^{-1}\gamma_1$ is the element corresponding to the first letter of $\gamma_0^{-1}\eta$, which means that $B_{\gamma_0^{-1}\eta}(o, \gamma_0^{-1}\gamma_1 o) = h(\gamma_0^{-1}\eta)$, it now follows that $\gamma_0^{-1} \cdot \widetilde{v} \in \mathcal{F}$.

If $\gamma_0 = id$, then apply γ_1^{-1} and consider $\gamma_1^{-1} \cdot \widetilde{v}$ instead of $\widetilde{v}$ in the above argument. This time, the new j_0 cannot be equal to 0 and we have that $\gamma_0^{-1} \cdot (\gamma_1^{-1} \cdot \widetilde{v}) \in \mathcal{F}$, where γ_1^{-1} is obtained by the above argument applied to $\widetilde{v}$, and γ_0^{-1} is obtained from a second application of this argument.

□

Thus, to show that the Liouville-Patterson measure is finite, it suffices to show that the following integral is finite:

$$\int_K |h(\xi_+)| \frac{d\mu_o(\xi_-)d\mu_o(\xi_+)}{|\xi_- - \xi_+|^{2\delta}} \tag{5.12}$$

Estimating the integral

The first step is to split the integral (5.12) into two parts which can then be treated separately. Namely, we rewrite integral (5.12) as follows:

$$\int_{L_0^H \times L_0^G} |h(\xi_+)| \frac{d\mu_o(\xi_-)d\mu_o(\xi_+)}{|\xi_- - \xi_+|^{2\delta}} \; + \; \int_{L_0^G \times L_0^H} |h(\xi_+)| \frac{d\mu_o(\xi_-)d\mu_o(\xi_+)}{|\xi_- - \xi_+|^{2\delta}}$$

Here, we will only show that the first of the two integrals in this sum is finite. The proof for the second integral can be carried out in a completely analogous manner with the roles of G and H exchanged. We start by using the fact that the set $L_0^H \times L_0^G$ can be written as the disjoint union:

$$\dot{\bigcup_{g \in G^*}} L_0^H \times gL_0^H$$

where $G^* := G \setminus \{id\}$. Therefore, we have:

$$\begin{aligned} & \int_{L_0^H \times L_0^G} |h(\xi_+)| \frac{d\mu_o(\xi_-)d\mu_o(\xi_+)}{|\xi_- - \xi_+|^{2\delta}} \\ = & \sum_{g \in G^*} \int_{L_0^H \times gL_0^H} |h(\xi_+)| \, \frac{d\mu_o(\xi_-)d\mu_o(\xi_+)}{|\xi_- - \xi_+|^{2\delta}} \\ = & \sum_{g \in G^*} \int_{gL_0^H} |h(\xi_+)| \int_{L_0^H} \frac{1}{|\xi_- - \xi_+|^{2\delta}} d\mu_o(\xi_-)d\mu_o(\xi_+) \end{aligned} \tag{5.13}$$

Since $\xi_- \in L_0^H$ and $\xi_+ \in L_0^G$, condition (2) in the definition of the $\mathcal{P}$-property implies that there exist constants $c_1, c_2 > 0$, independent of the particular choice of ξ_- and ξ_+, such that:

$$c_1 \leq \frac{1}{|\xi_- - \xi_+|^{2\delta}} \leq c_2$$

Thus, we have:

$$\begin{aligned}
&\sum_{g\in G^*}\int_{gL_0^H}|h(\xi_+)|\int_{L_0^H}\frac{1}{|\xi_- - \xi_+|^{2\delta}}d\mu_o(\xi_-)d\mu_o(\xi_+)\\
\asymp\ &\sum_{g\in G^*}\int_{gL_0^H}|h(\xi_+)|\int_{L_0^H}d\mu_o(\xi_-)d\mu_o(\xi_+) \qquad (5.14)\\
=\ &\sum_{g\in G^*}\int_{gL_0^H}|h(\xi_+)|\,\mu_o(L_0^H)d\mu_o(\xi_+)
\end{aligned}$$

Further, since $\mu_o(L_0^H)\leq 1$, we can bound the latter series by:

$$\sum_{g\in G^*}\int_{gL_0^H}|h(\xi_+)|\,d\mu_o(\xi_+)$$

By the definition of the function h this is equal to:

$$\sum_{g\in G^*}\int_{gL_0^H}\left|B_{\xi_+}(o,go)\right|d\mu_o(\xi_+)$$

Now we are going to apply the following three lemmata. The proofs of the first two are deferred to the end of this section, while the proof of the third one has been included in the Appendix.

Lemma 7. *For $\xi\in S^1$ and $z\in\mathbb{D}$, we have:*

$$|B_\xi(o,z)|\leq d(o,z)$$

Lemma 8. *There exist constants $M_1,M_2\in\mathbb{R}$ such that, for any $\xi\in L_0^H$ and $g\in G$, and for any $\xi\in L_0^G$ and $g\in H$, we have:*

$$M_1\leq B_\xi(g^{-1}o,o)-d(o,go)\leq M_2$$

Corollary 7. *Let M_1 and M_2 be given as in Lemma 8. We then have that, for any $\xi\in L_0^H$ and $g\in G$, and for any $\xi\in L_0^G$ and $g\in H$, the positive constants $c_1:=e^{\delta M_1}$ and $c_2:=e^{\delta M_2}$ satisfy:*

$$c_1\leq\frac{e^{-\delta d(o,go)}}{e^{-\delta B_\xi(g^{-1}o,o)}}\leq c_2$$

Lemma 9. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that a_n tends to infinity for n tending to infinity. Suppose that the series $\sum_n a_n^{-s}$ has abscissa of convergence equal to s_0. Then s_0 is also the abscissa of convergence for the series $\sum_n\log(a_n)\ a_n^{-s}$.*

Lemma 7 allows us to make the following estimate:

$$\begin{aligned}
&\sum_{g\in G^*}\int_{gL_0^H}\left|B_{\xi_+}(o,go)\right| d\mu_o(\xi_+)\\
\leq\ &\sum_{g\in G^*}\int_{gL_0^H} d(o,go)d\mu_o(\xi_+)\\
=\ &\sum_{g\in G^*} d(o,go)\int_{gL_0^H} d\mu_o(\xi_+)\\
=\ &\sum_{g\in G^*} d(o,go)\mu_o(gL_0^H)
\end{aligned}\tag{5.15}$$

Now, by the δ-harmonicity of the Patterson measures, see Section 2.0.8, the last expression is equal to:

$$\sum_{g\in G^*} d(o,go)\mu_{g^{-1}o}(L_0^H)$$

By the δ-conformality of the Patterson measures, this can be rewritten as:

$$\sum_{g\in G^*} d(o,go)\int_{L_0^H} p(g^{-1}o,\xi)^{\delta} d\mu_o(\xi)$$

Using the properties of the Poisson kernel $p(\cdot,\cdot)$, this is the same as:

$$\sum_{g\in G^*} d(o,go)\int_{L_0^H} e^{-\delta B_\xi(g^{-1}o,o)} d\mu_o(\xi)$$

From Corollary 7 it now follows that:

$$\begin{aligned}
&\sum_{g\in G^*} d(o,go)\int_{L_0^H} e^{-\delta B_\xi(g^{-1}o,o)} d\mu_o(\xi)\\
\asymp\ &\sum_{g\in G^*} d(o,go)\int_{L_0^H} e^{-\delta d(o,go)} d\mu_o(\xi)\\
=\ &\sum_{g\in G^*} d(o,go)e^{-\delta d(o,go)}\int_{L_0^H} d\mu_o(\xi)\\
=\ &\sum_{g\in G^*} d(o,go)e^{-\delta d(o,go)}\mu_o(L_0^H)
\end{aligned}\tag{5.16}$$

Since $\mu_o(L_0^H) \leq 1$, the last expression can be bounded by:

$$\sum_{g \in G^*} d(o, go) e^{-\delta d(o,go)}$$

Let us rewrite this expression as:

$$\sum_{g \in G^*} \log(e^{d(o,go)}) e^{-\delta d(o,go)}$$

Recall that here δ is equal to the Poincaré exponent $\delta(\Gamma)$ of the group Γ. It now follows from the fact that $\delta(G) < \delta(\Gamma)$ and Lemma 9 that the above series converges.

Proof. Lemma 7
It follows from the geometrical interpretation of the Busemann function that, for any point z in $\mathbb{D}$, the value of $B_\xi(o, z)$ ranges between $d(o, z)$ and $-d(o, z)$, depending on the choice of ξ. To see this, recall that the distance between two horocycles based at a point ξ in S^1 can be measured along any geodesic with endpoint at ξ. This gives a convenient interpretation of the value of $|B_\xi(o, z)|$. In particular, consider the horocycle based at ξ and containing z. Denote by x the point where this horocycle intersects the geodesic containing o with endpoint at infinity ξ. Then, the value of $|B_\xi(o, z)|$ is equal to the distance $d(o, x)$. Notice that any point of the horocycle based at ξ and containing z, in particular z itself, lies outside of the open hyperbolic ball of radius $|B_\xi(o, z)|$ centered at o; so $d(o, z) \geq |B_\xi(o, z)|$, see Figure 5.1. In fact, $B_\xi(o, z)$ varies continuously with ξ. It is equal to $d(o, z)$ precisely when ξ is the endpoint at infinity of the geodesic ray from o which contains z. It is equal to 0 precisely when ξ is a basepoint of one of the two horocycles containing both o and z. Finally, it is equal to $-d(o, z)$ precisely when ξ is the endpoint at infinity of the geodesic ray from z which contains o. □

Proof. Lemma 8
Let ω be a constant with respect to which Γ satisfies condition (2) in the definition of the $\mathcal{P}$-property. Fix a point ξ_0 in S^1 and let η_1 and η_2 be the two points in S^1 whose chordal distance to ξ_0 is equal to ω. Consider the horocycle based at ξ_0 and containing o. Let z_1 denote the point where this horocycle intersects the geodesic ray between o and η_1. Similarly, let z_2 denote the point where this horocycle intersects the geodesic ray between o and η_2. The location of these points has been depicted in Figure 5.2. Choose $R > 0$ large enough so that the open hyperbolic ball $B(o, R)$ contains the points z_1 and z_2. Put:

$$M_0 := \sup \left\{ -2d(g^{-1}o, o) : g \in G \cup H,\ d(g^{-1}o, o) \leq R \right\}$$

Now suppose that ξ is a point in L_0^H and g an element of G. If $d(g^{-1}o, o) \leq R$, then:

$$B_\xi(g^{-1}o, o) - d(o, go) = B_\xi(g^{-1}o, o) - d(g^{-1}o, o) \geq M_0$$

If $d(g^{-1}o, o) > R$, then consider the point z_g which is defined as the point of intersection of the geodesic ray between $g^{-1}o$ and ξ with the horocycle based at ξ containing o, see

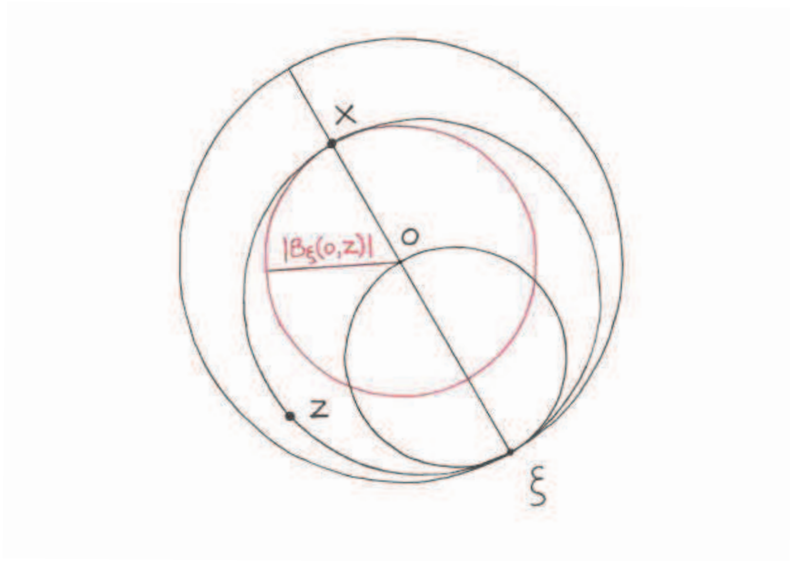


Figure 5.1:

Figure 5.3. Notice that, since $d(g^{-1}o, o) > R$, $\xi \in L_0^H$ and $g \in G$, the definition of R implies that $g^{-1}o$ lies outside the horocycle based at ξ containg o. So, z_g is well defined, and moreover, $d(z_g, o) \le R$. Observe that by the triangle inequality:

$$d(g^{-1}o, o) \le B_\xi(g^{-1}o, o) + d(z_g, o)$$

Hence, we have:

$$B_\xi(g^{-1}o, o) - d(o, go) = B_\xi(g^{-1}o, o) - d(g^{-1}o, o) \ge -R$$

The argument in the case $\xi \in L_0^G$ and $g \in H$ is completely analogous. By setting $M_1 := \min(M_0, -R)$, we obtain the first part of Lemma 8. The second part of Lemma 8 follows from the basic property $B_\eta(x, y) = -B_\eta(y, x)$ and Lemma 7, which allows us to set $M_2 := 0$. □

This finishes the proof of Theorem 13.

5.5 Examples for the failure of the $\mathcal{P}$-property

In Section 5.2 we described methods of constructing examples of groups in the $\mathcal{P}$-class. Now we turn to the complementary problem - what groups are not in the $\mathcal{P}$-class? We will describe some instances where either a geometric Schottky group does not have the $\mathcal{P}$-property or a group of Schottky type cannot be an element of the $\mathcal{P}$-class. We will first consider some simpler cases, and then address the question of whether normal subgroups of finitely generated groups of Schottky type can belong to the $\mathcal{P}$-class.

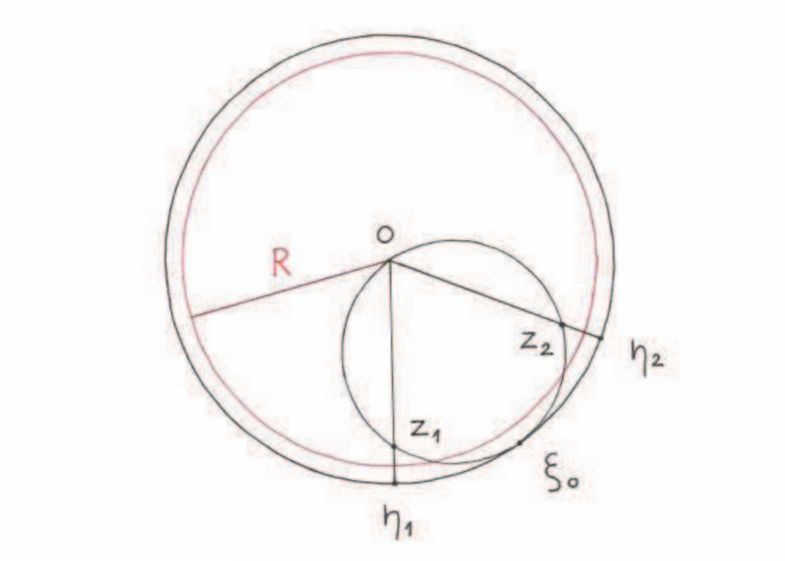


Figure 5.2:

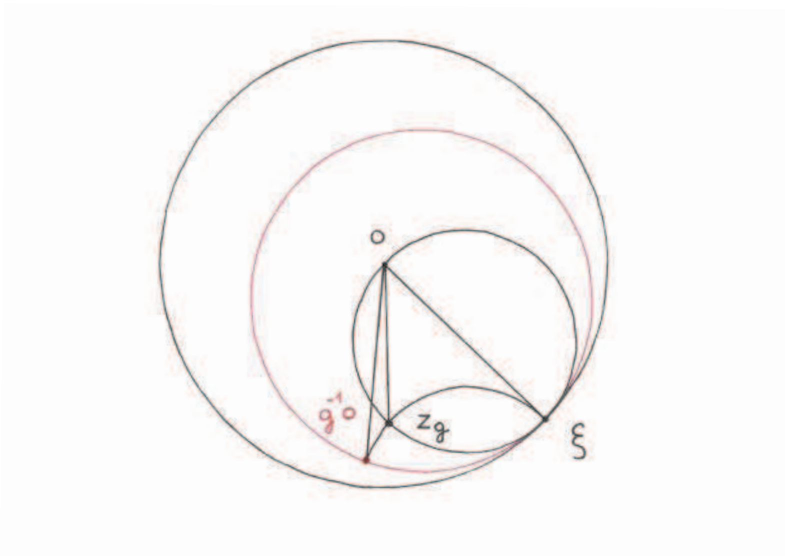


Figure 5.3:

5.5.1 Basic examples

We start with the geometric Schottky group Γ with two generators given in Section 5.2. So Γ has a Schottky description $(\{A_k\}, \{g_k\})_{i\in I}$ with $I = \{\pm 1, \pm 2\}$. To obtain our first examples for the failure of the $\mathcal{P}$-property, we are going to consider two infinitely generated subgroups of Γ. One will be the same as the normal group H_0 from Section 5.2;

the other group, denoted H_1, will be a slight variation on the construction of H_0, where one does not obtain a normal subgroup. We will consider these groups together with their Schottky descriptions, which in case of H_0 is the same as the one obtained in Section 5.2, and in the case of the group H_1 a Schottky description constructed in an analogous way as the one for H_0.

Remark 15. In fact, we can obtain further examples by considering other collections of intervals than the particular ones chosen in Section 5.2. But from a geometric point of view these examples would be very similar to the particular cases which we consider here.

First basic example for the failure of the $\mathcal{P}$-property

To obtain the first basic example, let us consider the following group, which already appeared in Section 5.2:

$$H_0 := \left\langle g_1^k g_2 g_1^{-k} : \quad k \in \mathbb{Z} \right\rangle$$

Let $(\{B_i\}, \{h_i\})_{i \in \mathbb{Z}^*}$ be the Schottky description for H_0 constructed in Section 5.2. In particular, the generators h_i in this Schottky description are of the form $g_1^k g_2^\epsilon g_1^{-k}$, with $k \in \mathbb{Z}$ and $\epsilon \in \{1, -1\}$. We will show that this geometric Schottky group cannot have the $\mathcal{P}$-property.

Consider an arbitrary partition $I_1 \dot{\cup} I_2$ of $\mathbb{Z}^*$ and put:

$$G := \langle h_i : i \in I_1 \rangle \quad \text{and} \quad H := \langle h_i : i \in I_2 \rangle$$

We now assume that condition (2) in the definition of the $\mathcal{P}$-property is satisfied and aim to show that $\delta(H_0) = max(\delta(G), \delta(H))$. Firstly, note that, because of how G and H were defined and since we are assuming that condition (2) in the definition of the $\mathcal{P}$-property is satisfied, we have that $L(G) \cap L(H) = \emptyset$. Observe that, for any increaing sequence $(a_k) \in \mathbb{N}^{\mathbb{N}}$, the sequence of points $g_1^{a_k} g_2 g_1^{-a_k} o$ accumulates at $\xi := \lim_{k \to \infty} g_1^k o$. This follows from our discussion of the group H_0 in Section 5.2. The geometric picture allows us to determine the location of the orbit points $g_1^{a_k} g_2 g_1^{-a_k} o$, so that we see that they are situated under geodesics that accumulate at the point ξ. So, if for some inreasing sequence $(a_k) \in \mathbb{N}^{\mathbb{N}}$ we have $g_1^{a_k} g_2 g_1^{-a_k} \in G$, then $\xi \in L(G)$; and similarly, if for some increasing sequence $(b_k) \in \mathbb{N}^{\mathbb{N}}$ we have $g_1^{b_k} g_2 g_1^{-b_k} \in H$, then $\xi \in L(H)$. It follows that there exists some $N \in \mathbb{N}$ such that either for all $k \geq N$, we have $g_1^k g_2 g_1^{-k} \in G$, or for all $k \geq N$, we have $g_1^k g_2 g_1^{-k} \in H$. Without loss of generality, assume that the former is the case. We will show that $\delta(H_0) = \delta(G)$.

In order to show this, we start by defining the following subgroup of G:

$$G_0 := \left\langle g_1^k g_2 g_1^{-k} : \; k \geq N \right\rangle$$

Proposition 32.

$$H_0 = \bigcup_{k\geq 0} g_1^{-k} G_0 g_1^k$$

Proof. Firstly, since H_0 is normal in Γ, we have that:

$$\bigcup_{k\geq 0} g_1^{-k} G_0 g_1^k \subseteq H_0$$

In order to obtain the other inclusion, consider an arbitrary element γ of $H_0 - \{id\}$. From the definition of H_0 we see that γ is of the form:

$$\gamma = g_2^{k_1} g_1^{t_1} g_2^{k_2} g_1^{t_2} \dots g_2^{k_m} g_1^{t_m} g_2^{k_{m+1}}$$

where k_1 and k_{m+1} might be equal to 0, whereas the remaining indices are non-zero, and where $\sum_{n=1}^{m} t_n = 0$. Choose $N_0 \in \mathbb{N}$ large enough so that:

$$N_0 + \inf_{1\leq i\leq m} \sum_{n=1}^{i} t_n > N$$

Put $K_1 := N_0$ and $K_i := N_0 + \sum_{n=1}^{i-1} t_n$, for $i \geq 2$, and consider the element:

$$g := g_1^{K_1} g_2^{k_1} g_1^{-K_1} g_1^{K_2} g_2^{k_2} g_1^{-K_2} g_1^{K_3} \dots g_1^{-K_m} g_1^{K_{m+1}} g_2^{k_{m+1}} g_1^{-K_{m+1}}$$

Now observe that:

$$\gamma = g_1^{-K_1} g g_1^{K_1}$$

Since $g_1^{K_i} g_2^{k_i} g_1^{-K_i} \in G_0$, for $1 \geq i \geq m+1$, we have $g \in G_0$, which finishes the proof. □

Now our general strategy is to apply the following theorem, which is a special case of a theorem due to Sullivan [Sullivan1979] and which we have already used in Section 4.4.

Theorem 14.
Let G ba a Fuchsian group and, for each $i \in \mathbb{N}_0$, let G_i be a subgroup of G such that

- $G_i < G_{i+1}$, *for all* $i \in \mathbb{N}_0$
- $G = \bigcup_i G_i$

Then the Poincaré exponent of G is given by:

$$\delta(G) = \sup_i \delta(G_i)$$

In order to apply the above theorem, we define for $i \in \mathbb{N}_0$:

$$\Gamma_i := g_1^{-i} G_0 g_1^i$$

Proposition 33. *For all $i \in \mathbb{N}_0$, we have:*

$$\Gamma_i < \Gamma_{i+1}$$

Proof. Let γ be an arbitrary element of Γ_i. We can express γ as a product of elements of the form $g_1^{-i} g g_1^i$, where g is some generator of G_0. Thus, it is enough to show that all elements of this form lie in Γ_{i+1}. So, let g be a generator of G_0, that is, $g = g_1^k g_2^\epsilon g_1^{-k}$ for some $k \geq N$ and $\epsilon \in \{1, -1\}$. Then we have:

$$g_1^{-i} g g_1^i = g_1^{-i} g_1^k g_2^\epsilon g_1^{-k} g_1^i = g_1^{-(i+1)} g_1^{k+1} g_2^\epsilon g_1^{-(k+1)} g_1^{i+1}$$

The latter element lies in Γ_{i+1}, since $g_1^{k+1} g_2^\epsilon g_1^{-(k+1)} \in G_0$. $\square$

Now, Proposition 32 and Proposition 33 allow us to apply Theorem 14, which gives:

$$\delta(H_0) = \sup_i \ \delta(\Gamma_i)$$

The following Lemma tells us that the value of $\delta(\Gamma_i)$ does not depend on $i \in \mathbb{N}_0$ and thus allows us to calculate $\delta(H_0)$.

Lemma 10. *Let G be a Fuchsian group. Then, for any $\gamma \in \mathrm{Iso}^+(\mathbb{D})$, we have:*

$$\delta(G) = \delta(\gamma G \gamma^{-1})$$

Proof. Consider the Poincaré series:

$$\sum_{h \in \gamma G \gamma^{-1}} e^{-sd(o,ho)} = \sum_{g \in G} e^{-sd(o,\gamma g \gamma^{-1} o)} = \sum_{g \in G} e^{-sd(\gamma^{-1} o, g \gamma^{-1} o)}$$

Clearly, this is the Poincaré series of G with $x = y = \gamma^{-1} o$. Since the abscissa of convergence is independent of the choice of x and y, the result follows. $\square$

Therefore, for all $i \in \mathbb{N}_0$, we have:

$$\delta(\Gamma_i) = \delta(g_1^{-i} G_0 g_1^i) = \delta(G_0)$$

Since $G_0 < G$, we have:

$$\delta(H_0) = \delta(G_0) \leq \delta(G)$$

Finally, since $G < H_0$ implies $\delta(G) \leq \delta(H_0)$, we can conclude that:

$$\delta(H_0) = \delta(G)$$

This shows that the group H_0 with its Schottky description $(\{B_i\}, \{h_i\})_{i \in \mathbb{Z}^*}$ does not have the $\mathcal{P}$-property.

Second basic example for the failure of the $\mathcal{P}$-property

To obtain the second basic example, let us consider the group:

$$H_1 := \left\langle g_1^k g_2 g_1^{-k} : \quad k \in \mathbb{N}_0 \right\rangle$$

Let $(\{B_i\}, \{h_i\})_{i \in \mathbb{Z}^*}$ be the Schottky description for H_1 constructed in an analogous way as the Schottky description of the group H_0. In particular, the generators h_i in this Schottky description are of the form $g_1^k g_2^\epsilon g_1^{-k}$, with $k \in \mathbb{N}_0$ and $\epsilon \in \{1, -1\}$. We will show that this geometric Schottky group cannot have the $\mathcal{P}$-property.

We will argue in a very similar way as in the case of the group H_0. Consider an arbitrary partition $I_1 \dot{\cup} I_2$ of $\mathbb{Z}^*$ and put $G := \langle h_i : i \in I_1 \rangle$ and $H := \langle h_i : i \in I_2 \rangle$. We now assume that condition (2) in the definition of the $\mathcal{P}$-property is satisfied and aim to show that $\delta(H_1) = max(\delta(G), \delta(H))$. Firstly, note that because of how G and H were defined and since we are assuming that condition (2) in the definition of the $\mathcal{P}$-property is satisfied, we have $L(G) \cap L(H) = \emptyset$. Observe that, for any increasing sequence $(a_k) \in \mathbb{N}^{\mathbb{N}}$, the sequence of points $g_1^{a_k} g_2 g_1^{-a_k} o$ accumulates at $\xi := \lim_{k\to\infty} g_1^k o$. So, if for some increasing sequence $(a_k) \in \mathbb{N}^{\mathbb{N}}$ we have $g_1^{a_k} g_2 g_1^{-a_k} \in G$, then $\xi \in L(G)$; and similarly, if for some increasing sequence $(b_k) \in \mathbb{N}^{\mathbb{N}}$ we have $g_1^{b_k} g_2 g_1^{-b_k} \in H$, then $\xi \in L(H)$. It follows that there exists some $N \in \mathbb{N}$ such that, either for all $k \geq N$, we have $g_1^k g_2 g_1^{-k} \in G$, or for all $k \geq N$, we have $g_1^k g_2 g_1^{-k} \in H$. Without loss of generality, assume that the former is the case. We will show that $\delta(H_1) = \delta(G)$.

In order to show this, we start by defining the following subgroup of G:

$$G_0 := \left\langle g_1^k g_2 g_1^{-k} : \ k \geq N \right\rangle$$

Proposition 34.

$$H_1 = g_1^{-N} G_0 g_1^N$$

Proof. Firstly, although the group H_1 is not normal in Γ, by considering the conjugates of generators of G_0, it is easy to verify that:

$$g_1^{-N} G_0 g_1^N \subseteq H_1$$

In order to obtain the other inclusion, consider an arbitrary element γ of H_1. We can express γ in terms of the generators of H_1 in the following way:

$$\gamma = g_1^{t_1} g_2^{k_1} g_1^{-t_1} \dots g_1^{t_m} g_2^{k_m} g_1^{-t_m}$$

where $t_i \geq 0$, and then express it as:

$$\gamma = g_1^{-N} (g_1^N g_1^{t_1} g_2^{k_1} g_1^{-t_1} g_1^{-N} g_1^N \dots g_1^{-N} g_1^N g_1^{t_m} g_2^{k_m} g_1^{-t_m} g_1^{-N}) g_1^N$$

The latter element clearly lies in $g_1^{-N} G_0 g_1^N$, which finishes the proof. $\square$

Now, Lemma 10 states that $\delta(g_1^{-N}G_0g_1^N)$ is equal to $\delta(G_0)$. Therefore, since $G_0 < G$, we have:

$$\delta(H_1) = \delta(G_0) \leq \delta(G)$$

and, since $G < H_1$ implies $\delta(G) \leq \delta(H_1)$, we can conclude that:

$$\delta(H_1) = \delta(G)$$

This shows that the group H_1 with its Schottky description $(\{B_i\}, \{h_i\})_{i\in\mathbb{Z}^*}$ does not have the $\mathcal{P}$-property.

Yet the above argument does not prove that H_1 does not belong to the $\mathcal{P}$-class, since we have chosen a particular Schottky description. It is possible to show that H_1 does not belong to the $\mathcal{P}$-class in an indirect way. Namely, we use a theorem of Matsuzaki and Yabuki [MatsuzakiYabuki2009], which relates the divergence type of a Fuchsian group to a purely algebraic property.

Theorem 15. ***Matsuzaki, Yabuki***
*Let G be a Fuchsian group. Suppose G has **proper conjugation**, that is, there exists $\gamma \in \mathrm{Iso}^+(\mathbb{D})$ such that the conjugate $\gamma G\gamma^{-1}$ is a proper subgroup of G. Then G is of convergence type.*

Proof. For the proof we refer the reader to the original paper [MatsuzakiYabuki2009], where this result appears in greater generality. □

In Section 5.3 we have shown that all groups in the $\mathcal{P}$-class are of divergence type. So, to show that the group H_1 does not belong to the $\mathcal{P}$-class, it is enough to show that H_1 has the proper conjugation property, as defined in Theorem 15.

Proposition 35. *The group H_1 has the proper conjugation property.*

Proof. Consider the conjugate $g_1H_1g_1^{-1}$. This is clearly a subgroup of H_1. It is also easy to see that $g_2 \notin g_1H_1g_1^{-1}$. This finishes the proof. □

Basic classes of non-examples

The approach used to show that the group H_1 does not belong to the $\mathcal{P}$-class is summarised in the following theorem, which provides a whole class of examples of groups not in the $\mathcal{P}$-class.

Proposition 36. *Let G be a Fuchsian group. Suppose G has the proper conjugation property. Then G does not belong to the $\mathcal{P}$-class.*

Proof. This follows immediately from Theorem 15 and the fact that all groups in the $\mathcal{P}$-class are of divergence type. □

In their paper Matsuzaki and Yabuki also give the following result (Corollary 4.3 in [MatsuzakiYabuki2009]), which allows us to deduce further results about the $\mathcal{P}$-class.

Proposition 37. *Let G be a Fuchsian group of divergence type. Suppose there exists a Fuchsian group Γ which contains G as a normal subgroup. Then $\delta(\Gamma) = \delta(G)$ and Γ is of divergence type.*

Proof. For the proof we refer the reader to the original paper [MatsuzakiYabuki2009], where a more general version of this result appears. □

With this result at hand, we can specify another class of Fuchsian groups which do not belong to the $\mathcal{P}$-class. Namely, we have the following proposition:

Proposition 38. *Let G be a normal subgroup of a Fuchsian group of convergence type. Then G does not belong to the $\mathcal{P}$-class.*

Proof. Suppose G is a normal subgroup of a Fuchsian group Γ, which is of convergence type. If G belonged to the $\mathcal{P}$-class, then it would be of divergence type. But then Proposition 37 would imply that the group Γ is of divergence type, which would be a contradiction. □

We obtained the above class of examples using only the second part of the statement in Proposition 37. Now we will also employ the information about the Poincaré exponents provided by Proposition 37. In particular, we will combine Proposition 37 with a theorem which describes the conditions under which a normal subgroup G of a Fuchsian group Γ satisfies $\delta(\Gamma) = \delta(G)$. Namely, we will appeal once again to the Brooks-Stadlbauer Theorem, which already appeared in Section 5.2.2 as Theorem 9.

Theorem 16. ***Brooks-Stadlbauer***
Let N be a normal subgroup of a finitely generated geometric Schottky group Γ. Then:

$$\delta(\Gamma) = \delta(N) \Leftrightarrow \Gamma/N \textit{ amenable}$$

The Brooks-Stadlbauer Theorem allows us to obtain a further class of Fuchsian groups which do not belong to the $\mathcal{P}$-class.

Proposition 39. *Let G be a normal subgroup of a finitely generated geometric Schottky group Γ such that Γ/G is non-amenable. Then G does not belong to the $\mathcal{P}$-class.*

Proof. If G belonged to the $\mathcal{P}$-class, then it would be of divergence type and Proposition 37 would imply that $\delta(\Gamma) = \delta(G)$. But then, by Theorem 9, we would have that Γ/G is amenable, which would be a contradiction. □

We now apply Proposition 39 to give an explicit construction of a class of groups which do not belong to the $\mathcal{P}$-class.

Let Γ be a finitely generated geometric Schottky group and $(\{A_i\}, \{g_i\})_{i\in I}$ its Schottky description, where $I = \{\pm 1, \ldots, \pm n\}$ with $n \geq 3$. Choose some set

$$W \subset \langle g_i : i \geq 3 \rangle$$

Now let G be the smallest normal subgroup of Γ containing W. The quotient Γ/G contains a free group isomorphic to the group $\langle g_1, g_2 \rangle$. Since a group which contains a free group with two elements is non-amenable, see [delaHarpe], it follows from Proposition 39 that G does not belong to the $\mathcal{P}$-class.

5.5.2 The case of normal subgroups

In the previous subsection we have already described several situations in which a normal subgroup of a finitely generated Fuchsian group of Schottky type is not in the $\mathcal{P}$-class. In this subsection we will treat the general case. Namely, we will address the question: When does a normal subgroup of a finitely generated Fuchsian group of Schottky type belong to the $\mathcal{P}$-class? We will provide a complete solution to this problem.

Theorem 17. *Let Γ be a finitely generated geometric Schottky group and let H be a normal subgroup of Γ. If the index $[\Gamma, H]$ is finite then H belongs to the $\mathcal{P}$-class, and if the index $[\Gamma, H]$ is infinite then H does not belong to the $\mathcal{P}$-class.*

Let Γ be a geometric Schottky group and $(\{A_k\}, \{g_k\})_{k \in I}$ its Schottky description where $|I| < \infty$. Let H be a normal subgroup of Γ. We start by distinguishing three cases:

Case 1. The index $[\Gamma : H]$ of H in Γ is finite.
Case 2. The index $[\Gamma : H]$ of H in Γ is infinite and H is of convergence type.
Case 3. The index $[\Gamma : H]$ of H in Γ is infinite and H is of divergence type.

Case 1.

Suppose the index of H in Γ is finite. We can construct a Schottky description for H as described in Section 3.3. Then the full set of right coset representatives obtained in this construction consists of finitely many elements. The fundamental domain for H obtained in this construction is obtained, by formula (3.2) in Section 3.3, from finitely many copies of the standard fundamental domain of Γ. Hence, this fundamental domain has a boundary consisting of finitely many geodesics and thus, the Schottky description for H, which we obtain in the construction, is of the form $(\{B_k\}, \{h_k\})_{k \in I_H}$, where I_H is a finite subset of $\mathbb{Z}^*$. This shows that H is a finitely generated group of Schottky type and thus it belongs to the $\mathcal{P}$-class.

Case 2.

We have shown in Section 5.3 that all groups in the $\mathcal{P}$-class are of divergence type. So, if H is of convergence type, it cannot belong to the $\mathcal{P}$-class.

Case 3.

Suppose that the index of H in Γ is infinite and that H is of divergence type. We are going to show that:

$$\nu(T^1(\mathbb{D}/H)) = \infty$$

where ν denotes the Liouville-Patterson measure associated to H. Since in Section 5.4 we have shown that for all groups in the $\mathcal{P}$-class their associated Liouville-Patterson measure is finite, it will then follow that H cannot belong to the $\mathcal{P}$-class.

The starting point of our argument is the following theorem of Matsuzaki and Yabuki which appears in [MatsuzakiYabuki2009] as Theorem 4.2.

Theorem 18. *Let G be a Fuchsian group of divergence type and let μ_z be the Patterson measure with respect to the point $z \in \mathbb{D}$. If an element $\gamma \in \mathrm{Iso}^+(\mathbb{D})$ satisfies $\gamma G \gamma^{-1} = G$, then:*

$$\gamma^* \mu_z = \mu_{\gamma^{-1}z}$$

Here $\gamma^*\mu_z$ is the measure defined by $\gamma^*\mu_z(E) = \mu_z(\gamma E)$.

If μ_z denotes the Patterson measure with respect to the point $z \in \mathbb{D}$ associated to the group H, then, from the above theorem and the fact that H is normal in Γ, it follows that for any $\gamma \in \Gamma$, we have:

$$\gamma^* \mu_z = \mu_{\gamma^{-1}z}$$

Now we prove the following Lemma, which is the core of our argument which shows that $\nu(T^1(\mathbb{D}/H))$ is infinite.

Lemma 11. *Let G be a Fuchsian group, μ_o its Patterson measure with respect to the point o, and $\widetilde{\nu}$ the associated measure which projects to its Liouville-Patterson measure, as defined in Section 2.0.12. If for some $\gamma \in \mathrm{Iso}^+(\mathbb{D})$ we have:*

$$\gamma^* \mu_o = \mu_{\gamma^{-1}o}$$

then the measure $\widetilde{\nu}$ is invariant under the action of γ, in the sense that for any Borel set $E \subseteq T^1\mathbb{D}$ we have:

$$\widetilde{\nu}(E) = \widetilde{\nu}(\gamma E)$$

Proof. Lemma 11 Let E be an arbitrary Borel subset of $T^1\mathbb{D}$ and let $\gamma \in \mathrm{Iso}^+(\mathbb{D})$ be an element satisfying $\gamma^*\mu_o = \mu_{\gamma^{-1}o}$. We can write E in terms of the Busemann parametrisation of $T^1\mathbb{D}$. In particular, there exist sets $X_-, X_+ \subseteq S^1$ and for each pair $(\eta, \xi) \in X_- \times X_+ \setminus diag$ a set $A_{\eta\xi} \subseteq \mathbb{R}$ (this set might be the empty set) such that:

$$E = \{(\eta, \xi, r) : \ \eta \in X_-, \ \xi \in X_+, \ r \in A_{\eta\xi}\}$$

Now, since $\gamma \cdot (\eta, \xi, r) = (\gamma\eta, \gamma\xi, r - B_\xi(o, \gamma^{-1}o))$, we have:

$$\begin{aligned} \gamma E &= \left\{(\gamma\eta, \gamma\xi, r - B_\xi(o, \gamma^{-1}o)) : \ \eta \in X_-, \ \xi \in X_+, \ r \in A_{\eta\xi}\right\} \\ &= \left\{(\gamma\eta, \gamma\xi, t) : \ \eta \in X_-, \ \xi \in X_+, \ t \in A_{\eta\xi} - B_\xi(o, \gamma^{-1}o)\right\} \end{aligned} \tag{5.17}$$

Here, for $Y \subseteq \mathbb{R}$ and $q \in \mathbb{R}$, we used the notation $Y - q := \{y - q : \ y \in Y\}$. We can now calculate $\widetilde{\nu}(\gamma E)$ as follows:

$$\widetilde{\nu}(\gamma E) = \int_{\gamma E} d\widetilde{\nu} = \int_{X_-} \int_{X_+} \int_{A_{\eta\xi} - B_\xi(o,\gamma^{-1}o)} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \ ds \ d\mu_o(\gamma\xi) \ d\mu_o(\gamma\eta)$$

$$= \int_{X_-} \int_{X_+} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \left(\int_{A_{\eta\xi} - B_\xi(o,\gamma^{-1}o)} ds \right) d\mu_o(\gamma\xi) \ d\mu_o(\gamma\eta)$$

$$= \int_{X_-} \int_{X_+} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \left(\int_{A_{\eta\xi}} ds \right) d\mu_o(\gamma\xi) \ d\mu_o(\gamma\eta) \tag{5.18}$$

$$= \int_{X_-} \int_{X_+} \int_{A_{\eta\xi}} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \ ds \ d\mu_o(\gamma\xi) \ d\mu_o(\gamma\eta)$$

$$= \int_{X_-} \int_{X_+} \int_{A_{\eta\xi}} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \ ds \ d\mu_{\gamma^{-1}o}(\xi) \ d\mu_{\gamma^{-1}o}(\eta)$$

Recall from Section 2.0.8 that for $x_1, x_2 \in \mathbb{D}$ and $\xi_0 \in S^1$ we have:

$$\frac{d\mu_{x_1}}{d\mu_{x_2}}(\xi_0) = \left(\frac{p(x_1, \xi_0)}{p(x_2, \xi_0)} \right)^\delta$$

where $p(x, \xi_0) := \frac{1-|x|^2}{|x-\xi_0|^2}$ denotes the Poisson kernel, which means that:

$$\frac{d\mu_{\gamma^{-1}o}(\xi_0)}{d\mu_o(\xi_0)} = \left(\frac{1 - |\gamma^{-1}o|^2}{|\xi_0 - \gamma^{-1}o|^2} \right)^\delta$$

Therefore, we have:

$$\widetilde{\nu}(\gamma E) = \int_{X_-} \int_{X_+} \int_{A_{\eta\xi}} \frac{1}{|\gamma\eta - \gamma\xi|^{2\delta}} \left(\frac{1 - |\gamma^{-1}o|^2}{|\xi - \gamma^{-1}o|^2} \right)^\delta \left(\frac{1 - |\gamma^{-1}o|^2}{|\eta - \gamma^{-1}o|^2} \right)^\delta ds \ d\mu_o(\xi) \ d\mu_o(\eta)$$

Finally, we make use of the following equality, which is obtained by combining two standard results, see [Nicholls] (Equation 1.3.2 and Theorem 1.3.4.):

$$|\gamma\eta - \gamma\xi|^{2\delta} = |\gamma'(\xi)|^\delta |\gamma'(\eta)|^\delta |\eta - \xi|^{2\delta} = \left(\frac{1 - |\gamma^{-1}o|^2}{|\xi - \gamma^{-1}o|^2} \right)^\delta \left(\frac{1 - |\gamma^{-1}o|^2}{|\eta - \gamma^{-1}o|^2} \right)^\delta |\eta - \xi|^{2\delta}$$

By substituting, we obtain:

$$\widetilde{\nu}(\gamma E) = \int_{X_-} \int_{X_+} \int_{A_{\eta\xi}} \frac{1}{|\eta - \xi|^{2\delta}} \ ds \ d\mu_o(\xi) \ d\mu_o(\eta) = \widetilde{\nu}(E)$$

□

Now recall the construction of a fundamental domain of a subgroup given in Section 3.3. Let $F := F(\Gamma)$ be the standard fundamental domain for Γ. The construction from Section 3.3 yields a set $S \subseteq \Gamma$ such that the following set is a fundamental domain for the normal subgroup H:

$$F_H = int\Big(\bigcup_{\gamma \in S} \overline{\gamma F} \Big)$$

The set S is a full set of right coset representatives for H, so, since the index of H in Γ is infinite, the set S is infinite. For a set $X \subseteq \mathbb{D}$, define:

$$T^1_X\mathbb{D} := \big\{ v \in T^1\mathbb{D} : \pi_b(v) \in X \big\}$$

Then, by the definition of the Liouville-Patterson measure ν, we have:

$$\nu(T^1(\mathbb{D}/H)) \geq \widetilde{\nu}(T^1_{F_H}\mathbb{D})$$

Observe that $T^1_{\gamma F}\mathbb{D} = \gamma \cdot T^1_F\mathbb{D}$, and therefore:

$$\widetilde{\nu}(T^1_{F_H}\mathbb{D}) \geq \sum_{s \in S} \widetilde{\nu}(T^1_{sF}\mathbb{D}) = \sum_{s \in S} \widetilde{\nu}(s \cdot T^1_F\mathbb{D}) = \sum_{s \in S} \widetilde{\nu}(T^1_F\mathbb{D})$$

Here, the last equality follows from Lemma 11, using the fact that $S \subseteq \Gamma$. Also, the last sum is infinite, since $\widetilde{\nu}(T^1_F\mathbb{D}) > 0$, which shows that $\nu(T^1(\mathbb{D}/H)) = \infty$.

This finishes the proof of Theorem 17.

5.6 Patterson measure of the uniformly radial limit set

In this section we will use the fact that for a group Γ in the $\mathcal{P}$-class, the geodesic flow on the manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure ν, and that this measure is finite to prove the following theorem:

Theorem 19. *Let Γ be an infinitely generated group in the $\mathcal{P}$-class. Then the Patterson measure μ_o of the uniformly radial limit set $L_{ur}(\Gamma)$ is equal to zero.*

This situation contrasts the case of a group of Schottky type Γ which is finitely generated, since for such a group we have $L_{ur}(\Gamma) = L(\Gamma)$ and thus $\mu_o(L_{ur}(\Gamma))$ equal to one.

For an infinitely generated group Γ in the $\mathcal{P}$-class we then have that $\mu_o(L_{ur}(\Gamma)) = 0$, while, as has been shown in Section 5.3, $\mu_o(L_r(\Gamma)) = 1$. This is an interesting observation in view of the fact that $\dim_{\mathrm{H}}(L_{ur}(\Gamma)) = \dim_{\mathrm{H}}(L_r(\Gamma))$, as shown by Stratmann in [Stratmann2004], since it means that the Patterson measure ignores one set of dimension $\delta(\Gamma)$ while assigning full measure to another set of dimension $\delta(\Gamma)$.

Throughout this section let Γ be a geometric Schottky group and $(\{A_k\}, \{g_k\})_{k \in I}$ its Schottky description. Assume that the indexing set I is infinite and that the group Γ

has the $\mathcal{P}$-property. Our aim is to prove that $\mu_o(L_{ur}(\Gamma)) = 0$ and we will show this in several steps. The starting point in our argument will be to employ the results of Section 3.2.3. Recall Theorem 3, which stated that the set $L_{ur}(\Gamma)$ consists of precisely those points ξ in the limit set $L(\Gamma)$, for which the coding sequence $\kappa(\xi) = [x_0, x_1, \ldots]$ satisfies that $x_i \neq 0$ and $|x_i| \leq N$, for all $i \in \mathbb{N}_0$ and for some $N \in \mathbb{N}$, where N might depend on the point ξ. If a coding sequence satisfies such condition, we will say that the sequence **contains no 0's** and is **bounded by N**. For each $N \in \mathbb{N}$, define the following subset of the limit set:

$$L_{ur}^N(\Gamma) := \{\xi \in L(\Gamma) : \kappa(\xi) \text{ contains no 0's and is bounded by } N\}$$

We then clearly have:

$$L_{ur}(\Gamma) = \bigcup_{N \in \mathbb{N}} L_{ur}^N(\Gamma)$$

We are going to consider the intersections of the sets $L_{ur}^N(\Gamma)$ with the intervals A_k and show that for every $k \in I$ and $N \in \mathbb{N}$ we have:

$$\mu_o(L_{ur}^N(\Gamma) \cap A_k) = 0$$

With this result we will be able to deduce that:

$$\mu_o\left(L_{ur}^N(\Gamma)\right) = \mu_o\left(\bigcup_{k \in I} L_{ur}^N(\Gamma) \cap A_k\right) = \sum_{k \in I} \mu_o\left(L_{ur}^N(\Gamma) \cap A_k\right) = 0$$

and hence that:

$$\mu_o\left(L_{ur}(\Gamma)\right) = \mu_o\left(\bigcup_{N \in \mathbb{N}} L_{ur}^N(\Gamma)\right) \leq \sum_{N \in \mathbb{N}} \mu_o\left(L_{ur}^N(\Gamma)\right) = 0$$

In order to show that $\mu_o(L_{ur}^N(\Gamma) \cap A_k) = 0$, for every $k \in I$ and $N \in \mathbb{N}$, we fix k and N and then consider the following subset of $T^1\mathbb{D}$, defined using the standard parametrisation of $T^1\mathbb{D}$:

$$\mathcal{C} := \left\{(\xi, \eta, s) \in T^1\mathbb{D} : \ \xi \in A_j, \ \eta \in L_{ur}^N(\Gamma) \cap A_k, \ s \in [-\epsilon, \epsilon]\right\}$$

Here, $j \in I$ is chosen such that $j \neq k$, and $\epsilon > 0$ is chosen small enough so that $\pi_b(\mathcal{C})$ is contained in the standard fundamental domain $F(\Gamma)$ of Γ. Then we have:

$$\nu\left(\pi(\mathcal{C})\right) = \widetilde{\nu}\left(\mathcal{C}\right) = \int_{-\epsilon}^{\epsilon} \int_{L_{ur}^N(\Gamma) \cap A_k} \int_{A_j} \frac{1}{|\xi - \eta|^{2\delta}} \ d\mu_o(\xi) \ d\mu_o(\eta) \ ds$$

Here, $\delta := \delta(\Gamma)$ is the Poincaré exponent of Γ. Note that for $(\xi, \eta, s) \in \mathcal{C}$ we have:

$$\frac{1}{|\xi - \eta|^{2\delta}} \geq \frac{1}{2^{2\delta}}$$

Thus we have the following estimate:

$$\nu\left(\pi(\mathcal{C})\right) \geq \int_{-\epsilon}^{\epsilon} \int_{L_{ur}^{N}(\Gamma)\cap A_k} \int_{A_j} \frac{1}{2^{2\delta}}\ d\mu_o(\xi)\ d\mu_o(\eta)\ ds$$

$$= \frac{\epsilon}{2^{2\delta-1}}\ \mu_o\left(L_{ur}^{N}(\Gamma)\cap A_k\right)\ \mu_o\left(A_j\right)$$

Note that by Sullivan's Shadow Lemma, see Section 2.0.9 Theorem 1, it follows that $\mu_o\left(A_j\right) > 0$, since A_j contains the shadow of $B(g_j go, \rho)$ for some $\rho > 0$ large enough, and for g chosen so that $g_j go$ lies close enough to a limit point in the interior of A_j. We are going to show that $\nu\left(\pi(\mathcal{C})\right) = 0$, which by the previous remark will imply that $\mu_o\left(L_{ur}^{N}(\Gamma)\cap A_k\right) = 0$.

Let $k_1, k_2 \in I$ be such that $k_2 > N$ and $k_1 \neq k_2$. Consider the following subset of $T^1\mathbb{D}$:

$$\mathcal{E} := \left\{(\xi, \eta, s) \in T^1\mathbb{D} :\ \xi \in A_{k_1},\ \eta \in A_{k_2},\ s \in [-\epsilon, \epsilon]\right\}$$

Here, $\epsilon > 0$ is chosen small enough so that $\pi_b(\mathcal{E})$ is contained in $F(\Gamma)$. In a similar manner as before, we can estimate:

$$\nu\left(\pi(\mathcal{E})\right) = \widetilde{\nu}\left(\mathcal{E}\right) \geq \frac{\epsilon}{2^{2\delta-1}}\mu_0\left(A_{k_1}\right)\mu_0\left(A_{k_2}\right)$$

From Sullivan's Shadow Lemma it again follows that $\mu_0\left(A_{k_1}\right) > 0$ and $\mu_0\left(A_{k_2}\right) > 0$, so we have $\nu\left(\pi(\mathcal{E})\right) > 0$. Since the geodesic flow on $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure ν and $\nu\left(T^1\left(\mathbb{D}/\Gamma\right)\right) < \infty$, it follows by Birkhoff's Ergodic Theorem that for ν-almost-every vector $v \in T^1\left(\mathbb{D}/\Gamma\right)$ there exists an unbouded sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers such that $g^{t_n}v$ lies in $\pi(\mathcal{E})$, for all $n \in \mathbb{N}$. In particular if we define:

$$\mathcal{E}' := \left\{v \in T^1\left(\mathbb{D}/\Gamma\right) :\ g^t v \notin \pi\left(\mathcal{E}\right)\ \text{ for all } t \in \mathbb{R}_+\right\}$$

Then, we have that:

$$\nu\left(\mathcal{E}'\right) = 0$$

It is easy to convince oneself that $\pi(\mathcal{C}) \subset \mathcal{E}'$, which immediately gives $\nu\left(\pi(\mathcal{C})\right) = 0$ as desired. To see this, first observe that, by the definition of $\mathcal{E}$, for any vector $v \in \pi(\mathcal{E})$, the ray defined by v will intersect the geodesic $\pi(\alpha_{k_2})$. Then observe that, for any vector $u \in \pi(\mathcal{C})$, by definition of $\mathcal{C}$, the ray defined by u can only intersect a geodesic $\pi(\alpha_i)$ if $i \leq N$. Since for any $t \in \mathbb{R}_+$ and any $v \in T^1(\mathbb{D}/\Gamma)$ the ray defined by $g^t v$ is contained in the ray defined by v, the claim follows.

5.7 Saturated coding sequences and the Myrberg property

For a geometric Schottky group Γ with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$, we will consider a certain property, which we call saturatedness, of the coding sequences of points in the limit set $L(\Gamma)$. We will show that if Γ has the $\mathcal{P}$-property, then μ_o-almost-every point in $L(\Gamma)$ has a coding sequence which is saturated. This property will be deduced from the ergodicity of the geodesic flow on $\mathbb{D}/\Gamma$ and the finiteness of the measure ν.

In the second part of this section, we will clarify the relationship between the saturatedness of coding sequences and the Myrberg property, as defined by Stratmann in [Stratmann1997]. This will allow us to reformulate Stratmann's refinement of Sullivan's theorem to give a new condition for the ergodicity of the geodesic flow on $\mathbb{D}/\Gamma$ in the case of a group of Schottky type.

5.7.1 Saturated coding sequences

The saturatedness property which we introduce here is defined in terms of the coding κ from Section 3.2. Recall that for a point ξ in the limit set $L(\Gamma)$, its coding sequence $\kappa(\xi)$ is a sequence over the alphabet $I \cup \{0\}$, which we write in the form $[x_0, x_1, \ldots, x_i, \ldots]$. Proposition 6 described all the sequences which can occur as codes of points in $L(\Gamma)$. In particular in the admissible sequences a letter k can not be followed directly by the letter $-k$ and the letter 0 can only appear in an infinite string of 0's at the end of the sequence.

Let $\mathcal{W}_\mathbf{I}$ denote the set of all finite words over the alphabet I, which are admissible, that is words in which a letter k is never followed by the letter $-k$. We will write a word w in $\mathcal{W}_\mathbf{I}$ in the form $[y_0, y_1, \ldots, y_m]$, refer to $m+1$ as its **length** and denote it by $|w|$. Further, we will say that a word $w = [y_0, y_1, \ldots, y_m]$ **appears** in the coding sequence $\kappa(\xi) = [x_0, x_1, \ldots]$ if there exists $n \in \mathbb{N}_0$ such that $[x_n, x_{n+1}, \ldots, x_{n+|w|-1}] = w$. If there exists a srictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}_0$ satisfying, for all $j \in \mathbb{N}$:

$$[x_{n_j}, x_{n_j+1}, \ldots, x_{n_j+|w|-1}] = w$$

then we will say that the word w **appears infinitely often** in the coding sequence $\kappa(\xi)$. The coding sequence $\kappa(\xi)$ will be called **saturated** if every word $w \in \mathcal{W}_\mathbf{I}$ appears infinitely often in $\kappa(\xi)$. Finally we define the following subset of the limit set:

$$L_s(\Gamma) := \{\xi \in L(\Gamma) : \ \kappa(\xi) \text{ is saturated}\}$$

The set $L_s(\Gamma)$ has full Patterson measure

We are now going to show that if Γ has the $\mathcal{P}$-property, then μ_o-almost-every point in $L(\Gamma)$ belongs to $L_s(\Gamma)$, that is, the set $L_s(\Gamma)$ is of full Patterson measure. We will prove the following theorem:

Theorem 20. *Let Γ be a geometric Schottky group for which the geodesic flow on the quotient manifolod is ergodic and the Liouville-Patterson measure is finite. Then the Patterson measure μ_o of the set $L_s(\Gamma)$ is equal to one.*

Corollary 8. *Let Γ be a group in the $\mathcal{P}$-class. Then the Patterson measure of the set $L_s(\Gamma)$ is equal to one.*

The proof of Theorem 20 bears some resemblance to the argument which we used in the previous section to show that $\mu_o(L_{ur}) = 0$ for a group Γ with the $\mathcal{P}$-property. The similarities in notation should serve to highlight this analogy.

We start by fixing a word $w = [y_0, y_1, \ldots, y_m] \in \mathcal{W}_I$. Let A_w denote the intersection:

$$L(\Gamma) \ \cap \ A_{y_0} \ \cap \ g_{y_0} A_{y_1} \ \cap \ g_{y_0} g_{y_1} A_{y_2} \ \cap \ldots \cap \ g_{y_0} \cdots g_{y_{m-1}} A_{y_m}$$

This is the set of all limit points η whose coding sequence $\kappa(\eta)$ is of the form:

$$[w, x_{m+1}, \ldots] = [y_0, y_1, \ldots, y_m, x_{m+1}, \ldots]$$

Choose some $k \neq y_0$ and define the set:

$$\mathcal{T} := \big\{(\xi, \eta, s) \in T^1\mathbb{D} : \ \xi \in A_k, \ \eta \in A_w, \ s \in [-\epsilon, \epsilon]\big\}$$

Here, $\epsilon > 0$ is chosen small enough so that $\pi_b(\mathcal{T})$ is contained in the standard fundamental domain $F(\Gamma)$ of Γ. In a similar way as is the proof of Theorem 19, one can use Sullivan's Shadow Lemma to deduce that $\nu\left(\pi(\mathcal{T})\right) > 0$. Due to the ergodicity of the geodesic flow and the finiteness of the measure ν, it then follows from Birkhoff's Ergodic Theorem that while flowing under the geodesic flow ν-almost-every vector v in $T^1(\mathbb{D}/\Gamma)$ enters the set $\pi(\mathcal{T})$ infinitely many times.

Now let E_w denote the set of all points η in $L(\Gamma)$ for which the word w does not appear infinitely often in the coding sequence $\kappa(\eta)$. Suppose that $\mu_o(E_w) > 0$. Then we have $\mu_o(E_w \cap A_j) > 0$, for some $j \in I$. Define the set:

$$\mathcal{E} := \big\{(\xi, \eta, s) \in T^1\mathbb{D} : \ \xi \in A_j, \ \eta \in E_w, \ s \in [-\epsilon', \epsilon']\big\}$$

The value of $\epsilon' > 0$ is again chosen small enough so that $\pi_b(\mathcal{E})$ is contained in the standard fundamental domain $F(\Gamma)$. As before, we can then show that $\nu\left(\pi(\mathcal{E})\right) > 0$. But this is impossible, since under the geodesic flow each vector in $\pi(\mathcal{E})$ can enter the set $\pi(\mathcal{T})$ only finitely many times. To see this, one has to consider the geodesics $\pi(\alpha_i)$, for $i \in I$, on the manifold $\mathbb{D}/\Gamma$ and the order in which a vector $v = \pi(\xi, \eta, s)$ cuts these geodesics. It follows from the discussion in Section 3.2.3 that the coding sequence of the point η describes precisely the order in which those geodesics are cut by the vector v. So, it is enough to consider for $v \in E_w$ the last occurence of the word w in the coding sequence of η to show that, after a certain point, the vector v cannot enter the set $\pi(\mathcal{T})$ (since every entering of this set corresponds to a new appearance of the word w in the coding sequence of η).

Thus, we have shown that for an arbitrary word w in $\mathcal{W}_I$ we must have $\mu_o(E_w) = 0$. Now notice that:

$$L_s(\Gamma) = L(\Gamma) \setminus \bigcup_{w \in \mathcal{W}_I} E_w$$

Since the set $\mathcal{W}_I$ is countable, we can conclude that $\mu_o(\bigcup_{w\in\mathcal{W}_I} E_w) = 0$, and thus, $\mu_o(L_s(\Gamma)) = 1$.

Notice that in our argument we only used the ergodicity of the geodesic flow and the finiteness of the measure ν. This finishes the proof of Thoerem 20.

5.7.2 The Myrberg property

In [Stratmann1997] Stratmann considered for a Kleinian group the subset of the limit set, consisting of all the points which are Myrberg. A version of the Myrberg property, formulated for geodesics rather than limit points and referred to as quasiergodicity, was first studied by Myrberg in [Myrberg1931]. The definition which we use here is the one given in [Stratmann1997]. Stratmann has proved there a sharpening of Sullivan's Theorem, the same one which appeared in Section 5.3. In particular, he has shown that for a Kleinian group Γ the geodesic flow is ergodic with respect to the Liouville-Patterson measure if and only if the set of Myrberg limit points is of full Patterson measure. In this Section we will clarify the relationship between the saturatedness of the coding sequence of a limit point and the Myrberg property of this point. This will allow us to give a new version of Sullivan's Theorem.

We start by defining the Myrberg property, which first requires us to introduce the notion of (T, ϵ)-close.

Definition 19. (T, ϵ)**-close**
Let α and β be hyperbolic geodesics or rays, z a point in $\mathbb{D}$ and $T,\ \epsilon > 0$. Then we say that α and β are (T, ϵ)-close at z if there exist $t_\alpha, t_\beta \in \mathbb{R}$ satisfying:

1. $d(z, \alpha(t_\alpha)) \leq \frac{\epsilon}{2}$
2. $d(z, \beta(t_\beta)) \leq \frac{\epsilon}{2}$
3. $d(\alpha(t_\alpha + t), \beta(t_\beta + t)) \leq \epsilon$ for all $t \in [-\frac{T}{2}, \frac{T}{2}]$

We will say that α and β are (T, ϵ)-close if α and β are (T, ϵ)-close at z for some $z \in \mathbb{D}$.

Definition 20. Myrberg
Let Γ be a Fuchsian group. We say that a point $\xi \in L(\Gamma)$ is Myrberg if, for every hyperbolic geodesic α with endpoints at infinity $\eta_-, \eta_+ \in L(\Gamma)$, every point z on α and every $T, \epsilon > 0$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of distinct elements in Γ such that, for all $n \in \mathbb{N}$, the ray s_ξ and the geodesic $h_n(\alpha)$ are (T, ϵ)-close at $h_n(z)$.

We will denote the set of all limit points which are Myrberg by $\mathbf{L_M(\Gamma)}$ and refer to it as the **Myrberg limit set**.

Remark 16. It is easy to show that $L_M(\Gamma)$ is a subset of the radial limit set $L_r(\Gamma)$. Simply consider the hyperbolic $\frac{\epsilon}{2}$-balls around the points $h_n(z)$. Since we can choose any $\epsilon > 0$, an even stronger statement holds, namely:

$$L_M(\Gamma) \subseteq \bigcap_{R>0} L_R(\Gamma)$$

Here, $L_R(\Gamma)$ denotes the set of those radial limit points which are radial with respect to the radius R.

We will prove that, for geometric Schottky groups, the Myrberg limit set is in fact equal to the set of limit points with saturated coding sequences. Namely, we will prove the following theorem:

Theorem 21. *Let Γ be a geometric Schottky group. Then the sets $L_s(\Gamma)$ and $L_M(\Gamma)$ are equal.*

The equality of the sets $L_s(\Gamma)$ and $L_M(\Gamma)$ immediately yields a reformulation of Stratmann's result from [Stratmann1997], and thus the following version of Sullivan's theorem for geometric Schottky groups:

Theorem 22. *Let Γ be a geometric Schottky group. Then the geodesic flow on the quotiend manifold $\mathbb{D}/\Gamma$ is ergodic with respect to the Liouville-Patterson measure ν if and only if the Patterson measure μ_o of the set $L_s(\Gamma)$ is equal to one.*

For geometric Schottky groups the above theorem gives an interesting, potentially more tractable condition for the ergodicity of the geodesic flow on the quotiend manifold. Due to the combinatorial nature of the saturatedness property of coding sequences, verifying this condition in terms of the set $L_s(\Gamma)$ might turn out to be easier than verifying the conditions appearing in other versions of Sullivan's theorem.

Saturatedness of the coding sequence implies the Myrberg property

Now we will show that for a geometric Schottky group Γ we have:

$$L_s(\Gamma) \subseteq L_M(\Gamma)$$

Let Γ be a geometric Schottky group with Schottky description $(\{A_k\}, \{g_k\})_{k\in I}$. We consider an arbitrary point ξ in the set $L_s(\Gamma)$ and aim to show that it is Myrberg. Let α be a geodesic in $\mathbb{D}$ with negative endpoint at infinity η_- and positive endpoint at infinity η_+ with $\eta_-, \eta_+ \in L(\Gamma)$. Also, let $z \in \mathbb{D}$ a point on α, that is, $z = \alpha(t_z)$, for some $t_z \in \mathbb{R}$ and $T,\ \epsilon > 0$.

Now we need to find a sequence $(h_n)_{n\in\mathbb{N}}$ of distinct elements in Γ such that for all $n \in \mathbb{N}$ the ray s_ξ and the geodesic $h_n\alpha$ are (T, ϵ)-close at $h_n z$. The first step is to find an element $h \in \Gamma$ such that that hz lies in the fundamental domain $F(\Gamma)$. This is easily done by finding the copy gF for which $z \in gF$ and putting $h := g^{-1}$. Next, let us consider the geodesic segment on $h\alpha$ between the points $h\alpha(t_z - \frac{T}{2} - \epsilon)$ and $h\alpha(t_z + \frac{T}{2} + \epsilon)$, that is the segment of length $T + 2\epsilon$ centered at hz, and place two open hyperbolic balls of radius ϵ at both ends of this segment. Denote by B_1 the ball at $h\alpha(t_z - \frac{T}{2} - \epsilon)$ and by B_2 the ball at $h\alpha(t_z + \frac{T}{2} + \epsilon)$. Using the same argument as in Lemma 2 from Section 4.1, we observe that any hyperbolic geodesic or geodesic ray which intersects first B_1 and then B_2 will be (T, ϵ)-close to $h\alpha$ at hz. But this behaviour can be forced by requiring that a geodesic or a geodesic ray intersects certain geodesics of the form $g\alpha_k$ in a given order. In particular, if we suppose that $\kappa(h\eta_-) = [x_0, x_1, \ldots]$ and $\kappa(h\eta_+) = [y_0, y_1, \ldots]$, then there exist m_1 and m_2 such that whenever a geodesic or ray intersects the geodesics:

$$g_{x_0} \cdots g_{x_{m_1 - 1}} \alpha_{x_{m_1}}, \ldots, \alpha_{x_0}, \alpha_{y_0}, \ldots, g_{y_0} \cdots g_{y_{m_2 - 1}} \alpha_{y_{m_2}}$$

and intersects them in that particular order, then it first enters the ball B_1 and then the ball B_2. So, let w be the word:

$$w := [x_{m_1}, x_{m_1 - 1}, \ldots, x_0, -y_0, \ldots, -y_{m_2}]$$

We will now use the fact that the word w appears infinitely often in the coding sequence of ξ. The infinite occurence of w means that there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of distinct elements in Γ such that the order in which s_ξ cuts geodesics of the form $g\alpha_k$ before entering $\gamma_n F$ corresponds to an appearance of $[x_{m_1}, x_{m_1 - 1}, \ldots, x_0]$ in the coding sequence. The order in which s_ξ cuts geodesics of the form $g\alpha_k$ after entering $\gamma_n F$ corresponds to an appearance of $[-y_0, \ldots, -y_{m_2}]$ in the coding sequence. In particular, for a given n, the element γ_n will be of the form $\gamma_n = g^* g_{-x_{m_1}} \cdots g_{-x_0}$, where g^* is a product of the generators in the Schottky description of Γ not ending in $g_{x_{m_1}}$. This implies that, for every $n \in \mathbb{N}$, the ray $\gamma_n^{-1} s_\xi$ will intersect the geodesics:

$$g_{x_0} \cdots g_{x_{m_1 - 1}} \alpha_{x_{m_1}}, \ldots, \alpha_{x_0}, \alpha_{y_0}, \ldots, g_{y_0} \cdots g_{y_{m_2 - 1}} \alpha_{y_{m_2}}$$

and so $\gamma_n^{-1} s_\xi$ will be (T, ϵ)-close to $h\alpha$ at hz. But then, for each $n \in \mathbb{N}$, the ray $h^{-1}\gamma_n^{-1} s_\xi$ will be (T, ϵ)-close to α at z, so we can put $h_n := h^{-1}\gamma_n^{-1}$. The existence of the sequence $(h_n)_{n \in \mathbb{N}}$ shows that ξ is indeed Myrberg.

The Myrberg property implies saturatedness of the coding sequence

We will show that we also have that:

$$L_M(\Gamma) \subseteq L_s(\Gamma)$$

Let Γ be as in the previous section and suppose that a point $\xi \in L(\Gamma)$ is Myrberg. Let $w = [y_0, \ldots, y_m]$ be a word in $\mathcal{W}_I$. In order to show that w appears infinitely often in the coding sequence $\kappa(\xi)$, we choose points $\eta_-, \eta_+ \in L(\Gamma)$ so that the coding sequence of η_+ is of the form $\kappa(\eta_+) = [w, x_{|w|}, x_{|w|+1}, \ldots]$. Then consider the geodesic α in $\mathbb{D}$ with negative endpoint at infinity η_- and positive endpoint at infinity η_+. Note that this geodesic intersects the following geodesics in the given order:

$$\alpha_{y_0}, g_{y_0}\alpha_{y_1}, \ldots, g_{y_0}\cdots g_{y_{m-1}}\alpha_{y_m}$$

Fix some point z on α satisfying $z \in F(\Gamma)$. It is clear that, for a $T > 0$ chosen large enough and $\epsilon > 0$ chosen small enough, any geodesic which is (T, ϵ)-close to α at z will also intersect the sequence of geodesics $\alpha_{y_0}, g_{y_0}\alpha_{y_1}, \ldots, g_{y_0}\cdots g_{y_{m-1}}\alpha_{y_m}$. For this choice of T and ϵ, the Myrberg property of the point ξ yields a sequence $(h_n)_{n\in\mathbb{N}}$ of distinct elements of Γ such that, for each $n \in \mathbb{N}$, the ray $h_n s_\xi$ and the geodesic α are (T, ϵ)-close at z. So, each h_n allows us to find a segment of of $h_n s_\xi$ which intersects the geodesics:

$$\alpha_{y_0}, g_{y_0}\alpha_{y_1}, \ldots, g_{y_0}\cdots g_{y_{m-1}}\alpha_{y_m}$$

By applying h_n^{-1} to such a segment, we obtain a segment of s_ξ which corresponds to an appearance of w in the coding sequence of ξ. Since the h_n's are distinct and Γ is a discrete group, we can pass to a subsequence of $(h_n)_{n\in\mathbb{N}}$ to ensure that the obtained segments of s_ξ are disjoint. This shows that w appears infinitely often in the coding sequence $\kappa(\xi)$, which implies that $\xi \in L_s(\Gamma)$.

Conclusion

This finishes the proof of Theorem 21 so we have shown that for a geometric Schottky group Γ we have:

$$L_s(\Gamma) = L_M(\Gamma)$$

This means that for Fuchsian groups in the $\mathcal{P}$-class we could have deduced that:

$$\mu_o(L_s(\Gamma)) = 1$$

directly from the ergodicity of the geodesic flow using Stratmann's sharpening of Sullivan's Theorem. On the other hand, our proof can be seen as a simple and very geometrical alternative argument showing that for groups in the $\mathcal{P}$-class the points with the Myrberg property are of full Patterson measure.

Appendix

Helly-Bray Theorem

Theorem (Helly-Bray) *If μ_1, μ_2,... are Radon measures on $\mathbb{R}^n$ with:*

$$\sup_i \{\mu_i(K)\} < \infty$$

for all compact subsets $K \in \mathbb{R}^n$, then the sequence $(\mu_i)_{i\in\mathbb{N}}$ has a vaguely convergent subsequence.

Proof. For a proof we refer the reader to [Mattila], where the above theorem appears as Theorem 1.23. . □

Here and in Section 2.0.8, by saying that a sequence $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to a measure μ, we mean that:

$$\int_{\mathbb{R}^n} f \; d\mu_n \longrightarrow \int_{\overline{\mathbb{D}}} f \; d\mu \quad \text{for all} \quad f \in \mathcal{C}_K(\mathbb{R}^n)$$

where $\mathcal{C}_K(\mathbb{R}^n)$ denotes the set of all continuous functions with compact support; while by saying that a sequence $(\mu_n)_{\in\mathbb{N}}$ converges weakly to a measure μ we mean that:

$$\int_{\mathbb{R}^n} f \; d\mu_n \longrightarrow \int_{\overline{\mathbb{D}}} f \; d\mu \quad \text{for all} \quad f \in \mathcal{C}_b(\mathbb{R}^n)$$

where $\mathcal{C}_b(\mathbb{R}^n)$ denotes the set of all bounded continuous functions on $\mathbb{R}^n$.

Proof of Lemma 3

Statement of the Lemma: *Let X be a topological space and m a finite Borel measure on X. Let $p > 0$ and $k \in \mathbb{N}$. Suppose $Y_1, \cdots, Y_N$ are Borel subsets of X such that for each $i = 1, \ldots, N$ we have $\mu(Y_i) \geq p$ and that each $x \in X$ belongs to at most k of the subsets $Y_1, \ldots, Y_N$. Suppose further that for each $j = 0, 1, \ldots, N$ the set*

$$A^j := \{x \in X : x \textit{ belongs to exactly } j \textit{ of the sets } Y_1, ..., Y_N\}$$

is Borel. Then we have:

$$N \leq \frac{m(X)}{p} \cdot k$$

Proof. Define the function $f : X \mapsto \{1, \cdots, N\}$ by setting $f(x) := j$ for $x \in A^j$. Since the sets A^j partition X this function is well defined and since the sets A^j are Borel the function f is measurable. So we can define a new measure $\widetilde{m}$ on X by:

$$\widetilde{m}(A) := \int_A f dm$$

It is easy to show that for any Borel set $A \subset X$ we have:

$$\widetilde{m}(A) = \sum_{i=1}^{N} m(Y_i \cap A)$$

Thus we have:

$$\widetilde{m}(X) = \sum_{i=1}^{N} m(Y_i \cap X) = \sum_{i=1}^{N} m(Y_i) \geq N \cdot p$$

On the other hand:

$$\widetilde{m}(X) = \int_X f dm \leq \int_X k \ dm = k \ m(X)$$

We conclude that:

$$N \cdot p \leq \widetilde{m}(X) \leq k \ m(X)$$

So as required:

$$N \leq \frac{m(X)}{p} \cdot k$$

□

Proof of Lemma 9

Statement of the Lemma: *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $a_n \nearrow \infty$. Suppose that the series $\sum_n a_n^{-s}$ has abscissa of convergence equal to s_0. Then s_0 is also the abscissa of convergence for the series $\sum_n \log(a_n)\ a_n^{-s}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $a_n \nearrow \infty$. Suppose that the series $\sum_n a_n^{-s}$ has abscissa of convergence equal to s_0. Let s_1 denote the abscissa of convergence of the series $\sum_n \log(a_n)\ a_n^{-s}$. We aim to show that in fact $s_1 = s_0$.

We will first show that $s_1 \leq s_0$. By the definition of s_0 we have that:

$$\sum_n a_n^{-(s_0+\epsilon)} < \infty \quad \text{for all} \quad \epsilon > 0 \tag{19}$$

Let $\epsilon_0 > 0$ be fixed. We now want to show that:

$$\sum_n \log(a_n)\ a_n^{-(s_0+\epsilon_0)} < \infty$$

First, we simply rewrite the series as:

$$\sum_n \log(a_n)\ a_n^{-(s_0+\epsilon_0)} = \sum_n \log(a_n)\ a_n^{-\frac{\epsilon_0}{2}} a_n^{-(s_0+\frac{\epsilon_0}{2})}$$

We put $c_n := \log(a_n)\ a_n^{-\frac{\epsilon_0}{2}}$ and observe that there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $c_n < 1$. So we can write:

$$\sum_n \log(a_n)\ a_n^{-(s_0+\epsilon_0)} = \sum_{n \leq N} c_n a_n^{-(s_0+\frac{\epsilon_0}{2})} + \sum_{n>N} c_n a_n^{-(s_0+\frac{\epsilon_0}{2})}$$

The first term is only a finite sum and thus finite. By the choice of N, the second series is bounded from above by $\sum_n a_n^{-(\delta+\frac{\epsilon_0}{2})}$ and so is also finite, which shows that (19) is satisfied. Since ϵ_0 was chosen arbitrarily this shows that $s_1 \leq s_0$.

Now we will show that $s_1 \geq s_0$. By the definition of s_0 we have that:

$$\sum_n a_n^{-(s_0-\epsilon)} = \infty \quad \text{for all} \quad \epsilon > 0 \tag{20}$$

Let $\epsilon_0 > 0$ be fixed. We now want to show that:

$$\sum_n \log(a_n)\ a_n^{-(s_0-\epsilon_0)} = \infty$$

First, we rewrite the series as:

$$\sum_n \log(a_n)\ a_n^{-(s_0-\epsilon_0)} = \sum_n \log(a_n)\ a_n^{\frac{\epsilon_0}{2}} a_n^{-(s_0-\frac{\epsilon_0}{2})}$$

We put $c_n := \log(a_n)\ a_n^{\frac{\epsilon_0}{2}}$ and observe that there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $c_n > 1$. So we can write:

$$\sum_n \log(a_n)\ a_n^{-(s_0-\epsilon_0)} = \sum_{n \leq N} c_n a_n^{-(s_0-\frac{\epsilon_0}{2})} + \sum_{n>N} c_n a_n^{-(s_0-\frac{\epsilon_0}{2})}$$

The first term is only a finite sum and thus finite. By the choice of N the second series is bounded from below by $\sum_n a_n^{-(\delta-\frac{\epsilon_0}{2})}$ and so it must diverge, which shows (20) as desired. Since ϵ_0 was chosen arbitrarily, this shows that $s_1 \geq s_0$. Therefore, we conclude that $s_1 = s_0$.

$\square$

Limit set of a normal subgroup

Proposition. *Let $G \subseteq \mathrm{Iso}^+(\mathbb{D})$ be a non-elementary Fuchsian group and let N be a normal subgroup of G. Then:*

$$L(N) = L(G)$$

Proof. From the definition of the limit set, it is clear that $L(N) \subseteq L(G)$. So it only remains to show that $L(G)$ is contained in $L(N)$. To do this we use the well known fact that for any Fuchsian group G its limit set $L(G)$ is a minimal closed subset of S^1 invariant under the action of G. Here we mean minimality with respect to inclusion, to be precise, if L' is any other closed subset of S^1 invariant under the action of G, then it must contain $L(G)$. In particular, this means that $L(G)$ cannot contain a non-empty proper subset which is closed and invariant under the action of G. Therefore, it is enough to show that $L(N)$ is invariant under the action of G.

Let ξ be a point in $L(N)$ and g an arbitrary element of G. By definition, there exists a sequence $(h_i)_{i\in\mathbb{N}}$ of elements of N such that:

$$h_i o \ \rightarrow \ \xi$$

Here the convergence is with respect to the Euclidean metric. By the continuity of the action of $\mathrm{Iso}^+(\mathbb{D})$ on $\mathbb{C}$ it follows that:

$$gh_i o \ \rightarrow \ g\xi$$

We can rewrite the sequence of points $(gh_i o)_{i\in\mathbb{N}}$ as $(gh_i g^{-1} go)_{i\in\mathbb{N}}$. Since N is a normal subgroup of G, the sequence $(gh_i g^{-1})_{i\in\mathbb{N}}$ is a sequence of elements in N. By putting:

$$z := go$$

we can view $(gh_i g^{-1} go)_{i\in\mathbb{N}}$ as a sequence of points in the orbit Nz. This shows that $g\xi$ belongs to $L(N)$. Since both the point ξ and the element g have been chosen arbitrarily, this shows that the set $L(N)$ is invariant under the action of G. □

Reverse triangle inequality

Proposition. Let α and β be two geodesic segments in $\mathbb{D}$ with a common endpoint x. Denote by y_α the other endpoint of α and by y_β the other endpoint of β. Let γ be the geodesic segment with endpoints y_α and y_β. For any $\theta > 0$ there exists a constant $C_\theta > 0$ depending only on θ such that, for any choice of α and β, if the internal angle between α and β at x is at least θ, then:

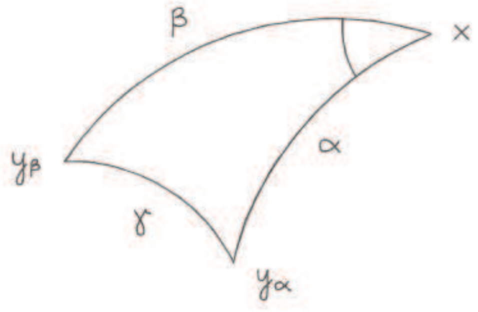
$$l(\alpha) + l(\beta) - C_\theta \leq l(\gamma)$$

Here by $l(\cdot)$ we denote the hyperbolic length.

Proof. This result can be deduced from the hyperbolic Cosine Rules, see [Beardon]. We present here an alternative argument, which due to its very gemetrical nature gives more insight into why the reverse triangle inequality holds.

We start by rephrasing the statement. To prove our proposition it is enough to find an upper bound, depending only on θ, for:

$$l(\alpha) + l(\beta) - l(\gamma)$$



This upper bound will then give us the required constant C_θ. Let p denote the point on the geodesic arc γ which lies closest to the point x. Then by a simple application of the usual triangle inequality we have:

$$\begin{aligned} l(\alpha) + l(\beta) - l(\gamma) &= d(x, y_\alpha) + d(x, y_\beta) - d(y_\alpha, y_\beta) \\ &\leq d(x,p) + d(p, y_\alpha) + d(x,p) + d(p, y_\beta) - d(y_\alpha, y_\beta) \\ &= 2d(x,p) \end{aligned} \tag{21}$$

So we have reuded our problem to that of bounding the distance between the point x and the arc γ. But this is easy! W choose an isometry $g \in \mathrm{Iso}^+(\mathbb{D})$ that maps the point x to the origin o, that is:

$$g(x) = o$$

Now, let us extend the geodesic arcs $g(\alpha)$ and $g(\beta)$ to geodesic rays issuing from x and let ξ_α and ξ_β denote the respective endpoints at infinity of these rays. Consider the geodesic with endpoints at infinity ξ_α and ξ_β, which we denote by γ'. The distance $d(x,p)$ is clearly bounded above by the distance between o and γ'. Moreover the distance between o and the geodesic γ' depends only on the internal angle at o between the arcs $g(\alpha)$ and $g(\beta)$, which is the same as the internal angle at x between α and β, and this distance decreases as the angle becomes larger. Thus we can simply take C_θ to be twice as large as the distance between the origin and a geodesic with endpoints at infinity ξ and η chosen so that the internal angle between the rays r_ξ and r_η is equal to θ. □

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