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The interconnective relationship of students' visualization and argumentation in geometry

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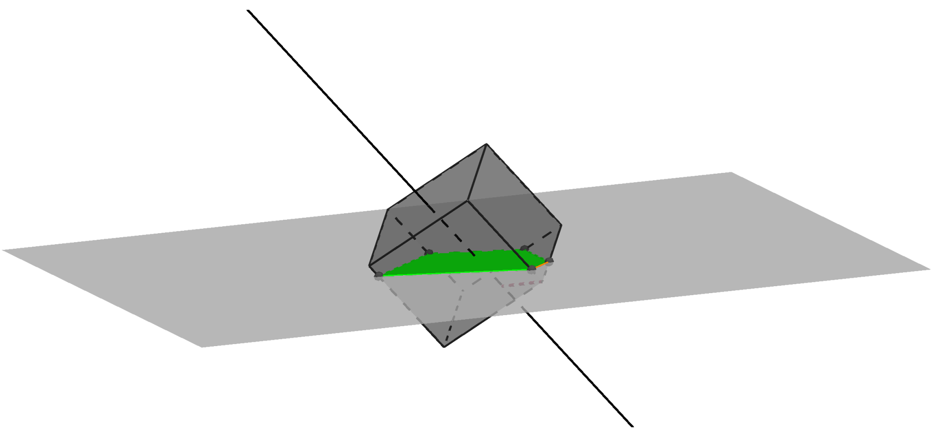

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This dissertation is dedicated
to all teachers and their students

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Abstract

One of the main subjects in school geometry involves learning about geometric objects and their properties. The ultimate goal would be for students to move beyond visual recognition of shapes and forms, towards grasping the structure of geometric objects and the organization of relations between their constitutive parts. This "grasping of the structure" and the correlation of properties of different parts of a geometric object, are the results of a cognitive process called visualization, which reveals what remains hidden from visual perception. In school, geometry is also the main field of mathematics in which students learn how to develop mathematical argumentation. Nevertheless, despite the significance of visualization in the process of learning geometry, it still does not seem to play a central role in the actual teaching of geometry or in the students' argumentations, at least not in a deliberate or pre-planned way.

In my work, I investigate the role of visualization in students' argumentations while they work on geometric tasks, whose specific design aims to promote students' visualization. The data presented here are from the implementation of my study in a 9^{th} grade school class in Germany. Two theories underlie this work: Toulmin's (1958) theory on argumentation and his functional model of argumentation, and Duval's (1999/2002) cognitive approach to visualization. These theories help me to describe the meaning of the argumentation and visualization processes in students' work. In order to observe these phenomena, I use two argumentation analysis methods to analyze my data and reconstruct students' argumentations. The two methods are Reid's (2002b) method of "patterns of reasoning" and Knipping's (2003, 2008) method of "argumentation structures". The two methods combined unwind the argumentation processes that take place while students work on given geometric tasks. Employing Duval's (1999/2002) visualization theory in the reconstructed argumentations reveals the roles of visualization in students' argumentations and, as a consequence, visualization's contribution in their learning of geometry.

The data analysis reveals five roles that visualization plays in students' argumentation (e.g. supporting the creation of a hypothesis) and also three functions that the students attribute to them in their arguments (e.g. as warrants). In the results, I also explain how visualization can be indicated in argumentation both through students' verbal descriptions (including metaphors) as well as their actions (e.g. gestures, use of drawings). Furthermore, I discuss how the task design as well as the organization of the learning social settings – students working in pairs and participating in classroom discussions – influence students' work and contribute to their learning in different ways.

This work concludes with a discussion in which all the important results are brought together and are also linked to the already existing literature. The contributions of my work are discussed and implications for the teaching of geometry and for further research are proposed.

Contents

1 See the transcript of the episode in the Digital Appendix H4.
2 See the transcript of the episode in the Digital Appendix H5.

1 Introduction

In her article *The World of Blind Mathematicians*, Allyn Jackson (2002) writes:

> “one thing that is difficult about visualizing geometric objects is that one tends to see only the outside of the objects, not the inside, which might be very complicated. By thinking carefully about two things at once, Morin has developed the ability to pass from outside to inside, or from one “room” to another” (p. 1248).[1]

In this quote, Jackson refers to visualization in geometry. She talks about the process of grasping the internal structure of geometric objects, in contrast to the mere visual perception of its outer “shell”. This is an important distinction and the role of visualization in learning geometry finds itself at the center of my work. More precisely, I am interested in the role that visualization can play in students' argumentations in geometry.

Visualization is an important, yet somewhat vague, notion in mathematics as well as in the field of didactics of mathematics. So, what do we mean exactly when we talk about visualization? What does this move from “outside to inside”, that also Jackson mentions, mean? And how is it linked to mathematics, or to geometry more specifically? These are only a few of the questions that are triggered every time somebody pulls up the word visualization when discussing in a mathematical context.

The notion of visualization, and its relation to argumentation in geometry stands in the center of the theoretical framework in this dissertation. Nevertheless, before I state the exact problematique that motivated this study and the subject of this dissertation, I would like to elaborate a little further on the two main theoretical notions: visualization and argumentation.

1.1 The notions of visualization and argumentation

Visualization has constituted a topic of research both for psychologists and mathematics education researchers. Nevertheless, to date researchers have not yet agreed on one unique definition regarding this notion. In the problem-solving context of this study, it is the cognitive view of visualization that will be taken into account, as it is related to argumentation. My understanding of visualization, building on Duval's (1998, 1999/2002) theory on the nature and character of it, brings together the two views of geometric objects Jackson talks about: the inside and the outside view.

[1] Jackson refers here to Bernard Morin, a blind French mathematician specializing in Topology.

For Duval (1998, 1999/2002) visualization is one of the three kinds of cognitive processes, involved in geometry. More specifically, Duval (1999) argues:

> "We have here the breaking point between visual perception and visualization. A semiotic representation does not show things as they are in the 3D environment or as they can be physically projected on a small 2D material support. That is the matter of visual perception. A semiotic representation shows relations or, better, organization of relations between representational units. [...] Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization. And that is why visualization should not be reduced to vision, that is to say: visualization makes visible all that is not accessible to vision. We can see now the gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary, visualization can get at once a complete apprehension of any organization of relations" (p.6).

Duval's (1999) distinction between visual perception and visualization is coherent to the distinction that Jackson (2002) draws between the "outside" and the "inside" of a geometric object. Visual perception reaches only the "outside" of a geometric object, while visualization gives us access to its "inside". Perceiving only the "outside" of a geometric object is lacking the understanding of the internal structure of the object, the organization of its component subparts, their properties and the relations between them. In contrast, visualization (or grasping the "inside") is the process that reveals all the aforementioned characteristics of a geometric object. In my work visualization is to be understood as this exact cognitive process, and its role in students argumentation is to be investigated.

The second significant notion in this work is argumentation in the context of geometry. In their book "*Proof in mathematics education*" Reid and Knipping (2010) present an overview on the notion of argumentation and the different meanings that have been attributed to it by different researchers, who look at argumentation through different theoretical prisms. In my work, the notion of argumentation is understood and used as the process of expressing one's reasons to support or reject a statement or an opinion, through verbal articulation, gestures or actions. Argumentation may take place between two or more participants, or a single student may perform it when expressing reasons out loud. This meaning of argumentation is based on the definitions for argumentation given by Douek (2002, 2005)[2]. My understanding of argumentation is also compatible with Toulmin's functional model of argumentation (Toulmin, 1958; see also Knipping, 2008), which is used here for the analysis and reconstruction of the students' argumentation processes.

1.2 Problematique and motivation for the study

According to what I have said above, I believe that either argumentation or visualization isolated, is not enough for students' meaningful engagement with

2 See details in Chapter 2, subsection 2.2.1.

geometric activities, such as solving a problem or proving a statement. The role of visualization in students' argumentation in geometry has not yet been given much attention in existing research. Therefore, one of my main theses in this study, is that there needs to be a continuous interplay between the two, for students' geometric reasoning to progress and evolve.

Geometry is the main field of mathematics, chosen in the school curriculum as the appropriate subject in which proof, deductive reasoning and argumentation should be taught (Herbst, 2002; Douady, 1998; Harel & Sowder, 1998; Hershkowitz, 1998). In geometry, students are usually asked to solve a problem, which is most of the times accompanied by a drawing. In many cases, the drawing acts as an obstacle for students' argumentations. Since the students are often able to see what they are asked to prove on the drawing, which is in many cases given to them, argumentation or even proof seems to be unnecessary for them (Mithalal, 2009). That is to say, an argumentation or proving context poor in visualization, in which only visual perception takes place, leads not only to a lack of visualization in geometry but as a result to a lack of, or the devaluation of, argumentation itself. This is another main thesis of this dissertation, which I demonstrate through my case study. Visualization seems to have been lost in the teaching of geometry. But if visualization is a fundamental part of argumentation or strongly linked to it, we need to turn our attention a little more to visualization and its relation to argumentation.

So, how could we use visualization in the teaching of geometry, in order to "challenge" students to engage in argumentations based on geometric relations and properties? Jackson's (2002) article inspired the idea behind my study, focusing on students working in the context of a Dynamic Geometry Environment (DGE): How would students argue mathematically, if they were given a task (designed in a DGE) in which the "outside" of a solid (in Jackson's terms) is invisible to them? My motivation was to challenge the students with such a task, where mere visual perception is limited or fails (Mithalal, 2009; Mithalal & Balacheff, 2019). My scenario was to create tasks in which three-dimensional geometric objects would be invisible and the students would have to identify these invisible solids, judging by its visible cross-sections with a plane. In this scenario students would have to turn to other more efficient ways in order to confirm a conjecture or justify an answer. In this context, two processes are necessary: visualization and argumentation. As Jackson (2002) remarks, it is the "inside" structure of a geometric object that is complicated, namely its component parts and their properties as well as the relationships between them. Understanding and grasping this underlying internal construction is essential and leads students into visualization and argumentation. How students succeed in this, is the topic of my dissertation study.

1.3 Design of the study

The aim of this study is to involve students in geometric tasks that challenge their argumentation and visualization competencies. Most of the research conducted in this area of the didactics of geometry, does not in fact involve classroom situations, but rather interviews with students or controlled "laboratory" conditions in which students work. In this study classroom contexts should be investigated.

I refer to the tasks used in my study as *D-transitional tasks.* These are geometric problems involving transitions from two-dimensional (2D) to three-dimensional (3D) geometric objects (and vice versa) and they are designed in a Dynamic Geometry Environment (GeoGebra 5). In the particular D-transitional tasks used in my work, the solid under investigation is always invisible. In D-transitional tasks, the correlation of properties between geometric objects of different dimensions is vividly present and challenges students to use their visualization and come up with arguments. This characteristic makes those tasks appropriate for the shift "from outside to inside", or from one "room to another" as Jackson (2002, p. 1248) comments in her article. When talking about the topologist Morin Bernard working on the topological structure of complicated geometric objects, this shift happens as well.

Inspired by the idea of Black Boxes (Laborde, 1998; Knipping & Reid, 2005), the D-transitional tasks designed in the DGE gained also the characteristic of "invisibility" [3]. The solids in the tasks are designed to be invisible for the students, so that they need to infer characteristics and properties of these objects based on limited three-dimensional visual evidence. It is also this nature of the tasks that can cause surprise or even cognitive conflict for the students, which motivates them to engage in argumentation.

However, not only cognitive challenges are built into those, but a social context is also constructed to engage students into argumentative tasks. Prusak et al. (2012; 2013) emphasize the importance of the social context when students engage in problem solving situations. The researchers present *design principles* for productive problem solving situations, which focus on two dimensions: the *task-design* and the organization of the *learning environment* (social settings). Hence, agreeing with Prusak et al's (2012; 2013) viewpoint that the social context in which learning is situated is one of the determining factors of the learning process, my study is designed and conducted in real-classroom situations as part of the normal geometry lesson routine. Furthermore, students work in two different social settings: in pairs and as members of a whole classroom discussion.

The design of the study, aims to challenge students to produce conjectures and examine their validity using strategies that go beyond naive empirical justifications, engaging naturally in mathematical activity that involves visualization and argumentation.

1.4 An introduction to my research questions

As I mentioned before, in this study I am interested in investigating the dialectic relationship between argumentation and visualization in geometry, in order to determine its characteristics. In that respect, I intend to provide some answers to the following main research questions, which are here introduced free from the detailed terminologies that I introduce in Chapters 2-5 of the theoretical background and methodology:

[3] See Chapter 5, subsections 5.2.2.1 and 5.2.2.2.

1. What strategies do students follow when they explore the tasks they are given in the Dynamic Geometry Environment?
2. What steps do students take in their argumentations? And how are these steps influenced by the design of the study?
3. What is the role of visualization in students' argumentation?

The first research question is discussed in Chapter 6. The second question shifts the focus from mere exploration to argumentation. Before I can investigate the role of visualization in argumentation, I need to get a good understanding of students' argumentation steps. The results related to this research question are in Chapter 7. Finally, in order to gain a deep insight into the process of visualization in argumentation it is necessary to analyze the fine structure of students' argumentations and look for the role of visualization in it. The results regarding the third research question are presented in Chapter 8.

1.5 Structure of the dissertation

The present chapter constitutes the introduction of the study. Here, I presented the problematique in the research of the didactics of geometry that triggered my research interest and led to this study. A description of the main theoretical notions that are used in the frame of my research, namely visualization and argumentation, are part of this chapter, as well as a sketch of the design of the study, including its aim.

Chapters 2 to 5 constitute the theoretical and methodological framework of my study, upon which the data analysis and results in Chapters 6 to 8 have been based and developed. In Chapter 2, I perform a literature review of the notion of argumentation in mathematics. I also present Toulmin's (1958) functional model of argumentation that is used to reconstruct students' arguments. Chapter 3 deals with the notion of visualization in geometry and the relationship between visualization and argumentation. In Chapter 4 a literature review of the teaching and learning of geometry in a school context is portrayed. I also mention results from empirical research regarding the use of DGE[4] in the teaching of geometry and the study of three-dimensional geometry. Chapter 5 is about my methods and methodology, and the theory underlying my design principles (Prusak et al., 2012; 2013) in two axes: the task-design and the organization of the learning environment in problem solving situations in the classroom. Also, the methodology of the argumentation reconstruction in portrayed.

Chapter 6 describes the exploration strategies that students follow in the DGE when dealing with the *D-transitional tasks*. These strategies are portrayed and deeper insights are provided into the ways in which these strategies occurred. To do this, I present specific examples and episodes of the students' work. Chapter 7 presents the results of the argumentation steps that students take during their work on the given tasks. In Chapter 8, I discuss the roles of visualization in argumentation as they are revealed and identified from the analysis of students' discussions. Chapter 9 summarizes the results of the study, its limitations and its contribution to the already

[4] Dynamic Geometry Environment

existing research, as well as its implications for teaching. I also propose some ideas for further research in the field.

2 Argumentation in the mathematics classroom

Argumentation and *visualization* in geometry are the two major theoretical notions playing the lead roles in this work. In this chapter, I focus on argumentation through a literature review that goes insofar as to serve my research and helps me to answer my research questions (see subsection 1.4). Further views on argumentation, different from the one I adopt here, are also mentioned shortly in order to provide a more complete literature review.

I start with a discussion on *reasoning* in mathematics (Section 2.1) and the way this is connected to *argumentation* (2.1.1). I then present some *types of reasoning* (2.1.2), which later constitute the base for the characterization of the *types of arguments* discussed in this chapter (2.2.3). The next section (2.2) begins with the description of the notions of *argument* and *argumentation* and the meanings I attribute to those terms in this work (2.2.1). I also mention different views of the notion of argumentation, and the reasons why these are not fitting to my research (2.2.2). Toulmin's (1958) functional model of argument is also presented here, as an illustration of how an argument can be reconstructed and represented schematically (2.2.3). In the last part of Section 2.2 I describe how different types of arguments can be reconstructed using Toulmin's model (2.2.4). The last section of the chapter is an epilogue, in which I also introduce the subject of the next chapter.

2.1 Reasoning in mathematics

There is an extensive literature, from Aristotle to today's research in philosophy, psychology and the didactics of mathematics, regarding *reasoning* and the *types of reasoning* taking place in mathematics (see for example Peirce, 1878; Polya, 1968; Balacheff, 1987; Reid & Knipping, 2010). Duval (2007) refers to reasoning as a cognitive process and Reid (1995) considers that reasoning arises as someone's response to his or her personal need to reason. Steen (1999) distinguishes between different ways in which the notion of mathematical reasoning is used in the research field of mathematics education:

> "Sometimes this phrase denotes the distinctively mathematical methodology of axiomatic reasoning, logical deduction and formal inference. Other times it signals a much broader quantitative and geometric craft that blends analysis and intuition with reasoning and inference, both rigorous and suggestive" (p. 270).

Regardless of the differences between the various descriptions of reasoning in mathematics, it seems that this notion is always used to express a special kind of thinking process that takes place when a person is dealing with a mathematical task. This means that reasoning is considered to be the psychological process of human thinking, in a mathematical context, taking place in the mind of the person who is reasoning. Since coming up with a specific definition of reasoning is beyond the aim of the present work, I will suffice in the aforementioned descriptions of it.

2.1.1 How is reasoning connected to argumentation in mathematics?

One of my foci is to observe and analyze students' argumentations. In this work, the difference between reasoning and argumentation lies in the nature of each process. *Reasoning* is an internal psychological process, happening in the mind of the reasoner, while *argumentation* is an external process. As I later explain in more detail (see subsection 2.2.1), I consider *argumentation* as a process during which students externalize their *reasoning* through articulation, gestures or actions. As a result, reasoning is a vital part of the mathematical activity that on the one hand precedes argumentation, and on the other hand it continues taking place during argumentation. My intention is to analyze what students say, trusting that during their discussions they express their reasoning, as accurately as they can. My aim is not to fully comprehend students' reasoning – I consider this to be an almost impossible endeavor – rather to examine their argumentations and the role that visualization plays in them.

Nevertheless, according to what I have said above, reasoning seems to always be the "trailhead" of students' arguments, and consequently of their argumentations as well. Therefore, I believe that reasoning and its types are what determine the types of arguments that are then created. In the next subsection, I present some of the main types of reasoning in mathematics. The discussion of the structure of these types of reasoning is necessary in order to later present and discuss the different types of arguments (subsection 2.2.4) that the students build in their argumentations. In the didactics of mathematics, an argument is characterized as, for example, deductive when the syllogism (Peirce, 1878) that it illustrates is deductive. Ergo, the *types of arguments* that students create may reflect the *types of reasoning* they follow.

2.1.2 Types of reasoning in mathematics

The *types of reasoning* first appear in the literature connected to the fields of logic and philosophy and they describe the different logic forms of reasoning. This means that they are not specifically focused on mathematical reasoning. In this work, I focus on the main types of reasoning that appear in a mathematics classroom, using the terminology and original examples used by the researcher who initially described them.

Charles Saunders Peirce (a chemist, philosopher, logician and mathematician) describes three *syllogisms* (logical forms of reasoning), which he considers "utterly irreducible" (37, 2.146, as mentioned in Pease & Aberdein, 2011, p. 3). These syllogisms are: *deduction*, *induction* and *abduction*. During his work on the types of reasoning (from 1860 to 1911), Peirce describes the syllogisms in logical form in 1867 (CP 2.474, 511; CE, Vol. 2, pp.27, 46, see also Reid & Knipping, 2010), while later in 1878 he describes them using three specific elements: *case, rule* and *result* (CP 2.623; CE, Vol. 3, pp. 325-326, see also Reid & Knipping, 2010). Here, I describe Peirce's syllogisms as he presented them in 1878 (see Table 2.1), since this latter description is not only more current, but also more befitting to the purposes of the representation of students' arguments in this work.

Reid and Knipping (2010) describe the three elements that build Peirce's syllogisms: a *case* is a specific observation that a condition (a characteristic of something or a relationship between things) holds (e.g. Maya is a ballerina). A *rule* is a general proposition that expresses a consequential relationship (e.g. All ballerinas are dancers), stating that if a condition occurs (ballerina) then another one will also occur (dancer). A *result* is a specific observation referring to a condition that is linked to the condition used in the *case*, by a *rule* (e.g. Maya is a dancer).

In *deduction* the inferred outcome is a *result*, in abduction it is a *case*, and in induction it is always a *rule*. Therefore, the distinction between induction and the other two types of *syllogism* is quite clear. Now, the way to distinguish between deduction and abduction is to consider how they connect a *case* to a *result* via a *rule*. In Table 2.1, I use the three syllogisms Peirce published in *Popular Science Monthly* in 1878, which are examples often cited in the mathematics education literature in order to present the three types of reasoning.

Deduction
Rule.—All the beans from this bag are white. *Case.*—These beans are from this bag. ∴ *Result.*—These beans are white.
Induction
Case.—These beans are from this bag. *Result.*—These beans are white. ∴ *Rule.*—All the beans from this bag are white.
Hypothesis. [Abduction.]
Rule.—All the beans from this bag are white. *Result.*—These beans are white. ∴ *Case.*—These beans are from this bag.

Table 2.1: Peirce's syllogisms for deduction, induction and abduction (1878, p. 472; CP 2.623)

In Peirce's syllogisms, in *deduction* (deductive reasoning), a general *rule* (for "all" beans) and a specific *case* (for "these" beans) lead to a specific *result* (these beans). In *induction* (inductive reasoning) a specific *case* (these beans) associated with a specific *result* (these beans), or many *cases* associated with many similar *results*, lead to the formulation of a general *rule* (all beans). In *abduction* (abductive reasoning, Peirce called it Hypothesis in 1878[1]) a general *rule* and a specific *result* lead to a specific *case*.

[1] Here, I use the term "abduction" to refer to Peirce's abduction/hypothesis. I will not use the term "hypothesis" to avoid confusion with the meaning I attribute to this word in this work (see Footnote 3 in 2.2.4)

From this example it is evident that abduction can be thought of as a reverse deduction, where the reasoning moves from a *result* via a *rule* to a *case* (abduction), rather than from a *case* via a *rule* to a *result* (deduction).

Another difference between the three types of reasoning is that abductive reasoning explains and explores, while inductive and deductive reasoning verify (Reid & Knipping, 2010). In Pease and Aberdein's (2011) words:

> "Peirce thought [37, 8.384] that there are two important characteristics in each type of reasoning – *security* (the level of confidence we have in an inference), and *uberty* (the value in productiveness). Roughly speaking these respectively decrease and increase from deduction to induction to abduction" (p.4).

Security, is related to the degree of certainty or probability for the validity of the outcome of the syllogism. *Uberty*, is related to the "newness" and originality of the outcome. For example, if the outcome is a *rule* drawn by induction or a *case* drawn by abduction that carries new information about the situation under examination, then the *uberty* of the syllogism is higher, than when we draw a *result* through deduction.

I now focus shortly on each of the three types of reasoning separately, as they have been observed in empirical research in the mathematics classroom.

Deduction

Deductive reasoning is the type of reasoning referred to most often in the teaching of mathematics at school. This is mainly because it is considered to be the basis of mathematical proof. As one of the goals in school mathematics is that the students learn to prove, the cultivation of deductive reasoning is a major goal (Reid & Knipping, 2010).

In deductive reasoning a general *rule* is applied in order to draw a specific *result*, which does not lead to further knowledge, and is used to establish certainty. The uses mainly assigned to deductive reasoning are verification and explanation (explaining something to an audience). However, deductive reasoning can also serve in the exploration of a situation. Although in deductive syllogisms the premises (*case* and *rule*) already provide all the necessary information, the process of deduction can shed light in a situation by drawing a conclusion (*result*), thus turning something that is "implicitly known into something that is explicitly known" (Reid & Knipping, 2010, p.88). So, although deductive reasoning does not lead to new knowledge per se, it helps to make more understandable that which is already known.

Reid (2002a) distinguishes three kinds of deductive reasoning employed by young students (age of seven and eight) in the mathematics classroom. These kinds of reasoning are differentiated based on the number and kind of *cases* from which they originate. The three kinds of deductive reasoning mentioned by Reid (2002a) are: *specialization* (one *case*), *simple* deductive reasoning (two or more *cases*), and *hypothetical* deductive reasoning (two or more *cases*, at least one hypothetical).

In *specialization*, the deductive syllogism uses a specific *case* and a general *rule* to draw a specific *result* (like in the example with Maya above and in Peirce's deductive syllogism with the beans in Table 2.1). In the other two kinds of deductive reasoning, the reasoner either uses more than one *case*, or the *case* that is used is general (instead of specific). If at least one of the *cases* used is hypothetical, then the reasoning is *hypothetical* deductive, otherwise it is *simple* deductive. Both *simple* deductive reasoning and *hypothetical* deductive reasoning may be performed in one-step or multiple-step forms. Deductive reasoning (whether *simple* or *hypothetical*) is called multiple-step when it is created with "chains of deductions" (Reid, 2002a, p. 4-108).

Here, I would like to expand more on the last two kinds of deductive reasoning (*simple* and *hypothetical* deductive reasoning). What follows applies both for single-step as well as for multiple-step deductions. For more details, please refer to Reid and Knipping (2010). As I mention above, both in *simple* as well as in *hypothetical* deductive reasoning, one deduces a result from two or more specific cases or from a general case. The difference between the two lies in the nature of those premises. In the simple deductive reasoning, the premises are established, while in hypothetical deductive reasoning one or more of those premises are hypothetical. That means that the reasoning stems from something that is not yet known to be true, rather it is only hypothesized to be true. This type of reasoning is used in two processes in mathematics; in *reductio ad absurdum* (proof by contradiction), when the reasoner shows that the hypothetical premise cannot be true by arriving at a contradictory result, and in *mathematical induction* (reasoning by recurrence), a process very much deductive, during which specific cases and a general rule are used in order to draw a specific result.

Induction

Peirce writes, "induction is the inference of the rule from the case and result" (1878, p. 471; CP 2.622). So, in Peirce's syllogisms, the outcome of an inductive syllogism is a *rule*. Inductive reasoning moves from specific cases to conclude a general rule, which is not certain, rather only a possible outcome. It also provides new knowledge, since it uses known facts to conclude something that was not known before (see for example Barnes, 1984; Reid & Knipping, 2010).

Reid and Knipping (2010) describe five types of inductive reasoning: *pattern observing*, *predicting*, *conjecturing*, *generalizing* and *testing*. Induction always originates from multiple specific *cases* or a general *case* but its outcome may differ each time. For example, in *pattern observing* the outcome of an induction is the observation that "several specific cases share a common feature" (ibid, p. 91; see also Reid, 2002b). In *predicting*, a hypothesis about another specific *case* is generated based on all the previous specific *cases*. In *predicting*, the outcome could also be a general rule (as in Peirce's syllogisms, see Induction in Table 2.1). If so, Reid and Knipping (2010) describe the outcome either as a *conjecture* or as a *generalization. Conjecturing* refers to the process using induction to make "a general statement" (a *conjecture*) that requires "additional verification" (p. 92). Contrary, in *generalizing* the general statement (*generalization*) "does not require additional verification" (p. 92). A *conjecture* explores, while a *generalization* both explores and verifies, but neither of them explains. Finally, *testing* is here "a specific type of reasoning which is used to test predictions and conjectures" (p. 92).

In mathematics, as well as in the teaching of mathematics, induction can be a way to create rules that can then be used in deductive reasoning, in order to reach new results.

Abduction

Pierce first introduced the term "abductive" reasoning in his work around 1867, giving to it its final from in 1878 (see Hypothesis/Abduction in Table 2.1). An *abduction* begins with the observation of a surprising fact (*result*), which is then explained with the help of a *rule*, leading to the initial *case* that caused the *result*.

The surprising nature of the *result* is of particular importance to Peirce. He ascribes special importance to the role of surprise in reasoning, as the threshold of exploration, which in turn leads to abductive reasoning. Peirce considers abductive reasoning to be "the cornerstone of all scientific discovery" (Pease & Aberdein, 2011, p.4). Based on his description of abduction (see Table 2.1), Peirce seems to have considered abductive reasoning "as possible on very limited evidence" (Reid & Knipping, 2010, p. 101). I discuss the relevance and the importance of those two points for the present work in Chapter 5, where I present the methods and methodology I have employed in this study.

Eco (1983) makes useful distinctions of abduction, based on Peirce's (1878) formulation of abductive reasoning. Eco presents abductive reasoning as the process of searching for a *rule* from which a *case* would follow, and he distinguishes between three kinds of abductive reasoning: overcoded, undercoded and creative abduction. Eco's overcoded abduction is the same as Pierce's (1878) abduction (or Hypothesis): there is a known *result* and the reasoner knows of only one possible *rule* that could lead to the *case*. When there are more than one possible *rules* that can lead to a *case*, and the reasoner chooses one of them, then the abduction is undercoded, in Eco's terms. Eco says that there are also cases, in which a reasoner may not be aware of any *rules* that could lead to a *case*. This means that the reasoner needs to invent a new *rule* and also formulate the *case* that follows from it. This is what Eco calls creative abduction. In this kind of abduction, the only given premise in the abductive reasoning is the *result,* and both the *rule* as well as the *case* follow from it.

Researchers in mathematics education see abduction in two different ways, either as a process of *reasoning backwards* (see for example Pedemonte, 2002; Knipping, 2003a), or as a process of *generating a hypothesis* (see for example Peirce, 1878; Pease & Aberdein, 2011; Papadaki et al., 2019). When seen as *reasoning backwards*, abduction is seen as a kind of reversed deduction, a means to *explain* a surprising fact. On the other hand, when it is seen as a way to *generate a hypothesis*, abduction is seen as the means to *explore* the surprising fact. Eco's three kinds of abduction are related to the two different operations of abduction (*explaining* and *exploring*). An overcoded abduction (single possible *rule*) *explains* a surprising *result*. An undercoded abduction (multiple possible rules) *explores*, and may at the same time *explain*, a surprising *result*. The same is true for a creative abduction, where not only the surprising *result* is *explored* (and *explained*), but also a new *rule* is invented (Reid & Knipping, 2010).

As presented later (see subsection 2.2.4) the distinction between the two *operations* of abduction (*explaining* and *exploring*) is particularly important when it comes to deciding *how* to reconstruct abductive arguments. Therefore, I revisit this matter in subsection 2.2.4, where I discuss the different types of arguments, and the different ways in which abductive arguments can be reconstructed based on the *operation* they perform.

In the next section I move on from reasoning to argumentation in mathematics, and from types of reasoning to types of arguments.

2.2 Argumentation in mathematics

As mentioned in the beginning of this chapter, in mathematics *reasoning* and *argumentation* are interconnected notions. Argumentation is in the spotlight in mathematics education mainly because "The process of generating an argument, individually or collectively, involves seeking an explanation/justification for a claim (idea, conclusion, verification, etc.). As such, argumentation encourages self explanation and learning" (Schwarz et al., 2010, p. 120). The benefits of argumentation for learning, have constituted an integral part of mathematics teaching at school.

Argumentation is one of the two theoretical tools that I use for the purposes of this research (the other one is visualization, and the theory around it is presented in Chapter 3). From now on, whenever I refer to argumentation in this work I mean argumentation in a mathematical context. In this section, I move from reasoning (see 2.1.1) to argumentation. I start with the description and meaning of argumentation that I follow in my work (2.2.1), and then I present other definitions of argumentation that have been widely used in literature but do not fit as well to the purposes of the present study. Then, I describe Toulmin's (1958) functional model of argument as the tool for reconstructing arguments (2.2.3). At the end of the section (2.2.4), I present different types of arguments used in mathematical argumentation, linking them with the types of reasoning that students employ in mathematics (see description in 2.1.2), and showing how they can be reconstructed using Toulmin's (1958) model of argument.

2.2.1 What is argumentation in the present work?

"Argumentation is a multifaceted term with different meanings" (Schwarz et al., 2010, p.116). In the present work, the notion of *argumentation* is related the process of expressing one's reasons to support or reject a statement or an opinion, through verbal articulation, gestures or actions. Argumentation may take place between two or more participants, or a single student may perform it when he/she expresses his/her reasons out loud (based on Douek's, 2002 and 2005 definitions, see below).

According to Knipping and Reid (2015), there are two distinct aims when considering argumentation in mathematics education. One of them is about learning mathematics in an argumentative setting, while the other is about learning argumentation within a mathematical context. In the first case, the focus lies on the mathematical content while argumentation is the frame in which this content is learned. In the second case, the focus lies on the learning of mathematical argumentation, with the mathematical content being the "background" enhancing the argumentation. My study belongs in the first category described by Knipping and Reid (2015) and the focus lies on the ways in which the students' learning of geometry can be promoted in an argumentative context.

This study is conducted in a real school classroom, in the form of experimental lessons (see Chapter 5). It would therefore be reasonable to adopt a description of argumentation that fits classroom situations. I find Douek's (1999, 2002, 2005) definition of an *argument* and *argumentation*, most fitting to my research. In the following paragraphs I explain the reasons for my choice.

Douek (2005) describes an *argument* as "a reason or reasons offered for or against a proposition, opinion or measure" (p. 152), which is *not* necessarily deductive. In Douek's (2002) work, the notion of *argumentation* "denotes the individual or collective process that produces a logically connected, but not necessarily deductive, discourse about a given subject" (p. 304). The notion of argument is linked to argumentation, in that the argumentation consists of one or more logically connected arguments. The connection may be by deduction, induction or analogy. The inclusion of various types of arguments, other that deduction, in the argumentation, is an important characteristic of Douek's definition, since in the reality of a school classroom students' arguments are rarely exclusively deductive. Douek (1999) also mentions that argumentation may include verbal arguments, numerical data, drawings etc. As I show in the results of my work, the students use visual representations generated by the Dynamic Geometry Environment (DGE), in which they work, as part of their argumentation (see *visual data* in Chapter 8). These representations are two-dimensional cross-sections of the solid that appear on the computer screen.

As Knipping (2012) also comments, "It is important to note that [...] mathematical argumentation does not follow explicit rules. Instead what is acceptable argumentation is negotiated in a social community" (pp. 3–4). In the school classroom one comes across Brousseau's (1984) notion of "didactical contract". In the micro-culture of a school classroom the students and the teacher constitute a community. This community establishes its own bases and rules for the mathematics in this classroom. In each classroom the rules may differ. What is accepted as "shared" knowledge in one classroom may not be as such in another one and hence may require explanation. In one classroom, to justify one's statement may mean to convince the rest of the members of the community only with deductive arguments, while in another classroom induction may also be accepted as a valid form of justification. The members of such a community are the validators of anything said in the community. This does not mean that a demonstration will be accepted as valid even if it is mathematically incomplete. It only means that the level of mathematical formality is every time adjusted to the knowledge and needs of the students. As I mention in the previous paragraph, Douek (2002) describes argumentation as possible both as an individual process, as well as a collective process. This portrays the social dimension of argumentation in Douek's (2002) definition.

The social dimension of Douek's (1999, 2002, 2005) definition of argumentation, as well as the inclusion of various types of arguments in it as parts of the argumentation, are two of the reasons why I chose to align with her definition. There is also a third reason; the compatibility of Douek's (1999, 2002, 2005) definitions of argument and argumentation with Toulmin's (1958) theory about arguments and his functional model of argument. In my study, I am interested in the structure of students' arguments, therefore I analyze them using, as methodological tools, Toulmin's (1958) functional model of argument and Knipping's (2008) method of argumentation reconstructions, which is an expansion of Toulmin's model (see also Knipping & Reid, 2019. See more details in 5.6.2 in Chapter 5). I discuss Toulmin's (1958) theory on reconstructing arguments, in subsections 2.2.3 and 2.2.4, below.

2.2.2 Other definitions of argumentation

In the research field of mathematics education, there are also other definitions of mathematical argumentation, which have been widely used. I would like to discuss some of them here. I also provide explanations as to why those definitions are not appropriate for my research, although they are perfect choices in other research contexts in the field. In Reid and Knipping (2010), the authors dedicate a chapter of their book to a review on the different definitions of argumentation used in the frame of mathematics and mathematics education (ibid, see Chapter 8). Here, I use this review as a reference for my further discussion on the various definitions of argumentation, as well as on the criteria upon which I have decided why those argumentation definitions are not appropriate for the purposes of the present work.

As Reid and Knipping (2010) mention, Balacheff (1999) points out that the differences between the approaches that researchers have taken regarding argumentation arise from the different theories upon which these approaches are based. Balacheff (1999) distinguishes between three such theoretical backgrounds, namely the works of: Perelman, Toulmin and Ducrot.

> "Briefly, Perelman sees argumentation as being about convincing. Toulmin sees it as being about the structure of the argument and its reference to premises accepted in a community. Ducrot places argumentation at the heart of the activity of discourse and focusses on grammatical structures. This gives a possible classification:
>
> - argumentation is what convinces another person
> - argumentation has a logical structure accepted in a community
> - argumentation is present in all discourse and founded on grammatical elements" (Reid & Knipping, 2010, p. 154)

In compliance with what I have said in subsection 2.2.1 regarding the purposes of my research, it would not make sense to adopt any definition of argumentation that focuses on strictly convincing others (Perelman's approach) or on the grammatical structure of argumentation (Ducrot's approach). Such definitions would for example be Duval's (1991) and Balacheff's (1991, 1999), both of whom consider argumentation to be a kind of non-deductive reasoning. Both Duval's and Balacheff's definitions are based on Perelman's approach to argumentation. This use of the notion of argumentation already comes in contradiction to the way it is used in my research. As I mention in subsection 2.2.1, for me argumentation is the related to the expression of one's reasons. It might even indicate the externalization of one's reasoning, but it is not a type of reasoning in itself. Furthermore, it includes even more than the mere articulation of reasons. It includes any type of action a person uses when expressing themselves (gestures, use or creation of drawings etc.).

As I discuss in subsection 2.2.1, Douek's (1999, 2002, 2005) definition, which I adopt in this work, is based on Toulmin's approach. But there are also other definitions of argumentation that are based on Toulmin's approach to argumentation. For example, Pedemonte (e.g. 2002) uses Toulmin's (1958) model to reconstruct students' arguments. At the same time though, Pedemonte (2007) defines argumentation as "the process connected with the conjecture" (p. 25). So, the reason I cannot adopt this definition, is that I consider it to be just one part of argumentation, which does not include the wide spectrum of processes that argumentation incorporates. I could associate such a view on argumentation only to the abductive arguments leading to the creation of a hypothesis (see the part on Abduction in subsection 2.2.4).

Krummheuer's (e.g. 1995, 1997, 2007) work on argumentation is extensive and it is based on both Toulmin's (1958) as well as Perelman's approaches (Reid & Knipping, 2010). He has used Toulmin's model of argument (1958) in order to reconstruct students' arguments (see e.g. Krummheuer, 1995). Krummheuer has a rhetorical understanding of argumentation in a social setting. In this setting, the participants in the argumentation interact with one another trying to convince each other. This later part of Krummheuer's description of argumentation echoes Perelman's approach to argumentation as being about convincing others. For Krummheuer, *argumentation* is a social phenomenon, a process of which *arguments* are the product of.

> "The final sequence of statements accepted by all the participants, which are more or less completely reconstructable by the participants or by an observer as well, will be called an argument" (1995, p. 247)

Krummheuer's (1995) view of argumentation is that it is a social phenomenon, accomplished mainly by more than one participant. Such cases of argumentations he characterizes as "collective argumentation".

In the frame of this work, I could have also adopted Krummheuer's understanding of argumentation. Nevertheless, I find Douek's (2002) definition the "necessary and sufficient" formulation of argumentation's definition for my work. Her definition includes both the individual and the collective view of argumentation as a process, it is in line with Toulmin's (1958) model of argument for the reconstruction of arguments, and it includes multiple types of arguments (deduction, induction, analogy etc).

Next, I move on to the description of Toulmin's (1958) theory on arguments (2.2.3), as well as the detailed description of the different types of arguments that appear in the mathematics class and how they can be reconstructed with Toulmin's model of argument (2.2.4).

2.2.3 Toulmin's functional model of argument

In his book *The uses of argument*, Toulmin (1958) presents a *functional model of argument* (see Figure 2.1). The applicability of Toulmin's model is not restricted to only one specific field, rather he talked about the layout of arguments in any context of discourse. He considered examples from law, mathematics, philosophy, medicine and more. Toulmin's model has also been used by researchers in the field of didactics of mathematics as a tool for the reconstruction of students' arguments (e.g. Knipping, 2003a, 2004; Cramer, 2018). Through Toulmin's work we can understand both the social as well as the structural characteristics of argumentation. Having already discussed the social dimension of argumentation (see 2.2.1), I would now like to focus on the structural part of argumentation through Toulmin's work.

Toulmin's idea was that in whichever context an argumentation takes place, it always maintains specific structural characteristics. Argumentation always revolves around a specific statement that expresses an assertion. This assertion "shows merits we are seeking to establish" (Toulmin, 1958, p. 97). Toulmin calls this assertion a *conclusion* or *claim* (C) (see Figure 2.1). He then asks, "What have you got to go on?" (ibid, p. 97) in order to establish a conclusion. One way "to go on" is to provide facts in order to support the conclusion. These facts are called *data* (D), and they are "the ground which we produce as support for the original assertion". Toulmin (1958) illustrates this by an example. If one's assertion is that "Harry's hair is not black", one can base this on the "personal knowledge that it is in fact red" (ibid, p. 97). Then the assertion is justified and becomes a conclusion, while the pair of the datum and the conclusion is the argument.

If the datum provided is not sufficient as evidence for the justification of the conclusion, and no additional data seem to be sufficient either, then "propositions of a rather different kind: rules, principles, inference-licenses (...)" (p. 98) are required. These types of propositions are called *warrants* (W) and their function is "to show that, taking these data as a starting point, the step to the original claim or conclusion is an appropriate and legitimate one" (ibid, p. 98). *Warrants* "can act as bridges, and authorize the sort of step to which our particular argument commits us" (ibid, p. 98). The warrant that Toulmin suggests for the above example is that "If anything is red, it will not also be black" (p. 98). Toulmin notes that although the distinction between a warrant and a datum may not always be easy to be made, their functions in an argument are different; a *datum* (D) is a fact, a piece of information, whereas a *warrant* (W) is a statement which justifies that the step from a datum to a claim is a legitimate one. He also notes: "data are appealed to explicitly, warrants implicitly" (p.100).

There are many kinds of warrants, which "may confer different degrees of force on the conclusions they justify" (p. 100). These different degrees of force of warrants, attribute in turn different degrees of certainty to the conclusions they justify. These degrees of certainty for the conclusion are called *qualifiers* (Q) and they are adverbs or expressions, such as "certainly", "probably", "presumably", "could be" etc.

Further elements of Toulmin's (1958) functional model of argument, are the *rebuttal* and the *backing*. A *rebuttal* (R) expresses the "circumstances in which the general authority of the warrant would have to be set aside" (p. 101). In other words, a rebuttal expresses the cases, which are exceptions to the rule of the warrant. A *backing* (B) is a statement, which acts as additional assurance "without which the warrants themselves would possess neither authority nor currency" (p. 103). The difference between *warrants* and *backings* is that "statements of warrants [...] are hypothetical, bridge-like statements, but the backing for warrants can be expressed in the form of categorical statements of fact quite as well as can the data appealed to in the direct support of our conclusions" (p. 105).

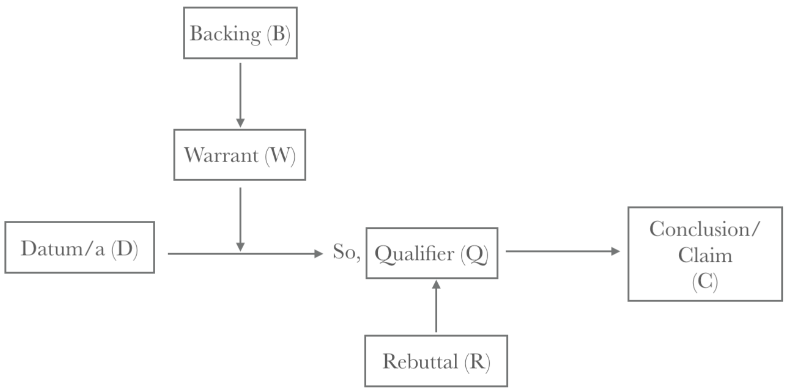


Figure 2.1: Toulmin's functional model of argument

Toulmin's functional model of argument is an excellent tool for the reconstruction of individual arguments occurring in argumentations that take place in the mathematics classroom. It has been used by many researchers in mathematics education in order to reconstruct students' arguments (e.g. Pedemonte (2002); Knipping (2003a, 2008)). It has also been used by researchers in mathematics and philosophy of mathematics in order to analyze arguments or even whole proofs, such as Aberdein (2006) and Banegas (2013). One limitation of the Toulmin model seems to be that it can only be used to reconstruct individual arguments, thus not providing a bigger image of the whole argumentation that takes place in a particular situation. This limitation has been overcome by Knipping (2003a and 2003b), who had the idea to connect individual arguments reconstructed with the Toulmin model, in order to illustrate whole argumentations. In my work, the reconstruction of students' whole argumentations is an important tool in order to analyze students' work. In Chapter 5, where I describe the methodology used in the present work, I present in details Knipping's methodology of argumentation reconstructions (see 5.6.2).

2.2.4 Types of arguments in mathematics

I would now like to present the types of arguments that occur in the mathematics classroom and how they can be reconstructed with Toulmin's (1958) functional model of argument. I examine in parallel both Peirce's (see Table 2.1 in 2.1.2) and Toulmin's (see Figure 2.1 in 2.2.3) terminologies in order to present how each type of reasoning is illustrated by the corresponding type of argument. I start with *deduction* and *induction*, as these are types of reasoning and arguments that both Peirce (1878) and Toulmin (1958) refer to. I also mention *hypothetical deduction*, and in particular how we can reconstruct a *Reductio ad absurdum* (RAA) with the Toulmin model. Then, I continue with Peirce's *abduction* and show how an abductive argument can be illustrated with Toulmin's model of argument.

Deduction

In his book, Toulmin (1958) focuses on two types of arguments, *deduction* and *induction*. He distinguishes between the two according to their relationship with warrants. He classifies *deductions* as *warrant-using* arguments and *inductions* as *warrant-establishing* arguments. In *warrant-using* arguments, "a single datum is relied on to establish a conclusion by appeal to some warrant whose acceptability is being taken for granted" (p. 120). Hence, "wherever there are established warrants or set procedures of computation by which to pass from data to a conclusion, there we may properly speak of 'deductions'." (p. 121). Toulmin (1958) symbolizes *deductions* as (D; W; so C) arguments. According to Peirce, the outcome of a *deduction* is a *result*, inferred by a *case* and a *rule*. In Toulmin's terminology this means that a conclusion (C) is inferred by a datum (D) and a warrant (W). The outcome of a deductive syllogism is a certain one. This "certainty" of the Result can be illustrated with a qualifier (Q). Table 2.2 shows the correspondence between Peirce's terms for the elements of deductive reasoning and Toulmin's terms for the elements of a deductive argument.

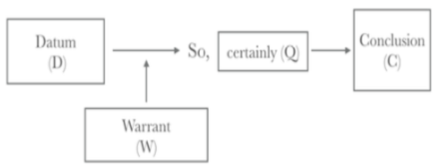
Deduction		
Peirce (1878)	**Toulmin (1958)**	
Case	Datum	
Rule	Warrant	
Result	Conclusion	

Table 2.2: Deduction in Peirce's and in Toulmin's terminology

So, if I would like to structure Peirce's example of a deduction (see Table 2.1) as an argument using Toulmin's model, it would look like this:

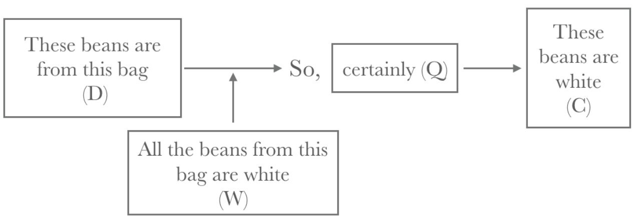


Figure 2.2: Peirce's deduction reconstructed with Toulmin's functional model of argument

Induction

On the other hand, *warrant-establishing* arguments are "such arguments as one might find in a scientific paper, in which the acceptability of a novel warrant is made clear by applying it successively to a number of cases in which both 'data' and 'conclusion' have been independently verified. In this type of argument the warrant, not the conclusion, is novel, and so on trial" (p.120). Therefore, *inductions* help us to establish a new warrant 'using our observations of regularities and correlations as the backing for a novel warrant' (p. 121). Naturally, in the case of *inductions* the warrant (W) that is used in the argument is under evaluation, until it is found to work for all the established data (D) and conclusions (C), for which it is under examination. As soon as the new warrant is established and "rendered general by induction" it can then be "applied as a rule of deduction in fresh situations to derive novel conclusions from our data" (Toulmin, 1958, p. 122). According to Peirce, the outcome of an *induction* is a *rule*, inferred by a *case* and a *result*. In Toulmin's terminology this means that a new warrant (W) is inferred by a datum (D) and a conclusion (C). This warrant, emerging from an inductive syllogism, is not certain rather only possible.

As I mention in the beginning of subsection 2.2.3, in Toulmin's terms argumentation always revolves around an assertion that "shows merits we are seeking to establish". This means that for Toulmin, arguments are about ways of establishing the truth of a claim (verification, convincing, etc.), not ways if generating claims. In subsection 2.1.2, I present *induction* as a type of reasoning (Peirce, 1878). There, I mention that *induction* is about all sorts of things, such as *pattern observing*, *predicting*, *conjecturing*, *generalizing* and *testing* (Reid & Knipping, 2010). What Reid and Knipping (2010) call *generalizing,* is closer to what Peirce calls *induction.* But *generalizing,* as well as *pattern observing*, *predicting*, *conjecturing* are all about generating a claim, not about establishing it. This may be one reason why it is not immediately obvious, how to illustrate Peirce's *induction* using Toulmin's (1958) model. *Testing* (Reid & Knipping, 2010) is the only sort of *inductive reasoning* that is about establishing the truth of a claim. But it is not in accordance with neither Peirce's (1878), nor Toulmin's (1958) description, since it goes from a *rule* (warrant) and a *result* (conclusion/claim) to a *case* (datum). This complexity of *induction* is a reason why it is not immediately obvious, how to illustrate Peirce's *induction* using Toulmin's (1958) model. Toulmin (1958) himself discusses *induction*, but unfortunately, he does not provide us with a specific example of an inductive argument reconstructed with his model. Also, I could not find an illustration of Peirce's *induction* (1878) with Toulmin's (1958) model in the literature, to use here as a reference. And since this endeavor is beyond the scope of the present work, I will conclude my description of *induction* here.

Although Toulmin (1958) only spoke of *deduction* and *induction* in his book, these are not the only types of arguments that take place in the mathematics classroom. *Abductions* and *hypothetical deductions* are two types of arguments that often appear in the teaching and learning of mathematics. And Toulmin has provided us with a tool that helps us to reconstruct other types of arguments as well; not just deductive arguments. Therefore, I would like to discuss *abductive* and *hypothetical deductive* arguments and show how these can be reconstructed using Toulmin's (1958) model of argument. Two more types of arguments that also appear often in the mathematics classroom; *hypothetical deduction* and *abduction.*

Hypothetical Deduction

A *hypothetical deduction* is a deductive argument that begins with a hypothetical statement (hypothetical fact), instead of a datum (known fact). It is an argument that is based on *hypothetical deductive reasoning* (Reid, 2002a), where we start to "reason from a hypothesis, something that is not known to be the case, either to show that it cannot be the case (as in proof by contradiction) or to show that if it were the case for one number it would also be true for the next number (as in a proof my mathematical induction)" [2] (Reid, 2002a, p. 110). "Such reasoning, because it involves a hypothesis, is called hypothetical deductive reasoning" (Reid, 2002a, p. 110). Here, I will only focus on one type of *hypothetical deduction*, namely *proof by contradiction*, otherwise known as *Reductio ad absurdum* (RAA). The tasks used in this study are not related to proof by mathematical induction, a process very much relevant in cases of algebraic tasks.

Figure 2.3 shows how an RAA-argument can be structured using Toulmin's functional model of argument. In *Reductio ad absurdum*, the argument starts with a supposition and ends with the negation of it (see Figure 2.3). I call *assumption* (As) this type of supposition that is meant to be refuted[3]. After the *assumption* (As) has been stated, a new conclusion (C) is inferred based on a known warrant (W) (additional data (D) may also contribute to the conclusion). Then a datum (known fact) comes in contradiction (red zigzag line in Figure 2.3) with this conclusion, thus leading to an *absurdity.* From that follows, according to the method of *Reductio ad absurdum*, that the initial *assumption* is not correct (arrow with RAA in Figure 2.3). Therefore, the assumption is rejected (negation of assumption, ¬As). Figure 2.3 shows an example of a RAA.

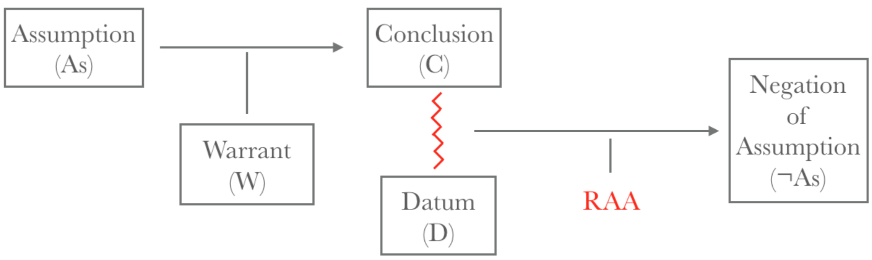


Figure 2.3: Reductio ad absurdum reconstructed with Toulmin's functional model of argument

[2] Mathematical induction is based on deductive reasoning and it is different from induction, which is based on inductive reasoning. To avoid confusion, if I need to refer to mathematical induction again in this work, I will use the alternative term of "reasoning by recurrence" (for more details see Reid & Knipping, 2010, p. 99).

[3] I do this, to distinguish it from other types of suppositions, that I use in this work and which are outcomes of abductions, such as the hypothesis (supposition of a possible result) and the claim (supposition of the most probable result). See more details on this in Chapter 5 (5.4.2 and Table 5.3)

Reductio ad absurdum (RAA) is basically a method for the rejection of a supposition. As shown above, Toulmin's model is very useful for the reconstruction of arguments that are generated based on RAA. In the present work, students use the RAA in their argumentations as well. In Chapter 8 I provide such an example (see one example in subsection 8.4.1).

Toulmin (1958) does not discuss *abduction* and the way abductive arguments can be reconstructed with his functional model. Nevertheless, abductive arguments appear often in the classroom reality, especially in explorative situations such as the one I implement in this work. Therefore, I would like to show how abduction can be reconstructed using Toulmin's model.

Here, I present how different researchers from the field of mathematics education (Pedemonte, 2002; Knipping, 2003a) and the field of philosophy of mathematics (Pease & Aberdein, 2011) use Toulmin's model in order to reconstruct abductive arguments. I also discuss the modifications that these researchers have performed on the structure of Toulmin's model, in order to reconstruct abductions and how their ways of reconstructing abductions differ from each other.

As I mention in ion 2.1.2, researchers in mathematics education see abduction in two different ways, either as a process of *reasoning backwards* (see for example Pedemonte, 2002; Knipping, 2003a), or as a process of *generating a hypothesis* (see for example Peirce, 1878; Pease & Aberdein, 2011; Papadaki et al., 2019). When seen as *reasoning backwards*, abduction is used as a kind of reversed deduction, as a means to *explain* a surprising fact. On the other hand, when it is seen as a way to *generate a hypothesis*, abduction is used as a means to *explore* a surprising fact. This difference in the processes (reasoning backwards or generating a hypothesis) and operations of abduction (explaining or exploring) in argumentation, leads to different ways of reconstructing abductive arguments using Toulmin's (1958) model (see Table 2.3).

In Papadaki, Reid and Knipping, 2019, we present a comparison of the different reconstructions of abductions. Table 2.3 (based on Table 2 in Papadaki et al, 2019) shows this comparison[4]. Peirce's terminology and his example of abduction (see example in Table 2.1) are used as a common reference in order to describe and depict the different ways in which abduction has been modelled by different researchers (Pease & Aberdein, 2011; Pedemonte, 2002; Knipping, 2003a) using Toulmin's terminology. Table 2.3 shows the way in which each researcher corresponds Peirce's terms for the elements of abduction (Case, Rule, Result) with Toulmin's terms for the elements of an abductive argument (Datum, Warrant, Conclusion/Claim). Also, the type of *process* that is performed by the abduction (generating hypothesis or reasoning backwards) in the argumentation, the *operation* of the abduction (explaining or exploring) and the *flow* of the abduction (forward or backward), are described for each reconstruction. The term *flow* refers to the direction of the arrow that connects the datum and the conclusion (Toulmin's (1958) terminology). When the arrow goes from the datum (D) to the conclusion (or claim, C), the abduction has a *forward flow*. When the arrow goes from the conclusion (C) to the datum (D), the

[4] Table 2.3 is an updated version of Table 2 in Papadaki et al. (2019). The term function of abduction has been replaced by the term "process" to avoid confusion with Toulmin's (1958) use of the term function for his functional model of argument. Also, Table 2.3 has one more row compared than Table 2, to show the operation of abduction for each of the described reconstructions.

abduction has a *backward flow*.

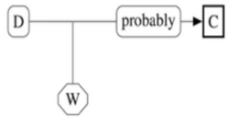
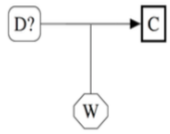
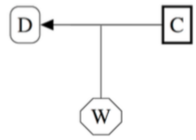
Peirce	**Pease & Aberdein (2011)**		**Pedemonte (2002)**		**Knipping (2003a)**	
Rule	W		W		W	
Result	D		C		C	
Case	C	Figure 2.4	D?	Figure 2.5	D	Figure 2.6
Process	Generating a hypothesis		Reasoning backwards		Reasoning backwards	
Operation of abduction	Exploring		Explaining		Explaining	
Flow of abduction	Forward		Forward		Backward	

Table 2.3: Three reconstructions of Peirce's abduction with Toulmin's terminology

In the first reconstruction (see Figure 2.4 in Table 2.3) Pease and Aberdein (2011) keep the forward flow in Toulmin's model (arrows pointing from datum to claim). They correspond the result of Peirce's syllogism with Toulmin's datum (D) because that is the known fact and it needs no justification. They correspond Peirce's *rule* with Toulmin's warrant (W), and Peirce's *case* with Toulmin's claim (C) as the statement that is inferred. They also express the uncertainty of the claim by employing a qualifier (the word "probably") and placing it before the claim (see Figure 2.4 in Table 2.3). Pedemonte (2002) was the first researcher in mathematics education to use Toulmin's model to describe abduction. She too models an abductive argument with a forward flow but reverses Pease and Aberdein's (2011) placement of data (D) and claim (see Figure 2.5 in Table 2.3). Pedemonte (2002) considers a claim to be the known target of the inference (Peirce's *result*) and the datum to be the unknown fact (Peirce's *case*), which must hold in order for the claim to stand. She denotes the "unknown" status of the datum with a question mark next to the letter D (which represents the datum).

The difference between Pease and Aberdein's, and Pedemonte's reconstructions of abduction may arise from the different *processes* and *operations* they attribute to it. Pease and Aberdein see abduction as a process of *generating a hypothesis*, whereas Pedemonte sees it as a reversed deductive process, as *reasoning backwards*. This means that Pease and Aberdein (2011) see abduction as a means to *explore* an interesting situation that represents Peirce's *result* and is reconstructed as Toulmin's datum. Pedemonte (2002) on the other hand, approaches abduction as a means to *explain* the result of a situation, which represents Peirce's *result* and is reconstructed as Toulmin's conclusion. In order to explain this result Pedemonte has to figure out what the unknown facts are (Peirce's *case*, Toulmin's datum) that would explain it (hence the "D?"). It is important to note here that these researchers model individual arguments, in isolation from other arguments.

Contrary to Pease and Aberdein (2011), and Pedemonte (2002), Knipping (2003a, 2008) describes a way of linking together arguments analysed using the Toulmin model to describe larger structures of argumentation. She reconstructs the whole argumentation of students, building a Global Argumentation Structure (GAS), which reveals a complete overview of the argumentation that took place (see more in Chapter 5, subsection 5.4.2). Knipping (2003a), like Pedemonte (2002), sees abduction in argumentation as a process of *reasoning backwards*. She reconstructs complex deductive argumentations, which may contain some abductive arguments. The *flow* of the overall GAS is *forward*, but the flow of the abductive arguments is *backward* (see Figure 2.6). Knipping (2003a, 2008), like Pedemonte (2002), considers the claim (Toulmin) to be the *result* (Peirce) of the inference, and the datum (Toulmin) to be the *case* (Peirce) that must hold in order for the claim to stand. The difference between Knipping's and Pedemonte's reconstructions lies in the *flow*. Knipping (2003a, 2008) marks the abduction differently, by reversing the arrow to indicate a *backward flow* from the conclusion (as the known *result*) to the datum (as the *case* that needs to be determined). Contrary to Pedemonte (2002), Knipping does not use a question mark next to the letter D to indicate the unknown *case*.

To sum up, abduction can have two operations in argumentation: *explaining* and *exploring*. Depending on that *operation* the process of abduction also differs. When abduction is used to *explain* then it is a process of *reasoning backward* from a known outcome to its undetermined cause. In this case, abduction can be reconstructed either with a forward *flow* (as in Pedemonte, 2002), or with a backward *flow* (as in Knipping, 2003a and 2008). When abduction is used to *explore* then it is a process of *generating a hypothesis* based on a known fact. In this case, it makes more sense to reconstruct abduction with a forward *flow* (as in Pease & Aberdein, 2011). An important element of this last type of structure is the use of qualifiers to mark the epistemic value (Duval, 1990, 2007) of the inference (claim or conclusion).

In this work I adopt Pease and Aberdein's (2011) way of reconstructing abduction. In my research the students are asked to *explore* a given situation. More precisely, they are given geometric tasks in which a three-dimensional geometric object is hidden. The students are asked to identify the hidden solid by its visible two-dimensional cross-sections with a plane (generated by the Dynamic Geometry Environment). Hence, the students have to work on an unknown and invisible solid. The focus of their argumentation is the *generation of hypotheses* regarding the form of the solid and the justification for their final conclusion. They are not asked to produce a mathematical proof for their conclusion. Therefore, I consider Pease and Aberdein's (2011) way of reconstructing abductions the most suitable in the context of the present research.

2.3 Epilogue

In this chapter, I have discussed the notions of *reasoning* (Section 2.1) and *argumentation* (Section 2.2) in the teaching of mathematics and the relationship between them. I have mainly focused on argumentation, as it is one of the two major theoretical notions used in the present work. Through the literature review I performed, I explained the meaning that I attribute to the notion of argumentation, which is that of "the individual or collective process that produces a logically connected, but not necessarily deductive, discourse about a given subject" (Douek, 2002, p. 304). This process may include reasons expressed both verbally as well as with the help of actions, such as gestures (Sections 2.2.1 and 2.2.2).

I also presented Toulmin's (1958) functional model of argument (Section 2.2.3) and the ways in which different types of arguments can be reconstructed using Toulmin's model (2.2.4). In this work, I use Toulmin's model as a tool for the reconstruction of students' arguments.

In the next chapter, I present the second major theoretical notion used in this work, namely *visualization*, and its relationship with argumentation in mathematics.

3 Visualization in geometry and its relation with argumentation

Visualization, in the context of geometry, is the second theoretical notion (in addition to argumentation) that plays a major role in the present work. This chapter is dedicated to *visualization* and it is based on a literature review of both theoretical as well as empirical research in the field of mathematics education. This literature review goes insofar as to serve my research and help me to answer my research questions (see subsection 1.4). Therefore, I focus here only on research that examines visualization in the context of geometry, and not in mathematics as a whole[1]. I discuss both the approach to visualization that I follow and the way I use it in this research, as well as other approaches, definitions and uses of visualization in geometry by other researchers in the field of mathematics education.

The chapter begins with a discussion on what visualization is in geometry and with the presentation of the specific approach followed here (Section 3.1). In Section 3.2, I give an overview of the empirical research that has been conducted in connection with the relationships between visualization and argumentation (see 3.2.1). Then, I explain what it is that I would like to examine further with regard to the relationship of the two notions, presenting the main aim of this work (3.2.2). Finally, the chapter closes with a short epilogue on visualization and a prologue on the subject of the next (and last) theoretical chapter of this work.

3.1 What is visualization in geometry?

Visualization in geometry (and mathematics in general) is a notion that has been described and defined in more than one way. This divergence depends mainly on the researcher's theoretical background through which visualization is approached (cognitive psychology, developmental psychology, pedagogy etc). What is common though in all the approaches is the expression of the importance of visualization in the teaching and learning of mathematics. According to Giaquinto (1992) "visualizing can be a means of discovering a geometrical truth" (p. 384), while for Duval (1999/2002), visualization is "at the core of understanding in mathematics" (p. 312). Since the 1980s, mathematics educators have focused on the importance of visualization in mathematics education (Dreyfus, 1994; Presmeg, 2008).

In this work, my main focus is to examine the way visualization is related to mathematical argumentation in the context of geometry. In order to do this, I first

1 In Chapter 2, I talk about argumentation in school mathematics, not only in geometry. In the case of visualization I focus specifically on its use in geometry, because the mathematical context seems to play a significant role in the meaning attributed to this notion in research.

need to discuss in more detail the meaning of the notion of visualization in the present work. In this section, I present a literature overview on the different theoretical approaches on visualization in geometry, based on both theoretical as well as empirical research. The plurality in the interpretations of what visualization *is* and how it can be used in research, creates a rather fruitful polyphony for the analysis of learning phenomena in mathematics classrooms. Nevertheless, consistency and explicitness regarding the way in which one uses this term is also important for a meaningful presentation of one's research. Therefore, for the purpose of my study I look deeper into the part of literature on visualization that is relevant to my work and can help me to investigate the role of visualization in students' argumentations. This is the literature in which visualization is seen as a cognitive process (see 3.1.1). Hence, in this work, visualization in argumentation is seen as a process externalized verbally, probably (but not necessarily) accompanied by actions such as gestures or metaphors.

In the first part of the section (3.1.1), I describe Duval's (1998, 1999/2002, 2005) theoretical approach to visualization, which is also the approach I adopt in this work. In the second part, I present the descriptions of visualization that other researchers have used and the way they approach visualization in the frame of spatial abilities (3.1.2). I also introduce the notion of *spatial manipulation*, related to spatial abilities, which I use in this work too (3.1.3).

3.1.1 Duval's cognitive approach to visualization

As I mention above, in this work I focus on examining and describing the role of visualization in students' argumentation in geometry. Therefore, it would make sense to choose an approach to visualization that allows me to identify it in the process of argumentation and describe its function in students' argumentation structures (see Chapter 2 and Chapter 5).

Duval (1999/2002), a cognitive psychologist whose work has been contributing to the research field of mathematics education since the 1970s, argues that there are many different models of visualization, which are used by researchers in mathematics education. He argues that most of those models of visualization are not suitable in the context of mathematics, since "these basic cognitive processes work quite differently in mathematics than in all the other fields of knowledge" (Duval, 1999/2002, p. 311). In this subsection, I present Duval's (1998) perspective on visualization and its relationship with argumentation (which he calls "reasoning"), explaining also why his approach to visualization is fitting to my study.

Before I begin, I think it is important to discuss some terms that Duval uses as well as their meanings, because these terms are mentioned in his description of visualization. Then, I move on with the description of what visualization *is* according to Duval. The way I apply his approach in order to reveal the role of visualization in students' argumentation is discussed in the methodological part of this research (see Chapter 5).

Drawings and figures

Duval (2017) uses the term "drawing" to refer to a representation or a depiction, of a geometric object. With the term "figure", Duval (2017) refers to the theoretical geometric object, that is a whole class of representations (drawings) of the geometric object, as well as to its properties. He writes:

> "The "drawing" is the particular configuration shown on the paper, on the blackboard or on the computer monitor, while the "figure" would be the object properties represented by the drawing or still, the class of all drawings that may be the visual representations of the object" (ibid, p.63).

Laborde (1993) also distinguishes between the two terms, in a similar way. Fo her, a "drawing refers to the material entity, while figure refers to a theoretical object" (p. 49).

Visual perception versus Visualization

At this point, it is important to address the distinction Duval (1999/2002) makes between *visual perception* (*vision*) and *visualization*. *Visual perception* is about perceiving and processing visual information by sensory and mental processes, and it provides a direct sensory access to the perceived (see also Gal & Linchevski, 2010). On the contrary, *visualization* (in Duval's sense, 1998; 1999/2002) is specific to mathematics and it is a more complex process: "Visualization refers to a cognitive activity that is intrinsically semiotic" (Duval, 1999/2002, p. 322). In Duval's (1999/2002) words:

> "We have here the breaking point between visual perception and visualization. A semiotic representation does not show things as they are in the 3D environment or as they can be physically projected on a small 2D material support. That is the matter of visual perception. **A semiotic representation shows relations or, better, organization of relations between representational units.** (...) Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization. And that is why visualization should not be reduced to vision, that is to say: **visualization makes visible all that is not accessible to vision**. We can see now the gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary, **visualization can get at once a complete apprehension of any organization of relations**"
> (pp. 321-322, emphases in the original).

Duval (1998) also describes *visualization* as one of the three cognitive processes that take place in the learning of mathematics, the other two being construction and reasoning. More precisely, according to Duval, *visualization* is the cognitive process that deals with the comprehension of drawings (Mithalal & Balacheff, 2019). By "construction processes", Duval (1998) refers to the construction of representations of geometric objects with specific tools. By "reasoning", he refers to the "*discursive processes* for extension of knowledge, for proof, for explanation" (p. 38, emphasis in the original), what I call here *argumentation* (see subsection 2.2.1). In this work I concentrate more on the relationship between Duval's processes of visualization and reasoning, and I leave "construction" out of my scope, since the tasks used in this research are not constructing tasks. The relationship between visualization and argumentation (or reasoning in Duval's terminology) is discussed further in Section 3.2.

The two types of *visualization*

According to Duval (1999/2002) *visualization* can help us gain access to the structure of a figure, its *figural units*[2] (constitutive parts of lower dimension), as well as the properties and inter-relationships that govern the figure's structure. Duval (2005) distinguishes between two types of visualization: *iconic* and *non-iconic visualization*, which guide the exploration and the way of comprehending[3] a figure.

Iconic visualization is a spontaneous approach to a figure (Mithalal & Balacheff, 2019). One recognizes it as the representation of a specific geometric object because "its shape is similar to an already known object" (ibid, p. 163). *Iconic visualization* is a *gestaltist* way of seeing a figure. An example of *iconic visualization* is the phenomenon of students falsely identifying a square as a rhombus, when its orientation is such as the one in Figure 3.1 below (Mithalal & Balacheff, 2019).

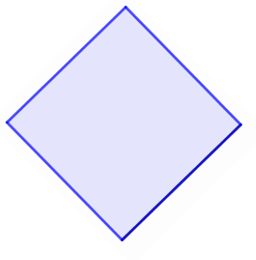


Figure 3.1: Square identified as a rhombus based on *iconic visualization*

Non-iconic visualization (NI-Visualization) is a property-based approach to a figure, where the figure is comprehended as a figural representation of the (theoretical) geometrical object, baring specific properties that determine it. Duval (2005, 2017) connects *NI-visualization* with three types of deconstructions of figures: *instrumental* deconstruction, *mereological* deconstruction and *dimensional*

[2] The figural units of a three dimensional (3D) figure can be: two dimensional (2D) such as plane cross sections or faces, one dimensional (1D) such as segments, inner lines, circles, or zero dimensional (0D), namely points, such as vertices and intersection points.

[3] Duval usually uses here the word "perceive" instead. I prefer to use the words "comprehend" or "apprehend" because they are not so closely related to the notion of simply seeing something with our senses (sight or touch). Both "comprehend" and "apprehend" refer to processes related to understanding and interpreting what is being observed with the senses.

deconstruction.

> "Visualization and deconstructions differ in nature but have strong links. Visualization deals with the perception[4] of drawings. Deconstructions are processes involved in their analysis" (Mithalal & Balacheff, 2019, p. 164. Footnote added by me)

More precisely about the different types of deconstructions:

- *Instrumental* deconstruction: refers to the steps that need to be followed in order to answer the question "How is it possible to produce this drawing with a given set of tools?" (Mithalal & Balacheff, 2019, p. 164). This type of reconstruction is an action-based process through which a new drawing must be created.
- *Mereological* deconstruction (or *mereological division,* Duval, 2017, p.61) [nD→ nD]: refers to the heuristic breakdown of a figure, "*into figural units of the same dimension* (2D→ 2D)" (Duval, 2017, p. 61, emphasis in the originals), i.e. the breaking down of a parallelogram in two triangles, by drawing one of the diagonals of the parallelogram. In the case of the mereological deconstruction of a solid the operation is a [3D→ 3D] one, i.e. breaking down a cone into 3D sub-parts by cutting it. The goal of this type of deconstruction is the reconfiguration of a drawing in order to solve a problem.
- *Dimensional* deconstruction [nD→ (n-1)D][5]: refers to the breaking down of the figure in its figural units, that is the lower-dimension parts which compose the figure, i.e. faces or cross-sections of solids (2D figural units), sides, diagonals, circles, ellipsis (1D), or points (0D). In this operation, the figural units of the figure are linked with each other through geometrical properties.

So, the *instrumental* deconstruction refers to a process of creating a drawing (or a figure), while *mereological* and *dimensional* deconstructions refer to two different processes of "breaking down" a figure into its sub-parts. In the present work, instrumental and mereological deconstructions do not play a further role, since the design of the tasks given to the students neither asks for a construction, nor allows the students to act heuristically (e.g. draw lines) on the provided figures. Here, I focus on students' *dimensional* deconstructions during their argumentation.

> "It is always the figural unit of the higher dimension that is perceptually recognized, and that blocks the recognition of all figural units of lower dimension, because it merges visually all these figural units potentially involved. *Seeing "geometrically" a figure is to operate a DIMENSIONAL DECONSTRUCTION OF THE SHAPES* that we recognize immediately into other shapes that are not seen at first glance, and this without changing anything in the figure displayed on the monitor or paper" (Duval, 2017, p. 59, emphasis in the original).

[4] See Footnote 3.

[5] In the symbolization [nD→ (n-1)D], the (n-1) actually refers to the maximum dimension figural unit that someone can get from a dimensional deconstruction. For a three-dimensional geometric object (3D) that means that it can be "broken" into two-dimensional (2D) figural units at most.

Consequently, through *non-iconic visualization* (*NI-visualization*) a figure is considered the result of a linking of its lower dimension sub-components, called *figural units,* through properties. Hence, with *non-iconic visualization* we move from a mere gestaltist apprehension of only the "surface", the "wrapping" of an object or of its representation, to a transparent view of the internal laws governing its construction. This is also the difference between the "inside" and the "outside" of a geometric object, which I discuss in Chapter 1 (see the beginning of the chapter, above subsection 1.1).

Mariotti (1989) also describes a similar connection between a geometric object and its subparts:

> "constructing the correct net of a solid implies coordination of a comprehensive mental representation of the object with the analysis of the single components (faces, vertices and edges)" (p. 263).

So, Mariotti (1989) talks about a process very much like the *dimensional deconstruction* of a mental representation, instead of a visible drawing like Duval (1999/2002) does. This could mean that the process of dimensional deconstruction is also possible in the physical absence of a drawing. I talk more about this phenomenon in Chapters 8 and 9.

Duval (2017) writes about the difference between the normal (as he calls it) and the mathematical way of seeing a figure in geometry. The usual way of seeing a figure is limited to an inflexible apprehension of it. On the contrary, seeing a figure mathematically means to be able to move beyond a rough apprehension of only its shape. It means to be able to recognize its components (0D, 1D, 2D, 3D figural units, as well as their various configurations) and the properties that govern them, and move between figural units of different dimensions "*spontaneously and quickly*" (ibid, p. 60, emphasis in the original). This is exactly the shift from *iconic* to *non-iconic* visualization.

Concluding remarks

Duval's (2017) approach to visualization is a cognitive one: "Cognitive analysis of the figures concerns *the way they need to be* "*seen*" *to be able to use them* to solve a problem or recognize the application of geometric properties in a real situation" (p. 58, emphasis in the original). I choose to adopt this theoretical approach to visualization, because it offers a framework that is practical and applicable in order to identify the role of visualization in students' argumentation. By "practical", I mean that I can develop specific indicators from Duval's theory, in order to identify the use of visualization in students' argumentations. These indicators are directly connected to the mathematical properties of geometrical objects, and this is manifested through its relationship with *dimensional* deconstruction. As Mithalal and Balacheff (2019) comment "Non-iconic visualization is a necessary condition for dimensional deconstruction to be operational" (p. 165).

As Hitt (2002) also explains,

> "Duval's research (1988, 1993, 1995, 2000) is centered around the idea that mathematical objects cannot be directly accessed by the senses, but only through semiotic representations, and that therefore the construction of a concept is only possible through the manipulation of its semiotic representations. Duval's approach is related to semiotic rather than mental representations. This makes his approach much more "graspable" because he restricts himself to materialistic observations and does not speculate about the possible mental processes these could indicate, or the kind of mental representations they could correspond to" (p. 246).

As I show in later chapters (Chapters 5 and 8) I could use very specific indicators for Duval's visualization in students' argumentations, exactly for the reasons Hitt (2002) gives (see quote above).

Next, I would like to mention another theoretical approach to visualization. Although I do not follow this approach for the purposes of my research, I would still like to discuss it shortly, as it is widely used in the research field.

3.1.2 Visualization in the frame of spatial abilities

Duval does not refer to the term "spatial abilities" when he talks about visualization. But, there are many other researches, in the field of didactics of mathematics, who do. In many cases visualization is presented in the more general frame of "spatial abilities", as just one of those abilities. Spatial abilities include a wide spectrum of "*imagining, constructing,* and *figuring*" abilities (Davis et al., 2015, p.5; Tahta, 1989). On one hand, the term "spatial abilities" is used by many researches with slightly different meanings or processes that integrate it. On the other hand, the same process may be described with different names by different people. Some examples of the terms used are: spatial abilities (Bishop, 1983; Presmeg, 1997), visualization and visualizing (Lean & Clements, 1981; Clements, 2012; Gutierrez, 1996), visual reasoning (Dreyfus, 1995) and spatial reasoning (Davis & Spatial Reasoning Study Group, 2015). As Sinclair et al. (2016) explain, what all these concepts and terms have in common is "the activity of imagining static or dynamic objects and acting on them (mentally rotating, stretching, etc.)" (p. 696).

It was in the 1980s that researchers such as Bishop (1980) and Presmeg (1986) initiated the discussion about spatial abilities in mathematics, bringing spatial abilities into the spotlight. In more recent research, the term "spatial reasoning" is preferred over that of "spatial abilities". Nevertheless, the meaning of the terms remains the same. Davis and the Spatial Reasoning Study Group (2015), describe spatial reasoning throughout their book (see especially p. 140), but Mulligan (2015) summarized this description in a kind of definition:

> "Spatial reasoning (or *spatial ability, spatial intelligence,* or *spatiality*) refers to the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects. Examples of spatial reasoning include: locating, orienting, decomposing/recomposing, balancing, diagramming, symmetry, navigating, comparing, scaling, and visualizing" (p. 513).

As seen in the above definition, visualization is considered as a type of spatial reasoning (or a spatial ability). Some examples of how visualization has been described in the frame of spatial abilities, are the following:

Visualization generally refers "to the ability to represent, transform, generalize, communicate, document, and reflect on visual information" (Hershkowitz, 1990, p. 75)

Visualization is "the process involved in constructing and transforming visual mental images..." (Presmeg, 1997, p. 304)

Bishop's (1983) Visual Processing ability, somehow corresponds to Duval's (1999/2002) description of visualization. Visual Processing "involves visualization and the translation of abstract relationships and nonfigural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery" (Bishop, 1983, p. 184).

Although spatial abilities (or spatial reasoning) are also related to the properties of the geometric objects, the definitions and descriptions of visualization in this context are multiple and not particularly precise as to the exact processes that take place in visualization. On the contrary, Duval's (1999/2002) definitions are much more precise, both for *iconic* and *non-iconic visualization.* Especially *non-iconic visualization* and the process of *dimensional deconstruction*, taking place in it, are described in a detailed and consistent way that makes them useful tools for the purposes of the present work.

3.1.3 Spatial manipulation

Besides visualization, there is another significant process to which I refer in the present work, and which is related to two other types of spatial abilities. I call this process *spatial manipulation* . In the present research, the students are given tasks in which they can manipulate an invisible solid using three sliders in a Dynamic Geometry Environment (DGE). Hence, I hypothesized that the students may, during their argumentation, describe the manipulations they imagine being performed on their mental image (Presmeg, 2006) of the invisible solid. Therefore, I created and use the term *spatial manipulation* (*Sp-manipulation*) in order to refer to students' processes of mentally manipulating the invisible solid in space (or rather, the mental manipulation of the mental image of the invisible solid). The term of *spatial manipulation* comprises processes distinguished in the literature by various spatial abilities, such as "spatial orientation" and "spatial visualization" (see McGee, 1979; Bishop, 1980). "Spatial orientation" refers to understanding and operating on the relationships between the positions of objects in space *with respect to one's own position,* while "spatial visualization" is the ability to mentally move a geometric object in space (see McGee, 1979; Bishop, 1980). Since I cannot see what is happening in students' minds, I include any kind of description the students use regarding their mental manipulation of a mental image, under a single term, hence *spatial manipulation*. I prefer to use my own term to avoid using any notions (such as *spatial visualization*) that might cause confusion with that of visualization (Duval, 1999/2002), which I already use.

3.1.4 Summing up

In the context of geometry, *visualization* may be seen as a cognitive process (Duval, 1998, 1999/2002; Mithalal, 2009; Mithalal & Balacheff, 2019) or as a type of spatial ability (e.g. Bishop, 1983; Presmeg, 1997; Battista, 2007). In the first approach (cognitive process), (*non-iconic*) *visualization* (Duval, 1999/2002) deals with the comprehension of drawings, and *dimensional deconstruction* is linked to it as an analysis process of the drawings. *Dimensional deconstruction* is based on the use of properties that connect the *figural units* (lower dimension subparts) of the figure. In the context of argumentation, *dimensional deconstruction* can be an observable phenomenon through students' discussions while they are solving a problem.

In the second approach (visualization as a spatial ability), *visualization* is described in different ways by different researchers, usually bearing the meaning of being able to "imagine" a geometric object and its transformations in space (e.g. Bishop, 1983). It may also include the ability to create and manipulate mental images of geometric objects in space (e.g. Presmeg, 1997).

For the purposes of my study, I need a definition of visualization that will provide me with the necessary tools in order to "pinpoint" it in students' argumentations. As I explain in subsection 3.1.1, Duval's (1999/2002) approach to *visualization*, and its link to *dimensional deconstruction* provides me with these tools (see details in Chapter 5). There are three factors that led me to decide that Duval's theory of visualization fits my work best: 1. Duval places *visualization* at the center of attention in geometry, assigning to it a central role in students' cognitive processes, 2. in his theory, *visualization* (in geometrical context) is directly and explicitly linked to the use of geometrical properties (through *dimensional deconstruction*), which provides me with a useful tool for the connection of students' visualization and argumentation based on the use of geometric properties, and 3. tracing stated (verbally or written) properties is a matter of tracing facts and interpreting them, instead of making speculations about students' possible unexpressed mental processes.

The tasks I designed and used in the present research have been constructed with the aim to promote students' visualization and *dimensional deconstruction* of visible 2D cross-sections and invisible 3D objects. As Duval (2017) argues:

> "The solution of a geometry problem 'in space' requires necessarily a dimensional deconstruction operation, i.e. *seeing the 2D form obtained by the intersection of a solid with any plane in space*, and not some spatial ability to see 'in space'" (p. 65).

Moving beyond a mere description of what *visualization* is in geometry, it is important to acknowledge the importance of visualization in the process of argumentation. My aim is to examine the exact relationship between visualization and argumentation, identifying the roles visualization plays in students' argumentations. Therefore, in the next section I discuss the findings of relative research.

3.2 The interplay between visualization and argumentation

Duval (1998) describes visualization and argumentation (which he calls reasoning), as two processes that are independent but "closely connected and their synergy is cognitively necessary for the proficiency in geometry" (p.38). In this section I would like to give some examples from empirical research, in which this link has been addressed (see 3.2.1). At the end of the section (see 3.2.2), I express my open question regarding the topic of the interplay between visualization and argumentation.

3.2.1 The interplay between visualization and argumentation in empirical research

The relationship between argumentation (and also proof) on one hand and diagrams (or figures) and visualization on the other, has been studied by researchers from various fields related to mathematics. Furthermore, many researchers have argued that visualization and imagery play important roles in the learning of geometry, in students' reasoning and in the strategies they employ when solving geometric tasks (e.g. Mithalal & Balacheff, 2019; Papadaki, 2015) Here, I present just some results regarding the interplay of visualization and argumentation from empirical research in the field.

Giaquinto (1992) argues that sometimes *visualizing leads to premises used in argumentation* (or verbal reasoning as he calls it, see ibid, p. 385). He writes:

> "The route to belief described above is quite mixed: part was valid verbal; part was the act of visualizing, which led to one of the premises of the verbal reasoning, namely the true belief that if y ('the inner square') is a square whose vertices are midpoints of the sides of a square x ('the original square'), then the parts of x beyond y ('the corner triangles') can be arranged to fit exactly to y, without overlap or gap, and without any changes of size or shape" (pp. 385-386).

This connection of verbal-reasoning and visualization (in the terms of Giaquinto) resembles the collaboration between argumentation and Duval's (1995, 1999/2002) *dimensional deconstruction* (a process involved in *non-iconic visualization*). Giaquinto explains this idea a bit better at the end of his article, when arguing:

> "...valid sentential reasoning can be regarded as an analytic procedure, one of unpacking what is implicit in prior beliefs, whereas visualizing [...] is synthetic, a putting together of the conceptual elements" (1992, p. 400).

This "putting together of conceptual elements" reminds me of Duval's (1999/2002) *figural units*, and the way he also links *visualization* with the use of properties and argumentation.

Mancosu (2005) too, argues that visualization enables us to discover something new or shape conjectures in mathematics, and is "a legitimate way to come to know a mathematical proposition" (Mancosu, 2005, p. 22). Hence, also Mancosu (2005) refers to a link between visualization and processes (such as conjecturing) that belong to argumentation (as this term is used in the present work). Moving one step further than the others, Giaquinto (1992) connects visualization to deductive reasoning and argues that in contrast to visual perception, visualizing is individually formed (depending on the person who forms it).

Pittalis and Christou (2010) distinguish four types of reasoning[6] in order to describe students' thinking in three-dimensional (3D) geometry. Their results also show that these four types of reasoning are linked to students' spatial abilities[7]. Although Pittalis and Christou (2010) do not use the same framework of visualization that I do (they use spatial abilities, not Duval's (1998) visualization), their results are both interesting as well as relevant to the subject of establishing a connection between visualization (or spatial abilities) and argumentation (or reasoning). That is because, in their framework this connection would be "translated", as a connection between spatial abilities and students' reasoning.

Mithalal and Balacheff (2019), use Duval's (1998, 1999/2002) theory on visualization to examine how "students' drawing perception has to evolve, from Iconic Visualization to Non-Iconic Visualization" (p. 161), since it is a process that they consider to be essential for mathematical proving. To do this, they use tasks designed in a 3D DGE, where students need to perform both *instrumental* and *dimensional deconstructions*, in order to solve them. The results of their research show, that it is possible to design tasks that "provoke the need for intellectual proof" (p. 175). With their tasks, the students shifted from *iconic* to *non-iconic* visualization through the process of *instrumental deconstruction.* When the *iconic visualization* was no longer reliable in order to solve the task, the students responded to this problem by turning to geometric properties of the three-dimensional figure and relations between its figural units. Therefore, Mithalal and Balacheff (2019) consider the use of *non-iconic* visualization by the students, a decisive step towards the learning of proving in geometry (which is a type of argumentation).

6 The four types of reasoning are: representing 3D objects, spatial structuring, conceptualising mathematical properties and measurement reasoning (in p. 206).

7 Pittalis and Christou (2010) define spatial abilities as follows: "Spatial abilities are considered as a form of mental activity that enables individuals to create spatial images and to manipulate them in solving various practical and theoretical problems (Hegarty & Waller, 2005; Koyhevnikov, Motes & Hegary, 2007)" (p. 191). More precisely they use the model of Lohman (1988) "who supported the existence of three major spatial ability factors: spatial visualization (Vz), spatial orientation (SO) and spatial relations (SR)" (p. 195).

In Papadaki (2015), I worked with blind and visually impaired students on geometric tasks using haptic tools. In one of the tasks the students were asked to identify the solid that would be created from the rotation of specific two-dimensional figures (e.g right-angle triangle). The students rotated the two-dimensional figures with their hands and identified the geometrical solids starting from their sub-parts (shape of the base, side surfaces etc.) and moving to the whole of the solid. This way the students managed to gradually build an image of the 3D object, which was the result of a synthesis of all its individual parts. This is a process of *non-iconic visualization* of the solid. Through *dimensional deconstruction* the solid is analyzed and identified by its figural units (Duval, 1999/2002). Here, the contribution of *visualization* was key both for the identification process of geometrical objects, as well as for the justifications of students' answers. Through *non-iconic visualization* (more specifically *dimensional deconstruction*), students created links between the solid and its figural units, which they then used in their argumentations to justify their answers. In that paper, I write that the "bidirectional relationship between visualization and geometrical reasoning can help students develop their geometrical thinking" (p. 569). To use the terminology that I have adopted in my current research, this study provided some evidence for the important role that visualization plays in students' argumentation in geometry.

3.2.2 Open question – What is still missing?

It is thanks to the contribution of all this research mentioned above (in 3.2.1) that we have gained an insight to the significance of visualization (and other spatial abilities) in students' argumentations (or reasoning). In this work though, my aim is to move one step further. I examine *how* exactly visualization contributes to students' argumentation and how its contributions can be depicted in students' argumentation structures (see Chapter 2 for details on the theory, Chapter 5 for the methodology and Chapters 8 and 9 for the results on this subject). This is a part that I feel is missing in the existing research on the interplay between visualization and argumentation, and I believe that shedding light on it will help us to gain a better insight into students' learning of geometry.

3.2.3 Epilogue

In the first section of this chapter (3.1), I have discussed the notion of *visualization* from a cognitive perspective (Duval, 1998,1999/2002) (see 3.1.1), as well as within the frame of *spatial abilities* (see 3.1.2). I have explained that in the present work I adopt the first approach (cognitive) and I have also presented the reasons for my decision. I also discussed the notion of *spatial manipulation* (see 3.1.3), which is a type of spatial ability that I consider useful and necessary in the frame of the present work, in order to describe students' actions (see more details in Chapter 8).

In the second section (3.2), I presented results from empirical research regarding the relationship between visualization and argumentation (see 3.2.1), taking into account studies that follow any one of the approaches to visualization presented in Section 3.1. I did this because regardless of the approach to visualization that I choose to adopt, the aim of the researchers has been common: to show the significance of visualization (and other spatial abilities) in the learning of geometry. At the end of the section I stated an open question in the research and the goal of the present study, which is to identify exactly how visualization functions in argumentation in geometry (see 3.2.2).

In the next chapter, I discuss important findings of the empirical research regarding the teaching and learning of geometry and how argumentation can be enhanced in geometry.

4 Empirical research on the teaching and learning of geometry

Geometry has, for years, constituted the main – and sometimes even the only – branch of mathematics used in school curricula as the appropriate one to introduce and engage students in deductive reasoning, argumentation and proof (Herbst, 2002; Douady, 1998; Harel & Sowder, 1998; Hershkowitz, 1998). Nevertheless, according to Hershkowitz (1998)

> "the product – a written proof – was more important than the process of proving, and thus teaching tended to neglect both the visual geometrical context (shapes and relations between them) and the learner" (p. 31).

In the last two decades, much research has been conducted focusing on the learning of geometry itself, and how to enhance it. Processes such as that of justification, explanation, argumentation and proving, as well as the use of tools that promote students' visualization are some of the main factors that are being considered (see for example Mithalal, 2009; Laborde, 2000; Hattermann, 2010).

Following that same line, I too focus on the teaching and learning of geometry through the examination of the role of visualization in students' argumentation, when they solve geometric tasks designed in a Dynamic Geometry Environment (see Chapter 5 for details on the research design). In the previous two chapters I focused on theoretical and empirical research related to argumentation (see Chapter 2) and visualization in geometry (Chapter 3). In this chapter, I would like to present some important findings of the empirical research regarding the teaching and learning of geometry and how argumentation can be enhanced in geometry. These results propose the use of tasks that promote argumentation (and proof), most of which are designed in Dynamic Geometry Environments (see Section 4.1). At the end of the chapter, I state once again the problematique I observe and explain how I intend to address it in the present study (see Section 4.2).

4.1 Empirical research on the teaching and learning of geometry

Argumentation is in the spotlight in both teaching and learning geometry (and mathematics education in general), and for a good reason, since

> "[t]he process of generating an argument, individually or collectively, involves seeking an explanation/justification for a claim (idea, conclusion, verification, etc.). As such, argumentation encourages self explanation and learning" (Schwarz et al., 2010, p. 120).

These benefits of argumentation in the process of learning geometry constitute it an integral part of its teaching at school. So, if argumentative settings are fruitful environments for the learning of geometry, how can we promote them and even enrich them?

To engage students in argumentation in geometry, in a way that they build arguments (or proofs) based on properties and not merely on visual inputs, we need to engage them in learning situations that will provoke them to move beyond what is visible and towards a "geometry of relations" (Laborde, 2000, p. 158). To achieve this, Mithalal (2009) stresses the significance of engaging students in tasks that support them to move beyond iconic visualization, and towards non-iconic visualization and the use of geometric properties. Sinclair et al. (2016) argue that, although visualization is relevant in all mathematical subjects, it is particularly significant in geometry and in the teaching and learning of geometry at schools.

So, to answer the question raised earlier, it seems that results from the empirical research suggest that the collaborative engagement of visualization and argumentation in geometry, would enrich students' argumentations and promote their learning. However, important aspects of geometry, such as visualization and the creation of hypotheses (or conjecturing) seem to be downgraded in the teaching. According to Reid and Knipping

> "[. . .] in traditional courses on Euclidean geometry the material is usually presented to students as a ready-made end product of mathematical activity. Hence, in this form, it does not fit well into curricula where pupils are expected to take an active part in the development of their mathematical knowledge" (2010, p. 339).

So, what does empirical research have to propose?

In the following two subsections (4.1.1 and 4.1.2) I would like to answer this question through a literature review on the subject of learning geometry by promoting argumentation and visualization. More precisely, in the past two decades, researchers have studied the benefits of three-dimensional (3D) geometry in students' learning for both two-dimensional and three-dimensional geometry (e.g. Markopoulos & Potari, 2005; Papadaki, 2015; Mithalal & Balacheff, 2019). Another aspect that has played an important role in research is the integration of technology in the design of geometric tasks (Laborde, 2000; Mithalal, 2009; Pittalis et al., 2010; Hattermann, 2010). Therefore, I discuss the results of empirical research regarding the benefits of using 3D geometric tasks and Dynamic Geometry Environments (DGE) in the teaching of geometry.

4.1.1 Three-dimensional geometry as an opportunity for argumentation and visualization

A most usual scenario in school geometry is when the students are given a two-dimensional geometric task, where a drawing is also provided, and they are asked to prove that a specific property is true. Mithalal (2009) argues that in such scenarios, the students may experience a conflict regarding the need to prove the property, since it is usually visible in the drawing making a proof look redundant. This phenomenon leads to the degradation of argumentation (and proof) in geometry and to superficial justifications. To overcome this obstacle and help students engage in fruitful argumentation using geometric properties, and even the production of formal proofs, Mithalal (2009) argues that we need to provide students with tasks where "visual information is no longer reliable" (ibid, p. 798). He proposes the use of three-dimensional geometric tasks, explaining that "in space geometry iconic visualization fails, and it is necessary to analyze the drawing in other ways", implying the use of geometric properties and theorems (see also Mithalal & Balacheff, 2019).

Nevertheless, Mithalal (2009) recognizes the challenges that bear the perspective representations of geometric solids, usually used in three-dimensional geometric tasks. He therefore suggests the use of Dynamic Geometry Environments (DGE) for such tasks:

> "Using 3D geometry computer environments may balance these difficulties, since the students could get more visual information, for instance by using various viewpoints as if the representations were models. It has to be noticed that, even in this kind of environment, visual information is usually not reliable, so that non-iconic visualization remains inadequate to solve geometry problems" (Mithalal, 2009, p. 798).

In the next subsection, I discuss the use of DGE in geometry further, both for two-dimensional as well as for three-dimensional geometric tasks.

4.1.2 The use of a Dynamic Geometry Environment in the teaching of geometry

Already in 1980, Tahta proposed the use of technology in the teaching of geometry as a possibly useful alternative to traditional teaching:

> "it would seem that geometry might be more successfully pursued if it were to be explored in its own terms. A possible way of doing this in the future lies in the field of computer graphics and automated control. Seymour Papert in reporting children's work with computers has emphasized that their activity cannot always be interpreted in traditional terms" (p. 7).

Since then, and mainly since the 1990s, there has been a shift of attention from traditional to dynamic Geometry, both in the research field of mathematics education as well as in the teaching of geometry at schools (see for example Mithalal, 2009; Hattermann, 2010; Sinclair & Robutti, 2013). During the last three decades Dynamic Geometry Environments (DGE[1]), such as GeoGebra and Cabri, are being used more and more in the teaching of geometry in secondary education. As Sinclair et al. (2016) comment:

> "The role of technology is just beginning to be understood, while, at the same time, it continues to evolve and rapidly change the world around us and in the classroom. Students and teachers are using digital tools throughout the day, and it is necessary to better understand how they can be used effectively for the teaching and learning" (p. 704).

The idea underpinning this attention to DGE, is that it can provide students with a more active role in the processes of making conjectures and evaluating them. The students can, for example, manipulate and explore the geometric situation under investigation freely using the tools of the DGE. Such a process immediately differentiates students' involvement in their learning from that in a traditional geometry lesson with only paper and pencil. Battista (2007) argues that DGE are tools which can improve the process of doing geometry not only for students, but rather for everyone, and that by using them we have the opportunity to explore many more geometrical ideas than we would have through the traditional paper and pencil exploration methods.

Reid and Knipping (2010) distinguish between "static" and "dynamic" geometry:

> "Also one may distinguish between a geometry which stresses "static" properties of geometric objects and a geometry where objects are considered in a "dynamic" setting, as they change under the effect of different types of transformations" (p.339).

According to Marrades and Gutierrez (2000):

> "The contribution of DGS is two-fold. First, it provides an environment in which students can experiment freely. They can easily check their intuitions and conjectures in the process of looking for patterns, general properties, etc. Second, DGS provides non-traditional ways for students to learn and understand mathematical concepts and methods" (p. 88).

In the above literature, a DGE is presented as a "rich" setting in which students can engage in conjecturing more actively and in an explorative and dynamic way.

But, although researchers in mathematics education have been particularly interested in the use of DGE and their role in the teaching and learning of geometry,

[1] Also referred, by other researchers, as DGS – Dynamic Geometry Softwares

some researchers express their concerns as to the proper use of DGE in teaching. These concerns regard the effect that DGE have on students' justifications and proving competencies. One such example, is the possibility that DGE may hinder students in the process towards more formal forms of mathematical justification. For example, Healy (2000) found out that her students could create a "robust construction" (ibid, p. 112) of a drawing in Cabri, using properties every step of the way. But, the same students after constructing a parallelogram in Cabri, could not prove a property that they had themselves stated as true (equal angles property). Healy (2000) draws the attention to the gap between construction and formal proof. She stresses the significance of the appropriate design of proving activities, as well as the way the DGE is used, as two factors that must be taken into great consideration in order to bridge the gap between construction and formal proof.

Marrades and Gutierrez (2000) point out another concern often brought up in research, regarding the use of DGE for proving in geometry. That is, that students get convinced about the validity of a conjecture through the results of exhaustive checking on the screen, and thus they "do not feel the necessity of more abstract justifications" (p. 96).

In response to the two concerns described above, further empirical research has brought forth encouraging results regarding the role of DGE in students' argumentations and proving, that "refute the current idea of proof being in danger by dynamic geometry environments" (Laborde, 2000, p. 152). Laborde (2000) argues that when working in a DGE, proof is not separated from action. On the contrary, there is interplay between "doing" on the DGE and justifying a conjecture using theoretical arguments. Jones (2000) also argues that in his research students work through the teaching unit shifting their thinking progressively "from imprecise 'everyday' expressions, through reasoning mediated by the software environment to mathematical explanations of the geometric situation" (p.80). Mariotti (2000), shows how the students can construct a system of axioms and theorems in a DGE, by adding new commands to it. The commands that are added build on previous commands. All of these commands represent axioms or properties in the traditional theory. In this process, proof is the means that is used in order to create an axiomatic system in the DGE, which is represented by the commands that the students create.

Other researchers (see for example Hadas et al., 2000; Marrades & Gutiérrez, 2000; Baki, Kosa & Guven, 2011) also stress the fact that what is particular to DGE, is that when elements of a constructed figure are dragged, the geometric properties that were employed in constructing it are maintained, which means, as Mariotti explains, that

> "[. . .] the logic of its construction; the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality" (2000, p. 27).

Furthermore, in response to the first concern stated above (the gap between construction and formal proof), Jones (2000) argues that DGE could play an important role in supporting students to formulate deductive explanations and therefore also in the development of students' deductive reasoning, as DGE:

> "appear to have the potential to provide students with direct experience of geometrical theory and thereby break down what can be an unfortunate separation between geometrical construction and deduction"
> (Jones, 2000, p. 56).

So, from the examples of empirical research discussed here, it seems that Laborde has a strong point when she writes that "DGS contain within them the seeds for a geometry of relations as opposed to the paper and pencil geometry of unrelated facts" (2000, p. 158). That is because in the process of the construction[2] of a geometric object in a DGE, it is necessary to be aware and able to use the properties of the geometric object as well as the properties connecting its constitutive parts.

We need to keep in mind though, that the connections between construction and deduction, as well as the transition from empirical experience to mathematical justifications and proving are not trivial processes. On the contrary, they require time and they should be "rooted on empirical methods used by students so far" (Marrades & Gutierrez, 2000, p. 119). In this way the DGE can provide students with "empirical explorations before trying to produce a deductive justification, by making meaningful representations of problems, experimenting, and getting immediate feedback" (ibid, p.119). So, in response to the second concern expressed earlier, instead of seeing the exhaustive checking of a situation (that is possible in a DGE) as a drawback for argumentation, we should use it as an advantage compared to non-computational or a static traditional pencil and paper environment, thus fostering the interaction between construction (in the DGE) and justification by means of arguments based on geometric properties and theorems.

A significant role in this direction is played by the appropriate task design (Mithalal, 2009; Marrades & Gutierrez, 2000; Healy, 2000; Mithalal & Balacheff, 2019). As I mention in subsection 4.1.1, Mithalal (2009) proposes the use of three-dimensional geometric tasks designed in a DGE, in order to bridge the gap between construction and proof. He draws the attention to the processes of dimensional and instrumental deconstructions (see details in Chapter 3), which help students move from iconic visualization to non-iconic visualization, fostering their argumentation by making use of geometric properties (see also Mithalal & Balacheff, 2019). Other researchers have also studied the relation between the use of DGE in geometry and students' visualization. Their studies have shown that the use of DGE can promote students' visualization skills. According to Christou, Jones, Mousoulides and Pittalis (2006, as mentioned in Baki, Kosa & Guven, 2011), "dynamic 3D applications would enhance the students' dynamic visualisation ability and enable them to acquire a greater understanding of 3D mathematical and spatial concepts" (p.294).

In Chapter 5, I discuss further issues related to the designing of tasks in order to promote students' engagement in argumentation and visualization in geometry. At this point, I would like to close this chapter by referring back to the main problematique underlining this work and how I intend to address it in my study.

[2] A construction in a DGE is the creation of a drawing only with tools corresponding to straightedge and compass. This process excludes any measurements or counting, and brings forth the use of properties in order to create the drawing.

4.2 Instead of an epilogue – The case of the present study

In this chapter, I conducted a literature review of the factors that researchers find most significant in order to improve the teaching of geometry and promote students' learning and their argumentation and proving competences. The factors mentioned in this literature have been, the use of Dynamic Geometry Environments (DGE) (e.g. Mithalal, 2009; Jones, 2000; Mariotti, 2000) and the use of appropriately designed tasks (e.g. Marrades & Gutierrez, 2000; Healy, 2000). Such tasks should help students create connections between constructions and argumentation (or deductive reasoning or proving) and also help them move beyond iconic visualization, towards non-iconic visualization (Mithalal, 2009; Mithalal & Balacheff, 2019; Duval 2017).

Although until now research has provided us with insights and results of great importance and value when it comes to the role of DGE in improving the teaching and learning of geometry, I believe there is still an important issue that requires our attention. From the literature review in this chapter (but also in Chapters 2 and 3), I can see connections that have been made between geometry and argumentation (e.g. Mariotti, 2000; Jones, 2000), as well as between geometry and visualization (e.g. Duval, 2017; Christou et al., 2006; Baki et al., 2011, see more in Chapter 3). However, the relationship between argumentation and visualization[3] in geometry, which I consider to be of decisive importance for the learning of geometry, has not yet been stressed enough. The works of Mithalal (2009) and Mithalal and Balacheff (2019), have offered valuable results in this direction and have opened a path towards a more precise description of the relationship between argumentation and visualization in geometry.

The main problematique I want to address in this work, is the lack of a more precise description regarding the possible roles of visualization in students' argumentations. The aim of the study conducted for this research, is to identify those roles of visualization in argumentation. By doing so, I intend to gain a deeper insight into how and at which points during their work, visualization supports students' argumentations in geometry, thus fostering their learning.

In the next chapter (Chapter 5), I discuss the methods and methodology of my research, before I move on with the presentation of the results (in Chapters 6, 7 and 8).

[3] Here, the word "visualization" is meant as described by Duval (1998; 1999/2002), referring to a cognitive process (taking place in the learning of mathematics) that deals with the apprehension of drawings (for details please refer to subsection 3.1.1 in Chapter 3).

5 Methodology and methods of the study

In this chapter, I present the background of the present study, from its design, to its implementation, and to the methods used in the data analysis. I begin with the aim of the study and the research questions that I intend to answer through my research (see Section 5.1). Next, I present the theory and the methodology underlying the design of the study (Prusak et al., 2012, 2013) and how I used it to create the tasks for the students. I describe the characteristics of the task-design and the reasons why these characteristics are important for the purposes of this research (see Section 5.2). Moving on, I present the steps of the implementation of the research (including the participants, teaching experiments etc.) (see Section 5.3). Then, follows the presentation of the methodology for the data analysis and the specific methods used for different levels of the data analysis (see Section 5.4). The methodology is comprised of three individual parts, based on: Reid's (2002b) method of *patterns of reasoning* (see subsection 5.4.2, Method #1), Knipping's (2003a, 2003b, 2008) method of building *argumentation structures* (see subsection 5.4.2, Method #2), and Duval's (1998, 1999/2002) theory of visualization in geometry (see Chapter 3). The different levels of the data analysis are also presented through a specific example arising from the collected data (see subsection 5.4.4). The chapter closes with an epilogue and a short introduction to the following chapters (see Section 5.5).

5.1 Aim and research questions of the study

In the present study, the focus lies on students' argumentations and the use of visualization while they work on geometric tasks. The study takes place within real classroom settings. My aim is to examine how the students use their visualization in geometry and what are the roles that visualization plays in their argumentations, thus promoting their learning.

The research questions of the present study have already been introduced in Chapter 1. Here, I would like to re-introduce them, refining them in the light of the theory that has been presented in the previous chapters (see Chapters 2, 3 and 4). Research question 3.2 is marked in bold, to represent the significance of the question for the aim of the study.

1. What *exploration strategies* do the students follow using the Dynamic Geometry Environment when they work on the given tasks?[1]

2.1 What are the observed *patterns of* students' *argumentations* while

[1] Research Question 1 is discussed in Chapter 6.

working on the given tasks?

2.2 How does the specific *design* of the given tasks influence the structure of students' patterns of argumentations?

2.3 How do students' patterns of argumentation differ in *pair-work* and in *classroom discussions*?[2]

3.1 How do *non-iconic visualization* and *spatial manipulation* manifest themselves in students' argumentations?

3.2 What are the *roles* of *non-iconic visualization* and *spatial manipulation* in students' argumentations?

3.3 How does the specific *design* of the given tasks influence students' *non-iconic visualization* and *spatial manipulation*?[3]

In the following sections, I present the methodology underlying my research, as well as the methods that later help me to analyze the collected data and answer my research questions. The answers to the research questions are discussed in the results chapters 6, 7 and 8 (see details in Footnotes for each research question).

5.2 Research methodology – The design of the study

In this section, I start by presenting the research design theory by Prusak et at. (2012, 2013) (see 5.2.1). I then explain how I applied this design theory in order to design my study (see 5.2.2). I also present the tasks used in the study (see 5.2.2.1), their design and specific characteristics (see 5.2.2.2) and the organization of the learning environment in which the study has been conducted (see 5.2.2.3).

5.2.1 Design Theory – Prusak, Hershkowitz and Schwarz (2012, 2013)

As discussed in the previous chapters, two of the factors that researchers consider vital in order to improve the teaching of geometry and promote students' argumentation and learning, are the use of appropriately designed tasks and the employment of DGE as tools in the teaching (see details in Section 4.2). Another factor that is very important when we want to engage students in argumentation is the organization of the learning environment in which students work.

Prusak, Hershkowitz and Schwarz (2012) suggest a design that promotes productive argumentation. Some years earlier, Andriessen and Schwarz (2009) had introduced the idea of argumentative design:

> "Argumentative design concerns the design, by a teacher, researcher, or educational professional, of collaborative situations in educational contexts

2 Research Questions 2.1 to 2.3 are discussed in Chapter 7.

3 Research Questions 3.1 to 3.3 are discussed in Chapter 8.

> in which participants take on productive argumentation, or the exploration of a dialogical space" (p. 145).

The design suggested by Prusak et al. (2012) has two design components, each including *design principles.* Those components are task-design and the role of the teacher as a facilitator of argumentative talk. The task-design principles that Prusak et al. (2012) suggest as promoting productive argumentation are: (a) *creating a situation of conflict*, (b) *creating a collaborative situation*, and (c) *providing tools for raising and checking hypotheses*.

Later on, Prusak, Hershkowitz and Schwarz (2013) examine how they can use the design principles to support students' mathematical argumentation within geometric problem-solving situations. In that work, they provide some further design principles enriching their theory. Prusak et al.'s (2012, 2013) theory is a *design theory,* that is a theory developed based on the results of a *design experiment*[4]. Instead of focusing on only one aspect of the teaching and learning procedure, design experiments take into consideration various aspects of the instructional practice: the classroom milieu, the tasks used, the types of discourse encouraged in the classroom, the didactical contract, the tools and materials used in the lesson; these are only some of those factors (Cobb et al., 2003; Kieran, 2019). This results in the shaping of theories (design theories) that consider all the above-mentioned aspects.

> "A design theory explains why designs work and suggests how they may be adapted to new circumstances" (Cobb et al., 2003, p. 9, as mentioned in Kieran, 2019, p. 269).

According to Prusak et al. (2013) principled design is effective in promoting a problem-solving culture, mathematical reasoning and conceptual learning, instilling argumentative norms. Prusak et al. (2013) describe again two design components, when designing a problem-solving situation. One is for the task-design and one for the design of the learning environment by the teacher. They add two more task-design principles and they also make more precise the design principles related to the organization of the learning environment, for the support of mathematical argumentation within geometric problem-solving situations. Through their work, it is made apparent how important it is that the design of the tasks is supported by the accompanying design of an instructional environment. This environment should involve such social norms that would support a collaborative problem-solving culture and students' augmentation. Although Prusak et al. (2013) use problem-solving situations, they did not only aim at promoting a problem-solving culture but also at "instilling inquiry learning and argumentative norms" (p. 266). Hence there is a shared objective in the two studies: to engage students in argumentation, whether is a problem-solving situation (Prusak et al., 2013) or in any type of mathematical activity (Prusak et al., 2012).

As I mention above, Prusak et al. (2013) enrich their design-principles, building on

[4] The term design experiment belongs to psychologist Ann Brown (2002) and emerged from her work on educational design (Kieran, 2019).

their previous work (Prusak et al, 2012). More specifically, they argue that:

> "five design principles could create productive problem solving, in the sense that students adopt a problem solving culture for the learning of mathematical ideas by:
>
> (a) creating multiple solutions;
>
> (b) creating collaborative situations;
>
> (c) engaging in socio-cognitive conflict;
>
> (d) providing tools for checking hypotheses; and
>
> (e) reflecting and evaluating the solutions created"
> Prusak et al. (2013, p.269).

Principles (b), (c) and (d) (collaborative situations, conflict and checking) are the same as the ones mentioned in Prusak et al. (2012) as the three design principles whose combination in designing tasks promotes productive argumentative talk.

In regard to the factor of the organization of the learning environment and the role of the teacher, Prusak et al. (2013) suggest five design principles. In their research they observed that these principles (together with the task-design principles) helped to create a classroom culture in which problem-solving practices were nurtured and the students developed "solving heuristics and engaged in multimodal argumentation, subsequently reaching deep understanding of a geometrical property" (p.282).

These five principles are:

> 1) "emphasis on processes rather than solely on results;
> 2) mathematical activities took place in various social settings (individual, small group, and whole-class);
> 3) development of a critical attitude towards mathematical arguments using prompts like: 'Is it airtight?' 'Does it convince me?'
> 4) students were encouraged to listen to and try to persuade each other, and thus develop ideas together;
> 5) students learned to report on what they did, first verbally, then in the written form, explaining their solutions to their teammates or to the entire class."
>
> (Prusak et al., 2013, p. 270)

Design theories, such as Prusak et al.'s (2012, 2013), have the characteristic of being able to function both as a *source* for the design of a research, as well as a *product* of design research. The way one uses them depends only on the aim one has for one's research. Hence, a well-developed and coherent theory, taking into account multiple factors of the instructional practice, can lead to further studies that will improve both the theory as well as the practice in more than one way.

If a design theory is used as a *source* then the researcher uses design as *intention.* In this case, the design theory is already developed and coherent and it is used in order to "provide theoretical tools and principles to support the design of a teaching sequence" (Kieran, 2019, p. 270). Therefore, the researcher who uses it should specify the characteristics of the design from the beginning; the design, as well as its intentions, should already be developed and clear.

But a design theory may also be the *product* of a design research. In that case, the researcher uses the design as *implementation* and the aim is to implement a designed sequence in the classroom and then refine the design based on the results of the implementation. The goal here is not merely to use a theory as a framework of a research, but rather to achieve the development and improvement of the theory itself. When design theories are used as an *implementation* they "inform us about both the process of learning and the means that have been shown to support that learning" (Kieran, 2019, p. 270).

In my research, I aim to observe students' argumentations when they solve geometric tasks, and examine the role of visualization in their argumentations. Therefore, I chose to use a design theory as *intention.* More precisely, I chose to use Prusak et al.'s (2012, 2013) design theory as the methodology that underpins the design of my study, because it is a coherent design theory that takes into account many aspects of the process of teaching and learning. In the diagram below (see Figure 5.1), I have summarized the two design components proposed by Prusak et al. (2012, 2013), as well as the design principles for each component, which promote students' engagement in argumentation. In the next subsection, are presented the reasons that make Prusak et al.'s (2012, 2013) design theory fitting for the design of my study and the way it is applied in the study.

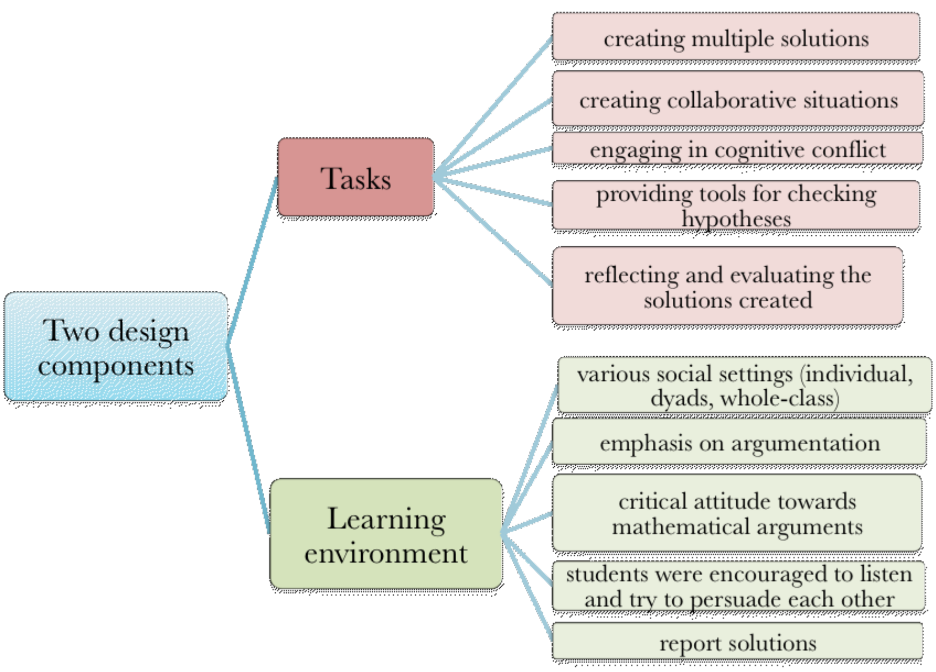


Figure 5.1: The design principles by Prusak et al. (2012, 2013)

5.2.2 The present study – Application of Prusak et al.'s (2013) design theory

Much of the empirical research carried out to date, regarding students' argumentation and visualization in geometry, is in the frame of interviews outside the classroom reality. Nevertheless, in our societies where students study in classrooms with many other students and not alone, the classroom reality is an influential factor in their learning process. Therefore, I am interested in conducting my research in real classroom conditions.

Prusak et al. (2013) emphasize the importance of the social context in students' learning processes. As I mention in Chapter 2, I am interested in observing and analyzing students' discussions in order to examine their argumentation. Therefore, I chose Prusak et al.'s (2012, 2013) design theory to underpin my methodology for the design of the study, because it focuses on promoting argumentation arising from collaborative work and discussions between the students.

I use this theory and the design principles it introduces as tools for the design of my study. The tasks used in the study as well as the organization of the learning environment in which the students work are designed according to Prusak et al.'s (2013) design principles. Their task-design principles allow for the creation of tasks that are interesting, original and challenging for the students. And their learning environment-principles enable the researcher to design a study in real classroom conditions.

The present study is a qualitative research that has been conducted in the frame of two *experimental lessons* in geometry classrooms. The term *experimental lessons* is used to describe lessons during which no traditional instructional teaching takes place, rather the students work together on tasks specifically designed for the purposes of the learning goals set by the teacher and the researcher. In this study the students engage in explorative geometric tasks. The design of the study, aims at encouraging the students to produce conjectures and examine their validity, going beyond iconic visualization (see Chapter 3) and naïve empirical justifications, and towards non-iconic visualization, the use of properties and argumentation.

Next, I present the way I applied Prusak et al.'s (2012, 2013) design in my study. First, the tasks used in the study are presented and then their exact design and their characteristics are described and explained. Then, I describe the various social settings of the learning environment in which the study takes place.

5.2.2.1 The tasks

For the purposes of the present study five geometric tasks are designed in a Dynamic Geometry Environment (DGE). More precisely, they are designed in the *3D Graphics* environment of GeoGebra 5. In the first task (Task 1), a visible cylinder is designed in a three-dimensional Cartesian coordinate system in the DGE (see Figure 5.2). In each of the other four tasks (Tasks 2, 3A, 3B and 3C), one *invisible* solid is designed in the three-dimensional Cartesian coordinate system.

In each task the GeoGebra window is divided into two sub-windows (see Figure

5.2). In the right sub-window (labeled 3D Graphics) there is a three-dimensional coordinate system in which a solid was designed, and a plane defined by the axes x and y (plane xOy). In the left sub-window (Graphics) there are three sliders: h (named after the German word Höhe, meaning height), n (named after the German word Neigung, meaning tilt), and d (named after the German word Drehung, meaning spin/rotation). The students can manipulate these sliders in order to move the solid in space (see for example Figure 5.3). The variation of the h-slider, with values in the interval [-4, 4], moves the solid up and down on the z'z axis, dragging it above or under the plane xOy. The variation of the n-slider with degree-values in the interval [0°, 360°], tilts the solid sideways parallel to plane yOz. The d-slider, also with degree-values in the interval [0°, 360°], rotates the solid around its symmetry axis that goes through its center (for the sphere in Task 2) or through the center point of its base (for the cylinder in Task 1, the cone in Task 3A, the pyramid in Task 3B and the cube in Task 3C). In the left sub-window there is also a two-dimensional depiction of the cross-section that is created when the solid is intersected by the xOy-plane. Each task is accompanied by a worksheet with instructions for the specific task (see all worksheets in Appendix B).

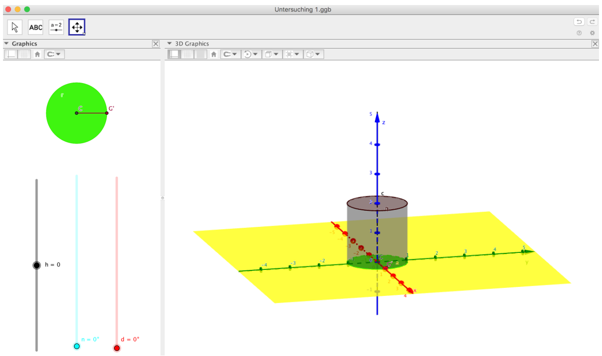


Figure 5.2: Task 1 – Visible Cylinder. Position of the solid and its cross-section at initial position (h=0, n=0°, d=0°)

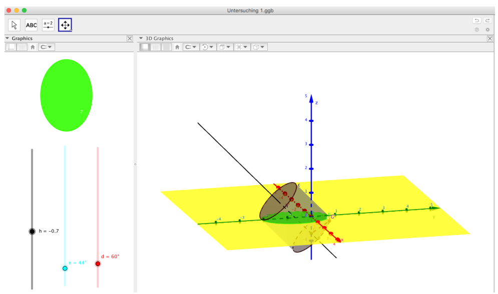


Figure 5.3: Task 1 – Visible Cylinder. Position of the solid and its cross-section at the lowered, tilted and rotated position (h=-0,7, n=44°, d=60°)

I would now like to present each type of task.

In Task 1, the solid (a cylinder) is visible. This is a preliminary task through which the students were introduced to the set-up. First, students are asked to experiment with the three sliders and describe what is the function of each slider and how this function affects the position of the solid in space (see Part a in Worksheet of Task 1 in Appendix B1). They are also asked to name some properties of the cylinder. In the second part of the worksheet (Part b, Worksheet of Task 1, Appendix B1), there is an "Exploration Matrix" with specific (h, n, d)-positions for the cylinder. The students are asked to re-create these positions in the DGE and then describe the shape of the cross-section and its properties. They are also asked if they can relate any of the cross-section's properties with the properties of the solid.

The purpose of Task 1 has been to introduce the students to the 3D-Graphics of the DGE and to its functions and help them make sense of the sliders' functions in relation to the movement of the solid in space. On account of this, the solid is visible in this task. Another purpose of this task was to give the students the opportunity to observe the effects that the movement of the cylinder has on the shapes of the cross-sections.

In Tasks 2, 3A, 3B and 3C, the set-up is exactly the same as in Task 1, but the solids are *invisible* (see for example Figures 5.4a and 5.7a). The students are asked to explore the situation and identify the form of the invisible solids. For each task, a worksheet is provided. In each worksheet there is again an "Exploration Matrix" (see Part a, Worksheets for Tasks 2, 3A, 3B, 3C, Appendices B2 to B5), with given (h, n, d)-cases and positions[5], as well as extra rows to note further cases or positions of their choice. The Exploration Matrix is provided to the students with the intention, on the one hand, to support them if the task proves to be too challenging, and, on the other hand, to enrich their exploration with some suggested cases that they may not have explored on their own. Nevertheless, the students were told that they are not obliged to use the Exploration Matrix in their work, and that they could explore the situation any way they preferred. In Part b, the students are asked if they can identify the form of the invisible solid by the "clues" they have gathered during their explorations. They are also asked to write down a justification for their answer.

In the tasks with the invisible solids, the identification of the solid's form is based on the cross-sections emerging by the intersection of the solid with the (visible) plane xOy. The students can move the solid in space, using the three sliders. The figures below demonstrate examples of (h, n, d)-positions, for the tasks of the invisible sphere and the invisible pyramid (see examples for all tasks in Appendix B). The students only worked with the solids being invisible during their explorations. Here, I present the positions also with the solids being visible, for the purposes of task presentation.

As I explain above, the purpose of Task 1 is to introduce the students to the DGE and to the set-up of the main tasks. The tasks I focus on for the purposes of this study are the ones with the invisible solids (Tasks 2, 3A, 3B and 3C). Therefore, I would now like to focus on these tasks and explain the principles behind their design. Then, I move on to the description of the design principles related to the learning environment organization, in which the study has been conducted.

5 I refer to the triad (h, n, d) as "case" when at least one factor is not constant, but rather varying. I refer to triad (h, n, d) as "position" if all three factors have a specific value, e.g. position (h=0, n=0°, d=0°) is always the initial position, (h=-2, n=0°, d=0°) is a position in which the solid has been pulled downwards on the z-axis by 2, but has not been tilted (n=0°) or rotated (d=0°). The case (h, n=45°, d=0°) represents the exploration when the solid is tilted 45°, not rotated at all and the height can be explored in the whole range of its values in the interval [-4, 4].

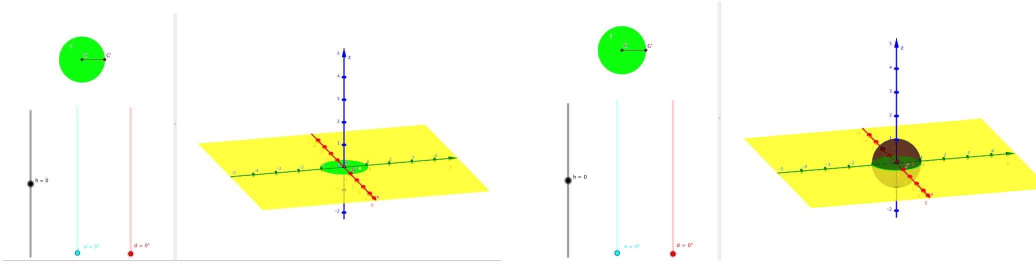


Figure 5.4: a (left) and b (right). Task 2 – Sphere at initial position (h=0, n=0°, d=0°) as seen by the students (left) and as it appears when the solid is visible (right)

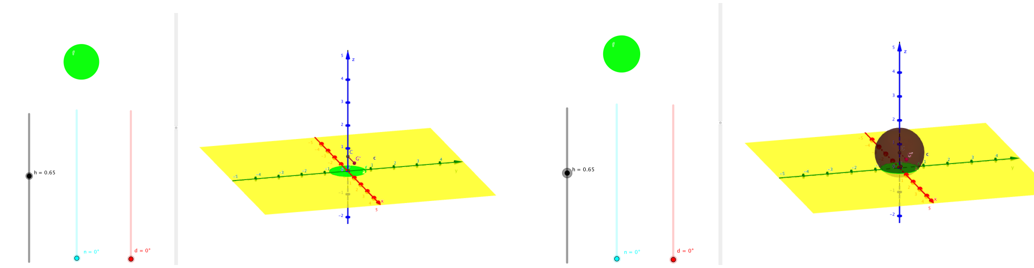


Figure 5.5: a (left) and b (right). Task 2 – Sphere in lifted position (h=0,65, n=0°, d=0°) as seen by the students (left) and as it appears when the solid is visible (right)

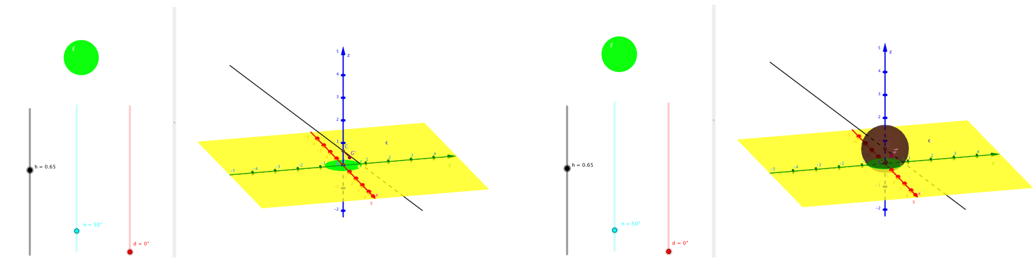


Figure 5.6: a (left) and b (right). Task 2 – Sphere in lifted and tilted position (h=0,65, n=50°, d=0°) as seen by the students (left) and as it appears when the solid is visible (right)

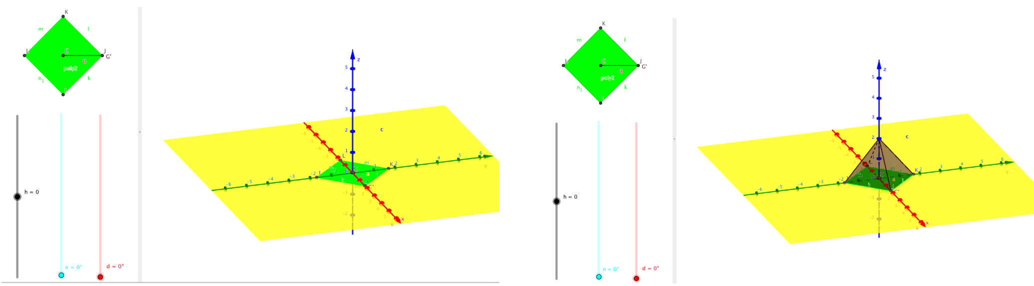


Figure 5.7: a (left) and b (right). Task 3B – Pyramid at initial position (h=0, n=0°, d=0°) as seen by the students (left) and as it appears when the solid is visible (right)

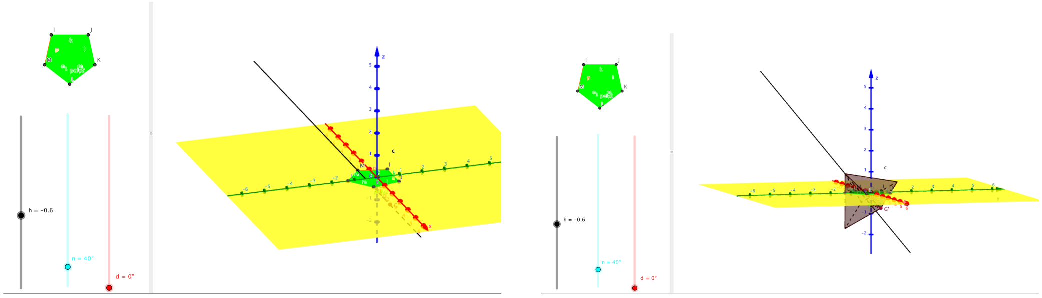


Figure 5.8: a (left) and b (right). Task 3B – Pyramid at lowered position (h=-0,6, n=40°, d=0°) as seen by the students (left) and as it appears when the solid is visible (right)

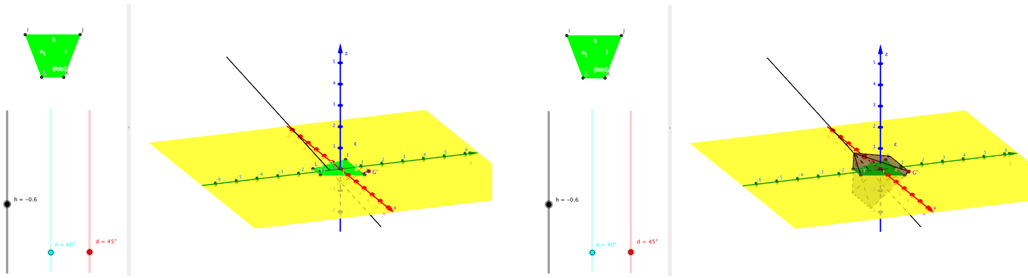


Figure 5.9: a (left) and b (right). Task 3B – Pyramid at lowered and tilted position (h=-0,6, n=40°, d=45°) as seen by the students (left) and as it appears when the solid is visible (right)

5.2.2.2 Task-design and the characteristics of the tasks

As I explain at the beginning of subsection 5.2.2, the design of the study is based on the design principles by Prusak et al. (2012, 2013). Figure 5.1 (at the end of subsection 5.2.1) shows the principles for the two design components (tasks and learning environment) of Prusak et al.'s (2012, 2013) design theory. Figure 5.10 below, shows the characteristics of the tasks in my study (DGE, D-transitional, Black Box).

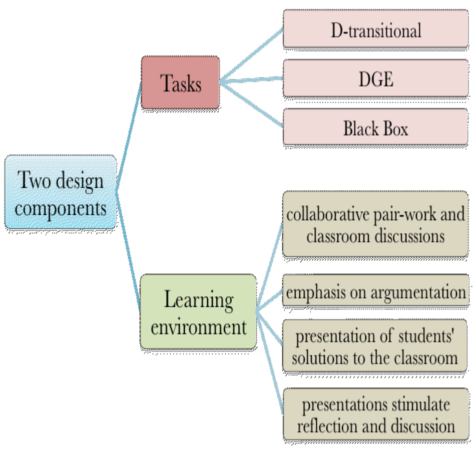


Figure 5.10: Application of Prusak et al.s' (2012, 2013) design principles in my study

These characteristics are used in order to meet Prusak et al.'s (2012, 2013) task-design principles. The figure also shows the characteristics of the learning environment (collaborative pair work and classroom discussions etc.) in which the study is conducted. I discuss this part of the design later on.

In Chapter 4 (see 4.1.1), I mention Mithalal (2009) who argues that in order to help students engage in fruitful argumentation using geometric properties, we need to provide them with tasks where "visual information is no longer reliable" (ibid, p. 798). Mithalal (2009) proposes to use three-dimensional geometric tasks in the teaching of geometry, explaining that "in space geometry iconic visualization fails, and it is necessary to analyze the drawing in other ways", implying the use of geometric properties and theorems. According to Markopoulos (2003), understanding the properties of a solid is equivalent to understanding the characteristic parts of a three-dimensional shape, the comparative relations between the same or different structural parts and how the elements of the solid are interrelated. That is the same idea with what Duval (2011) calls "figural units" of a shape. That means that the properties of the component parts (figural units) of a geometric object are also properties of the three-dimensional geometric object itself. This correlation of properties between geometric objects of different dimensions is vividly present in tasks that involve both two-dimensional and three-dimensional geometric objects.

The aforementioned literature, as well as my previous research experience using tasks that enable transitions from two-dimensional to three-dimensional geometric objects (see Papadaki, 2015), constituted the inspiration behind the idea of the tasks designed for the purposes of the present study. These tasks have three characteristics: they are D-transitional, they are designed in a DGE and they are designed based on the idea of Black Box tasks. Following, I describe each of these characteristics in more detail, explaining also how each of them is connected to Prusak et al.'s (2012, 2013) task-design principles.

D-transitional tasks

The term *D-transitional*[6] is used to refer to geometric tasks involving transitions from two-dimensional to three-dimensional geometric objects (and vice versa). In such tasks, the correlation of properties between geometric objects of different dimensions is vividly present. The difference between these tasks and other ones that involve transitions between geometric objects of different dimensions is that in D-transitional tasks students are asked to recognize a three-dimensional geometric object by having access only to two-dimensional parts of it and not the other way around. Usually the task is: "This solid is a cylinder. What kind of cross-sections with a plane could it have?". On the contrary, in D-transitional tasks the process is a reversed one with the main question set to the students being: "What solid do you think this could be, judging by its cross-sections?". The transitions that students need to perform in order to solve a D-transitional task do not only involve moving from two to three dimensions, rather also vice versa. But the visible parts of the geometrical object under exploration are only its two-dimensional components.

[6] The letter D stands for the word dimension.

As Laborde argues:

> "If properties of figures are not conceived as dependent, a deductive reasoning has no meaning. The question of the validity of a property conditional on the validity of other properties would not arise in a world of unrelated properties" (2000, p. 157).

Therefore, the purpose of D-transitional tasks is to provide the students with the opportunity to create relations between a three-dimensional geometric object and its lower dimension figural units, by identifying relationships, connections and dependencies between their properties.

Designed in a DGE

The second characteristic of the tasks used in this study, is that they are designed in a Dynamic Geometry Environment (DGE). The nature of D-transitional tasks is such, that it would be impossible to design them in a traditional way with paper and pencil. Dynamic Geometry Environments (DGE) enable the creation of D-transitional tasks, in which the students can interact with the geometric objects they deal with. This interaction is realized with the manipulation of the GeoGebra sliders that can move the solid in space (see details in subsection 5.2.1).

As discussed in Chapter 4 (see subsection 4.1.2), the use of DGE in the teaching of geometry promotes students' argumentation and use of non-iconic visualization. DGE is also a teaching and learning tool that can fulfill multiple of Prusak et al.'s (2012, 2013) task-design principles (see Figure 5.10). For example, the principle of "creating multiple solutions" can be translated here as different explorations of (h, n, d)-positions and cases, or different exploration strategies employed by the students. The DGE also provides the opportunity for "collaborative situations", because it is possible for students to use the DGE and work on the task together (e.g. in pairs).

A DGE also provides the opportunity to engage "in cognitive or socio-cognitive conflict". For example, there may appear cross-sections that the students cannot immediately explain, or that they find unexpected or even confusing. According to Healy and Hoyles (2002), and Laborde (2000), students feel the need for explanation when what they observe on the computer screen gives them a feedback that is surprising or is in conflict with what they expected. Hadas et al. (2000) suggest that DGE may create situations of uncertainty, leading students to seek for explanations. Therefore, in cases of cognitive conflict, the students have the opportunity to re-examine their hypotheses or solutions in the DGE. In cases of socio-cognitive conflict, in which the difference arises between the opinions of two or more students, there is a chance for negotiation of ideas. In such a case, it is important to create a collaborative situation (e.g. a classroom discussion with the teacher as facilitator), in order to ensure that productive argumentation is taking place.

Finally, the use of a DGE here also fulfills the principle of "providing tools for checking hypotheses", such as the h, n, d-sliders in the tasks of the present study. The students also have the opportunity to "reflect and evaluate the solutions created" with the use of the DGE (Prusak et al., 2012, 2013). For example, if the students have concluded that the invisible solid in a task is a pyramid, they can use the DGE to check if the results of specific slider-movements agree with their solution.

A way to surprise: Black Box

Black Box is the third characteristic of the tasks designed for the present study. Black Box activities were designed by Laborde (1998) in the context of geometry teaching. They have also been used by Knipping and Reid (2005) in geometric context, where they used the DGE *Cabri Geometry*. In Black Box tasks, a geometric construction is offered to the students but the properties and rules on which this construction is based are hidden. The students are asked to find out the rules that underline the construction. The Black Box activities give students the opportunity of interesting and productive explorations, because "When students' predictions turn out to be wrong, this is a good opportunity for asking 'Why is it so?' and calling for an explanation or even proof" (Laborde, 2002, p. 311).

Inspired by the idea of Black Boxes (Laborde, 1998; Knipping & Reid, 2005) I designed the D-transitional tasks in GeoGebra 5, so that the three-dimensional geometric objects are *invisible*. My idea has been that the Black Box characteristic, could be one more factor to cause surprise or even cognitive conflict to students, giving the students the opportunity to engage in argumentation, negotiating their ideas.

To sum up, the task-design fulfills Prusak et al.'s (2012, 2013) principles for the component of tasks, through three characteristics: they are D-transitional tasks, they are designed in a DGE and they are based on the idea of Black Box tasks. Following, I would like to explain how the design-principles of the learning environment component are applied in the present study (see also Figure 5.10, at the beginning of subsection 5.2.2.2).

5.2.2.3 Learning environment organization: Working in various social settings

The four principles for the design of the learning environment, are: using various social settings (individual, dyads, whole-class), place emphasis on argumentation, having a critical attitude towards mathematical arguments, encourage students to listen and try to persuade each other, and report solutions (Prusak et al., 2013. See Figure 5.1 at the end of subsection 5.2.1). In the present study, the learning environment and the application of the experimental lessons are organized according to those design principles (see Figure 5.10, at the beginning of subsection 5.2.2.2).

In this study, during the experimental lesson, the students have the opportunity to work in multiple social settings, taking advantage of different opportunities offered by the different settings. They first work in pairs using the DGE and the worksheet provided by the researcher and the teacher. In this setting they can work on the task intensively, with only one more classmate. This gives them enough time to discuss their ideas and follow which even solution process they prefer, engaging in argumentation with their classmate. Later the students have the opportunity to share and negotiate their solutions with the rese of their peers during the classroom-discussions with the whole class. In the classroom-discussions the emphasis is on students' argumentations during the presentation of their solutions. The students can listen to their peers' arguments and they are asked by the teacher to keep a critical attitude towards these arguments. The discussion provides a breeding ground for the critical negotiations of students' solutions through argumentation.

5.3 The implementation of the study

The present study was designed based on Prusak et al.'s (2012, 2013) principles and it was conducted in the frame of two *experimental lessons.* As I explain in subsection 5.2.2, the term *experimental lessons* refers to lessons during which no traditional instructional teaching takes place. The students worked together on tasks specifically designed for the purposes of the learning goals set by the teacher and the researcher, in this case learning the properties of geometric solids and their lower dimension figural units. The study was conducted as a part of students' geometry lesson, in the time when they learned about three-dimensional objects and their properties.

The study took place in three schools in northern Germany. All three are "Gymnasium" schools, which is a type of school in Germany that prepares students for university entrance. From each school one class participated in the study. The participants were 9^{th} and 10^{th} grade students. The duration of the study was two weeks. The exact organization of the phases of the study is further explained in subsection 5.3.3.

The students worked on geometry tasks designed in GeoGebra. The subject of these tasks was the recognition of three-dimensional invisible solids by their two-dimensional cross-sections with a plane. There was also a preliminary task carried out at the beginning of the first day (see Task 1 in 5.2.2.1). First, the students worked on each task in pairs on one computer. For each task, they were given a worksheet that mainly fulfilled a supportive role. The students could use it to the extent that they wanted (see details in 5.2.2.1). Before the end of each lesson, a discussion with the whole classroom took place, in which the students discussed their solutions to the tasks.

Following, I discuss all the aforementioned elements of the study in more detail.

5.3.1 The experimental lessons

At school students are usually taught about two-dimensional shapes and their properties separately from three-dimensional geometrical objects. In the rare case that these two ever come together, it is by trying to recognize any two-dimensional shapes in a solid. As I have discussed previously (see Chapter 4 and subsection 5.2.2.2), I suggest that two-dimensional shapes and three-dimensional geometric objects should be seen as related and connected through properties and not as individual disconnected entities.

From the teaching perspective, this has been the purpose of the D-transitional tasks and the lessons performed in the frame of the study. I call the types of lessons that took place in the study, *experimental lessons.* I decided to give to these lessons a specific name because they were different from what the teachers usually did in their every-day teaching on the subject of geometrical shapes and objects. The design of the lessons differed from students' usual teaching, mainly because of the task-design. But, the teachers participating in the study informed me, the organization of the learning environment in the *experimental lessons* was quite similar to the one in their usual teaching practice. I felt that this was a positive factor in the research since I wanted to intervene as little as possible to the teaching and learning norms of the classrooms.

The expression *experimental lessons* may remind of "teaching experiments" (see for example, Cobb, 2000), but they should not be confused with them. I do not call the lessons in my study a "teaching experiment" because the aim was not to carry out research in order to develop and improve any tasks or research design. The tasks in the study were designed in order to research students' argumentations and visualizations within a specific framework (Prusak et al.'s (2012, 2013) design principles). The focus lies on the students and their mathematical work. The creation of the tasks on which they work, has been only one part of the research design, and not the main focus of my research.

5.3.2 Preparation for the study

Before starting the main study, a preparation phase took place. This phase had two purposes: a) to introduce students to GeoGebra, its tools and functions, and b) to reassure that the equipment for the data collection (microphones, cameras, screen-recordings, etc.) would work as planned during the main study.

The preparation phase was conducted in a two-hour lesson slot, two weeks before the main study. I call the week when the preparation phase took place, "Week 0". The data collected in that week will not be analyzed or used in this study. The purposes of the preparation were merely didactical (for the students) and practical (for the secure flow of the research). The two tasks used in the preparation phase can be seen in Appendix B0.

5.3.3 Phases and participants of the main study

The study was conducted in three schools in northern Germany, one in Bremen and two in Hamburg. In the study participated students and their teachers from one class in each of these schools. The participants were 9^{th} and 10^{th} grade students, respectively. The total number of participants is seventy-two students and three teachers. However, the data reported and analyzed in this work are from only one of the schools (see more details in 5.4.1). The duration of the study in each school was two weeks and the experimental lessons were conducted in the two phases of the study (Table 5.1), both of which started with the students working in pairs. The two experimental lessons were planned in collaboration with the teachers and carried out by them.

Table 5.1 shows the phases of the main study and the exact tasks on which they worked in each phase. In the first phase all students worked on the same tasks (Task 1 and Task 2). After the pair work on each task, followed a whole classroom discussion. As is discussed in detail in 5.2.2.1, the purpose of the first task was to introduce the 3D-graphics of the DGE and the task set-up to the students. The rest of the tasks (2, 3A, 3B and 3C) were the D-transitional tasks designed for the research purposes of the study.

Phases of the main study	**Steps of the Phases**
1st Phase - Students work on the same tasks (Week 1 – Day 1 – 90')	Task 1 - Visible cylinder
	Pair work *Classroom-discussion*
	Task 2 - Invisible sphere
	Pair work *Classroom-discussion*
2nd Phase - Work in parallel Students work on different tasks (Week 2 – Day 2 – 90')	Task 3A - Invisible cone Task 3B - Invisible pyramid Task 3C - Invisible cube
	Pair work *Classroom-discussion* on all three tasks

Table 5.1: Phases of the main study

In the second phase the students were working again in pairs. For about an hour the students worked on at least one task. Before the end of the lesson, all three tasks and students' results were discussed in a classroom discussion.

In order to observe students' argumentations as closely as possible, three pairs of pupils were selected (in each class) to work together over the entire period of the study. The work of these three pairs of students has been video-recorder in order to be analyzed. As the focus in this work lies on observing students' processes in depth and not on generalizing results based on large number of observations, a decision was made in favor of the close observation of a limited number of students during their whole work.

The choice of the students was made according to students' willingness to have their whole working process video recorded. Only students that agreed to the video recording of their whole work were selected to participate in this process. The factor of heterogeneity was also taken into account to the extent possible. In each class, the students that were video recorded were both boys and girls. In two of the schools, the classes in which the study was conducted were inclusive classrooms for students with visual impairments. In either of the two classes there had been one student with a type of visual impairment. Those students were also willing to participate in the study as members of the pairs that were video recorded.

5.3.4 Data collection

The guiding factor in collecting the data in this study was to be able to observe students' discussions during their pair work and also to capture all the interactions taking place during the whole classroom discussions. These aspects were important in order to gather the necessary data to answer my research questions.

Therefore, three video cameras were used to record the three pairs of students whose work would be more closely observed (in each classroom) during the study. To avoid technical problems with the sound, due to the challenging acoustic conditions of the classroom, each camera was connected to a highly sensitive table microphone that was placed between the two students of each pair. This way, the statements made by the two students are clearer and easier to understand. Each camera was placed on a tripod next to the pair of students, recording them from the side. The angle of the video recording had to be such that both students were in the scope of the camera, to document their gestures and their work on the worksheet. Furthermore, it was important that the screen of the computer be also in the scope of the video recording, in order to be able to follow students' gestures and references to what happens in the DGE during their explorations.

Since it would not always be possible to see all the details of students' work in the video (i.e. their exact writings on the worksheet), their worksheets[7] were also collected at the end of each phase of the study. For the same reason, students' work on the computer was also recorded with VLC-screen recording program. These screen recordings complement the rest of the data and their role has been to provide information as to students' actions in the DGE, when this information was not absolutely clear from the video recordings and the worksheets. Each screen recording was synchronized with the pair's work, to make it possible to trace any moment of the their work in the video recording with the actions captured in the screen recording. Furthermore, the researcher was present in the classroom during the whole study and took field notes.

The classroom discussions were documented with one of the three cameras previously used for the pair work recordings. During the discussions, the students used a computer at the front of the classroom, which was connected to the projector. This way they could use the DGE to present something to the rest of the class. Therefore, during the classroom discussions the camera was always placed at a position close to the projection on the wall. At the same time this made it possible to capture any student that would contribute to the discussion. The researcher operated the camera changing its focus when the action in the classroom shifted during the discussion.

In the worksheets (and in the data analysis of the videos), students' personal data have been anonymized. The students used pseudonyms, instead of their real names, in their worksheets. Because some of them used very complicated pseudonyms or codes, these pseudonyms were changed to simpler ones by the researcher in the phase of the data analysis. Apart from students' genders, the pseudonyms do not reveal any other personal data.

[7] I collected the worksheets from all the students participating in the study, not only of those that were video-recorded during the pair work. I did this for two reasons. First, any student could participate in the classroom discussion. In this case, the worksheet constitutes a source of complementary (written) data for their work, to which I can refer in the data analysis. Secondly, being able to get some sort of insight into the work of all the students provides the opportunity to shape a more complete image of students' work. However, taken individually, the data from the worksheets are not subjected to any analysis.

To sum up, in this study are gathered anonymized data from three pairs of students (in each of the three classes where the research took place) and from multiple classroom discussions. These data include: Video-recordings of the pair works on the tasks and the classroom discussions that followed, screen recordings of the students' use of the DGE during pair work, worksheets with the students' notes and field notes.

In the next section, follows the description of the data analysis.

5.4 Data-analysis methodology

This section is about the methodology upon which is based the analysis of the gathered data. It begins with the presentation and explanation for the data selection for the analysis (see 5.4.1). Then, the two methods of argumentation analysis are presented (see 5.4.2). These methods are used in order to get an insight into students' argumentations. Then, follow the levels of data analysis and the individual processes taking place in each of the analysis levels, including the method of identifying the use of visualization in students' argumentations (see 5.4.3). Last but not least, an example of data analysis is provided. The entire data analysis process is presented using, as an example, an episode from the study (see 5.4.4).

5.4.1 Data selection

In this study, I am interested in observing how students use visualization in their work and in determining the roles of visualization in their argumentations, when they solve D-transitional tasks. In his work, Duval (1998) explains the importance of visualization in students' argumentation in geometry. Here, using Duval's (1998) theory, I attempt to describe students' visualization in argumentation in a concrete way, identifying its exact positions and functions in argumentation. My intention is to observe the processes of visualization and argumentation in depth attempting to understand and describe them as precise as possible. The purpose here is not to generalize my results.

Therefore, in this work I have chosen to report only on data collected in one of the three schools in which the study was conducted. The reported data were selected after watching all the videos from the three schools. The intention has been to use enough, but not more than the necessary amount of data to illustrate the interplay between visualization and argumentation. The school that was selected was the one that provided the richest data in terms of the use of visualization in argumentation both in pair works as well as in the classroom discussions.

The selected class had twenty-five 9th grade students and their teacher was Frau Karl (equivalent to Ms Karl, in English). In this class, four boys and two girls worked in pairs and their work was recorded in video. The girls worked in the same pair and the four boys formed the other two pairs. In the one pair of boys, one of the students (Tom) has a visual impairment, which nevertheless does not prevent him from working on the computer, or writing in hand. However, the use of a larger letter font is helpful for his reading. Therefore, Tom and his classmate's worksheet was printed out in a larger font (22pt instead of 12pt). The colors used for the task in GeoGebra were chosen purposefully to be quite vivid (bright yellow for the intersection plane and neon green for the cross-sections), so that they are easier to see and distinguish (especially for Tom).

In one case the video recording of the observed students was affected during the data collection. During the second phase of the study, the camera that was recording the pair work of the two female students stopped working. This resulted in lack of data (in phase two) for this pair of students. Therefore, here are reported and analyzed the data from the two other pairs of students.

As is explained in the description of the tasks (see subsection 5.2.2.1), Task 1 is a preliminary task that was used to introduce students to the 3D-Graphics of GeoGebra and help them understand how the sliders function and how they move the solid in space. Therefore, no data regarding this task are used in the analysis. The data emerge from students' work on Tasks 2, 3A, 3B and 3C (see details in 5.2.2.1).

Table 5.2 shows the students whose data are reported and analyzed in the present work. The tasks the students worked on were divided into the two days of the study. On the first day, all the students worked on the same task, namely Task 2 (the invisible sphere). On the second day, each pair of students began with a different task (Task 3A, 3B or 3C). When a pair of students completed their work on a task they could continue with another one. For example, Tom and Lukas worked on Tasks 3A and 3C on that day, while Axel and Dave started with task 3C, then moved to 3B and finally worked also on task 3A. Unfortunately, Tom and Lukas' pair work on Task 3C cannot be analyzed due to lack of discussion between the two students. Some results of their work are observable during their presentation in the classroom discussion on Task 3C[8]. For each group, the Latin number (I, II or III) next to the name of each task denotes the order in which the group worked on the task. The minutes noted in the table, show the duration of the students' pair work or of the classroom discussion for each task (e.g. Axel and Dave worked on Task 3A for fifteen minutes and this was the third task they worked on (hence, 3A.III)).

Students working on the task	Tasks used in the main study			
	Day 1	Day 2		
	2 – Sphere	3A – Cone	3B – Pyramid	3C – Cube
Axel and Dave (GR1AD)[9]	10'	3A.III 15'	3B.II 8'	3C.I 30'
Tom and Lukas (GR2TL)	10'	3A.I 31'		
Classroom-discussions (CD)	8'	10'	10'	10'

Table 5.2: Selected data

During the time the students worked on the tasks, the teacher and the researcher were merely observers of the situation. The teacher would only intervene in students' work, if they asked her a question. The researcher only intervened a couple of times, to see how the students proceeded with the tasks. During the classroom discussions, the teacher was the orchestrator and facilitator of the discussion.

8 See the global argumentation structure of episode CD3C-TL in Appendix E11.

Next, I present the analysis methodology followed for the analysis of the selected data.

5.4.2 Two analysis methods brought together

As I have mentioned before, in this study the focus lies on the role of visualization in students' argumentation. In order to observe this, it is first necessary to analyze students' argumentations. In this work, two methods of argumentation analysis are combined for this purpose: Reid's (2002a, 2002b) method of building *patterns of reasoning* and Knipping's (2003a, 2003b, 2008) method of building *argumentation structures.* The different methods of argumentation analysis offer two different but equally important perspectives of how an argumentation may flow in a student-centered learning environment, in which students work in pairs or small groups and the role of the teacher is limited to that of the facilitator of students' discussions. The role of visualization in argumentation is examined, based on Duval's (1998) theory, in the frame of the argumentation structures created with the Knipping's (2003a, 2003b, 2008) method when analyzing classroom argumentations.

Method #1: Building *patterns of argumentation*

Reid (2002a, 2002b) observes students' work while solving mathematical tasks in groups. He analyzes students' discussions and describes what he believes they reveal about their reasoning. Reid (2002b) creates diagrams of students' reasoning, which he calls *patterns of reasoning*, illustrating what students' reasoning "looks like in school contexts" (ibid, p. 6) (see Figure 5.11).

A pattern consists of *elements*, which describe "*ways* of reasoning (deductive, by analogy, etc.), *needs* to reason (to explain, explore, verify), and the degree of *formulation* or awareness of reasoning" (Reid, 2002b, p. 9). Constitutive elements of a pattern of reasoning can be: observing a pattern (or pattern observation, PO), conjecturing (Conj), testing (Test) the conjecture, generalizing (Gen) the conjecture, using the generalization to make simple deductions (1D) or to extract special cases from it (Sp), contradictions (Contra) or counter examples (CE).

An example of a pattern of reasoning by Reid (2002b) is quoted below and it explains the pattern seen in Figure 5.11. In this case the students are in the process of refuting the -never explicitly stated- conjecture ($Conj_1$).

> "The implicit generalization (Gen) is specialized to two cases (Sp_1, Sp_2) that lead to a pattern of reasoning [...] (PO –> Conj –> Test). In this case the test results in a contradiction (Contra) that is taken as a counter-example (CE). One of the specializations (Sp_2) also gives rise to a third specialization (Sp_3) by way of either an analogy ($Analogy_2$) or an implicit conjecture ($Conj_2$)" (Reid, 2002b, p.15).

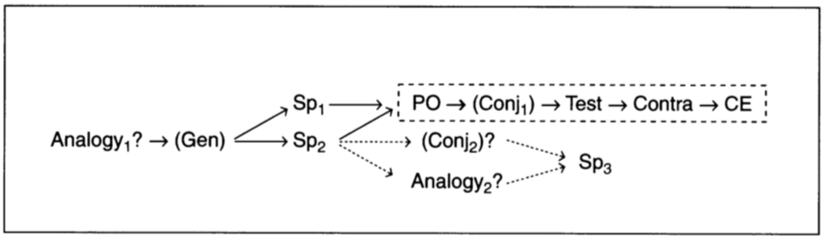


Figure 5.11: Pattern of reasoning: Case 2 (Reid, 2002b, p.15)

A pattern may consist of one or more *paths.* A *path*, refers to a part of the pattern that begins at the pattern's first element and ends at one of its final elements, without any junctions or splits. For example, a path in the Pattern in Figure 5.11 is the following: "Analogy$_1$? –> (Gen) –> Sp$_2$ –> PO –> (Conj$_1$) –>Test –> Contra –> CE". The whole structure of the pattern does not reflect the events that took place in a chronological order, but each path of it does. That means, that paths that run in parallel in a pattern, do not happen simultaneously. The vertical dimension of the patterns does not mean "simultaneously", rather "also". But, the arrows inside the paths have a chronological connotation and can be read as "and then".

In Reid's (2002a and b) research, as in mine, students work on mathematical problems in various social settings: "When working on a problem, they [*the students*] first work in pairs or triples, then come together in their groups to compare solutions, and then report to the half-class" (Reid, 2002b, p. 10, comment in *italics* by myself). Reid (2002b) analyzes students' discussions to find out what their statements may reveal about their reasoning. Reid (2002b) reconstructs students' argumentations and provides insight into their reasoning, by revealing the thinking processes that may underlie students' arguments. Hence, the *patterns* in his method illustrate students' reasoning; ergo *patterns of reasoning.*

In my research, I also analyze students' utterances. However, I focus only on what students say out loud, and what their utterances reveal about their argumentation. The difference between the two approaches lies on the extent in which is assumed what thoughts may be "hidden" behind students' utterances and actions. The reason for this is that I assume as little as possible about what or how the students may have been thinking when that is not made explicit by their own words.

Therefore, in this work Reid's (2002b) method is used to describe students' argumentation; not their reasoning. Hence, the *patterns* in this work are *patterns of argumentation* (not patterns of *reasoning*). Furthermore, due to different mathematical contexts and types of the tasks used in Reid's work and in mine, in some cases the names of some of the elements in the patterns of argumentation differ slightly from Reid's (2002b). For example, Reid (2002b) uses tasks in which students are supposed to observe patterns. As a result, it makes sense to talk about "observing a pattern". In the tasks used in my work, students are not supposed to observe patterns, rather data about two-dimensional cross-sections and movements of the sliders, geometric properties etc.. Hence, the term "observing a pattern" has been replaced by the more fitting term "observing data". In the table below, are presented the different terminologies. The meanings of the elements in this table become clearer during the presentation of the data analysis, where explicit examples from this study are provided (see 5.4.4, Level 2 Analysis).

Elements of **Pattern of Argumentation**	**Elements of** **Pattern of Reasoning (Reid, 2002b)**
DO (Observing data/Data Observation)	**PO** (Observing a Pattern/Pattern observation)
Hyp (stating a Hypothesis) a supposition created by the students, suggesting a possible case based on the available data. This is a case, which at the moment looks plausible, and whose validity is not yet confirmed.	**Conj** (Conjecturing) Something is considered neither true nor false, rather it is subject to testing.
Clai or **Conc** (stating a Claim or Conclusion) A *claim* is more than a hypothesis; it is more than just a possible case or solution. It is *the* possible case which one considers as the most probable and shows the intention to confirm it or argue in favor of it. A *conclusion* is a statement that is accepted by all as true.	**Gen** (Generalizing) Here generalizing is used in the sense of uttering a statement that is accepted by someone or a social group as true but still not by all, so it may be a generalization for some and a conjecture for others.
Contra (stating a Contradiction)	**Contra** (Contradiction)
1D (drawing a conclusion with simple deduction)	**1D** (simple deduction)
Test (testing)	**Test** (testing)
CE: using a counter-example	**CE**: using a counter-example

Table 5.3: Terminological comparison of *elements* in patterns of argumentation and patterns of reasoning

There are both similarities as well as some differences between Reid's (2002b) terminology and mine. Reid makes a distinction between *conjecture* and *generalization.* I make the same distinction between the terms *hypothesis* and *claim.* Reid's *conjecture* and my *hypothesis* are etymologically and epistemically equivalent. The same stands for what Reid (2002b) calls *generalization* and what I call *claim.* That means that both Reid and I differentiate the two elements based on their epistemic values. Both *conjecture* and *hypothesis*, express a supposition that is put forth in order to be tested, without baring a specific initial belief (by the student) on its validity. On the contrary, a *generalization* and a *claim* represent something that is believed by the student to be true, but is not yet confirmed.

As is mentioned above, students' patterns of argumentation are based on what the students say in their discussions with their peers while working on a mathematical task. Therefore, students' written justifications are not analyzed with this method. The written justifications are the products of students' oral argumentations. They are individual arguments (or series of arguments), justifying in short, their final conclusions. Students' written justifications do not describe the actions they took while solving a task, rather the statements they consider necessary in order to verify their final conclusion.

The written justifications are analyzed with the second analysis method used here: Knipping's (2003a, 2003b, 2008) method of illustrating students' argumentations with *argumentation structures*. As is explained next, the *argumentation structures* are based on the *functions* of students' statements in their argumentation and are therefore a subject for functional argumentation analysis (Toulmin, 1958).

Method #2: Building *argumentation structures*

In order to explore what is the role of students' visualization in their argumentation, an insight into the finer structure of students' argumentation is necessary. To achieve this, an argumentation analysis method is needed that enables the reconstruction of such a detailed structure. As discussed above, Reid's (2002a, 2002b) method provides information about the processes that students follow in their argumentations, which help to create *patterns* that reveal the structures of the argumentations. Knipping's (2003a, 2003b, 2008) method examines students' argumentation from another point of view, that of functional analysis of the argumentation, revealing the function of each statement in the argumentation structure.

Knipping (2003a, 2003b, 2008) reconstructs argumentations taking place in mathematical discourse in school classrooms. Her research data arise in deductive mathematical situations, mainly during proving a theorem with the participation of the whole classroom under the supervision or even the guidance of the teacher. The students are given a mathematical statement, which they are asked to prove. In such situation, is needed "a conception of 'rational argument' that does not cut off students' rationality" (Knipping, 2008, p. 429), which of course is not the same as the formal rationality of a mathematician. The *functional model of argument* by Toulmin (1958) is suited for this kind of analysis, because it provides us with a tool that can indeed "go beyond dismissing it [*students' thinking*] as "illogical" (ibid, p. 429, comment in italics added by myself), in contrast to a pure logical analysis of students' argumentations.

Knipping's method for the reconstruction of argumentations is developed based on Toulmin's (1958) idea of the structure of arguments and how this structure could be symbolized and depicted graphically. In Chapter 2, I present Toulmin's (1958) *functional model of argument.* Toulmin (1958) created a model for the reconstruction of individual arguments, focusing on the functional – and not only on the logical – structure of rational arguments. His model is applicable to arguments emerging from any field and it can also be used to reconstruct students' arguments in school contexts, using diagrams with specific elements. In this process, students' utterances are analyzed based on the *function* (data, conclusion, warrant, etc.) attributed to each of them, by the students. This kind of argumentation analysis is called functional analysis of argumentation.

Knipping (2008) goes one step further than Toulmin (1958), extending his model of argument, by creating connections between one-step individual arguments. Knipping (2008, see also Knipping & Reid, 2019) calls the individual arguments, *local arguments* or *argumentation steps.* The connections between local arguments lead to a network of arguments, a wider reconstruction of the whole argumentation that takes place in a classroom. The result of this reconstruction is what she calls *global* *argument* or the *argumentation structure* (see Figure 5.14). The argumentation structure allows Knipping to see at a glance the whole argumentation that has taken place in the classroom. In the present work, for reasons of consistency, I will only use the notions *local argument* and *argumentation structure*. There is also an intermediate-level argumentation step, between the levels of a *local argument* and an *argumentation structure*, which is the *argumentation stream* (see Figures 5.12 and 5.13). An argumentation stream consists of a number of local arguments that are linked, and lead to a final conclusion (Knipping & Reid, 2019). Next, I elaborate on Knipping's (2003a, 2003b, 2008) method of argumentation reconstruction.

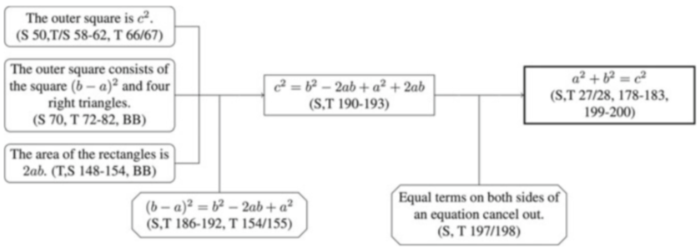


Figure 5.12: An *argumentation stream* from an oral argumentation (Knipping & Reid, 2019, p. 8, Fig. 1.6)

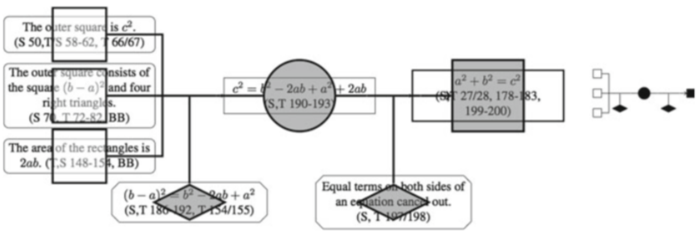


Figure 5.13: The argumentation stream from Fig. 5.12, reduced to functional schematic (Knipping & Reid, 2019, p. 8, Fig. 1.7)

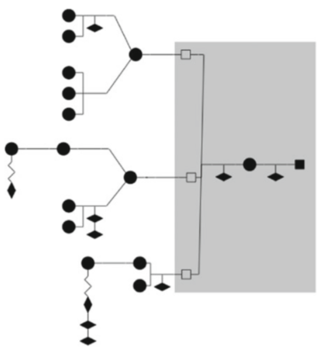


Figure 5.14: The argumentation structure of an entire proving process in classroom discussion (Knipping & Reid, 2019, p. 9, Fig. 1.8)

Knipping (2003a, 2003b, 2008, see also Knipping & Reid, 2019) follows a three-stage process in order to analyze classroom argumentations: reconstructing the sequencing and meaning of classroom talk, analyzing arguments and argumentation structures, and comparing local arguments and argumentation structures. The first stage in Knipping's method includes identifying "the general topics emerging in the classroom talk" (Knipping & Reid, 2019, p. 10) and reconstructing their sequence. This provides an overview of the whole argumentation that takes place. Then, follows the second stage of the analysis, which is to reconstruct students' *local arguments*, using Toulmin's (1958) functional model of argument. The local arguments that are logically linked are then connected together, thus creating *argumentation streams*. For example, Figure 5.12 shows an *argumentation stream* from Knipping and Reid (2019). The statements in the boxes, are students' statements taken from the transcript. Here, three data lead to the conclusion "$c^2 = b^2 - 2ab + a^2 + 2ab$", using as warrant the algebraic identity "$(b - a)^2 = b^2 - 2ab + a^2$". From this conclusion, follows then another conclusion, namely "$a^2 + b^2 = c^2$", based on warrant "Equal terms on both sides of an equation cancel out". This process is continued until the whole argumentation is analyzed and the *argumentation structure* of the whole discussion has been reconstructed.

In the third stage of her method, Knipping (2003a, 2003b, 2008, see also Knipping & Reid, 2019) compares local arguments and argumentation structures in order to reveal their rationale. This is a step that will not be used in the present work, since comparing the structure of students' argumentations is out of my scope in the present work. However, I would like to explain shortly this final stage in Knipping's method. Knipping represents local arguments, argumentation streams and argumentation structures using specific shapes as codes to represent different functional elements of argument. For example (see Figure 5.13), the data are symbolized with white squares, the warrants with rhombuses, the intermediate conclusions in every stream with circles and the *final* conclusion in the argumentation is symbolized with a black square. The zigzag lines denote a contradiction This is done for the whole argumentation structure, resulting to a coded representation of it (see Figure 5.14), which is much easier to evaluate, observe its rationale and compare it with other argumentation structures (for more details please refer to Knipping & Reid, 2019).

As with Reid's (2002b) method, there are both similarities as well as some differences between Knipping's (2003a, 2003b, 2008) research design and mine. Knipping's (2003a, 2003b, 2008) research has also been in the context of geometry and she reconstructs students' argumentations based on their uttered statements. However, in her work the learning environment organization during classroom discussions is different than that in my research. In Knipping's (2003a, 2003b) research the teacher's role is central for the evolution of the discussion and it is a determining factor for the turns that students' argumentations take. In my work, the role of the teacher is limited to that of a facilitator and advisor. The students in my research work in small groups without the teacher's constant intervention. The teacher acts only as a supporter whenever the students ask for help, or as the orchestrator of the discussion during the classroom discussions. She becomes more involved in the discussion only to ask some questions, if she considers it necessary, in order to help their discussion to evolve.

Furthermore, the tasks used in this study do not reveal the final result, as they are not tasks that ask the students to prove a given statement. Rather the students are asked to find out for themselves and justify their findings. These tasks promote exploration, experimentation and abductive arguments, with less cases of deduction emerging. Visualization also plays an important role in students' solving processes.

However, these differences do not constitute an obstacle for the use of this method for the analysis of my data. Knipping's (2003a, 2003b, 2008, and Knipping & Reid, 2019) method has been used in various mathematical and social contexts in research (see for example Shinno, 2017; Cramer, 2018; Potari & Psycharis, 2018). Knipping's method (2003a, 2003b, 2008, and Knipping & Reid, 2019) allows us to shed light to the detailed functional structure of students' argumentations. Therefore, it is in the frame of this method that I also examine the role of students' visualization in their argumentation. I discuss this further in the next section, in which I present my data analysis methodology.

The bottom line as to the use of the two argumentation analysis methods

Both Reid's (2002b) and Knipping's (2003a, 2003b, 2008) methods of argumentation analysis are useful tools in the process of understanding better the structure of students' argumentations in multiple mathematical and social contexts. Their combined use offers analyses of the argumentations from two different perspectives: a perspective on the processes the students follow in their argumentations (Reid, 2002b) and a perspective on the functional structure of their arguments and argumentations (Knipping, 2003a, 2003b; Knipping, 2008). The elements in Knipping's (2003a, 2003b, 2008) argumentation reconstructions (local arguments, argumentation streams and argumentation structures) are static and what makes the whole reconstruction dynamic are the arrows creating relationships between the noun-elements of the model (datum, warrant, conclusion, etc.). We could say that the arrows in Knipping's model could be read as "and so". On the contrary, in Reid's (2002b) method the elements are not static, rather they carry the argumentation's evolution in them. But the arrows are only there to denote the end of one step and the beginning of another, as processes. These arrows could be read as "and then", with chronological connotation. Hence, Reid's (2002b) verb-based model is a valuable tool in order to get a complete and detailed enough overview of the flow of a whole argumentation. On the other hand, Knipping's (2003a, 2003b, 2008) reconstructions help to reveal the fine structure of students' argumentations and examine the role of visualization in argumentation.

5.4.3 The three levels of my data analysis

As discussed in the previous subsection (see 5.4.2), I analyze my data combining two methods to reconstruct students' argumentations; Reid's (2002b) *patterns of reasoning* (corresponding to my *patterns of argumentation*) and Knipping's (2003a, 2003b, 2008) *argumentation structures*. I also use Duval's (1998, 1999/2002) theory on visualization (see Chapter 3) in order to analyze the use of visualization in argumentation. More precisely, the data analysis has three levels with the following composition:

Level 1 Analysis – Structure and summary of the episode

Level 2 Analysis – *Pattern of argumentation*

(a) Identifying elements of the pattern of argumentation
(b) Reconstructing the pattern of argumentation
(c) Explaining the pattern of argumentation

Level 3 Analysis – *Argumentation structures* and *visualization*

(a) Identifying hypotheses and claims
(b) Identifying other functional elements of argument
(c) Reconstructing local arguments
(d) Reconstructing argumentation streams and the argumentation structure
    (i) Oral argumentation
    (ii) Written justification
    (iii) Comparing the oral and the written argumentation structures
(e) Identifying and explaining the roles of visualization and spatial manipulation in the argumentation

In Level 1, the structure of the episode is presented in a table and a summary of the whole episode is provided. My aim here is to give a short overview of the whole episode, based on what I observed the students do during their work on the task, without interpreting their actions. My intention is to act as a mere observer and narrator of the actions taking place. Therefore, each episode has a "map"-like feature; I divide the episode into thematic sub-parts, each of which has a title (see Knipping, 2003a, 2003b; Knipping & Reid, 2019). In this level of analysis I also observe the exploration strategies that the students follow while solving a task. I report the results of Level 1 analysis in Chapter 6.

In Levels 2 and 3 I introduce my interpretations for the argumentations. Firstly, in Level 2 I analyze students' argumentations in a chronological order, reconstructing their *pattern of argumentation*. As discussed earlier (see Method # 1 in subsection 5.4.2), this analysis is based on Reid's (2002b) method of "patterns of reasoning" and it provides me with an overview of the processes involved in students' argumentation. I begin by identifying the different pattern-elements of students' argumentation. These are elements such as data observations (DO), stating hypotheses (H), stating claims (Cl), drawing conclusions (C), refuting (R), testing of a hypothesis/claim (Test), and others (see Table 5.3 in subsection 5.4.2). Then I link those elements with each other following the chronological sequence in which they emerge in the argumentation. The arrows linking each element to the next can be read as "and then". The *patterns of argumentation* illustrate the flow of students' argumentation from the perspective of the chronological order of students' actions. The results of this analysis level are reported in Chapter 7.

In Level 3, the logical structure of students' argumentations is reconstructed. This analysis is based on Knipping's (2003a, 2003b, 2008) methodology for reconstructing argumentations in classroom situations (see also Papadaki, Reid & Knipping, 2019; and Knipping & Reid, 2019). This argumentation analysis focuses on the identification of every functional element in students' argumentation and their links to each other. The process begins with the identification of students' initial hypotheses (H) and claims (Cl). These are the first suppositions students make for the possible forms of the invisible solid. Knowing the starting point of students' argumentation and their initial suppositions provides me with a good starting point in my attempt to follow the flow of their argumentation. Then, I continue with the identification of the rest elements of the argument, such as data (D), warrants (W), conclusions (C), backings (B), qualifiers (Q) and refutations (R) (see more details in Chapter 2, subsection 2.2.3). After having identified all the elements of the argumentation, I reconstruct the *local arguments*. Then, I examine how the individual local arguments are linked, or not, with each other. The product of the process of linking the local arguments together is the *argumentation structure* of the students' argumentation.

By reconstructing students' arguments through this method, I aim at examining the structure of the overall argumentation in greater detail. Another important contribution of this analysis level is that it allows me to examine the function and role of students' visualization in their argumentations. Trying to observe only the existence of visualization in an argumentation does not require a detailed argumentation analysis. However, if one is interested in the details of the interplay between argumentation and visualization, one has to "dig deeper". Since the aim of the present work is to provide an insight into the role of visualization in argumentation, it is important to illustrate the specific points, in students' arguments, in which visualization takes place.

For this purpose, I introduce two additional elements in the argumentation reconstructions, which have not been previously used in the process of reconstructing argumentations in research. These elements are *non-iconic visualization* (see subsection 3.1.1) and *spatial manipulation* (see subsection 3.1.3), and they show the places in argumentation in which students have employed their visualization. The functions[10] and roles of *non-iconic visualization* and *spatial manipulation* in the argumentation are explained in Chapter 8 of the results. For the identification of *non-iconic visualization*, I examine how the students use the properties of the cross-sections, whether they identify figural units of the solid and how they relate these figural units with the solid itself. Hence, the use of *dimensional deconstruction* is a clear indicator of *non-iconic visualization* (see also subsections 3.1.1 and 3.2.1). As I mention in subsection 3.1.3, I identify students' use of *spatial manipulation* examining their verbal description about the movement and orientation of the invisible solid in space, as well as the accompanying gestures or metaphors. The indicators used to identify these two new elements in students' argumentations are also described in subsection 5.4.4 (see Level 3 Analysis, part e) and explained in detail in Chapter 8.

In the worksheets given to the students with every task, there is a final question (see subsection 5.2.2.1 and in Appendix B the Worksheets for Tasks 2, 3A, 3B, 3C, Part b), where they are asked to give a written justification regarding the form of the solid. In the cases where the students have provided such a written justification, I reconstruct the local arguments and the argumentation streams of their written justification, following the same steps I described previously for the reconstruction

10 The word function is used here in Toulmin's (1958) sense (see subsection 2.2.3).

of the oral argumentation. Then, I describe the *argumentation structure* of both the oral argumentation, as well as that of the written justification. I also compare the two argumentation structures (the oral and the written), revealing the differences between the two processes.

All the transcripts of the episodes are provided in the electronic Appendix of the thesis, as they are too long to be presented in their entirety in the main text. Nevertheless, I provide excerpts of the transcripts that I consider vital for the understanding of the analysis and the presented results, in the main body of text. The transcripts are originally in German. The excerpts used in the text are provided both in the original as well as translated in English (translated by the author). The statements used to reconstruct the *argumentation structures* are also in English.

5.4.4 Data analysis example: Analysis of an episode

In this subsection, I analyze one episode as an example of the way I have analyzed all my data following the three levels of data analysis presented in the previous subsection. The code-name of the episode used here is GR1AD-2[11], and it is from Axel and Dave's work on the invisible sphere task, on the first day of the main study (see Table 5.2). Next follows the presentation of all three levels of analysis for this episode, as well as the results of each analysis level.

Level 1 Analysis – Structure and summary of episode GR1AD-2

In this step I start by presenting the structure of the episode in a table-format. Then follows a summary of the episode, where I give a short overview of the students' actions.

<u>Structure of the episode</u>

The following table presents the structure of Axel and Dave's work on Task 2. Each part of the structure represents a different focus in students' work on the task. Each time their focus, intentions or actions change I represent this by creating a new part in the structure. Each part of the structure is numbered. For each part, the utterances (numbers of the transcript lines), as well as the time in the video are mentioned in the table.

My aim with Table 5.4 is to create a quick overview and an orientation for the episode. Nevertheless, the table is quite laconic. Then follows a summary with some more information about the episode. This is a necessary preliminary step for my analysis as it helps me to obtain a better understanding of the situation, before I start interpreting what has happened.

11 The students participating in it are Axel and Dave (AD), who are Group 1 (GR1). The task they are working on here is Task 2 (2), which is the invisible sphere.

GR1AD-2: Axel and Dave's pair work on the invisible sphere task				
Structure			**Utterances**	**Video Minutes**
1	Is the solid a cone, a circle or a sphere?		1-9	00:59:43 – 01:00:08
2	Observing the cross-section for various values of n, h and d. The solid is a sphere.		10-12	01:00:09 – 01:00:33
3	Further exploration with the use of the Exploration Matrix on the worksheet.		13-29 and 43-61	01:01:07 – 01:02:33 and 01:03:20 – 01:06:03
	3.1	Exploring case (h, n=0°, d=0°).	13-20.1	01:01:07 – 01:02:33 and 01:03:20 – 01:06:03
	3.2	Specifying the length of the radius of the sphere.	20.1 – 29	01:02:00 – 01:02:33
4	Discussion with a third student, Jacob, about his conjecture.		30-42	01:02:33 – 01:03:20
5	Continuing the exploration using the Exploration Matrix.		43-61	01:03:20 – 01:06:03
	5.1	Exploring further the position (h=0, n=0°, d=0°)	43-45	01:03:20 – 01:03:50
	5.2	Exploring the case (h=0,4, n, d=0°)	46-50	01:04:27 – 01:05:29
	5.3	Exploring the case (h=0,4, n=0°, d)	51-61	01:05:29 – 01:06:03
6	The solid is a sphere – A and D write their justification on the worksheet.		63-74	01:07:35 – 01:09:10

Table 5.4: The structure of Axel and Dave's work on Task 2

Summary of the episode

On the first day of the experimental lesson, Dave and Axel work on Task 2, exploring various cross-sections of an invisible solid. The two students start exploring the situation (utterances 1-9) by changing the values of the height-slider (h) and then the tilt-slider (n). First, while moving the h-slider Dave says that the solid is a cone (utterances 1-3). Then Axel moves the n-slider and says that the solid is a circle (utterance 4). In a short discussion that follows, Dave (utterance 5) explains to Axel that the solid is a three-dimensional object, and that therefore it cannot be a circle (which is a two-dimensional geometric object). He also says that the circles that Axel observes are only the cross-sections of the invisible solid.

After this clarification Axel says that the solid is a sphere (and not a cone as Dave had proposed). From that moment on the two students engage in a discussion over the validity of their last supposition (the solid is a sphere). Axel gives a first explanation on why the solid is a sphere and not a cone by saying that "yes, because how else could the tilt?" (utterance 10, my translation); at this moment he moves the n-slider up and down observing that the shape of the cross-section does not change. And he continues by saying "This way" (by reducing the value of h under zero) "it gets smaller and smaller, because the sphere goes out and so pop!". When Alex says "out" he drugs the h-slider downwards until there is no more cross-section to be seen (see Figure 5.15). When he says the word "pop", a cross-section reappears as he draws the h-slider upwards again (see Figure 5.16).

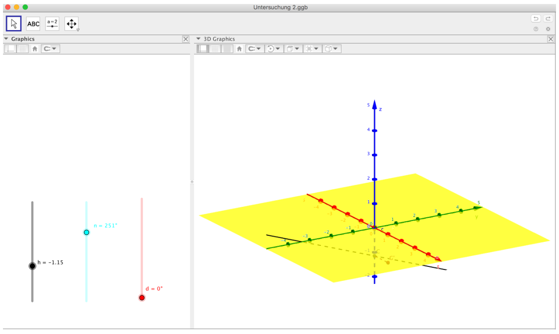


Figure 5.15: No cross-section at position (h=-1,15, n=251°, d=0°)

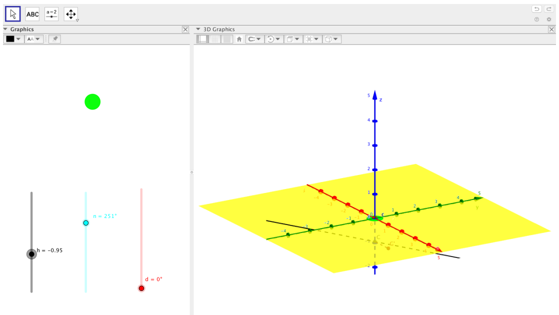


Figure 5.16: Cross-section at position (h=-0,95, n=251°, d=0°)

After this short discussion, the students move on with an exploration based on the given positions in the Exploration Matrix. During this exploration phase they test various cases, each time examining if the results emerging from the manipulations of the sliders fit to their sphere-supposition. When they finish with the Exploration Matrix, and having found no clue to contradict their supposition, they write a justification on their worksheet.

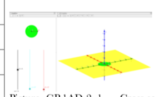
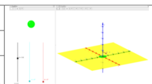
In their written justification the students write that the solid is a sphere. They provide two reasons for this: there are only circular cross-sections, and the tilt (the change of the tilt-slider) has no impact on the shape of the cross-section, because in a sphere the segment from its center to any point of its surface is always the same. In their oral discussion, while writing down this justification they call this segment the "radius of the sphere".

As I mention above, Dave and Axel begin to explore the situation without referring to any strategy or plan for the choice of movements they perform on the invisible solid (see parts 1 and 2 in the episode structure, in Table 5.4). From now on, I will refer to this kind of exploration as *free exploration* of the situation. Later, the two students continue their explorations using the cases and positions provided by the Exploration Matrix (see parts 3 and 5 in the episode structure, in Table 5.4). I will call this kind of exploration as *guided exploration*. I discuss all the kinds of explorations that I identified based on the analysis of all the data, in Chapter 6.

Level 2 Analysis – *Pattern of argumentation* for episode GR1AD-2

a. Identifying elements of the pattern of argumentation

In this level of analysis, I first identify in the transcript the elements of the *pattern of argumentation* for this episode (see for example Figure 5.17). The subscript, on the bottom right of a letter representing an element of the argumentation, indicates the number of the utterance in which the statement is made, unless stated otherwise. The same stands for all the steps of the data analysis. For example, the notation Hyp_3 represents a student's action of "stating a hypothesis" in utterance 3. In the transcript excerpt below the coding of the statements is shown. This coding is based on the identification of the elements of the argumentation (the whole transcript can be seen in the Digital Appendix H1, because it is too long to be presented in the text).

Utterance #		Transcript	Notes
1.1 1.2	DO_{1-2}	Axel: Oh, der [*the solid*] ist unsichtbar. [*Axel performs the exploration (h, n=0°, d=0°). See pictures GR1AD-2_1 to GR1AD-2_3II*] #00:59:46-2#	 Picture GR1AD-2_1 – Cross-section at initial position (h, n=0°, d=0°)
2		Dave: Ja, die Stelle. (?) #00:59:51-5#	
3	Hyp_3	Dave: Ist ein Kegel. (unverständlich)	
4.1 4.2	Hyp_4	Axel: Oah, die Neigung ist awesome [*Exploration of the case (h=0, n, d=0°)*]. Ist ein Kreis. [*Pictures GR1AD-2_4I and GR1AD-2_4II*] #00:59:53-3#	
5.1 5.2	$Contra_{5-7}$	Dave: Ne, das ist ein Kegel. Ist ja ein dreidimensionaler Körper und das ist nur die Schnittstelle, weil der Körper unsichtbar ist. #01:00:02-5#	
6		Axel: Ja, und das ist ein Kreis. #01:00:03-0#	
7		Dave: Ja, die Schnittstelle ist ein Kreis. #01:00:05-4#	
8	$Clai_8$	Axel: Ne, das ist eine Ku-, also das ist ein Kugel insgesamt. #01:00:08-2#	 Picture GR1AD-2_2I – Cross-section at position (h=0,8, n=0°, d=0°)
9		Dave: Sicher? #01:00:08-7#	
10.1 10.2 10.3	$Test_{10-12}$	Axel: Ja, normal, weil sonst- wie würde denn sonst die Neigung- [*varying the n. He observes that the cross-section does not change*]? (..) So [*varying the h*] wird es immer kleiner, weil die Kugel raus geht [see picture *GR1AD-2_10I*] und so- bopp [*a cross-section appears again and it is like the sphere is coming back again*, see picture *GR1AD-2_10II*]. *#01:00:19-1#*	
11.1 11.2	$Test_{11}$	Dave: Geht das in beide Richtungen [*for h<0 and for h>0. See pictures GR1AD-2_11I and GR1AD-2_11II*]? Nur so als Test. #01:00:26-2#	

1

Figure 5.17: Element identification for the *pattern of argumentation* in episode GR1AD-2

b. Reconstructing the pattern of argumentation

As a next step, I reconstruct Axel and Dave's argumentation by linking the elements together in a chronological sequence, creating and explaining their *pattern of argumentation*. In the diagram below (see Figure 5.18), you can see the *pattern of argumentation* for Axel and Dave's oral argumentation. As I explain in subsection 5.4.2 (see Method #1), I only create patterns for students' oral argumentations, not for their written justifications.

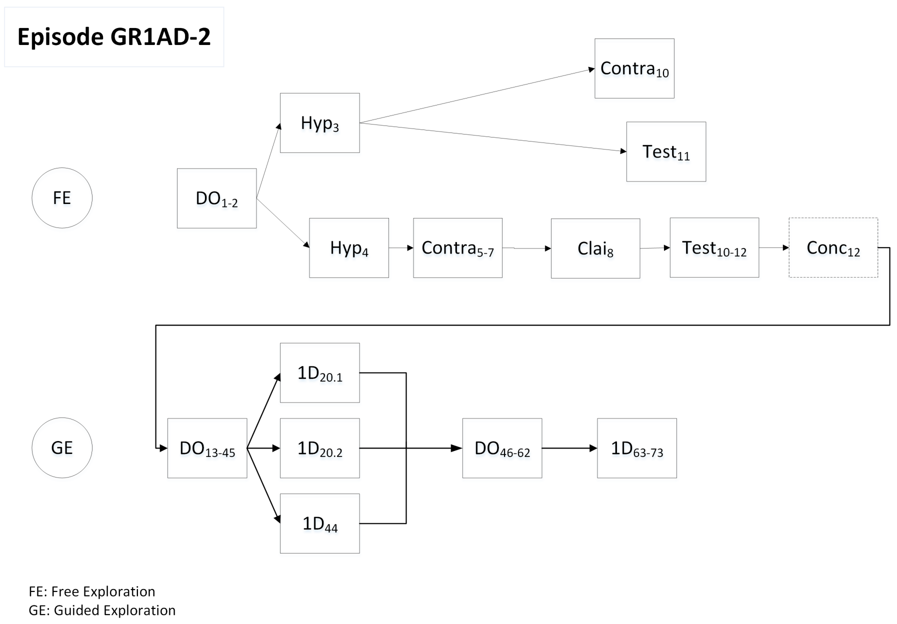


Figure 5.18: Axel and Dave's pattern of argumentation for Task 2

Before going into more detail about the *pattern of argumentation* in Figure 5.18, I would like to present the way this pattern can be read:

> "The pattern begins with observing data ($DO_{1\text{-}2}$), followed by the creation of two hypotheses (Hyp_3 and Hyp_4). Then, the second hypothesis (Hyp_4) is contradicted (Contra $_{5\text{-}7}$). Afterwards, a new claim is stated ($Clai_8$). The claim and the remaining initial hypothesis (Hyp_3) are tested ($Test_{10\text{-}12}$ and $Test_{11}$, respectively), The initial hypothesis is contradicted ($Contra_{10}$) but the claim passes the test. After that, follows the implicit conclusion that the claim stands ($Conc_{12}$). Then the pattern continues with further data observations ($DO_{13\text{-}45}$), which lead to three new conclusions drawn by simple deductions ($1D_{20.1}$, $1D_{20.2}$ and $1D_{44}$). Then follow more data observations ($DO_{46\text{-}62}$) and a final conclusion is drawn by simple deduction ($1D_{63\text{-}73}$)"

The words "then" or "follows" in the above text denote the chronological function of the arrows in *patterns of argumentation.* In Reid's (2002b) method, the whole structure of the pattern does not reflect the events that took place in a chronological order, but each *path* of it does (see pattern in Figure 5.11). That means, that in Reid (2002b) paths that run in parallel (one under the other) in a pattern, do not happen simultaneously. Here, in order to illustrate both cases: when paths are built one after the other chronologically or simultaneously, I use the length of the arrows to introduce the precise chronological order of students' actions. More precisely, the chronological order of the elements of the pattern is denoted in two ways: by the numbers of the elements' subscripts, and by the length of the arrows. Different *paths* (see 5.4.2, Method # 1) of the pattern may either be completed consecutively in time, or be -in parts- built in parallel. Here for example, hypothesis Hyp_3 and claim $Clai_8$ have been tested simultaneously ($Test_{11}$ and $Test_{10\text{-}12}$). Therefore, in this work, the vertical dimension of the patterns will not only mean "also" (as in Reid, 2002b) but show the chronological flow of students' argumentation as well.

c. Explaining the pattern of argumentation

I would now like to explain the above pattern providing more details, revealing the process I follow for the identification of the elements of the pattern and the construction of the pattern.

As mentioned in Level 1 Analysis, Axel and Dave begin with a *free exploration* of the task. During this process they gather data from their observations and perform tests ($DO_{1\text{-}2}$, $Test_{10\text{-}12}$, $Test_{11}$). During their data observation, they explore the case (h, n=0°, n=0°). They drag the height slider under and over zero (h=0) and on the screen appear circular cross-sections that get smaller as the slider moves away from h=0, in both directions. Based on these observations they create two hypotheses about the possible shape of the solid (Axel's Hyp_3: cone (in utterance 3) and Dave's Hyp_4: circle (in utterance 4). Then, they negotiate the second hypothesis (Hyp_4). Dave contradicts this hypothesis ($Contra_{5\text{-}7}$), saying that a solid is a three-dimensional object and that what is observed on the screen in only a cross-section, because the solid is hidden. He also insists on the first hypothesis (Hyp_3), that the solid is a cone (utterance 5). After the students agree that it is the cross-section that is a circle, Axel claims[12] that the solid is "altogether a sphere" ($Clai_8$, utterance 8: "also das ist ein Kugel insgesamt"). This claim is an inferential statement that does not just express a possible case, but rather a case that is believed by Axel to be the most probable, hence a claim. Then Axel and Dave perform a test, that contributes to their discussion for both the cone-hypothesis ($Test_{11}$) and the sphere-claim ($Test_{10\text{-}12}$). During the test Axel varies the tilt-slider (n) while keeping the other two values (height and spin) constant (here starts $Test_{10\text{-}12}$ of the sphere-claim). On the computer screen we can see that the circular cross-section does not change. This seems to contradict Dave's cone-hypothesis ($Contra_{10}$ for Hyp_3). Then Axel moves the height-slider down again and he says that the cross-section "gets smaller, because the sphere goes out" (utterance 10 "wird es immer kleiner, weil die Kugel raus geht") (still $Test_{10\text{-}12}$). Dave asks Axel to test if this is true "in both directions" (utterance 11 "Geht das in beiden Richtungen?", here is $Test_{11}$ of the cone-hypothesis). Axel moves the h-slider both over and under h=0 and on the screen appear cross-sections in both cases. Axel comments on this that "look, you can see the sphere. Almost." (utterance 12 "guck Mal, so kannst du schon die Kugel sehen. Fast."). At this point the cone-hypothesis is silently abandoned and the students implicitly conclude that their sphere-claim is valid (implicit $Conc_{12}$). This becomes evident from the way they continue their argumentation.

Axel and Dave move on with a *guided exploration*, that is an exploration based on the (h, n, d)-positions proposed in the Exploration Matrix. At this stage, their exploration aims at verifying their sphere-claim. During their guided exploration, the students observe data that appear on the computer screen. The first given exploration case is (h, n=0°, d=0°). Dave and Axel observe what happens on the screen when they move the height slider (h) over and under zero again ($DO_{13\text{-}45}$). From that Axel draws three conclusions by single-deduction ($1D_{20.1}$, $1D_{20.2}$, and $1D_{44}$). More specifically, Axel

12 For the difference between a hypothesis and a claim, please refer to 5.4.2 and also Table 5.3.

says that the sphere[13] bounces in and out[14] (when varying the height, $1D_{20.1}$). He also says that the he can find out the radius of the sphere. He drags the height slider over zero, stops at h=1 and says that the radius is equal to one ($1D_{20.2}$). Axel does not explicitly explain how he got to this conclusion. In the exploration of the case (h, n=0°, d=0°) we can observe on the screen that at position (h=0, n=0°, d=0°) there is the biggest circular cross-section (see Figure 5.19).

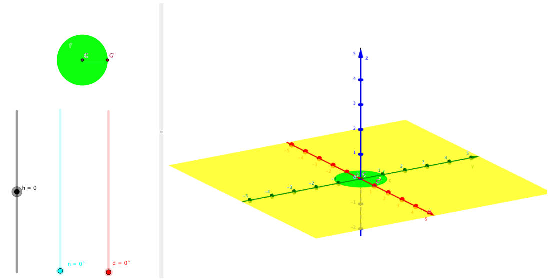


Figure 5.19: The cross-section at position (h=0, n=0°, d=0°) in Task 2 (invisible sphere)

When the h-slider is drugged either over or under h=0, the circular cross-sections get smaller (see Figures 5.20 a and b).

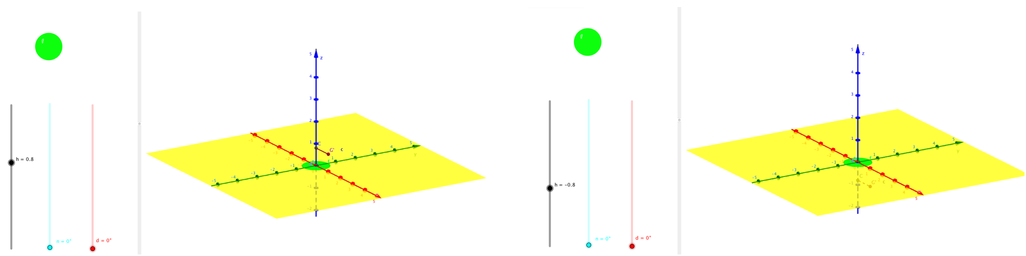


Figure 5.20: a (left) and b (right). Cross-sections at positions (h=0,8, n=0°, d=0°) and (h=-0,8, n=0°, d=0°) in Task 2 (invisible sphere)

The cross-sections are single points at h=1 (see Figure 5.21a) and h=-1, while they disappear completely over h=1 (see Figure 5.21b) and under h=-1. When he says that the radius is 1, he stops the dragging of the h-slider at position (h=1, n=0°, d=0°). Axel's last conclusion ($1D_{44}$) is that during the height exploration "[*the cross-section*] is always circular. Only the diameter [*of the cross-section*] becomes smaller, diminishes" (utterances 43-44, "Ja es ist immer kreisförmig. Nur der Durchmesser verkleinert sich, verringert sich").

[13] Axel misspeaks and says "circle" instead of sphere. This is a repeating misspeaking mistake that Axel does. I believe this to be only a mistake, because he has said the same sentence before, calling the object a sphere at that point (utterance 10).

[14] Here, Axel most probably refers to the plane of intersection xOy, through which the sphere "bounces in and out" when dragging the h-slider up and down.

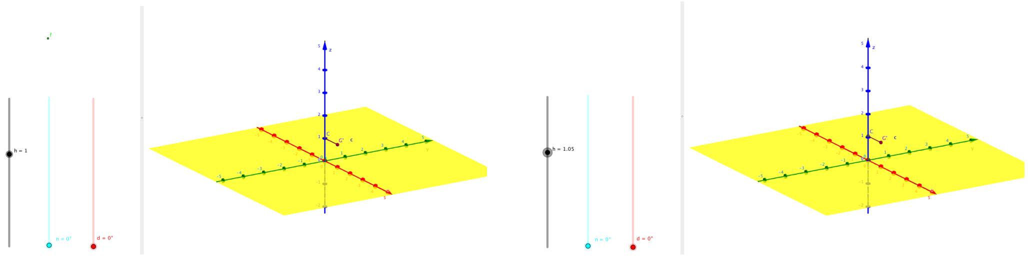
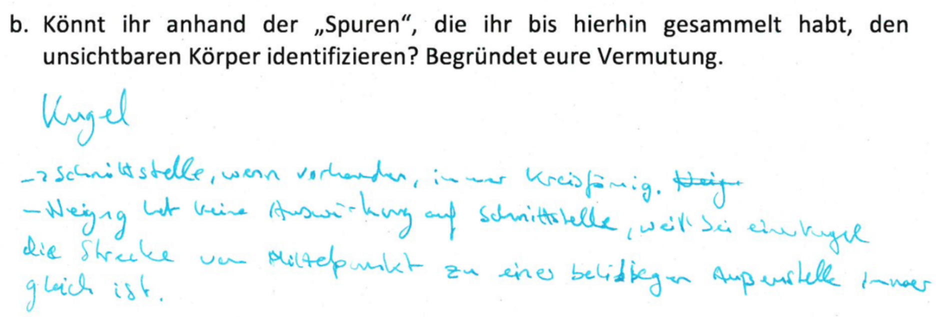


Figure 5.21: a (left) and b (right). Single-point cross-section at position (h=1, n=0°, d=0°) and absence of cross-section at (h=1,05, n=0°, d=0°) in Task 2 (invisible sphere)

Next, Axel and Dave explore two more given cases from the Exploration Matrix, namely (h=0,4, n, d=0°), which is a tilt exploration (n) and (h=0,4, n=0°, d), which is a spin exploration (d). This leads to the observation of data $DO_{46\text{-}62}$ (Figure 5.18). During the tilt exploration they observe what appears on the screen – the cross-section remains the same during the tilt rotation – and Axel only says "and then is.. because it is a sphere" (utterance 49, "und dann ist.. weil es eine Kugel ist"). The students do not discuss further on the matter. Dave writes down on their worksheet: "circular. it does not change when the n is moved" ("kreisformig. veraendert sich nicht beim Bewegen von n", see Appendix G1, Worksheet GR1AD-2, second row in the Exploration Matrix). During the spin exploration (h=0,4, n=0°, d), the two students agree that the cross-section does not change (utterances 51-61). During their guided exploration, Axel and Dave observe the data appearing on the screen ($D_{13\text{-}45}$ and $DO_{46\text{-}62}$, in Figure 5.18). They are looking at the results of each (h, n, d)-case, considering if they fit to their claim that the solid is a sphere ($Clai_8$). Since this seems to be so for all the given cases, Axel and Dave complete their exploration without questioning the validity of their claim anymore.

At the last step of their argumentation pattern, Axel and Dave turn the claim into their final conclusion ("This was a sphere", utterance 63 "Das war eine Kugel"), with a simple (one-step) deduction ($1D_{63\text{-}73}$, in Figure 5.18). Their argument is expressed both in written form, as well as orally while discussing it before they write it down. Figure 5.22 below shows Axel and Dave's original answer on their worksheet. It is followed by its translation in English.



b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Kugel

-> Schnittstelle, wenn vorhanden, immer kreisförmig. ~~Neig~~

- Neigung hat keine Auswirkung auf Schnittstelle, weil sie eine Kugel die Strecke von Mittelpunkt zu einer beliebigen Außenstelle immer gleich ist.

Figure 5.22: Axel and Dave's written justification in Task 2

Translation:

> b. Could you identify the invisible solid, based on the ”clues” that you have gathered until now? Justify your supposition.
>
> “*Sphere:*
>
> *-cross-section, when provided, is only circular*
>
> *-Tilt has no effect, because in a sphere the distance from the center to any outside location is always the same.*”

Axel and Dave say that the solid is a sphere (utterance 63, “Das war eine Kugel”) because all the possible cross-sections of the solid are circles (utterances 65-66.1, “Schnittstelle immer kreisförmig. – Ja, immer kreisförmig von 1 bis -1”) and varying the tilt (n) (for given h and d) does not influence the shape of the cross-section (utterances 66.1-66.2, “Neigung hat keine Auswirkung auf die Schnittstelle”), because in a sphere the segment connecting its middle point with any point of its surface is always of the same length (utterance 69, “.. weil bei eine Kugel der Radius von Mittelpunkt zu jedem Außenpunkt gleich ist”). Their simple deduction appears in their written justification, on their worksheet (see Appendix G1).

The method of *patterns of argumentation* helps us to observe the structure of students’ argumentation in the flow of time. It is interesting to observe Axel and Dave’s oral argumentation and their written justification. The written justification is much shorter than the oral argumentation. In their written answer (see Figure 5.22), the students have only used the elements that they consider absolutely necessary in order to justify their conclusion. Therefore, it is significant to analyze the whole of students’ argumentation process and not only their final answers, as they would only provide us with a very limited idea of their argumentation and would conceal the most fruitful and interesting parts of their work, both from a mathematical and a didactical perspective.

Next, I present the final level of the data analysis for Axel and Dave’s episode. In order to be able to observe the exact structure of students’ argumentations, as well as the role that visualization plays in them, we need to analyze them further. That is the purpose of the second argumentation analysis method (Knipping, 2003a, 2003b, 2008).

Level 3 Analysis – *Argumentation structure* and *visualization* in episode GR1AD-2

I begin this analysis level by identifying the argumentation elements in the transcript. First, I identify students’ hypotheses and claims and then I move on to the rest of the argumentation elements, such as data, warrants, conclusions etc. In this step, I also identify the indicators of visualization and its function in argumentation, including it in the local arguments as well. I then move on to the reconstruction of *local arguments* (one-step individual arguments, see more in subsection 5.4.2, Method # 2). Then, I link into *argumentation streams* all the *local arguments* that are logically linked together. The following step is the creation of the *argumentation structure*, where *argumentation streams* and other *local arguments* (that may stand alone) are brought together into a united argumentation, in the extent possible. At the end of the argumentation analysis process, I observe the functions of visualization in the argumentation, describing its role.

Contrary to Level 2 Analysis, here I apply this argumentation analysis method in order to reconstruct not only students' oral argumentations, but the arguments of their written justifications as well. In steps (a) to (d) in descriptions below, I use an example from an oral argumentation in order to explain the way I apply Knipping's (2003a, 2003b, 2008) argumentation analysis method. At the end of step (d), I also discuss the analysis of Axel and Dave's written justification in this episode.

a. Identifying hypotheses and claims

In this research the students are asked to explore situations in geometric tasks in which they need to identify an invisible solid by its visible two-dimensional cross-sections with a plane. Hence, the students have to work on an unknown and invisible solid and the focus of their argumentation lies on the *generation of hypotheses* regarding the form of the solid and the justification for their final conclusion. As a result, the main process of students' argumentation is abductive (see subsection 2.2.4 in Chapter 2). Because what the students do is to *explore* a new situation in which the conclusion is not pre-given (in contrast to processes of proving a theorem, e.g. in Knipping, 2003a, see also here Table 2.3 in 2.2.4), I consider Pease and Aberdein's (2011) way of reconstructing abductions the most suitable in the context of the present research (see Figure 5.23 and Table 2.3 in 2.2.4). Figure 5.23 shows how Pease and Aberdein (2011) model abductive arguments using Toulmin's (1958) terminology:

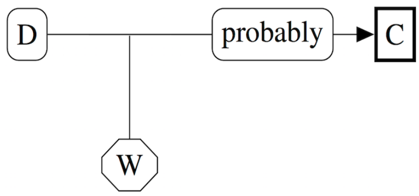
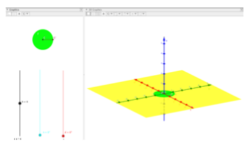


Figure 5.23: Pease and Aberdein's (2011) reconstruction of abduction

Given that the outcome of abduction is a supposition (or more suppositions) expressing the probable or possible conclusion of the argument, I will not immediately use the element of conclusion (C) as the outcome of an abductive argument. A statement will be referred to as a *conclusion*, only if the validity of the statement has been justified by the students. Before this happens, a supposition will be referred to either as a *hypothesis* (H) or as a *claim* (Cl). These two notions bear the same meaning in both argumentation analysis methods used here (Reid, 2002b, and Knipping, 2003a, 2003b, 2008), as tools in Level 1 Analysis and Level 2 Analysis (respectively)[15]. As I explain in Table 5.3 in subsection 5.4.2, there is a significant difference between a *hypothesis,* which is a supposition of a possible result, and a *claim,* which is a supposition of the most probable result.

[15] In Level 1 Analysis, the elements of a pattern of argumentation represent processes and actions that the students follow. I use the symbols "Hyp" and "Clai" to represent the processes of hypothesizing (stating a hypothesis) and claiming (stating a claim), respectively. In Level 2 Analysis, the elements of the arguments are not processes, rather statements. To make the distinction between processes and statements clear, I use slightly different symbols than in Level 1 Analysis in order to symbolize hypotheses and claims. In Level 2 Analysis a hypothesis is symbolized as "H" and a claim as "Cl".

Therefore, my first step is to identify as elements of their argumentation the initial hypotheses (H) and claims (Cl) made by the students (see Figure 5.24). The distinction between what makes a supposition a *hypothesis* and what makes it a *claim*, lies within the indicators provided by the students. Axel and Dave's attitude towards a supposition is an indicator that can be used in order to characterize it as a hypothesis or a claim. Judging by the oral expression of this attitude, I consider a supposition to be a claim when I identify a student's intention to support and prove this supposition. Some examples of this intention lie in expressions such as: "because the sphere comes out and so - pop" (utterance 10.2-3, see Figure 5.25), "look, this way you can almost see the sphere" (utterance 12.1, see digital Appendix H1), "has a radius of 1" (utterance 20.2, see digital Appendix H1), "lol, are you stupid? No, it's not a cone. No, it's not a cone. It is no cone. This is a sphere to 100% . What do we want to bet on?" (utterance 32, see digital Appendix H1). The powerfulness of these expressions reveals the amount of faith the students have in their claim. Especially the last utterance, although strong and perhaps slightly impolite, is an indicator of the epistemic value of the statement that the solid must be a sphere, making it a claim.

Utterance #		Transcript	Notes
1.1 1.2		Axel: Oh, der [*the solid*] ist unsichtbar. [*Axel performs the exploration (h, n=0°, d=0°). See pictures GR1AD-2_1 to GR1AD-2_3II*] #00:59:46-2#	 Picture GR1AD-2_1 – Cross-section at initial position (h, n=0°, d=0°)
2		Dave: Ja, die Stelle. (?) #00:59:51-5#	
3	**H_3**	Dave: **Ist ein Kegel**. (unverständlich)	
4.1 4.2	**H_4**	Axel: Oah, die Neigung ist awesome [*Exploration of the case (h=0, n, d=0°)*]. **Ist ein Kreis.** [*Pictures GR1AD-2_4I and GR1AD-2_4II*] #00:59:53-3#	
5.1 5.2		Dave: Ne, das ist ein Kegel. Ist ja ein dreidimensionaler Körper und das ist nur die Schnittstelle, weil der Körper unsichtbar ist. #01:00:02-5#	
6		Axel: Ja, und das ist ein Kreis. #01:00:03-0#	
7		Dave: Ja, die Schnittstelle ist ein Kreis. #01:00:05-4#	

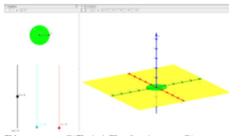
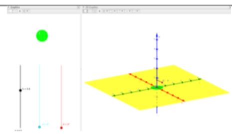
1

8	**Cl_8**	Axel: Ne, das ist eine Ku-, also **das ist ein Kugel insgesamt**. #01:00:08-2#	

Figure 5.24: Identification of initial hypotheses and claims – Episode GR1AD-2

b. Identifying other functional elements of argument

Next, I move on to the identification of other *functional elements of argument*, for the whole transcript. That includes identifying data (D), visual data (VD)[16], warrants (W), qualifiers (Q), backings (B) and conclusions (C)[17], as well as further hypotheses (H) and claims (Cl). Figure 5.25 shows the same transcript excerpt as Figure 5.24, including the identification of all the argumentation elements in this excerpt.

Utterance #		Transcript	Notes
1.1 1.2	**$D_{1.1}$** VD_{1-4}	Axel: Oh, **der [*the solid*] ist unsichtbar**. [*Axel performs the exploration (h, n=0°, d=0°). See pictures GR1AD-2_1 to GR1AD-2_3II*] #00:59:46-2#	 Picture GR1AD-2_1 – Cross-section at initial position (h, n=0°, d=0°)
2	VD_{1-4}	Dave: Ja, die Stelle. (?) #00:59:51-5#	
3	**H_3** VD_{1-4}	Dave: **Ist ein Kegel**. (unverständlich)	
4.1 4.2	**H_4** VD_{1-4}	Axel: Oah, die Neigung ist awesome [*Exploration of the case (h=0, n, d=0°)*]. **Ist ein Kreis.** [*Pictures GR1AD-2_4I and GR1AD-2_4II*] #00:59:53-3#	
5.1 5.2	*$D_{5.1}$* **$C_{5.2}$**	Dave: Ne, das ist ein Kegel. *Ist ja ein dreidimensionaler Körper* und **das ist nur die Schnittstelle**, weil der Körper unsichtbar ist. #01:00:02-5#	
6	VD_6	Axel: Ja, und das ist ein Kreis. #01:00:03-0#	
7	**C_7**	Dave: Ja, **die Schnittstelle ist ein Kreis**. #01:00:05-4#	
8	**Cl_8**	Axel: Ne, das ist eine Ku-, also **das ist ein Kugel insgesamt**. #01:00:08-2#	 Picture GR1AD-2_2I – Cross-section at position (h=0,8, n=0°, d=0°)
9		Dave: Sicher? #01:00:08-7#	
10.1 10.2 10.3	**$VD_{10.1}$** $VD_{10.2}$≡ $C_{10.2}$ **$W_{10.2-3}$**	Axel: Ja, normal, weil sonst- **wie würde denn sonst die Neigung- [*varying the n. He observes that the cross-section does not change*]**? (..) So [*varying the h*] wird es immer kleiner, **weil die Kugel raus geht** [see picture *GR1AD-2_10I*] **und so- bopp** [*a cross-section appears again and it is like the sphere is coming back again,* see picture *GR1AD-2_10II*]. *#01:00:19-1#*	
11.1 11.2	VD_{11}	Dave: Geht das in beide Richtungen [*for h<0 and for h>0. See pictures GR1AD-2_11I and GR1AD-2_11II*]? Nur so als Test. #01:00:26-2#	

1

Figure 5.25: Identification of argumentation elements – Episode GR1AD-2

This excerpt begins with the students observing what happens on the computer screen when Axel drags the height-slider up and down. The students gather visual data VD_{1-4}: When the h-slider is dragged over and under zero (utterances 1-3), the cross-sections are circular and they change size. When the n-slider (tilt) is dragged over zero degrees, while the height and the spin remain constant (utterance 4), the size of the circular cross-sections remains the same. Axel and Dave do not express any of this verbally. From the visual data VD_{1-4} follows Dave's hypothesis "It is a cone" (H_3) and Axel's hypothesis "It is a circle" (H_4). Using as a datum ($D_{5.1}$) the fact that the solid is a three-dimensional object, which is also invisible ($D_{1.1}$), Dave concludes that the two-dimensional shapes that they see are the cross-sections of the

16 Data that are visible on the computer screen and which emerge from the manipulation of the sliders.

17 For a reminder of the meaning of Toulmin's (1958) terminology, please refer to subsection 2.2.3 in Chapter 2.

solid ($C_{5.2}$). And since these shapes are circles (which is a visual datum VD_6), he concludes that the cross-section is a circle (C_7). Then Axel states a new claim, namely that the solid is a sphere (Cl_8). He supports his claim using the visual data ($VD_{10.1}$) that the size of the circular cross-sections does not change when dragging the tilt-slider. Axel then moves the height-slider up and on the screen appear diminishing circular cross-sections ($VD_{10.2}$). He explains this as the result (conclusion $C_{10.2}$) of the sphere moving up and out of the plain of intersection xOy and back in when the h-slider is dragged down again (warrant $W_{10.2\text{-}3}$). Dave then asks if "it goes on both directions" (utterance 11), Axel drags the h-slider both over and below zero and on the screen appear circular cross-sections both over and under zero (VD_{11}).

The rest of the transcript is also analyzed following this method (see Transcript GR1AD-2 in Digital Appendix H1).

c. Reconstructing local arguments

The next step is to link the identified elements of argument together creating *local arguments*. A local argument consists of at least one datum leading to a conclusion (C) or a supposition (claim (Cl) or hypothesis (H)), through the use of an explicit or implicit warrant (W) that supports their connection.

Such an argument is the one in Figure 5.26. In the diagrams of the local arguments (and argumentations streams and structures later on), I use the English translations of students' utterances. In cases where the utterance is too vague when seen out of the transcript context, I rephrase the utterance based on my interpretation of what the student refers to. For example, the utterance "der ist unsichtbar" (utterance 1, see Figure 5.25), which is translated "it is invisible" is quite vague out of the transcript context. But, from the rest of what Axel says this "it" can be interpreted as "the solid". Therefore, in the local argument the datum $D_{1.1}$ becomes "the solid is invisible".

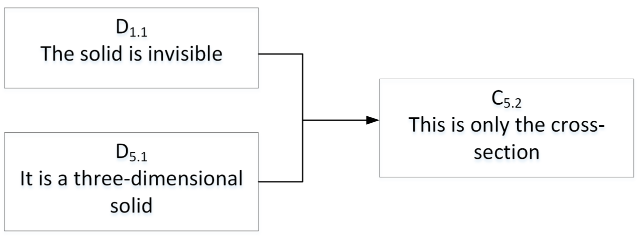


Figure 5.26: A local argument in Axel and Dave's oral argumentation – Episode GR1AD-2

Before this local argument, Axel has said that the solid is a circle (H_4, see Figure 5.25). In this argument of Figure 5.26, Dave moves from two data to a conclusion. More precisely, he uses the fact that the solid is invisible ($D_{1.1}$) and the fact that the solid is a three-dimensional object ($D_{5.1}$), to draw the conclusion ($C_{5.2}$) that the circle Axel refers to is only a cross-section (and not the solid).

I reconstruct in the same way all the (individual) local arguments occurring from the transcript.

d. Reconstructing argumentation streams and the argumentation structure

Here, I reconstruct both the oral argumentation as well as the written justification by Axel and Dave.

i. Oral argumentation

The next step in this analysis is to reconstruct *argumentation streams.* An *argumentation stream* is a chain of local arguments by which a goal statement is justified (Knipping, 2003a, 2003b). An argumentation stream may even consist of only one local argument, if the goal-conclusion is reached in a single argumentation step.

An example of an argumentation stream is shown in Figure 5.27. This argumentation stream consists of two local arguments. The first local argument is the one depicted in Figure 5.26 above. The conclusion $C_{5.2}$, that the geometric object observed on the screen is a cross-section, is now used as a datum (hence the symbolism $C_{5.2}$/D). Another visual datum is also used: “this [*the cross-section*] is a circle” (VD_6). The two data together lead to the conclusion that “the cross-section is a circle” (C_7). This conclusion completes the argumentation stream in Figure 5.27, which ultimately refutes the hypothesis that the solid is a circle (H_4) (see argumentation structure GR1AD-2 in Appendix E2 and in Figure 5.28).

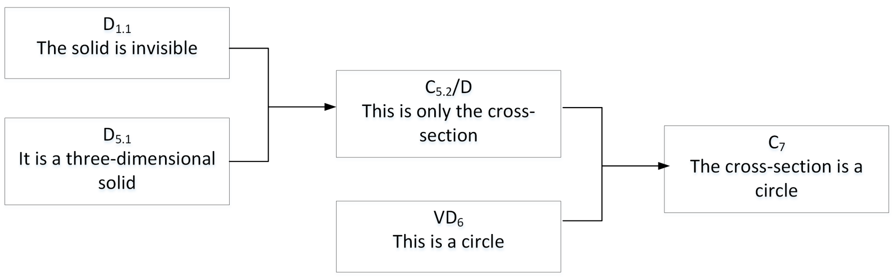


Figure 5.27: An argumentation stream in Axel and Dave’s oral argumentation – Episode GR1AD-2

The reconstruction of students’ argumentation moves from the identification of individual elements of argumentation, to (individual) local arguments, to argumentation streams and then to the “whole” *argumentation structure.* The set of all the argumentation streams of an episode constitutes the *argumentation structure* (see Figure 5.28). The argumentation structure of the episode GR1AD-2 can also be seen in Appendix E2. After the reconstruction of the oral argumentation stricture, I also analyze and reconstruct students’ written justification. Later, I talk more about the written justification (see ii. Written justification), where I compare the oral and the written argumentation structures.

Figure 5.28 below, shows the argumentation structure of Axel and Dave’s work on the sphere task (episode GR1AD-2).

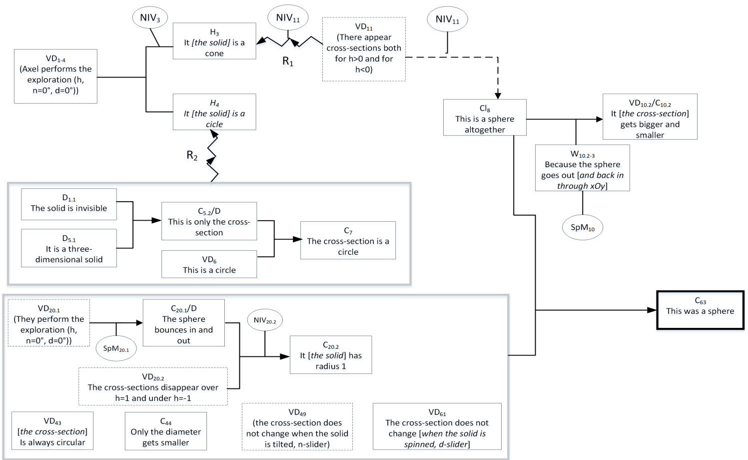


Figure 5.28: Global argumentation structure of the oral argumentation in episode GR1AD-2

I would now like to explain the argumentation structure in Figure 5.28. During their discussion, Axel and Dave start by exploring a situation, which is mostly unknown to them. The only thing they know for sure is that they are asked to identify the form of an invisible solid, which is hidden in a 3D coordinate system, in which only the cross-sections of the solid with the plane xOy are visible. Based on their observations, during their free exploration (utterances 1-12), the students start by gathering data. They then use abduction to create two hypotheses (H_3: cone and H_4: circle) about the possible forms of the invisible solid. Abductive arguments are "something, which looks as if it might be true and were true and which is capable of verification or refutation by comparison with facts" (as described by Peirce in CP 1.120 and mentioned in Reid, 2018, p.3). And that is exactly what they do by examining the situation further. Thus, later the two hypotheses are rejected by the refutations R_1 and R_2 (R_2 is a stream on its own) respectively. The two refutations lead the students to a new claim (Cl_8), that the solid is a sphere. Until now, the argumentation structure provides the reconstructed local arguments arising from students' free exploration (utterances 1-12). Next the students carry on with their guided explorations (utterances 13-62).

After having established their claim, Axel and Dave engage in the "Test" process (see $Test_{10\text{-}12}$ and $Test_{11}$ in the *pattern of argumentation* in Figure 5.18 in Level 2 Analysis). They they carry on with a guided exploration[18] (see GE in Figure 5.28), discussing what happens with the cross-section in these positions. Here, Axel and Dave name some more data they observe without connecting them with their previous arguments. Therefore, some data stand alone in the argumentation. In this phase, the students simply examined all the cases in the Exploration Matrix and verified that the results of all of them agree with their claim (Cl_8: the solid is a sphere). After having found no case that gives a contradictory outcome they conclude that the invisible solid is a sphere ($C_{49/63}$).

The first part of the argumentation leads to the claim (Cl_8) that "the solid is a sphere". The same statement "the solid is a sphere", is also the final outcome of the whole oral argumentation, now having the status of a conclusion ($C_{49/63}$). Although the

18 Exploration based on the positions and cases provided by the Exploration Matrix in the task worksheet.

statement sounds the same, the difference is that in the first case it has the epistemic value of a *claim,* while in the second one it has the epistemic value of a *conclusion*. This change of epistemic value from a *claim* to a *conclusion*, is the result of students' process of gathering data through guided exploration and using them to verify and justify their claim through argumentation.

ii. Written justification

In their written justification (see Figure 5.22) Axel and Dave write that the solid is a sphere, giving two reasons for this: 1. because all the cross-sections of the solid are circles, and 2. because when varying the tilt (n-slider), keeping the height and the spin constant, the shape of the cross-section does not change. They explain their second reason, by saying that in a sphere the segment connecting its center with any point of its surface is always of the same length. I have reconstructed this justification as seen in Figure 5.29.

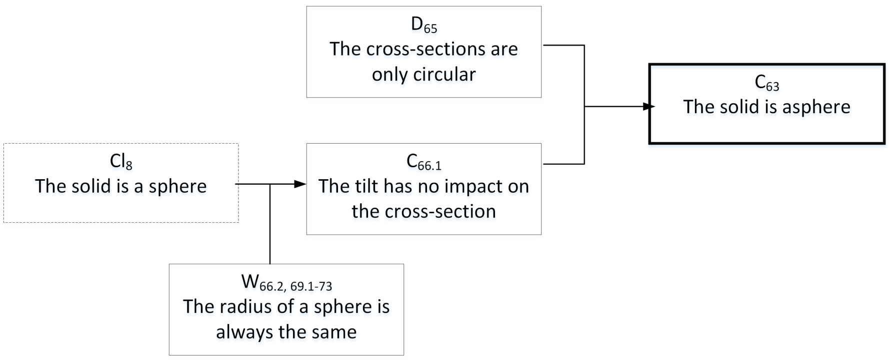


Figure 5.29: Argumentation structure of the written justification in episode GR1AD-2

Axel and Dave's justification is based on deduction. In this deductive argument, the fact that "the cross-sections are always circular" is a datum (D_{65}) and the statement that "varying the tilt has no impact to the shape of the cross-section" is a conclusion ($C_{66.1}$). This conclusion arises from a local argument with implicit data and an explicit warrant. More precisely, I argue that Axel and Dave implicitly use the *visual datum* VD_{49} that nothing changes when the tilt is changed, together with the *claim* Cl_8 that the solid is a sphere, and from those two they draw the conclusion $C_{66.1}$ that the change of tilt has no impact on the shape of the cross-sections. They explicitly support this step by calling on warrant $W_{66.2/69}$, that the radius of a sphere is always the same. They then use their conclusion $C_{66.1}$ as a datum, together with datum D_{65} in order to draw their final conclusion C_{63} that the solid is a sphere. The fact that they express their justification the other way around – starting by naming their conclusion first, before they give their reasons – does not change the fact that the nature of their argument is deductive.

iii. Comparing the oral and the written argumentation structures

As with the patterns of argumentation, the differences between Axel and Dave's oral argumentation and written justification are also reflected in their argumentation structures[19]. The reconstructed argumentation structure for the written justification is much simpler than the one for their oral argumentation. There are some important structural as well as content-relevant differences between the oral and the written argumentation structures.

Already from a first glance at the two argumentation structures, one can see that the oral argumentation structure is much richer in local arguments compared to the written argumentation structure, with more complex connections. On the contrary, the argumentation steps in the written argumentation structure do not contain multiple intertwined local arguments. Furthermore, the oral argumentation structure includes different types of arguments (both abductive and deductive), instead of only deductive arguments. In the oral argumentation, Axel and Dave gather data and use abduction in order to shape initial hypotheses (H_3 and H_4) and claims (Cl_8). Using further data, they then build arguments in order to refute or verify them. At the end, all the non-contradictory arguments lead to the verification of the claim and to the final conclusion ($C_{49/63}$). On the contrary, in the written justification, the students use only the elements of their oral argumentation that they consider necessary, in order to justify their conclusion deductively.

Consequently, there is lots of interesting information that is lost, if we only examine students' written argumentation. But it is not only information that vanishes, but also processes: the process of the creation of a hypothesis is gone (abductive arguments), and the use of visualization is absent and untraceable (see part (e) below). Nevertheless, the "minimalism" of written justification shows us the mathematical principles and elements that the students consider necessary and sufficient in order to justify their answer, which is another important process that we want our students to acquire when learning mathematics.

Hence, I consider it very important to look at both argumentation processes (the oral as well as the written), in order to form a more accurate view of students' argumentations.

e. Identifying and explaining the roles of visualization and spatial manipulation in students' argumentation

What is particularly interesting to me at this stage of the analysis, is the nature of statements such as: "So wird es immer kleiner, weil die Kugel raus geht und so-bopp" ("So it gets smaller and smaller [*the cross-section*] because the sphere goes out and so pop", Axel in utterance 10.2-3, see Figure 5.25), "Guck Mal, so kannst du schon die Kugel sehen. Fast." ("Look, this way you can see the sphere. Almost." in utterance 12.1), "der Kreis bouncst rein und raus" ("the circle [*meaning sphere*] bounces in and out", Axel in utterance 20.1), "Hat einen Radius von 1" ("[*the sphere*] has a radius of 1" in utterance 20.2). These expressions have a shared characteristic and they act as indicators of students' use of visualization and spatial manipulation in their argumentation. Let us, for example, consider the argumentation stream in Figure 5.30.

19 For reasons of brevity, I will from now on refer to the argumentation structure for the students' oral argumentation as "oral argumentation structure" and to the argumentation structure for their written justification as "written argumentation structure".

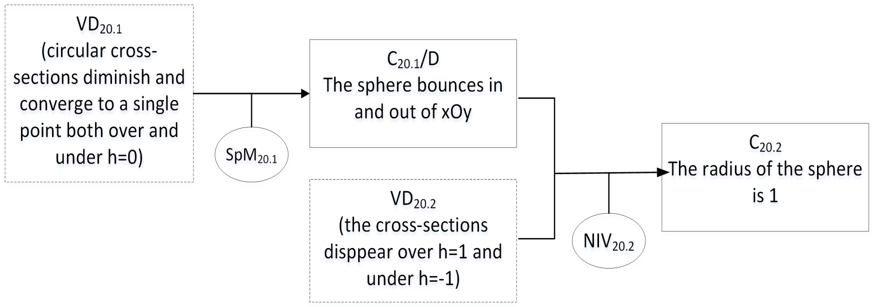


Figure 5.30: *Non-iconic visualization* and *spatial manipulation* in the argumentation stream

In the argumentation stream shown in Figure 5.30, there appear two elements ($SpM_{20.1}$ and $NIV_{20.2}$) that are different from the argumentation elements mentioned by Toulmin (1958) or Knipping (2003a, 2003b, 2008). These are *spatial manipulation* $SpM_{20.1}$ and *non-iconic visualization* $NIV_{20.2}$. As I have already mentioned before, *spatial manipulation* and *non-iconic visualization* are the two processes, whose function and role in students' argumentation I am interested to examine.

In subsection 5.4.3 (see also Chapter 8, Introduction), I explain that for the identification of *non-iconic visualization*, I examine how students use the properties of the cross-sections, whether they identify figural units of the solid and how they related these figural units with the solid itself. This is why I consider students' use of *dimensional deconstruction* as an indicator of *non-iconic visualization* (see also subsections 3.1.1 and 3.2.1). For the identification of *spatial manipulation* in students' argumentation, I examine their verbal description about the movement and orientation of the invisible solid in space, as well as the accompanying gestures or metaphors they employ.

In Figure 5.30, the argumentation stream begins with an implicit visual datum ($VD_{20.1}$), which consists of what Axel and Dave observe on the screen during the height-exploration (-4<h<4, n=0°, d=0°). What is shown on the screen, are circular cross-sections that diminish when the height-slider is dragged over or under h=0 (see Figures 5.31a, b and c). The cross-sections become single points for the values h=1 and h=-1, while over h=1 and under h=-1 the cross-sections disappear completely (see Figures 5.32a and b).

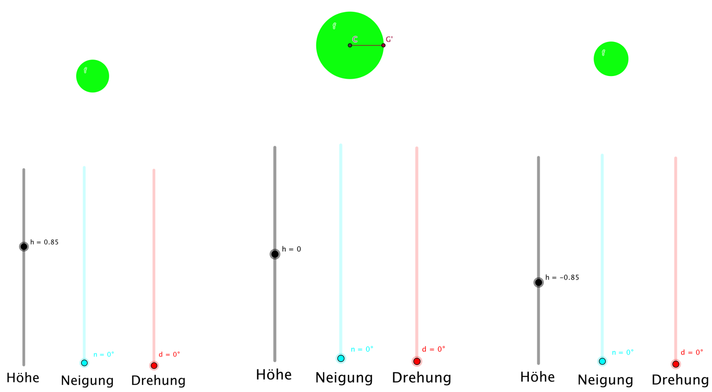


Figure 5.31: a (left), b (middle) and c (right). The circular cross-sections at positions (h=0,85, n=0°, d=0°), (h=0, n=0°, d=0°) and (h=-0,85, n=0°, d=0°), respectively

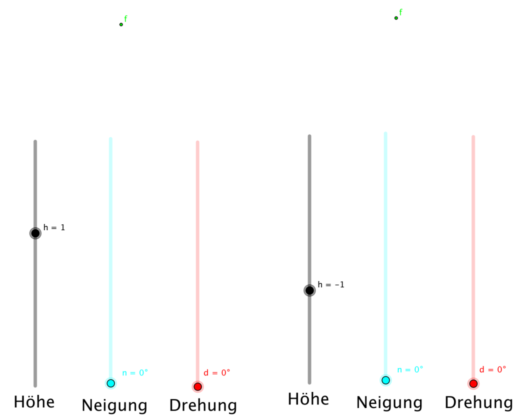


Figure 5.32: a (left) and b (right). The single-point cross-sections at positions (h=1, n=0°, d=0°) and (h=-1, n=0°, d=0°)

Transcript 5.1 shows the part of Axel and Dave's discussion that corresponds to the argumentation stream in Figure 5.30.

Utterance	Codes	Original German transcript	English translation
20	$VD_{20.1}$ **$C_{20.1}$** $VD_{20.2}$ **$C_{20.2}$**	Axel: Oh. Ja, **der Kreis** [*Axel misspeaks. He means sphere*] **bouncst rein und raus. Bounce, bounce!** Ah guck mal, man kann wieder den Radius bestimmen [*the radius of the sphere*]. **Hat einen Radius von eins** [*Axel stops the height slider at h=1, at position (h=1, n=0°, d=0°)*].#01:02:01-1#	Axel: Oh. Yes, **the circle** [*Axel misspeaks. He means sphere*] **bounces in and out. Bounce, bounce!** Oh, look, you can determine the radius [*the radius of the sphere*]. **Its radius is one** [*Axel stops the height slider at h=1, at position (h=1, n=0°, d=0°)*].

Transcript 5.1: Axel's statement after the height-exploration (-4<h<4, n=0°, d=0°) during pair-work with Dave on Task 2 (invisible sphere)

As Dave drags the height-slider up and down, Axel says that the sphere (he misspeaks saying "circle") "bounces in and out" (utterance 20, Transcript 5.1). By that I believe that he refers to the sphere is passing in and out through the plane of intersection xOy. Axel uses this metaphor of a sphere "bouncing", in order to describe his reasoning. He does not really see a sphere bouncing in an out of anywhere, since during the height-exploration, what is visible on the screen are only circular cross-sections diminishing and getting bigger. Axel's metaphor indicates that he has created a mental image (Presmeg, 2006; see also Chapter 3, subsection 3.1.3) of a sphere. Furthermore, since his metaphor is referring to a movement performed by the sphere ("bouncing" is not a static condition), it also indicates that he can manipulate his mental image and imagine it moving in space. Based on his metaphor, I argue that he imagines the sphere moving vertically above, through and under the xOy plane, while the height-slider is dragged up and down. Therefore, I argue that Axel employs his *spatial manipulation* ($SpM_{20.1}$) in order to explain what he sees on the screen.

I would now like to discuss the *function* and the *role* of $SpM_{20.1}$ in the first reconstructed argument of the stream (see Figure 5.30). The element $SpM_{20.1}$

functions as a warrant, linking an implicit visual datum ($VD_{20.1}$: the circular cross-sections of the solid with the plane xOy diminish until they converge to single point both over and under h=0) to a conclusion ($C_{20.1}$: the sphere bounces in and out of plane xOy). $SpM_{20.1}$ plays a double role in the argument; it helps Axel to draw a conclusion ($C_{20.1}$), while at the same time it also helps him to explain the visual data he has observed on the screen ($VD_{20.1}$).

Next, I analyze the function and role of *non-iconic visualization* $NIV_{20.2}$ in the argumentation. In the second argumentation step of the stream (see Figure 5.30), the previous conclusion $C_{20.1}$ is used as a datum ($D_{20.1}$), together with the new visual datum $VD_{20.2}$, leading to a new conclusion, namely $C_{20.2}$. More precisely, after having drawn the conclusion that the sphere bounces in and out ($C_{20.1}$), Axel stops the height-slider at h=1. At this moment, what appears on the screen is a single point of intersection between the invisible solid and the plane xOy (see Figure 5.32a). This happens while Axel says "Oh, look, you can determine the radius" (Transcript 5.1). Axel then says that the radius of the sphere "is one". Therefore, the visual data that appear on the computer screen (single point of intersection at h=1) come into the stream as $VD_{20.2}$. From the conclusion $C_{20.1}$ (now used as a datum $D_{20.1}$) and the new visual data $VD_{20.2}$, Axel draws the final conclusion $C_{20.2}$ of the stream, that the radius of the sphere is "one".

When Axel says "Oh, look, you can determine the radius" (Transcript 5.1), he seems to observe that the cross-sections disappear when he drags the height-slider over h=1 and under h=-1 ($VD_{20.2}$ in Figure 5.30) and from that he draws the conclusion $C_{20.2}$ that the radius of the sphere is one. Here, Axel moves from thinking about the solid (sphere), to seeing the cross-sections as its two-dimensional (2D) figural units. Axel determines the radius of the sphere by the values of the height-slider above (h>1) and under (h<-1) which the cross-sections disappear. This transition from the solid to its height is a transition from the three-dimensional sphere (or at least its mental image) to an one-dimensional figural unit of it, namely its radius. This process involves both the *dimensional deconstruction* of the sphere into its cross-sections (2D figural units), as well as the use of a property of its radius (1D figural unit), even if that is done implicitly. This property could for example be, that the radius of a sphere is the distance from the center of the sphere, which coincides with the center of its biggest circular cross-section, to the circumference of the sphere, which here would be the point at which the cross-section converges to a single point (this being for h=1 and h=-1). Therefore, I argue that the transition from $D_{20.1}$ (the sphere bounces in and out of xOy when the height-slider is dragged over and under zero) and $VD_{20.2}$ (the cross-sections disappear over h=1 and under h=-1) to $C_{20.2}$ (the radius of the sphere is one) is supported by Axel's use of *non-iconic visualization* ($NIV_{20.2}$).

In this episode *dimensional deconstruction* and the relations of properties between the solid and its figural units of lower dimension, are indicators of the use of *non-iconic visualization* by Axel. The function of $NIV_{20.2}$ in the argument is that of a warrant, through which a conclusion is linked to previously attained data. The role of $NIV_{20.2}$ is to help Axel draw a new conclusion.

In conclusion, I can point out one *function* and two different *roles* that *spatial manipulation* and *non-iconic visualization* have played in students' argumentation in the argumentation stream shown in Figure 5.30. Both *spatial manipulation* and *non-conic visualization* function in AS-5 as warrants in the argumentation. Spatial manipulation $SpM_{20.1}$ has played both the role of *explaining a visual datum* ($VD_{20.1}$), as well as *drawing a conclusion* ($C_{20.1}$). Non-iconic visualization $NIV_{20.2}$ has also played the role of *drawing a conclusion* ($C_{20.2}$).

As can be see in the argumentation structure of the episode in Figure 5.30, there are more cases in which Axel and Dave employ their spatial manipulation and non-iconic visualization in their argumentation. I analyze and present these cases in Chapter 8, together with examples from other episodes. In the stream presented in Figure 5.30 we only saw cases in which spatial manipulation and non-iconic visualization function as warrants. These warrants are used not only in order to move from a datum to a conclusion, but also in order to explain a phenomenon. I elaborate more on further functions and roles of these two processes in Chapter 8.

5.5 Epilogue

In this chapter, I have presented both the methods and the methodologies that underline the design of my research, the interpretation of the study and the collection of the data, as well as the levels of the data analysis.

I have also presented all the steps of the data-analysis methodology I follow, through a specific example, using one of the episodes of the study in which two students (Dave and Axel) work together on the task of the invisible sphere.

The results from the whole data-analysis are presented in the next three chapters, based on the three data-analysis levels. In Chapter 6 are presented the results of Level 1 Analysis and the Exploration strategies that the students followed while working on the tasks. Chapter 7 discusses the results of Level 2 Analysis, which are students' *patterns of argumentation.* Finally, in Chapter 8 I present the results of Level 3 Analysis, focusing particularly on the discussion of the functions and roles on non-iconic visualization and spatial manipulation in students' argumentations.

6 Students' exploration strategies

In this chapter I use the results of my Level 1 Analysis (structures of the episodes, see Appendix C), in order to answer my first research question:

1. What *exploration strategies* do the students follow using the Dynamic Geometry Environment when they work on the given tasks?

By *exploration strategies*, I refer to the way in which the students approach the tasks that they are given, in terms of the way they use the DGE[1] (GeoGebra 5). These strategies provide me with a first impression of how the students work on these explorative geometric tasks. The use of the word "strategy" here, does not bear the implication of a pre-decided set of exploration-steps. The *exploration strategies* refer only to the decisions the students make regarding the use of the DGE, both during their pair-work on the tasks as well as during the classroom-discussions.

In the Level 1 Analysis, I observed three types of *exploration strategies* followed by the students. I categorize these strategies based on two factors: whether or not there is *initiative*, on the part of the students, for the choice of the (h, n, d)-cases and positions to be explored, and whether or not there is a specific *intention* behind the initiative. I name the three categories I observed: *free exploration*, *guided exploration*, and *structured exploration*. In each of the following sections, I present the three *exploration strategies* one-by-one.

6.1 Free exploration strategy

Free exploration is used to indicate an exploration during which the students examine a situation, using the three GeoGebra sliders (h, n, and d) and moving the invisible solid in space, without expressing a specific plan upon which they are acting. This type of exploration strategy is characterized by *initiative* taken by the students, while at the same time lacking specific *intention* or expectation for the performed action. That means, that a specific case or position is explored in order to "see what will happen", without having any expectation about the outcome.

This type of exploration was used only by one group of students (group 1), specifically Axel and Dave (see codes starting with GR1AD, in Table 6.1). They used it in the first two tasks they worked on. The first one was Task 2 – Invisible sphere (episode GR1AD-2 in Table 6.1), on the first day of the study. On the second day, Axel and Dave worked on three more tasks. In their first task for the day, which was Task 3C – Invisible cube (episode HGR1AD-3C.I in Table 6.1), they also perform free exploration to some extent. Following, are two examples of *free exploration* from the aforementioned episodes.

[1] Dynamic Geometry Environment

6.1.1 Free exploration in Axel and Dave's work on the invisible sphere (episode GR1AD-2)

Axel and Dave use *free exploration* at the beginning of their work (utterances 1-12), and *guided exploration* later on (from utterance 13 on) until the end of their oral argumentation (see Figure 6.1). During their *free exploration*, Dave and Axel move the height-slider (h) and the tilt-slider (n) up and down, watching the changes in the cross-sections that appear on the screen (see Transcript in Figure 5.15). During the height-exploration (h, n=0°, d=0°), the tilt and spin of the solid are at zero degrees, while the height varies from h=0 to h=-4 and then back to h=0 until h=4. On the screen circular cross-sections appear (see Figures 6.1a, b and c). At (h=0, n=0°, d=0°) there is a circle, which diminishes as the height is dragged below zero until at some point it disappears. The same occurs when the h-slider is dragged from h=0 upwards. During the tilt-exploration (h=0, n, d=0°), the height is at zero and the spin is at zero degrees, while the tilt varies from n=0° to n=360°. On the screen appears a circular cross-section appears that does not change during the tilt-exploration (see Figures 6.2a and b[2]).

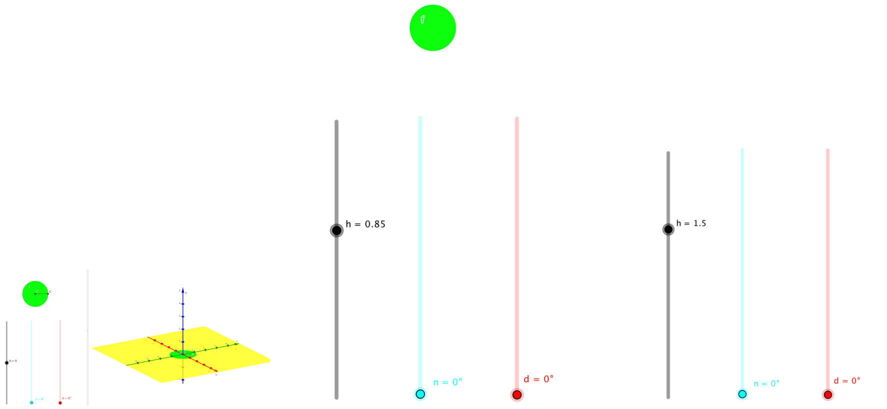


Figure 6.1: a (left), b (middle) and c (right). Task 2 (invisible sphere) – Cross-sections at positions (h=0, n=0°, d=0°), (h=0,85, n=0°, d=0°) and (h=1,5, n=0°, d=0°)

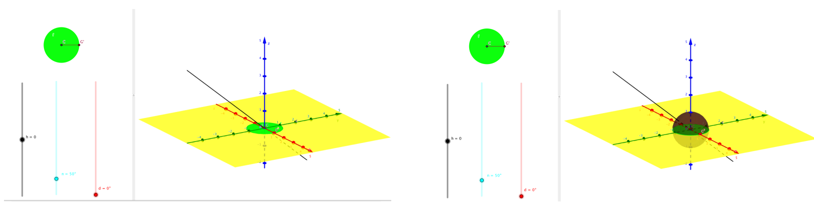


Figure 6.2: a (left) and b (right). Task 2 (invisible sphere) – Cross-section at position (h=0, n=50°, d=0°) and visible sphere at position (h=0, n=50°, d=0°)

This exploration results in the creation of two hypotheses (H_3: the solid is a cone, and H_4: it is a circle), both of which are then refuted by the students. The refutation of the second hypothesis, leads them to a new claim (Cl_8: the solid is a sphere). So,

2 In Figure 6.2b, the solid is visible for the purposes of demonstration of the analysis. The solids were invisible in all tasks during students' work.

free exploration in this case has helped Axel and Dave to gather information, create hypotheses and a claim, and refute one of the hypotheses.

Next follows another example of the use of the *free exploration* strategy by Axel and Dave.

6.1.2 Free exploration in Axel and Dave's work on the invisible cube (episode GR1AD-3C.I)

In this episode, Axel and Dave use *free exploration* for longer than in the previous episode (utterances 1-76). Then they continue the rest of their work with *guided exploration* (utterances 77-262) and *structured exploration* (utterances 227-239). Here I only discuss the part of their work where they employ *free exploration.*

Before they start with their explorations, Axel and Dave make two hypotheses about the form of the solid, based only on the picture from their worksheet (see Figure 6.3). They say that the solid may be a pyramid (hypothesis H_3) or a cube (H_5).

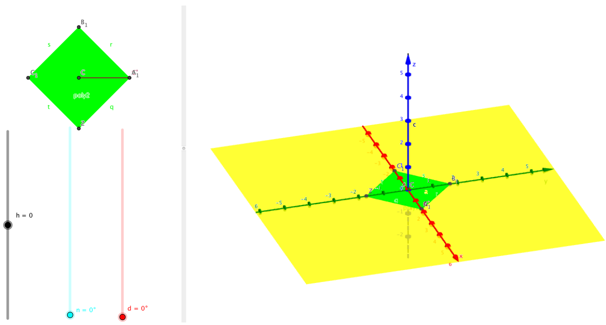


Figure 6.3: Task 3C (invisible cube) – Cross-section at position (h=0, n=0°, d=0°)

Then Dave begins a *free exploration.* He performs the height-exploration (h, n=0°, d=0°), dragging the height-slider under and over zero. Dave then states a new hypothesis, that the solid could also be a cuboid (H_9). He then performs a tilt exploration (h=-0,8, n, d=0°) and says that the solid could also be a prism (H_{14}).

At this point Dave and Axel do not have a specific plan about the way in which they explore the situation in the task. Later, they also move the spin-slider (d). Then, they continue just dragging all the sliders and observing what happens, without expressing any expectations from their actions.

By the end of their *free exploration* they have rejected the pyramid-hypothesis (H_3) and the prism hypothesis (H_{14}). They also re-state the cuboid-supposition this time as a claim (Cl_{28})[3]. Next, the students continue their *free exploration*, as a process of testing the validity of their claim. Observing multiple (h, n, d)-cases they check whether their observations confirm or contrast their claim, like they did in Task 2. But this is not an intention they explicitly express. Therefore, there is *initiative* in their actions, but still no specific intention to their action or expectation for the outcome. Dave and Axel simply use their observations in order to learn more about the solid and gather

[3] The claim in this case has not been expressed explicitly, because of the interruption of Dave's syllogism by Axel's affirmative response to what Dave was meaning to say. But, it was easy to suspect what Dave meant to say by the further flow of their discussion.

information. They also draw conclusions, both about the cross-sections as well as about the solid. For example, they conclude that the solid in its initial position (h=0, n=0°, d=0°)), is placed in the three-dimensional Cartesian coordinate system with its lowest point on plane xOy.

To sum up, at the start of their work the *free exploration* was quite fruitful for Axel and Dave. It aided them to gather information and generate hypotheses and claims, as well as to reject some of them. Nevertheless, this strategy became quite overwhelming for the students at some point (see utterance 77 from the transcript of the episode):

Axel: "Das ist voll kompliziert irgendwie. Lass erstmal die Aufgabe machen. n null, d null, erkunde die Werte für h zwischen minus vier und vier."

Axel says here: "It's kind of complicated. Let's do the exercise first. n zero, d zero, explore the values for h between minus four and four."

He refers to the task and the results from their free exploration, and he suggests that they continue by exploring the cases provided in the Exploration Matrix on the worksheet, to which he refers as "the exercise". Then he reads the first case to be explored, namely (-4<h<4, n=0°, d=0°). This phrase by Axel reveals, that the *free exploration*, although helpful and fruitful at the beginning, it may also have become a bit overwhelming.

6.1.3 Epilogue

The two examples presented in this section, is shown how the use of *free exploration* can both aid students' work and argumentation, as well as overwhelm them if used for too long. *Free exploration* helps students gather information about the situation they work in, generate hypotheses and claims, as well as refute some of these hypotheses and claims (see Table 6.1).

Table 6.1 shows the episodes in which students use *free exploration* and the processes in which this type of exploration has supported them.

I believe that the feeling of being overwhelmed by the gathered information may be caused by a lack of specific *intention*, on the students' side, for the actions they perform. Although their actions, when employing *free exploration*, are characterized by taking *initiative*, these actions still lack *intention*. The students explore a case without having an expected outcome in mind, and not with the intention to test if something they may have in mind will actually occur. They simply pick a (h, n, d)-case to explore and see what happens on the screen. Without a specific aim towards which they want to work, the plethora of information offered by a *free exploration* can become meaningless and impossible to use. It is the lacking *intention* that can provide them with actual control over their actions, and it is what is still missing from this type of exploration strategy.

Episodes	Uses of *free exploration*		
	Gather information	Testing validity, verify, refute hypothesis/claim	Create hypothesis/claim
Axel and Dave Task 2 – invisible sphere GR1AD-2	✓	✓	✓
Axel and Dave Task 3C – invisible cube GR1AD-3C.I	✓	✓	✓
Axel and Dave Task 3B – invisible pyramid GR1AD-3B.II			
Axel and Dave Task 3A – invisible cone GR1AD-3A.III			
Tom and Lukas Task 2 – invisible sphere GR2TL-2			
Tom and Lukas Task 3A – invisible cone GR2TL-3A.I			

Table 6.1: *Free exploration* in pair-work episodes

Another interesting result is that students only used *free exploration* during their pair-works. As is shown in the following sections, during the classroom-discussions the students employ *guided* and *structured explorations*.

6.2 Guided exploration strategy

I refer to *guided exploration* as the type of exploration strategy that the students follow when they choose to explore a task by using the given (h, n, d)-cases and positions provided in the Exploration Matrix in the worksheet. Firstly, I present the results from students' pair-works (see subsection 6.2.1) followed by the results from the classroom-discussions (see subsection 6.2.2).

6.2.1 Guided exploration in pair-work

Both groups, whose work has been analyzed in this study, have used this type of exploration during their pair work. Axel and Dave (Group 1) use this *guided exploration* both partially, as part of their overall explorations (in Task 2 and Task 3C), as well as exclusively, during the whole exploration phase (in Tasks 3B and 3A). Tom and Lukas (Group 2) use *guided exploration* exclusively, in Task 2, and partially in a combination of multiple strategies in Task 3A.

The cases in this category are more than those found in *free exploration.* Therefore, I present already at the beginning of this subsection the Table 6.2, which shows the different uses of *guided exploration* in students' argumentations during pair-work, and which of them appear in each episode (marked with ✓). Under the table, follow examples of episodes in which the uses of *guided exploration* appear.

Episodes	Uses of *guided exploration*				
	Organize the exploration	Gather information	Generate hypothesis/claim	Testing validity, verify, refute hypothesis/claim	Draw final Conclusion
Axel and Dave Task 2 – invisible sphere GR1AD-2		✓		✓	✓
Axel and Dave Task 3C – invisible cube GR1AD-3C.I	✓	✓		✓	✓
Axel and Dave Task 3B – invisible pyramid GR1AD-3B.II		✓	✓	✓	✓
Axel and Dave Task 3A – invisible cone GR1AD-3A.III		✓	✓	✓	✓
Tom and Lukas Task 2 – invisible sphere GR2TL-2		✓	✓	✓	✓
Tom and Lukas Task 3A – invisible cone GR2TL-3A.I		✓	✓	✓	✓

Table 6.2: *Guided exploration* in pair-work episodes

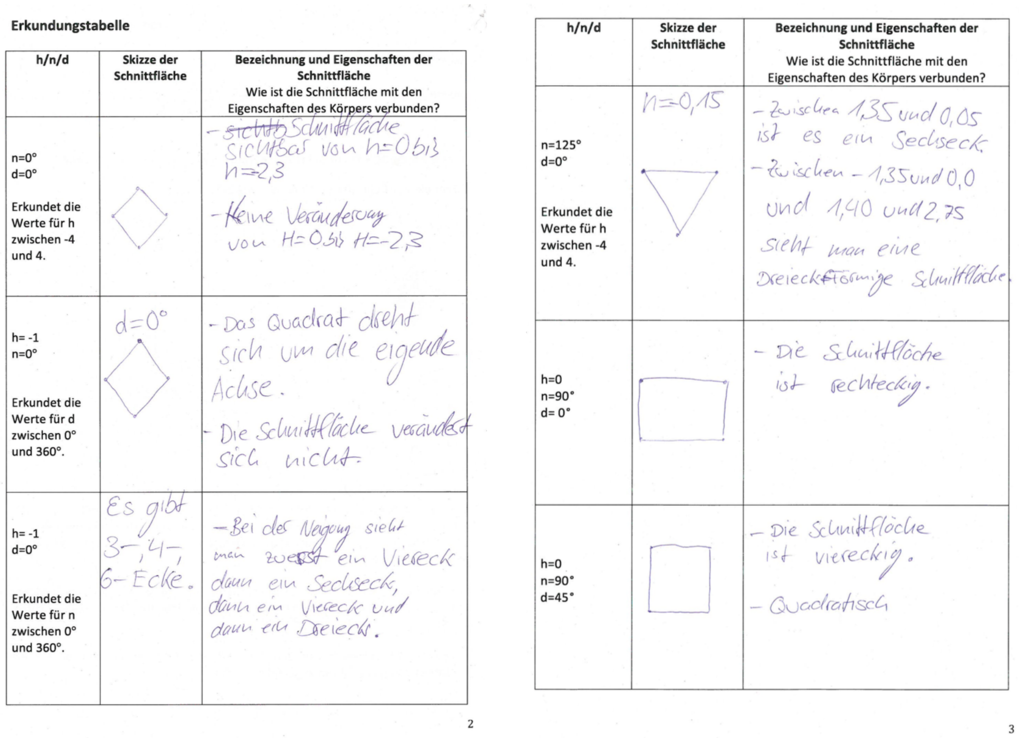
6.2.1.1 Partial use of guided exploration in Axel and Dave's work on the invisible cube (episode GR1AD-3C.I)[4]

As I mentioned in subsection 6.1.2, Axel and Dave start their work on the task of the invisible cube (Task 3C) with *free exploration.* Although their exploration is fruitful in the beginning, leading them to create hypotheses and check their validity (see 6.1.2), it later becomes quite overwhelming. This is mainly due to the large amount

[4] See the transcript of the episode in the Digital Appendix H4.

of new information that Axel and Dave do not seem to know how to make use of. Axel explicitly expresses his confusion in utterance 77 ("*It's kind of complicated*", see subsection 6.1.2), and suggests turning to the examination of the given cases in the Exploration Matrix.

Therefore, at this point Axel and Dave stop their *free explorations* and continue their work performing *guided exploration*, examining the cases that are provided in the Exploration Matrix (see Figures 6.4a and b).



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		- ~~sichtb~~ Schnittfläche sichtbar von h=0 bis h=2,3 - Keine Veränderung von H=0 bis H=-2,3
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.	d=0°	- Das Quadrat dreht sich um die eigende Achse. - Die Schnittfläche verändert sich nicht.
h= -1 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Es gibt 3-, 4-, 6-Ecke.	- Bei der Neigung sieht man zuerst ein Viereck dann ein Sechseck, dann ein Viereck und dann ein Dreieck.

2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=125° d=0° Erkundet die Werte für h zwischen -4 und 4.	h=-0,15	- Zwischen 1,35 und 0,05 ist es ein Sechseck. - Zwischen -1,35 und 0,0 und 1,40 und 2,75 sieht man eine Dreieckförmige Schnittfläche.
h=0 n=90° d= 0°		- Die Schnittfläche ist rechteckig.
h=0 n=90° d=45°		- Die Schnittfläche ist viereckig. - Quadratisch

3

Figure 6.4: a (left) and b (right). Exploration Matrix of Task 3C – Dave and Axel's notes during their guided exploration

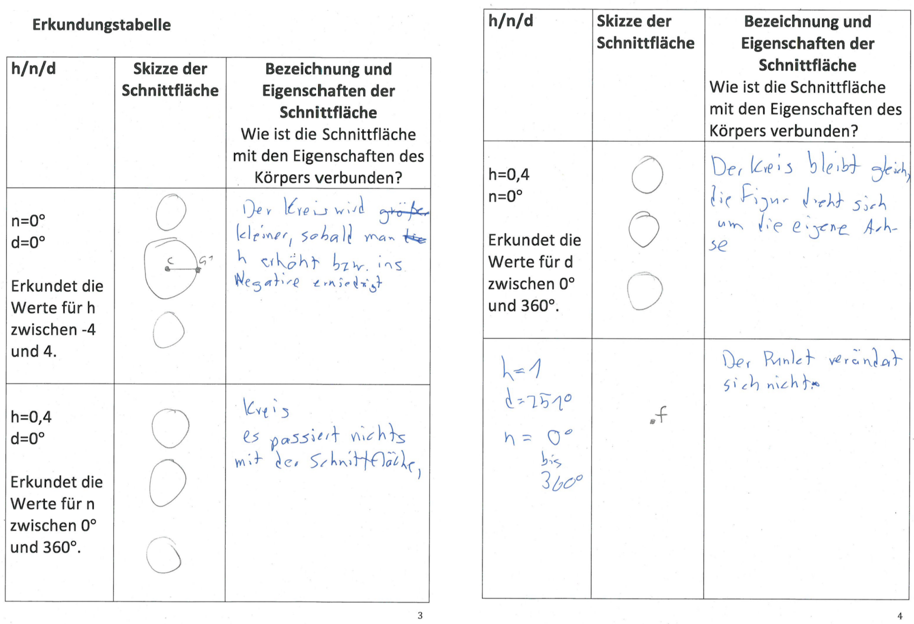
Using the Exploration Matrix, Axel and Dave examine specific cases and positions in an organized order. At the end of the *guided exploration* they have gathered enough information on the situation they explore. Next, they reject in one step (utterances 262-264) all their previous hypotheses about the form of the solid (and more forms not mentioned previously), which they had considered, that do not fit with their observations. Those hypotheses are: sphere, cylinder, pyramid and cone. In this way they draw their final conclusion that the solid is a cuboid, or possibly a cube (utterance 286).

As shown in Table 6.2 (see the row of episode GR1AD-3C.I), through the *guided exploration,* Axel and Dave engage in an organized exploration, getting out of the chaotic situation in which they had found themselves at the end of their *free exploration* (see 6.1.2). Furthermore, they gather information that helps them reject any hypotheses (created during their *free exploration*) that do not fit with the results of their *guided exploration.* By the end of their *guided exploration,* Axel and Dave also draw their final conclusion about the form of the invisible solid.

6.2.1.2 Exclusive use of guided exploration in Tom and Lukas' work on the invisible sphere (episode GR2TL-2)[5]

In this example, Tom and Lukas (group 2) work on Task 2 (invisible sphere). In contrast to Axel and Dave's partial use of the *guided exploration* strategy, in the previous example (see 6.2.1.1), Tom and Lukas exclusively follow a *guided exploration.* Tom and Lukas use the given (h, n, d)-cases and positions provided in the Exploration Matrix (see Figures 6.5a and b).

Their notes on the Exploration Matrix, reveal the students observations. For example, in the first case (h, n=0°, d=0°), the two students note that "*the circle becomes smaller as soon as someone raises h up or lowers it towards the negative*" ("Der Kreis wird kleiner, sobald man h erhöht, bzw. ins Negative erniedrigt"). As shown in Table 6.2 (see the row of episode GR2TL-2), during their *guided exploration* Tom and Lukas gather information about the cross-sections. They also create hypotheses and claims about the form of the solid ($H_{49/51/71}$: truncated double cone, Cl_{64}: sphere), and refute the ones that do not fit with the results of their exploration. By the end of their *guided exploration*, Tom and Lukas verify their claim that the solid is a sphere and state it as their conclusion.


Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Der Kreis wird ~~größer~~ kleiner, sobald man h erhöht bzw. ins Negative erniedrigt
h=0,4 d=0° Erkundet die Werte für n zwischen 0° und 360°.		Kreis es passiert nichts mit der Schnittfläche,

3

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0,4 n=0° Erkundet die Werte für d zwischen 0° und 360°.		Der Kreis bleibt gleich, die Figur dreht sich um die eigene Achse
h=1 d=251° n = 0° bis 360°	f	Der Punkt verändert sich nicht.

4

Figure 6.5: a (left) and b (right). Exploration Matrix of Task 2 – Tom and Lukas' notes during their guided exploration

Next, I present some examples of students' exploration strategies during the classroom-discussions.

6.2.2 Guided exploration in classroom-discussion

In the classroom-discussions, the situation is different than in students' pair-works. The teacher, Frau Karl, organizes the discussion. On the first day of the study, she

5 See the transcript of the episode in the Digital Appendix H5.

chooses to let the students decide how they are going to present their work. This decision was made, because all the students had previously worked on the same task (Task 2 – invisible sphere), so they were all familiar with the task. I discuss this case in section 6.3 (*structured exploration*), because the students followed a *structured exploration* in the discussion of that task. On the second day, Frau Karl asks two students to present their work following a *guided exploration.* Her decision was based on the fact that on this day the student had previously worked on different tasks (Tasks 3A, 3B, or 3C). That means that for each task there have been students who had not explored one –or even two– of the three tasks.

Hence, there is a double role that the *guided exploration* (using the cases from the Exploration Matrix) plays in the presentations in the classroom-discussions. On the one hand, its role is to familiarize all students with the situation in the task. On the other hand, it is to give the students some time to "meet" the rest of their classmates, who may have had already worked on this task, at a point at which they all have enough experience of the situation and maybe a shaped hypothesis, or a conclusion about the form of the invisible solid. Below I present as example, the episode of the classroom-discussion on the task of the invisible pyramid (Task 3B).

Jacob and Michael – The task of the invisible pyramid[6]

In this episode, Jacob and Michael present their work on the invisible pyramid task to the rest of the classroom. Following the teacher's instructions, they use *guided exploration* to go through all the cases and positions given in the Exploration Matrix (see Figures 6.6a and b). Michael manipulates the sliders in the DGE, while Jacob presents their work.

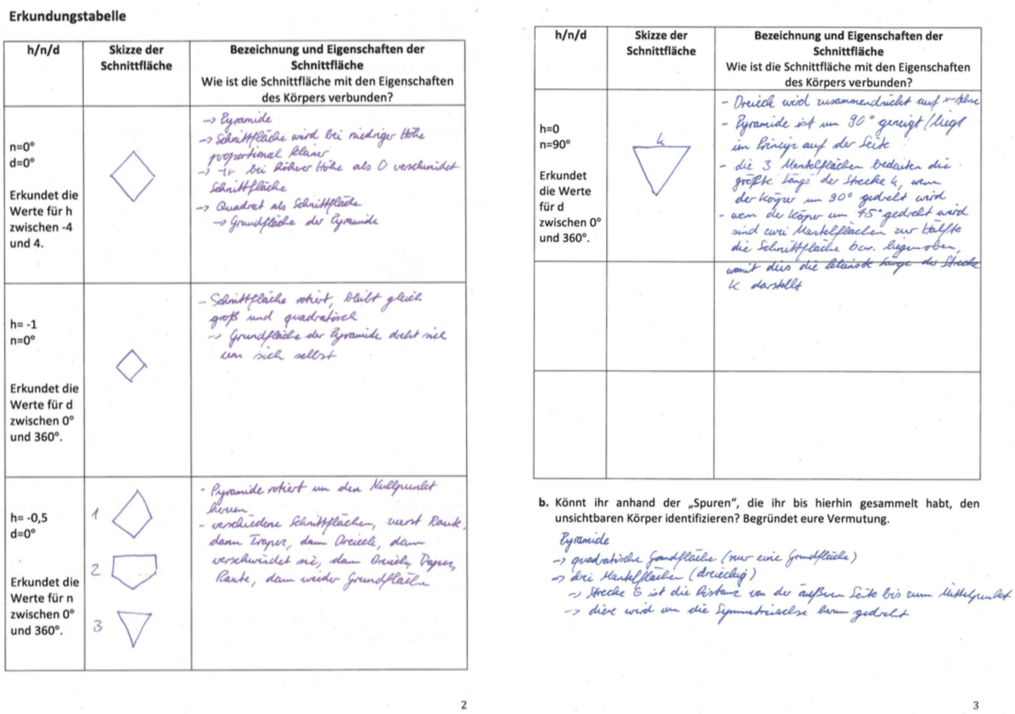
The two students start with the exploration of the first case in the Exploration Matrix (see Figures 6.6a and b). They perform the height exploration (h, n=0°, d=0°) and Jacob says (utterance 1.1-1.13 in the German original and English translation below)[7]:

> "Wir haben halt erkannt, dass es, em, dass die Schnittfläche, wenn man die Höhe niedriger macht, also verringert, dass die Schnittfläche proportional kleiner wird, aber immer noch ein Quadrat bleibt, das halt so gedreht ist um 45 Grad soweit. Em genau. Das heißt das wird halt- mach mal (unverständlich) nach unten - das wird proportional kleiner, hier dann quasi nach unten, bis es irgendwann verschwindet, bis es irgendwann keine Schnittfläche mehr gibt, und wenn man das nach oben macht - mach mal nach oben - (..) oben, dann verschwindet es sofort. Das heißt wir haben halt du hast irgendwie noch gesagt, wir haben halt vermutet, dass es eine Pyramide ist, das heißt, dass halt die quadratische Grundfläche bleibt, und wenn man es nach unten bewegt, em, dass sie halt kleiner wird, weil das ja drei- äh dreieckige Seiten sind, drei das, oder? Also die meinten die bilden vier?"
>
> "We realized that it, em, that the cross-section, if you make the height lower, so decreased, that the cross-section becomes proportionally smaller, but still remains a square that is rotated 45 degrees so far. Em exactly. That

6 See transcript of episode CD3B-JM in the Digital Appendix H9.

7 The statements in italics in the parentheses are explanatory comments added by me, for the presentation purposes of students' actions.

means it will stop - go down (incomprehensible) [*he talks to Michael asking him to drag the h-slider downwards*]- it will be proportionally smaller, here then quasi downwards, until it disappears at some point, until at some point there is no more cross-section, and if you do that up - go up [*he talks to Michael asking him to drag the h-slider upwards*] - (..) above, then it [*the cross-section*] disappears immediately. That means you said somehow, we just assumed that it is a pyramid, that means that the square base remains, and when you move it down, it just gets smaller, because that yes, there are three- uh triangular sides, three, right? So they said they make four?"



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		-> Pyramide -> Schnittfläche wird bei niedriger Höhe proportional kleiner -> -"- bei höher Höhe als 0 verschwindet Schnittfläche -> Quadrat als Schnittfläche -> Grundfläche der Pyramide
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.		- Schnittfläche rotiert, bleibt gleich groß und quadratisch -> Grundfläche der Pyramide dreht sich um sich selbst
h= -0,5 d=0° Erkundet die Werte für n zwischen 0° und 360°.	1 2 3	- Pyramide rotiert um den Nullpunkt herum - verschiedene Schnittflächen, zuerst Raute, dann Trapez, dann Dreieck, dann verschwindet sie, dann Dreieck, Trapez, Raute, dann wieder Grundfläche

2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=90° Erkundet die Werte für d zwischen 0° und 360°.	k	- Dreieck wird zusammengedrückt auf x-Achse - Pyramide ist um 90° geneigt (liegt im Prinzip auf der Seite - die 3 Mantelflächen bedeuten die größte Länge der Strecke k, wenn der Körper um 90° gedreht wird - wenn der Körper um 45° gedreht wird sind zwei Mantelflächen zur Hälfte die Schnittfläche bzw. liegen oben, womit dies die kleinste Länge der Strecke k darstellt

b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Pyramide
-> quadratische Grundfläche (nur eine Grundfläche)
-> drei Mantelflächen (dreieckig)
-> Strecke k ist die Distanz von der äußeren Seite bis zum Mittelpunkt
-> diese wird um die Symmetrieachse herum gedreht

3

Figure 6.6: a (left) and b (right). Michael and Jacob's Exploration Matrix

In the transcript above, Jacob says that when the height is reduced, then the cross-section gets smaller, always remaining a square, and when the height is dragged over h=0, the cross-section disappears completely. He also says that they claim that the solid is a pyramid. From the second exploration on, they consider their claim to be true and they go through the rest of the cases gathering data and drawing further conclusions about the solid. This is partly a process of testing their claim, and partly a process of gathering more information about the characteristics of the solid. Jacob and Michael complete the *guided exploration* of all the cases in the Exploration Matrix and then they draw their final conclusion that the solid is indeed a pyramid (for more details see the global argumentation structure of the episode in Appendix E10).

The rest of the students did not have much to add in the classroom-discussion. At the request of the teacher for comments and questions, the students appeared to be content with Jacob and Michael's presentation.

The *guided exploration* here has served the same purposes as in the pair-works: gathering information, generating hypotheses and claims, refuting and/verifying them, and drawing conclusions. But, it also seems to have served one more purpose. The use of *guided exploration* in Michael and Jacob's presentation has given them the opportunity to organize their arguments around the provided (h, n, d)-cases. This has resulted in an argumentation with a logical structure that was easy to follow for the rest of the class. So, *guided exploration* in classroom-discussion also aids students to organize their presentations and arguments, and consequently their overall argumentation. On the other hand, it seems to have left little room for discussion, exchange of ideas and negotiation of those ideas.

6.2.3 Epilogue

As shown in this section, *guided exploration* is a strategy that can be used in pair-work and in classroom-discussions. In pair-works, it has appeared both as a strategy used as a part of students' work (see Dave and Axel's work in 6.2.1.1), as well as an exclusive strategy (see Tom and Lukas' work in 6.2.1.2). In both cases, its uses remain similar: gathering information for the situation under examination, drawing conclusions about the characteristics and/or the form of the invisible solid, refuting hypotheses or claims, and drawing a final conclusion (see Table 6.2). When *guided exploration* is used as an exclusive strategy though, then the generation of hypotheses and claims is also part of this procedure (see Table 6.2). In the case that another exploration strategy has preceded, the process of generating hypotheses may have been part of the previous exploration strategy. This is, for example, the case in Axel and Dave's work on the task of the invisible cube (see 6.1.2 and 6.2.1.1).

In the pair-works, when the students use *guided exploration* they do not seem to have a specific expectation regarding the outcome of each exploration. After having shaped their initial hypotheses (or claims), the students focus on testing them by examining the given cases and positions in the Exploration Matrix and observing whether the outcomes confirm or contradict those hypotheses and claims. This process leads, on the one hand to refutations of the hypotheses, which do not fit the new acquired information, and on the other hand, to the validation of the fitting hypothesis or claim.

In the classroom-discussions, the use of *guided exploration* is a decision made by the teacher. Frau Karl also chooses to have one pair of students present each task, following the cases in the Exploration Matrix. On the one hand, the *guided exploration* provides an order to students' presentation, and gives a chance to other students, who haven't previously worked on the task, to have the time to reason for themselves and shape their own ideas regarding the situation. On the other hand, students' presentations tend to mainly become monologues. Each time that two students present their work, they present simultaneously the cases of the Exploration Matrix and their argumentation. This results in little interaction with the rest of the class, except for minor corrections made by their classmates during the presentation or a short negotiation of the final conclusion. The lack of vivid interaction between the students, leads to a lack of polyphony and negotiation of different ideas, which is usually the main point of a classroom-discussion. Nevertheless, and in defense of Frau Karl's choice, this monologue-type of classroom-discussion is also the result of the rich and coherent descriptions and justifications that the students provided during their presentations, leaving little room for doubts or need for additional comments on their classmates' side.

I move on now to the next section, where I present the last type of exploration strategy that I identified in my data.

6.3 Structured exploration strategy

I refer to *structured exploration* as the type of exploration strategy in which the students choose which (h, n, d)-case or position they want to explore having a specific aim or expectation already in mind. Unlike in *free exploration*, in which students make their choices impulsively, in *structured explorations* students' choices are characterized not merely by taking *initiative*, but also by taking an initiative with a specific *intention*. The choices of the explored cases are not impulsive, rather purposeful. In what follows, I present the results emerging from the use of *structure exploration*, both in students' pair-works (see 6.3.1), as well as in classroom-discussions (see 6.3.2).

6.3.1 Structured exploration in pair-work

From the two pairs of students observed, only Axel and Dave (group 1) uses *structured exploration*, while working on the task of the invisible cube (Task 3C). As discussed in the previous section (see 6.1.2 and 6.2.1.1), Axel and Dave start with a *free exploration* (utterances 1-76) and then continue with *guided exploration* (utterances 77-262). During the *guided exploration*, they perform two *short structured explorations* (first in utterances 234-239 and then in 270-279). Here, I discuss the first of those two *structured explorations*.

I would like to start my description a little before the *structured exploration* begins. At utterance 227 (see Transcript 6.1 below), Axel and Dave examine the given position (h=0, n=90°, d=0°) (see Figure 6.7), and they agree that the cross-section is a rectangle (utterance 230, Transcript 6.1).

The first intentional decision that Axel makes, is to ask Dave (who manipulates the DGE sliders) to change the position of the n-slider, bringing the tilt back to zero degrees (utterance 234.1). As soon as Dave has brought the sliders into position (h=0, n=0°, d=0°), Axel also asks Dave to change the orientation of the view in the DGE, so that he can see the cross-section from the top of the z-axis (the blue axis in Figure 6.8a, utterance 234.2). Axel knows from a previous stage of their work, that the cross-section at position (h=0, n=0°, d=0°) is the base of the solid. When Dave changes the view (see Figure 6.8b), Axel says that this rectangular cross-section is not the half of the area that appears as cross-section in position (h=0, n=0°, d=0°) (utterance 234.3).

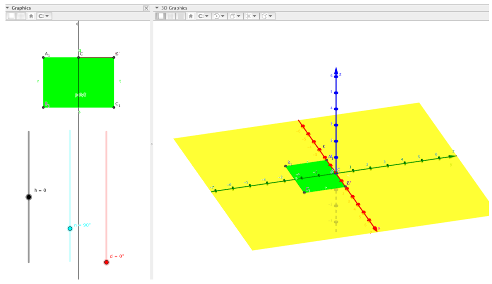


Figure 6.7: Task 3C – Cross-section at position (h=0, n=90°, d=0°)

Utterance	Original German transcript	English translation
234.1 234.2 234.3	Axel: Axel: Warte und jetzt. Warte, geh bitte einmal bei Neigung auf Null [*see Figure 6.8a*]. Von oben. Also das, die Ansicht von oben [*see Figure 6.8b*] und Dreieck auf nu- äh Neigung auf null. Ok, es ist aber nicht genau die Hälfte. Es ist nicht die Hälfte der Fläche [*he refers to the cross-section in position (h=0, n=0°, d=0°)*] (..) #00:25:11-8#	Axel: Wait and now. Wait, please go to tilt at zero [*see Figure 6.8a*]. From above. So that, the view [*is*] from above [*see Figure 6.8b*] and the tri- at eeh, the tilt at zero. Ok, but it's not exactly half. It's not half of the area [*he refers to the cross-section in position (h=0, n=0°, d=0°)*]

Transcript 6.1: Axel's *structured exploration* in Task 3C (invisible cube)

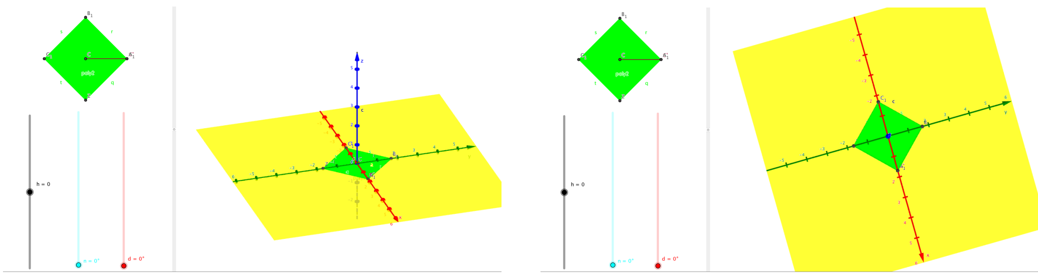


Figure 6.8: a (left) and b (right). Task 3C – Cross-section at position (h=0, n=0°, d=0°) with view from in front of x-axis and from the top of z-axis

Position (h=0, n=0°, d=0°), is the first position chosen by Axel himself and with a very specific intent and purpose: to check the validity of his hypothesis that the cross-section in position (h=0, n=90°, d=0°) may be the half of the cross-section in position (h=0, n=0°, d=0°). Position (h=0, n=0°, d=0°), is a position that has been explored before, both during the *free exploration* (utterances 1-10) as well as during the *guided exploration* (uttarances 77-96). But this time it is re-visited by Axel with a specific aim. In addition, Axel's choice to change the orientation of the view in the DGE has been *intentional* for the same purpose (to check the validity of the hypothesis). Therefore, I classify those two actions as *structured exploration,* since they are characterized by *initiative* taken by Axel and specific *intention.*

6.3.2 Structured exploration in classroom-discussion

Unlike on day 2 of the study, when not all pairs of students worked on the same tasks, on day 1 the whole class worked on Task 2 (invisible sphere). Hence, at the end of the pair-work phase, all the students were familiar with the task of the invisible sphere, and had shaped their own opinions about the situation. Therefore, on that day Frau Karl (the teacher) chose to discuss Task 2 with her students in a whole classroom-discussion without a presentation.

Below, I include the structure of this episode (from Level 1 data analysis, see Table 6.3, shown also in Appendix C7), as a tool for the better understanding of this episode. The teacher (Frau Karl) is the orchestrator of the classroom-discussion. First, she asks the class to give an answer and their justification regarding the form of the invisible solid (number 1 in the structure in Table 6.3). Next, she poses questions regarding other possible solutions to the task, or other possible hypotheses about the form of the solid (number 2 in the structure in Table 6.3). The classroom-discussion ends when every other hypothesis has been refuted, and the sphere-hypothesis is verified and presented as the final conclusion.

Here, I discuss some examples of *structured explorations* in this episode.

Episode CD2 – Classroom-discussion on the invisible sphere task				
Structure			**Utterance numbers in the transcript**	**Video Minutes**
1	Justifying the sphere-hypothesis		1-17	01:15:55 – 01:17:43
1	1.1	Niko's justification The solid is a sphere because all its cross-sections are circles. Rejecting the idea of the solid being a cuboid via Reductio ad absurdum	1-12	01:15:55 – 01:16:59
1	1.2	Jacob's justification The lack of influence of the n-variation to the cross-sections of the solid is a decisive factor, proving that the solid is a sphere.	14-17	00:17:00 – 01:17:43
2	Stating and rejecting other hypotheses judging only by the (h, n=0°, d=0°) exploration		18-54	01:17:44 – 01:23:02
2	2.1	The solid could be a cylinder	23-33	01:17:44 – 01:19:20
2	2.2	The solid could be cone	25 and 34-40	01:18:33 and 01:19:22 – 01:20:03
2	2.3	The solid could be a double cone	41-53	01:20:03 – 01:23:02

Table 6.3: Structure of classroom-discussion episode on Task 2 (invisible sphere)

The classroom-discussion starts with Niko arguing that the solid is a sphere (see 1.1 in Table 6.3, utterance 6). The reason he gives for this, is that no matter how one tilts or spins the solid, all the cross-sections are circles. This explanation suggests that Niko considers the tilt-exploration (n-slider) and the spin-exploration (d-slider) as being decisive factors for his conclusion. Niko's conscious choice of these two factors in order to justify his conclusion is what classifies them as a *structured exploration.*

Another example of *structured explorations* is found in the second part of the classroom discussion (see number 2 in Table 6.3). In utterances 18-53, the class takes on another approach to the situation, initiating the argumentation about the form of the solid from the start. This time they start the argumentation with the height-exploration. Frau Karl poses the question:

"*What else could the solid be, other than a sphere, judging only by the case (h, n=0°, d=0°)?*" (utterance 18)

The class first performs a height exploration, examining the case (h, n=0°, d=0°). One student, Victor, has already said that taking into consideration only this data emerging from this case, the solid could also be a cylinder (utterance 23). Dave then refutes this hypothesis (utterance 33, see Transcript 6.2). Dave chooses the height exploration (h, n=0°, d=0°) and he argues that if the solid were a cylinder, then during the height variation all the cross-sections should be circles of the same size. But he says that this does not happen here so the solid is not a cylinder.

Utterance	Original German transcript	English translation
33	Dave: Ja, aber dann kann es beides [*cylinder or cone*] eigentlich nicht sein, weil wenn man die Schnittfläche betrachtet, die nimmt ja halt immer proportional ab, wenn man nach oben oder unten geht [*when the h-slider is dragged over and under zero*], das heißt em, ein Zylinder wär, wenn man nach unten oder nach oben gehen würde, komplett irgendwann nichts. Und das ist hier ja nicht und beim Zylinder wäre es [*the cross-section*] ja immer ein gleich großer Kreis und das halt aber nicht (unverständlich) #01:19:11-4#	Dave: Yes, but then it can't actually be any of them [*cylinder or cone*], because if you look at the cross-section, it always decreases proportionally when you go up or down [*when the h-slider is dragged over and under zero*], which means, em, a cylinder would, when one goes down or up, at some point, completely nothing. And that is not do here and with the cylinder it [*the cross-section*] would always be a circle of the same size and that is not so (incomprehensible)

Transcript 6.2: Dave's *structured exploration* in Task 2 (invisible sphere)

In this episode, students' choices of the cases and positions to be explored have always constituted conscious *initiatives*. Furthermore, these choices always indicated the intentions of the student who chose them. Therefore, I classify those explorations performed by the students in this episode as *structured explorations*. The uses of *structured explorations* in the classroom-discussions have been to gather information, create hypotheses, refute hypotheses and draw conclusions.

6.3.3 Epilogue

Structured exploration is an exploration strategy in which the student takes a conscious *initiative*, regarding the (h, n, d)-case or position to be explored, assigning to it a specific *intention* or *expectation*. In the examples presented in this section, *structured explorations* are used both in the pair-work, as well as in the classroom-discussion.

In pair-work, *structured exploration* appears as a strategy only partially, with the single use of checking the validity of a hypothesis (see subsection 6.3.1). On the contrary, in the classroom-discussion (see subsection 6.3.2) it is used exclusively having multiple uses: gathering information, creating and/or refuting hypotheses, and drawing conclusions. In the classroom-discussion, *structured exploration* seems to be a strategy that follows naturally after a pair-work phase, in which all the students have worked on the task and have shaped their own ideas on the situation.

The whole classroom-discussion in the form of a dialogue between the classmates, in contrast to a presentation from a single pair of students, gives the opportunity to the whole class to participate and contribute to the discussion. In the classroom-discussion episode presented in this section (see 6.3.2), the structure of the classroom-discussion has given the students the opportunity to interact, negotiate their ideas and challenge or enrich each other's arguments in a frame of polyphony and collaboration.

6.4 Conclusion

In this chapter, I focused on the different exploration strategies that students follow with the use of the DGE when working on a task, both during their pair-work as well as in their classroom-discussions. I also described the various uses of these strategies in students' work. In the next three subsections, I summarize the results of this chapter and I answer the research question I stated at the beginning of the chapter:

1. What *exploration strategies* do the students follow using the Dynamic Geometry Environment when they work on the given tasks?

6.4.1 The three types of exploration strategies

From the Level 1 data analysis in my study, three types of exploration strategies arise, that students follow while working on the given tasks. All the strategies are related to the way students use the Dynamic Geometry Environment (DGE). Those three types of exploration strategies are: *free exploration*, *guided exploration* and *structured exploration.*

The differences between the three types of explorations, depend on two factors: conscious *initiative* and *intention* or *expectation* of their actions. In *free* and in *structured* explorations the students take *initiative* about the (h, n, d)-cases and positions they want to examine. But, in *free exploration* the *initiative* taken by the student is *impulsive* and lacks aim or expectation. On the contrary, in *structured exploration* the student chooses *consciously* the case he or she wants to explore, and has a specific aim or expectation in mind. In *guided explorations* the student takes no initiative, rather prefers to examine positions of the solid following the cases and positions provided in the Exploration Matrix in the task's worksheet.

6.4.2 The use of exploration strategies in the episodes

Table 6.4 below shows which exploration strategies were used in each of the episodes, both during pair-work, as well as during the classroom-discussion. Axel and Dave (group 1) used multiple exploration strategies in their first two tasks (Task 2 – invisible sphere, on day 1, and Task 3C – invisible cube, on day 2), while they continued with exclusive use of *guided explorations* in the other two tasks (Task 3B – invisible pyramid and Task 3A – invisible cone, on day 2). Tom and Lukas (group 2) used *guided exploration* exclusively in both tasks they worked on (Task 2 on day 1 and Task 3C on day 2).

During the classroom-discussions, the students used *structured exploration* exclusively on day 1 for Task 2, and *guided exploration* exclusively on day 2 for all the tasks (Tasks 3A, 3B and 3C). The choice of the exploration strategy followed in each classroom-discussion was influenced by the teacher's decision on the way the classroom-discussion would be organized. On the first day, the teacher organized a classroom-discussion in which an open dialogue between the whole class (the teacher included) took place. This decision was made upon the fact that all the students had already worked on Task 2 in the preceding pair-work. These two parameters (dialogue between all the students and experience on the task) led to the use of *structured exploration*. Each student who contributed to the discussion argued using a specific (h, n, d)-case or position, of his or her own choice (*initiative*) and with a specific *intention*, which he/she made explicit. This led to an interactive discussion, with many students contributing to the argumentation.

On the second day of the study, not all the students had worked on the same tasks. Therefore, the teacher decided to follow another format for the classroom-discussion. First a pair of students who had worked together during the pair-work phase, presented the task, going through all the given cases in the Exploration Matrix on the worksheet. At the same time, or after the presentation of the whole *guided exploration*, the two students presented their conclusion and a justification for it. In this case, the classroom-discussions were not as interactive as the classroom-discussion on the first day. The presentations dominated in the discussions, and although the argumentation presented by the pairs was not less rich or coherent than the one built on day 1 by the whole class, it was nevertheless mainly the product of a monologue by the pair of students who held the presentation.

<table>
<tr><th>Strategy</th><th>Free Exploration</th><th>Guided Exploration</th><th>Structured Exploration</th></tr>
<tr><td rowspan="6">Pair-work Episodes</td><td colspan="2">Axel and Dave – Task 2: Invisible sphere (GR1AD-2)</td><td></td></tr>
<tr><td colspan="3">Axel and Dave – Task 3C: Invisible cube (GR1AD-3C.I)</td></tr>
<tr><td></td><td>Axel and Dave – Task 3B: Invisible pyramid (GR1AD-3B.II)</td><td></td></tr>
<tr><td></td><td>Axel and Dave – Task 3A: Invisible cone (GR1AD-3A.III)</td><td></td></tr>
<tr><td></td><td>Tom and Lukas – Task 2: Invisible sphere (GR2TL-2)</td><td></td></tr>
<tr><td></td><td>Tom and Lukas – Task 3A: Invisible cone (GR2TL-3A.I)</td><td></td></tr>
<tr><td>Total # of pair-work episodes with this strategy</td><td>2/6</td><td>6/6</td><td>1/6</td></tr>
<tr><td>Total # of groups using this strategy</td><td>1/2</td><td>2/2</td><td>1/2</td></tr>
<tr><td rowspan="4">Classroom-discussion Episodes</td><td></td><td></td><td>Task 2: Invisible sphere (CD2)</td></tr>
<tr><td></td><td>Task 3A: Invisible cone (CD3A-AD)</td><td></td></tr>
<tr><td></td><td>Task 3B: Invisible pyramid (CD3B-JM)</td><td></td></tr>
<tr><td></td><td>Task 3C: Invisible cube (CD3C-TL)</td><td></td></tr>
<tr><td>Total # of Classroom-discussion episodes with this strategy</td><td>0/4</td><td>3/4</td><td>1/4</td></tr>
</table>

Table 6.4: Use of strategies in episodes

6.4.3 The uses of exploration strategies in students' work

Table 6.5 below shows the uses of the exploration strategies in all the episodes. Each color in the table represents one use. For example, the use "Create a hypothesis or a claim" is marked green. Each use has the same color along all the strategies in which it appears.

So, we can see that there are three uses that appear in all three exploration strategies. These uses are: Create a hypothesis/claim, Test the validity of an idea/Verify or refute a hypothesis or a claim, and Gather information. *Guided* and *structured explorations* have one more shared use: Draw a final conclusion. *Guided* *exploration* has one more function, not shared by the other two strategies: Organize the exploration.

6.4.3.1 Partial versus exclusive use of exploration strategies

In episodes where an exploration strategy is exclusively used, all the uses of the exploration are covered by this one type of strategy. Observe, for example, episode GR1AD-3A.III in Table 6.5. In this episode, *guided exploration* is used for everything, from the creation of hypotheses, to the gathering of information and the testing of the hypotheses, their validation or refutation, and the reach of the final conclusion. The same is true for classroom-discussion episodes, in which only one strategy is followed, such as in episode CD3B-JM.

When multiple strategies are used in one episode, two or even all three strategies are employed. Such a case is, for example, episode GR1AD-3C.I (see also subsections 6.1.2 and 6.2.1.1). Here, Axel and Dave started with *free exploration*, and when the outcomes of their exploration became overwhelming they switched to *guided exploration*. They also shortly used *structured exploration* in order to check the validity of an idea Axel had. In this case, all three exploration strategies share common uses in different moments of the students' work. For example, in three different moments in their work all three strategies have functioned as a mean to refute a hypothesis or a conclusion.

6.4.3.2 Uses of exploration strategies in pair work versus in classroom-discussions

In the classroom-discussion the strategies that have been used, are the *guided* and the *structured exploration*. Therefore, I compare the uses of each of these two strategies in pair-work and in classroom-discussions.

In the *guided exploration*, the use "organize the exploration" appears in all the classroom-discussion episodes in which it has been used. This is due to the fact that in classroom-discussion the choice of the strategy has always been made by the same person, the teacher, and with the same *intention*, to help the students who had not worked on that particular task to familiarize themselves with the situation. This does not happen with all the pair-work episodes in which this strategy is used. Not even in both pair-work episodes, in which *guided exploration* is partially used (see GR1AD-2 and GR1AD-3C.I). It only happens in episode GR1AD-3C.I, after the students express their confusion following a long free exploration, thus switching to *guided exploration*. To be more precise, this is the only episode, in which this use is explicitly expressed.

Episode		*Free exploration*			*Guided exploration*					*Structured exploration*			
		Create hypothesis or claim	Testing validity, verify/refute hypothesis or claim	Gather info	Organize the exploration (De-chaotication)	Create hypothesis or claim	Testing validity, verify/refute hypothesis or claim	Gather information	Draw final conclusion	Create hypothesis or claim	Refute hypothesis/ Check validity of idea	Gather information	Draw final conclusion
Pair-work	GR1AD-2	✓	✓	✓			✓	✓	✓				
	GR1AD-3C.I	✓	✓	✓	✓		✓	✓	✓		✓		
	GR1AD-3B.II					✓	✓	✓	✓				
	GR1AD-3A.III					✓	✓	✓	✓				
	GR2TL-2					✓	✓	✓	✓				
	GR2TL-3A.I					✓	✓	✓	✓				
Class-discussion	CD2									✓	✓	✓	✓
	CD3A-AD				✓	✓	✓	✓	✓				
	CD3B-JM				✓	✓	✓	✓	✓				
	CD3C-TL				✓	✓	✓	✓	✓				

Table 6.5: The uses of the exploration strategies in students' work

I therefore argue that, although not explicitly expressed by the participants in the rest of the episodes, the feelings of "security" and organization provided by the given cases and positions in the Exploration Matrix, may be one of the reasons why the students have either exclusively, or at least in part, used *guided exploration* as their exploration strategy in each task.

The *structured exploration* is even more interesting. It reveals only a single function when used during pair-work, while it functions in all possible ways when used in the classroom-discussion (see Table 6.5). This is probably so, mainly due to the fact that *structured exploration* is used in pair-work (episode GR1AD-3C.I) only as a partial exploration strategy, while it is used exclusively in the case of the classroom-discussion (episode CD2). In episode GR1AD-3C.I *structured exploration* emerges only towards the end of students' explorations, when all the other uses have been covered by the preceding *free* and *guided explorations.*

6.5 Epilogue

In the present chapter I presented the results of my Level 1 data analysis, providing a first overview on the ways that the students deal with the tasks they are given and the exploration strategies they develop using the Dynamic Geometry Environment.

Next, I would like to present the results arising from Levels 2 (Chapter 7) and 3 (Chapter 8) of my data analysis.

7 Students' patterns of argumentations

In the present study I observe students' argumentations from three different perspectives (exploration strategies, patterns of argumentation and argumentation structures). In the previous chapter I examined the exploration strategies that the students follow in D-transitional tasks. In this chapter, I examine the patterns that students follow in their argumentations, illustrating them schematically with diagrams.

I reconstruct the argumentation focusing on the actions the students take during their discussions. This process belongs to Level 2 Analysis of students' argumentations, and the products of this analysis-process are students' *patterns of argumentation* (for more details on the analysis method see in Chapter 5 subsection 5.4.2 and for the Level 2 Analysis see subsection 5.4.4). The reconstruction of patterns of argumentation is a method that allows the observation of students' actions (e.g. observing data, generating hypothesis, drawing a conclusion etc.) as their argumentations develop. In this study, I reconstruct the patterns of argumentation taking place during students' pair-work, as well as during whole classroom discussions.

Furthermore, in this chapter I provide answers to the following three research questions:

> 2.1 What are the observed *patterns of* students' *argumentations* while working on the given tasks? (Sections 7.1 to 7.4)
>
> 2.2 How does the specific *design* of the given tasks influence the structure of students' patterns of argumentation? (Section 7.5)
>
> 2.3 How do students' patterns of argumentation differ in *pair-work* and in *classroom discussions*? (Section 7.6)

Following that, I present students' patterns of argumentation. I begin with an overview of the results emerging from the Level 2 Analysis and answer to my research question 2.1 (Section 7.1). Through my analysis, I identified six types of patterns of argumentation, three for students' pair-works and three for the classroom discussions. I begin by presenting a table of all the types of patterns that I identified (see Table 7.2). Then I move on to a detailed presentation of each type of pattern, using specific examples. The presentations are performed in two parts. I start with the patterns observed during students' pair-work (Section 7.2) and then I move on to the ones from the classroom discussions (Section 7.3). Bringing the results together, I comment again the research question 2.1 stated above (Section 7.4). Finally, I discuss and answer the other two research questions (2.2 and 2.3) in two separate sections (7.5 and 7.6, respectively). The chapter closes with a short epilogue and an introduction to the next chapter (Section 7.7).

7.1 The patterns of argumentation – An overview

As discussed in Chapter 5 (see subsection 5.4.2), Level 2 Analysis is based on Reid's (2002b) method of "patterns of reasoning". Nevertheless, in the present work instead of reconstructing students' work based on their reasoning, I use their oral argumentation. Therefore, I refer to the patterns I reconstruct as patterns of *argumentation.*

I firstly identify the different elements of the "patterns" of students' argumentations. These elements are actions students take while solving a task. As a result, the abbreviations of these elements represent verbs, expressing those actions, such as: Observing data (DO)[1], hypothesizing (Hyp), claiming (Clai), drawing a conclusion (Conc), contradicting (Contra), testing a hypothesis/claim (Test), and others. Table 7.1 shows all the elements that may be found in a pattern of argumentation (left column), as well as their correspondence to the elements of patterns of reasoning (right column), as they were defined and used by Reid (2002b)[2]. For the purposes of my study, I needed to adapt some of the elements that Reid uses in his patterns of *reasoning*, in order to give them characteristics that fit to the purposes of my specific patterns of *argumentation.*

After having identified the elements, I link them to each other following the sequence in which they emerge in time during students' discussions. The *arrows* ($\longrightarrow$) linking each element with the next one can be read as "and then (follows)", in the chronological sense, not in the sense that something "follows" from something else as a result/consequence (denoting causality).

Before moving ahead to present the results, I would like to remind the reader that in this study, the students worked on D-transitional tasks on two different days. On the first day all students worked on the same task, namely Task 2 (invisible sphere). On the second day, they worked on one, or more, of three tasks: Task 3A (invisible cone), Task 3B (invisible pyramid), Task 3C (invisible cube). Not all students worked on all three tasks on day 2. On each day, the students worked first in pairs (for about 35'-40'), and then followed a classroom discussion moderated by their teacher, Frau Karl, during which all the tasks where discussed.

In the data analysis, I decided to separate the patterns of argumentation for pair-work and classroom discussions, instead of trying to create common patterns for the two different settings. The reason for this is that argumentations in pair-work take place while students work on the task and are in search of their answer. On the contrary, the argumentations taking place in classroom discussions are built after the students have worked on the tasks, shaped their ideas and concluded about their answers.

[1] Reid (2002b) uses tasks in which students are supposed to observe patterns. As a result, he talks about "observing a pattern" (or pattern observation), which he symbolizes as PO. In the tasks used in the present work, students are not supposed to observe patterns, rather data about two-dimensional cross-sections and movements of the sliders, geometric properties etc. Hence, the term "observing a pattern" has been replaced by the more fitting term "observing data" (or data observation). Following Reid's (2002b) symbolization method I symbolize this as DO (and not as OD).

[2] For more details on how Table 7.1 emerges, please refer to Chapter 5, subsection 5.4.2, Table 5.3.

Elements of Pattern of Argumentation	Elements of Pattern of Reasoning (Reid, 2002b)
DO (Observing data/Data Observation)	**PO** (Observing a Pattern/Pattern observation)
Hyp (stating a Hypothesis) a supposition created by the students, suggesting a possible case based on the available data. This is a case, which at the moment looks plausible, and whose validity is not yet confirmed.	**Conj** (Conjecturing) Something is considered neither true nor false, rather it is subject to testing.
Clai or **Conc** (stating a Claim or Conclusion) A *claim* is more than a hypothesis; it is more than just a possible case or solution. It is *the* possible case which one considers as the most probable and shows the intention to confirm it or argue in favor of it. A *conclusion* is a statement that is accepted by all as true.	**Gen** (Generalizing) Here generalizing is used in the sense of uttering a statement that is accepted by someone or a social group as true but still not by all, so it may be a generalization for some and a conjecture for others.
Contra (stating a Contradiction)	**Contra** (Contradiction)
1D (drawing a conclusion with simple deduction)	**1D** (simple deduction)
Test (testing)	**Test** (testing)
CE: using a counter-example	**CE**: using a counter-example
Q? (The teacher poses a question in order to provoke students' argumentation)	

Table 7.1: Elements of patterns of argumentation (left) and patterns of reasoning (right)

Table 7.2 shows the six types of patterns of argumentation that I identified through the Level 2 data analysis. In this table, I have gathered and presented all the patterns of argumentation that appear in the episodes of this study. There are two categories of patterns: the patterns observed in students' pair-work (symbolized as PW), and patterns observed in the classroom discussions (symbolized as CD). For each pattern, I describe the sequence of its elements and the episodes in which it is observed. Furthermore, each pattern has a number (e.g. pattern 1PW) and descriptive name (e.g. "Direct" pattern) that expresses the main characteristic of the pattern.

All the types of patterns in Table 7.2 begin with the initial DO-element (observing data) and are completed with a conclusion (Conc). The characteristics that differentiate the patterns from another, are the pattern elements that lie between the initial DO-element (observing data) and the last Conc-element. These are always marked in bold.

In the case of the pair-work episodes, I observed three patterns. In the first pattern identified in students' pair-work argumentations, Pattern 1PW, the students complete their whole exploration and observation of data (DO) before they state their hypothesis (Hyp). Then this hypothesis turns into the final conclusion because it is accepted as true from both students (Conc). Because the process followed in this pattern is quite direct, I refer to it as "Direct" pattern.

In Pattern 2PW the students first observe a limited number of data (first DO-element in the pattern), during their exploration, and then they generate one or more hypotheses (Hyp). They then use the rest of their exploration to contradict some hypotheses, confirm another, or accept one as the most probable case by eliminating the rest (second DO-element in the pattern). The final conclusion that is drawn (Conc) is one of the initial hypothesis. It is the initial hypothesis that has not

Patterns of argumentation in pair-work			
	Pattern 1PW – "Direct" pattern	Pattern 2PW – "Narrowing down" pattern	Pattern 3PW – "Testing" pattern
	DO⟶ **Hyp**⟶ Conc	DO⟶ **Hyp**⟶ **DO**⟶ Conc	DO⟶ **Hyp**⟶ **Test**⟶ **DO**⟶ Conc
Episodes	GR2TL-2	GR1AD-3B.II GR1AD-3A.III GR2TL-3A.I	GR1AD-2 GR1AD-3C.I

Patterns of argumentation in classroom discussion			
	Pattern 1CD – "Confirming" pattern	Pattern 2CD – "Question-provoking" pattern	Pattern 3CD – "Reverse debate" pattern
	DO⟶ **Clai**⟶ **DO**⟶ Conc	(Full)DO⟶ **1D**⟶ **Q?** ⟶ Conc	DO⟶ **1D**⟶ **Q?**⟶ **Hyp**⟶ **Contra**⟶ **Test** (of Hyp)⟶ Conc
Episodes	CD3B-JM CD3C-TL	CD3A-AD	CD2

Table 7.2: Patterns of argumentation identified in pair-work and in classroom discussions

been contradicted by any of the data, rather it agrees with all of them. Therefore, I refer to this pattern as the "Narrowing down" pattern. This pattern is observed in three episodes, arising from both of the two student pairs that were observed in the study (Axel and Dave, and Tome and Lukas).

Pattern 3PW is the most "complex" one. The students begin by observing data (first DO-element in the pattern), they then generate multiple hypotheses (Hyp), some of which they later contradict. The remaining hypothesis becomes a claim (Clai), whose validity is then checked both by testing it (e.g. Test$_{10\text{-}12}$ in episode GR1AD-2), as well as by observing further data (second DO-element in the pattern). The students draw their final conclusion based on data they have gathered from both processes (Conc). Because this pattern includes a test process that is not included in either of the previous two patterns, I call it the "Testing" pattern. This pattern is observed in two episodes by Axel and Dave (Group 1).

Patterns 1CD and 2CD emerge from classroom discussion episodes that took place on day 2, when not all students had worked on every task. This created the need for a presentation of the task, before the actual discussion. A pair of students would present their work on a task and then this task would be discussed further with the entire classroom. On the contrary, on day 1 all the students had worked on the same one task provided (Task 2). This meant that everyone had experience of the task-situation, and before the discussion started they had the opportunity to shape their ideas and their arguments. Hence, the teacher saw no need for a presentation and simply immediately initiated a discussion with the whole classroom directly. Therefore, the structure of the pattern has been fairly influenced by the structure of the discussion.

Nevertheless, Patterns 1CD and 2CD have an important difference. In Pattern 1CD the presentation begins with observing data (first DO-element in the pattern) from only some of the given cases in the Exploration Matrix. After this short observation phase, the students make a claim (Clai), which is then confirmed during the rest of the data observation process (second DO-element in the pattern), turning it into their final conclusion (Conc). I call this the "Confirming" pattern, because a supposition that was believed to be true (a Claim), is indeed confirmed.

In Pattern 2CD, the presentation starts with a full observation of all the cases given in the Exploration Matrix. Based on some of the data gathered during this process, the students then draw their conclusion (1D, simple deduction), without going through a "claiming" phase first. Therefore, the teacher steps in to the discussion, asking questions (Q?-element in the pattern) that will enrich the argumentation, by asking students to explain why the cross-sections that appear throughout the explorations befit to their conclusion. Therefore, I call this pattern, the "Question-provoking" pattern. The discussion ends when the students have explained adequately how all the cross-sections occur.

Pattern 3CD is quite different from the previous two patterns. Here the discussion and consequently the pattern as well, begin with an argument. A student states his conclusion, which he has drawn from data he has observed previously during the pair-work. The connection between the data and the conclusion is performed through a rule (see $1D_{6.1\text{-}6.4}$, in episode CD2, subsection 7.3.3). Although this beginning of Pattern 3CD seems to resemble the beginning of Pattern 2CD, it is not the same. Here, the whole set of data has not yet been revealed. The observation of the data (DO) is only limited to the data that the student needs to mention in order to build his simple deduction argument (1D-element in the pattern). The actual exploration and gathering of data will happen during the next steps of the pattern. The teacher triggers the discussion by asking questions (Q?). These questions lead to further hypotheses (that were not mentioned before) and their negotiation (**Hyp⟶ Contra⟶ Test** (of Hyp)). The discussion ends when all the hypotheses have been contradicted, and the initial conclusion, seems to be the only valid solution (implicit $Conc_{54}$, see episode CD2 in subsection 7.3.3). I call this pattern the "Reverse debate" pattern.

In the next two sections (7.2 and 7.3) I explain each of the six patterns in more detail.

7.2 Patterns of argumentation in pair-work

In students' pair-work, I reconstruct the patterns of students' *oral* argumentations. As I explain in Chapter 5 (see subsection 5.4.2), students' patterns of argumentation are based on what the students say in their discussion while working on a mathematical task. Therefore, their written justifications are not analyzed with this method, as they do not represent actions, rather they are written arguments through which the students justify their conclusions at the end of their discussions. Nevertheless, when the final justification is also orally expressed (some times students talk about it, as they are writing it down), then it becomes part of the pattern of argumentation.

The episodes described here emerge from the observation of the work of two groups (pairs of students) on both days of the study. Both groups worked on Task 2 on day 1. On day 2, Axel and Dave (Group 1) worked on Tasks 3C, 3B and 3A (in this order), while Tom and Lukas (Group 2) worked on Task 3A. As I mention in Section 7.1, three patterns of argumentation arise from the analysis of the data, each of which has specific characteristics.

Table 7.3 shows the three patterns of argumentation identified in students' pair-work episodes (same as the first three patterns in Table 7.2). The descriptions of the pattern-types in the table do not include all the details of the individual patterns that belong to each type but are more general descriptions of them. The names of the patterns include a number and the abbreviation PW, denoting that a pattern is observed during pair-work. Each pattern also has a descriptive name that helps to associate it with its main characteristic. In the subsections that follow I present the three observed patterns of argumentation through specific examples.

Patterns of argumentation in pair-work			
	Pattern 1PW – "Direct" pattern	Pattern 2PW – "Narrowing down" pattern	Pattern 3PW – "Testing" pattern
	DO⟶ **Hyp**⟶ Conc	DO⟶ **Hyp**⟶ **DO**⟶ Conc	DO⟶ **Hyp**⟶ **Test**⟶ **DO**⟶ Conc
Episodes	GR2TL-2	GR1AD-3B.II GR1AD-3A.III GR2TL-3A.I	GR1AD-2 GR1AD-3C.I

Table 7.3: Patterns of argumentation identified in students' pair-work

7.2.1 Pattern 1PW – "Direct" pattern: DO⟶ Hyp⟶ Conc

The "Direct" pattern is observed in episode GR2TL-2, in which Tom and Lukas work on Task 2 (invisible sphere). As I mention in Section 7.1, in this type of pattern the whole exploration and data observation is done before the students state their suppositions (hypotheses and claims). Then follows the step of drawing the final conclusion. The structure of this episode is presented in Appendix C5.

Figure 7.1 illustrates the pattern of argumentation for this episode. This pattern consists of five elements: DO (observing data), Hyp (hypothesizing or generating a hypothesis), Contra (contradicting a statement), Clai (stating a claim) and 1D (drawing a conclusion by a simple deduction).

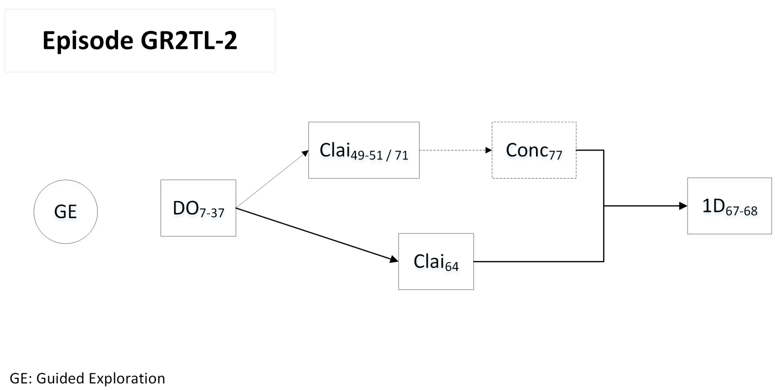
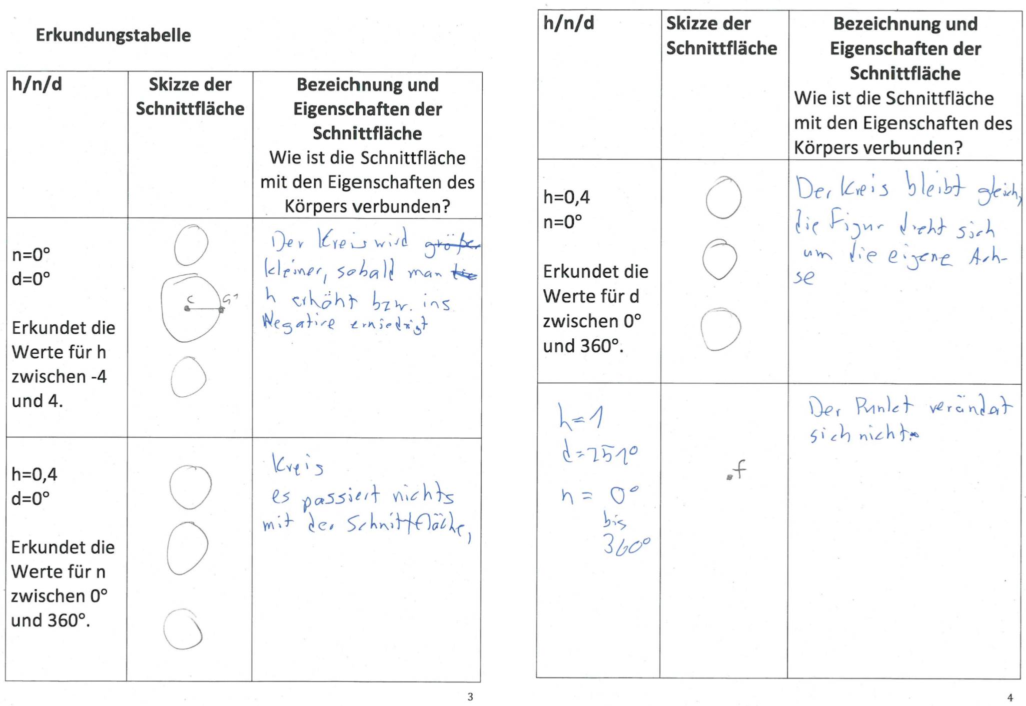


Figure 7.1: Pattern of argumentation in episode GR2TL-2

Starting with data observation during utterances 7-37 ($DO_{7\text{-}37}$), Tom and Lukas observe the data provided from the explorations they perform, using the cases in the Exploration Matrix (Figures 7.2a and b). This means that they perform a guided exploration[3].



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Der Kreis wird ~~größer~~ kleiner, sobald man h erhöht bzw. ins Negative erniedrigt
h=0,4 d=0° Erkundet die Werte für n zwischen 0° und 360°.		Kreis es passiert nichts mit der Schnittfläche,

3

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0,4 n=0° Erkundet die Werte für d zwischen 0° und 360°.		Der Kreis bleibt gleich, die Figur dreht sich um die eigene Achse
h=1 d=251° n = 0° bis 360°	f	Der Punkt verändert sich nicht.

4

Figure 7.2: a (left) and b (right). Exploration Matrix from Tom and Lukas' worksheet in episode GR2TL-2

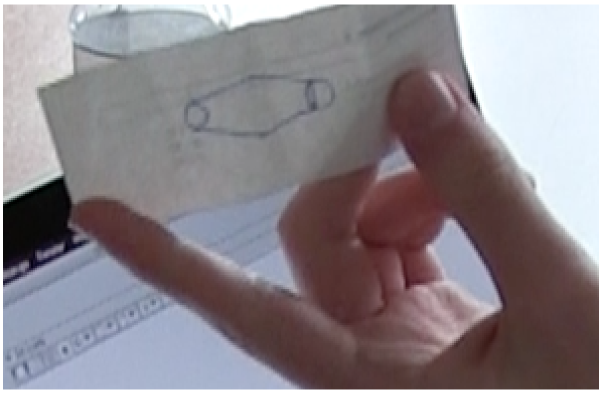


Figure 7.3: Lukas' illustration of a truncated bicone

[3] This is the type of exploration strategy that the students follow when they choose to explore a task by using the given (h, n, d)-cases and positions provided in the Exploration Matrix in the worksheet (for more details see Chapter 6, Section 6.2).

After the students have completed the explorations of all the cases in the Exploration Matrix, they state their claims. The pattern is divided into two *paths*[4]. In the one path Lukas states the claim that the invisible solid has the form shown in Figure 7.3 (utterance 49 in Transcript 7.1, see also $\text{Clai}_{49\text{-}51/71}$ in Figure 7.1). Later on (in utterance 51) Lukas calls the solid in his drawing a "truncated cone", although what he has drawn is actually a truncated bicone (Figure 7.3).

In the second *path* of the pattern of argumentation, Tom states his claim that the solid is a sphere (Clai_{64}), rejecting Lukas' claim. As soon as Tom states his sphere-claim, Lukas abandons his own claim and accepts Tom's idea. Since both students agree with Tom's claim, this claim becomes the final conclusion of their argumentation ($1\text{D}_{67\text{-}68}$).

Utterance	Original German transcript	English translation
47	Lukas: Weißt du wie ich mir vorstelle? #01:04:39-9#	Lukas: Do you know how I imagine it?
48	Tom: Was denn? #01:04:39-0#	Tom: What then?
49	Lukas: Hast du mal einen Zettel, so einen ganz kleinen? (...) So. Ich stelle mir das Ding so vor. So irgendwie, ist ja eigentlich 3D, schlecht 3D gemalt [*see Figure 7.3*] - So ist die (unverständlich) Form [*of the solid*]. #01:04:56-7#	Lukas: Do you have a piece of paper, a small one? (...) So. I imagine the thing like this. Something like this, it is of course three-dimensional, badly three-dimensionally drawn [*see Figure 7.3*] – Such is the form [*of the solid*].
50	Tom: Weißt du was es ist? [*He refers to the form of the invisible solid*] #01:04:59-3#	Tom: Do you know what it is? [*He refers to the form of the invisible solid*]
(...)		
53	Lukas: Ein abgeschnittener Kegel. #01:05:02-6#	Lukas: A truncated cone.
54	Tom: Du nicht. #01:05:05-3#	Tom: No.
(...)		
64	Tom: Kugel! #01:05:30-7#	Tom: Sphere!
65	Lukas: Was? Ne Kugel, das hier? #01:05:32-9#	Lukas: What? A sphere, this one?
(...)		
67	Lukas: Ah, das ist eine Kugel! #01:05:47-9#	Lukas: Oh, it is a sphere!
(...)		

continued on next page

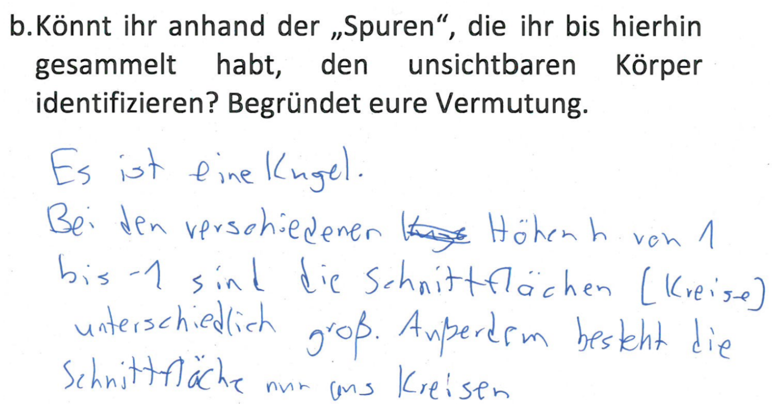
4 A path is a part of the pattern that begins at the pattern's element and ends at one of its last elements, without any junctions or splits (see Chapter 5, subsection 5.4.2).

71	Lukas: Ich dachte das Ding sieht so aus [*points at his drawing, see Figure 7.3*]. Ich dachte das sind zwei Kegel, die abgeschnitten werden. Ich hab mich schon gewundert warum sich das [*the shape of the cross-section*] nicht ändert, wenn man jetzt diesen äh, die Neigung ändert. Ja, das Programm ist kaputt, ganz klar. #01:06:18-2#	Lukas: I thought this thing looked like this [*points at his drawing, see Figure 7.3*]. I thought it was two cones that are truncated. I was wondering why that [*the shape of the cross-section*] doesn't change if you change this, uh, the tilt. Yes, the software is broken, of course.
(...)		
77	Lukas: Hätte ja sein können. Aber wenn man den mit der Neigung hätt's, hätt's nicht gepasst [*he refers to Tom's claim*]. #01:06:50-2#	Lukas: Could have been. But if you had the thing with the tilt, it [*he refers to Tom's claim*] would not have worked.

Transcript 7.1: Part of Tom and Lukas' discussion in episode GR2TL-2

Nevertheless, even after the final conclusion has been stated, Lukas explains under which conditions his claim would have been valid and why it no longer is. In utterances 71 and 77, Lukas says that he thought the solid looks like two cones with their bases attached to each other and truncated tops (see Figure 7.3). With this claim in mind, he had already wondered why the cross-section did not change shape during the tilt-variation. He then jokes that this probably happened because the software is broken. Finally, in utterance 77, he says that his claim could have been true, and that if the cross-section indeed changed shape during the tilt-variation then Tom's claim about the sphere would not align with the observation. Lukas refers to a condition that if applied here, then it would befit his claim, while at the same time contradicting Tom's claim. Nevertheless, the shape of the cross-sections does not change during n-variation, a fact that contradicts his claim, while at the same time it fits with Tom's claim. This comment (see utterances 71 and 77) by Lukas allows me to consider it the origin of an implicit contradiction (Contra_{77}) to his own claim, and the reason why he accepted Tom's claim (Clai_{64}) as valid.

Therefore, both *paths* of the pattern arrive at the same conclusion ($1\text{D}_{67\text{-}68}$). The fact that the final conclusion of the argumentation is drawn by a simple deduction (1D) is not visible in Tom and Lukas' oral argumentation, but it is evident in their written justification (see Figure 7.4 and its translation). This fact shows the importance of the observation of both argumentation forms (oral and written), in order to attain a complete picture of students' argumentation as is possible.


b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Es ist eine Kugel.
Bei den verschiedenen ~~Lage~~ Höhen h von 1 bis -1 sind die Schnittflächen (Kreise) unterschiedlich groß. Außerdem besteht die Schnittfläche nur aus Kreisen

Figure 7.4: Tom and Lukas' written justification in episode GR2TL-2

Translation:

b. Could you identify the invisible solid, based on the "clues" that you have gathered until now? Justify your supposition.

"*It is a sphere.*

For the various heights h from 1 to -1 the cross-sections (circles) are of different size. In addition the cross-sections consist only of circles".

7.2.2 Pattern 2PW – "Narrowing down" pattern: DO⟶ Hyp⟶ DO⟶ Conc

The "Narrowing down" pattern of argumentation is met in three episodes, namely GR1AD-3B.II[5], GR1AD-3A.III[6], and GR2TL-3A.I[7] (see all episode structures in Appendices C3, C2, and C6 respectively). I present this pattern, using episode GR1AD-3A.III as the main example. I also present the patterns of argumentation of the other two episodes, as well as some interesting aspects.

Episode GR1AD-3A.III

The structure of this episode is presented in Appendix C2 and Figure 7.5 below illustrates the *pattern of argumentation* (also in Appendix D3). In this episode Axel and Dave work on Task 3A (invisible cone), their third task on the second day of the study. The pattern of argumentation consists of one main path and two small divisions that lead to contradictions and refutations of some hypotheses.

In short, in this episode Axel and Dave perform guided exploration using the cases and position in the Exploration Matrix. During the exploration of the first case, they create four hypotheses, three of which (cylinder, bicone, and half-sphere) are contradicted and one (cone) which becomes a claim. They then explore the rest of the cases in the Exploration Matrix and afterwards draw the conclusion that the solid is a cone, without explicitly connecting it to the data they previously observed. At

[5] Group 1, Axel and Dave working on Task 3B as their second task on Day 2 of the study.
[6] Group 1, Axel and Dave working on Task 3A as their third task on Day 2 of the study.
[7] Group 2, Tom and Lukas working on Task 3A as their first (and only) task on Day 2 of the study.

the end they discuss what will be their written justification, agreeing to write that the solid is a cone, because it has a circular base and converges to a point.

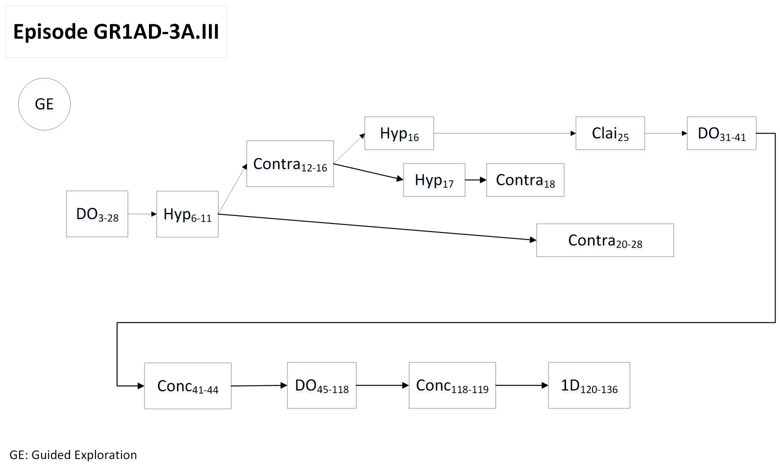
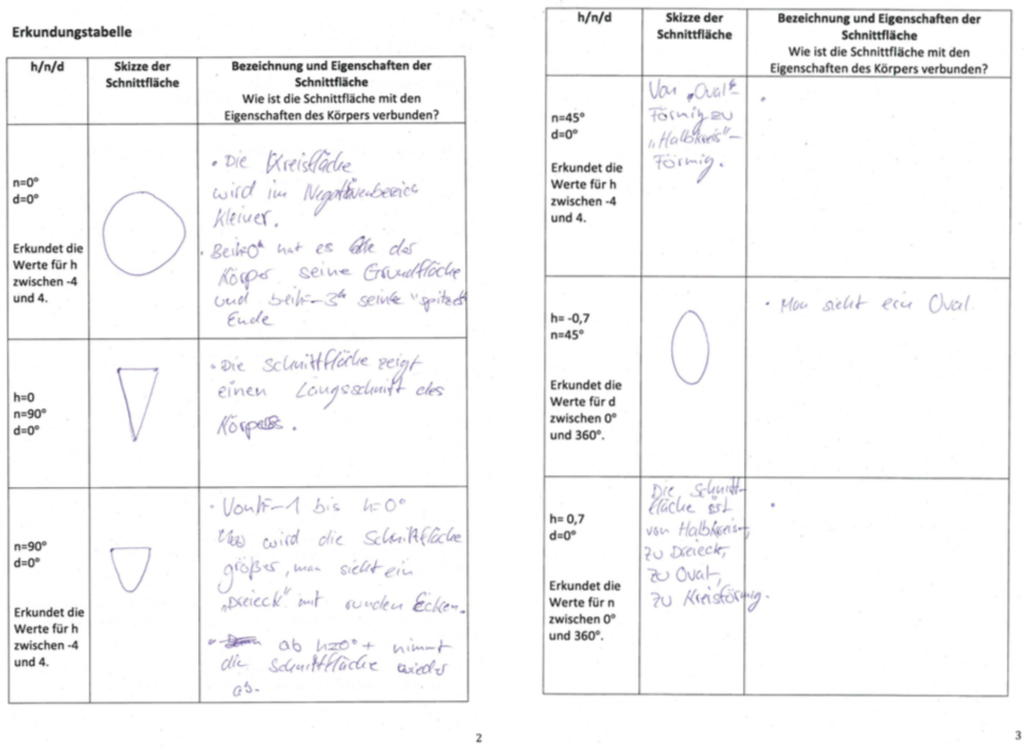


Figure 7.5: Pattern of argumentation in episode GR1AD-3A.III

The pattern begins with an element of observing data ($DO_{3\text{-}28}$). Axel and Dave explore the situation using the Exploration Matrix given in their worksheet (see Figures 7.6a and b). Their guided exploration begins with the case (h, n=0°, d=0°) (see Figure 7.6a, the first raw in the Exploration Matrix). During this exploration they state their first two hypotheses (during $Hyp_{6\text{-}11}$): the solid is either a *half-sphere* or a *cylinder* (utterance 11, see Transcript 7.2 below). The cylinder-hypothesis, generated by Dave (utterance 11) is contradicted immediately ($Contra_{12\text{-}16}$), as soon as Axel questions it (utterance 12: "it can't be a cylinder"). While Axel tries to explain the his reasoning behind his contradiction he says that the solid could be a cone (utterance 14). Then, Dave also says that he "meant cone, not cylinder" (utterance 16). This way Dave contradicts his own hypothesis (cylinder) by correcting his misspeaking and the two students together create a new hypothesis in step Hyp_{16} of the pattern, namely that the solid is a cone.

At the same time, Axel generates another hypothesis (utterance 17, Hyp_{17}). He describes the solid as two cones with common bases. That is a bicone (see Figure 7.13). Dave contradicts Axel's hypothesis ($Contra_{18}$) by saying that it is a "normal cone", because for n=0° and d=0°, when the slider is moved over h=0, then the cross-section disappears. This statement by Dave implies that this happens because the base of the cone is on xOy at position (h=0, n=0°,d=0°), which means that as soon as the solid is lifted over h=0, the cross-sections will disappear. Nevertheless, Dave does not say this explicitly.



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Die Kreisfläche wird im Negativenbereich kleiner. • Bei h=0° hat es die des Körper seine Grundfläche und bei h=-3° seine "spitzes" Ende
h=0 n=90° d=0°		• Die Schnittfläche zeigt einen Längsschnitt des Körpers.
n=90° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Von h=-1 bis h=0° wird die Schnittfläche größer, man sieht ein „Dreieck" mit runden Ecken. • ~~Dann~~ ab h=0°+ nimmt die Schnittfläche wieder ab.

2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=45° d=0° Erkundet die Werte für h zwischen -4 und 4.	Von „Oval"-Förmig zu „Halbkreis"-Förmig.	•
h= -0,7 n=45° Erkundet die Werte für d zwischen 0° und 360°.		• Man sieht ein Oval
h= 0,7 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Die Schnittfläche ist von Halbkreis-, zu Dreieck, zu Oval, zu Kreisförmig.	•

3

Figure 7.6: a (left) and 7.6b (right). Exploration Matrix from Axel and Dave's worksheet in episode GR1AD-3A.III

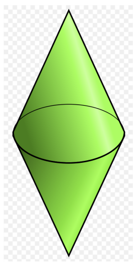


Figure 7.7: Bicone

Utterance	Original German transcript	English translation
5	Dave: Kreis, wird kleiner und ist weg [*Dave continues moving the h-slider, now from 4 down to almost -4. He stops a little before -4, after there is no more cross-section to be observed. See Figures 7.8 and 7.9*]. #00:43:30-1#	Dave: Circle, getting smaller and goes away [*Dave continues moving the h-slider, now from 4 down to almost -4. He stops a little before -4, after there is no more cross-section to be observed. See Figures 7.8 and 7.9*].
6	Axel: Ist es ne halbe Kugel? #00:43:32-4#	Axel: Is it a half-sphere?
7	Dave: Nein. #00:43:34-4#	Dave: No.

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8	Axel: Ja, aber drei- minus zwei Komma fün- #00:43:36-5#	Axel: Yes but three- minus two point fi-
9	Dave: Warte das ent- (unverständlich) #00:43:37-6#	Dave: Wait this - (incomprehensible)
10	Axel: Größer? #00:43:37-9#	Axel: bigger?
11	Dave: Entweder ist das ne halbe Kugel oder es ist ne Zylinder. #00:43:44-4#	Dave: It is either a half-sphere, or a cylinder
12	Axel: Es kann kein Zylinder sein. #00:43:45-1#	Axel: It cannot be a cylinder
13	Dave: Warum? #00:43:48-2#	Dave: Why?
14	Axel: Da musste ein- Dann müsste in Zylinder- (..) Dann, es müsste einen Kegel; ein Kegel der auf beiden Seiten- #00:43:56-6#	Axel: There had to be a- Then in cylinder- (..) Then, it would have to be a cone; a cone that on both sides-
15	Dave: (unverständlich, Axel und Dave reden gleichzeitig). #00:43:57-1#	Dave: (incomprehensible, Axel and Dave talk simultaneously)
16	Dave: Kegel meine ich, keinen Zylinder. #00:44:01-9#	Dave: I mean a cone, not a cylinder
17	Axel: Ein Kegel sein, der auf beiden Seiten- Also es ist ein Kegel, der mit den- mit der Grundfläche aneinander ist [*what Axel describes is a bicone*]. Mach nochmal [*Axel asks Dave to perform the height-exploration once again*]. #00:44:11-2#	Axel: A cone, that on both sides- So, it is a cone that- has the bases on each other [*what Axel describes is a bicone*]. Do it again [*Axel asks Dave to perform the height-exploration once again*]
18	Dave: Nein. Ganz normaler Kegel. Weil, guck mal, bei null [*h=0*] verschwindet es wieder nach null [*there is no cross-section for h>0*]. (..) Mal doch mal einen Halbkreis. Oder mal ein Kreis quasi einfach [*Dave refers to the sketch Axel should draw for this case on the Exploration Matrix*]. #00:44:25-9#	Dave: No. A completely normal cone. Because, look, at zero [*h=0*] it disappears again after zero [*there is no cross-section for h>0*] (..) Draw a half-circle. Or draw a circle simply [*Dave refers to the sketch Axel should draw for this case on the Exploration Matrix*]
19	Axel: Ist ja wie ein- äh ja Kreis. (..) #00:44:35-7#	Axel: It is like a – ehh yes circle.
20	Dave: Kreisfläche wird immer kleiner. (..) Bis drei Komma, bis drei, minus drei. Von null bis minus drei [*(-3<h<0, n=0°, d=0°)*]. #00:44:54-5#	Dave: The circular surface is getting smaller. (..) Up to three point, up to three, minus three. From zero until minus three [*(-3<h<0, n=0°, d=0°)*]

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21	Axel: Ok. Ja, ok. #00:44:59-8#	Axel: Ok. Yes, ok.

Transcript 7.2: Axel and Dave on Task 3A (invisible cone)

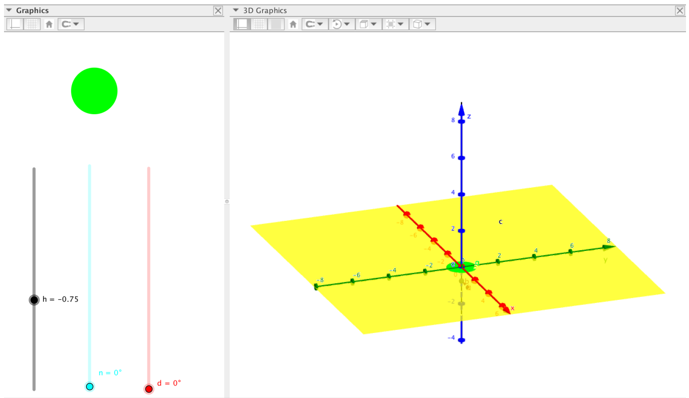


Figure 7.8: Position (h=-0,75, n=0°, d=0°)

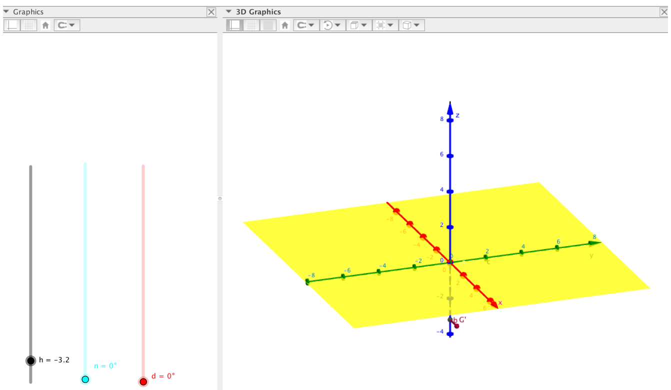


Figure 7.9: Position (h=-3,2, n=0°, d=0°)

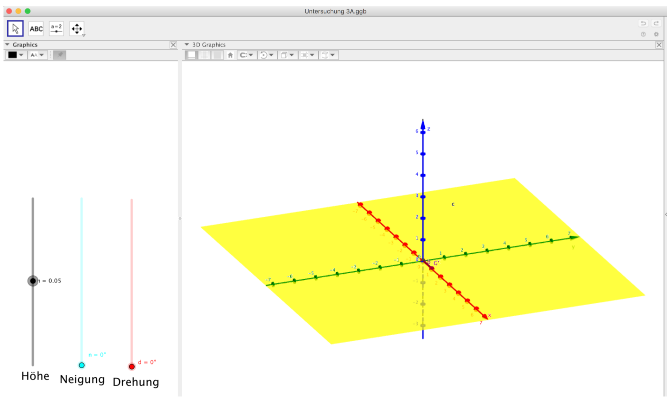


Figure 7.10: Position (h=0,05, n=0°, d=0°)

In this part of the transcript (see Transcript 7.3), Dave and Axel are re-examining their half-sphere hypothesis ($Hyp_{6\text{-}11}$, in Figure 7.5), which occurred at the beginning of their discussion, together with the cylinder hypothesis. Dave starts by saying that the solid is not a half-sphere[8] (utterance 22), and Axel agrees with him, adding that the solid is a cone[9] (utterance 25). At this point the cone hypothesis becomes a claim ($Clai_{25}$), because the students seem certain for the validity of their supposition and they set off to strengthen it through further observation.

Utterance	Original German transcript	English translation
22	Dave: Warte, wenn das ein Kreis ist. (..) Axel, das ist kein, das kann kein Halbkreis sein [*Dave misspeaks. He meant to say half-sphere (Halbkugel)*]. #00:45:14-1#	Dave: Wait if that's a circle. (..) Axel, that is not a semicircle [*Dave misspeaks. He meant to say half-sphere (Halbkugel)*].
23	Axel: Ne, ist es auch nicht. Das ist ne Pyramide. Äh ne, ne wie heißt denn. #00:45:17-1#	Axel: No, it is not. This is a Pyramid, eeh no, no how is it called?
24	Dave: Ku-, Kegel. #00:45:18-0#	Dave: Sphe-, cone.
25	Axel: Kegel. #00:45:18-5#	Axel: Cone
26	Dave: Man hat einen Halbkreis [*Dave again means half-sphere (Halbkugel)*]. Guck mal der ist hier, ein, ein, ein met-, zwei Meter Durchmesser, ne? [*The sliders are at position (h=0, n=0°, d=0°). Dave points at the screen on the two points where the circumference of the cross-section meets the y-axis and measures the length of the segment between these two points. See Figures 7.11 and 7.12*] #00:45:21-6#	Dave: You have a semicircle [*Dave again means half-sphere (Halbkugel)*]. Look, it is here, one, one met-, two meters diameter, right? [*The sliders are at position (h=0, n=0°, d=0°). Dave points at the screen on the two points where the circumference of the cross-section meets the y-axis and measures the length of the segment between these two points. See Figures 7.11 and 7.12*]
27	Axel: Ja. #00:45:22-2#	Axel: Yes.
28	Dave: Das heißt dann muss der auch bis hier zwei, bei minus zwei aufhören [*Dave points at the screen, at point (0,0,-2) on the z-axis. See Figure 7.13*]. Aber der geht dann bis minus drei runter. #00:45:26-6#	Dave: This means, then this must also until two, quit until minus two [*Dave points at the screen, at point (0,0,-2) on the z-axis. See Figure 7.13*]. But it goes down until minus three.
29	Axel: Ja, ich weiß. Ja, ist es auch nicht [*a half-sphere*]. #00:45:31-3#	Axel: Yes, I know. Yes, it is not [*a half-sphere*].

Transcript 7.3: Axel and Dave on Task 3A (invisible cone)

Next, Dave explicitly explains why the solid is not a half-sphere, contradicting the ormer hypothesis ($Contra_{20\text{-}28}$). In the half-sphere hypothesis, it is implied that the base of the solid "sits" on plane xOy. Dave says (utterance 26, see Transcript 7.3)

8 In this part of the transcript, Dave repeatedly misspeaks, referring to the half-sphere (Halbkugel) as a half-circle (Halbkreis). The fact that he actually does this, and that he doesn't really mean a half-circle is apparent from the transcript, where he talks about the diameter of the **solid** and its circular cross-section, as well as about the height of the **solid**.

9 In utterance 23 Axel misspeaks too, calling the solid a pyramid but he immediately corrects himself.

that if the solid is a half-sphere, then its diameter is two (meters), something he observes at position (h=0, n=0°, d=0°) (see Figures 7.11, 7.12 and 7.13). Axel agrees, and Dave continues by arguing that in that case, when dragging the solid downwards (h<0), there should be cross-sections until h=-2 and then they should disappear. But this is not so, since the cross-section only disappears after h=-3[10]. Axel agrees and thus in step $\text{Contra}_{20\text{-}28}$ the half-sphere hypothesis is refuted as well. This contradiction ($\text{Contra}_{20\text{-}28}$) is based on hypothetical deductive reasoning (see Chapter 2, subsection 2.1.2). Dave contradicts the half-sphere hypothesis, starting with the *assumption* that this is a case and reaching a contradiction, which leads to the rejection of the assumption by *reduction ad absurdum.*

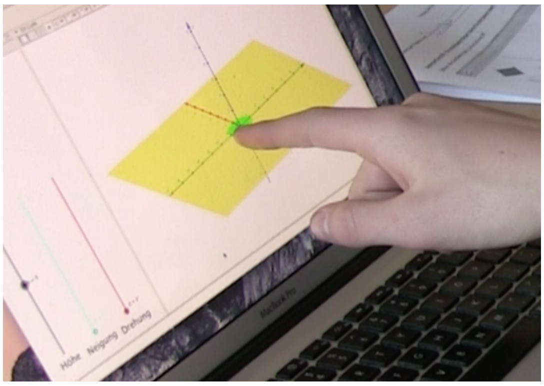


Figure 7.11: Position (h=0, n=0°, d=0°). Dave points at the *left point of intersection between the cross-section and the y-axis.*

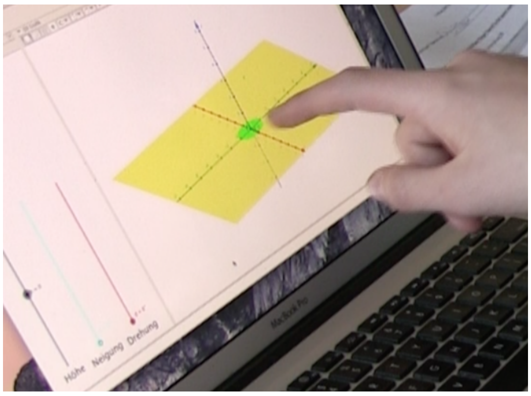


Figure 7.12: Position (h=0, n=0°, d=0°). Dave points at the *right point of intersection between the cross-section and the y-axis.*

[10] During his argument in Contra20-28, Dave makes one mistake. He says that the cross-sections should appear only until h=-2. This is not correct because, if the diameter of the cross-section in (h=0, n=0°, d=0°) is 2, and we consider this cross-section to be the base of the half-sphere, then the height of the half-sphere is equal to the radius of the whole sphere, which equals with half the length of its diameter. This means, that the cross-sections should appear only until h=-1. Nevertheless, the exploration shows that the cross-section continue to appear until h=-3, and since both h=-2 and h=-1 are before h=-3, this mistake does not invalidate his argument.

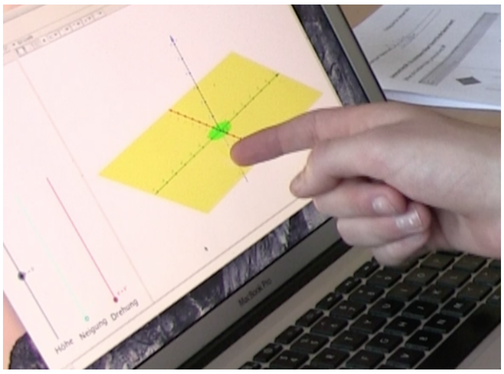


Figure 7.13: Position (h=0, n=0°, d=0°). Dave points at the point (0, 0, -2).

The pattern of argumentation continues again in the main path, with element $DO_{31\text{-}41}$. From utterance 30 on, the students continue their guided exploration going over the rest of the cases in the EM keeping in mind their claim that the solid is a cone. The students observe the data emerging from the exploration of the case (h=0, n=90°, d) ($DO_{31\text{-}41}$), and they draw the conclusion that the cross-section is a cross-cut (Queerschnitt) of the cone ($Conc_{41\text{-}44}$). With the word "Queerschnitt" they refer to the cross-section that goes through the top of a cone, ending vertically (on the diameter of) its base (see Figures 7.14a and b).

The pattern continues with one more step of data observation ($DO_{45\text{-}118}$). The students explore the rest of the cases in the Exploration Matrix and they come to the conclusion that the solid is a cone ($Conc_{118\text{-}119}$). In utterances 118-119 their conclusion is only announced, without being justified.

But then, while Axel starts to orally formulate the written justification in order to note it on the worksheet, the two students discuss how their argument should be stated (see Transcript 7.4). Hence, the pattern ends with a conclusion drawn by simple deduction ($1D_{120\text{-}136}$). Axel and Dave say that the solid is a cone (utterance120, see Transcript 7.4), because it has one round base (utterance126) and it converges to a point (utterance134). Dave also adds that the solid is "longer that three" (utterance135), but then he says that the previous two statements are enough and that they do not need this. This element of the pattern is a simple deduction, in which a conclusion (the solid is a cone) is drawn based on specific data (the solid has a round base and it converges to a point). Normally, in deductive arguments, there is a rule connecting the data to the conclusion. Here, the rule is left unstated.

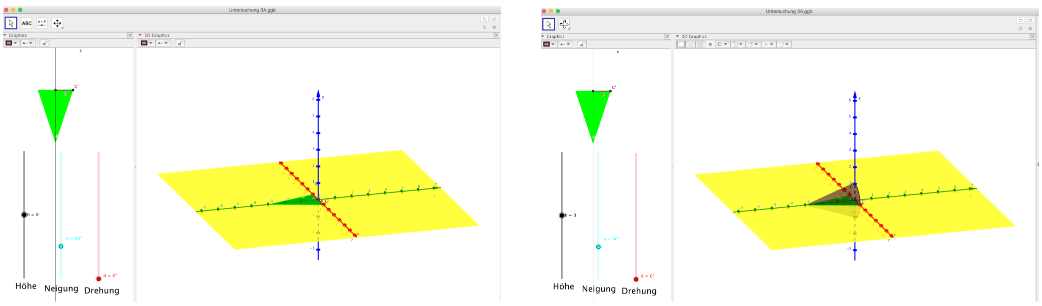


Figure 7.14: a (left) and b (right). Position (h=0, n=90°, d=0°) as observed by the students (left) and as it looks if the solid is visible (right)

Utterance	Original German transcript	English translation
120	Axel: Kegel. #00:56:57-6#	Axel: Cone.
121	Dave: Ja. Runde, schreib runde Grundfläche #00:57:00-4#	Dave: Yes. Round, write round base.
122	Axel: Ja. #00:57:01-4#	Axel: Yes.
123	Dave: Eine! Schreib eine runde Grundfläche. #00:57:04-6#	Dave: One! Write down one round base.
124	Axel: Ja. #00:57:07-5#	Axel: Yes.
125	Dave: Unterstreich mal „eine“. (..) Schreibst runde Grundfläche zusammen. Und „runde“ groß. #00:57:17-4#	Dave: Underline the “a”. (..) Write down a round base. And "round" in capital letters.
(...)		
134	Axel: Ja. Ich und (unverständlich). Lass mich in Ruhe. Eine runde Grundfläche, läuft spitz zu. #00:57:42-5#	Axel: Yes. I and (incomprehensible). Leave me in peace. One round base, converges to a point.
135	Dave: Äh, ja. Und es ist länger als drei. Zumindest (unverständlich) passt schon, läuft spitz zu, reicht schon. #00:57:54-2#	Dave: eh, yes. And it is longer than three. At least (incomprehensible) it is enough. Converges to a point, it is enough.
136	Axel: Ja. Fertig. (unverständlich)	Axel: Yes. Done. (incomprehensible)

Transcript 7.4: Axel and Dave on Task 3A (invisible cone)

Episode GR1AD-3B.II

In this episode, Axel and Dave work on the task of the invisible pyramid (Task 3B). The structure of this episode is presented in Appendix C3. The transcript of the episode is provided in the digital Appendix of this dissertation. Figure 7.15 illustrates the *pattern of argumentation* for this episode (also in Appendix D4).

This pattern has a single path, because the students agree on the same claim (that the solid is a pyramid), and therefore, there is no division in their pattern. In this pattern the students start by observing data (DO), then they generate their claim (Clai), and afterwards they continue with their observations (DO), before they finally draw their conclusion (Conc) about the form of the solid in question.

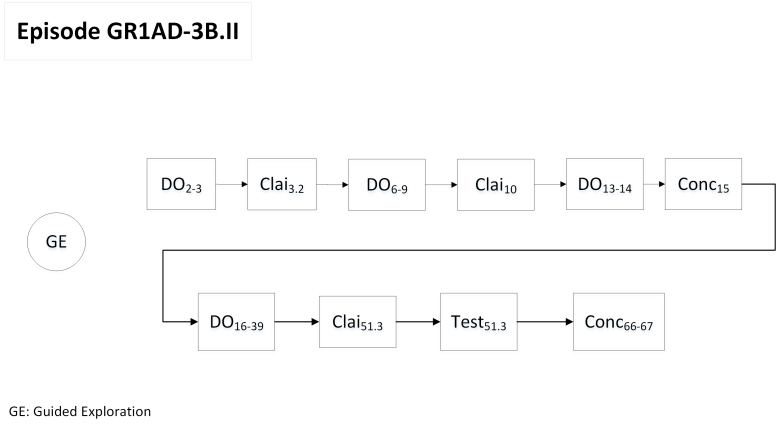
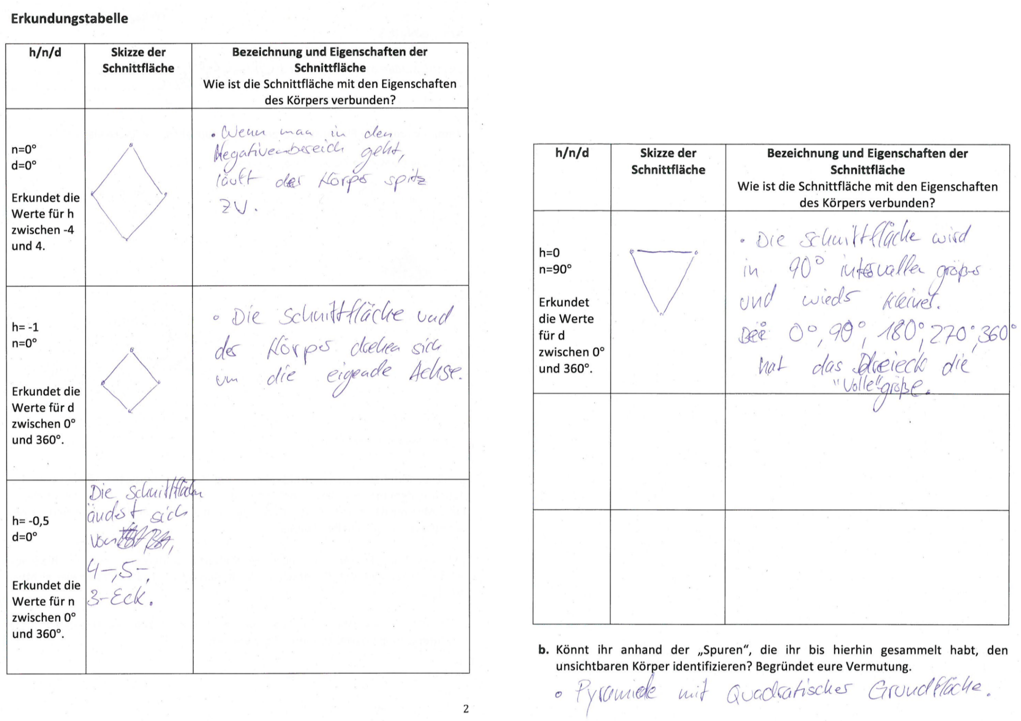


Figure 7.15: Pattern of argumentation in episode GR1AD-3B.II

The pattern begins with data observation ($DO_{2\text{-}3}$). Axel and Dave perform a guided exploration for all the cases provided in the Exploration Matrix in their worksheet (Figures 7.16a and b). This means, that Axel and Dave perform here a guided exploration of the situation. The students begin observing data ($DO_{2\text{-}3}$) while exploring the case (h, n=0°, d=0°) (see the first case in Figure 7.16a). From this observation they claim that the solid is a pyramid ($Clai_{3.2}$). They continue exploring the same case ($DO_{6\text{-}9}$) and Dave claims that the pyramid is inverted ($Claim_{10}$). Originally, Axel wonders why Dave thinks so, but then he seems to agree with Dave. No explicit explanation is given by the students for the inverted pyramid claim and at the time the matter is not discussed further.

In contrast to Tom and Lukas, in the case presented for Pattern 1PW – "Direct" pattern, here Axel and Dave state their first claims ($Clai_{3.2}$ and $Clai_{10}$) already from the beginning of their explorations. They then continue exploring and gathering data that support their claim (that the solid is a pyramid, $Clai_{3.2}$).

The pattern continues with another data observation step ($DO_{13\text{-}14}$). Axel and Dave go back to observing data, exploring the next cases in the Exploration Matrix (see Figure 7.16a and b). During the next explorations, the students observe what occurs with the cross-sections having in mind that the solid is a pyramid. From what they observe in case (h=-1, n=0°, d) (see Figures 7.17a and b), Dave draws the conclusion ($Conc_{15}$) that the pyramid spins around itself (utterance 15, "*Dreht sich um sich selbst die Pyramide*" , Translation: the pyramid rotated around itself). Dave does not explain how he arrived to this conclusion about the movement of the solid.



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Wenn man in den Negativenbereich geht, läuft der Körper spitz zu.
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.		• Die Schnittfläche und der Körper drehen sich um die eigende Achse.
h= -0,5 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Die Schnittflächen ändert sich von ~~[illegible]~~, 4-, 5-, 3-Eck.	

2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=90° Erkundet die Werte für d zwischen 0° und 360°.		• Die Schnittfläche wird in 90° intervallen größer und wieder kleiner. Bei 0°, 90°, 180°, 270°, 360° hat das Dreieck die "Volle" größe.

b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

• Pyramide mit Quadratischer Grundfläche.

Figure 7.16: (left) and b (right). Exploration Matrix from Axel and Dave's worksheet in episode G1AD-3B.II

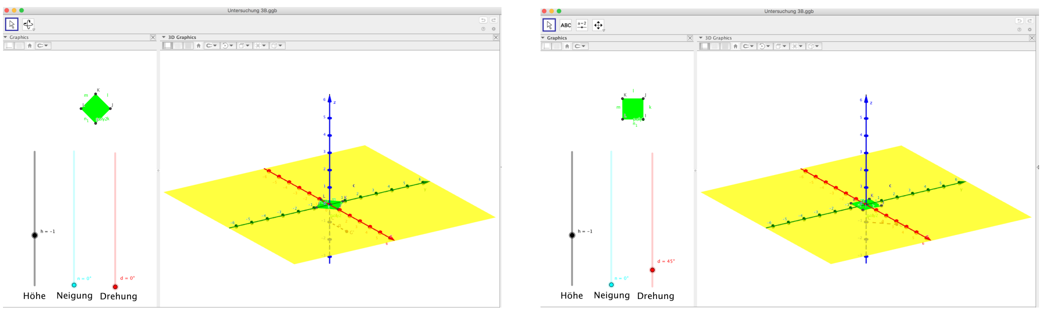


Figure 7.17: a (left) and b. Position (h=-1, n=0°, d=0°) and position (h=-1, n=0°, d=45°)

After this, the students move on to the next cases ($DO_{16\text{-}50}$). During the observation of the new data emerging from the exploration of the case (h=-0,5, n, d=0°) ($DO_{16\text{-}39}$) they only describe the cross-sections they see on the screen (triangles, quadrilaterals and pentagons), without explicitly connecting them with the characteristics and properties the solid should have in order to have such cross-sections.

In their next exploration (h=0, n=90°, d), students observe what happens when they spin the solid around its axis while on height zero and tilted at 90° ($DO_{40\text{-}50}$, see Figures 7.18a and b).

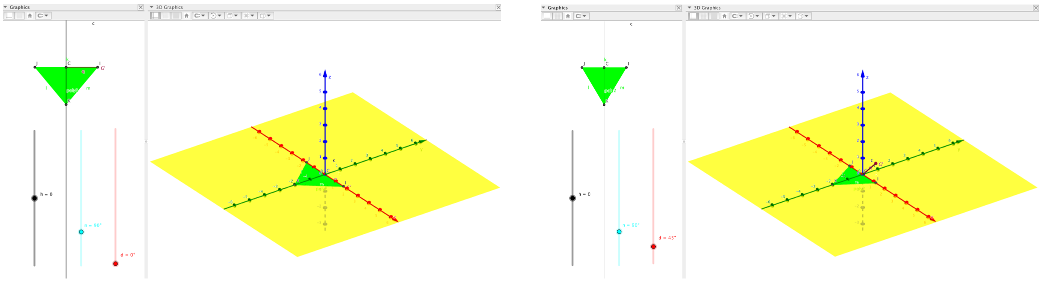


Figure 7.18: a (left) and b (right). Position (h=0, n=90°, d=0°) and position (h=0, n=90 °, d=45°)

Axel and Dave agree that what they observe happening with the cross-section while spinning the solid in case (h=0, n=90°, d), "is logical" [11] and "makes absolute sense" [12] to them. By that they probably mean that what they observe befits to their claim that the solid is a pyramid (Clai$_{3.2}$). What actually happens during this observation is that the base of the triangular cross-section (segment IJ) gets shorter and longer as the values of the spin vary while they move the d-slider.

Then, Axel asks Dave, who manipulates the slider, to check once more the values of d (spin) for which IJ is the longest. Before Dave completes the d-variation that Axel asked him to do, Axel makes a prediction (Clai$_{51.3}$). He predicts that one of the values of d for which IJ is the longest, is at 90° and that then it will get smaller immediately over 90°. This is a step of "stating a claim", which is then followed by a "testing" step (Test$_{51.3}$). Dave moves the d-slider accordingly to test the validity of Axel's prediction. Although it is not observable in the transcript, in the video of the group one can see that Axel's predictions are verified as soon as Dave moves the d-slider at position d=90° and then dragging it over 90°. This is also observable in Figures 7.19a and b. Segment IJ is longer in position (h=0, n=90°, d=90°) than it is in position (h=0, n=90°, d=95°).

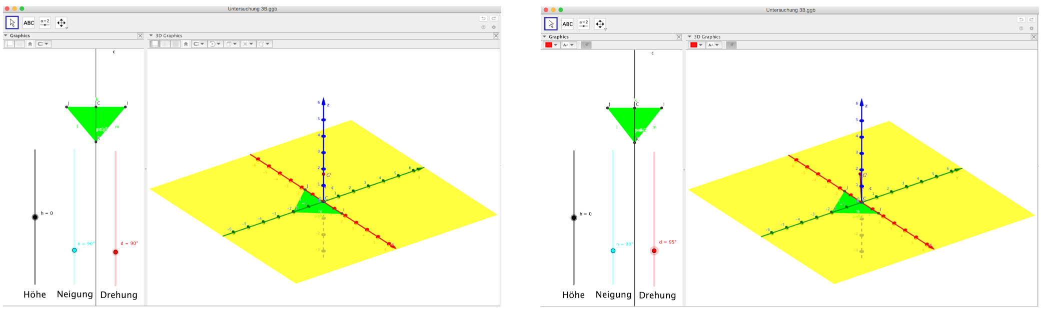


Figure 7.19: a (left) and b (right). Position (h=0, n=90°, d=90°) and position (h=0, n=90 °, d=95°)

The last step in this pattern is Conc$_{66\text{-}67}$ (see Figure 7.15). During this step, the two students draw their final conclusion (see Transcript 7.5).

[11] Utterance 42, Dave: "(...) das ist logisch"

[12] Utterance 47, Axel: "macht es voll Sinn"

Utterance	Original German transcript	English translation
66	Axel: [*He reads question b on the worksheet*] Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Pyramide. #00:41:24-5#	Axel: Could you identify the invisible solid, based on the "clues" that you have gathered until now? Pyramid,
67	Dave: Pyramide mit einer quadratischen Grundfläche. #00:41:26-4#	Dave: Pyramid with a square base.
68	Axel: Ja. #00:41:30-5#	Axel: Yes.

Transcript 7.5: Axel and Dave's conclusion on Task 3B (invisible pyramid)

Axel and Dave agree that the solid is a pyramid with a square base. Nevertheless, they do not provide an explicit justification for this, neither orally, nor in writing. In their worksheet they state that the solid is a "Pyramid with square base" (see their answer in question b at the end of Figure 7.16b). Furthermore, the students do not specify the orientation of the pyramid in their final answer given. When they generate their claims they argue that the solid is a pyramid ($\text{Clai}_{3.2}$), and later they add that the pyramid is inverted (Clai_{10}). Nevertheless, at the end of their discussion they do not refer to the initial orientation[13] of the pyramid again. They only discuss about the form of the solid, and not about its initial orientation.

Episode GR2TL-3A.I

The structure of episode GR2TL-3A.I (see Appendix C6) is similar to that of episode GR1AD-3B.II presented above. Therefore, I discuss it here without diving into too much detail. Figure 7.20 illustrates the pattern of argumentation (see Appendix D7). Tom and Lukas work on Task 3A and they mainly perform guided exploration, using the Exploration Matrix. At the end of their argumentation, they perform one last exploration of their own choice (free exploration), which I discuss later.

This pattern of argumentation begins with observing data (DO_1) and creating hypotheses and claims (Hyp_2: sphere, $\text{Claim}_{3\text{-}5}$: cone). In this process, the sphere hypothesis is refuted, while the claim about the cone is strengthened and its orientation is specified ($\text{1D}_{7\text{-}9}$: upward cone). Then the students continue their guided exploration, going through all the cases in the Exploration Matrix. Their discussion is interrupted in utterance 194 by the researcher, where a discussion between the researcher and the two students takes place (utterances 194–213). After the discussion with the researcher the students explore one last position ($\text{DO}_{217\text{-}225}$) chosen by Tom in the form of free exploration[14], and then draw their final conclusion (Conc_{237}).

[13] That is the orientation of the solid at the initial position (h=0, n=0°, d=0°).

[14] This is a case of free exploration, because it entails initiative from the side of the students, but it does not involve any intension, such as to test or verify a hypothesis.

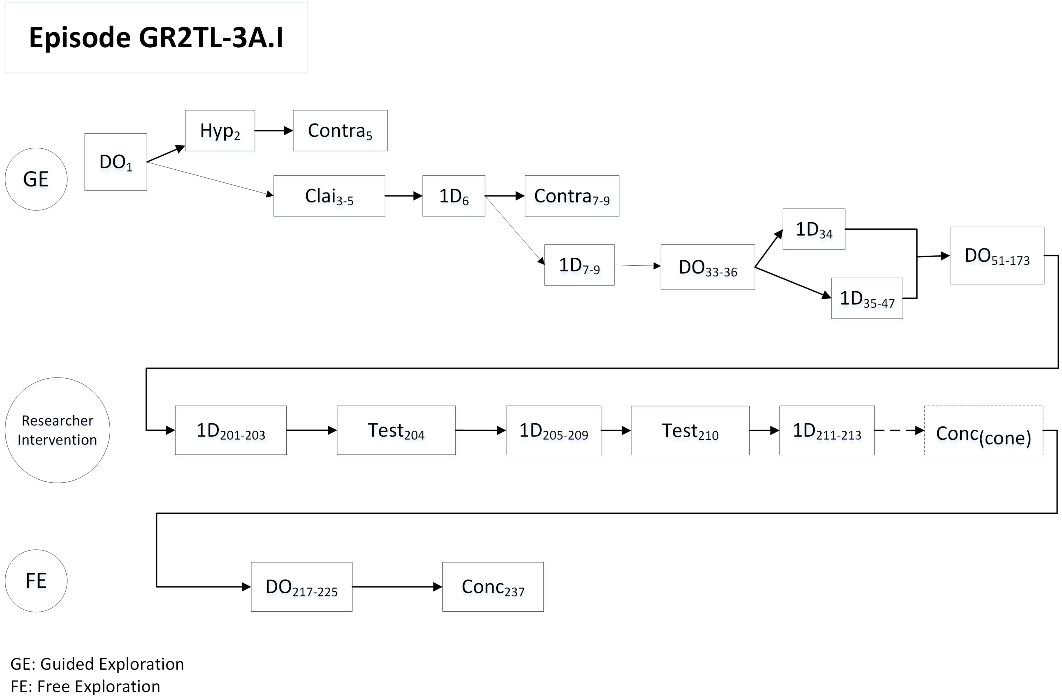


Figure 7.20: Pattern of argumentation in episode GR2TL-3A.It

As in the previous episodes, also here the students start with minor observations and create their first hypotheses. Then they continue observing data, which refute all but one of the hypotheses (this is the "narrowing down" process). The hypothesis (or claim) that is not contradicted by the observed data (here Clai$_{3\text{-}5}$: the solid is a cone), rather actually supported by it is considered a conclusion at the end of the argumentation. As a result, the conclusion is the result of confirming a hypothesis through data observation.

I would now like to comment on the researcher's intervention during Tom and Lukas' work. If, at first, we do not take into consideration what has been discussed during the researcher intervention (utterances 194-213), then the students' final conclusion about the form of the solid (Conc$_{237}$: cone) seems to be based on multiple "fitting" data. But, when we examine what happens during the researcher intervention, we can see that there is a bit more to their argumentation than just that. The students begin with a simple deduction (1D$_{201\text{-}203}$): They say that in case (h, n=0 °, d=0 °), when they move the h-slider under h=0, this results in the cross-sections getting smaller and smaller until they end up at a point and then vanish completely. Also, when they move the h-slider over h=0 the result is that the cross-section vanishes directly, without having other smaller cross-sections first, like previously. These two data, lead the students to the conclusion that the solid is a cone. Then the students add more data, through testing, in order to support their conclusion. Lukas says (utterance 204) that you can see if the solid is a cone, if you tilt the solid to position (h=0, n=90°, d=0°). At this position the cross-section is a triangle, exactly like the "Querschnitt" of a cone, which is the cross-section of a cone that emerges if you "lie the solid on its side"[15], or if you "cut a cone from top to

[15] "Der Figur quasi auf die Seite gelegt", utterance 207.

bottom, so to say"[16]. Tom and Lukas continue with another test ($Test_{210}$: (h=0, n=90°, d) exploration) that provides one more supportive datum through simple deduction ($1D_{211\text{-}213}$: no changes in the cross-section means that the solid is symmetrical).

Although this process seems inductive, as is also the process of reaching their final conclusion without the researcher's intervention, it nevertheless involves several deductions, which would have remained hidden without the intervention of the researcher. This makes evident that students' reasoning may be more precise or rich than their argumentations will allow us to think, and that therefore the negotiation of their arguments in a more provocative environment, such as that of a classroom discussion, may give the students the opportunity to enrich their arguments.

7.2.3 Pattern 3PW – "Testing" pattern: DO⟶ Hyp⟶ Test⟶ DO⟶ Conc

The "Testing" pattern occurs in episodes GR1AD-3C.I and GR1AD-2. In this type of pattern, the students examine the validity of their suppositions (hypotheses and claims) through "Testing". I use episode GR1AD-3C.I as an example in order to present this type of pattern in detail. The pattern of argumentation of episode GR1AD-2 is presented in Chapter 5 (see Figure 5.18 in subsection 5.4.4), as an example for the presentation of the methodology I use in this work.

The structure of episode GR1AD-3C.I is in Appendix C4. Here, Axel and Dave work on Task 3C (invisible cube). Figure 7.21 illustrates the pattern of argumentation for this episode (see also Appendix D5).

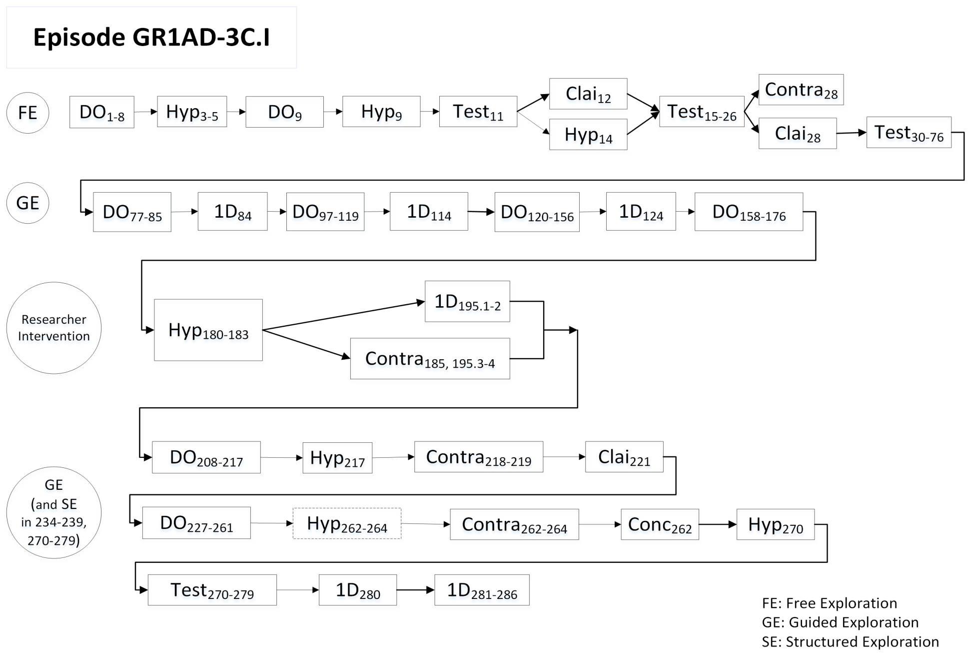


Figure 7.21: Pattern of argumentation in episode GR1AD-3C.I

[16] "Von oben nach unten ein Kegel aufgeschnitten sozusagen", utterance 209.

On the left side of the pattern you can see the types of exploration strategies[17] that the students used in the various phases of the pattern. In this episode, Axel and Dave mainly employ free and guided explorations, but they also shortly use structured exploration. The pattern of argumentation in Figure 7.21 is separated in parts, according to the exploration strategy used each time. The pattern has one main path and a few short divisions.

Here, contrary to all the previous episodes, Axel and Dave begin with a *free exploration* (FE, utterances 1-76, see Transcripts 7.6, 7.7 and 7.8). Thus, the pattern begins with Axel and Dave making their first observations of data ($DO_{1\text{-}8}$, DO_9) and generating multiple hypotheses ($Hyp_{3\text{-}5}$, Hyp_9, Hyp_{14}) and claims ($Clai_{12}$, $Clai_{28}$). They also perform tests for these hypotheses and claims ($Test_{11}$, $Test_{15\text{-}29}$, $Test_{30\text{-}76}$). These testing steps for the verification of the hypotheses and claims, are the main characteristic of this third type of patterns and the main difference between this type and the previous two ("Confirming" pattern and "Narrowing down" pattern).

The first element in this pattern is $DO_{1\text{-}8}$ (see Figure 7.21 and Transcript 7.6). The students observe the picture on their worksheet, showing the cross-section at the initial position (h=0, n=0°, d=0°) (see Figure 7.22). From this, they hypothesize that the solid can be a pyramid or a cube ($Hyp_{3\text{-}5}$). As soon as they observe the data from their first exploration though (DO_9: h, n=0°, d=0°), they implicitly abandon the pyramid hypothesis and form a new one, namely Hyp_9: "this is a cuboid".

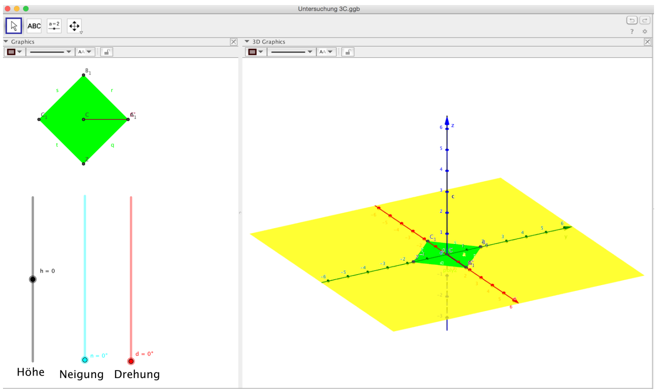


Figure 7.22: Picture of position (h=0, n=0°, d=0°) on the worksheet of Task 3C

The pattern continues with a "Testing" step for the cuboid hypothesis ($Test_{11}$). This is the part of the pattern that differentiates this type from the previously presented patterns, and makes it a "Testing" pattern. Dave explores the case (h=-0,8, n, d=0°), and they observe the hexagonal cross-sections, such as the one appearing at n=60° (Figure 7.23).

Dave starts exploring the case (h=-0,8, n, d=0°) but he is too fast. Axel asks him to re-explore it, in order to test if their cuboid hypothesis fits with the polygonal cross-sections that emerge (utterance 11, $Test_{11}$, see also the cross-section in Figure 7.23). Dave claims that the solid must be a cuboid ($Clai_{12}$), otherwise what they observe would not be possible. But he also says that it may be a prism (Hyp_{14}). Axel

[17] For a concise overview of the types of exploration strategies, refer to subsection 6.4.1 in Chapter 6.

is not so sure about the prism hypothesis, and so the students perform another test (Test$_{15\text{-}26}$). To do that, they explore again the case (h, n=0°, d=0°). The students imply, that the test will verify the validity of their cuboid claim, if they find one more rectangular base (utterances 21-22, see Transcript 7.7), since they know already that there is at least one rectangular base (DO$_{1\text{-}8}$). Moving the h-slider downwards they see that the solid is "at the beginning and at the end a rectangle" (utterance 26), which leads them to claim that "then it can only be" a cuboid.

Utterance	Original German transcript	English translation
2	Axel: Boah, das sieht (unverständlich) aus. [*They are looking at the picture on the worksheet and make conjectures based only on that. See Figure 7.22*] #00:03:08-4#	Axel: Boah, this looks (incomprehensible). [*They are looking at the picture on the worksheet and make conjectures based only on that. See Figure 7.22*]
3	Dave: Ja. (..) Das ist 'ne Pyramide. #00:03:12-7#	Dave: Yes. (..) That is a pyramid.
4	Axel: Ja, wahrscheinlich. #00:03:15-6#	Axel: Yes, perhaps.
5	Dave: Oder ein Würfel. #00:03:17-5#	Dave: Or a cube.
6	Axel: Ich denke eine Pyramide, weil das andere Achse ohne ist (unverständlich). #00:03:20-2#	Axel: I thing a pyramid, because the other axis is not (incomprehensible)
7	Dave: Kann es auch etwas anderes sein? (...) #00:03:32-5#	Dave: Can it also be something else?
8	Axel: (unverständlich) #00:03:31-5#	Axel: (incomprehensible)
9	Dave: Ja, ne. Das ist ein Quader. [*He explores the case (h, n=0°, d=0°)*] #00:03:38-3#	Dave: Yes, no. This is a cuboid. [*He explores the case (h, n=0°, d=0°)*]
10	Axel: Ja. (..) #00:03:40#	Axel: Yes.

Transcript 7.6: Axel and Dave's discussion on Task 3C (invisible cube)

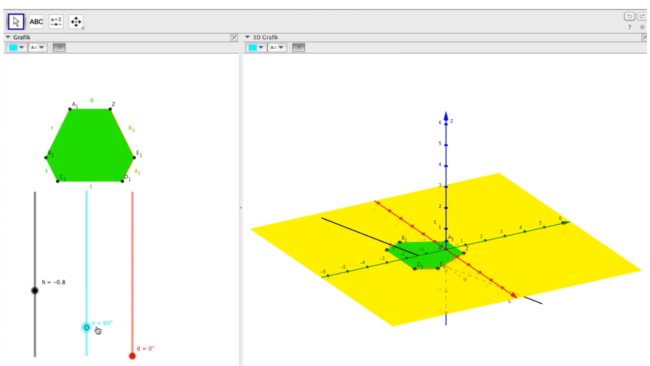


Figure 7.23: Hexagonal cross-section at position (h=-0,8, n=60°, d=0°)

Although Dave does not finish his sentence in utterance 28, leaving the name of the solid unstated, it becomes apparent later in the discussion (utterance 43), where the students refer to the solid as "the cuboid". Their observations at $\text{Test}_{15\text{-}26}$, lead simultaneously to the implicit contradiction of the prism hypothesis (Contra_{28}) and to their claim that the solid can only be a cuboid (Clai_{28}), which are the next two steps in the pattern.

Utterance	Original German transcript	English translation
11	Axel: [*Dave explores the case (h=-0,8, n, d=0°)*] Ne. Ja, oder? Doch. Warte, neig den nochmal soweit, dass mehr Ecken entstehen. [*See Figure 7.23*] (...) #00:04:02-7#	Axel: [*Dave explores the case (h=-0,8, n, d=0°)*] No. Yes, right? Indeed. Wait, tilt it again enough to create more vertices.
12	Dave: Das ist ein Quader. Sonst geht das nicht, oder? #00:04:05-9#	Dave: This is a cuboid. Otherwise it won't work, right?
13	Axel: Ja. (..) #00:04:11-4#	Axel: Yes. (..)
14	Dave: Oder ein Prisma. #00:04:12-3#	Dave: Or a prism.
15	Axel: Das werden aber nie mehr als (..) vier Ecken sichtbar sein, oder? Egal wie herum du es- #00:04:21-1#	Axel: But there will never be more than (..) four vertices visible, right? No matter how you turn it-
16	Dave: Eigentlich ja. #00:04:22-3#	Dave: Actually yes.
17	Axel: Die obere Schnittfläche. Ne, es können nie mehr-. Nicht auf der Schnittfläche. [*Axel here misspeaks, he means the bases (Grundflächen) of the solid, not its cross-sections (Schnittflächen)*]#00:04:31-9#	Axel: The top cross-section. No, it can never again-. Not on the cross-section. [*Axel here misspeaks, he means the bases (Grundflächen) of the solid, not its cross-sections (Schnittflächen)*]
18	Dave: Warte Mal, ist die Schnittfläche bei nur- (?) (..) Null, weißt du? #00:04:37-2#	Dave: Wait, is the cross-section at just- (?) (..) Zero, you know?
19	Axel: Ja #00:04:39-4#	Axel: Yes.
20	Dave: Und wenn wir hoch gehen, ist komplett weg [*For the case (h>0, n=0°, d=0°)*]. Das ist die obere Fläche [*he referes to the top face of the solid*]. #00:04:43-6#	Dave: And when we move upwards, it goes away completely [*For the case (h>0, n=0°, d=0°)*]. This is the top surface [*he referes to the top face of the solid*].
21	Axel: Wir brauchen gleich auf jeden Fall noch ein- #00:04:45-0#	Axel: We definitely need another-
22	Dave: Rechteck. #00:04:46-5#	Dave: Rectangle

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23	Axel: Ja #00:04:47-0#	Axel: Yes
24	Dave: Mal runter gehen? [*For the case (h<0, n=0°, d=0°)*] Bleibt's Rechteck? #00:04:57-1#	Dave: Go down? [*For the case (h <0, n = 0 °, d = 0 °)*] Does it remain a rectangle?
25	Axel: Auch wieder weg. #00:04:57-1#	Axel: Again away.
26	Dave: Einfach weg. Das heißt es ist am Anfang und am Ende ein Rechteck. #00:05:01-6#	Dave: Simply away. This means, it is at the beginning and at the end a rectangle.
27	Axel: Ja. #00:05:03-6#	Axel: Yes
28	Dave: Dann kann es doch eigentlich nur ein- [*he means a cuboid*] #00:05:04-2#	Dave: Then it can actually only be a- [*he means a cuboid*]
29	Axel: Ja. #00:05:05-1#	Axel: Yes.

Transcript 7.7: Axel and Dave's discussion on Task 3C (invisible cube)

The final step, in the frame of the free exploration (utterances 30-76), in the pattern is another "testing" process ($Test_{30\text{-}76}$). This time the students test their latest claim ($Clai_{28}$: cuboid) to see if what they observe befits to it. During this test step, Axel and Dave try to make sense of all their observations, also explaining to each other how the cross-sections occur and how the solid is placed in space when a specific cross-section emerges. I demonstrate here only one part of the testing process $Test_{30\text{-}76}$. For example, in utterances 41-58 Dave and Axel explore the case (h=0, n=35°, d) (see Transcript 7.8 and Figures 7.24a and b). In this part of their discussion, Dave says that the variation of d-slider in this case "turns the cuboid upwards completely" (utterance 43). He does not specify what he means by "upwards". Then Axel asks: "how can you have five points?" (utterance 44). By "five points" he means the points A_1, B_1, C_1, Z, and C, observed on the cross-section in Figures 7.24a and b. Dave shows where each of these points stand in the three dimensional Cartesian coordinate system, and explains that C is the "Hauptpunkt" (utterance 45). Axel then says that "ah so, this is the "Hauptpunkt", this does not spin around", which is the same with what Dave describes several lines later as "C is the point in the middle simply" (utterance 55). Point C is actually the center of the base of the solid, which for h=0 coincides with point O(0,0,0).

From utterance 48 on, Axel tries to make sense of the next position (h=0, n=35°, d=270°) that he observed during the exploration of case (h=0, n=35°, d), where a triangular cross-section occurs. Dave though goes on with his explanation about the five points of the cross-section of the previous position (h=0, n=35°, d=70°), until he explains also the orientation of the solid is space. His gesture while talking in utterance 57 is seen in Figure 7.26, and it reveals more about the position of the solid compared to his words. From his gesture we can understand that the vague word "gedingst" here means that the solid is *tilted.* With that, Dave explains the position of the solid in space for any position of the case (h=0, n=35°, d).

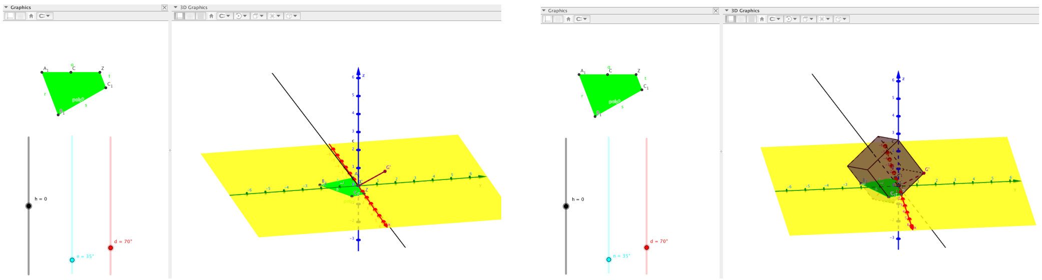

Figure 7.24: a (left) and b (right). Cross-section at position (h=0, n=35°, d=70°) as observed by the students and as it looks when the cube is visible

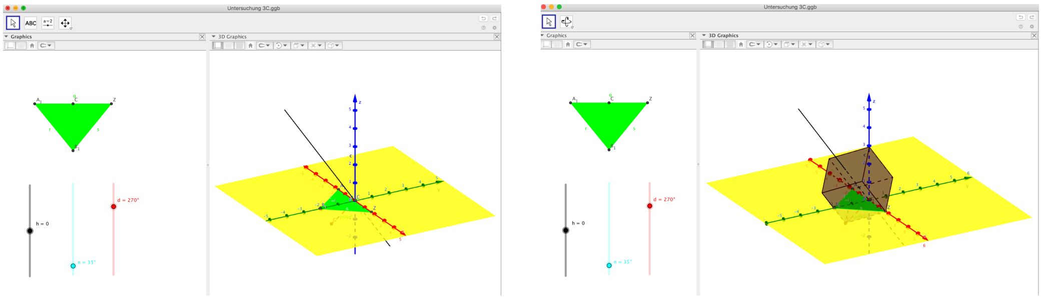

Figure 7.25: a (left) and b (right). Triangular cross-section at position (h=0, n=35°, d=270°) as observed by the students and as it looks when the cube is visible

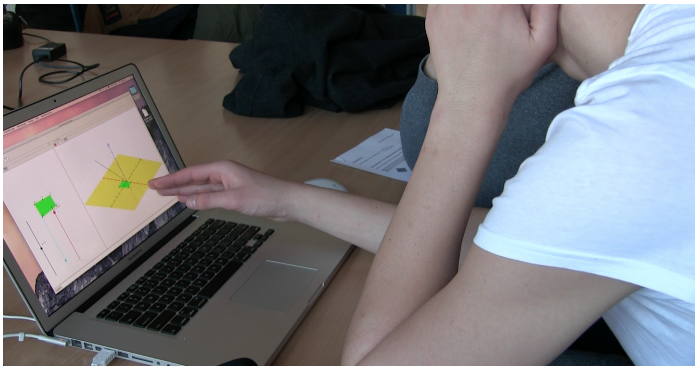

Figure 7.26: Dave's gesture showing how the solid is tilted sideways at (h=0, n=35°,d)

Utterance	Original German transcript	English translation
40	Dave: Und das ist ja der Mittelpunkt, weißt du? (unverständlich) [*he is at position (h=0, n=35°, d=0°) and he points at point C on the 3D-coordinate system*] #00:06:02-3#	Dave: And this is the center, you know? (incomprehensible) [*he is at position (h=0, n=35°, d=0°) and he points at point C on the 3D-coordinate system*]
41	Dave: Warte Mal kurz. Wenn wir den hier verändern [*talks about varying the d-slider*]? Der ist der Durchmesser, weißt du? #00:06:06-6#	Dave: Wait a minute. If we change this [*talks about varying the d-slider*]? That's the diameter,you know?

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42	Axel: Aha, jetzt bringt das hier auch Mal was [*observing what happens when varying the spinning d*]. #00:06:13-4#	Axel: Aha, now it brings something [*observing what happens when varying the spinning d*]
43	Dave: Das dreht jetzt nämlich den Quader nach oben komplett, weißt du? Aber jetzt ist ja nicht mehr- #00:06:17-2#	Dave: It turns the cuboid completely upwards, you know? But now it's no longer-
44	Axel: Wie kannst du denn fünf Eckpunkte haben? [*See cross-section in Figures 7.24a and b*] #00:06:19-6#	Axel: Then, how can you have five corner-points? [*See cross-section in Figures 7.24a and b*]
45	Dave: Das ist ja der Hauptpunkt [*he refers again to point C, which in utterance 40 he called "Mittelpunkt"*] #00:06:19-9#	Dave: That's the main point [*he refers again to point C, which in utterance 40 he called "Mittelpunkt"*]
46	Axel: Ach so ja das ist der Hauptpunkt, der wird nicht mit gedreht. #00:06:21-7#	Axel: Oh yes, that's the main point, it won't be rotated.
47	Dave: Weißt du? #00:06:23-3#	Dave: You know?
48	Axel: Ja. [*he says he understands the problem with the five point*] Hä? Ich verstehe das nicht [*He refers to the triangular cross-section at position (h=0, n=35°, d=270°)*]. #00:06:29-3#	Axel: Yes. [*he says he understands the problem with the five point*] Huh? I don't understand this [*He refers to the triangular cross-section at position (h=0, n=35°, d=270°)*].
49	Dave: Also C1 ist der Punkt. [*See Figure 7.24a. Dave points at point C1 both on the green cross-section on the left side of the screen as well as on the same point in the coordinate system on the right side of the screen*] #00:06:31-4#	Dave: So, C1 is this point. [*See Figure 7.24a. Dave points at point C1 both on the green cross-section on the left side of the screen as well as on the same point in the coordinate system on the right side of the screen*]
50	Axel: Ja, ich weiß. #00:06:32-3#	Axel: Yes, I know.
51	Dave: Z ist der Punkt. #00:06:33-2#	Dave: Z is this point.
52	Axel: Ja. #00:06:33-7#	Axel: Yes.
53	Dave: B1 ist der Punkt,- #00:06:35-5#	Dave: B1 is this point.
54	Axel: Aber ich verstehe nicht wie da ein Dreieck draus werden kann. [*See Figures 7.25a and b*] #00:06:37-4#	Axel: But I don't understand how it can become a triangle [*See Figures 7.25a and b*]
55	Dave: C ist der Punkt in der Mitte einfach (..) #00:06:40-7#	Dave: C is simply the point in the middle
56	Axel: Ja, ich weiß. #00:06:42-3#	Axel: Yes, I know

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57	Dave: Also ist das nur gedingst. [*See Figure 7.26*] #00:06:43-3#	Dave: So it is only tilted. [*See Figure 7.26*]
58	Axel: Ich verstehe es, ja. Aber- (...) #00:06:52-3#	Axel: I understand this yes. But- (..)

Transcript 7.8: Axel and Dave's discussion on Task 3C (invisible cube)

This has been one example of how the students tried to check whether their claim agrees with their observations, during the process of testing it ($\text{Test}_{30\text{-}76}$). Until now the students have seen many kinds of cross-sections: triangular, quadrilateral, pentagonal and hexagonal. For some of these observations, they can see how they fit to their cuboid claim, but others confuse them (e.g. triangular cross-section, not understood by Axel, utterance 54), because they cannot see how they could fit with their claim.

Hence, by the end of the testing step, the students say that the situation has become too complicated, and that therefore they decide to continue their observations using the Exploration Matrix. This brings us to the second phase of pattern of argumentation (see Figure 7.21), in which Axel and Dave use *guided exploration* (GE), examining the cases and position given in the Exploration Matrix (see Figures 7.27a and b). During the guided exploration they observe the data emerging from all the given cases, drawing some conclusions about the solid (e.g. $1D_{84}$: so long is the solid) or about the cross-sections (e.g. $1D_{114}$: the square cross-section spins around itself). This part starts with element $DO_{77\text{-}85}$, during which the students observe what happens during the h-variation in the case (h, n=0°, d=0°). On the screen, cross-sections appear between the height values $-2{,}3<h<0$. From that they infer that "so long is the solid" (utterance 84), meaning that the length of the solid is 2,3. This is a conclusion drawn by simple deduction ($1D_{84}$).

The guided exploration continues and the students explore the first four cases of the Exploration Mattrix (see Figures 7.27a and b).

At that point, the researcher intervenes, in order to see how the students' work is progressing. What is discussed is indicative of students' understanding of the situation they explore, up until this point (see Transcript 7.9). The researcher asks the students which solid they explore (utterance 179). Axel says that they thought that the solid is a cuboid (utterances 180-183), but "sometimes this does not work, because some times it [*the cross-section*] becomes a hexagon" (utterance 185). The researcher asks Axel why having hexagonal cross-section is a problem if the solid is a cuboid. Axel responds (utterances 195):

> "One thinks in the head, that he cross-section can either only be a triangle or a quadrilateral. Or a rectangle. But not a hexagon. And also if you change the diameter [*he actually means the d-slider and points at it*], then it [*the cross-section*] becomes a pentagon."

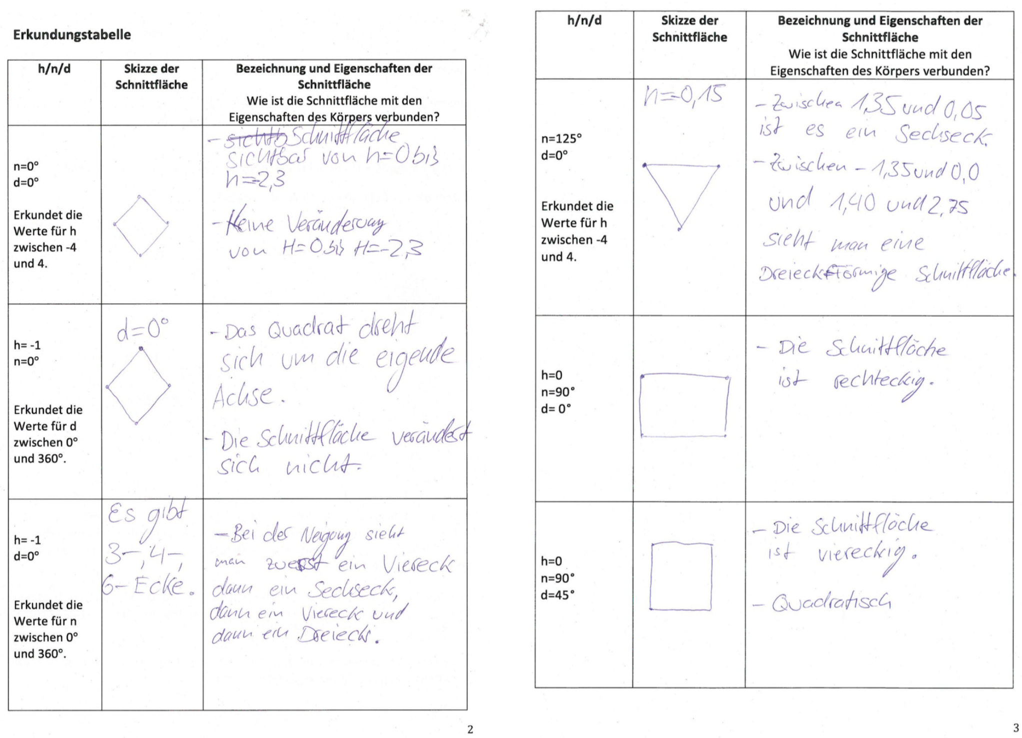

Figure 7.27: a (left) and b (right). Exploration Matrix from Axel and Dave's worksheet in episode G1AD-3C.I

The pattern of argumentation continues as shown in Figure 7.21. Axel's statement is a hypothetical deductive argument (see Chapter 2, subsection 2.1.2), with which he hypothesizes that the solid is a cuboid ($\text{Hyp}_{180\text{-}183}$), and from that he infers two things. On the one hand, through a simple deduction ($1\text{D}_{195.1\text{-}2}$), he infers that the cuboid can have triangular and quadrilateral cross-sections. On the other hand, he expresses as a contradiction to his hypothesis ($\text{Contra}_{195.3\text{-}4}$), the existence of pentagonal and hexagonal cross-sections. In the researcher's question, "if the solid is a cuboid, can't the cross-section be a hexagon?" (utterances 198-200), Axel responds that this is what they are thinking all the time, because they are not completely sure (utterances 201 and 203).

Utterance	Original German transcript	English translation
177	P: Wie geht's bei euch? #00:20:09-1#	P: How is it going here?
178	Dave: Gut. #00:20:09-9#	Dave: Well.
179	P: Gut. Was habt ihr denn da? Für einen Körper meine ich. #00:20:16-1#	P: Nice. What do you have here? I mean what type of a solid.
180	Axel: Ihm, wir glauben eigent-, also eigentlich dachten wir einen Würfel #00:20:20-6#	Axel: Hmm, we think act-, so actually we thought we have a cube
181	P: mhm (bejahend)	P: mhm (affirmative)

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182	Dave: Quader, Quader. #00:20:22-0#	Dave: Cuboid, cuboid
183	Axel: Quader. #00:20:22-5#	Axel: Cuboid
184	P: mhm (bejahend) #00:20:23-7#	P: mhm (affirmative)
185	Axel: Aber das kommt manchmal nicht hin, weil es ja manchmal ein Sechseck wird, #00:20:27-6#	Axel: But, sometimes this does not work, because some times it [*the cross-section*] becomes a hexagon
186	P: mhm (bejahend) #00:20:28-8#	P: mhm (affirmative)
187	Axel: Wie (..) #00:20:30-2#	Axel: How (..)
188	P: Also eine Schnittfläche meinst du? #00:20:32-7#	P: You mean the cross-sectiom?
189	Axel: Ja. #00:20:32-6#	Axel: Yes.
190	P: Wie hier zum Beispiel? #00:20:33-3#	P: Like here for example?
191	Axel: Und hier hat es eine Schnittfläche von einem Sechseck #00:20:35-6#	Axel: And here it has a hecagonal cross-section
192	P: Und warum ist das ein Problem, wenn der Körper ein Quader oder ein Würfel wäre? #00:20:42-6#	P: And why would this be a problem, if the solid were a cuboi or a cube?
193	Axel: Weil wir dachten, dass eigentlich immer wenn man den neigt, dass immer nur ein- ja, man kann sich das nur im Kopf ja so vorstellen #00:20:51-8#	Axel: Becasue we though that actually always when we tilt it, that always only a-, yes, one can only imagine it in one's head
194	P: mhm (bejahend) #00:20:52-4#	P: mhm (affirmative)
195	Axel: Im Kopf denkt man sich eigentlich das kann entweder nur ein Dreieck oder ein äh Viereck von der Schnittfläche sein. Oder ein Rechteck. Aber kein Sechseck. Und es geht auch wenn man die- den Durchmesser [*points with his finger at the spinning/Drehung slider*] verändert, dann wird es auf einmal ein Fünfeck. #00:21:07-2#	Axel: One thinks in the head, that he cross-section can either only be a triangle or a quadrilateral. Or a rectangle. But not a hexagon. And also if you change the diameter [*he actually means the d-slider and points at it*], then it [*the cross-section*] becomes a pentagon
196	P: mhm (bejahend) #00:21:08-0#	P: mhm (affirmative)

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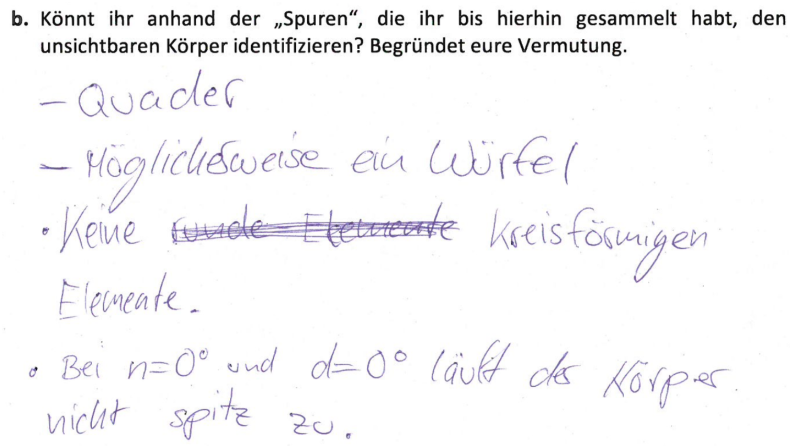
197	Axel: Also- wissen wir noch nicht ganz. Aber eigentlich müsste es ein Quader sein. #00:21:14-3#	Axel: So- we don't know exactly yet. But it should actually be a cuboid.
198	P: Ok. Wenn das ein Quader wäre, oder wenn das ein Quader ist (..) #00:21:20-3#	P: Ok. If it were a cuboid, or if it is a cuboid (..)
199	Axel: Ja #00:21:21-3#	Axel: Yes
200	P: Könnte das nicht- die Schnittfläche ein Sechseck sein? #00:21:24-6#	P: Could this- the cross-section be a hexagon?
201	Axel: Das überlegen wir ja die ganze Zeit. #00:21:26-8#	Axel: That is what we think about the whole time.
202	P: mhm (bejahend) #00:21:27-9#	P: mhm (affirmative)
203	Axel: Weil wir uns da nicht ganz sicher sind. #00:21:29-3#	Axel: Because we are not completely sure about it.

Transcript 7.9: Axel and Dave's discussion on Task 3C (invisible cube)

When the researcher leaves Axel and Dave, they continue with their guided exploration starting where they had left off before. Thus, the pattern continues with another DO step ($DO_{208\text{-}217}$). The students explore again the fourth case (h, n=125°, d=0°) of the Exploration Matrix once again, observing hexagonal and triangular cross-sections. These observations lead them to hypothesize (Hyp_{217}) that the solid could also be a prism, a hypothesis which is then directly rejected by both of them. Dave also explains why he rejects it, by saying that the solid "*cannot be a prism*" (utterance 217) because since "*it has a quadrilateral surface, then it is a cuboid (...) It must be a cuboid. After all that we have as* [observations?]" (utterances 219 and 221). In utterances 218-221 Dave simultaneously contradicts the prism hypothesis ($Contra_{218\text{-}219}$) and returns to their initial claim that the solid is a cuboid ($Clai_{221}$).

Next, Axel and Dave explore the two last given positions of the Exploration Matrix (see Figure 7.27b), observing the data that emerge ($DO_{227\text{-}261}$). After this step of observations Axel and Dave go back to their worksheet in order to answer question b (see Figure 7.28): "Can you identify the invisible solid, with the "clues" you have gathered until now? Justify your supposition". Before they write down their answer, they discuss it. As their next step of the pattern, Dave says that the solid is a cuboid ($Conc_{262}$) and both students together name and reject several solids that do not fit their observations: no sphere, no cylinder, no pyramid, no cone, no cube. The students do not explicitly name their hypotheses, before they start rejecting them, rather they do both of those things in one step. Therefore, in the pattern of argumentation I illustrate the cases they reject as an implicit step of hypothesizing, symbolizing that by putting element $Hyp_{262\text{-}264}$ in a dotted box. The step of contradicting all the hypotheses is illustrated in the pattern of argumentation by the element $Contra_{262\text{-}264}$ (see Figure 7.21).



b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

– Quader

– Möglicherweise ein Würfel

• Keine ~~runde Elemente~~ kreisförmigen Elemente.

• Bei n=0° und d=0° läuft der Körper nicht spitz zu.

Figure 7.28: Axel and Dave's answer in Question b in episode GR1AD-3C.I

Translation for Figure 7.28:

> b. Could you identify the invisible solid, based on the "clues" that you have gathered until now? Justify your supposition.
>
> " *– Cuboid*
>
> *– Possibly a cube.*
>
> - *No circular elements.*
> - *For n=0° and d=0° the solid does not converge to a point*"

But right after they have concluded that the solid is a cuboid, Axel asks Dave why the solid cannot be a cube. Axel's question puzzles Dave who wonders now too if the solid could be a cube. Hence, a new "hypothesizing" element is added in the pattern (Hyp$_{270}$). This triggers a new testing step (Test$_{270\text{-}279}$), during which the students measure all three dimensions of the solid: its length, its width and its height, in order to check if the solid can be a cube. First, they position the solid at (h=0, n=0°, d=45°) (see Figure 7.29). In this position, the cross-section of the solid with plane xOy is its base and the edges of the cross-section are parallel to the x-axis and y-axis. So, the students measure the edges of the cross-section by looking at the points of intersection of the cross-section with the axes. They say that all sides of the cross-section are equal to "two point something" (utterance 272). This way, the students measure the length and with of the solid by measuring the sides of the cross-section.

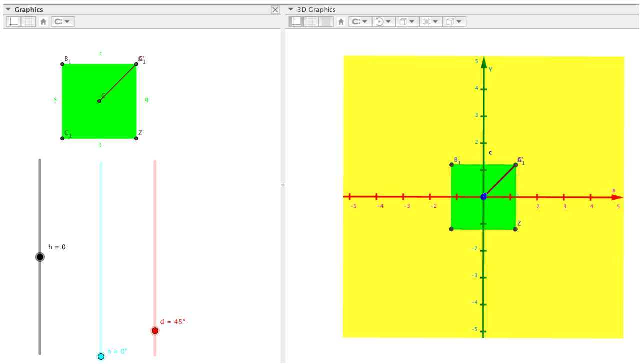


Figure 7.29: View of the cross-section at position (h=0, n=0°, d=45°), from the top of z-axis

Then they move the solid downwards with the h-slider, and observe that the cross-section is there until h=-2,35 and from h=-2,4 on it disappears. As they had already established earlier (1D$_{84}$) this is how tall the solid is. So, they have seen that the length and the width are "two point something" and the height is 2,35. So, they infer that it is also possible that the solid is a cube (1D$_{280}$). After this, the students conclude that the solid "cuboid, possibly a cube" (1D$_{281\text{-}286}$).

The final conclusion ("cuboid, possibly cube") is the result of all the DO-steps together with the initial and final Test-steps (Test$_{11}$, Test$_{15\text{-}26}$, Test$_{30\text{-}76}$, Test$_{270\text{-}279}$). This is the difference between this pattern of argumentation, and Patterns 1PW – "Direct" pattern and Pattern 2PW – "Narrowing down" pattern. In the "Testing" pattern, the confirmation of the claim is the result of both observing data and testing the claim.

7.3 Patterns of argumentation in classroom discussions

In this section I present the three types of *patterns of argumentation* identified through the Level 2 Analysis of the classroom discussions. The episodes used here emerge from the observation of the classroom discussions, which took place on both days of the study, after the pair-work phase on each day. At the end of each day, all the tasks were discussed in the frame of a whole classroom discussion. On day 1 all the students worked on Task 2, which was then discussed amongst the teacher and the students. On day 2 each pair of students work on one or more of the Tasks 3A, 3B and 3C. That means, that at the end of the pair-work phase on day 2 not every pair had worked on all three tasks. During the classroom discussions, a computer that was connected to the classroom projector, was used in order for the whole classroom to be able to observe the explorations that are performed.

Three types of *patterns of argumentation* arise from the analysis of the classroom discussion data (see Table 7.4). Each of these pattern-types has specific characteristics. In this section, I present these three types using episodes as examples. As in the case of pair-work, here too, the characteristics of the patterns are determined by the actions that the students follow in their argumentations. The descriptions of the pattern-types in Table 7.4 do not include all the details of the individual patterns that belong to each type; rather they are more general descriptions of them. The names of the patterns include a number and the abbreviation CD, denoting that a pattern is observed during classroom discussions. Each pattern also has a descriptive name that helps to associate it with its main characteristic. Next, I describe each pattern in more detail using episodes as examples.

Patterns of argumentation in classroom discussion			
	Pattern 1CD – "Confirming" pattern	Pattern 2CD – "Question-provoking" pattern	Pattern 3CD – "Reverse debate" pattern
	DO⟶ **Clai**⟶ **DO**⟶ Conc	(Full)DO⟶ **1D**⟶ **Q?** ⟶ Conc	DO⟶ **1D**⟶ **Q?**⟶ **Hyp**⟶ **Contra**⟶ **Test** (of Hyp)⟶ Conc
Episodes	CD3B-JM CD3C-TL	CD3A-AD	CD2

Table 7.4: Patterns of argumentation identified in classroom discussions

7.3.1 Pattern 1CD – "Confirming" pattern: DO ⟶ Clai⟶ DO⟶ Conc

The "Confirming" pattern is observed in episodes CD3B-JM and CD3C-TL[18]. In this type of pattern, the students perform part of the data observation, then they state a claim and then they continue the data observation, through which they verify their claim. I will elaborate further on the pattern, using mainly episode CD3B-JM as an example. I will also shortly discuss the pattern of argumentation of episode CD3C-TL and some interesting aspects of it.

Episode CD3B-JM

In this episode, Jacob and Michael present their work on Task 3B (invisible pyramid). This episode is from day 2 of the study. On that day, before the classroom discussion begins, the teacher (Frau Karl) explains to the students how she would like to organize the discussion. She says that for each episode, one pair of students can present their work in front of the classroom, before they all discuss it together. She explains that this will help the students who have not worked on a particular task to gain some insight and then be able to follow the discussion. Therefore, the students that hold the presentations are asked to begin by going through all the cases and positions in the Exploration Matrix, before they give their final answer and justification regarding the form of the invisible solid. In this episode Jacob and Michael stand in the front of the entire classroom to present their work on it.

Table 7.5 shows the structure of the episode, and Figure 7.30 illustrates the *pattern of argumentation.*

[18] CD3B-JM: Classroom discussion on Task 3B (invisible pyramid), Jacob and Michael present their work. CD3C-TL: Classroom discussion on Task 3C (invisible cube), Tom and Lukas present their work. (See detailed *patterns of argumentation* in Appendices D10 and D11, respectively)

CD3B-JM				
Structure			**Utterance numbers**	**Video Minutes**
1	Jacob and Michael's presentation		1-7	01:10:53 – 01:15:43
	1.1	Going through the positions and cases of the Exploration Matrix. Supposition: The solid is a pyramid	1-7.8	01:10:53 – 01:15:00
	1.2	Justification of the conclusion that the solid is a pyramid.	7.9-7.16	01:15:00 – 01:15:43
2	Ella's correction of Jacob's misspeaking mistake: The pyramid has four triangular faces, not three.		8-11	01:15:51 – 01:16:03
3	Presentation of the visible cone in GeoGebra, by Jacob.		12-18	01:16:06 – 01:19:18

Table 7.5: Structure of episode CD3B-JM

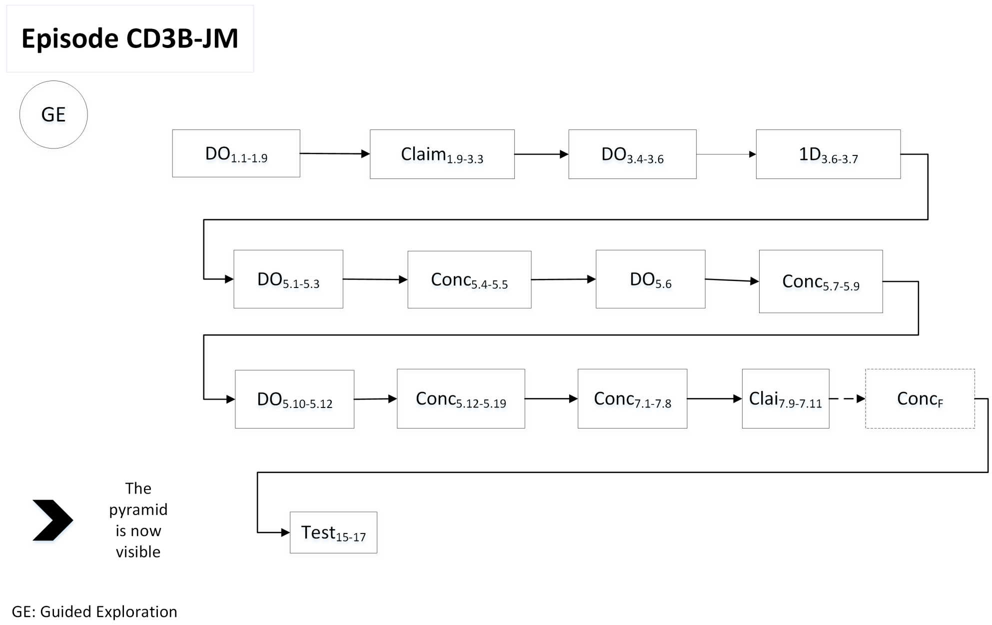


Figure 7.30: Pattern of argumentation for episode CD3B-JM

In this episode, Jacob and Michael present Task 3B. Michael manipulates the sliders in GeoGebra, on the computer that is connected to the classroom projector. At the same time Jacob presents their work. With the help of the projector everybody can watch Michael's explorations.

The pattern begins with observation of data ($DO_{1.1-1.9}$). Jacob and Michael begin with the exploration of the case (h, n=0°, d=0°), which is the first case in the Exploration Matrix (see Figure 7.31a)[19]. After this observation they claim ($Clai_{1.9-3.3}$) that the solid is a pyramid with a square base and four triangular faces, which build the side surface of the pyramid.

[19] Figures 7.31a and b, show the notes that Jacob and Michael kept on their worksheet during their pair-work.

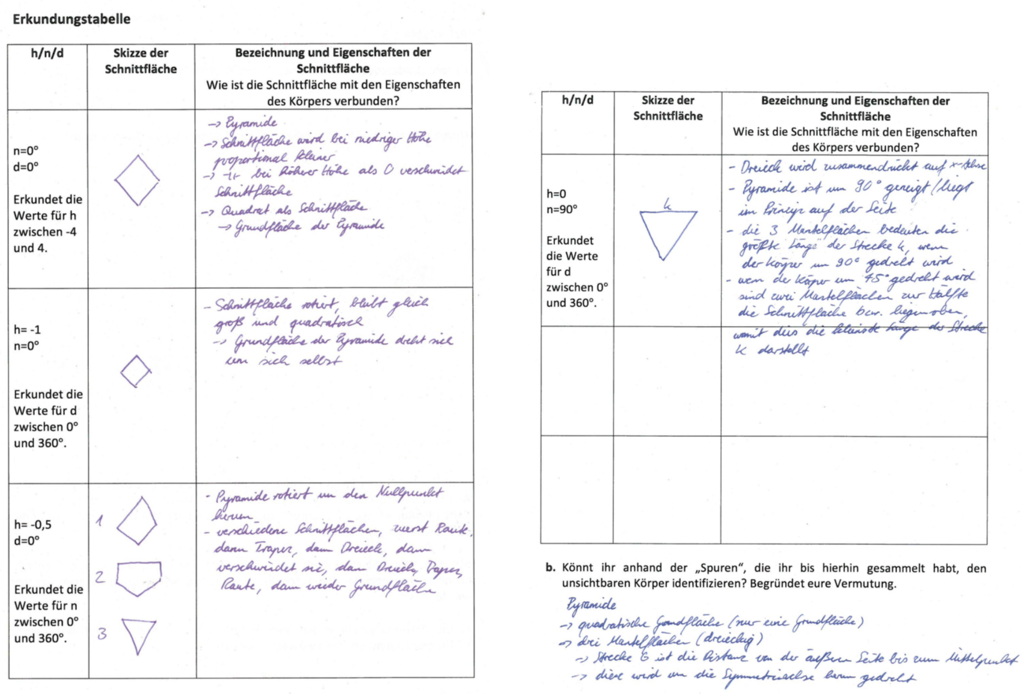


Figure 7.31: a (left) and b (right). Jacob and Michael's worksheet for Task 3B

They then move on to explore the rest of the cases in order to verify their claim. They observe the data emerging from the second case (h=-1, n=0°, d) ($DO_{3.4\text{-}3.6}$) and they say that here the base of the pyramid spins around itself, because the cross-section at (h=-1, n=0°, d) spins around itself ($1D_{3.6\text{-}3.7}$). This suggests a simple deductive syllogism that connects the two statements, while leaving the link between them implicit.

The pattern of argumentation continues with another DO-step. From the observation of the next case (h=-0,5, n, d=0°) ($DO_{5.1\text{-}5.3}$), Jacob draws the conclusion that the pyramid turns around the plane xOy ($Conc_{5.4\text{-}5.5}$) (see Figures 7.32a and b). the pattern continues with further observations and conclusions. Jacob says that when the cross-section disappears ($DO_{5.6}$) it means that the pyramid is under the plane xOy ($Conc_{5.7\text{-}5.9}$) (see Figures 7.33a and b).

The observation of data continues with the exploration of the last case (h=0, n=90°, d) in the Exploration Matrix ($DO_{5.10\text{-}5,12}$). Jacob says that in this case the pyramid lies "on the side", not on its side surface, rather through with xOy plane running through the middle of it ($Conc_{5.12\text{-}5.19}$). He also says that the side surface spins around its symmetry axis ($Conc_{7.1\text{-}7.8}$). By symmetry axis he refers to the symmetry axis running perpendicularly through the middle point of the square base. This symmetry axis is part of the design of the task in GeoGebra and it is visible, even though the solid is not visible (see for example Figures 7.32a and 7.33a).

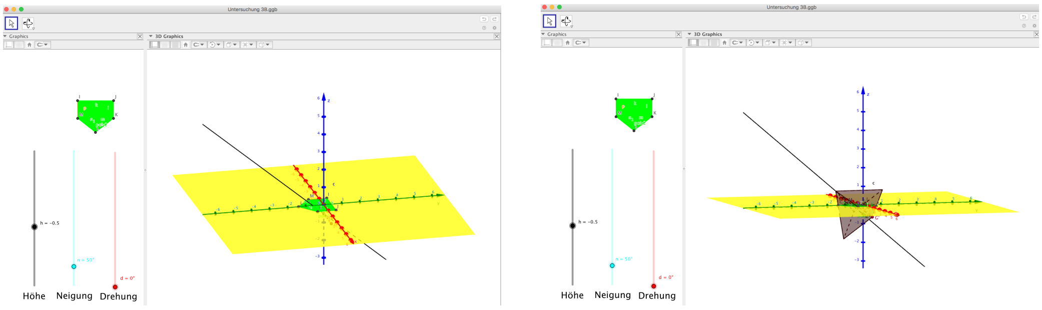


Figure 7.32: a (left) and b (right). Position (h=-0,5, n=50°, d=0°) as observed by the students and as it looks when the pyramid is visible

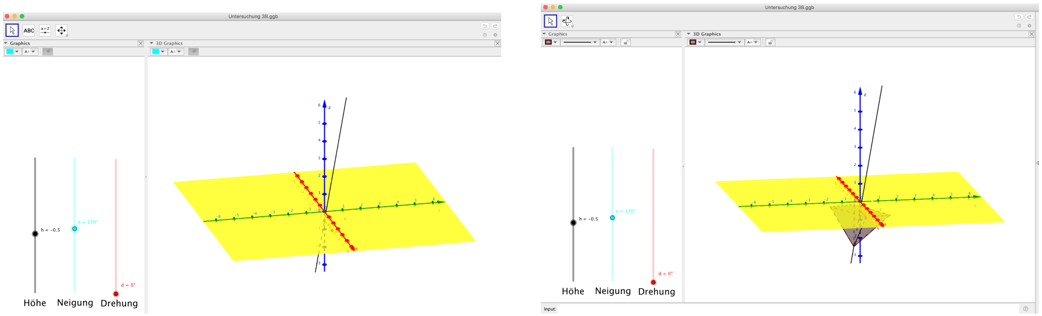


Figure 7.33: a (left) and b (right). Position (h=-0,5, n=170°, d=0°) as observed by the students and as it looks when the pyramid is visible

At the end of the presentation, Jacob claims again that the solid is a pyramid with a square base, because it has one square base and four triangular faces ($\text{Clai}_{7.9\text{-}7.10}$). I classify this action as claiming, rather than drawing a conclusion, because he uses the words "we think the solid is a pyramid" (utterances 7.9-7.10) for his statement, rather than something like "the solid is a pyramid". This claim is actually identical to the one he uses in $\text{Clai}_{1.9\text{-}3.3}$, at the beginning of his presentation. Nevertheless, it is their final answer for which they have also offered a justification. Therefore, I believe it may be an implicit conclusion as well.

As soon as Jacob has finished with the presentation, Frau Karl comments that this has been a very detailed explanation and she asks the rest of the students, whether they have something to add or ask (utterance 8). Only Ella makes a short comment that the side faces of the pyramid are four, and not three as Jacob falsely said on two occasions during their presentation. Michael explains that Jacob only misspoke, and that he actually meant four triangular faces. The absolute absence of any kind of objection from the rest of the classroom on Jacob's final claim, implies that it is accepted by everyone as the final conclusion (Conc_{F}, the dots denote the implication).

The last element shown in the pattern of argumentation (see $\text{Test}_{15\text{-}17}$ in Figure 7.30), represents a "testing" process that followed afterwards; the repetition of the exploration of all the cases in the Exploration Matrix with the solid this time being visible. Frau Karl's and the researcher's intention for this, was to make visible what Jacob had been describing before, in a situation in which it is much easier to grasp the connection between what is observed about the cross-section and how this is related to the movement of the solid. Our aim was to give the students the opportunity to actually see what they have been working on, understand what has been difficult to visualize and allow for the co-relation between the two-dimensional and the three-dimensional parts of the solid to become more prominent.

In this episode, Jacob and Michael who present their work to the rest of the classroom mainly hold the classroom discussion. The participation of the rest of the students is limited to one comment by a third student (Ella) and by the confirmation of the teacher that the presentation of the two students is detailed and thorough.

Episode CD3C-TL

In this episode the same type of pattern is observed, with only a few differences in some elements (see Figure 7.34). Here, Tom (manipulating the sliders) and Lukas (presenting) present their work. The two students present the first three cases of the Exploration Matrix ($DO_{1.1\text{-}1.9}$) and then Lukas claims that the solid is a cuboid ($Clai_{1.9}$). From that point on Lukas explains what is observed during the explorations by connecting the movements of the sliders and the shapes of the cross-sections with the movement of the so claimed cuboid. Their final conclusion is never stated explicitly, but the final conclusion regarding the form of the solid ($1D_{29.5\text{-}29.6}$) is stated clearly a little later, after the contribution of one more student.

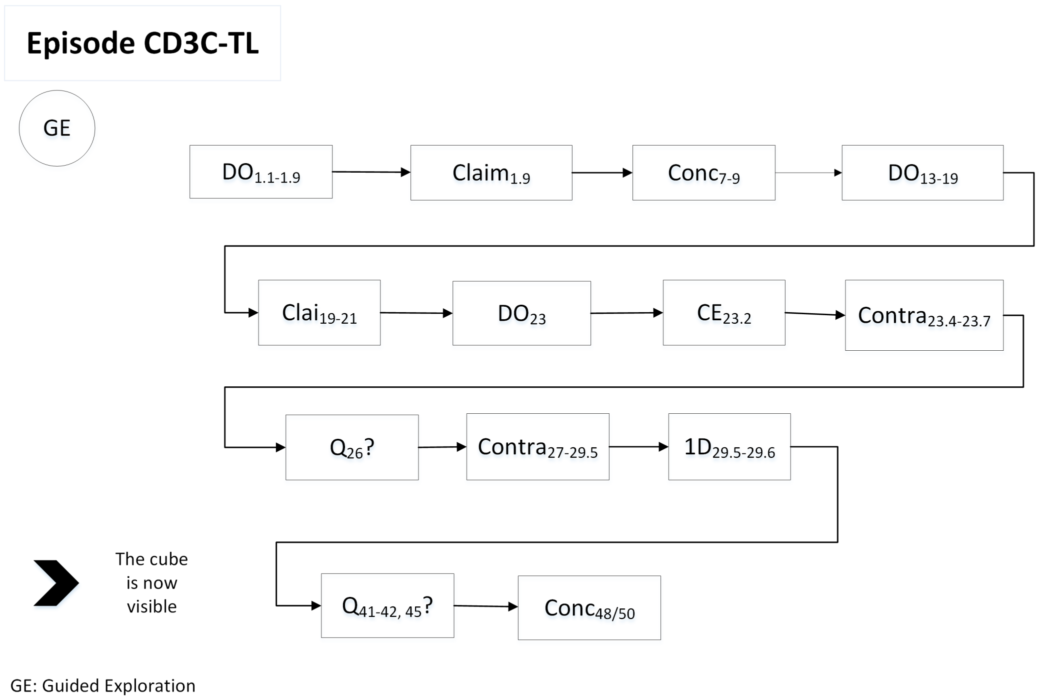


Figure 7.34: Pattern of argumentation for episode CD3C-TL

An interesting phenomenon in this pattern, is Lukas' choice to use a counter-example ($CE_{23.2}$: cylinder) during the observation of position (h=0, n=90°, d=45°) in order to explain that the invisible solid is not symmetrical:

> Lukas: Ja und dann noch einmal h null, n 90 Grad, und d 45, (...) ja, und da sieht man halt, dass es em, jetzt kein zum Beispiel (..) kein Zylinder oder so ist, weil beim Zylinder zum Beispiel wär jetzt der Querschnitt, würde es immer gleich bleiben, wenn man den drehen würde, um die eigene Achse, und jetzt ist ja die Umdrehung *(unverständlich)* wird immer größer und kleiner und daran sieht man, dass es halt vier Kanten hat, weil da wo das immer nicht die Kante ist, da wird es wieder ein bisschen dünner, und dann wieder, ja. Auseinander. Daran sieht man eigentlich, dass es nicht gleichmäßig ist.
>
> Translation: Yes and then again h zero, n 90 degrees, and d 45, (...) yes, and then you can see that it is em, now not for example (..) not a cylinder or something, because with the cylinder for example, the cross-cut would be now, it would always remain the same if you turned it around its own axis, and now the rotation *(incomprehensible)* is getting bigger and smaller and you can see that it has four edges, because where that is not always the edge, it gets a little thinner again, and then again, yes. Apart. From this you can actually see that it [*the solid*] is not even [*meaning symmetrical*].

The use of the counter-example is an effort to make their claim (cuboid) stronger, by eliminating another possible solid, while at the same time emphasizing a characteristic of the invisible solid, namely that *it is not symmetrical.*

The elements marked as "Q?" symbolize the teacher's questions to the students. In this episode the teacher asks two questions. The first time she asks the classroom, whether anybody wants to add or comment something (Q_{26}) and the next time she asks if anyone could explain why there are pentagonal and hexagonal cross-sections ($Q_{41\text{-}42,\ 45}$). The first question, leads to the contribution of one more student, Theo, to the discussion. Theo says that he and his classmate with whom he worked, found that all the edges of the solid are equal, which means that the solid is not a cuboid ($\text{Contra}_{27\text{-}29.5}$), rather a cube ($1D_{29.5\text{-}29.6}$). Theo also shows the rest of the classroom the observations, which lead them to their conclusion that the solid is a cube (see Figures 7.35a and b). He showed that in position (h, n=0°, d=45°) one can measure the length of all four sides of the base of the solid, making use of the x-axis and the y-axis, as well as its height, by moving the h-slider downwards and seeing until which value of h there are cross-sections. All sides of the solid are of length 2,4 so they are all equal, which means that the solid is a cube.

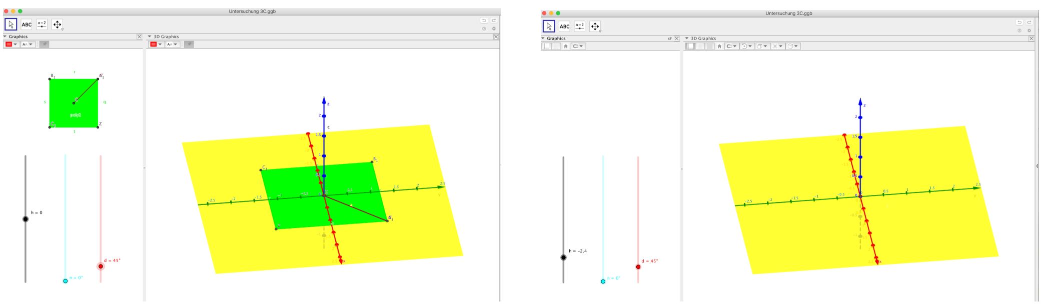


Figure 7.35: a (left) and b (right). Positions (h=0, n=0°, d=45°) and (h=-2,4, n=0°, d=45°) respectively

The last question in the classroom discussion is about the pentagonal and hexagonal cross-sections ($Q_{41\text{-}42,\,45}$). Students have seemed puzzled about how these cross-sections occur. Therefore, the teacher wants to come back to them and discuss it with the whole classroom. Also, the solid is made visible in order to help the students understand when the cross-section occur and how they are related to the parts of the solid. Theo tries to give an answer, but he says that he does not know how to explain his idea. The researcher rephrases the teacher's question, in an effort to help Theo and the rest of the students, by asking "How is the plane xOy related to the solid? So, how does this plane come through the solid? Here for example?" (utterance 45). At this point the projection on the wall shows the solid in position (h=-0,35, n=65°, d=87°), having a hexagonal cross-section with plane xOy (see Figures 7.36a and b). Michael struggles to explain his thinking, but he actually says that every face of the cube touches the plane xOy, and this is why the cross-section is a hexagon (utterances 47,49).

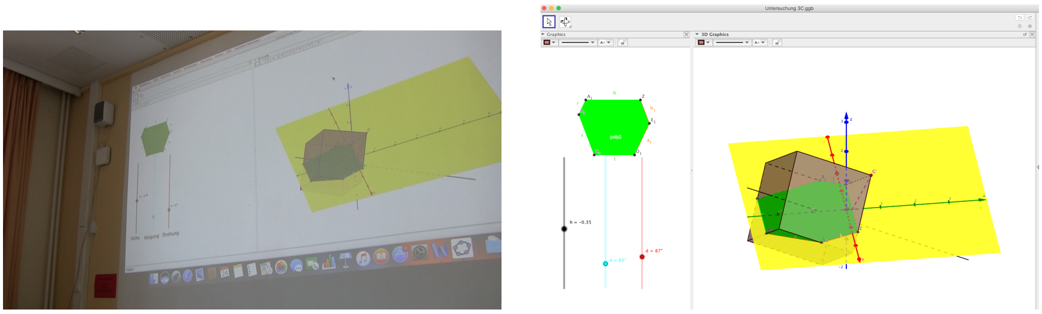


Figure 7.36: a (left) and b (right). Visible cube at position (h=-0,35, n=65°, d=87°). On the left snapshot from the video-recording, on the right snapshot from GeoGebra

The patterns of argumentation in both episodes presented here are of the same type (Pattern 1CD – "Confirming" pattern). They both start with observations of data from one or more explorations, upon which a claim is stated. Then this claim is used in order to explain the rest of the observations. At the end of the explorations the claim is stronger and it turns to an implicit conclusion.

Also, in both episodes the students follow a guided exploration for the presentations of their work. This is, nevertheless, the result of the instructions given by the teacher at the beginning of the classroom discussion. The teacher had told them to present their work by first going over the Exploration Matrix cases and then give their final answer and their justification. Although the students followed the structure of the Exploration Matrix for their presentations, they did not wait until the end of the explorations, before they presented their ideas about the form of the solid. On the contrary, they revealed their claim during the explorations and they also used their claim to explain the observed data.

7.3.2 Pattern 2CD – "Question-provoking" pattern: (Full)DO⟶ 1D⟶ Q? ⟶ Conc

The "Question-provoking" pattern is observed in episode CD3A-AD (see Figure 7.37). This type of pattern of argumentation begins with a complete observation of data ("(Full)DO") during guided exploration. Then the students who hold the presentation of the task draw their conclusion (Conc or 1D) regarding the form of the invisible solid. The teacher then poses some questions (Q?) asking for further clarifications on students' justification for their answer. This leads to further argumentation and discussion with the rest of the classroom, until the conclusion is fully justified.

The episode presented here took place on the second day of the study. Hence, the aforementioned teachers' instructions about the presentations also apply here. Axel and Dave present their work on Task 3A (invisible cone). In this episode Dave manipulates the sliders in GeoGebra and Axel presents the task. Table 7.6 shows the structure of the episode and Figure 7.44 illustrates the pattern of argumentation.

<table>
<tr><th colspan="5">CD3A-AD</th></tr>
<tr><th colspan="3">Structure</th><th>Utterance numbers</th><th>Video Minutes</th></tr>
<tr><td rowspan="5">1</td><td colspan="2">Axel and Dave's presentation</td><td>1-18</td><td>00:58:34 – 01:03:48</td></tr>
<tr><td>1.1</td><td>Going through the positions and cases of the Exploration Matrix.</td><td>4-6.7</td><td>00:59:46 – 01:02:35</td></tr>
<tr><td>1.2</td><td>The solid is a cone.</td><td>6.8-8</td><td>01:02:35 – 01:03:16</td></tr>
<tr><td>1.3</td><td>The solid is an inverted cone.</td><td>10-12</td><td>01:03:20 – 01:03:35</td></tr>
<tr><td>1.4</td><td>The solid is not an inverted cone. It is a cone standing with its base on plane xOy.</td><td>13-18</td><td>01:03:38 – 01:03:46</td></tr>
<tr><td rowspan="4">2</td><td colspan="2">How do the curved cross-sections occur?</td><td>19-41</td><td>01:03:50 – 01:07:30</td></tr>
<tr><td>2.1</td><td>Jacob's explanation.</td><td>20</td><td>01:04:19</td></tr>
<tr><td>2.2</td><td>Jacob's explanation with use of gestures.</td><td>21-35</td><td>01:04:50 – 01:06:28</td></tr>
<tr><td>2.3</td><td>Jacob's explanation with the use of a haptic cone-model.</td><td>36-41</td><td>01:06:30 – 01:07:28</td></tr>
<tr><td>3</td><td colspan="2">Presentation of the visible cone in GeoGebra.</td><td>42-56</td><td>01:07:30 – 01:10:10</td></tr>
</table>

Table 7.6: Structure of episode CD3A-AD

The pattern of argumentation begins with an element of observing data ($DO_{4\text{-}6.8}$, see Figure 7.37). Axel and Dave explore *all* the cases and positions in the Exploration Matrix (guided exploration), describing what occurs with the cross-sections in each of them. Their descriptions are based on their observation during their pair-work. You can see in Figures 7.38a and b, the notes they had kept on their worksheet during the pair-work phase. These are their drawings of the cross-sections they observed in the cases also presented during the classroom discussion.

The next element in the pattern is $1D_{6.8\text{-}6.9}$ and it comes only at the end of the explorations. Axel and Dave draw (by simple deduction) the conclusion that the solid is a cone and also offer their reasons for it. Axel says that the solid is a cone, because it has a circular base and it converges to a point:

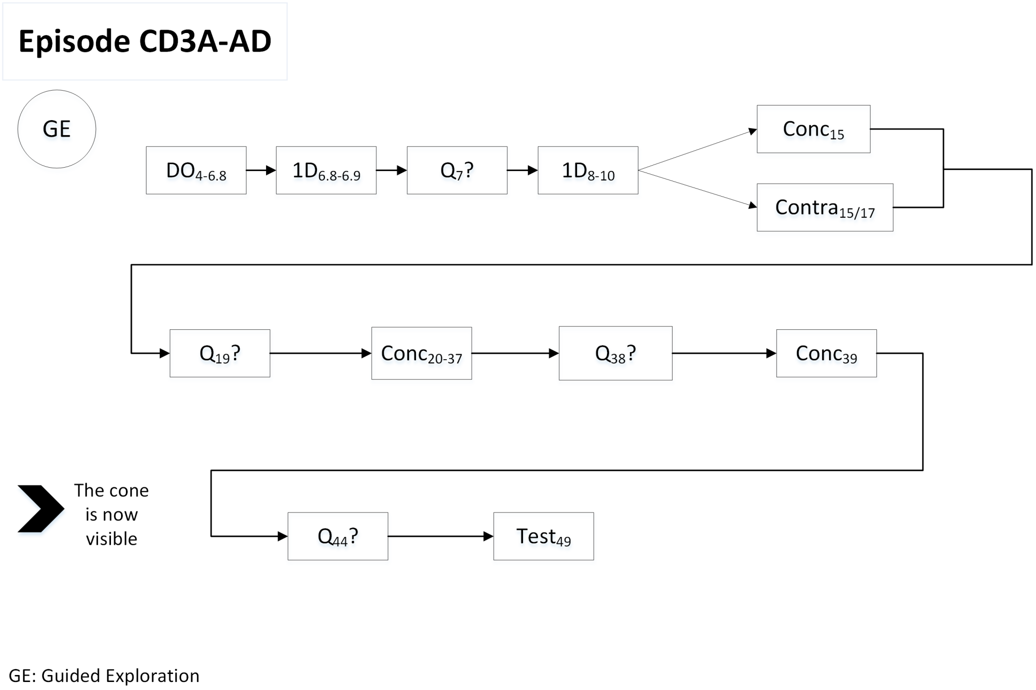
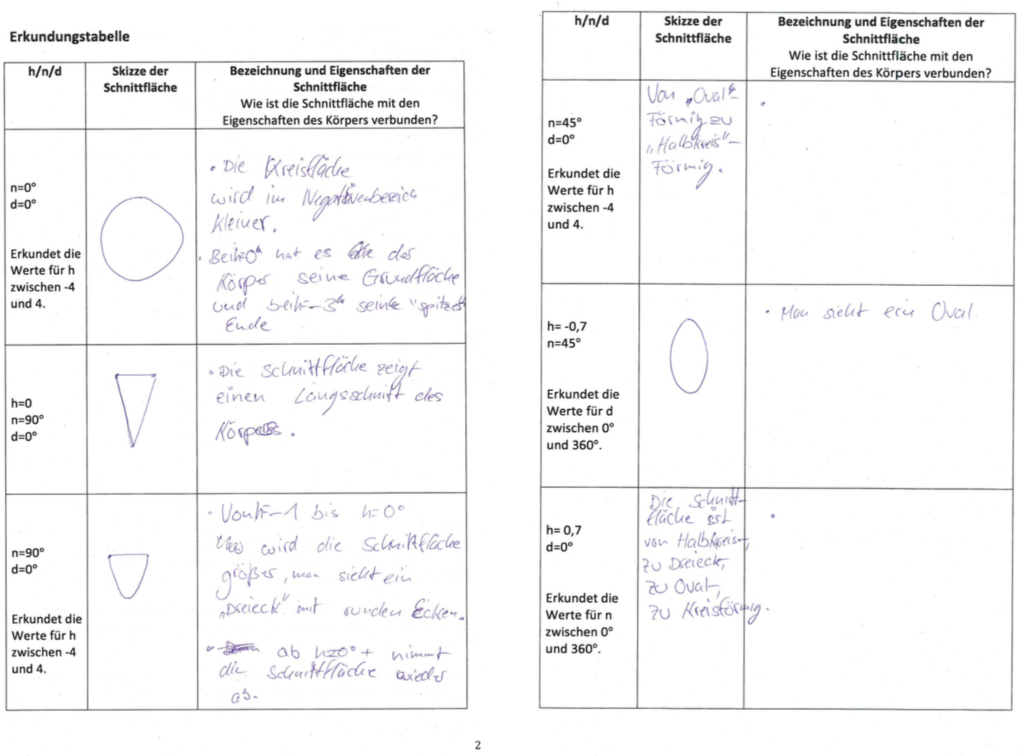


Figure 7.37: Pattern of argumentation for episode CD3A-AD



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Die Kreisfläche wird im Negativenbereich kleiner. • Bei h=0 hat es die Ecke des Körpers seine Grundfläche und bei h=3 seine "spitzes" Ende
h=0 n=90° d=0°		• Die Schnittfläche zeigt einen Längsschnitt des Körpers.
n=90° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Von h=-1 bis h=0° wird die Schnittfläche größer, man sieht ein „Dreieck" mit runden Ecken. • ab h=0° + nimmt die Schnittfläche wieder ab.

2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=45° d=0° Erkundet die Werte für h zwischen -4 und 4.	Von „Oval"-Förmig zu „Halbkreis"-Förmig.	•
h= -0,7 n=45° Erkundet die Werte für d zwischen 0° und 360°.		• Man sieht ein Oval
h= 0,7 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Die Schnittfläche ist von Halbkreis, zu Dreieck, zu Oval, zu Kreisförmig.	•

3

Figure 7.38: a (left) and b (right). Exploration Matrix from Axel and Dave's worksheet for Task 3A

> Axel (utternaces 6.8 and 6.9): Daran hatten wir herausgefunden, was fur eine Form es ist, es ist ein Kegel, weil er eine runde Grundflache hat und spitz zusammenlauft.
>
> Translation: From this we found out what kind of shape it is, it is a cone because it has a round base and it converges to a point.

The next element of the pattern is "Q_7?", which represents the teacher's action at that moment. As soon as the students have stated their conclusion, the teacher asks them whether they could justify it in a little more detail, since they presented many positions. This element in the pattern is what differentiates it from the "Confirming" pattern (Pattern 1CD, see Table 7.4). It also gives the pattern its name, as a "Question-provoking" pattern of argumentation.

As a response to Frau Karl's question, Axel provides some more explanation for their answer, thus enriching their justification (element $1D_{8\text{-}10}$). He explains that in position (h=0, n=0°, d=0°) the cross-section is the base of the solid, the biggest cross-section you can have[20], and as you go to the negative values of h, the cross-section gets smaller until it vanishes completely (utterance 8, see Figures 7.39a, b, c and d). He also says that the cone is inverted (utterance 10) and explains what he means by "inverted" by gesturing with his hands, a cone that is oriented with its top pointing downwards and its base above (see Figure 7.40).

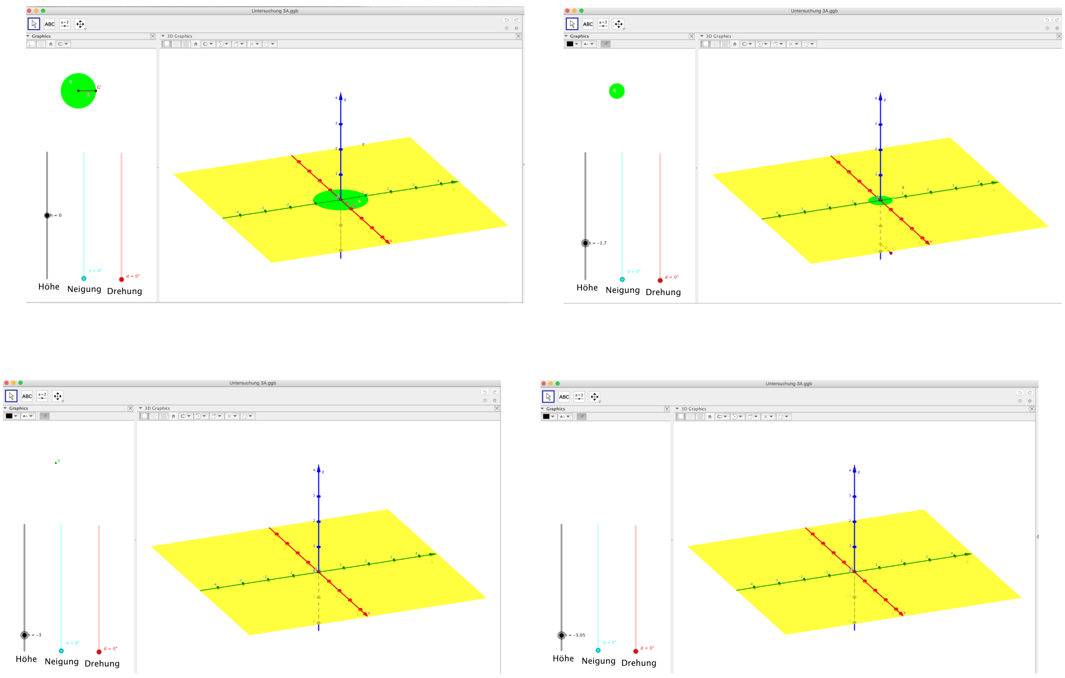


Figure 7.39: a (top left) and b (top right). Positions (h=0, n=0°, d=0°) and (h=-1,7, n=0°, d=0°) respectively. c (bottom left) and d (bottom right), positions (h=-3, n=0°, d=0°) and (h=-3,05, n=0°, d=0°) respectively

[20] Here, he most probably meant the biggest circular cross-section, and not the cross-section of the cone with the biggest area.

Figure 7.40: Axel's gesture for the inverted cone

The pattern continues with Dave's contradiction ($Contra_{15/17}$) to Axel's statement that the cone is inverted and the justification of his conclusion that the cone stands with its base on plane xOy. Dave says that the cone stands with its base on the xOy plane ($Conc_{15}$) and contradicts Axel's statement that the cone is inverted ($Contra_{15\text{-}17}$). He says that when dragged downwards, it converges more and more:

> Dave (utterance 17): "Und wenn man den quasi immer weiter runter fahrt, dann sieht man halt, dass der immer spitzer zusammenlauft"
>
> Translation: "And if you keep driving it downwards, you can see that it is getting more and more pointed"

By dragging it "downwards", Axel refers to the vertical dragging of the solid along the z-axis, which happens when the h-slider is dragged downwards (towards its negative values, see Figures 7.39a to d).

After the initial orientation of the cone has been determined, the classroom discussion becomes more vivid and more students participate in the argumentation. The pattern continues with another Q?-element. Frau Karl asks the classroom how the oval and half-oval cross-sections occur (Q_{19}?)[21]. A third student, Jacob, responds to that question (element $Conc_{20\text{-}37}$). He says that the oval and curved cross-sections emerge when the curved side surface of the cone touches the plane of intersection xOy. He also says that the bigger the part of the side surface that touches the plane is, the bigger the oval cross-section is ($Conc_{20\text{-}37}$). The two students also use gestures in order to describe what Jacob says. Michael, the classmate who sits next to Jacob, gestures a cone with his hands and Jacob describes again what he said using gestures (see Figures 7.41a and b).

[21] Under the terms "oval" and "half-oval", are meant here any observed curved cross-sections, whether these are ellipsoid cross-sections, or a part of an ellipsis (see the third case in Figure 7.38a and the second case in Figure 7.38b).

Later the students are also given an improvised paper model of a cone, made on spot with an A4 page, in order to help them present their ideas better (see Figures 7.42a and b, and 7.43a and b). In Figures 7.41and 7.42, Jacob describes the way an oval cross-section occurs on a cone, using Michael's hands and a paper cone as haptic tools representing the cone.

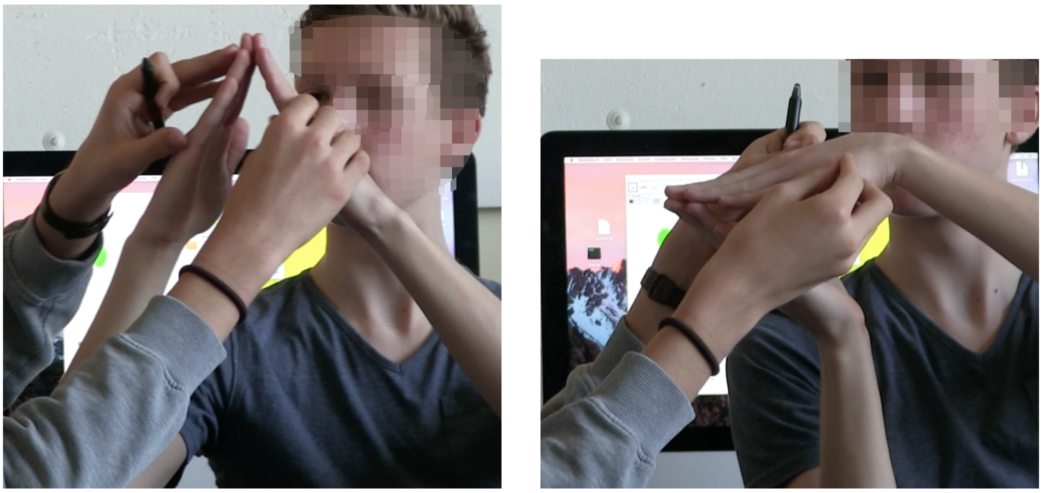


Figure 7.41: a (left) and b (right). Jacob shows the initial position of the cone (left) and the place where the oval cross-section is on the cone

Jacob explains, that when the cone touches the plane xOy, a cross-section is created. The "points of contact" (utterance 33.6: Berührungspunkte) of the cone with the plane, are the ones that constitute the cross-section. These "points of contact", are points on the side surface of the cone (Figure 7.42a), as well as points inside the cone (Figure 7.42b), because the solid has a certain mass (utterance 37.4: "weil der Körper eine bestimmte Masse darstellt"). This is how Jacob completes his justification in $Conc_{20\text{-}37}$.

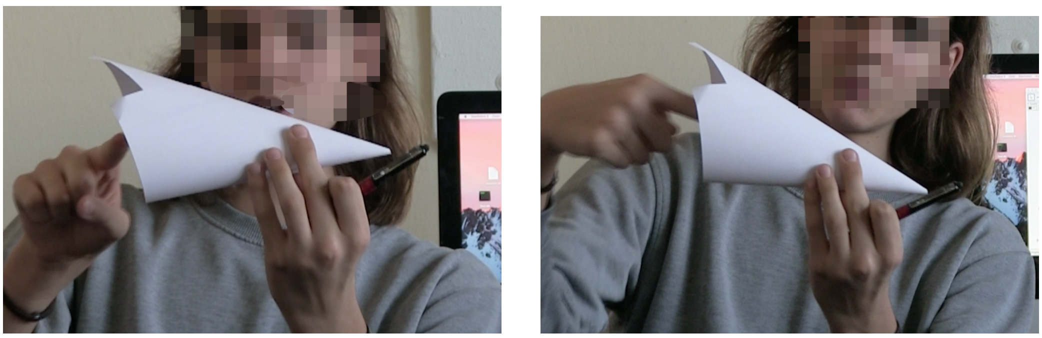


Figure 7.42: a (left) and b (right). Jacob shows how an oval intersection can occur on a cone (left) and that the surface of the cross-section is the "inside" part of the cone (right)

Then the pattern continues with one last Q?-element. Frau Karl asks if anyone wants to add something (Q_{38}?). Michael then explains how the circular cross-sections of the cone occur ($Conc_{39}$, also see Figures 7.43a and b). Although Michael does not state this explicitly, nevertheless the positions that he shows with the paper cone, correspond to those cases in which the cone intersects the plane with its symmetry axis perpendicular to xOy. On the DGE[22], those cases are $(-2{,}4<h<0,\ n=0^\circ,\ 0^\circ<d<360^\circ)$, $(-2{,}4<h<0,\ n=360^\circ,\ 0^\circ<d<360^\circ)$ and $(0<h<2{,}4,\ n=180^\circ,\ 0^\circ<d<360^\circ)$.

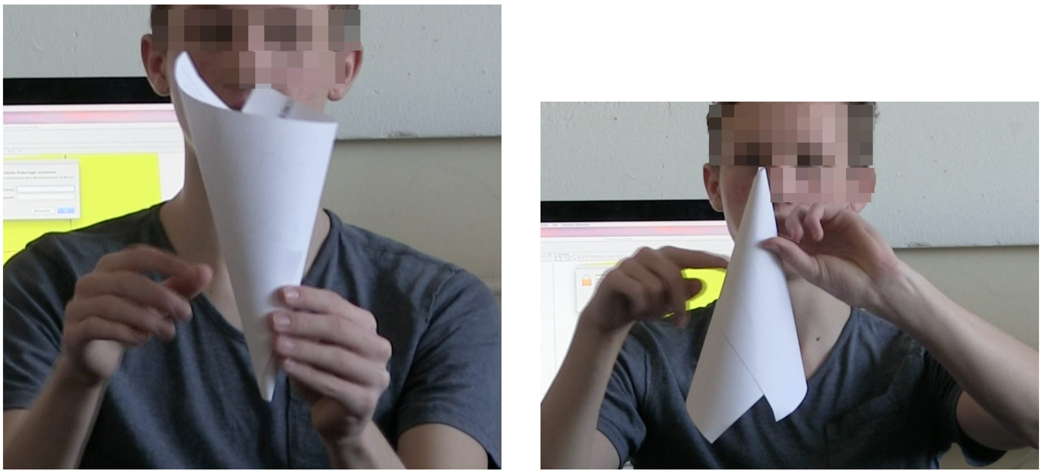


Figure 7.43: a (left) and b (right). Michael shows different positions where circular cross-sections occur

The last part of the pattern of argumentation (see element in Figure 7.37) takes place after the solid is set to be visible in the DGE. After Frau Karl makes sure that no one else wants to add anything to the discussion, I set the cone to be visible in GeoGebra and the teacher asks Axel and Dave to go through the cases and positions of the Exploration Matrix once again (Q_{44}?), so that the students can see all the phenomena that were discussed before in the software (Test$_{49}$). Although, the opinions of the students have been shared and discussed before the cone becomes visible, I consider that this "visible-check" is one more test for the students towards a more robust confirmation of the validity of all the ideas previously discussed.

In conclusion, in this episode, the pattern of argumentation starts with a complete observation of data, moving to a justified conclusion, followed by a classroom discussion in which the connections and co-relations between the two-dimensional cross-sections and the three-dimensional solid are discussed. This discussion, and hence the argumentation that takes place in it as well, are not focused solely on the conclusion about the form of the solid. On the contrary, after this has been settled amongst the students, the focus shifts towards the transition from two dimensions (2D cross-sections, 2D/3D sub-parts and figural units of the solid) to three dimensions (the form of the solid, its orientations and movement in space).

The structure of the classroom discussion is mainly determined by the teacher's decisions. To some degree, this also influences the structure of the pattern of argumentation. For example, the way Axel presents all the observations and then justifies his conclusion could have been much different if the teacher had not told them to do their presentation in this way. Furthermore, the teacher's questions triggered an interesting argumentation, within which the students had the opportunity to share, negotiate, explain and elaborate further on their ideas and arguments.

[22] Dynamic Geometry Environment

7.3.3 Pattern 3CD – "Reverse debate" pattern: DO⟶ 1D⟶ Q?⟶ Hyp⟶Contra ⟶Test (of Hyp)⟶Conc

The "Reverse debate" pattern is observed in episode CD2. Unlike all the episodes discussed up until this point, this took place on the first day of the study. On this day, all the students had worked in pairs on the same task (Task 2, invisible sphere). The classroom discussion that takes place in this episode is very different to the ones in the rest of the episodes. First of all, it is less strictly structured. Due to the fact that all the students have worked on the same task during the pair-work phase, everybody is familiar with it. Therefore, there is no need for an initial presentation and all the students can participate in the discussion directly from the start. This results in a discussion in which the students use the observation of the data in order to make a point. That means that the students choose for themselves the case they want to explore in order to create a hypothesis or draw a conclusion, engaging in structured exploration instead of guided exploration[23].

The role of the teacher here is to be the orchestrator of the discussion and also pose questions that will challenge the students' argumentation, engaging them in more complex geometrical relationships between two-dimensional and three-dimensional parts of the solid. Also, more students participate in the discussion of this episode, than in the other episodes. As a result, the pattern of argumentation in this one is also quite different from those in the rest of the episodes.

The pattern begins with a short observation of data ($DO_{2\text{-}6.1}$) that is used in order to draw the conclusion that the solid is a sphere ($1D_{6.1\text{-}6.4}$). It is only after the form of the solid is established that the actual exploration of alternative possibilities starts. The teacher's questions (Q_7?, Q_{18}?, Q_{52}?) bring a structure to the argumentation, so that arguments do not just "flow" unlinked in the classroom. While students answer the teacher's questions, new hypotheses are generated, some of which ($Hyp_{23\text{-}24}$, $Hyp_{25\text{-}29}$) are contradicted more easily, while others (Hyp_{42}) demand more explorations and data, or further justification in order to be refuted. So, in this pattern of argumentation the flow of events seems to be the opposite to those in Patterns 1CD and 2CD (see Table 7.4).

Table 7.7 shows the structure of the episode and Figure 7.44 illustrates the pattern of argumentation.

[23] For a concise overview of the types of exploration strategies, refer to subsection 6.4.1 in Chapter 6.

CD2				
Structure			**Utterance numbers**	**Video Minutes**
1	Justifying the sphere-hypothesis		1-17	01:15:55 – 01:17:43
	1.1	Niko's justification The solid is a sphere because all its cross-sections are circles. Rejecting the idea of the solid being a cuboid via RAA.	1-12	01:15:55 – 01:16:59
	1.2	Jacob's justification The lack of influence of the n-variation to the cross-sections of the solid is a decisive factor, proving that the solid is a sphere.	14-17	00:17:00 – 01:17:43
2	Stating and rejecting other hypotheses judging only by the (h, n=0°, d=0°) exploration		18-54	01:17:44 – 01:23:02
	2.1	The solid could be a cylinder	23-33	01:17:44 – 01:19:20
	2.2	The solid could be cone	25 and 34-40	01:18:33 and 01:19:22 – 01:20:03
	2.3	The solid could be a double cone	41-53	01:20:03 – 01:23:02

Table 7.7: Structure of episode CD2

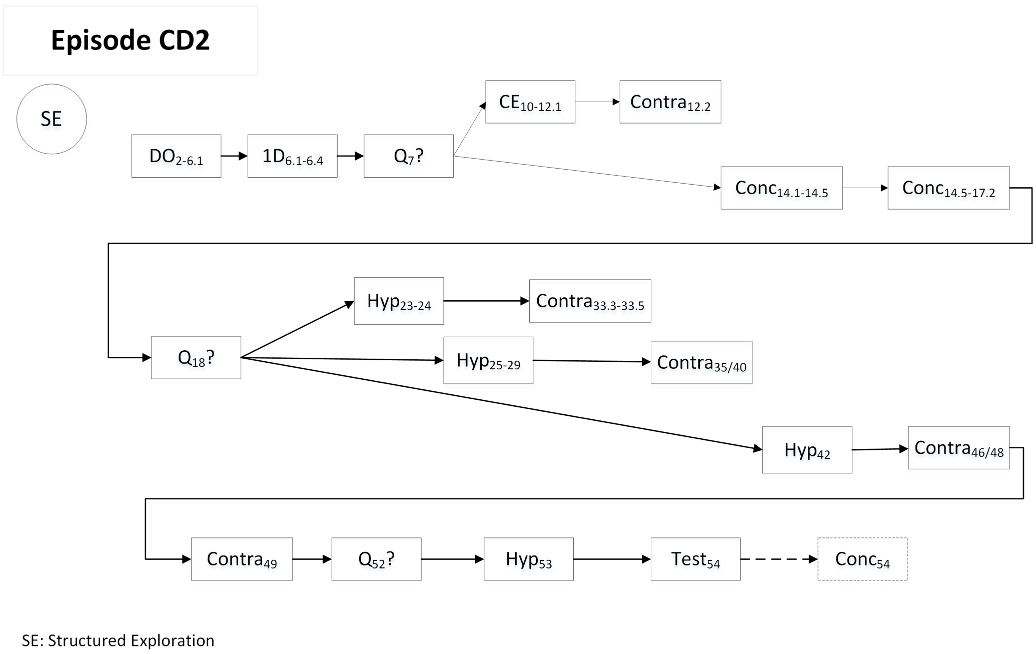


Figure 7.44: Pattern of argumentation for episode CD2

The teacher initiates the discussion by saying (utterance 1):

> Frau Karl: Ihr habt ja jetzt eine etwas andere Aufgabe, weil jetzt der Körper nicht direkt gesehen war. Em (..) viele von Euch sind relativ schnell drauf gekommen, aber ich denke auf die Begründungen sollten wir auch nochmal

> gucken, em, was habt ihr da ausprobiert, was ist euch dabei aufgefallen? Und wie kann man das entsprechend begründen? (..) Niko
>
> Translation: You have a slightly different task now because the solid was not to be seen directly. Em (..) many of you came up with it relatively quickly, but I think we should look at the justification, em, what did you try, what did you notice? And how can you justify that? (..) Niko

Niko is the first student to contribute to the discussion. Therefore, the pattern begins with his data observation and conclusion. With a simple deduction ($1D_{6.1\text{-}6.4}$), he argues that no matter how he has tilted (n-slider) or spun the solid around (d-slider), the cross-sections have been circles ($DO_{2\text{-}6.1}$, in Figure 7.44). He argues that since the cross-sections always remain always circles and a sphere consists of many circles, which all together produce the sphere, then the solid is certainly ("auf jeden Fall", utterance 6.4) a sphere. Niko's conclusion is a simple deductive syllogism (1D), connecting observed data (always circular cross-sections, no matter how you tilt or spin the solid), through a rule (a sphere is a solid created by many circles), to a result (the solid is a sphere).

Frau Karl agrees with Niko's justification and poses a question "*And what should have changed, if this was not a sphere and you changed the tilt* [n-slider]*?*" (Q_7?). The pattern is now divided into two paths The first path corresponds to Niko's answer, while the second one to Jacob's answer. In order to respond to Frau Karl's question, Niko uses a counter-example. He says that if the solid was not a sphere, and instead was - for example - a cuboid ($CE_{10\text{-}12.1}$), then the cross-sections would have lines [edges] and points [vertices]. This counter example (CE) is contradicted by the observed data ($Contra_{12.2}$), since none of the cross-sections he has observed have any straight lines or points, or are not circular. In the frame of the argumentation, the contradiction of this counter-example makes his conclusion stronger.

Jacob enters the discussion giving another answer to Frau Karl's question. He says that the tilt (n-slider) is the *decisive factor* ("der ausschlaggebende Punkt", utterance 14.1), because it has no influence on the cross-sections ($Conc_{14.1\text{-}14.5}$). He also says that the variation of height only makes the circular cross-sections proportionally smaller, until they disappear, when the sphere is over or under the plane xOy ($Conc_{14.5\text{-}17.2}$).

The pattern continues with another question by Frau Karl. In response to Jacob's reference to the height-exploration, Frau Karl asks the students, what other solid this could be, if they judged only by the case (h, n=0°, d=0°) (Q_8?). This means, that the solid is moved only up and down on z-axis, by varying the height slider. The solid is neither tilted nor spun at all. Victor and Axel generate two different hypotheses, leading to two different paths in the pattern. Victor says that the solid is a cylinder because in this case its cross-section is also circular ($Hyp_{23\text{-}24}$). Axel says that the solid is a cone because it is also a solid that converges to a point and its cross-section would be circular ($Hyp_{25\text{-}29}$).

Dave enters the discussion and objects to both hypotheses. He says that the solid cannot be a cylinder or a cone. His intervention contributes to the first sub-path of the pattern, with a contradiction of Victor's hypothesis. Dave explains that the solid could not be a cylinder, because here the circular cross-sections change size when you move up and down with the h-slider, but in the case of a cylinder the cross-section would always be a circle of the same size ($Contra_{33.3\text{-}33.5}$). Then Jacob also enters the conversation and he contradicts the cone hypothesis, thus contributing to the second path of the pattern. He explains that it could not be a cone either, because in that case there would be no cross-section at all when you move it upwards (for h>0). Here, Jacob implies that the cone stands with its base on the plane xOy. Ella explains this, in order to help Jacob with the formulation of his explanation.

After both hypotheses have been refuted, a new path begins in the pattern, as a third answer to Frau Karl's question Q_{18}?. Lukas joins the discussion with a new hypothesis: *bicone*, a double cone with common base (Hyp_{42}, see Figure 7.7). Michael and Ella express two different opinions on Lukas' hypothesis. First, Michael contradicts Lukas' hypothesis ($Contra_{46/48}$). He says that the solid could not be a bicone, even if we only consider the case (h, n=0°, d=0°), because in the case of a bicone when moving towards any of its pointed tops, the solid gets smaller *suddenly*, while in the case of the sphere, it gets smaller more *proportionally* (utterances 46 and 48). In response to both Lukas' and Michael's statements, Ella says that we cannot tell if this is the case only by looking at (h, n=0°, d=0°), because in both cases (sphere and bicone) the cross-sections are only circles (during only h-variation, with zero tilt and spin), whose radii change when moving them over or under their center (utterances 49 and 51). Thus, Michael contradicts Lukas' hypothesis ($Contra_{46/48}$) and then Ella contradicts Michael's contradiction ($Contra_{49/51}$).

This leads to Frau Karl's final question in the pattern of argumentation (utterance 52, Q_{52}?):

> Frau Karl: So we can keep from here that you can recognize that the radius is changing, then you can, so, Dave had previously used the word proportionally, and you could do that if you calculated it probably, but at first glance it is difficult to see a difference [*between proportional and non-proportional change, so also between a bicone and a sphere*]. But how could you refute that that there are two cones attached to each other, if you no longer keep the settings? (...) Tom?

By "not keeping the settings", Frau Karl means that the students are now allowed to manipulate anyone of the three sliders, and not only h. This question leads the discussion further with the generation of a new hypothesis, a testing process and a final conclusion. More precisely, Tom says that we can simply tilt (n-slider) the solid and then we will see that the cross-section remains a circle (Hyp_{53}). Frau Karl then changes the tilt in order to test Toms hypothesis ($Test_{54}$), and everyone can see that his hypothesis is confirmed. Frau Karl says that what Tom suggests has made the situation clear. Hence, the unstated conclusion of the argumentation is that the solid is indeed a sphere (implicit conclusion $Conc_{54}$)[24].

[24] The implicit nature of the conclusion is denoted in Figure 7.44, by the dotted box in which it is placed.

7.4 Conclusions about students' patterns of argumentation

I would now like to point out some interesting aspects about the types of *patterns of argumentation*, thus drawing a bottom line for what has been discussed until now. Therefore, I refer back to my research question 2.1 (see below and in the introduction of Chapter 7), which I have already discussed in Section 7.1. Table 7.8 summarizes the identified *patterns of argumentation* both during pair-work as well as in classroom discussions. It also includes the *exploration strategies* that were used in each type of pattern.

2.1: "What are the observed *patterns of* students' *argumentations* while working on the given tasks?"

Patterns of argumentation in pair-work			
	Pattern 1PW – "Direct" pattern	Pattern 2PW – "Narrowing down" pattern	Pattern 3PW – "Testing" pattern
	DO⟶ **Hyp**⟶ Conc	DO⟶ **Hyp**⟶ **DO**⟶ Conc	DO⟶ **Hyp**⟶ **Test**⟶ **DO**⟶ Conc
Episodes	GR2TL-2	GR1AD-3B.II GR1AD-3A.III GR2TL-3A.I	GR1AD-2 GR1AD-3C.I
Types of Exploration Strategies	Guided Exploration	Guided Exploration in all and also minor Free Expl. in GR2TL-3A.I	Free and Guided Expl. in both and also Structured Expl. in 3C.I
Patterns of argumentation in classroom discussion			
	Pattern 1CD – "Confirming" pattern	Pattern 2CD – "Question-provoking" pattern	Pattern 3CD – "Reverse debate" pattern
	DO⟶ **Clai**⟶ **DO**⟶ Conc	(Full)DO⟶ **1D**⟶ **Q?** ⟶ Conc	DO⟶ **1D**⟶ **Q?**⟶ **Hyp**⟶ **Contra**⟶ **Test** (of Hyp)⟶ Conc
Episodes	CD3B-JM CD3C-TL	CD3A-AD	CD2
Types of Exploration Strategies	Guided Exploration	Guided Exploration	Structured Exploration

Table 7.8: *Patterns of argumentation* and *exploration strategies* identified in pair-work and in classroom discussions

As shown in Table 7.8, three types of patterns have been identified in each social setting (pair-work and classroom discussions). Neither of the two groups of students that were filmed during their pair-work used only one pattern. Axel and Dave followed the "Narrowing down" pattern[25] and the "Testing" pattern[26], Tom and Lukas followed the "Direct" pattern and the "Narrowing down" pattern. Nevertheless, the only pattern used by both groups was the "Narrowing down" pattern.

The table also shows the exploration strategies that where followed in each type of pattern. The "Direct" pattern is observed only in one episode (GR2TL-2), in which the Tom and Lukas perform a guided exploration. In the "Narrowing down" patterns, in all the episodes the students followed guided explorations. In one of the episodes (GR2TL-3A.I) there is also a short use of free exploration. The students employed the free exploration when they decided which extra position to add to the Exploration Matrix on their worksheet. The "Testing" pattern is identified in two episodes by Axel and Dave (Group 1). In both episodes the students employ free exploration for the part "DO⟶ **Hyp**⟶ **Test**" of the pattern, while they continue with guided exploration for the part "**DO**⟶ Conc". In episode GR1AD-3C.I, Axel and Dave also use structured exploration in two cases (see $\text{DO}_{227\text{-}261}$ and $\text{Test}_{270\text{-}279}$, in Figure 7.21, subsection 7.2.3).

In general, Axel and Dave seemed to use exploration strategies that gave them more freedom in their explorations. They use free exploration as much as guided exploration, and also structured exploration in some parts. On the other hand, Tom and Lukas preferred the guided exploration and rarely took initiatives in their explorations.

Three more types of patterns of argumentation were identified in the classroom discussions. As we saw in the previous sections, the structure of the discussion influences not only the types of exploration strategies that take place, but also the whole pattern of argumentation that takes place in the classroom. Of course, in the case of this study, the structure of the discussion is much influenced by the design of the study. The fact that on day 1 all students worked on the same task, while on day 2 students worked on different tasks, lead the different approaches of the classroom discussion. On the first day the discussion started directly, without a presentation from a pair of students. This lead to a discussion, in which all the students could participate already from the beginning, and the explorations that the students would choose in order to justify their answers were not "restricted". As a result, the students employed structured explorations and the discussion had polyphony. Consequently, the "Reverse debate" -type of pattern that has been created through this process is fruitfully complex.

On the second days, the discussions of all the tasks began with presentations of the explorations in the Exploration Matrix. This resulted in a much more structured discussion with limited opportunity for vivid discussion after the thorough presentations. Nevertheless, the aim of these approaches in both cases was to aid all the students in their participation in the discussion, and not only those who were familiar with the task that was being discussed.

[25] Episodes GR1AD-3B.II and GR1AD-3A.III in Table 7.8.

[26] Episodes GR1AD-2 and GR1AD-3C.I in Table 7.8.

7.5 Influence of the task-design to the patterns of argumentation

The design and implementation of the present study is based on Prusak et al.'s *design principles* (see Chapter 5). These principles focus on two dimensions: the *task-design* and the organization of the *learning environment* during the implementation of the study. Here, my aim is to answer my research question:

2.2 "How does the specific *design* of the given tasks influence the structure of students' *patterns of argumentations*?"

Because of their design in a DGE the tasks have given the students the opportunity to explore the situation in various ways. The three sliders, with which they could move the solid, gave them infinite possible positions they could examine. The fact that they could manipulate it freely, provided them also with the opportunity to test their hypotheses about what would happen if a specific movement of a slider was to be conducted. In terms of patterns of argumentation, the DGE supported students' actions of observing data (DO), creating hypotheses or claims (Hyp, Clai), and testing a statement (Test).

The D-transitional[27] nature of the tasks, gave the students the opportunity to argue about the solid, its figural units (faces, edges, vertices), and their properties by observing its sub-parts cross-sections. The students had the opportunity to link properties of two-dimensional parts of the solid, with its three-dimensional characteristics, and create hypotheses (Hyp, Clai) or draw conclusions (Conc, 1D).

The combination of D-transitional tasks and their design in a DGE, gave the students the opportunity – and challenge at the same time – to come across possibilities they had not thought before, such as the hexagonal cross-sections in the case of the cube. The surprise and cognitive conflict was a challenge the students had to overcome. Sometimes this was successful and at other times it was not. This is why it was important that the tasks were also discussed in the frame of a whole classroom discussion. I discuss this matter further in the next section.

7.6 Patterns of argumentation in pair-work and in classroom discussion

I would now like to discuss my research question:

2.3 "How do students' patterns of argumentation differ in *pair-work* and in *classroom discussions*?"

As I explain in Section 7.1, I decided to create separate *patterns of argumentation* for pair-work and for the classroom discussions. I did this because in the case of students' pair-work, the argumentation takes place while students work on the task. That means that they begin with no initial impression on what the situation they have in front of them is, and familiarize themselves with it step-by-step. On the contrary, argumentations taking place in classroom discussions are built after the students have worked on the tasks, shaped their ideas and concluded about their answers. These

[27] For the description of D-transitional tasks, refer to Chapter 5, subsection 5.2.2.2.

different origins may create different conditions for argumentation. How much of this difference in students' argumentations, is illustrated in the patterns of argumentations though?

The answer is "not much". The main difference between the pair-work patterns and the classroom discussion patterns, is that in the second ones there is a new element, which does not appear in the pair-work patterns, and this is "Q?". This element represents the teacher's action of raising a question to the classroom, and one could say that this is quite an expected difference. Apart from this though, there is not much difference between the patterns in the two categories (pair-work and classroom discussions).

However, there are important differences in students' argumentations in the two cases (pair-work and classroom discussions), even though these are not mirrored in the patterns of argumentation. To gain a more precise insight into the detailed structure of students' argumentations I need to perform an even "deeper" argumentation analysis of students' work. That is what I do in Level 3 Analysis of the data, the results of which I discuss in Chapter 8.

7.7 Epilogue

In this chapter I have presented the results from the Level 2 Analysis of my data. I analyzed students' argumentations both during their pair-work, as well as in the classroom discussions. From the analysis, six types of *patterns of argumentation* emerged: three of them in their pair-work argumentations, and three in the classroom discussions. The observation of these patterns provide me with a view of students' overall argumentation and helped me organize their work in step-by-step actions they took from the beginning of their work until reaching their final conclusions.

Looking at both students' pair-works, as well as at the classroom discussions has revealed the different challenges and opportunities each of these learning environment settings brings to the argumentation. In pair-work, students' patterns of argumentation are mainly oriented around exploration (DO, Hyp, Clai), taking initiative, experimentation (DO and Test) and drawing their conclusion (Conc, 1D), based on their own thinking. In argumentations taking place in classroom discussions the students collaborate, they negotiate, exchange ideas, make statements (Hyp, Clai, Conc, 1D), argue for or against a statement (Conc, 1D, Contra), they provide alternative hypotheses or solutions, and they help each other explain or understand challenging phenomena (e.g. how the pentagonal and hexagonal cross-sections of a cube occur).

Nevertheless, the method of Pattern of argumentation has its own limitations. As I commented in Section 7.6, it does not promote a deep insight into the fine structure and the detailed characteristics of the argumentations. It also does not provide me with insights into students' visualization and the way it may influence their argumentation. The Pattern of argumentation is an action-oriented perspective on argumentation. It focuses on argumentation elements that represent students' actions and the chronological sequence of these actions.

In the next chapter I present the results of my Level 3 analysis of students work, with the aim to address the issues raised in the paragraph above.

8 The role of visualization in students' argumentations

Introduction

Until now I have presented results about the exploration strategies students employ (see Chapter 6), as well as about their patterns of argumentations while dealing with a D-transitional task (see Chapter 7).

In this chapter, I present and discuss the way in which students use *visualization* in their work and how it influences their argumentations. In order to do this, I "zoom-in" on the detailed structure of their argumentations, in order to identify their functional elements (Toulmin, 1958, see also Chapter 2) and to examine the roles that *visualization* plays in their argumentations as well as its functions (warrant, backing, etc) in the argumentation structures.

For the purposes mentioned above, I analyzed students' work in two steps: 1. I reconstructed their argumentations using Knipping's (2003a, 2003b, 2008) methodology of *argumentation reconstruction* (see also Knipping & Reid, 2019) and Toulmin's (1958) *functional model of argumentation*, and then 2. I identified the uses of *visualization* in students' argumentation by looking for specific indicators (see more details further below) that hint to those uses. Finally, based on this analysis I represented schematically the use of *visualization* in the *argumentation structures* (see for example Figure 8.4 in subsection 8.1.1). I call this the "Level 3 Analysis" of my data. In Chapter 5, where I present the method and methodology of the present work, I explain the processes of my "Level 3 Analysis" in full detail (see subsections 5.4.3 and 5.4.4).

Knipping's (2003a, 2003b, 2008) methodology of *argumentation reconstruction* and Toulmin's (1958) *functional model of argumentation* have been widely used in empirical research (e.g. Cramer, 2018; Potari & Psycharis, 2018) in order to analyze students' argumentations within various learning situations. I am using Knipping's and Toulmin's approaches as part of my analysis, extending their use for the illustration of *visualization* in students' argumentations. Therefore, in this chapter, I focus on the presentation of the results from step 2 (see paragraph above) of the "Level 3 Analysis", which is about the roles of *visualization* in students' argumentations and its function in the argumentation structures.

In Papadaki et al. (2019) we have already presented results from the argumentation analysis (Step 1 in "Level 3 Analysis") of the data of the present work. The results presented in that paper concern the modeling of abductive arguments in the settings of the present work. Here, I focus on visualization and in this chapter, I discuss in more detail the results of Step 1 of the "Level 3 Analysis", whenever I consider that it adds valuable information for a complete description of students' use of *visualization* in their argumentation.

The chapter is structured around the following three research questions:

3.1 How do *non-iconic visualization* and *spatial manipulation* manifest themselves in students' argumentations? (Subsection 8.6.1)

3.2 What are the *roles* of *non-iconic visualization* and *spatial manipulation* in students' argumentations? (Subsection 8.6.2)

3.3 How does the specific *design* of the given tasks influence students' *non-iconic visualization* and *spatial manipulation*? (Subsection 8.6.3)

Below, I briefly describe the five roles of *visualization* in students' argumentation that I identified in my analysis, as well as students' actions, which indicate the use of *visualization* in their argumentation. In Sections 8.1 to 8.5, I present those roles in more detail through various examples from students' pair-works and from the classroom discussions. In Section 8.6 I summarize the results and answer the research questions of this work that are relevant to this chapter (namely, research questions 3.1, 3.2 and 3.3). The chapter closes with an epilogue (Section 8.7).

Spatial manipulation and _non-iconic visualization_ in students' argumentation

By *spatial manipulation* of a mental image, I refer to the mental manipulation of an invisible geometric object in space by a student, as well as to its mental movement in respect to other objects in its environment (see also chapter 3). Here, manipulation involves any type of movement in space: rotation or tilt around any symmetry axis or point of the solid, e.g. rotation of a pyramid around its axis that goes through its top point and the center of its base. Regarding the movement of a geometric solid in respect to another object, this could be for example the change of orientation of a pyramid placed in a three-dimensional coordinate system in respect to the plane xOy defined by axes x and y: is the pyramid over it? under it? does it stand on it on one of its side surfaces or with its base? is it tilted towards it? Here, the students work with three-dimensional geometric objects that are invisible, as well as with the mental images of them that the students create and manipulate in their minds. Hence, *spatial manipulation* refers to the students' cognitive processes of mentally manipulating invisible solids in space, anticipating their orientation and position with respect to: their symmetry axis, the three-dimensional coordinate system, the plane xOy, or the movement of a slider.

As I discuss in Chapter 3, in this work *visualization* refers to a cognitive process and is a means to explore and interpret a mental image (or a drawing). Because *visualization* works both in the presence, as well as in the absence of a physical object, it also works with a mental image of it. Duval (2005) distinguishes two modalities of *visualization*, namely "iconic" and "non-iconic" *visualization* (see Chapter 3, subsection 3.1.1). *Iconic visualization* is a spontaneous and superficial, kind of "looks-like" approach of perceiving a drawing or a mental image, where the shape and the appearance of the object are fundamental for its interpretation. In contrast, through *non-iconic visualization* one recognizes the figural units (components of lower-dimension, such as vertices, edges, diagonal planes etc.) of an object or of its mental image. It also moves further than that. Using *non-iconic visualization* means to relate properties of sub-parts of different dimensions.

Hence, it is the use of *non-iconic visualization* that can be expressed through relationships and properties, and it can play an important role in argumentation.

Iconic visualization is not a process of making connections or "creating relationships" type of process. This means that it cannot play a role in an argumentation that is based on properties and statements linked with consequential relationships. Therefore, in this work I focus on the use of *non-iconic visualization* in students' argumentations.

Although both *spatial manipulation* and *non-iconic visualization* are cognitive processes related to geometric objects, there is an important difference between the two. *Non-iconic visualization* involves relationships only between a geometric object and its sub-parts, while through *spatial manipulation* a geometric object is associated with its environment. More specifically, in *non-iconic visualization* one relates a geometric object with its constitutive parts (its figural units), and also the properties of the object with the properties of its constitutive parts. Hence, the entire reasoning happens within the frame of the geometric object and not outside of it. In the case of *spatial manipulation* the cognitive process taking place, associates a geometric object with its environment. So, in this work for example, the association may be between the invisible solid and its position with respect to the plane of intersection xOy, or between the sliders moving the invisible solid and its resulting movement in space.

For the sake of brevity, from this point on I refer to *spatial manipulation* with the term *Sp-manipulation* and to *non-iconic visualization* with the term *NI-visualization.*

Indicators of *spatial manipulation* and *non-iconic visualization* in argumentation

My aim in this chapter is to reveal the *roles* of *Sp-manipulation* and *NI-visualization* in students' argumentations. As main *indicators* of *Sp-manipulation*, I use students' *verbal descriptions* and *gestures,* either of solid's movement in space during the movement of a slider, or of the solid's position when a specific cross-section occurs. Duval's (2005) *NI-visualization* goes hand in hand with the use of geometric properties. On account of this, I follow students' use of properties as a possible *indicator* of their *NI-visualization.* As I mention in Chapters 3 and 5, *NI-visualization* is closely related to *dimensional deconstruction*, which involves identifying and using sub-parts of a geometric object that are of lower dimensions. *NI-visualization* is a prerequisite for someone to be able to perform *dimensional deconstruction* of a geometric object (Mithalal & Balacheff, 2019). Therefore, in the present work I use students' performance of *dimensional deconstruction* as an *indicator* that the students employ their *NI-visualization* in argumentation. For more details about the relationship between *NI-visualization* and *dimensional deconstruction*, refer to Chapter 3.

The roles and functions of *spatial manipulation* and *non-iconic visualization* in argumentations

The "Level 3 Analysis" of students' argumentations has revealed the different roles that *Sp-manipulation* and *NI-visualization* play in them, as well as the way they function in the argumentation structures. In Table 8.1 below I provide a concise overview of the different roles and functions of *Sp-manipulation* and *NI-visualization* in students' argumentations, before I move to their detailed description.

<table>
<tr><th colspan="2">Roles of Sp-manipulation and NI-visualization in argumentation</th><th>Functions of Sp-manipulation and NI-visualization in the argumentation structure</th></tr>
<tr><td>1</td><td>Creating a hypothesis or a claim</td><td rowspan="3">Warrant</td></tr>
<tr><td>2</td><td>Drawing a conclusion</td></tr>
<tr><td>3</td><td>Explaining visual data</td></tr>
<tr><td>4</td><td>Creating a refutation</td><td>Warrant, Backing or Refutation</td></tr>
<tr><td>5</td><td>Backing a warrant</td><td>Backing</td></tr>
</table>

Table 8.1: Roles and functions of *Sp-manipulation* and *NI-visualization*

From the data analysis, it becomes apparent that *Sp-manipulation* and *NI-visualization* can play five roles in students' argumentations. The left column of Table 8.1 shows those five roles, presented in detail in Sections 8.1 to 8.5.

With regard to the contents of the right column of the table, a little further explanation is required. The right column shows the "functions" of *Sp-manipulation* and *NI-visualization* in the argumentations. Some functions are shared by more than one role, for example Roles 1,2 and 3 share the function "warrant". Other times a single role may have a different function depending on the situation, such as Role 4 in which *Sp-manipulation* and *NI-visualization* may function as warrants, backings or elements in a refutation (for more details refer to Section 8.4).

In those roles, *Sp-manipulation* and *NI-visualization* function in argumentation in the same way that warrant-statements and backing-statements do (Toulmin, 1958, see also Chapter 2), as I demonstrate later in this chapter. From now on I will refer to the function called "warrant" in Table 8.1, as *Sp-manipulation* and *NI-visualization* functioning as warrants in the argumentation. When *Sp-manipulation* and *NI-visualization* function as warrants, they facilitate the step from a datum towards a hypothesis, a claim or a conclusion. This can happen while creating a hypothesis or a claim (Role 1), while drawing a conclusion (Role 2), or while explaining visual data (Role 3). I explain those cases in more detail through examples in the following sections. The warrant–function takes place when the students do not state explicitly the warrant that they use in order to connect the data with the conclusion (hypothesis or claim).

Similarly, from now on I will refer to the "backing"-function called (see Table 8.1) when *Sp-manipulation* and *NI-visualization* function as backings in the argumentation. This happens when *Sp-manipulation* and *NI-visualization* are observed to support an explicit warrant provided by the students. Their purpose in this case, is to support the warrant, exactly as a backing-statement (e.g. a geometric property, a theorem etc.) would do. This function appears when *Sp-manipulation* and *NI-visualization* fulfill the Roles 4 and 5, namely when they support the creation of a refutation or when they back a warrant.

Finally, *Sp-manipulation* and *NI-visualization* take place in the argumentation in refutations or as refutations. This function ("refutation" in Table 8.1) appears when *Sp-manipulation* and *NI-visualization* support the creation of a refutation by the students (Role 4). I illustrate and discuss this as well through examples in Section 8.4.

In the following examples, I first present my observations of students' work and the parts of it, which I identify as *indicators* of *Sp-manipulation* and *NI-visualization*. I then lay out my interpretation about the roles of *Sp-manipulation* and *NI-visualization* in students' work, representing them in the argumentation structure as well.

In Toulmin's (1958) functional model of argument, each element of the argument represents a specific statement. For example, a warrant may be a specific statement, such as the Pythagorean theorem. In the classroom reality though, students rarely express their arguments explicitly enough, providing complete statements. In most cases students do not articulate parts of their reasoning process at all, making their argumentations even less explicit. In my data, the processes of *Sp-manipulation* and *NI-visualization* are indicated by other processes followed by the students (e.g. *dimensional deconstruction*), actions they perform or gestures they do in order to express and describe their thinking. Therefore, in this work, *Sp-manipulation* and *NI-visualization* are not represented in the argumentation structures with just one statement. On the account of that, I symbolize the processes of *Sp-manipulation* and *NI-visualization* in the argumentation structures with the codes SpM (for *Sp-manipulation*) and NIV (for *NI-visualization*), and I describe them in detail providing information about the processes, actions, gestures or verbal description that indicate them, in the text description of each episode.

8.1 Role 1: Supporting the creation of a hypothesis or a claim

There are cases in students' argumentations, where *Sp-manipulation* and *NI-visualization* support the students to shape a hypothesis or a claim, originating from the data they observe. In these cases, I say that the role of *Sp-manipulation* and *NI-visualization* is that of supporting the creation of a hypothesis or a claim (Role 1). Recall, that the difference between a hypothesis and a claim lies in the *epistemic value* (Duval, 2007) attributed to a statement by the students, whether explicitly or implicitly (see Chapter 5, subsection 5.4.2). A hypothesis is a supposition stated by the students without a conviction from their part about its validity, it is merely the statement of a possible case under specific circumstances (data). A claim, on the other hand, is a supposition that is believed by the students to be the most possible case, and consequently true.

As I show in the examples that follow, students' hypotheses and claims may be about the form of the hidden solid, its orientation or the evolution of their cross-section's shape during the manipulation of a slider (height, tilt or spin). In all the cases though, when either *Sp-manipulation* or *NI-visualization* play the role of supporting the creation of a hypothesis or a claim, they function in argumentation as warrants. That means, that they stand between the datum (or data) and the hypothesis or claim, leading the argument from the former to the latter. As I mention in the introduction of the chapter, the employment of *Sp-manipulation* and *NI-visualization* is indicated by processes the students follow (e.g. *dimensional deconstruction*), actions they perform or gestures they make in order to express and illustrate their thinking.

In Table 8.2 I describe the first role of *Sp-manipulation* and *NI-visualization* in argumentation. In the first column of the table I present three *modes*, ways in which

Sp-manipulation and *NI-visualization* take place in argumentation, namely each one alone or both in *synergy*. As I have mentioned, in the first Role *Sp-manipulation* and *NI-visualization* function in the same way as warrant-statements would. Therefore, in the argumentation structures that are illustrated in the examples that follow, *Sp-manipulation* and *NI-visualization* are always found functioning as warrants. In the second column, I present the phenomena (processes or actions) that I consider as *indicators* of *Sp-manipulation* and *NI-visualization* in students' work, such as the performance of *dimensional deconstruction*, referring to relationships between properties of different figural units of the solid, using gestures to illustrate the movement of the solid, and others.

Role 1: Creating a hypothesis or a claim	
Mode	**Indicators**
NI-Visualization	• Performing *dimensional deconstruction* (recognizing 2D figural units) • Referring to relations between properties of objects of different dimensions
Synergy of *NI-visualization* and *Sp-manipulation*	• Performing *dimensional deconstruction* (recognizing 2D figural units) • Referring to relations between properties of objects of different dimensions • Connecting properties of cross-sections with the movement of the slider • Connecting the movement of the slider with the evolution of the solid
Sp-manipulation	• Describing the movement of the solid and the consequential change of the cross-sections • Describing the way the solid is oriented in space

Table 8.2: Role 1 of *NI-visualization* and *Sp-manipulation* – Creating a hypothesis or a claim

8.1.1 *Non-iconic visualization* supporting the creation of a claim

In the episode described here, two cases of *NI-visualization* occur, supporting the creation of a hypothesis and a claim (see Transcript 8.1[1]). Axel and Dave are working on Task 2[2] (invisible sphere). Recall, that the symbols for the GeoGebra sliders: h (for the German word Höhe), n (for the German word Neigung) and d (for the German word Drehung), represent the operations on height, tilt and spin, respectively.

<u>Episode description</u>

This episode starts with Axel moving the height-slider (h-slider) up and down (exploration of the case (h, n=0°, d=0°)). The visible cross-section is circular and decreases in size as Axel moves the slider above zero (see Figures 8.1 and 8.2a. The circular cross-section in Figure 8.1 for h=0,85, is smaller than that in 8.2a at the initial position where h=0). Dave then states, "It is a cone" (utterance 3, H_3). Axel then puts the h-slider back to zero and changes the tilt-slider (n-slider, exploration of the case (h=0, n, d=0°)). The cross-section appearing on the computer screen is a large circle and it remains unchanged while varying the tilt (n-slider) (see Figures 8.2a and b. The circular cross-sections are of the same size). Axel then states, "It is a circle" (utterance 4, H_4). Dave contradicts him, repeating that it is a cone, and the circle is only the cross-section. Axel then states, "this is a sphere" (utterance 8, code Cl_8) "because otherwise… the tilt" (utterance 10). He varies the tilt again, and then the height, and as he increases the h-slider over zero he notes "it is always smaller, because the sphere comes out" (utterance 10, code $W_{10.2\text{-}3}$). Dave asks "Does it go in both directions?" and Axel moves the h-slider down below zero and back above zero. On the screen appear circular cross-sections that decrease in size as the h-slider moves away from zero, in both directions (see Figures 8.1, 8.2a and 8.3).

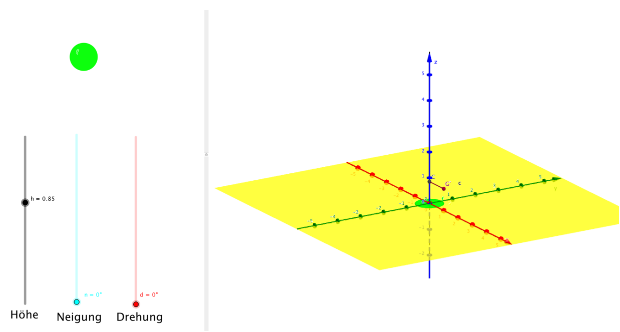


Figure 8.1: The circular cross-section at position (h=0,85, n=0°, d=0°)

[1] For the meanings of the codes in Transcript 8.1 (and all further transcripts) please see the table in Appendix E1.

[2] See the global argumentation structure and the worksheet for Episode GR1AD-2 in Appendices E2 and G1, respectively.

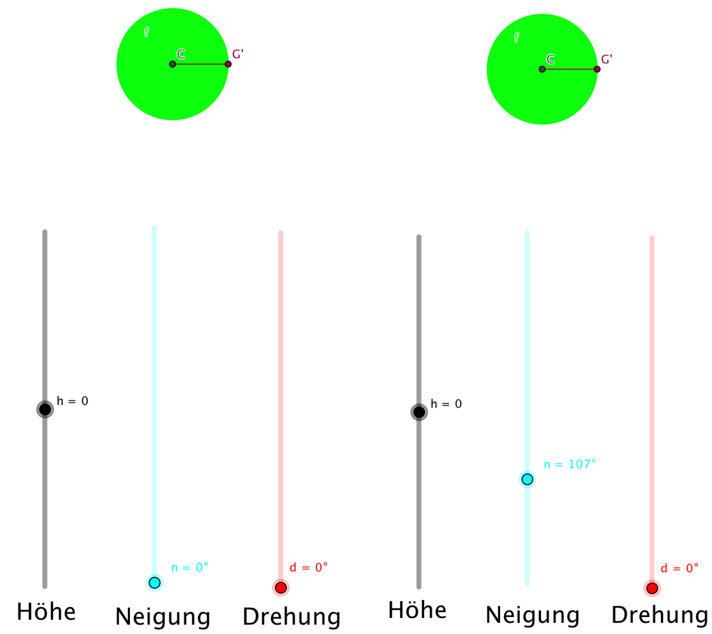


Figure 8.2: a (left) and b (right). The circular cross-sections at initial position (h=0, n=0°, d=0°) and at position (h=0, n=107°, d=0°) are of the same size

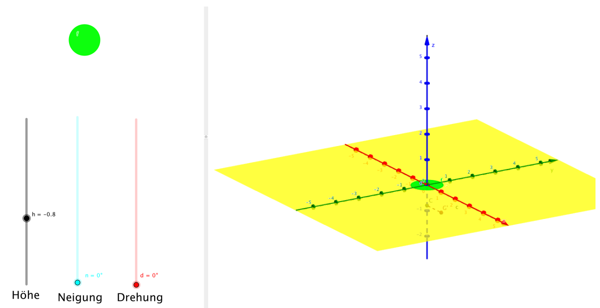


Figure 8.3: a (left) and b (right). The circular cross-sections at initial position (h=0, n=0°, d=0°) and at position (h=0, n=107°, d=0°) are of the same size

Utteran.	Codes	Original German transcript	English translation
1	$\mathbf{D_{1.1}}$ $VD_{1\text{-}4}$	Axel: Oh, **der [*the solid*] ist unsichtbar**. [*Axel moves the h-slider up and down for the exploration of the case (h, n=0°, d=0°)*] #00:59:46-2#	Axel: Oh, **it [*the solid*] is invisible** [*Axel moves the h-slider up and down for the exploration of the case (h, n=0°, d=0°)*].
2	$VD_{1\text{-}4}$	Dave: Ja, die Stelle.. (?) #00:59:51-5#	Dave: Yes, the position..
3	$VD_{1\text{-}4}$ $\mathbf{H_3}$, NIV_3	Dave: **Ist ein Kegel**. (unverständlich)	Dave: **It is a cone**.
4	$VD_{1\text{-}4}$ $\mathbf{H_4}$	Axel: Oah, die Neigung ist awesome [*Axel moves the n-slider up and down for the exploration of the case (h=0, n, d=0°)*]. **Ist ein Kreis**. #00:59:53-3#	Axel: Wow, the tilt is awesome [*Axel moves the n-slider up and down for the exploration of the case (h=0, n, d=0°)*]. **It is a circle.**
5	$D_{5.1}$ $\mathbf{C_{5.2}}$	Dave: Ne, das ist ein Kegel. Ist ja ein dreidimensionaler Körper und **das ist nur die Schnittstelle**, weil der Körper unsichtbar ist. #01:00:02-5#	Dave: No, this is a cone. It is a three-dimensional solid and **this is only the cross-section**, because the solid is invisible.
6	VD_6	Axel: Ja, und das ist ein Kreis. #01:00:03#	Axel: Yes, this is a circle.
7	$\mathbf{C_7}$	Dave: Ja, **die Schnittstelle ist ein Kreis**. #01:00:05-4#	Dave: Yes, **the cross-section is a circle**.
8	$\mathbf{Cl_8}$	Axel: Ne, das ist eine Ku-, also **das ist ein Kugel insgesamt**. #01:00:08-2#	Axel: No, this is a sphe-, so **this is a sphere altogether**.
9		Dave: Sicher? #01:00:08-7#	Dave: Sure?
10	$VD_{10.2}$ $\mathbf{W_{10.2\text{-}3}}$	Axel: Ja, normal, weil sonst- wie würde denn sonst die Neigung- [*Axel varies the n. The cross-section does not change*]? (..) So [*Axel varies the h*] wird es immer kleiner, **weil die Kugel raus geht und so- bopp** [*a cross-section appears again*] *#01:00:19-1#*	Axel: Yes, normal, because otherwise- how would otherwise the tilt- [*Axel varies the n. The cross-section does not change*]? This way [*Axel varies the h*] it always gets smaller, **because the sphere comes out and so pop- bopp** [*a cross-section appears again*]
11	VD_{11} NIV_{11}	Dave: Geht das in beide Richtungen [*Axel moves the height slider over and under zero*]? Nur so als Test. #01:00:26-2#	Dave: Does it go in both directions [*Axel moves the height over and under zero*]? Just so as a test.
12		Axel: (unverständlich) guck Mal, so kannst du schon die Kugel sehen [*Axel varies the rotation slider d*]. Fast. (..) #01:00:33-5#	Axel: Look, this way you can already see the sphere [*Axel varies the rotation slider d*]. Almost..

Transcript 8.1: Dave and Axel on Task 2 (invisible sphere)

Reconstruction of the argumentation

Figure 8.4[3] shows the first part of Axel and Dave's argumentation where the two *NI-visualizations* take place in it (NIV_3, NIV_{11}). The argumentation structure begins with visual data $VD_{1\text{-}4}$, which are data visible on the computer screen and they emerge from the manipulation of the sliders. When the h-slider (height) is dragged over and under zero (utterances 1-3), the cross-sections are circular and they change size. When the n-slider (tilt) is dragged over zero degrees, while the height and the spin remain constant (utterance 4), the size of the circular cross-sections remains the same. Axel and Dave do not express any of this verbally. From the visual data $VD_{1\text{-}4}$ follows Dave's hypothesis "It is a cone" (code H_3) and Axel's hypothesis "It is a circle" (code H_4). From the additional visual data $VD_{10.2}$ (the size of the circular cross-sections does not change when manipulating the tilt-slider) and VD_{11} (there are cross-sections both over and under zero) follows the refutation R_1 of Dave's hypothesis H_3 "It is a cone". The refutation R_2 is a sub-argument in which Dave refutes Axel's hypothesis H_4 "It is a circle". From the refutation R_2, and the additional visual data $VD_{10.2}$ and VD_{11}, follows the claim Cl_8 "this is a sphere". The two *NI-visualizations* (NIV_3, NIV_{11}) taking place in this argumentation stream, join some of its elements together. NIV_3 occurs in the argument joining $VD_{1\text{-}4}$ and H_3. NIV_{11} connects $VD_{10.2}$, VD_{11} and Cl_8. NIV_{11} supports also the refutation R_1 of hypothesis H_3 by the visual data $VD_{10.2}$ and VD_{11}. Following, is a more detailed description of this process.

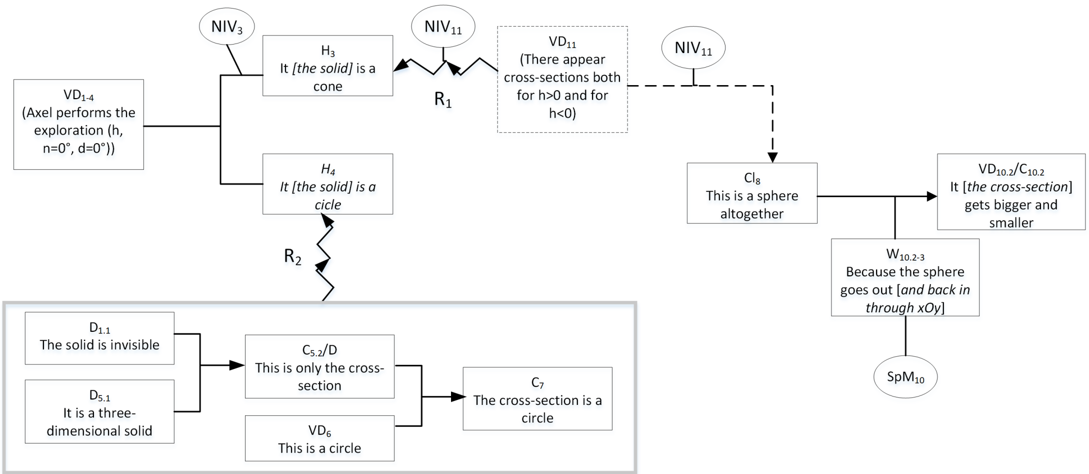


Figure 8.4: *NI-visualization* in Dave and Axel's argumentation structure (Task 2 – invisible sphere)

Interpretation

NIV_3: Towards Dave's hypothesis via *dimensional deconstruction*

Dave arrives at the hypothesis H_3 "It is a cone" on the basis of the visual data $VD_{1\text{-}4}$ (the diminishing circular cross-sections when h is increased over zero). He does not state a warrant for his argument. Given the nature of the data, I infer that he is connecting properties of the cross-sections with properties of the cone. More precisely, he seems to connect the fact that the cross-sections are circles that get smaller, with

[3] For the meanings of the symbols used in Figure 8.1 (and all the further figures of argumentation structures) please refer to the table in Appendix E1.

the property of the cone, that it is a solid whose horizontal cross-sections are circles that get smaller until they converge to a single point.

I interpret this as Dave performing a *dimensional deconstruction* of the cone. The diminishing circular cross-sections seem to be key for the formulation of his hypothesis H_3 (the solid is a cone). When Dave sees the diminishing circular cross-sections as the two-dimensional (2D) figural units (see Table 8.2 in the previous pages) of the invisible solid (which he says to be a cone), he relates what he can see (the diminishing 2D circular cross-sections) with the cross-sections of a cone.

The fact that the solid is invisible makes this *dimensional deconstruction* particularly interesting. In the empirical research literature, *dimensional deconstructions* are usually performed in the presence of a figure or a real object that represents the geometric object (Duval, 2017; Mithalal & Balacheff, 2018). Here, Dave manages to "see" the cross-sections as lower-dimensional figural units of an invisible solid, inferring from this that the solid may be a cone. Using *NI-visualization* means to relate properties of a geometric object's sub-parts of different dimensions. Therefore, *NI-visualization* is necessary in order to be able to perform the *dimensional deconstruction* of a geometric object (Mithalal & Balacheff, 2018). Here, Dave performs a *dimensional deconstruction* of the cone (or more precisely of his mental image of a cone), which means that he can visualize the cone in a non-iconic way, and connect the properties of the cone with those of the visible circular cross-sections. Hence, I consider the performance of *dimensional deconstruction* by Dave to be an *indicator* of his use of *NI-visualization*, which is symbolized in this argumentation structure as NIV_3.

NIV_{11}: Creating a new claim and refuting Dave's hypothesis

The next use of *NI-visualization* takes place when Axel and Dave use the visual data VD_{11} (moving the h-slider up and down, below and above zero) to support Axel's claim that the solid is a sphere (Cl_8).

More precisely, after his hypothesis H_4 has been rejected by Dave, Axel says that the solid is a sphere (utterance 8). When Dave asks Axel if he is sure about his statement, Axel responds that he is sure (utterance 10), thus expressing his belief that his new supposition is not just one valid possibility, rather the only valid possibility. Furthermore, after Dave's question "Does it go in both directions?" (utterance 11), both students engage in a process to validate Axel's supposition. Both Axel's verbal expressions as well as the students' conscious intention to prove the validity of Axel's supposition, are factors that indicate the epistemic value of the statement "this is a sphere" as a claim (Cl_8, see Figure 8.4).

What Axel and Dave do in utterance 11, is that they identify the circular cross-sections of different sizes they see during the height-exploration, as two-dimensional figural units of the sphere. This is a process of *dimensional deconstruction* of the sphere. It is on these observations that they base their claim that the solid is a sphere (Cl_8), although they do not explain why and how they move from their observation to the inference of their claim. Because they perform the *dimensional deconstruction*, in place of the missing warrant, I infer that the students use *NI-visualization* (NIV_{11}). Via NIV_{11} the students move from the visual data of the circular cross-sections that get smaller both over and under xOy (VD_{11}), to the claim that the solid is a sphere (Cl_8).

NIV_{11} is also used to refute Dave's initial hypothesis H_3 that the solid is a cone (see refutation R_1 in Figure 8.4). I discuss the role of *NI-visualization* in refuting a statement in more detail in section 8.5.

8.1.2 *Synergy* of *spatial manipulation* and *non-iconic visualization* supporting the creation of a hypothesis or claim

In this subsection I focus on the role of the *synergy* of *Sp-manipulation* and *NI-visualization* in Dave and Axel's argumentation, while they work on Task 3B (invisible pyramid)[4]. Transcript 8.2 shows the part of students' discussion on which I focus here.

Episode description

In this part, Dave performs the height-exploration of the case (h, n=0°, d=0°), dragging the height-slider (h-slider) above and below zero. In the software, the cross-section disappears as soon as the h-slider is dragged over zero (see Figure 8.5), while when dragging the h-slider under zero, the cross-sections are squares that diminish in size (see Figure 8.6) until they converge to a single point. Then, Axel says that the solid is a pyramid ($Cl_{3.2}$). Dave agrees with Axel, saying that it is "pretty sure" (utterance 4). Then, Dave says that the cross-section he sees while dragging the h-slider up and down "is definitely a quadrilateral" ($C_{6.1}$, utterance 6). Then they say that the pyramid "runs up to a point when one goes to the negative area", that is when the h-slider is dragged under zero ($C_{7\text{-}8}$). Dave also points out that when the value of height is equal to minus two (h=-2), the cross-section "is only a point" ($C_{8.1}$). Finally, Dave makes a statement about the orientation of the solid in space stating, "this is a reverse pyramid" (Cl_{10}).

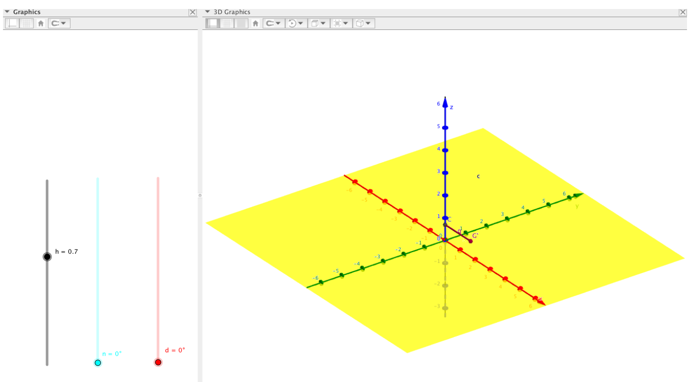


Figure 8.5: No cross-section at position (h=0,7, n=0°, d=0°)

[4] See the global argumentation structure and the worksheet for Episode GR1AD-3B.II in Appendices E4 and G3, respectively.

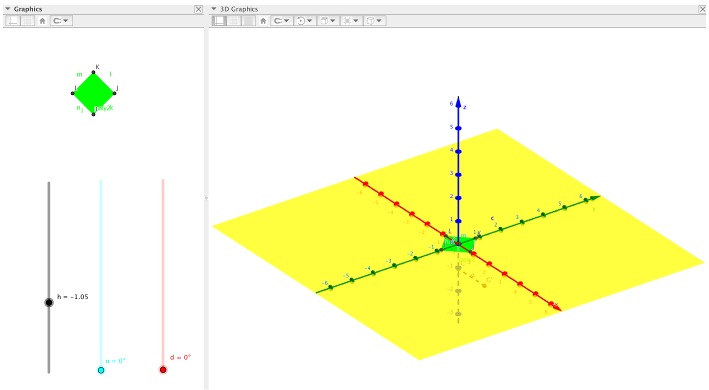


Figure 8.6: Smaller square cross-section at position (h=-1,05, n=0°, d=0°)

Utterance	Codes	Original German transcript	English translation
3	*VD_3* **$Cl_{3.2}$**	Axel: Ja. Ja [*Axel says that as he watches Dave dragging the h-slider over and under zero, for the case (h, n=0°, d=0°)*]. **Pyramide**. #00:33:36-2#	Axel: Yes. Yes [*Axel says that as he watches Dave dragging the h-slider over and under zero, for the case (h, n=0°, d=0°)*]. **Pyramid**.
4	**Q_4**	Dave: **Ziemlich sicher**, oder? #00:33:38-0#	Dave: **Pretty sure**, right?
5		Axel: Ja. #00:33:40-9#	Axel: Yes
6	**$C_{6.1}$/D**	Dave: **Das ist auf jeden Fall ein Viereck** [*Dave refers to the shapes of the cross-sections emerging on the screen when the h-slider is dragged under zero*]. Mal ein Viereck. (..) Habt ihr B? B? [*Dave talks to a different student*] #00:33:55-5#	Dave: **This is definitely a quadrilateral** [*Dave refers to the shapes of the cross-sections emerging on the screen when the h-slider is dragged under zero*]. Draw a quadrilateral. Do you have B? B? [*Dave talks to a different student*]
7	$C_{7\text{-}8}$/D	Axel: Wenn man in den negativen Bereich geht [*for h under zero*] oder in den positiven [*for h over zero*] läuft es spitz zu? #00:34:00-2#	Axel: Does it run up to a point when one goes to the negative area [*for h under zero*] or to the positive [*for h over zero*]?
8	**$C_{8.1}$/D**	Dave: In den negativen. (..) Und **h bei minus zwei ist nur noch ein Punkt da**. #00:34:23-2#	Dave: To the negative. (...) And **h at minus two there is only a point there**.
9		Axel: Der [*Axel mumbles while he is keeping notes on the worksheet*].. #00:34:27-8#	Axel: The [*Axel mumbles while he is keeping notes on the worksheet*]..
10	*Q_{10}* **Cl_{10}**	Dave: *Das ist* ein **umgedrehte Pyramide**. #00:34:30-1#	Dave: *This is* a **reverse pyramid**.

Transcript 8.2: Axel and Dave on Task 3B (invisible pyramid)

Reconstruction of the argumentation

Next, I discuss the structure of a part of Axel and Dave's argumentation from this episode and the role of the *synergy* of *Sp-manipulation* and *NI-visualization* taking place in it (see Figure 8.7). The argumentation starts with the visual data $VD_{2\text{-}3.1}$ where the cross-section disappears as soon as the height-slider (h-slider) is dragged above zero, while the quadrilateral cross-sections diminish when dragging the h-slider below zero. From that, follows Axel's claim $Cl_{3.2}$ that the solid is a pyramid, to which Dave agrees (utterance 4). From the same visual data, three more explicit conclusions, which are then used as data, follow describing what happens with the cross-sections when dragging the h-slider: when dragging the h-slider down the cross-section "is definitely a quadrilateral" ($C_{6.1}$/D), the pyramid "runs up to a point" "when one goes" "to the negative area" ($C_{7\text{-}8}$/D), and when the value of height is equal to minus two (h=-2), the cross-section "is only a point" ($C_{8.1}$/D). From these three data, follows the claim Cl_{10} that the solid "is a reverse pyramid". The *synergy* of *Sp-manipulation* (SpM_{10}) and *NI-visualization* (NIV_{10}) occurs in the argument joining the conclusions $C_{6.1}$/D, $C_{7\text{-}8}$/D and $C_{8.1}$/D, with the claim Cl_{10}.

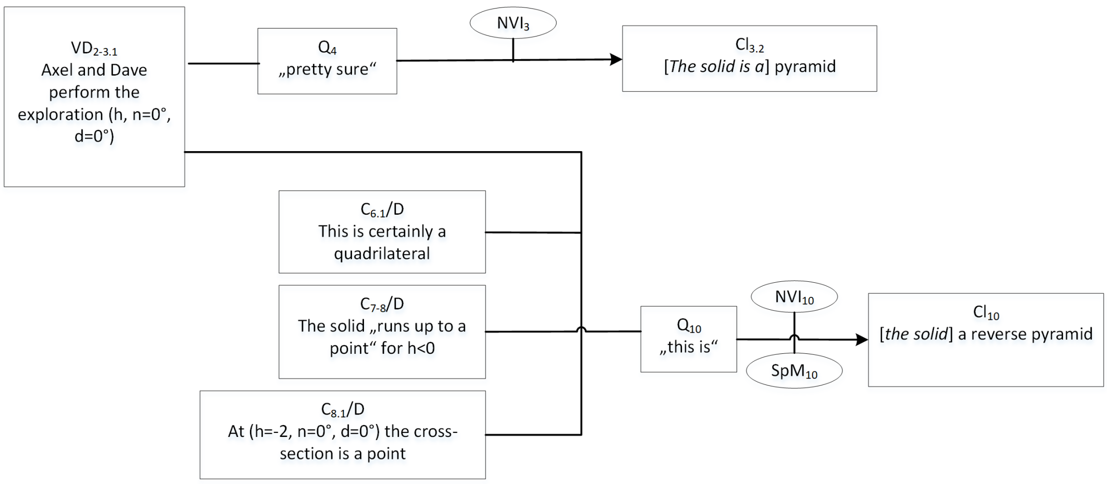


Figure 8.7: Synergy of NIV_{10} and SpM_{10} in Dave and Axel's argumentation structure (Task3B – invisible pyramid)

Interpretation

Dave's argumentation consists of different paths. First, he moves from the three conclusions that are now used as data $C_{6.1}$/D, $C_{7\text{-}8}$/D and $C_{8.1}$/D to his claim Cl_{10}. All three data consider the cross-sections of the solid, which is claimed to be a pyramid. Dave and Axel first identify the shapes and properties of the cross-sections when Dave says that they are quadrilaterals (utterance 6). Dave uses the word "quadrilateral", but Axel draws a square in the worksheet. I infer, that the fact that all the cross-sections are squares (they say "quadrilateral", $C_{6.1}$) that get smaller when dragging the h-slider under zero ($C_{7\text{-}8}$), while they disappear completely as soon as h is dragged over zero ($VD_{2\text{-}3.1}$), leads the two students to the claim that the solid is a pyramid ($Cl_{3.2}$), which is also part of the claim that the solid is a reverse pyramid (Cl_{10}). This means that Axel and Dave see the cross-sections as 2D figural units of the pyramid. This is a process of *dimensional deconstruction* and it is the *indicator* of the use of *NI-visualization* both in NIV_{10} as well as in NIV_3 in the argumentation (Figure 8.7).

But, the transition from the three data to claim Cl_{10} (reverse pyramid) does not only involve *NI-Visualization.* Axel asks: "Does it run up to a point when one goes to the negative area or to the positive?" (utterance 7). To this Dave responds that the cross-sections get smaller when he decreases h under zero ($C_{7\text{-}8}$), making the cross-section to converge to a point ($C_{8.1}$ in utterance 8). After this question, Dave claims that the pyramid is reversed at the initial position (h=0, n=0°, d=0°) (Cl_{10}). This means that he imagines the pyramid standing with its base on plane xOy and its top on z-axis, pointing towards the subspace under the plane.

Dave arrives at the claim Cl_{10}, following a process, during which he combines the movement of the slider (dragging h-slider downwards under zero) with the diminishing of the cross-sections. Seeing that the square cross-section "shrinks" when the height-slider is moved downwards under zero, he claims that the pyramid is placed reverse (like standing on its top point). That implies that Dave imagines the top of the solid being in the "negative" sub-space under plane xOy. Axel's question, is relative to the position of the pyramid in space, since one can determine whether the pyramid is upward or reverse by finding out for which values of h (height) there are cross-sections with the plane xOy, and for which there are not. The mental process that takes place then is *Sp-manipulation.* Dave's ability to imagine the solid and its orientation in space is the result of using *Sp-manipulation.* Therefore, I infer that this question and the conclusion that emerges after it ($C_{7\text{-}8}$), led Dave to his claim that the pyramid is reversed via the use of *Sp-manipulation.*

Dave's claim Cl_{10}, is the result of the *synergy* of both *NI-visualization* (NIV_{10}) and *Sp-manipulation* (SpM_{10}). *NI-visualization* is behind the parts of Dave's and Axel's argumentation in which they use geometric properties (in the process of *dimensional deconstruction*) in order to connect the form of the cross-sections with the form of the solid. This means that the use of geometric properties is the *indicator* of *NI-visualization.* On the other hand, *Sp-manipulation* is the process that helps Dave to connect his action on the height-slider with the orientation of the solid in space at its initial position (h=0, n=0°, d=0°). It is this connection between the external factor influencing the movement of the solid (the movement of the slider) and the solid's movement that indicates the use of *Sp-manipulation* here.

Dave's claim Cl_{10} (that the solid is a reverse pyramid) is not entirely correct. The solid is indeed a pyramid, but it is not reversed in its initial position (h=0, n=0°, d=0°) (see Figure 8.8). Based on the above description of how *NI-visualization* and *Sp-manipulation* operate in Dave's argument, I infer that what constitutes a challenge for Dave is imagining the movement of the invisible pyramid in space. NIV_{10} helps Dave to shape the part of his claim Cl_{10} that is relevant to the form of the invisible solid, namely that the solid is a pyramid. Therefore, it is not an ignorance of the properties (related to his *NI-visualization* NIV_{10}) that challenges him.

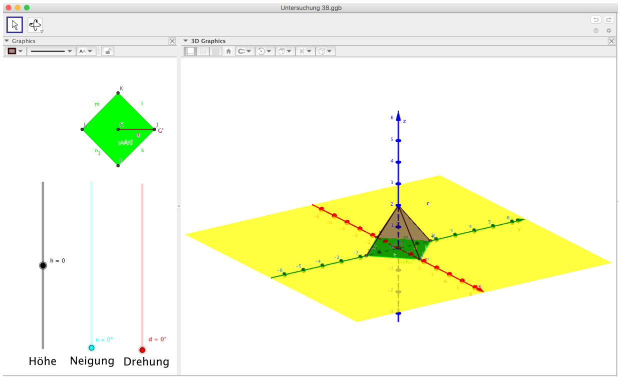


Figure 8.8: The pyramid at its initial position (h=0, n=0°, d=0°)

One possible explanation of Dave's misconception could be the following. The cross-sections get smaller as the height-slider is moved downwards under zero. One may falsely imagine that it is the plane of intersection that is dragged downwards "scanning the solid" as it moves downwards on the z-axis. In this case, the solid would have to stand on xOy with its top point on the negative side of the z-axis at the initial position (h=0, n=0°, d=0°). On the contrary though, it is the solid that moves downwards passing through the plane of intersection as the height-slider is dragged downwards under zero. The value h=0 represents the position (h=0, n=0°, d=0°), at which the base of the solid touches the xOy (and consequently the point O (0, 0, 0)). This means that the cross-sections will get smaller as the top point of the pyramid moves downward approaching the plane of intersection. Therefore, it is SpM_{10} that leads Dave to the (incorrect) second part of his claim, namely that this pyramid is reversed. Knowing that the misconception lies in Dave's *Sp-manipulation* can help us to choose the appropriate arguments with which to negotiate Dave's claim.

The distinction between what constitutes *Sp-manipulation* and what constitutes *NI-visualization* at each point in the argumentation, can help us (as researchers and as teachers) to recognize whether it is *NI-visualization*, *Sp-manipulation* or their *synergy* that operate, as well as the instances in students' argumentations in which they take place. This knowledge can assist us to unravel the cause of students' misconceptions and help them to overcome them.

8.1.3 *Spatial manipulation* supporting the creation of a hypothesis or claim

In this subsection I discuss the role of *Sp-manipulation*. First, I present two cases of *Sp-manipulation* in Dave and Axel's argumentation during their pair-work on Task 3B (invisible pyramid)[5]. Then, I present an example of *Sp-manipulation* during the classroom discussion on Task 3C (invisible cube)[6].

[5] See the global argumentation structure and the worksheet for Episode GR1AD-3B.II in Appendices E4 and G3, respectively.

[6] See the global argumentation structure for Episode CD3C-TL in Appendix E11.

SpM_{51} and SpM_W: Imagining the movement of the pyramid in space

Episode description

Here, I use an example from the same episode as in subsection 8.1.2. In this part of the episode, two cases of *Sp-manipulation* occur, supporting the creation of a claim and a conclusion. Let me start with a short description of what the students did before I move to the presentation of my interpretation. Axel and Dave have already claimed that the solid is a pyramid and they now perform the spin-exploration of the case (h=0, n=90°, d) (see Transcript 8.3). The pyramid is initially at position (h=0, n=0°, d=0°). Axel reads the information on the worksheet and tells Dave what values to put in the sliders. Dave places the height-slider at zero and the tilt-slider (n-slider) at 90° (see Figure 8.9 and utterances 40-41). Dave says that what he sees seems logical to him (utterance 42).

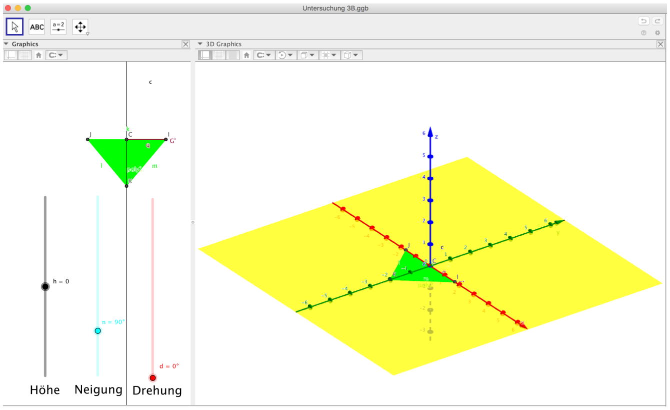


Figure 8.9: Cross-section at position (h=0, n=90°, d=0°)

Then, Dave drags the spin-slider (d-slider) up and down (utterances 43-44) and Axel says that "it is getting bigger and smaller" (D_{45}, utterance 45). Dave says that the length of segment IJ (ICJ in the transcript, utterance 46) changes and Axel says that this "makes total sense" (utterance 47). Then, Axel asks every how many degrees of spin, "it gets bigger and smaller" (utterance 49), wondering if this happens every 45°. He asks Dave to explore again slowly from d=0°. Then Axel says "it jump[s] big again" (in the sense of "expanding"), not at d=45°, rather at 90° (see last sentence in utterance 49, code D_{49}). He then says that he thinks that after d=90° "it will get smaller" at d=95° ($Cl_{51.3}$, in utterance 51).

Figures 8.9 and 8.10 show the position of the pyramid before the beginning of the spin-exploration. Figure 8.9 shows what the students could see at position (h=0, n=90°, d=90°), while in Figure 8.10 we see the same position when the pyramid is visible. Figure 8.11 shows a stream from the argumentation structure of the discussion presented in Transcript 8.3, as well as the cases of *Sp-manipulation* that take place in it.

Utterance	Codes	Original German transcript	English translation
40		Dave: h null #00:37:56-7#	Dave: h zero
41		Axel: n neunzig und d erkunden. [*Dave puts the h-slider at zero, the n-slider at ninenty degrees and drags the d-slider up*] #00:38:04-7#	Axel: n ninety and explore d [*Dave puts the h-slider at zero, the n-slider at ninenty degrees and drags the d-slider up*] .
42		Dave: Ja, da war ich gerade. Wo bei der n 90 ist, dann ist es nicht mehr da, das ist so [*Dave gestures the position and the orientation of the solid with his hands*], das ist logisch. #00:38:12-5#	Dave: Yes, that's where I just was. Where the n is 90, it is no longer there, it is so [*Dave gestures the position and the orientation of the solid with his hands*], that's logical.
43		Axel: Und d erkunden? d (unverständlich) #00:38:17-5#	Axel: And explore d? d
44		Dave: Ne. #00:38:17-2#	Dave: No
45	**D$_{45}$**	Axel: Eee.. **Es wird kleiner und größer**, ne? #00:38:21-7#	Axel: Eee.. **It gets smaller and bigger**, right?
46		Dave: Also- die Strecke ICJ (unverständlich) [*By "Strecke ICJ" Dave refers to segment IJ on the cross-section. C is a point on this segment, between points I and J, see Figures 8.9 and 8.10*] #00:38:24-5#	Dave: So- the segment ICJ [*By "Strecke ICJ" Dave refers to segment IJ on the cross-section. C is a point on this segment, between points I and J, see Figures 8.9 and 8.10*]
47		Axel: (unverständlich) macht es voll Sinn. #00:38:28-1#	Axel: it makes absolute sense.
48		Dave: Ja. #00:38:29-9#	Dave: Yes.
49	**D$_{49}$**	Axel: Bei den- was für Abständen wird das größer und kleiner? 45 Grad? Bei d, bei d, bei d. Geh mal auf null [*d=0* °], so langsam hoch [*d>0* °], so und jetzt wird - du bist zu schnell. Ja mach mal fünfundvi- also langsam. Wird- **springt das** jetzt auf fünfundvierzig **wieder groß**? Ne, erst **auf neunzig**; mach mal weiter #00:39:18-7#	Axel: With what kind of distances does that get bigger and smaller? Forty-five degrees? At d, at d, at d. Go again at zero [*d=0* °], slowly upwards [*Axel asks Dave to slowly drag the d-slider over 0* °], so and now it will – you are too fast. Yes go again at forty-five and slowly. Will- Does **it jump big again** now at forty-five? No, **only at ninety**, keep going
50		Dave: Glaubst du es sieht sich so kleiner? #00:39:21-1#	Dave: Do you think it looks smaller?
51	SpM$_{51}$ *Q$_{51.3}$* **Cl$_{51.3}$** **VD$_{51.3}$**	Axel: Ja, ich will wissen wann es wieder diesen großen Sprung macht. Wann es wieder so eine Ecke erreicht. Eigentlich, wahrscheinlich auch neun- ne, geh mal, ja neunzig und jetzt geh mal **auf fünfundneunzig.** *Ich glaube* **es wird hier kleiner**. #00:40:23-0#	Axel: Yes, I want to know when it does this jump again. When it reaches such an angle again. Actually, probably also nine- no, go, yes ninety and now go to **ninety-five**. *I believe* **it will get smaller here.**

Transcript 8.3: Axel and Dave on Task 3B (invisible pyramid)

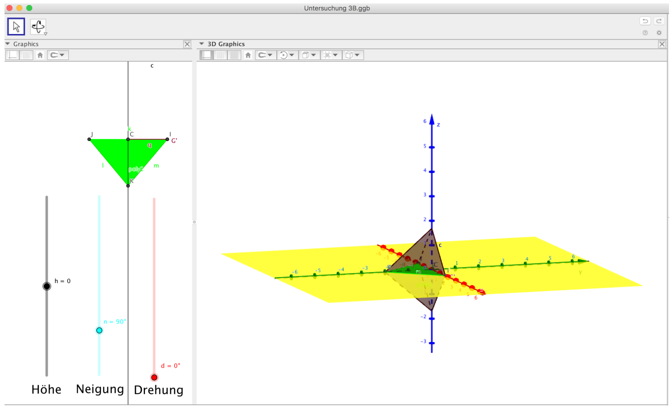


Figure 8.10: Position of the pyramid and the cross-section at position (h=0, n=90°, d=0°)

<u>Reconstruction of the argumentation</u>

Here, I describe the structure of the argumentation stream where the *Sp-manipulations* (SpM_{51} and SpM_W) I discuss later, take place. The argumentation stream starts with datum D_{45} that the cross-section gets smaller and bigger as Dave drags the spin-slider (d-slider) over 0° in the case (h=0, n=90°, d) (see Figures 8.12a and b). Then, follows Axel's observation (D_{49}) that the cross-section "jumps big again" at d=90° (see Figure 8.13a). From that Axel claims that the cross-section will get smaller again after 90°, specifically at 95° ($Cl_{51.3}$). As soon as Dave moves the d-slider from 90° to 95° the software shows the triangular cross-section getting smaller again ($VD_{51.3}$), confirming his claim ($Cl_{51.3}$). From his claim $Cl_{51.3}$ that the cross-section diminishes at d=95° and its verification by $VD_{51.3}$, Axel draws the conclusion C_W that "The cross-section reaches its full size at 90° and then gets smaller again. At 0°, 90°, 180°, 270°, 360° the triangle [*he refers to the cross-section*] has its "full" -size" (see Figure 8.13 for the original text in German that Axel wrote on the worksheet). *Sp-manipulation* occurs in the argument twice: SpM_{51} joins the datum D_{49} with the claim $Cl_{51.3}$, and SpM_W joins the claim $Cl_{51.3}$ and the visual data $VD_{51.3}$ with the conclusion C_W.

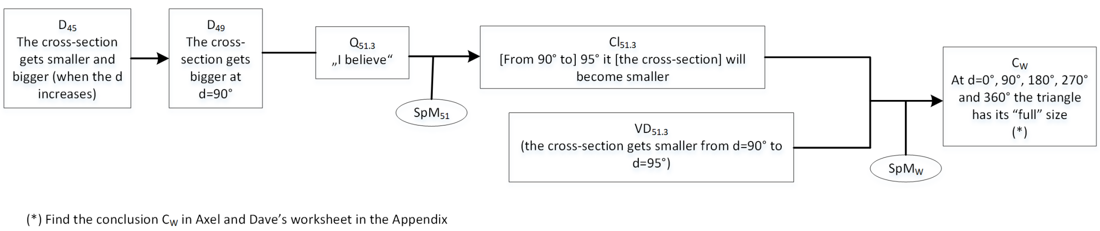


Figure 8.11: Sp-manipulations SpM_{51} and SpM_W in Dave and Axel's argumentation structure (Task3B – invisible pyramid)

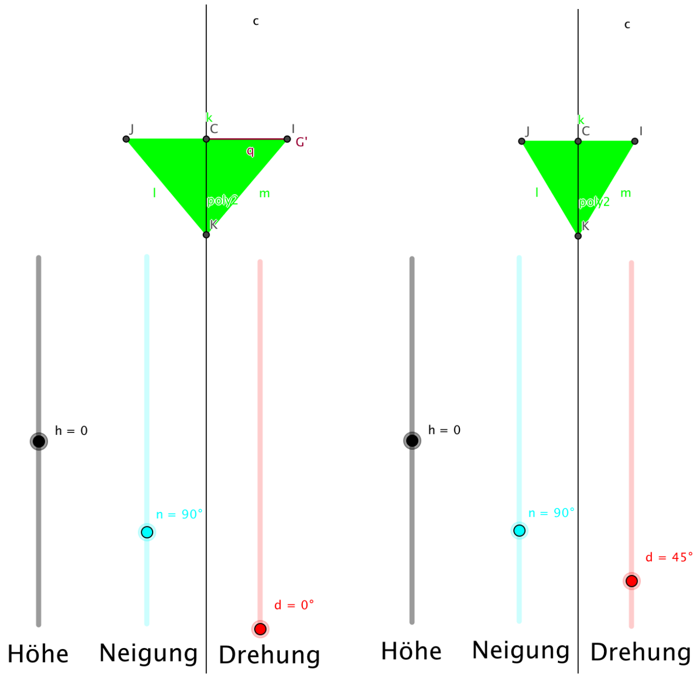
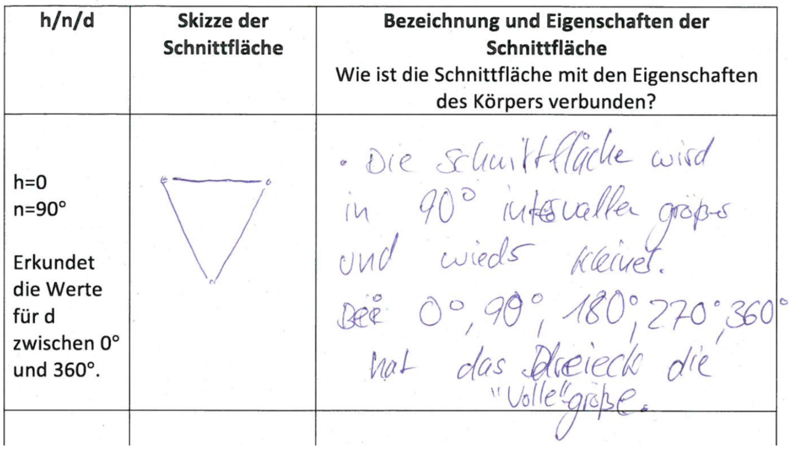


Figure 8.12: a (left) and b (right). The triangular cross-section diminishes from position (h=0, n=90°, d=0°) to position (h=0, n=90°, d=45°)



h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=90° Erkundet die Werte für d zwischen 0° und 360°.		• Die Schnittfläche wird in 90° intervallen größs und wieder kleiner. ~~Bei~~ 0°, 90°, 180°, 270°, 360° hat das Dreieck die "Volle" größe.

Figure 8.13: Conclusion C_W – Notes from Axel and Dave's worksheet on the exploration of the case (h=0, n=90°, d)

Translation for students' note in Figure 8.13:

"*The cross-section reaches its full size at 90° and then gets smaller again. At 0°, 90°, 180°, 270°, 360° the triangle has its "full" -size*"

Interpretation

In the argumentation structure in Figure 8.11 The d-slider has a step of 5°. That means that after 90° the d-value would directly move to 95° and then to 100°, etc. Therefore, Axel says that the cross-section will get smaller at d=95° ($Cl_{51.3}$, utterance

51). As I mentioned previously, as soon as Dave moves the d-slider from 90° to 95° the software shows the triangular cross-section getting smaller again ($VD_{51.3}$), confirming Axel's prediction ($Cl_{51.3}$).

In utterance 49, Axel wonders every how many degrees of spin, the cross-section reaches its maximum size, wondering if this happens every 45°. The fact that by "it" he refers to the cross-section is not explicit in the transcript, but it becomes clear in his written notes, where he explicitly mentions the cross-section (see Figure 8.14 and its translation under the figure). Axel asks Dave to explore again slowly from d=0° until the cross-section reaches its maximum size again. Axel observes that this does not happen at d=45°, rather at 90° (D_{49}, see last sentence in utterance 49).

Then Axel wonders when the cross-section will make the "jump" again (utterance 51). I infer that by "jump" Axel refers to the spin value (d) at which the cross-section "expands" reaching its full size. Axel then says, "When it reaches such an angle again". I believe that by "angle" here, Axel refers to the vertices (or else angles) of the square base of the pyramid. Axel tries to figure out at which degrees of spin, the cross-section has the biggest size. When the cross-section is bigger the segment IJ is also longer, something that the students have also noticed (utterances 45-46, see also Figures 8.12a and b). The cross-section is at its fullest size every time the vertices of the pyramid's base touch the plane xOy (more specifically the x-axis, see Figure 8.10). Therefore, I infer that Axel's phrase "When it reaches such an angle again" is an indicator of *Sp-manipulation* and that Axel uses his *Sp-manipulation* imagining the pyramid spinning in space (SpM_{51}). Then, based on D_{49} (the cross-section reaches its full size again at d=90°) he predicts that after 90° the cross-section will start to diminish again. The d-slider has a step of 5°. That means that after 90° the d-value would directly move to 95° and then to 100°, etc. Therefore, Axel says that the cross-section will get smaller at d=95° ($Cl_{51.3}$, utterance 51). This way, his *Sp-manipulation* SpM_{51} joins the datum D_{49} (the cross-section reaches its full size again at d=90°) with his claim $Cl_{51.3}$ (the cross-section will get smaller at d=95°). His prediction ($Cl_{51.3}$) is verified by the software ($VD_{51.3}$).

Directly after he sees that his claim is correct (IJ and the cross-section indeed gets smaller as soon as d goes from 90° to 95°, see Figures 8.13a and b), he writes down in the worksheet his conclusion C_W, that the triangular cross-section reaches its full size in case (h=0, n=90°, d) for the spin values 0°, 90°, 180°, 270°, 360° (see Figure 8.13). He draws this conclusion without checking it by using the software. Therefore, I argue that Axel draws his conclusion using his *Sp-manipulation,* imagining the way that the solid spins in space in case (h=0, n=90°, d). I consider as an indicator of SpM_W still the phrase "When it reaches such an angle again" in utterance 51.

Although Axel refers to figural units of the pyramid (he refers to the vertices of its base), he does not use any properties – at least not explicitly – to relate the pyramid and its figural units with the cross-sections. The difference between the use of *Sp-manipulation* and *NI-visualization* lies in the use of the properties of the geometric object and its lower dimension sub-parts. Therefore, from Axel's phrase "When it reaches such an angle again", I can only infer the use of his *Sp-manipulation* for the creation of his claim $Cl_{51.3}$ and for the inference of his conclusion C_W.

In this episode, *Sp-manipulation* has assisted Axel to imagine the way the pyramid spins in space and draw a conclusion about how this spin influences the size of the triangular cross-sections that are created. In the next example, I describe a somewhat different operation of *Sp-manipulation*, moving from a cross-section of the invisible solid to the determination of its orientation in space.

$SpM_{21.2}$: Determining the orientation of the cube in relation to plane xOy

This example is from the classroom discussion on the task of the invisible cube (Task 3C). Lukas presents the work he did on the task with his classmate Tom. During the presentation, Tom manipulates the sliders in the software, while Lukas presents. Until this moment, Lukas has presented previous positions and also his and Tom's supposition that the solid is a cuboid. In Transcript 8.4 Lukas presents what happens at the position (h=0, n=90°, d=0°). Here, *Sp-manipulation* $SpM_{21.2}$ supports the formulation of two claims by Lukas.

Episode description

Tom places the sliders in the position (h=0, n=90°, d=0°) and Lukas says that the created cross-section is a square ($D_{19.2}$), that the solid "is askew" ($D_{19.3a}$), that "the y-axis has become the height of the cuboid" ($D_{19.3b}$) and that the cross-section in this position is the "cross-cut of the cuboid". Then he points at the projection on the wall and he gestures a triangular sub-part of the solid under the plane xOy saying that he and Tom presume ($Q_{21.2}$) that "now this goes down into the negative area" ($Cl_{21.1}$). Then he gestures another triangle over the plane xOy and says "here this goes into the positive area, over the x-axis" ($Cl_{21.2}$). From there, he says that this means that the cross-section is a cross-cut of the cuboid ($C_{21.3\text{-}4}$).

Utterance	Codes	Original German transcript	English translation
19	VD_{19} **$D_{19.2}$** $C_{19.3a}/D$ *$C_{19.3b}/D$* **$C_{19.4}/D$**	Lukas: Einmal, ah ne, hast du, da kommt noch was, ne? Ach so, genau h null und n neunzig (..) und d null **ist auch ein Quadrat** und das ist jetzt so, haben wir ja gesagt, das ist jetzt, das liegt jetzt einfach nur schief, auf und zwar (unverständlich) *das ist die y-Achse zur Höhe geworden ist, des Quaders*, und **das ist der Querschnitt** und zwar (..) ja, egal. Und zwar geht es hier, der Querschnitt halt des Quaders, geht dann hier in den negativen Bereich im Grunde #01:25:09-4#	Lukas: Once, ah no, there is something more, right? Oh yes, exactly h zero and n ninety (..) and d zero **is also a square** and that is now, so, we said, it is now, it's just askew, and *the y-axis has become the height of the cuboid*, and **that is the cross-cut** and (..) yes, it doesn't matter. And here it goes, **the cross-cut of the cuboid**, then basically goes into the negative area.
20		P : mhm (bejahend) #01:25:09-4#	P: mhm
21	**$Cl_{21.1}$** *$Q_{21.1}$* **$Cl_{21.2}$** **$C_{21.3\text{-}4}$**	Lukas: Und zwar geht der **jetzt hier noch in den negativen Bereich runter**, also *vermuten wir* jetzt mal, und **hier geht er in den positiven Bereich**, also auf der x-Achse nach oben, das heißt jetzt einmal in der Mitte bei den beiden Kanten, nur das jetzt so, halt, halt einmal durchgeschnitten worden. Und zwar, oder **das ist jetzt der Querschnitt davon**. #01:25:39-6#	Lukas: And **now this** [*Lukas uses a masculine article*] **goes down into the negative area**, *we presume*, and **here this** [*he uses a masculine pronoun*] **goes into the positive area, over the x-axis**, that means now in the middle at the two edges, only that now, stop, stop cut through. Namely, **that's the cross-cut of it now**.

Transcript 8.4: Lukas and Tom's presentation in classroom discussion on Task 3C (invisible cube)

Reconstruction of the argumentation

Figure 8.14 shows Lukas' argumentation stream for the Transcript 8.4. There are multiple cases of *NI-visualization* and *Sp-manipulation* in this stream. Here, I focus only on one of the *Sp-manipulation* cases, namely $SpM_{21.2}$, which leads to the creation of two claims ($Cl_{21.1}$ and $Cl_{21.2}$). The stream starts with the visual data VD_{19} that emerge at position (h=0, n=90°, d=0°) (see Figure 8.15a[7]). From there follows the datum $D_{19.2}$ that the cross-section at position (h=0, n=90°, d=0°) is a square. From there Lukas draws the conclusion $C_{19.3a}$ that the solid is "askew" and he gestures with his hand that the solid tilts sideways. He then draws his next conclusion ($C_{19.3b}$) that now the height of the solid lies on the y-axis. Another conclusion follows, that the cross-section in position (h=0, n=90°, d=0°) is a cross-cut through the middle of the cube (*Queerschnitt*)

[7] Figure 8.15a shows position (h=0, n=90°, d=0°) as the students see it, with the solid being hidden. In Figure 8.15b, the cube is visible for the purposes of my presentation.

($C_{19.4}$).

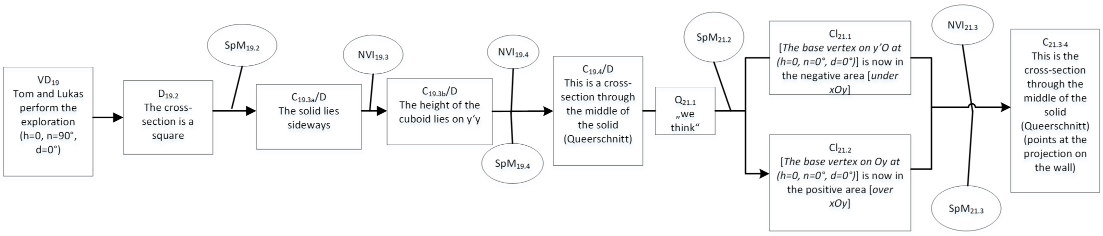


Figure 8.14: SpM21.2 in the argumentation stream from the classroom discussion on Task 3C (invisible cube)

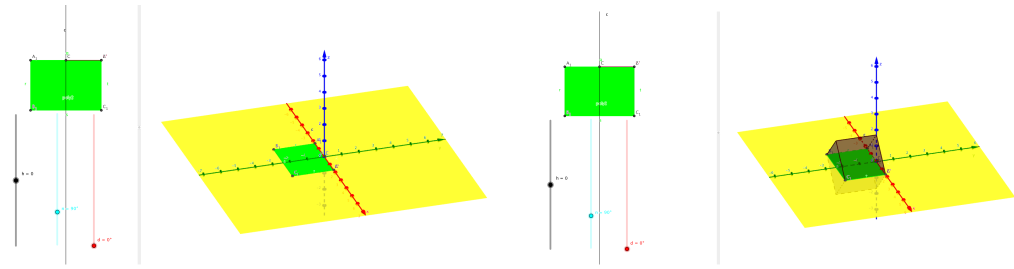


Figure 8.15: a (left) and b (right). The solid at position (h=0, n=90°, d=0°) as the students see it (left) and as it appears when it is visible (right)

Interpretation

Lukas then states two claims. He says that a part of the solid sinks under the plane xOy ($Cl_{21.1}$), while at the same time another part of it lifts from the plane xOy moving over it ($Cl_{21.2}$). To move from the statement that the cross-section in this position is a cross-cut through the middle of the cuboid, to the two claims about the orientation of the solid, Lukas employs his *Sp-manipulation* ($SpM_{21.2}$). In the video-recording of the classroom-discussion Lukas is seen pointing on the projection on the wall, gesturing the two parts of the cuboid that move under and over the plane. He gestures two triangular sub-parts of the solid, one moving over the xOy plane and one moving under it. In inference, the top points of the triangles that move over and under the plane, represent the two base vertices of the cuboid that previously – at position (h=0, n=0°, d=0°) – lie on the y-axis (see points P and Q in Figure 8.16). Then, the other two points of the triangles must be in both cases the edges of the cross-section that are on the x-axis (points R and G, Figures 8.15b and 8.16).

Lukas' gestures and exact illustration of the way he imagines the solid oriented in space at position (h=0, n=90°, d=0°) are indicators of the employment of *Sp-manipulation*. He relies on the way he imagines the solid moves in space from position (h=0, n=0°, d=0°) (see Figure 8.16) to position (h=0, n=90°, d=0°) (see Figure 8.15b), in order to connect the cross-section he observes, with the orientation of the solid in a specific position. In this example, *Sp-manipulation* $SpM_{21.2}$ takes place in the argumentation structure, functioning as a warrant would, and operates moving the argumentation from a statement about the cross-section of the solid ($C_{19.4}$: the cross-section is a cross-cut through the middle of the cuboid) towards two statements that describe the orientation of the solid ($Cl_{21.1}$ and $Cl_{21.2}$).

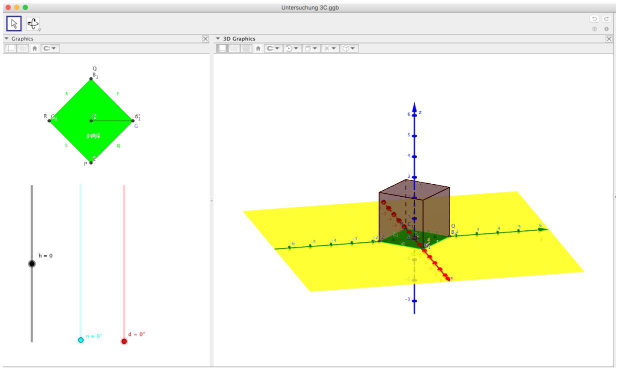


Figure 8.16: The square base RQCP of the solid as cross-section at position (h=0, n=0°, d=0°)

Although one could argue that Lukas also employs his *NI-visualization* in the step from conclusion $C_{19.4}$ to the two claims, I argue that the research data show that he employs his *Sp-manipulation*. That is because Lukas does not state any properties or connections between properties of the solid's figural units and the cross-section. He merely describes the movement of the solid from one position to another, using gestures only for the purposes of his description about the way that the sub-parts of the solid are placed over and under the plane of intersection at position (h=0, n=90°, d=0°).

To conclude, in this example $SpM_{21.2}$ operates in the opposite direction to that of SpM_{51} in the previous example. Here, the transition is from a statement about the cross-section ($C_{19.4}$: this is a cross-cut) to two statements about the orientation of the solid ($Cl_{21.1}$ and $Cl_{21.2}$: a part of the solid under the plane xOy and another part of it lies over the plane). In the previous example, SpM_{51} operates in a transition starting from imagining the way the pyramid spins in space, moving to a conclusion about how this spin influences the size of the triangular cross-sections that are created (C_W).

8.1.4 Summing up on Role 1

To identify the use of *Sp-manipulation* and *NI-visualization* in students' argumentations, I use various *indicators*. As shown in Table 8.2, at the beginning of section 8.1, the *indicators* for *NI-visualization* are related to the use of properties that connect the cross-section to the solid, as well as to performing *dimensional deconstruction*, thus relating the solid with the properties of its lower-dimensional sub-parts. In contrast, *Sp-manipulation* is based on imagining the movement of the invisible solid in space. Therefore, the phenomena that indicate it are the use of gestures and verbal descriptions regarding the movement or the orientation of the solid, as well as seeing the relationship between the movement of the sliders and the resulting movement of the solid.

In this section, I have discussed the use of *Sp-manipulation* and *NI-visualization* in students' argumentation. Particularly, I have examined them when they play the role of supporting the creation of a hypothesis or a claim (Role 1). As seen in all the examples presented here, in this role both *Sp-manipulation* and *NI-visualization* function in the argumentation as warrants, joining data with a hypothesis or a claim. S*p-manipulation* and *NI-visualization* also take place in argumentation in different *modes* (see Table 8.2), either alone or in *synergy*.

8.2 Role 2: Drawing a conclusion

Another role that *Sp-manipulation* and *NI-visualization* play in students' argumentation is that of supporting the creation of a new conclusion. This is Role 2, which I present in this section, through various examples from the students' work. In this role, belong all the cases where *Sp-manipulation* and *NI-visualization* operate in the argumentation as a means to move from observed data, or former conclusions, to a new conclusion. This means that, as in the case of Role 1, here too *Sp-manipulation* and *NI-visualization* function in the argumentation as warrants. The conclusions that are drawn with the help of *Sp-manipulation* and *NI-visualization* may be about the hidden solid, its form, its orientation or any part of it, or about the cross-section, the evolution of the shape of the cross-section during the manipulation of a slider (height, tilt or spin) etc. The difference between Role 1 and Role 2, is that in Role 2 the data leads to a conclusion, instead of a hypothesis or a claim (see distinction in Chapter 5, subsection 5.4.2).

Table 8.3 shows the *modes* (each one alone or in *synergy*) in which *Sp-manipulation* and *NI-visualization* function in argumentation, as well as their indicators. The table has the same structure as Table 8.2 in section 8.1. Please refer to Table 8.2 for more details on the symbolizations used in Table 8.3.

Role 2: Drawing a conclusion	
Mode	**Indicators**
NI-Visualization	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Transitioning from the solid to its lower dimension figural units • Transitioning from lower dimension figural units to the solid • Transitioning between figural units of the same dimension • Transitioning between figural units of different dimensions
Synergy of *NI-visualization* and *Sp-manipulation*	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Describing verbally the solid's orientation • Transitioning from the solid to its lower dimension figural units • Transitioning from lower dimension figural units to the solid • Transitioning between figural units of different dimensions • Transitioning from the movement of a slider to the orientation of the solid
Sp-manipulation	• Using gestures • Using metaphors • Describing verbally the solid's orientation • Transitioning from the two-dimensional cross-sections to the orientation of the solid • Transitioning from the movement of a slider to the orientation of the solid

Table 8.3: Role 2 of *NI-visualization* and *Sp-manipulation* – Drawing a conclusion

8.2.1 *NI-visualization* leading to a conclusion

In this subsection, I present four examples from different episodes, in which students' use of *NI-visualization* helps them to draw a conclusion. Each example is about different kinds of *indicators* and characteristics of *NI-visualization* (see the first cell in the second column of Table 8.3).

$\text{NIV}_{20.2}$: From the sphere to its radius

The instance described here, is from an episode of Axel and Dave's work on the task of the invisible sphere (Task 2)[8]. I focus on the role of *NI-visualization* when it links visual data observed by the students, with a new conclusion. Let me start with a short description of what the students did, before I move on to my interpretation of the situation.

[8] See the global argumentation structure and the worksheet for Episode GR1AD-2 in Appendices E2 and G1, respectively.

Episode description

In subsection 8.1.1, I have already presented Axel and Dave's work up to the point that they reach their claim that the solid is a sphere (Cl_8). In this excerpt of the episode (see Transcript 8.5), Axel and Dave once again perform a height-exploration (-4<h<4, n=0°, d=0°). Dave moves the height-slider over and under zero, while the other two sliders remain at zero degrees. On the screen appear multiple circular cross-sections that get smaller, both when moving from h=0 upwards (for h>0), as well as from h=0 downwards (for h<0) (see Figures 8.17a, b and c). The circular cross-sections diminish until they finally converge to single points both over and under zero, before they disappear completely. After that, Axel says that the sphere "bounces in and out" ($C_{20.1}$). Then he stops the height-slider at h=1 and he says that the radius of the sphere "is one" ($C_{20.2}$).

Utterance	Codes	Original German transcript	English translation
20	$VD_{20.1}$ **$C_{20.1}$/D** $VD_{20.2}$ **$C_{20.2}$** **$SpM_{20.1}$, $NIV_{20.2}$**	Axel: Oh. Ja, **der Kreis** [*Axel misspeaks. He means sphere*] **bouncst rein und raus. Bounce, bounce!** Ah guck mal, man kann wieder den Radius bestimmen [*the radius of the sphere*]. **Hat einen Radius von eins** [*Axel stops the height slider at h=1, at position (h=1, n=0° , d=0°)*].#01:02:01-1#	Axel: Oh. Yes, **the circle** [*Axel misspeaks. He means sphere*] **bounces in and out. Bounce, bounce!** Oh, look, you can determine the radius [*the radius of the sphere*]. **Its radius is one** [*Axel stops the height slider at h=1, at position (h=1, n=0° , d=0°)*].

Transcript 8.5: Axel's statement after the height-exploration (-4<h<4, n=0°, d=0°) during pair-work with Dave on Task 2 (invisible sphere)

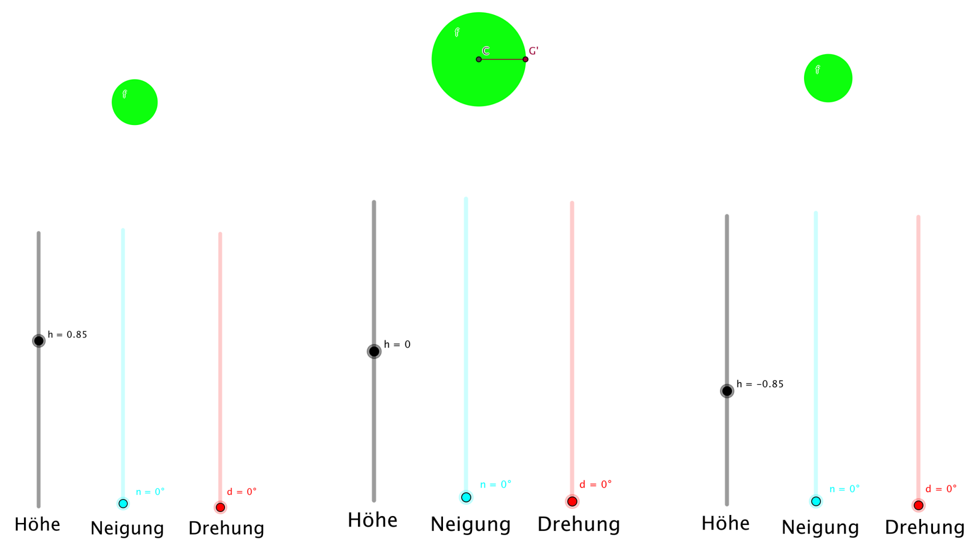


Figure 8.17: a (left), b (middle) and c (right). The circular cross-sections at positions (h=0,85, n=0°, d=0°), (h=0, n=0°, d=0°) and (h=-0,85, n=0°, d=0°), respectively

Reconstruction of the argumentation

Figure 8.18 shows Axel and Dave's argumentation stream for the excerpt in Transcript 8.5. The structure starts with the visual data $VD_{20.1}$ that appear on the screen during the height-exploration. More precisely, what can be seen on the computer screen are circular cross-sections that diminish until they converge to single points, both when dragging the height over zero, as well as when dragging it under zero. From that, follows Axel's conclusion ($C_{20.1}$) that the sphere "bounces in and out" of the plane xOy when they drag the height-slider over and under zero. Then the visual data $VD_{20.2}$ that appear on the computer screen come into the stream. What can be seen on the screen are the circular cross-sections becoming single points at h=-1 and at h=1 (see Figures 8.19a and b). This happens while Axel says "Oh, look, you can determine the radius" (Transcript 8.5). From the conclusion $C_{20.1}$ (now used as a datum $D_{20.1}$) and the new visual data $VD_{20.2}$, follows the final conclusion $C_{20.2}$ of the stream, that "its radius is one", that is the radius of the sphere is one.

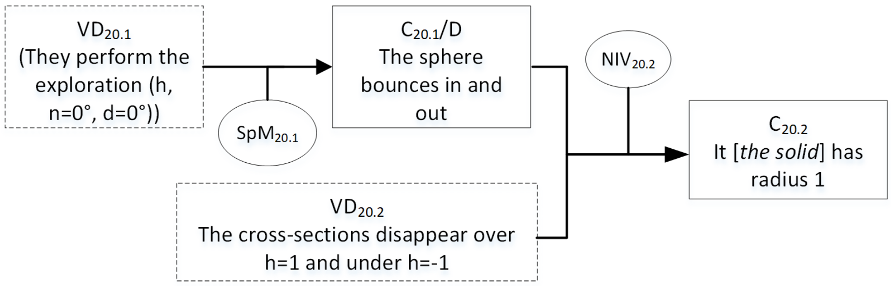


Figure 8.18: NI-visualization $NIV_{20.2}$ and Sp-manipulation $SpM_{20.1}$ in Dave and Axel's argument

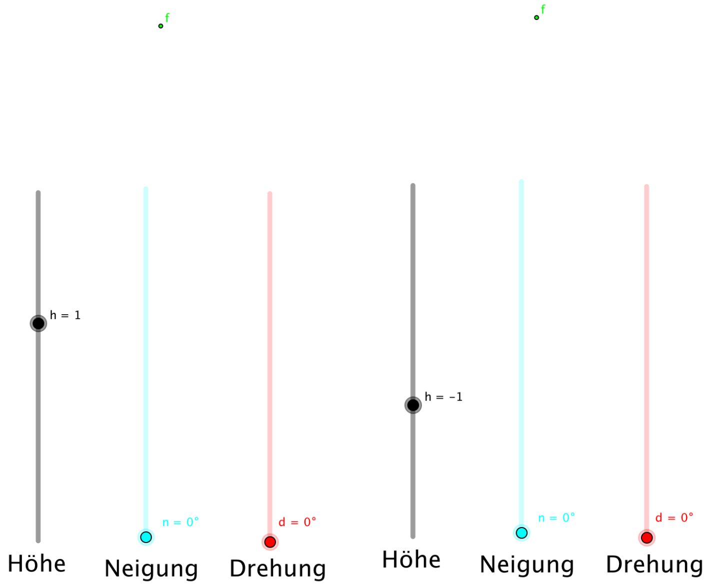


Figure 8.19: a (left) and b (right). The single-point cross-sections at positions (h=1, n=0°, d=0°) and (h=-1, n=0°, d=0°)

Interpretation

In the argumentation stream of Figure 8.18, there is a case of *Sp-manipulation* ($SpM_{20.1}$) and a case of *NI-visualization* ($NIV_{20.2}$). Here, I shortly comment on $SpM_{20.1}$, and I mainly focus on $NIV_{20.2}$, which leads from multiple data ($D_{20.1}$ and $VD_{20.2}$) to the final conclusion ($C_{20.2}$) of the stream.

From his observations during the height-exploration, Axel forms a conclusion ($C_{20.1}$, in Figure 8.18). He says that during the height-exploration, the sphere "bounces in and out" of the plane xOy (see Transcript 8.5). This metaphor that Axel uses here in order to describe the movement of the sphere in space during the dragging of the height-slider, is an indicator of the use of *Sp-manipulation* ($SpM_{20.1}$, in Figure 8.18). I elaborate on $SpM_{20.1}$ further in subsection 8.2.3, where I discuss the role of *Sp-manipulation* in drawing a conclusion.

Before Axel and Dave started this height-exploration, they had already stated their claim that the solid is a sphere (see Cl_8 in Transcript 8.1). This claim remained in their argumentation from that point on during their explorations. When Axel says "Oh, look, you can determine the radius" (Transcript 8.5), he seems to observe that the cross-sections disappear when he drags the height-slider over h=1 and under h=-1 ($VD_{20.2}$ in Figure 8.18) and from that he draws the conclusion $C_{20.2}$ that the radius of the sphere is one.

Here, Axel moves from thinking about the solid (sphere), to seeing the cross-sections as its two-dimensional (2D) figural units. Axel determines the radius of the sphere by the values of the height-slider above (h>1) and under (h<-1) for which the cross-sections disappear. This transition from the solid to its height, is a transition from the three-dimensional sphere (or at least its mental image) to a one-dimensional figural unit of it, namely its radius. This process involves both the *dimensional deconstruction* of the sphere into its cross-sections (2D figural units), as well as the use of a property of its radius (1D figural unit), even if done implicitly. This property could for example be, that the radius of a sphere is the distance from the center of the sphere, which coincides with the center of its biggest circular cross-section, to the circumference of the sphere, which here would be the point at which the cross-section converges to a single point (this being for h=1 and h=-1). Therefore, I argue that the transition from $D_{20.1}$ (the sphere bounces in and out of xOy when the height-slider is dragged over and under zero) and $VD_{20.2}$ (the cross-sections disappear over h=1 and under h=-1) to $C_{20.2}$ (the radius of the sphere is one) is supported by Axel's use of *NI-visualization* ($NIV_{20.2}$).

So, again in this episode, *dimensional deconstruction* and the relations of properties between the solid and its figural units of lower dimension, are indicators of the use of *NI-visualization* by Axel.

$NIV_{W7\text{-}8}$: Perceiving a parabolic cross-section as a figural unit of the cone

The example described here is from Tom and Lukas' work on the task of the invisible cone (Task 3A)[9]. I present a single argument from their argumentation and the role that *NI-visualization* plays in it, by drawing two new conclusions from an existing, previous conclusion. This example is from the students' written notes on their worksheet. I begin with the description of Tom and Lukas' discussion while

[9] See the global argumentation structure and the Worksheet for Episode GR2TL-3A.I in Appendices E7 and G6, respectively.

Tom writes down notes on the worksheet, before I move on to my interpretation.

Episode description

Transcript 8.6 shows the discussion between the Tom and Lukas, during the exploration of the position (h=0, n=80°, d=30°) and while Tom keeps notes of the situation in the worksheet. In Figure 8.20 you can see the orientation of the cone in position (h=0, n=80°, d=30°), as it shows when the solid is visible.

Before the beginning of the discussion in utterance 215, Tom and Lukas have set the sliders in position (h=0, n=80°, d=30°). The discussion starts with Lukas saying that in this position he sees a parabola (C_{215}). Then, Tom asks if he should note this down in their worksheet and he starts by writing down the position under exploration with help from Lukas (utterances 216-222). Then Lukas says that he does not understand how the cone is oriented in position (h=0, n=80°, d=30°) (utterances 223-225). Directly afterwards though, Lukas says that the cone "lies crooked", gesturing the orientation of the solid with his hand (see Figure 8.21).

Utterance	Codes	Original German transcript	English translation
215	$\mathbf{C_{215}}$	Lukas: Ok. Machen wir das. Tom. **Eine Parabel**. Ja [*see Figure 8.20*] #00:26:37-5#	Lukas: Ok. We'll do that. Tom. A parabola. Yes [*see Figure 8.20*]
216		Tom: Soll ich es mal aufschreiben? Was sind die Werte? #00:26:41-7#	Tom: Should I write it down? What are the values?
217	$VD_{217\text{-}223}$	Lukas: Null, achzig, dreizig. Obwohl dreizig ist auch (unverständlich) #00:26:46-1#	Lukas: Zero, eighty, thirty. Although thirty is also
218		Tom: h null, das nächste, n #00:26:48-9#	Tom: h zero, the next, n
219		Lukas: achzig #00:26:50-0#	Lukas: Eighty
220		Tom: achzig Grad. Und was ist das letzte? d #00:26:54-1#	Tom: Eighty degrees
221		Lukas: 30. #00:26:55-1#	Lukas: Thirty
222		Tom: 30 Grad. #00:26:56-1#	Tom: Thirty degrees
223		Lukas: Was ich gerade nicht verstehe, #00:26:57-3#	Lukas: What I do not understand-
224		Tom: Was denn? #00:26:58-2#	Tom: What then?
225	$\mathbf{C_{225}}$	Lukas: Wie liegt denn jetzt der Kegel? Ah, **der liegt fast schief**. **Der liegt so ungefähr** [*see* ***Lukas' gesture*** *in Figure 8.21*]. #00:27:04-0#	Lukas: How does the cone lie now? Ah, **it lies almost crooked**. **It lies approximately like this** [*see* ***Lukas' gesture*** *in Figure 8.21*].

Transcript 8.6: Tom and Lukas on Task 3A (invisible cone)

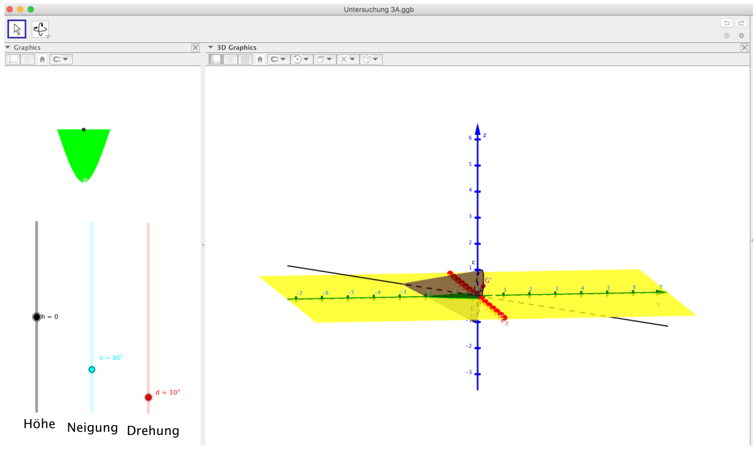
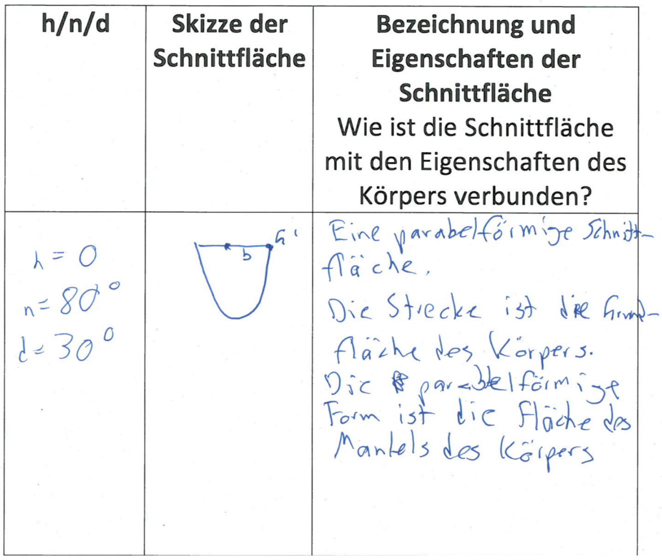


Figure 8.20: The orientation of the cone and the parabolic cross-section in position (h=0, n=80°, d=30°)

Figure 8.21: Lukas' gesture for the orientation of the cone in position (h=0, n=80°, d=30°)

After the two students have completed their discussion, Tom makes some notes on the Exploration Matrix on the worksheet that he does not state verbally (see Figure 8.22 and Table 8.4 for the translation of Tom's notes in English). Tom writes that at position (h=0, n=80°, d=30°) there is "a parabolic cross-section" (C_{W6}), that "the segment is the base of the solid" (C_{W7}) and that "the parabolic shape is the side surface of the solid" (C_{W8}).



h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h = 0 n = 80° d = 30°		Eine parabelförmige Schnitt-fläche. Die Strecke ist die Grund-fläche des Körpers. Die parabelförmige Form ist die Fläche des Mantels des Körpers

Figure 8.22: Tom's notes on the worksheet for the exploration of the position (h=0, n=80°, d=30°)

Codes in the argument (see Figure 8.23)	German original	English translation
C_{W6}	Eine parabelförmige Schnittfläche.	A parabolic cross-section.
C_{W7}	Die Strecke ist die Grundfläche des Körpers.	The segment is the base of the solid.
C_{W8}	Die parabelförmige Form ist die Fläche des Mantels des Körpers.	The parabolic shape is the side surface of the solid.

Table 8.4: Translation of Tom's notes in Figure 8.22

Reconstruction of the argumentation

Figure 8.23 shows the reconstructed argument from Tom's notes on the worksheet (see Figure 8.22 and Table 8.4). For the reconstruction of the argument I used Transcript 8.6 and the information I extracted from the students' notes from their worksheet (see codes in Table 8.4).

The argument starts with the visual data that emerge on the screen at position (h=0, n=80°, d=30°) (see the cross-section on the left window of Figure 8.20). From there on, follows Tom's conclusion C_{W6} that there is "a parabolic cross-section". From this, come three more conclusions. Two of them are expressed by Tom in writing: "The segment is the base of the solid" (C_{W7}) and "The parabolic shape is the side surface of the solid" (C_{W8}). The third conclusion (C_{225}) is drawn by Lukas, who says that the cone "lies almost crooked" (see Transcript 8.6 and Lukas' gesture of the orientation of the cone in Figure 8.21).

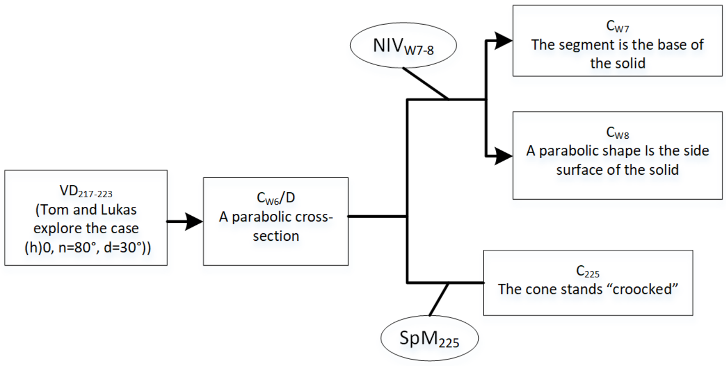


Figure 8.23: Tom and Lukas' argumentation stream about the cross-section in position (h=0, n=80°, d=30°)

Interpretation

In this argument a case of *NI-visualization* ($NIV_{W7\text{-}8}$) and a case of *Sp-manipulation* (SpM_{225}) appear in different places. Here, I describe SpM_{225} briefly and then I mainly focus on $NIV_{W7\text{-}8}$.

Lukas explores the position (h=0, n=80°, d=30°) and in utterance 215 (see Transcript 8.6), he characterizes the cross-section as "parabolic". Tom makes a note of this in the worksheet (C_{W6}). From there, Lukas draws the conclusion that in this position the cone "lies crooked" (225). He uses a gesture to describe the orientation of the cone (see Figure 8.21). Lukas' gesture "shows" the solid tilted with its axis (the line that goes through the top of the cone and the center of its base) almost parallel to the desk. The desk is probably used by Lukas as reference to the xOy plane. Lukas' gesture and also his use of a metaphor (the word "crooked") in order to describe the orientation of the cone, are indicators of his ability to manipulate the mental image of the cone in his mind. Therefore, I argue that at this point Lukas employs *Sp-manipulation* in order to draw conclusion C_{225}.

Tom draws two more conclusions (C_{W7} and C_{W8}) from the previous conclusion C_{EM6} (Figure 8.23). More specifically, Tom says that "The segment is the base of the solid" (C_{W7}) and "The parabolic shape is the side surface of the solid" (C_{W8}). These expressions are not quite correct mathematically, but I believe that Tom had the right idea in mind, which he simply did not formulate in a precise manner. I argue that when he says, "The segment is the base of the solid", he means that the straight segment of the cross-section (see Figure 8.20 and Tom's drawing on Figure 8.22) is part of the base of the solid. This segment is actually the part of the base that coincides with the plane xOy in this position (see the right window of Figure 8.20, which shows the plane xOy coinciding with the base of the cone). Respectively, when he says, "The parabolic shape is the side surface of the solid", I believe he means that the parabolic limit-line of the cross-section is part of the side surface of the solid. This parabolic part is the intersection of the side surface of the cone with the plane xOy (see right window in Figure 8.20).

Therefore, I argue that although the formulation of his statements is not entirely correct, mathematically speaking, he nonetheless still manages to connect the properties of the cross-section with those of the two-dimensional figural units of the cone. I believe that Tom sees the cross-section not merely as a two-dimensional object, but rather as a two-dimensional figural unit of the cone. This means that while remaining in the two dimensions, he manages to relate:

- the shape of the cross-section with the solid: the parabolic cross-section is perceived as a two-dimensional figural unit of the cone.
- the properties of the cross-section with the properties of the solid: the straight segment of the cross-section is created by the base of the solid coinciding with the plane of intersection xOy, and the parabolic line is created by the curved side surface of the cone when intersected by xOy.

These processes are characteristic of *dimensional deconstruction* and act as indicators of the use of *NI-visualization* ($NIV_{W7\text{-}8}$) in order to draw new conclusions (C_{W7} and C_{W8}). This has been an example of both a transition from the cross-section to the solid, as well as of "transitions between figural units of the same dimension" (see Table 8.3), because the transition also happens from the cross-section to the surfaces of the solid, which are all two-dimensional subparts of the cone.

NIV_{84}: From the cross-sections to the height of the cube

This example is taken from Axel and Dave's discussion while working on the task of the invisible cube (Task 3C)[10]. Here, I focus on a single argument and on the role of *NI-visualization* when it supports drawing a new conclusion.

Episode description

Transcript 8.7 shows the discussion of Axel and Dave in the part of the episode I present here. Before I present what happens in it, I would like to give some background of the work of Axel and Dave so far. To this point, the two students have performed height-explorations and spin-explorations and after negotiating multiple hypotheses and claims as to what the form of the solid is (the solid is a pyramid, or a cube, or a cuboid), they conclude with their final claim, which is that the solid is most probably a cuboid.

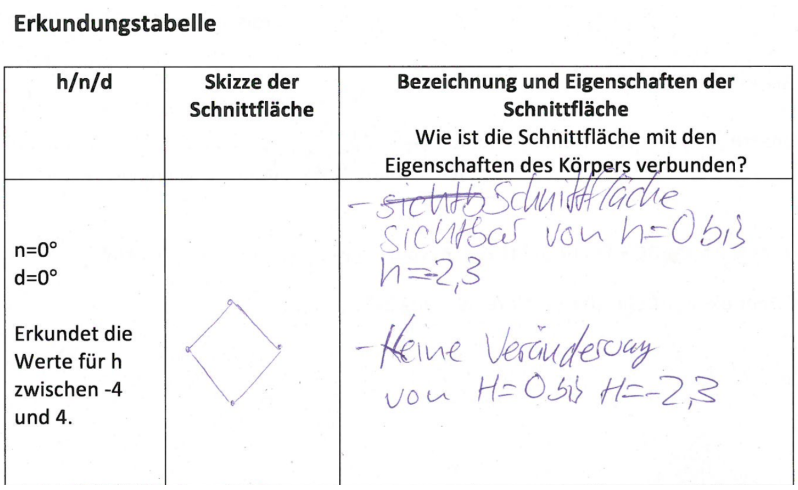
In this part of the episode, the students perform a height-exploration (case (h, n=0°, d=0°)). The discussion in Transcript 8.7 starts with Dave and Axel dragging the height-slider over and under zero. Dave says that the cross-section is visible only between h=-1 and h=-2,35 (utterance 80). After a question by Axel (utterance 81), Dave then corrects himself, saying that the cross-section is visible only between h=0 and h=-2,3 (utterances 82 and 84). He then says that the solid is "so long" (utterance 84). Axel agrees, saying the cross-section is visible "there", referring to the interval from h=0 till h=-2,3.

[10] See the global argumentation structure and the Worksheet for Episode GR1AD-2 in Appendices E2 and G1, respectively.

Utterance	Codes	Original German transcript	English translation
80	$VD_{80\text{-}84}$	Dave: Nichts, nichts. Guck mal, es ist nur zwischen eins (..) und zwei Komma fünfunddreisig. Zwischen minus eins und minus zwei Komma fünfunddreisig sieht man eins. Sieht man eine Schnittfläche. #00:08:43-0#	Dave: Nothing, nothing. Look, it is only between one and (...) two point thirty-five. Between minus one and minus two point thirty-five you see one. You see a cross-section.
81		Axel: Minus eins? #00:08:45-6#	Axel: Minus one?
82		Dave: Bei minus eins bis minus- ne. Von ne natürlich von null #00:08:48-1#	Dave: At minus one until minus- no. From, no, of course from zero.
83		Axel: Null. Ja. #00:08:49-5#	Axel: Zero. Yes.
84	$\mathbf{C_{84}}$	Dave: Ok, null bis minus zwei Komma drei. **So lang ist das** [*the solid*]. #00:08:54-0#	Dave: Ok, zero until two point three. **So long is it** [*the solid*].
85		Axel: Ja. Da [*between h=-2,3 and h=0*] ist Schnittfläche sichtbar. #00:09:00-5#	Axel: Yes. There [*between h=-2,3 and h=0*] is the cross-section visible.

Transcript 8.7 :Axel and Dave on Task 3C (invisible cube)

Next, the students keep notes of what they observed on their worksheet (see Figure 8.24 and the translation in English in Table 8.5).


Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		- ~~sichtb~~ Schnittfläche sichtbar von h=0 bis h=2,3 - Keine Veränderung von H=0 bis H=-2,3

Figure 8.24: Dave and Axel's notes on the worksheet from the height-exploration of the case (h, n=0°, d=0°)

Reconstruction of the argumentation

The reconstruction of Axel and Dave's argument is based both on their discussion in Transcript 8.7, as well as on the notes from their worksheet (see Figure 8.24). Table 8.5 shows the notes of the students both in the German original and translated into English. The codes in Transcript 8.7 and in Table 8.5 are used to reconstruct the argument in Figure 8.25.

Codes in argument (see Figure 8.25)	German original in the Worksheet	English translation
C_{W1}	Schnittfläche sichtbar von h=0 bis h=-2,3.	Cross-section visible from h=0 until h=-2,3.
C_{W2}	Keine Veränderung von h=0 bis h=-2,3.	No change from h=0 until h=-2,3.

Table 8.5: Translation of the students' notes in their worksheet (see Figure 8.24) and coding of the notes for the reconstruction of the argument (see Figure 8.25)

The argument starts with the visual data $VD_{80\text{-}84}$ of the cross-sections that appear on the screen while Axel and Dave drag the height 22 over and under zero. Cross-sections appear on the screen only between the height-values h=0 and h=-2,3. From that, follow two conclusions that the students note on their worksheet, namely that the "cross-section [*is*] visible from h=0 until h=-2,3" (C_{W1}) and that there is "no change from h=0 until h=-2,3" (C_{W2}). From those two conclusions, follows the final conclusion of the argument, that "it is so long".

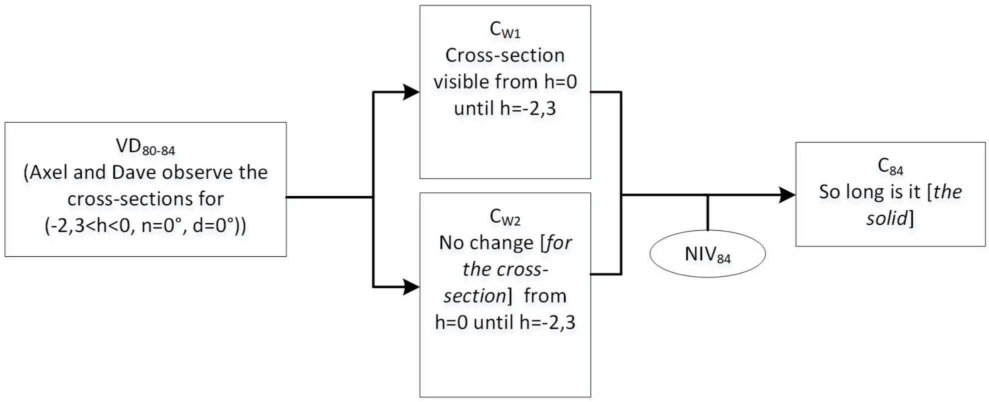


Figure 8.25: Dave and Axel's argument during the height-exploration (h, n=0°, d=0°)

Interpretation

There is one case of use of *NI-visualization* in this argument (NIV_{84}), when the students draw their final conclusion based on two previously existing conclusions, which is what is discussed below.

During the height-exploration, Axel and Dave observe that there is a cross-section only for -2,3<h<0. They write this observation down in their worksheet, as a conclusion (code C_{W1}). Under -2,3 and over zero there is no cross-section to be seen. They also draw and note in their worksheet another conclusion: that the cross-sections do not change at all between h=-2,3 and h=0 (code C_{W2}). From these two conclusions, they then draw the next conclusion: "it" is "so long" (C_{84}).

I argue that by "it" they refer here to the solid, drawing a conclusion about the length of its height. Thus, the students make a transition from the cross-sections and their properties – only visible between zero and -2,3, and always the same – to a property of the solid – the height of the cuboid. The connection of the properties between two objects of different dimensions: the two-dimensional cross-sections that appear in the interval from h=0 to h=-2,3, and the height as a one-dimensional (1D) figural unit of the cube, is an indicator of the use of *NI-visualization* in this argument. This *NI-visualization* (NIV_{84}) helps Dave and Axel draw their final conclusion (C_{84}) about how high the cuboid is, relating properties of the cross-sections and the height of a cuboid.

NIV_{295} and NIV_{297}: From properties of the cross-sections to properties of the solid

This example is taken from the same episode as the previous one (NIV_{84}). Dave and Axel work on the invisible cube task (Task 3C). This time though, the students have finished exploring the cases given in the Exploration Matrix in the worksheet, and move on to the second part of the task, where they are asked whether they can identify the form of the invisible solid, and provide a justification for their answer (see Question b in Figure 8.26).

Episode description

Transcript 8.8 shows a part from Dave and Axel's discussion while working on Question b in the worksheet of the task of the invisible cube. In this example, the discussion starts with Axel reading Question b out loud (utterance 261). Dave tells him to write down that the solid is a "cuboid". He says that this is the only possible form the solid can have, because they have already had a sphere in another task. He also says that the solid cannot be a cylinder (utterance 262). Axel adds that the solid cannot be a pyramid (utterance 263).

Then the two students discuss whether they should also write that the solid could be "even a cube" (utterances 280-282). Axel decides to write that the solid could "possibly [*be*] a cube" (utterance 283). Dave says to Axel to also write down the forms that the solid cannot be (utterance 288).

Then the students decide to write also that the solid has "no round element" and Dave says that this is the reason why "so many fall away" (utterance 295). After that Dave says that only the possibilities of the solid being either a pyramid or a prism, namely a cuboid, remain. Then he immediately adds that the solid cannot be a pyramid because "it does not converge to a point" (utterance 297). In utterances 302-305 the two students say that the solid cannot be a pyramid, because when they vary the height-slider in case (h, n=0°, d=0°) the solid "does not converge to a point" (utterance 302).

Utterance	Codes	Original German transcript	English translation
261		Axel: So, jetzt haben wir nur noch zwei [*Axel refers to Question b on the worksheet*]. Hö? Dürfen wir uns das jetzt ausdenken oder wie? Könnt ihr anhand der „Spüren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Ah. [*Axel is reading Question b out loud*] #00:27:51-7#	Axel: So, now we have two [*Axel refers to Question b on the worksheet*]. Heh? Can we think this now or how? Can you identify the invisible solid based on the "clues" you have collected so far? Ah. [*Axel is reading Question b out loud*]
262		Dave: Schreib Quader hin. Das kann jetzt einfach, das ist jetzt die Möglichkeit die wir haben. Wir haben einmal Kugel [*Dave refers to tone of the previous tasks they worked on*], das kann es nicht sein. Zylinder kann es nicht sein.. [*While he talks, Dave also counts the solids he names with his left-hand fingers*] #00:28:00-0#	Dave: Write down cuboid. It can now only, that is the possibility that we have now. We have had a sphere [*Dave refers to tone of the previous tasks they worked on*], it can't be that one. It can't be a cylinder.. [*While he talks, Dave also counts the solids he names with his left-hand fingers*]
263		Axel: Pyramide kann es nicht sein. #00:28:01-4#	Axel: It cannot be a pyramid.
(...)			
280		Dave: Es könnte sogar ein Quadrat sein [*Dave misspeaks, he means cube. Axel corrects him in the next utterance*]. #00:28:51-1#	Dave: It could actually even be a square [*Dave misspeaks, he means cube. Axel corrects him in the next utterance*].
281		Axel: Ein Würfel. (..) Möglicherweise ein Würfel. Soll ich das schreiben? Weil wir können es ja nicht genau ablesen [*Axel refers to the length to of edges of the base of the cube*]. #00:29:03-2#	Axel: A cube. Possibly a cube. Should I write this? Because we cannot read it in detail [*Axel refers to the length to of edges of the base of the cube*].
282		Dave: Ich würde schreiben es ist auf jeden Fall ein Würfel oder ein Quader. #00:29:06-4#	Dave: I would write that it is certainly either a cube or a cuboid.
283		Axel: Ja. Möglicherweise ein Würfel. #00:29:08-8#	Axel: Yes. Possibly a cube.
(...)			
288		Dave: Aber schreib doch, es kann ja die anderen Sachen nicht sein.	Dave: But write, it can't be the other things.
(...)			
294		Axel: Ne, ne, ne. Sollen wir noch schreiben "keine Runden Elemente" ? Sowas? #00:29:46-9#	Axel: No, no, no. Should we also write "no round elements" ? Something like that?

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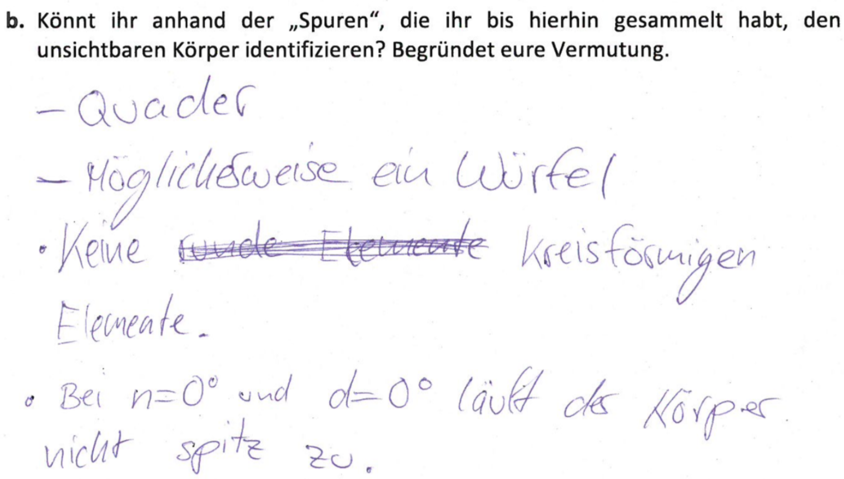
295	C_{295}	Dave: Ja, ja. Keine runden Elemente, deswegen fallen schonmal ganz viele [*other types of solids*] weg. #00:29:50-4#	Dave: Yes, yes. No round elements, therefore so many [*other types of solids*] fall away.
297	C_{297}	Dave: Wenn es keine runden Elemente hat, kann es- bleibt nur noch Kegel- äh bleibt nur noch Pyramide oder Prisma. Reicht doch- Quader ist ja ein Prisma. So gesehen. Und eine Pyramide ist es auch nicht, weil es nicht spitz zuläuft. (...) #00:30:29-6#	Dave: If it does not have any round elements, it can only- only the cone remains- eeh only the pyramid or the prism remain. It is enough- cuboid is a prism. Seen this way. And it's not a pyramid either, because it doesn't converge to a point.
(...)			
302		Axel: Ihm (..) Bei ihm [*Axel refers to the h-slider*] - mach mal die beiden Sachen auf null. Wenn man nur die Höhe verändert läuft es auch nicht spitz zu. #00:31:01-3#	Axel: On it (..) On it [*Axel refers to the h-slider*] – turn these both things to zero [*n=0° and d=0°*]. When you only change the height it [*the solid*] does not converge to a point.
303		Dave: Ne #00:31:02-7#	Dave: No.
304		Axel: Ja, eben. Deswegen kann es schonmal keine #00:31:03-9#	Axel: Yes, exactly. That's why it cannot be a-
305		Dave: Pyramide sein. #00:31:04-9#	Dave: pyramid

Transcript 8.8: Axel and Dave on Task 3C (invisible cube)

Figure 8.26 is a snapshot of Axel's notes on the worksheet. These notes emerged from their discussion with David shown in Transcript 8.8. The translation of these notes can be seen in Table 8.6.


b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

- Quader
- Möglicherweise ein Würfel
- Keine ~~runde Elemente~~ kreisförmigen Elemente.
- Bei n=0° und d=0° läuft der Körper nicht spitz zu.

Figure 8.26: Axel and Dave's written justification in Task 3C (invisible cube)

Reconstruction of the argumentation

The argumentation structure I present here (Figure 8.27) is the result of the combined examination of Dave and Axel's oral argumentation during their discussion (see codes in Transcript 8.8), as well as of their written justification (see Figure 8.26 and the translation in Table 8.6).

Codes (Figure 8.28)	German original in the Worksheet	English translation
C_{W6}	Quader. Möglicherweise ein Würfel	Cuboid. Possibly a cube
D_{W1}	Keine kreisförmigen Elemente.	No circular elements.
D_{W2}	Bei n=0° und d=0° läuft der Körper nicht Spitz zu.	For n=0° und d=0° the solid does not converge to a point.

Table 8.6: Translation of the students' notes from Figure 8.26 and codes of the argument

The argumentation in Figure 8.27 starts with two data (D_{W1} and D_{W2}), each leading to a different conclusion (C_{295} and C_{297}, respectively). Then these conclusions are used in combination as new data to draw their conclusion (C_{W6}). Datum D_{W1} (Table 8.6 and Figure 8.26) says that there are no circular cross-sections. From that follows the conclusion C_{295} that the solid cannot be a sphere, a cone or a cylinder (Transcript 8.8). Datum D_{W2} (Table 8.6 and Figure 8.26) says that the solid does not converge to a single point during the dragging of the height-slider in case (h, n=0°, d=0°), which leads to the conclusion C_{297} (Transcript 8.8) that the solid cannot be a pyramid.

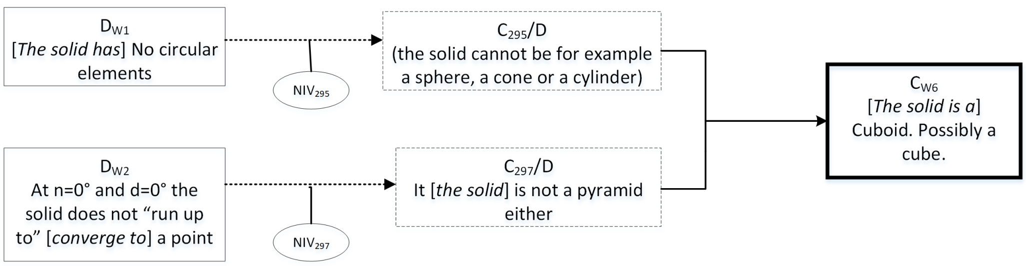


Figure 8.27: Argumentation structure of Axel and Dave's written justification for Task 3C (invisible cube)

Interpretation

There are two cases of use of *NI-visualization* in this argumentation (see NIV_{295} and NIV_{297} in Figure 8.27). Here, both are described. To do so, I need to describe their argumentation in greater detail.

In this example, Axel and Dave reach their conclusion about the form of the solid (C_{W6}) by eliminating other forms of solids. They begin by identifying the types of properties that the cross-sections do *not* have: "no round elements" and "the solid does not converge to a point". These are statements regarding properties that the one-dimensional (1D) figural units and zero-dimensional (0D) figural units of the cross-sections do *not* have. More precisely, a 1D figural unit of the cross-section that is "round", could be a curved border, while a 0D figural unit of the cross-section could be the single point, to which the cross-sections would converge. Then, they transfer these properties to the invisible solid. The lack of "round elements" (see datum D_{W1}), leads to the conclusion (C_{295}) that the solid cannot be a geometric object with curved surfaces (these surfaces would be 2D figural units of the solid), such as a cone, a sphere or a cylinder. The fact that the cross-sections do not shrink to a single point when the height is changed (D_{W2}), leads Axel and Dave to the conclusion that the solid cannot have a pointy top (0D figural unit of the solid), such as a pyramid (C_{297}).

We have here two transitions between geometric objects of different dimensions. The one is from 1D figural units ("round" elements) of the solid to 2D figural units of the solid (curved surfaces of the solid, such as those of a cone, a cylinder, or a sphere). The other is from a 0D figural unit of the cross-section (a single point cross-section) to a 0D figural unit of the solid (the pointy top of a pyramid). Both these transitions require that one moves between geometric objects of different dimensions, sees the lower-dimension objects as constitutive parts of the higher-dimension objects, and correlate their properties.

Axel and Dave connect, for example, the "roundness" of the edges of the cross-section (D_{W1}) with the "roundness" of the surfaces or edges of the solid (C_{295}). This is a process of *NI-visualization,* in which the students perform a *dimensional deconstruction* of multiple solids (cone, cylinder and sphere). Therefore, I argue that NIV_{295} helps Axel and Dave transit from D_{W1} to C_{295}. Similarly, NIV_{297} supports students to draw the conclusion C_{297} from the datum D_{W2}.

Beyond the use of *non-iconic visualization,* Axel and Dave's justification is particularly interesting for another reason. Axel and Dave justify their conclusion (the solid is a cuboid and possibly a cube) indirectly by eliminating other possible (or not so possible) cases (cone, cylinder, sphere and pyramid).

8.2.2 *Synergy* of *NI-visualization* and *Sp-manipulation* leading to a conclusion (NIV_{207} and SpM_{207})

In the previous subsection (8.2.1) I presented an example of use of *NI-visualization* from Tom and Lukas' work on the task of the invisible cone (see $NIV_{W7\text{-}8}$). There, I focused on a single argument from their argumentation (Figure 8.23). Here, I present another example from the same episode.

Episode description

This part of Tom and Lukas' argumentation (see Transcript 8.9) is from a short discussion between the students and the researcher (person P in transcript), which took place during the students' pair-work. In the discussion the students tell the researcher that the solid is a cone. In this part of the discussion, they present the researcher with their thoughts about the form of the cone and its orientation in position (h=0, n=90°, d=0°) (see Figure 8.28).

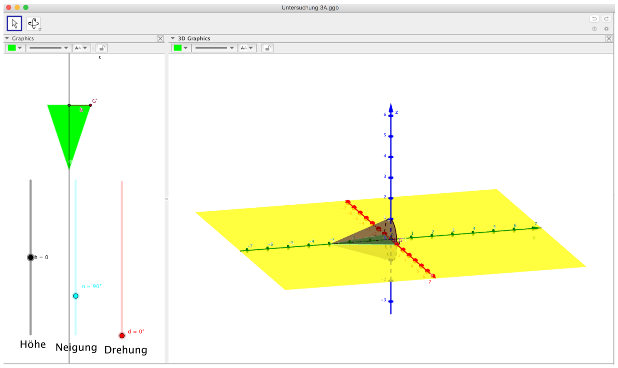


Figure 8.28: The triangular cross-section and the orientation of the cone at position (h=0, n=90°, d=0°)

The episode starts with Lukas manipulating the tilt-slider (n) in the case (h=0, n, d=0°), moving it from n=0° to n=90°. At the same time, he says "if you turn [*n*] to zero and then turn it, you can recognize that a bit" (utterance 204). Leaving the sliders in position (h=0, n=90°, d=0°) he calls the cross-section there a "cross-cut" (utterance 206). Tom then says that the solid in this position is "almost put on its side" and that the cross-section "here is a triangle" (utterance 207). Tom also adds that the cone is "cut from top to bottom" (utterance 209).

Utterance	Codes	Original German transcript	English translation
204	VD$_{204}$	Lukas: Und wenn man auf null [*h=0 and d=0°*] und den dann dreht [*Lukas moves the tilt slider to n=90°*], kann man das auch bisschen mit erkennen. Da erkennst du nicht genau, ob es ein Kegel ist, aber- #00:23:06-3#	Lukas: And if you turn to zero [*h=0 and d=0°*] and then turn it [*Lukas moves the tilt slider to n=90°*] you can recognize that [*the cone*] a bit. You don't exactly know if it's a cone, but-
205		Tom: Genau. Hier ist das einmal der - #00:23:08-6#	Tom: Exactly. Here it is again, the-
206		Lukas: Der Querschnitt. #00:23:09-2#	Lukas: the cross-cut
207	**C$_{207.1}$/D** C$_{207.2}$/D	Tom: **Die Figur** [*Tom refers to the solid*] **quasi auf die Seite gelegt**, und da sieht man das eigentlich auch ganz gut, dass das da hier ein Dreieck ist [*Tom refers to the shape of the cross-cut at (h=0, n=90°, d=0°)*]#00:23:18-6#	Tom: **The figure** [*Tom refers to the solid*] **almost put on its side**, and you can see that this here is a triangle [*Tom refers to the shape of the cross-cut at (h=0, n=90°, d=0°)*]
208		P: mhm (bejahend) #00:23:18-7#	P: mhm [*affirmative*]
209	**C$_{209}$**	Tom: Und dass dann die, ja. **Von oben nach unten ein Kegel aufgeschnitten** sozusagen. #00:23:26-3#	Tom: And that then, yes. **A cone cut from top to bottom**, so to speak.

Transcript 8.9: Tom and Lukas on Task 3A (invisible cone)

Reconstruction of the argumentation

Figure 8.29 shows the reconstructed argument from Tom and Lukas' discussion with the researcher in Transcript 8.9. The argument starts with the visual datum VD_{204}, which emerges from Lukas' dragging of the tilt-slider from n=0° to n=90° in *case* (h=0, n, d=0°) and shows a triangular cross-section (see the cross-section in the link window in Figure 8.28). From this, comes the conclusion that the cross-section "is a triangle" (C_{207}). This conclusion is then used as a datum that leads to the next conclusion, that the solid is "put on its side" ($C_{207.1}$). From that follows then the final conclusion of the argument that in position (h=0, n=90°, d=0°) the cone is "cut from top to bottom" (C_{209}).

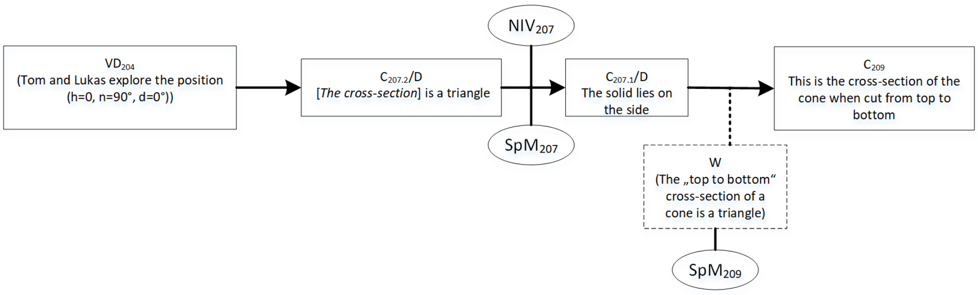


Figure 8.29: *NI-visualization* and *Sp-manipulation* in Tom and Lukas' argument

Interpretation

In this episode, Tom and Lukas explain to the researcher what they observe at position (h=0, n=90°, d=0°). In the argument in Figure 8.29 there takes place a *synergy* of *NI-visualization* and *Sp-manipulation* (NIV_{207} and SpM_{207}) and a *NI-visualization* alone (NIV_{209}). Here, I focus on presenting the *synergy.*

Tom and Lukas say that when the solid is tilted at n=90°, the cross-section is a triangle (conclusion C_{207}), and that the solid in this position lies on its side (conclusion $C_{207.1}$). From this, they conclude that this triangle is the cross-cut ("Queerschnitt", utterance 206) of the solid, which they describe as the cross-section that emerges when "a cone is cut from top to bottom" (conclusion C_{209}).

I argue that the students' transition from $C_{207.2}$ to $C_{207.1}$, is supported by the use of both *Sp-manipulation* (SpM_{207}) and *NI-visualization* (NIV_{207}). The students recognize the triangular cross-section ($C_{207.2}$), as the "cross-cut" of a cone (utterance 206). This indicates that they perceive the triangular cross-section as a two-dimensional (2D) figural unit of the solid, relating the properties of the cross-section to those of the cone. This is a process of *dimensional deconstruction* and a transition from the cross-section as a two-dimensional geometric object to the solid and its properties. Therefore, I argue that this transition is based on the use of *NI-visualization* (NIV_{207}) that is indicated by their performance of *dimensional deconstruction.*

Tom and Lukas also seem to translate the 90°-movement of the tilt-slider (VD_{204}), to a 90°-tilting of the solid (see utterance 204). From that, they conclude that the solid lies on its side ($C_{207.1}$). This metaphor used as a verbal description of the solid's orientation is an indicator of the use of *Sp-manipulation* (SpM_{207}). That means that Tom and Lukas connect the movement of the slider with the consequent movement of the solid, estimating its resulting orientation in relation to plane xOy.

In this example, NIV_{207} supports the students to link the triangular cross-section and its properties with the properties of the cone, while SpM_{207} helps them imagine the movement and the orientation of the solid in space. The two processes act in *synergy*, functioning as a warrant, in order to draw conclusion $C_{207.1}$.

8.2.3 *Sp-manipulation* leading to a conclusion

In this subsection I present an example of the use of *Sp-manipulation* in argumentation to draw a conclusion. In the example, *Sp-manipulation* takes place alone, that is without the simultaneous employment of *NI-visualization.*

I have used parts of the same episode also in previous examples. Axel and Dave work on the task of the invisible sphere (see Task 2)[11]. Figure 8.30 shows the global argumentation structure of the whole episode. I have already presented part of the same excerpt I use here (see Transcript 8.10), in subsection 8.2.1 (see Transcript 8.5), in order to discuss the use of *NI-visualization* in the students' argument (see $NIV_{20.2}$: From the sphere to its radius in subsection 8.2.1). In that example, I discussed the same part of the argumentation that I will be discussing here (see red marked part in Figure 8.30 and Figure 8.31). Then I focused on the use of *NI-visualization* in the argument ($NIV_{20.2}$). This time I focus on the *Sp-manipulation* taking place ($SpM_{20.1}$). Also, in subsection 8.1.1, I have presented the part of Axel and Dave's discussion that preceded the excerpt I use here (Transcript 8.10). There, I discussed two cases of *NI-visualization* used in order to create a hypothesis (see NIV_3) and a claim (see NIV_{11}) (see green marked part in Figure 8.30 and also Figure 8.4).

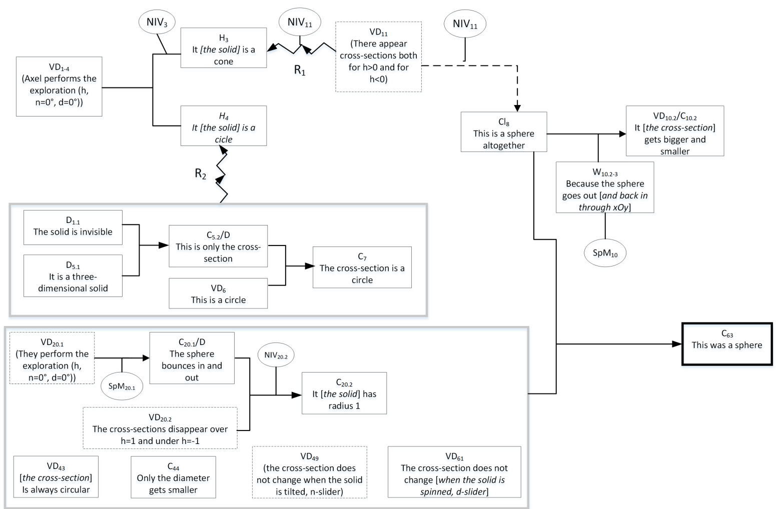


Figure 8.30: *NI-visualization* and *Sp-manipulation* in the global argumentation structure of Dave and Axel's work on the invisible sphere task (Task 2)

Next, I give a description of the students' work in this example, before I move on with the reconstruction of the argumentation and my interpretation of the students' actions.

11 See the global argumentation structure and the Worksheet for Episode GR1AD-2 in Appendices E2 and G1, respectively.

Episode description

The episode starts with the students preparing the *case* (h, n=0°, d=0°). Dave reads the values and Axel moves the sliders accordingly. He sets the tilt-slider (n) and the spin slider (d) at zero degrees. Then Dave says that for the height-slider the task asks to "Explore the values for h between minus four and four". As soon as Axel drags the height slider above and under zero, he says "Yes, the circle bounces in and out. Bounce, bounce! Oh, look, you can determine the radius. Its radius is one".

Utterance	Codes	Original German transcript	English translation
13		Dave: n null Grad, d null Grad. #01:01:08-1#	Dave: n zero, d zero degrees
14		Axel: Und h? #01:01:07-8#	Axel: And h?
15		Dave: Steht hier nicht. (..) #01:01:14-2#	Dave: It doesn't say (..)
16		Axel: Ja. (..) #01:01:23-8#	Axel: Yes (..)
17		Dave: Also, n null Grad. #01:01:29-9#	Dave: So, n zero degrees
18		Axel: Ach so. (..) Ja. #01:01:36-1#	Axel: I see.. Yes
19		Dave: Erkundet die Werte für h zwischen minus vier und vier. #01:01:41-8#	Dave: Explore the values for h between minus four and four.
20	**$C_{20.1}$** $\underline{C_{20.2}}$	Axel: Oh. Ja, der Kreis [*Axel miss-speaks. He means sphere*] bouncst rein und raus. Bounce, bounce! Ah guck mal, man kann wieder den Radius bestimmen [*the radius of the sphere*]. Hat einen Radius von eins [*Axel stops the height slider at h=1, at position (h=1, n=0° , d=0°)*]. #01:02:01-1#	Axel: Oh. Yes, **the circle** [*Axel miss-speaks. He means sphere*] **bounces in and out**. **Bounce, bounce!** Oh, look, you can determine the radius [*the radius of the sphere*]. <u>Its radius is one</u> [*Axel stops the height slider at h=1, at position (h=1, n=0° , d=0°)*].

Transcript 8.10: Axel and Dave on Task 2 (invisible sphere)

Reconstruction of the argumentation

Figure 8.31 shows Axel and Dave's argumentation on which I focus here. In the discussion that has preceded this excerpt (Transcript 8.10), the students have arrived at the claim that the solid is a sphere (see Cl_8, in subsection 8.1.1). In subsection 8.2.1 I presented the role of $NIV_{20.2}$ in the same argument I discuss here. There, I have also shortly commented on $SpM_{20.1}$ shortly. Here, I focus specifically on the role of $SpM_{20.1}$ in drawing a conclusion.

The structure of the argument in Figure 8.31 starts with the visual data $VD_{20.1}$ (visible on the computer screen) that there are circular cross-sections which diminish until they converge to single points, both when dragging the height over zero, as well as when dragging it under zero. From that, follows Axel's conclusion ($C_{20.1}$) that the sphere "bounces in and out" of the plane xOy when they drag the height-slider over and under zero. Although Axel says "the circle bounces in and out", I believe that what he means to say is "sphere", since this is the form of the invisible solid which they claim (Cl_8) earlier. Then the visual data $VD_{20.2}$ that appear on the computer screen come into the stream. What can be seen on the screen are the circular cross-sections becoming single points at h=-1 and at h=1 (see Figures 8.19a and b). This happens while Axel says "Oh, look, you can determine the radius" (Transcript 8.10). From the conclusion $C_{20.1}$ (now used as a datum $D_{20.1}$) and the new visual data $VD_{20.2}$, follows the final conclusion $C_{20.2}$ of the stream, that "its radius is one", implying that the radius of the sphere is one.

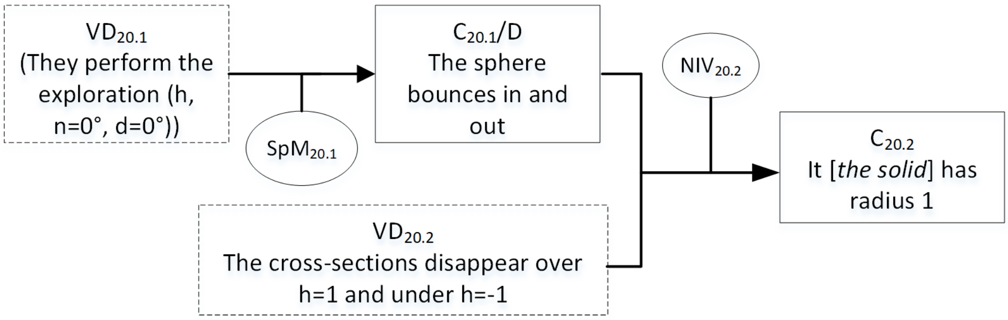


Figure 8.31: The structure of Axel and Dave's argument in which $SpM_{20.1}$ takes place

Interpretation

In the argumentation stream of Figure 8.31, there is a case of *Sp-manipulation* ($SpM_{20.1}$) and a case of *NI-visualization* ($NIV_{20.2}$). Here, I only discuss $SpM_{20.1}$ that leads from the visual data $VD_{20.1}$ to the conclusion $C_{20.1}$.

From the visual data ($VD_{20.1}$) that the circular cross-sections (of the solid with the plane xOy) diminish until they converge to single point in both directions of the h-slider (over and under h=0), Axel forms the conclusion ($C_{20.1}$) that the sphere bounces in and out of the plane xOy. With this statement Axel gives his explanation for the emergence of the cross-sections during the variation of the height-slider, which are circles getting smaller as one moves away from zero in both directions (h<0 and h>0).

Here, Axel talks about the sphere as if he actually sees it, although the solid is invisible. Axel's expression is a metaphor that he uses in order to describe the movement of the sphere in respect to the plane of intersection xOy and explain the cross-sections that occur. This metaphor is an indicator of the use of *Sp-manipulation* ($SpM_{20.1}$) in order draw a conclusion about the movement of the solid in space ($C_{20.1}$). Drawing from those indicators, I argue that Axel has a mental image of a sphere and can actually imagine it moving in space and going up-and-down through a surface (the plane xOy), as he drags the height-slider up-and-down.

$SpM_{20.1}$ plays two roles in this situation. On the one hand, it functions as a warrant, helping the students to draw a conclusion (Role 2) and on the other hand, it serves as a means to explain some visual data (the emerging circular-sections) (Role 3, see also subsection 8.3.3).

Axel does not explicitly express the reasoning that leads him to his conclusion $C_{20.1}$. Therefore, it is not possible to say whether his conclusion is based solely on his use of *Sp-manipulation* about the way the sphere moves in respect to the variation of the height-slider, or if he has also taken into consideration any properties of the cross-section and/or the solid. Hence, I consider only the use of *Sp-manipulation* as the means that leads him from the implicit visual datum ($VD_{20.1}$) to his conclusion ($C_{20.1}$).

8.2.4 Epilogue

In this section, I showed examples from the students' work, in which they employ their *Sp-manipulation* and *NI-visualization* when they move from a datum (or data) to a conclusion. In the examples we have seen three different *modes* in which these two processes appear: *NI-visualization* alone (subsection 8.2.1), both together in *synergy* (8.2.2), or *Sp-manipulation* alone (8.2.3). In all cases, the processes of *Sp-manipulation* and *NI-visualization* function in the argumentation as warrants.

The decision of the *mode* in which they appear, depends always on the indicators that I trace in students' words and actions (for example, gestures, drawings, manipulations of the sliders). As shown in Table 8.3 (at the beginning of Section 8.3), *NI-visualization* is indicated by the performance of *dimensional deconstruction* (see $NIV_{20.2}$ and $NIV_{W7\text{-}8}$ in 8.2.1, and NIV_{207} in 8.2.2), relating properties of geometric objects of different dimensions (see $NIV_{W7\text{-}8}$ in 8.2.1), transitioning from the solid to lower dimension figural units different dimensions (see $NIV_{20.2}$ in 8.2.1) and vice versa (see $NIV_{W7\text{-}8}$ in 8.2.1), and by transitioning between different figural units of the solid (of the same or different dimensions) (see NIV_{84} in 8.2.1). The common characteristic of all those indicators is that they are all processes that are based on the use of properties. They also demand that one has (and can use) the knowledge of geometric objects' structure, as well as their properties.

The indicators for *Sp-manipulation* are also varied, but contrary to those of *NI-visualization* they are not based on properties. Indicators of *Sp-manipulation* include (see also Table 8.3): using metaphors (see $SpM_{20.1}$ in 8.2.3), using gestures (see SpM_{225} in the example for $NIV_{W7\text{-}8}$ in 8.2.1), verbally describing the movement and the orientation of the solid in space (see SpM_{225} in the example for $NIV_{W7\text{-}8}$ in 8.2.1), and relating the movement of the slider to the movement or the orientation of the solid (see $SpM_{20.1}$ in 8.2.3).

8.3 Role 3: Explaining visual data

In some cases, *spatial manipulation* (*Sp-manipulation*) and *non-iconic visualization* (*NI-visualization*) have played a significant role in students' efforts to explain either for themselves, or to others (to the classmate they work with or to the whole classroom), the things or phenomena that occur during the explorations they performed. In students' argumentations, *Sp-manipulation* and *NI-Visualization* are used as visual data (code VD in the argumentation structures). When the students use them to explain the visual data, then I say that they play the role of supporting the process of explaining these visual data (Role 3).

Table 8.7 shows the *modes* (used alone or in *synergy*) in which *Sp-manipulation* and *NI-visualization* function in argumentation, as well as their indicators. The table has the same structure as Table 8.2 in section 8.1. Please refer to Table 8.2 for more details on the terms used in the table below.

Role 3: Explaining visual data	
Mode	**Indicators**
NI-visualization	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Transitioning from the solid to its lower dimension figural units
Synergy of *NI-visualization* and *Sp-manipulation*	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Transitioning from the solid to its lower dimension figural units • Transitioning from lower dimension figural units to the solid • Describing verbally the solid's orientation • Using gestures • Relating the movement of a slider with the movement of the solid
Sp-manipulation	• Using gestures • Describing verbally the solid's orientation • Relating the movement of a slider with the movement of the solid

Table 8.7: Role 3 of *NI-visualization* and *Sp-manipulation* – Explaining visual data

In this section I present some examples from students' work, in which *Sp-manipulation* and *NI-visualization* play the role of supporting the process of explaining visual data (Role 3). As with Roles 1 and 2, in Role 3 too, *Sp-manipulation* and *NI-visualization* function in the arguments like warrants.

8.3.1 *NI-visualization* explaining visual data

In this subsection, I present an example from a whole classroom-discussion. The discussion is about the task of the invisible cone (Task 3A)[12]. Axel and Dave have already presented the task and the explorations they performed, and they have argued that the solid is a cone. After that, two more students, Jacob and Michael, join the conversation. Jacob and Michael describe the way the cone moves in space using hand-gestures. In the part of the discussion presented here, the researcher provides Jacob and Michael with a paper-model of a hollow cone, made out of a rolled-up A4-sheet (see Figures 8.32 and 8.33). Below, I start with the description of the

[12] See the global argumentation structure for Episode CD3A-AD in Appendix E9.

episode, moving then to the presentation of the reconstruction of the argumentation and finally offering my interpretation of what has happened.

Episode description

Transcript 8.11 shows exactly the way in which Jacob explains how the "oval" cross-section (as the students call it), at position (h=-0,85, n=46°, d=0°) (see cross-section at the left window of Figure 8.34) and other partially curved cross-sections are created. He also uses the paper-cone to present his thoughts. Jacob starts by pointing all around the side-surface of the cone (see Figure 8.32) saying, that the surface "is rounded" ($D_{37.2\text{-}3}$, Table 8.8). Then he says that "what lies within this cone, also belongs to this cross-section" ($C_{37.3\text{-}4}$, Table 8.8) because the solid has a mass ($D_{37.4}$, Table 8.8). He then argues that from that he can infer that the points of the cone that touch the plane xOy (he calls it "floor"), will also appear on the cross-section ($C_{37.5\text{-}6}$, Table 8.8). From there he argues that if one tilts the cone a bit forward ($D_{37.7}$) then the plane (he refers to it as "bottom" and he gestures with his hand an imaginary flat surface under the cone) will touch both the side-surface of the cone as well as "something inside the cone" ($C_{37.8\text{-}9}$, in Table 8.8, see also Figure 8.33 where he uses a gesture to point to the inside of the paper-cone). Jacob says that that is how the oval cross-section is created. Finally, he adds that all the "oval rounded" cross-sections emerge ($C_{37.10\text{-}11}$), because "this" (he points at the side surface of the cone) "is all rounded" ($W_{37.10}$, Table 8.8).

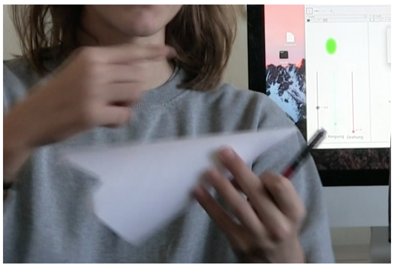


Figure 8.32: Jacob shows the side surface (Mantelfläche) of the cone

Figure 8.33: Jacob shows the tilted position of the cone and the points of the "inside" part of the cone

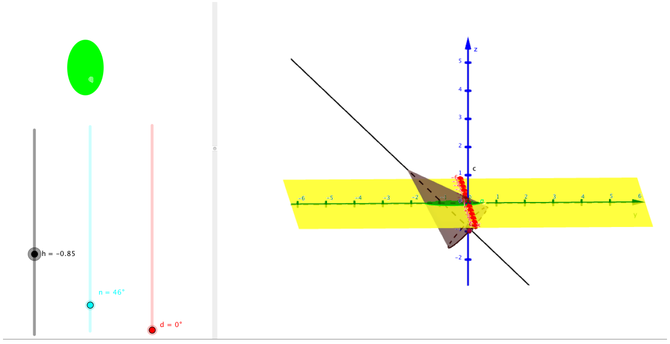


Figure 8.34: Oval cross-section in position (h=-0,85, n=46°, d=0°)

Utterance	Original German transcript	English translation
37	Jacob: Also das ist halt ja die Mantelfläche [*Jacob gestures around the curved surface of the cone. See Figure 8.32*], das weiß man ja allgemein. So, das ist die Mantelfläche, die ist ja abgerundet. Und dann was innerhalb dieses Kegels liegt, gehört ja auch zu dieser Schnittfläche, weil die, weil der Körper ja eine bestimmte Masse darstellt. Das heißt der, die Punkte, wo der Körper genau diesen Boden berührt [*he refers to the xOy plane*], werden ja aufgezeigt durch diese Schnittfläche. Und das heißt, wenn hier jetzt, wenn das jetzt hier im Prinzip der Boden ist, dann wenn das neigt sich so ein bisschen nach vorne [*for n>0, see Figure 8.34*] und dann berührt ja das [*the plane xOy*] diese abgerundete Fläche [*he points at the side surface of the cone*], und etwas innerhalb des Kegels [*see Figure 8.33*], wo, also das heißt so kann man das im Prinzip sehen [*e.g. the cross-section in Figure 8.34*]. Das ist ja alles abgerundet, deshalb entstehen auch diese ganzen ovalen abgerundeten Formen. #01:07:17-8#	Jacob: So that's the side-surface [*Jacob gestures around the curved surface of the cone. See Figure 8.32*], we know that in general. So, that's the side surface, it is rounded. And then what lies within this cone, also belongs to this cross-section, because it, because the solid represents a certain mass. That means that the points where the solid touches exactly this floor [*he refers to the xOy plane*], are shown by this cross-section. And that means, if here now, if this is basically the bottom here, then if it is tilted a little bit forward [*for n>0, see Figure 8.34*] and then it [*the plane xOy*] touches this rounded surface [*he points at the side surface of the cone*], and something inside the cone [*see Figure 8.33*], where, so that means that this is how you can see that [*e.g. the cross-section in Figure 8.34*], in principle. This is all rounded, that is why all these oval rounded shapes emerge.

Transcript 8.11: Jacob on Task 3A (invisible cone)

Reconstruction of the argumentation

Figure 8.35 shows the reconstruction of Jacob's argumentation. In Table 8.8, I present the coding of the argumentation elements that are used in Jacob's argumentation stream. As seen in Figure 8.35, Jacob's argumentation involves multiple cases of *NI-visualization* and *Sp-manipulation*. In this subsection I discuss the case of *NI-visualization* $NIVz_{37.5}$, which supports the explanation of a visual datum, in this case being the oval cross-sections (see Figure 8.34).

Codes (see Figure 8.35)	German original	English translation
$D_{37.2\text{-}3}$	die Mantelfläche, die ist ja abgerundet	side surface, it is rounded
$D_{37.4}$	der Körper ja eine bestimmte Masse darstellt	the solid has a certain mass
$C_{37.3\text{-}4}$	dann was innerhalb dieses Kegels liegt, gehört ja auch zu dieser Schnittfläche	what lies within this cone, also belongs to this cross-section
$C_{37.5\text{-}6}$	die Punkte, wo der Körper genau diesen Boden berührt, werden ja aufgezeigt durch diese Schnittfläche	the points where the solid touches exactly this floor are shown by this cross-section
$D_{37.7}$	dann neigt sich das so ein bisschen nach vorne	it is tilted a little bit forward
$C_{37.8\text{-}9}$	berührt ja das diese abgerundete Fläche, und etwas innerhalb des Kegels	then it touches this rounded surface and something inside the cone
$W_{37.10}$	Das ist ja alles abgerundet	This is all rounded
$C_{37.10\text{-}11}$	entstehen auch diese ganzen ovalen abgerundeten Formen	all these oval rounded shapes occur

Table 8.8: Coding of elements in Jacob's argumentation (Figure 8.35)

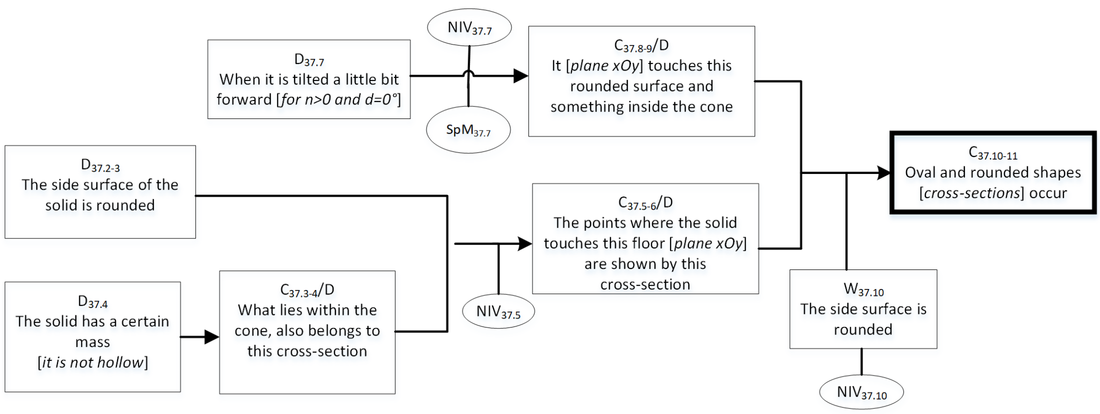


Figure 8.35: *NI-visualization* and *Sp-manipulation* in the Jacob's argumentation

In the first step of his argumentation, Jacob provides two data: he points at the side-surface of the paper-cone, which he characterizes as "rounded" (see Figure 8.32, and $D_{37.2\text{-}3}$ in Figure 8.35 and in Table 8.8), and he also says that the solid has a specific mass (see Figure 8.33, where Jacob points at the internal points of the cone, and $D_{37.4}$ in Figure 8.35 and in Table 8.8). From the second datum ($D_{37.4}$), follows that some internal parts of the cone will also be elements of a cross-section ($C_{37.3\text{-}4}$, Figure 8.35 and in Table 8.8). Then from the combination of the first datum ($D_{37.2\text{-}3}$: the side surface of the cone is rounded) and the conclusion ($C_{37.3\text{-}4}$: what lies within this cone, also belongs to this cross-section), follows the new conclusion that the points of the solid that touch the plane xOy are the ones constituting its cross-section ($C_{37.5\text{-}6}$, in Figure 8.35 and Table 8.8).

I stop the description of the argumentation here, as I just want to focus on the use of *NI-visualization* ($NIV_{37.5}$ in the last step I describe in the paragraph above (from $D_{37.2\text{-}3}$ and $C_{37.3\text{-}4}$, to $C_{37.5\text{-}6}$). I continue the description of the argumentation in the next subsection (8.3.2), where I discuss the *synergy* of *NI-visualization* and *Sp-manipulation* ($NIV_{37.7}$ and $SpM_{37.7}$). Next, I present my interpretation of what happens in this episode.

Interpretation

In the argument discussed above, there is one case of *NI-visualization* ($NIV_{37.5}$). Before I elaborate on it, I would like to discuss the argument in more general terms.

In his argument described above, Jacob first refers to the roundness of the cone's side surface ($D_{37.2\text{-}3}$). This is a property of a two-dimensional (2D) figural unit (the side surface) of the cone. Then he says that the cone "represents a certain mass" ($D_{37.4}$). Judging from his gesture, pointing at the inside of the cone (see Figure 8.33), I believe that with this phrase he means that the cone is compact (not hollow as the paper model is). This "compactness" is a property of the specific cone under examination in Task 3A. From that property, Jacob draws his conclusion that it is also the cone's internal points that belong to its cross-section ($C_{37.3\text{-}4}$). Finally, combining this conclusion with the first datum he uses ($D_{37.2\text{-}3}$: the side surface of the solid is rounded), he infers that what is shown as a cross-section are the exact points of the cone that touch the plane xOy of intersection ($C_{37.5\text{-}6}$).

In this last step of his argument, Jacob moves from a figural unit of the solid to its cross-section. More precisely, when he says that the internal points of the solid are the elements that constitute the cross-section when the plane xOy intersects the solid ($C_{37.5\text{-}6}$), he transitions from the internal parts of the cone, to its cross-section. The internal parts he refers to ("what lies within the cone" at $C_{37.3\text{-}4}$) are points or surfaces of the cone, representing the 0D (zero-dimensional) and 2D (two-dimensional) figural units of the cone, respectively. Jacob maps these figural units of the solid to the corresponding figural units of the cross-section. Therefore, I argue that ultimately Jacob perceives the cross-section as a lower-dimension figural unit of the solid. This is a process of *dimensional deconstruction*, which is an indicator of the use of *NI-visualization* ($NIV_{37.5}$).

This is an example of how *NI-visualization* operates in order to explain a visual datum. Here, Jacob manages to explain the exact way in which the oval and other curved cross-sections emerge, by connecting the properties of the cone and its figural units, with the properties of the cross-section. Both the interconnection of properties between geometric objects of different dimensions, as well as the *dimensional deconstruction*, are indicators of the use of *NI-visualization* in Jacob's argumentation. $NIV_{37.5}$ functions here both as a means to explain visual data (Role 3), as well as a warrant that leads from data ($D_{27.2\text{-}3}$ and $C_{37.3\text{-}4}$) to a new conclusion ($C_{37.5}$) (Role 2).

8.3.2 *Synergy* of *NI-visualization* and *Sp-manipulation* in the process of explaining visual data

Here, I continue the presentation of Jacobs' argumentation stream that I started in the previous subsection (8.3.1). I now focus on the operation of the *synergy* of *NI-visualization* and *Sp-manipulation* in Jacob's next argumentation step (Figure 8.35). Since I have already described the whole episode, I will move directly to the description of the rest of the argumentation reconstruction and then to my interpretation of it.

Reconstructing the argumentation

In Figure 8.35 you can see the second part of Jacob's argumentation, which I discuss here (from $C_{37.5\text{-}6}$ and $D_{37.7}$, until $C_{37.10\text{-}11}$). Table 8.8 shows the codes used in the argumentation, which represent the functions of Jacob's statements (see Transcript 8.10) in his argumentation.

After Jacob has concluded that the points of the solid that touch the plane xOy are the ones constituting its cross-section ($C_{37.5\text{-}6}$, in Figure 8.35 and Table 8.8), he considers the case of tilting the solid a bit forward ($D_{37.7}$, Table 8.8 and Figure 8.35). In GeoGebra that means to drag the tilt-slider (n-slider) over zero degrees (e.g. position seen in Figure 8.34). From this new datum he draws the conclusion that then the plane of intersection xOy, will touch the rounded side surface of the cone, as well as the inside of the cone ($C_{37.8\text{-}9}$). Next, Jacob combines his two last conclusions ($C_{37.5\text{-}6}$ and $C_{37.8\text{-}9}$) in order to draw his final conclusion ($C_{37.10\text{-}11}$). He argues that the oval cross-sections occur ($C_{37.10\text{-}11}$), because the side surface of the cone (which is intersected by the plane xOy when the cone is tilted) is rounded ($W_{37.10}$). The latter statement is the warrant ($W_{37.10}$) that Jacob uses in order to pass from his two previous conclusions to his final conclusion ($C_{37.10\text{-}11}$).

Interpretation

In this part of Jacob's argumentation stream, there is a place where he employs both his *NI-visualization* and *Sp-manipulation* ($NIV_{37.7}$ and $SpM_{37.7}$). Jacob decides to examine what happens when he tilts the cone forwards ($D_{37.3}$). He does that in order to explain what happens in position (h=-0,85, n=46°, d=0°), where the solid has an oval cross-section. In GeoGebra, the solid was first in position (h=-0,85, n=0°, d=0°) before it was tilted 46°, to end up in position (h=-0,85, n=46°, d=0°).

Jacob manipulates the paper-cone accordingly, tilting its top slightly forward. Then, with the use of gestures he describes how the plane xOy will be passing through the cone, cutting through its rounded side surface ($C_{37.8\text{-}9}$). Based on his manipulation of the haptic cone and his gestures, I argue that Jacob can imagine the movement of the solid in space from position (h=-0,85, n=0°, d=0°) to position (h=-0,85, n=46°, d=0°), when the tilt-slider is moved. He represents this by moving the paper cone accordingly. I believe that this is an indicator that he employs his *Sp-manipulation* ($SpM_{37.7}$).

During this process, Jacob passes from the movement of the tilt-slider to the cone (the haptic model), projecting the consequences of the dragging of the slider, on the orientation of the cone. At the same time, he can explain what happens with the cone and the plane xOy when the solid is tilted ($C_{37.8\text{-}9}$: the plane xOy touches the rounded side surface and something inside the cone), thus transitioning from the cone and its movement, to its lower-dimension figural units (rounded side surface and the inside of the cone). Jacob also reflects the property of the roundness of the cone's side surface to the roundness of the boundary of the cross-section (oval cross-section). Through this process of interrelating properties of the solid and its figural units, as well as projecting properties of the solid, to its cross-section, Jacob performs the *dimensional deconstruction* of the cone, which indicates the use of *NI-visualization* in his argumentation ($NIV_{37.7}$).

Therefore, it is here the *synergy* of *NI-visualization* and *Sp-manipulation* that brings forth this argumentation step in Jacob's argumentation stream. The *synergy* functions here as a warrant that links a datum with a new conclusion, and plays two roles; the role of supporting the explanation of the visual data of the rounded cross-sections (Role 3), as well as the role of supporting the creation of a new conclusion (Role 2).

8.3.3 *Sp-manipulation* in the process of explaining visual data

In Subsection 8.2.3, I discuss an example of *Sp-manipulation* in the role of drawing a conclusion ($SpM_{20.1}$, see Figure 8.31). In that example, Dave and Axel work on Task 2 (invisible sphere)[13], exploring the vertical movement of the solid by moving the height-slider over and under zero (see Transcript 8.10). Please refer back to Subsection 8.2.3 for the detailed episode description, reconstruction of argumentation and the interpretation. Next, I discuss in more detail the use of the same *Sp-manipulation* $SpM_{20.1}$ acting in the role of explaining a visual datum.

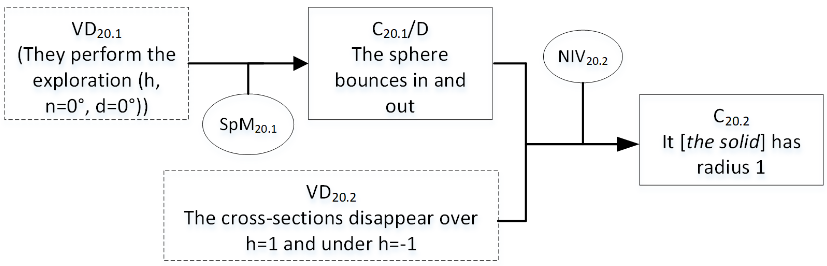


Figure 8.36: The structure of Axel and Dave's argument in which $SpM_{20.1}$ takes place

Interpretation (continued from Subsection 8.2.3)

In the episode (described in Subsection 8.2.3 and also discussed here), while Dave drags the height-slider up and down he says that the sphere (he miss-speaks saying "circle") "bounces in and out" (utterance 20, Transcript 8.10). By that I believe that he refers to the sphere passing in and out through the plane of intersection xOy.

Dave uses a metaphor in order to describe his thought: the sphere "bounces in and out". He does not really see a sphere bouncing in an out of anywhere. During the height-exploration, what are visible on the screen are only circular cross-sections diminishing and getting bigger. Dave's metaphor indicates that he has created a mental image of a sphere. Furthermore, since his metaphor is referring to a

13 See the global argumentation structure and the Worksheet for Episode GR1AD-2 in Appendices E2 and G1, respectively.

movement performed by the sphere (bouncing is not a static condition), it also indicates that he can manipulate his mental image and imagine it moving in space. Based on his metaphor, I argue that he imagines the sphere moving vertically above, through and under the xOy plane, while the height-slider is dragged up and down. Therefore, I argue that Dave employs his *Sp-manipulation* ($SpM_{20,1}$) in order to explain what he sees on the screen.

In the reconstructed argument (see Figure 8.36), $SpM_{20.1}$ functions as a warrant, linking a visual datum ($VD_{20.1}$) to a conclusion ($C_{20.1}$). Dave expresses his explanation as a conclusion in his argument ($C_{20.1}$). As I also mention in Subsection 8.2.3, $SpM_{20.1}$ plays a double role in the argument; it helps Dave to draw a conclusion ($C_{20.1}$: the sphere bounces in and out of plane xOy) (Role 2), while at the same time it also helps him to explain the visual data he has observed on the screen ($VD_{20.1}$: the circular cross-sections of the solid with the plane xOy diminish until they converge to a single point both over and under h=0) (Role 3).

8.3.4 Epilogue

In Sections 8.1 to 8.3 I have discussed three roles for *NI-visualization* and *Sp-manipulation* taking place in the argumentation: creating a hypothesis or claim (Role 1), drawing a conclusion (Role 2), and explaining visual data (Role 3). Through all these roles, *NI-visualization* and *Sp-manipulation* contribute to students' argumentation always functioning as warrants, linking visual data to hypotheses (Role 1), supporting students to draw new conclusions (Role 2), or explaining how some visual data emerge (Role 3). In all the roles, *NI-visualization* and *Sp-manipulation* have taken place either each on its own or both together in *synergy*.

In the next two sections I present two more roles of *NI-visualization* and *Sp-manipulation* in argumentation, in which they function either as warrants, as backings or as elements in refutations.

8.4 Role 4: Supporting a simple contradiction or a refutation by Reductio ad absurdum

In this section, I present examples in which *spatial manipulation* (*Sp-manipulation*) and *non-iconic visualization* (*NI-visualization*) contribute to students' argumentation by taking place in the process of refuting specific statements. I have identified two types of refutations taking place in students' argumentations in this work: *simple contradiction* and *refutation by reductio ad absurdum*. With the term *simple contradiction*, I refer to cases in which a statement is refuted because it is contradicted by a single datum. In *refutations by reductio ad absurdum* (RAA), the students provide a whole argument, which leads to the contradiction of a statement or of a step in the argumentation, thus leading to its rejection. In the role I present here, *Sp-manipulation* and *NI-visualization* function in argumentation in two ways, either as warrants or as backings.

Table 8.9 shows how *Sp-manipulation* and *NI-visualization* contributed to these two different types of refutations. Contrary to the similar tables in previous sections (see Tables 8.2, 8.3 and 8.7), Table 8.9 has only two modes (*NI-visualization* and *synergy*). The reason for this is that my analysis of the data did not reveal any case of *Sp-manipulation* acting alone in a *simple contradiction* or a *refutation by RAA*. I comment on this further in Subsection 8.4.3. In the next subsections, I provide examples of the two modes described in Table 8.9.

Role 4: Creating a refutation by a *simple contradiction* or by *Reductio ad absurdum*			
Mode	**Type of refutation**	**Function in argumentation**	**Indicators**
NI-visualization	Simple contradiction	Contradiction	• Performing *dimensional deconstruction* • Transitioning from the solid to lower dimension figural units • Referring to relations between properties of objects of different dimensions
	Refutation by Reductio ad absurdum	Warrant	
Synergy of *NI-visualization* and *Sp-manipulation*	Simple contradiction	Contradiction	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Using gestures • Relating the movement of a slider to the movement and orientation of the solid
	Refutation by Reductio ad absurdum	Warrant or Backing	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Using gestures • Describing verbally the solid's movement • Relating the movement of a slider to the movement of the solid

Table 8.9: Role 4 of *NI-visualization* and *Sp-manipulation* – Creating a refutation

8.4.1 *NI-visualization* supporting a refutation

As shown in Table 8.9 (see first *mode* in the third and fourth row), *NI-visualization* supports the refutation of a statement in students' argumentation, either through a *simple contradiction* or by contributing to a *refutation by Reductio ad absurdum*. Here, I present one example for each of these types of refutations. Both examples are from the same episode of Axel and Dave's pair-work on the task of the invisible cone (Task 3A)[14]. Figure 8.37 shows the part of Axel and Dave's argumentation structure that I present in the episode. This part of the argumentation is presented in Transcript 8.12.

[14] See the global argumentation structure and the Worksheet for Episode GR1AD-3A.III in Appendices E3 and G2, respectively.

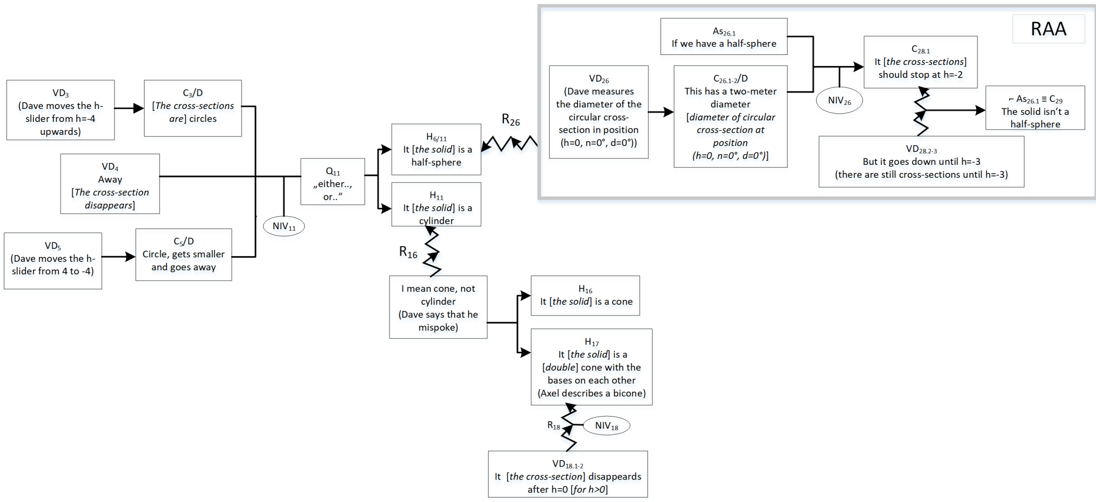


Figure 8.37: The beginning of Axel and Dave's argumentation structure on the invisible cone task (Task 3A)

<u>Episode description</u>

Transcript 8.12 (see next pages) shows Axel and Dave's discussion from the beginning of their work on the task of the invisible cone. The students start with a height exploration. Dave manipulates the sliders in GeoGebra. Leaving the tilt (n) and spin (d) sliders at zero degrees, he moves the height-slider (h) first from zero downwards to -4. What appears on the screen are circular cross-sections that get smaller and converge to a point, before they disappear completely (before the slider reaches the value h=-4). Dave then drags the slider upwards again, over h=-4. This time, first emerges a point as a cross-section and then circular cross-sections that get gradually bigger. At that point Dave says "Circle" (utterance 3). Dave drags the h-slider up over h=0, after which the cross-sections disappear again. At this point Axel says "Away". Dave keeps dragging the h-slider upwards until h=4 and says: "Circle, getting smaller and goes away" (utterance 5).

Then Axel asks if the solid is a half-sphere (utterance 6) and Dave says "no" (utterance 7). But then Dave says, "It is either a half-sphere, or a cylinder" (utterance 11). Axel says that the solid cannot be a cylinder and Dave asks why. Then Axel starts saying that "There had to be a- Then in cylinder- (..) Then, it would have to be a cone; a cone that on both sides-" (utterance 14), to which Dave responds that he did not mean to say cylinder, rather cone (utterance 16). Axel continues his description (of a bicone), saying: "A cone, that on both sides- So, it is a cone that- has the bases on each other" (utterance 17), and he asks Dave to perform the height-exploration again. Dave drags the slider from h=4 downwards and on the screen there appear no cross-sections until h=0. Dave also says at the same time, "No. A completely normal cone. Because, look, at zero it disappears again after zero" (utterance 18). When he says "after" here, he has the h-slider above (not below) zero. He then continues, dragging the h-slider further, through and below h=0 and says "The circular surface is getting smaller. (..) Up to three point, up to three, minus three. From zero until minus three" (utterance 20). What appears on the screen are circular cross-sections getting smaller from h=0 until h=-3 (-3<h<0, n=0° , d=0°) and then disappearing. After this, Axel agrees with Dave, saying "Ok. Yes, ok" (utterance 21).

Next, Dave says: "Wait if that's a circle. (..) Axel, that is not a semicircle[15]" (utterance 22). Axel responds, "No, it is not" (utterance 23) and says that the solid is a cone (utterance 25). But Dave does not stop there. He continues, "You have a semicircle[16]. Look, it is here, one, one met-, two meters diameter, right?" (utterance 26). The sliders are at position (h=0, n=0° , d=0°) and Dave points at the screen on the two points where the circumference of the cross-section meets the y-axis (see Figures 8.38a and 8.38b) and he measures the length of the segment between these two points. Axel agrees with what Dave says (utterance 27) and Dave continues, "This means, then this must also until two, quit until minus two" (utterance 28). Here, Dave points at the screen, at point (0,0,-2) on the z-axis (see Figure 8.39). Then he says, "But is goes down until minus three" (utterance 28). Axel responds to Dave "Yes, I know. Yes, it is not".

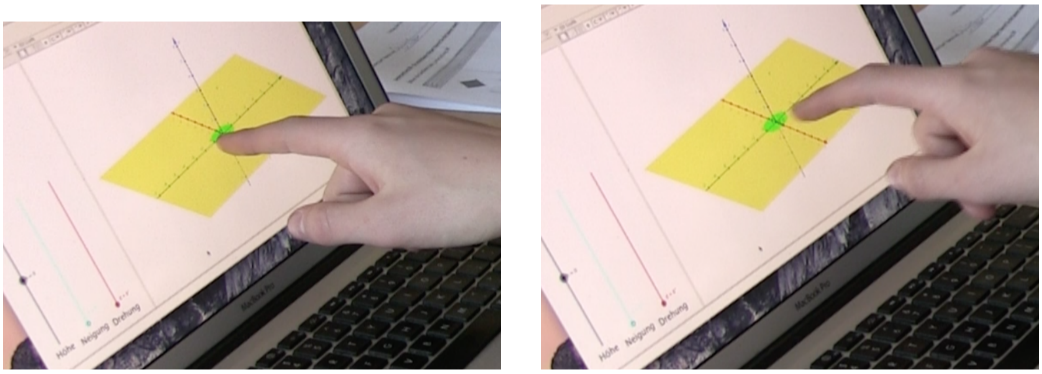


Figure 8.38: a (left) and b (right). Dave points at the left and right points of intersection between the cross-section (at position (h=0, n=0° , d=0°) and the y-axis.

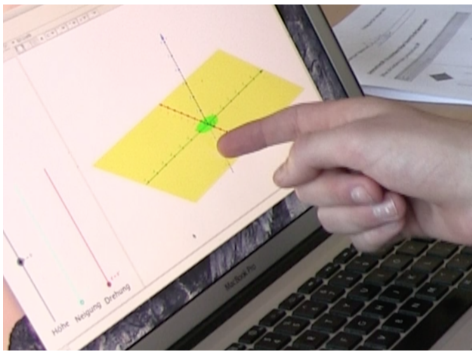


Figure 8.39: Dave points at point (0,0,-2) on the z-axis

15 Dave misspeaks. He meant to say half-sphere. See Transcript 8.12, utterance 22.
16 Dave misspeaks again. He means half-sphere. See Transcript 8.12, utterance 26.

Utterance	Original German transcript	English translation
3	Dave: [*Dave moves the height-slider to h=-4*] Kreis [*Dave says that, during the height-exploration as he moves the height-slider from h=-4 upwards*] #00:43:24-2#	Dave: [*Dave moves the height-slider to h=-4*] Circle [*Dave says that, during the height-exploration as he moves the height-slider from h=-4 upwards*]
4	Axel: Weg #00:43:28-1#	Axel: Away
5	[*Dave moves the h-slider from 4 down to almost -4. He stops a little before -4, as soon as there is no more cross-section to be seen*] Dave: Kreis, wird kleiner und ist weg #00:43:30-1#	[*Dave moves the h-slider from 4 down to almost -4. He stops a little before -4, as soon as there is no more cross-section to be seen*] Dave: Circle, getting smaller and goes away
6	Axel: Ist es ne halbe Kugel? #00:43:32-4#	Axel: Is it a half-sphere?
7	Dave: Nein. #00:43:34-4#	Dave: No
8	Axel: Ja, aber drei- minus zwei Komma fün- #00:43:36-5#	Axel: Yes but three- minus two point fi-
9	Dave: Warte das ent- (unverständlich) #00:43:37-6#	Dave: Wait this - (incomprehensible)
10	Axel: Größer? #00:43:37-9#	Axel: bigger?
11	Dave: Entweder ist das ne halbe Kugel oder es ist ne Zylinder. #00:43:44-4#	Dave: It is either a half-sphere, or a cylinder
12	Axel: Es kann kein Zylinder sein. #00:43:45-1#	Axel: It cannot be a cylinder
13	Dave: Warum? #00:43:48-2#	Dave: Why?
14	Axel: Da musste ein- Dann müsste in Zylinder- (..) Dann, es müsste einen Kegel; ein Kegel der auf beiden Seiten- #00:43:56-6#	Axel: There had to be a- Then in cylinder- (..) Then, it would have to be a cone; a cone that on both sides-
15	Dave: (unverständlich, Axel und Dave reden gleichzeitig). #00:43:57-1#	Dave: (incomprehensible, Axel and Dave talk simultaneously)
16	Dave: Kegel meine ich, keinen Zylinder. #00:44:01-9#	Dave: I mean a cone, not a cylinder
17	Axel: Ein Kegel sein, der auf beiden Seiten- Also es ist ein Kegel, der mit den- mit der Grundfläche aneinander ist [*what Axel describes is a bicone*]. Mach nochmal [*Axel asks Dave to perform the height-exploration once again*]. #00:44:11-2#	Axel: A cone, that on both sides- So, it is a cone that- has the bases on each other [*what Axel describes is a bicone*]. Do it again [*Axel asks Dave to perform the height-exploration once again*]

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18	Dave: Nein. Ganz normaler Kegel. Weil, guck mal, bei null [*h=0*] verschwindet es wieder nach null [*there is no cross-section for h>0*]. (..) Mal doch mal einen Halbkreis. Oder mal ein Kreis quasi einfach [*Dave refers to the sketch Axel should draw for this case on the Exploration Matrix*]. #00:44:25-9#	Dave: No. A completely normal cone. Because, look, at zero [*h=0*] it disappears again after zero [*there is no cross-section for h>0*] (..) Draw a half-circle. Or draw a circle simply [*Dave refers to the sketch Axel should draw for this case on the Exploration Matrix*]
19	Axel: Ist ja wie ein- äh ja Kreis. (..) #00:44:35-7#	Axel: It is like a – ehh yes circle.
20	Dave: Kreisfläche wird immer kleiner. (..) Bis drei Komma, bis drei, minus drei. Von null bis minus drei [*(-3<h<0, n=0 ° , d=0 °)*]. #00:44:54-5#	Dave: The circular surface is getting smaller. (..) Up to three point, up to three, minus three. From zero until minus three [*(-3<h<0, n=0 ° , d=0 °)*]
21	Axel: Ok. Ja, ok. #00:44:59-8#	Axel: Ok. Yes, ok.
22	Dave: Warte, wenn das ein Kreis ist. (..) Axel, das ist kein, das kann kein Halbkreis sein [*Dave misspeaks. He meant to say half-sphere (Halbkugel)*]. #00:45:14-1#	Dave: Wait if that's a circle. (..) Axel, that is not a semicircle [*Dave misspeaks. He meant to say half-sphere (Halbkugel)*].
23	Axel: Ne, ist es auch nicht. Das ist ne Pyramide. Äh ne, ne wie heißt denn. #00:45:17-1#	Axel: No, it is not. This is a Pyramid, eeh no, no how is it called?
24	Dave: Ku-, Kegel. #00:45:18-0#	Dave: Sphe-, cone.
25	Axel: Kegel. #00:45:18-5#	Axel: Cone
26	Dave: Man hat einen Halbkreis [*Dave again means half-sphere (Halbkugel)*]. Guck mal der ist hier, ein, ein, ein met-, zwei Meter Durchmesser, ne? [*The sliders are at position (h=0, n=0° , d=0°). Dave points at the screen on the two points where the circumference of the cross-section meets the y-axis and measures the length of the segment between these two points. See Figures 8.38a and b*] #00:45:21-6#	Dave: You have a semicircle [*Dave again means half-sphere (Halbkugel)*]. Look, it is here, one, one met-, two meters diameter, right? [*The sliders are at position (h=0, n=0° , d=0°). Dave points at the screen on the two points where the circumference of the cross-section meets the y-axis and measures the length of the segment between these two points. See Figures 8.38a and b*]
27	Axel: Ja. #00:45:22-2#	Axel: Yes.
28	Dave: Das heißt dann muss der auch bis hier zwei, bei minus zwei aufhören [*Dave points at the screen, at point (0,0,-2) on the z-axis. See Figure 8.39*]. Aber der geht dann bis minus drei runter. #00:45:26-6#	Dave: This means, then this must also until two, quit until minus two [*Dave points at the screen, at point (0,0,-2) on the z-axis. See Figure 8.39*]. But it goes down until minus three.

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29	Axel: Ja, ich weiß. Ja, ist es auch nicht [*a half-sphere*]. #00:45:31-3#	Axel: Yes, I know. Yes, it is not [*a half-sphere*].

Transcript 8.12: Axel and Dave on Task 3A (invisible cone)

Reconstruction of the argumentation

Figure 8.40 (see next page) illustrates the structure of Axel and Dave's argumentation based on Transcript 8.12. Table 8.10 below, shows in detail the coded elements used in the structure in Figure 8.40. The argument starts with three visual data that are visible on the computer screen during the height-exploration performed by the students:

- VD_3: the cross-sections when dragging the h-slider from h=-4 upwards (utterance 3)
- VD_4: the cross-sections that disappear over h=0 (utterance 4)
- VD_5: the change of the cross-sections when dragging the h-slider downwards from h=4 until h=-4 (utterance 5)

From those three visual data, follow two conclusions. From VD_3 follows conclusion C_3 that the cross-sections in this case are circles. From VD_5 follows conclusion C_5 that the cross-sections are circles that diminish and then disappear. Then, from the two conclusions and VD_4 follow two hypotheses; Axel's hypothesis $H_{6/11}$: the solid is a half-sphere, and Dave's hypothesis H_{11}: the solid is a cylinder. After Axel's objection (utterances 12 and 14) to Dave's hypothesis, Dave rejects his cylinder hypothesis himself saying he meant to say "cone" (utterance 16, see refutation element R_{16} in Figure 8.40 and Table 8.10). Therefore, from the refutation of H_{11} follow two new hypotheses: Dave's new hypothesis H_{16}: the solid is a cone, and Axel's new hypothesis H_{17}: "A cone (...) that has the bases on each other" where he describes a bicone.

Then follow two refutations. The *simple contradiction* R_{18} of hypothesis H_{17} (bicone) and the *refutation* of hypothesis $H_{6/11}$ (half-sphere) *by Reductio ad absurdum* (see RAA in Figure 8.40). In those two refutations take place the two cases of *NI-visualization* that I want to discuss here (NIV_{18} and NIV_{26}). Therefore, I present the structure of each refutation below separately, providing also my interpretation for the *NI-visualization* that takes place within it.

Codes (Figure 8.40)	German original	English translation
C_3/D	Kreis	Circle
VD_4	Weg	Away
C_5/D	Kreis, wird kleiner und ist weg	Circle, gets smaller and goes away
Q_{11}	entweder..., oder	either..., or
$H_{6/11}$	halbe Kugel	half-sphere
H_{11}	Zylinder	cylinder
H_{16}	Kegel	cone

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R_{16}	Kegel meine ich, keinen Zylinder	I mean cone, not cylinder
H_{17}	Ein Kegel (...) der mit den- mit der Grundfläche aneinander ist [*what Axel describes is a bicone*]	A cone (...) that has the bases on each other [*what Axel describes is a bicone*]
$VD_{18.1\text{-}2}$	verschwindet es wieder nach null [*there is no cross-section for h>0*]	it disappears again after zero [*there is no cross-section for h>0*]
$As_{26.1}$	Man hat einen Halbkreis [*again means half-sphere (Halbkugel)*]	You have a semicircle [*again means half-sphere (Halbkugel)*]
$C_{26.1\text{-}2}$/D (conclusion based on VD_{26})	zwei Meter Durchmesser	two meters diameter
$C_{28.1}$	muss der auch bis hier zwei, bei minus zwei aufhören	this must also until two, quit at minus two
$VD_{28.2\text{-}3}$	Aber der geht dann bis minus drei runter	But it goes down until minus three.
$As_{26.1} \equiv C_{29}$	ist es auch nicht [*a half-sphere*]	it is not [*a half-sphere*]

Table 8.10: Coding of elements in Axel and Dave's argumentation (Figure 8.40)

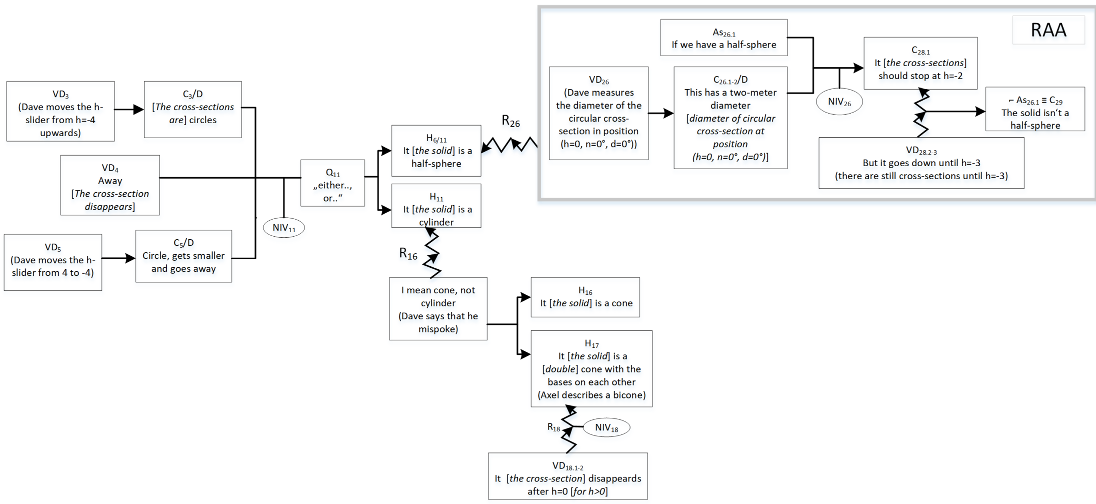


Figure 8.40: Axel and Dave's argumentation on the invisible cone task (utterances 3-29, see also Table 8.10)

NIV_{18}: *Simple contradiction* refuting a hypothesis

Description of the refutation in the argumentation

I now continue the description of Axel and Dave's argumentation structure, discussing the role of NIV_{18} in the refutation of Axel's bicone-hypothesis (H_{17}). Dave refutes (R_{18}) Axel's hypothesis (H_{17}) by saying that the cross-section "disappears again after zero" ($VD_{18.1\text{-}2}$), while he drags the h-slider over h=0. Axel accepts Dave's refutation (see utterance 21), without providing any further information as to why this is an actual refutation.

I call this refutation a *simple contradiction*, because an argumentation element (H_{17}) is refuted by another *single* element ($VD_{28.1\text{-}2}$) of the argumentation.

Interpretation

In his bicone-hypothesis, Axel implies that the bicone is positioned with its middle part (the common base of the two cones) on plane xOy. Dave's refutation is based on the fact ($VD_{18.1\text{-}2}$) that there are no cross-sections when the height-slider is dragged over zero. The link between this fact and the reason *why* it contradicts the bicone-hypothesis remains implicit. Below, I discuss what Dave's thinking may have been, judging by some of his arguments on the same matter during the classroom discussion on Task 3A.

In the classroom discussion on Task 3A, Dave says: "Ja, also der Kegel steht erstmal auf der (...) Achse. Und wenn man den quasi immer weiter runter fährt, dann sieht man halt, dass der immer spitzer zusammenläuft" [*Yes, so the cone stands initially on the (...) axis. And when you drag it further downwards, then you see that it gets more and more pointed*] (utterances 15 and 17 in episode CD3A-AD)[17]. Dave talks about the initial orientation of the cone at position (h=0, n=0°, d=0°). He says that at that position the cone stands on plane xOy plane (which he mistakenly calls here "the axis"), meaning that it has its base on the plane xOy and its top upwards (in the subspace above xOy). By the "dragging of the cone downwards" he refers to the dragging of the height-slider under zero, during which the cross-sections diminish on the screen.

Therefore, I argue that Dave bases his conclusion about the orientation and the form of the solid (upward cone) on the condition that if there are no cross-sections when the solid is dragged over xOy (for h>0), then there can't be a part of the solid under xOy at (h=0, n=0°, d=0°). As a result, the solid cannot be a bicone as Axel thinks, because then the solid would be on both sides of the plane xOy (both over and under it). The rest of Dave's statement says that when the solid is dragged under xOy (h<0) (VD_{17}, in CD3A-AD) the solid becomes more and more pointed (C_{17}, in CD3A-AD). So, the solid is a cone standing upwards on xOy.

If the thinking process I describe above is indeed the reasoning that lies behind Dave's statement $VD_{18.1\text{-}2}$, and I do believe it is, then Dave achieves the refutation of Axel's hypothesis by employing his *NI-visualization* (NIV_{18}). Dave connects the orientation of the solid with the absence of circular cross-sections when the height-slider is dragged over zero ($VD_{18.1\text{-}2}$). Thus, he connects a property of the two-dimensional figural unit of the solid with the solid itself and its properties. Therefore, I argue that *NI-visualization* NIV_{18} supports Dave's refutation R_{18} (see Figure 8.40).

[17] See elements C15-17, VD17 and C17 in the argumentation structure in Figure 8.42 in next subsection 8.4.2. Also the global argumentation structure of episode CD3A-AD in Appendix E9.

NIV_{26}: Refuting a hypothesis by *Reductio ad absurdum*

Description of the refutation

Next, Dave refutes his previous hypothesis that the solid could be a half-sphere ($H_{6/11}$), thus leaving the cone to be the only valid hypothesis in the end (H_{16}). In utterance 22, Dave says "Axel, that is not a semicircle". Dave consistently calls the half-sphere a semicircle by mistake. Here and in his next statements what he means by "semicircle" is a half-sphere.

Dave then explains why he thinks the solid cannot be a half-sphere. He starts with an assumption "You have a semicircle" (half-sphere, $As_{26.1}$, Figures 8.40 and 8.41, and Table 8.10). At this moment, the sliders are at position (h=0, n=0°, d=0°) and Dave points at the screen to the two points where the circumference of the circular cross-section meets the y-axis (see Figures 8.38a and b). He measures the length of the segment between these two points (see Figure 8.41 and Table 8.10), saying: "Look, it is here, one, one met-, two meters diameter, right?" (VD_{26} and $C_{26.1\text{-}2}$). Axel agrees (utterance 27 in Transcript 8.12) and Dave continues. He points at the screen, at point (0,0,-2) on the z-axis (see Figure 8.39) and says, "This means, then this must also until two, quit until minus two" ($C_{28.1}$, Figure 8.41 and Table 8.10). "But it goes down until minus three" ($VD_{28.2\text{-}3}$, Figure 8.41 and Table 8.10). Here Dave drags the height-slider downwards and shows that there are cross-sections to be seen until h=-3. To that Axel replies, "Yes, I know. Yes, it is not *[a half-sphere]*" ($As_{26.1} \equiv C_{29}$, Figure 8.40 and 8.41, and Table 8.10).

Reconstruction of the refutation in the argumentation

Figure 8.41 shows the structure of Dave's refutation of the half-sphere hypothesis ($H_{6/11}$). In contrast to his previous refutation (R_{18}, see previous example) that consisted of a single argumentation element ($VD_{18.1\text{-}2}$), here Dave provides a detailed refutation by *Reductio ad absurdum* (see Figure 8.41 and also the position of the RAA box on the top right in Figure 8.40).

As I mention in Chapter 2 (see subsection 2.2.4), I use the term *assumption* to describe the hypothetical proposition with which a *Reductio ad absurdum* begins, and with the negation of which it ends. An *assumption* (As) is a supposition that is meant to be refuted at the end of the argument.

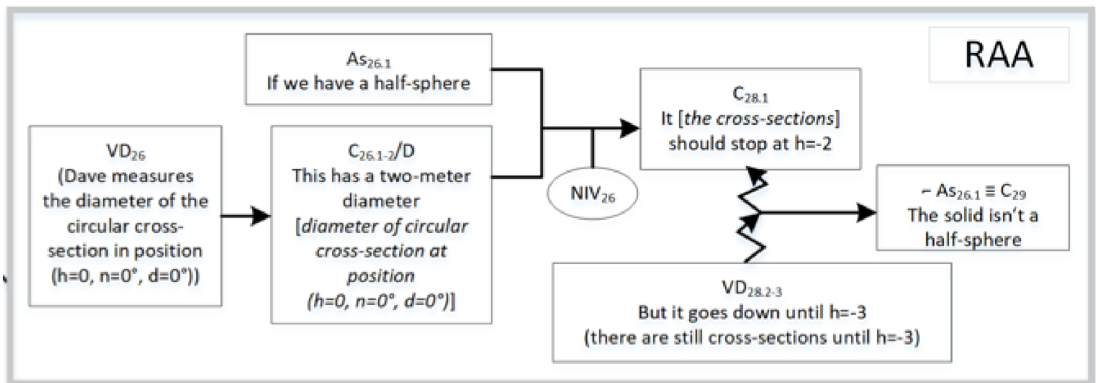


Figure 8.41: *Reductio ad absurdum* refuting the half-sphere hypothesis $H_{6/11}$

The RAA starts with the assumption, that the solid is a half-sphere ($As_{26.1}$). From the visual datum VD_{26} (see Figure 8.41), follows then the conclusion $C_{26.1\text{-}2}$ that the diameter of the base of the solid is two meters. From the combination of the assumption (the solid is a half-sphere) and the conclusion (circular base with diameter 2), follows the conclusion that in order for the solid to be a half-sphere, the cross-sections should stop after the height-slider reaches the value h=-2 ($C_{28.1}$). Then Dave checks this conclusion by dragging the height-slider downwards. The visual data $VD_{28.2\text{-}3}$ that emerge show that there are still cross-sections until h=-3. Thus, $VD_{28.2\text{-}3}$ contradicts the conclusion $C_{28.1}$ (no cross-sections under h=-2). This contradiction leads to the conclusion that the initial assumption must be wrong and from there to its negation ($\neg$ $As_{26.1}$), which is expressed by Axel as conclusion C_{29} (the solid is not a half-sphere).

Interpretation

In this refutation, *NI-visualization* links the combination of assumption $As_{26.1}$ (the solid is a half-sphere) and conclusion $C_{26.1\text{-}2}$ (the diameter is two meters), with conclusion $C_{28.1}$ (the cross-sections should stop below the height-slider value h=-2).

At the beginning of his refutation, Dave assumes that the invisible solid is a half-sphere ($As_{26.1}$). Previously, Dave and Axel have seen that as soon as they lift the solid above h=0 the cross-sections disappear (see utterance 18). From this and also from Dave's assumption ($As_{26.1}$), I argue that the students implicitly consider the cross-section at position (h=0, n=0°, d=0°) to be the base of the half-sphere. Therefore, according to Dave's assumption $As_{26.1}$, the solid is a half-sphere that stands with its base on xOy at position (h=0, n=0°, d=0°). Then, Dave measures the diameter of the circular cross-section at position (h=0, n=0°, d=0°) (that is, the diameter of the half-sphere's base) and he says that the diameter is two meters long. From that, he then draws the conclusion $C_{28.1}$, that the cross-sections of the solid should disappear when the height-slider is dragged below h=-2.

From Dave's argument so far, it seems that he argues that if the solid is a half-sphere ($As_{26.1}$) with a base-diameter equal to two meters ($C_{26.1\text{-}2}$), then the height of the half-sphere must also be two meters (my interpretation of his conclusion $C_{28.1}$). The warrant that leads to this conclusion though remains unstated. I argue that here Dave employs his *NI-visualization* in order to draw the conclusion $C_{28.1}$. In this argumentation step Dave relates the diameter of the half-sphere's base (in $C_{26.1\text{-}2}$) with the height of the half-sphere (in $C_{28.1}$), as well as the properties of the one with the properties of the other, as figural units of the half-sphere. Therefore, the implicit warrant here seems to be that the diameter of a half-sphere is equal to its height. This is the implicit link that leads to conclusion $C_{28.1}$. This process during which Dave relates the figural units of the solid with each other (the diameter and the height of the half-sphere) as well as their properties (equal lengths), is a process of *dimensional deconstruction* of the half-sphere and it is an indicator of the use of *NI-visualization* at this step of the argumentation. The NIV_{26} functions here as a warrant that links an assumption and a former conclusion ($As_{26.1}$ and $C_{26.1\text{-}2}$) to a new conclusion ($C_{28.1}$). Before I discuss the role of NIV_{26} in this refutation, I would like to comment on the mathematical correctness of Dave's conclusion $C_{28.1}$, and also complete the interpretation of his refutation by *Reductio ad absurdum* (RAA).

Dave's conclusion $C_{28.1}$ (the cross-sections should stop after the height-slider reaches h=-2) is not entirely correct. This mistake emerges when Dave confuses the diameter with the radius of the sphere. Luckily for his argument though, the invisible solid (which is a cone) is taller than 2 meters, which does not contradict his flawed conclusion.

As a next step in the refutation, Dave tests his conclusion $C_{28.1}$ by dragging the height-slider further below h=-2. He observes that the cross-sections continue to appear until h=-3 ($VD_{28.2\text{-}3}$). Using this visual datum ($VD_{28.2\text{-}3}$), Dave refutes his $C_{28.1}$ conclusion that the cross-sections should disappear below h=-2. This refutation leads to an "absurdity" that in turn leads to the refutation (negation) of the initial assumption ($\neg$ $As_{26.1}$), which agrees with Axel's conclusion "Yes, I know. Yes, it is not [*a half-sphere*]" (C_{29}).

In this *Reductio ad absurdum* argument, *NI-visualization* has played a crucial role in helping Dave draw a conclusion (Role 2, see section 8.2) functioning as a warrant (see Figures 8.40 and 8.41), and ultimately supporting him to build a refutation for the half-sphere hypothesis $H_{6/11}$ (Role 4). Consequently, NIV_{26} plays two roles in the argumentation; from the perspective of the whole argumentation it plays the role of supporting a refutation (Role 4), and inside the RAA argument it plays the role of supporting the creation of a conclusion (Role 2, see Section 8.2).

8.4.2 *Synergy* of *NI-visualization* and *Sp-manipulation* supporting a refutation

During the data analysis, I observed three cases in which *NI-visualization* and *Sp-manipulation* appear together at the same step in a refutation. Here, I present two of them. As shown in Table 8.9 (see second *mode* in the fifth and sixth row), the *synergy* of *NI-visualization* and *Sp-manipulation* supports the refutation of a statement in students' argumentation, either through a *simple contradiction* or by contributing to a *refutation by Reductio ad absurdum.* In the first example I present here ($NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$), the *synergy* supports a simple contradiction, while in the second example ($NIV_{23.5a}$ and $SpM_{23.5a}$) it takes place in a refutation by *Reduction ad absurdum* (RAA) (for more details on the difference between the two kinds of refutation, please refer to the first paragraph of Section 8.4).

$NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$: *Simple contradiction* refuting a conclusion

This is an example from the classroom discussion on the task of the invisible cone (Task 3A)[18]. Axel and Dave present their work to the whole class. In this presentation, Dave manipulates the sliders in GeoGebra and Axel describes their work. All of Dave's actions in GeoGebra appear on the classroom-wall with the help of a projector, so everyone can see.

The part of Axel and Dave's argumentation that I describe here is reconstructed as shown in the argumentation stream in Figure 8.42. I focus particularly on refutation $R_{15\text{-}17}$, where the *synergy* of $NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$ takes place. Before I move to the description and interpretation of refutation $R_{15\text{-}17}$ though, I would like to provide a background of what has happened so far. I therefore start with the description of the argumentation stream (see Figure 8.42) from the visual data $VD_{1\text{-}4}$ until the conclusion

[18] See the global argumentation structure for Episode CD3A-AD in Appendix E9.

C_{10} (which is then refuted by $R_{15\text{-}17}$).

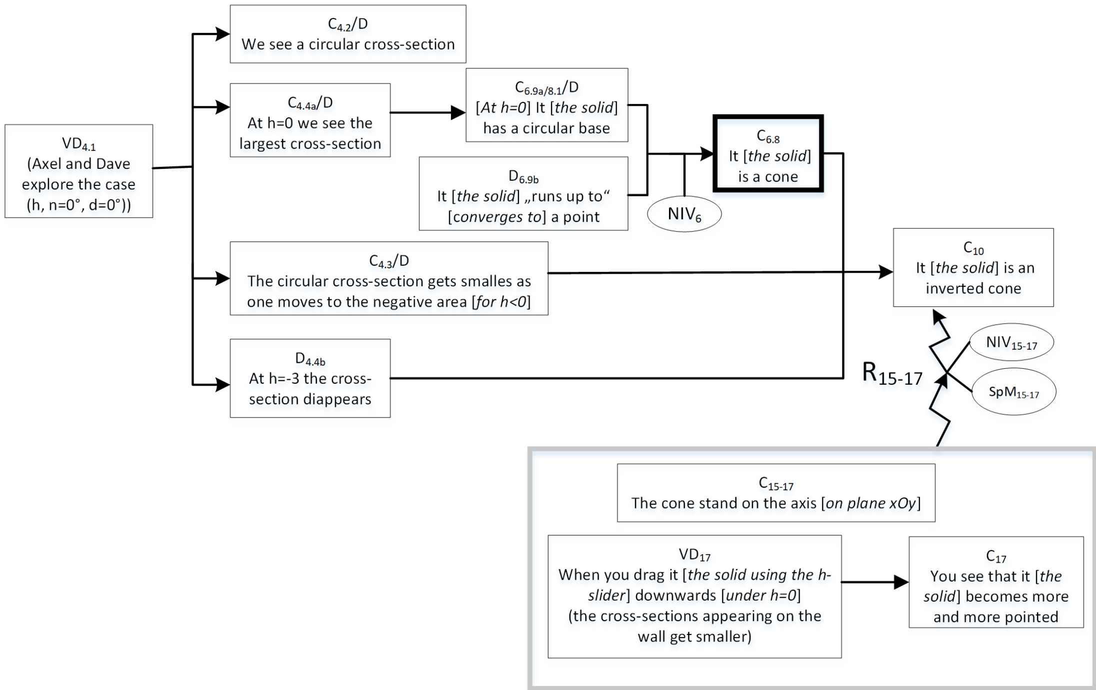


Figure 8.42: Axel and Dave's argumentation stream during the whole classroom discussion on the invisible cone task (Task 3A)

The presentation started with Axel describing what happens during the height-exploration ($VD_{4.1}$), when the height-slider is dragged above and under h=0 in the *case* (h, n=0°, d=0°). From those visual data Axel draws the following conclusions (see Figure 8.42):

$C_{4.2}$: the cross-sections are always circular

$C_{4.4a}$: the circular cross-section is largest when the height-slider is at h=0

$C_{4.3}$: when the height-slider is dragged below h=0, the circular cross-sections get smaller, and

$C_{4.4b}$: the cross-section disappears when the height-slider reaches h=-3

Then, from the conclusion that the biggest cross-section is at h=0 ($C_{4.4.a}$), Axel infers that this cross-section is the base of the solid ($C_{6.9a/8.1}$). He also says that the solid converges to a single point ($C_{6.9b}$). From these last two conclusions, Axel draws a new conclusion ($C_{6.8}$) that the solid is a cone. Finally, Axel combines his last conclusion ($C_{6.8}$) that the solid is a cone, with the previous conclusions, that the cross-sections get smaller when the height-slider is dragged below h=0 ($C_{4.3}$) and that the cross-section disappears completely when it reaches the value h=-3 ($C_{4.4b}$). From this combination Axel draws the final conclusion that the cone is "inverted" (C_{10}). Using gestures (see Figure 8.43), Axel explains that by "inverted" he means that the base of the cone is on the plane xOy and the top of the cone is under the plane (on the negative side of the z-axis).

Next, Dave refutes ($R_{15\text{-}17}$) Axel's conclusion C_{10}. Below, I present how Dave formed his refutation, as well as the role of the *synergy* of $NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$ in the argumentation reconstruction of the refutation.

Figure 8.43: Axel gestures the *inverted* orientation of the cone in the initial position (h=0, n=0°, d=0°)

Description of the refutation

After Axel has completed his presentation of the height exploration and he has concluded that the solid is a cone ($C_{6.8}$) that is inverted (C_{10}), Dave objects. Transcript 8.13 shows what Dave says about the orientation of the cone.

Utterance	Original German transcript	English translation
13	Dave: mhm (verneinend) Der [*Kegel*] steht auf einem- #01:03:33-3#	Dave: mhm (disagreeing) It [*the cone*] stands on a-
14	Frau Karl: Dave, sprichst du ein bisschen lauter bitte? #01:03:35-1#	Frau Karl: Dave, could you talk a little louder please?
15	Dave: Ja, also der Kegel steht erstmal [*at position* (*h=0, n=0°, d=0°*)]auf der- #01:03:38-0#	Dave: Yes, so the cone stands initially [*at position* (*h=0, n=0°, d=0°*)] on the-
16	Axel: Ja, ja, ja [*He gestures with the hands showing the cone standing on its base. See Figure 8.44*] #01:03:39-1#	Axel: Yes, yes, yes [*He gestures with the hands showing the cone standing on its base. See Figure 8.44*]
17	Dave: ... Achse. Und wenn man den quasi immer weiter runter fährt, dann sieht man halt, dass der immer spitzer zusammenläuft. #01:03:45-0#	Dave: ... axis. And when you keep dragging it [*the cone*] down, you can see that it is getting more and more pointed.
18	Axel: Stimmt. #01:03:46-8#	Axel: Right.

Transcript 8.13: Dave's objection on Axel's conclusion of the inverted cone

Dave says that, in the initial position (h=0, n=0°, d=0°), the cone stands on the "axis" (utterances 13, 15 and 17). At this point Axel also agrees with Dave (utterance 16) and he makes a gesture showing the cone standing upwards (see Figure 8.44). Then Dave also says that as the cone is dragged downwards, it gets more and more pointed (utterance 17).

Figure 8.44: Axel gestures the *upward* orientation of the cone in the initial position (h=0, n=0°, d=0°)

Reconstruction of the refutation in the argumentation

In Figure 8.42, the refutation $R_{15\text{-}17}$ shows the reconstruction of Dave's objection in the argumentation. Dave refutes Axel's conclusion (C_{10}) that the cone is inverted using:

- conclusion $C_{15\text{-}17}$: at position (h=0, n=0°, d=0°) the cone stands with its circular base on plane xOy. Here, Dave uses this conclusion as a datum.
- and a single-step argument from visual data VD_{17} to conclusion C_{17} (the cone gets more pointed): Dave drags the height-slider below h=0 and the circular cross-sections diminish.

I would like to comment here, that Dave talks about an "axis" (utterance 17), on which the cone stands at position (h=0, n=0°, d=0°). By "axis" he refers to the plane xOy. This is something that is left here implicit, yet becomes established later on in the discussion, by the students' gestures and descriptions of how the plane xOy intersects the cone. So, here Dave describes the orientation of the solid as standing with its circular base on the plane xOy, and its top over xOy, on the opposite part of axis z'z.

Interpretation

I argue that Dave's refutation is supported by the use of *NI-visualization* and *Sp-manipulation* (see $NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$ in Figure 8.42). In his single-step argument (from VD_{17} to C_{17}, see description in the paragraph above and in Figure 8.42), Dave manipulates the height-slider and talks about the movement of the solid, saying "when you keep dragging it [*the cone*] down, you can see that it is getting more and more pointed". He says that we can "see" that the cone gets more pointed, although the cone is invisible. What we can actually see is that as the height-slider moves downwards, the circular cross-sections are getting smaller and smaller, converging to a single point. Therefore, I argue that Dave's conclusion C_{17} is the result of his joint use of *NI-visualization* and *Sp-manipulation* (see $NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$ in Figure 8.42).

More precisely, when passing from the visual data VD_{17} (Dave drags the height-slider below h=0 and the cross-sections diminish) to conclusion C_{17} ("you can see that it is getting more and more pointed"), Dave implies that as he drags the height-slider downwards, the cone is also dragged downwards. This means, that Dave's conclusion C_{17} indicates that he perceives the movement of the cone in space as a consequence of the movement of the height-slider. Therefore, I argue that Dave imagines that as the cone is dragged downwards below h=0, it goes through the plane xOy (which remains stable at h=0). And that is actually exactly what happens . As the solid is dragged downwards passing through plane xOy, it is "scanned" by the plane, revealing smaller and smaller circular cross-sections, which converge to a point. Dave's realization of the interrelation between the slider's movement and the consequence it has on the movement of the cone in space indicates the use of *Sp-manipulation.*

As I mentioned earlier, another interesting point of Dave's argumentation, is that in his conclusion (C_{17}) he describes how the shape of the solid evolves: "it is getting more and more pointed" (although he cannot see it) based only on the visible diminishing circular cross-section. This indicates that Dave relates the evolution of the cross-sections during the dragging of the height-slider (see VD_{17}, the circular cross-sections get smaller and smaller), with the properties of the solid (see C_{17}, the cone gets more and more pointed). Dave relates the shrinking of the cross-sections with the narrowing of the cone as its top moves closer to the plane xOy. This means that he perceives these cross-sections as two-dimensional (2D) figural units of the cone, relating the properties of the 2D figural units with the properties of the cone. The process of identifying lower-dimension figural units of the solid is called *dimensional deconstruction.* This process and also the relation between properties of the figural units and properties of the cone are indicators of the use of *NI-visualization* in argumentation.

Dave's refutation is the result of the combined use of *Sp-manipulation* (SpM) and *NI-visualization.* Here, $SpM_{15\text{-}17}$ connects the movement of the slider with the movement of the cone, while $NIV_{15\text{-}17}$ helps Dave to relate the properties of the cross-sections as 2D figural units of the cone, with the properties of the cone itself, thus supporting his refutation. Therefore, both processes ($NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$) take place in a refutation, enhancing it.

$NIV_{23.5a}$ and $SpM_{23.5a}$: Refutation by *Reductio ad absurdum*

This example is from the classroom discussion on the invisible cube (Task 3C)[19]. In this classroom discussion, Lukas and Tom present their work. Lukas presents what they did, while Tom manipulates the GeoGebra sliders. In the specific excerpt that I focus on here, Lukas refutes the possibility of the solid being a cylinder. Previously, he has already argued that the solid is a cuboid. He now uses the cylinder as a counter-example, in order to explain why the solid in this task could not be one with curved surfaces. Transcript 8.14 shows Lukas' rejection of the cylinder.

Description of the refutation

The refutation of the cylinder-assumption (As_{23} in Table 8.11 and Figure 8.46), takes place during the exploration of the position (h=0, n=90°, d=45°) (see Lukas' first sentence in Utterance 23). What is visible at this position is illustrated in Figure 8.45.

[19] See the global argumentation structure for Episode CD3C-TL in Appendix E11.

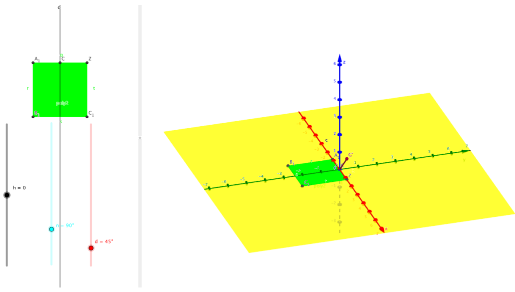


Figure 8.45: Position (h=0, n=90°, d=45°)

Utterance	Original German transcript	English translation
23	Lukas: Ja und dann noch einmal h null, n neunzig Grad, und d fünfundvierzig, (...) ja, und da sieht man halt, dass es em, jetzt kein zum Beispiel (..) kein Zylinder oder so ist, weil beim Zylinder zum Beispiel wär jetzt der Querschnitt, würde es [*the cross-section*] immer gleich bleiben, wenn man den [*the solid*] drehen würde, um die eigene Achse. Und jetzt ist es- Dreh mal bisschen hin und her [*Tom drags the spin-slider (d-slider) that rotates the solid*]. Ja, bei der Umdrehung [*he points at the cross-section on the wall*] wird es immer größer und kleiner und daran sieht man, dass es [*the cross-section*] halt vier Kanten hat, weil da wo das immer nicht die Kante [*of the solid*] ist, da wird es [*the cross-section*] wieder ein bisschen dünner, und dann wieder, ja. Auseinander. Daran sieht man eigentlich, dass es [*the solid*] nicht gleichmäßig ist. #01:26:21-6#	Lukas: Yes and then again h zero, n ninety degrees, and d forty-five, (...) yes, and then you can see that it is emm, not for example (..) not a cylinder or something, because with the cylinder for example, the cross-section would be, it [*the cross-section*] would always remain the same if you rotated it [*the solid*] around its own axis. And now it is- Rotate it a bit back and forth [*Tom drags the spin-slider (d-slider) that rotates the solid*]. Yes, during the rotation it [*he points at the cross-section on the wall*] gets bigger and smaller and you can see, that it [*the cross-section*] has four edges, because always where the edge [*of the solid*] is not, it [*the cross-section*] gets a little thinner again, and then again, yes. Apart. From this you can actually see that it [*the solid*] is not even.

Transcript 8.14: Lukas talks in the classroom discussion on Task 3C (invisible cube)

Lukas starts by asking Tom to drag the sliders into position (h=0, n=90°, d=45°). Then he says that the solid could not, for example, be a cylinder because in that case, if one rotated the solid around its axis (spin rotation using the d-slider), the cross-section would not change. He then asks Tom to rotate the solid around its axis and Tom drags the spin-slider (d-slider) up and down. While Tom does this, Lukas points at the cross-section on the wall and says "during the rotation it [*the cross-section*] gets bigger and smaller". He also says that the cross-section "has four edges, because always where the edge [*of the solid*] is not, it [*the cross-section*] gets a little thinner again, and then again". Finally, he says that from his last sentence he infers that the solid "is not even".

<u>Reconstruction of the refutation in the argumentation</u>

Figure 8.46 illustrates Lukas' refutation of the cylinder (see also Table 8.11 for the elements of the arguments). It is a refutation by *Reductio ad absurdum* (RAA). Here, I present the structure of Lukas' *Reductio ad absurdum* (RAA), and the roles that $NIV_{23.5a}$ and $SpM_{23.5a}$ play in it.

Codes (Figure 8.46)	German original	English translation
VD_{23}	h null, n neunzig Grad, und d fünfundvierzig	h zero, n ninety degrees, and d forty-five (h=0, n=90°, d=45°)
As_{23}	[*der Körper ist ein Zylinder*]	[*the solid is a cylinder*]
$W_{23.4}$	weil beim Zylinder (...) der Querschnitt, würde (...) immer gleich bleiben, wenn man den [*the solid*] drehen würde, um die eigene Achse	because in the case of a cylinder (...) the cross-section would (...) always remain the same, if you rotated it [*the solid*] around its own axis
$C_{23.3}$	der Querschnitt, würde (..) immer gleich bleiben [*during the rotation*]	the cross-section would always remain the same [*during the rotation*]
$VD_{23.4\text{-}5}$	bei der Umdrehung wird es [*he points at the cross-section on the wall*] immer größer und kleiner	during the rotation it [*he points at the cross-section on the wall*] gets bigger and smaller
As_{23}	[*The solid*] ist nicht ein Zylinder	[*The solid*] is not a cylinder

Table 8.11: Coding of elements in Lukas' refutation (see Figure 8.46)

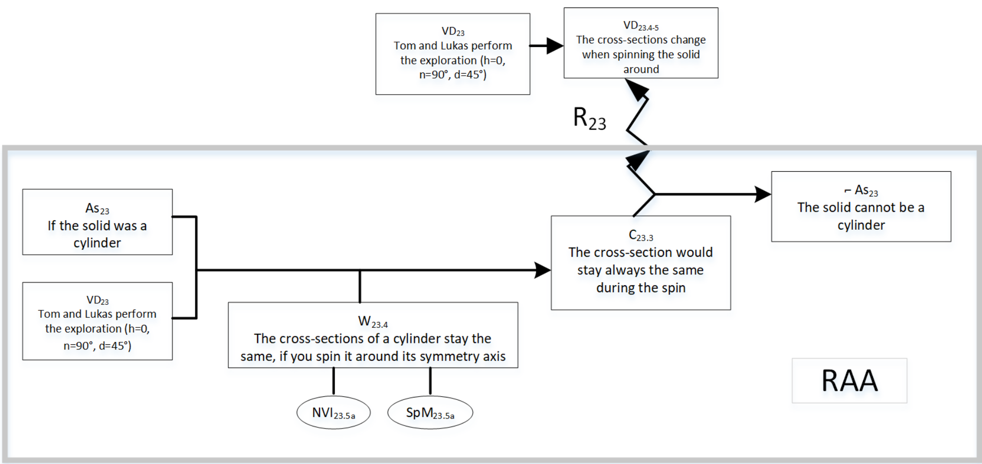


Figure 8.46: Lukas' refutation of the cylinder (see Transcript 8.14 and Table 8.11)

In this part of his presentation, Lukas excludes a solid (cylinder) by the process of *Reductio ad absurdum.* His argument starts with the assumption that the solid is a cylinder (As_{23}) and he concludes that, if this is so, then during the spin-exploration in case (h=0, n=90°, d) the cross-section should always remain the same as the one seen in position (h=0, n=90°, d=45°) ($C_{23.3}$). Lukas uses a warrant in order to link his initial cylinder assumption ($As_{23.3}$) with his conclusion ($C_{23.3}$). This warrant says "because in the case of a cylinder (...) the cross-section would (...) always remain the same, if you rotated it [*the solid*] around its own axis" ($W_{23.4}$).

Lukas' conclusion is tested when he asks Tom to move the spin-slider and it is contradicted (R_{23}): the visual data $VD_{23.4\text{-}5}$ show that when the spin-slider (d-slider) is dragged up and down, the cross-section changes (compare Figures 8.45 and 8.47). More precisely, the cross-section transforms from a square to a longer rectangle (e.g.

Figure 8.47) and then it narrows again slowly until it becomes a square again, and so on. This contradiction leads to the negation of the initial assumption As_{23} by *Reductio ad absurdum* (RAA).

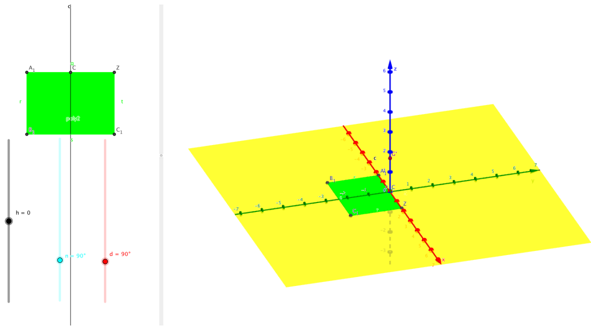


Figure 8.47: Position (h=0, n=90°, d=90°)

Interpretation

Figure 8.46 shows that both *NI-visualization* and *Sp-manipulation* take place in this refutation ($NIV_{23.5a}$ and $SpM_{23.5a}$). I would now like to discuss the role of each one of them in the refutation.

In order to move from his assumption ($As_{23.3}$) that the solid is a cylinder to the conclusion ($C_{23.3}$) that the cross-section should not change when the spin-slider (d-slider) is dragged up and down, Lukas uses a warrant ($W_{23.4}$). This warrant says that when a cylinder is rotated around "its own axis" the cross-section should stay the same. By "its own axis", Lukas refers to the line that goes through the centers of the cylinder's bases, something that is apparent from his gestures during the description of rotation of the assumed cylinder. Therefore, from the Lukas's gestures and verbal descriptions of the movement of the invisible solid in the case (h=0, n=90°, d), I infer that he can imagine the exact way that the invisible solid moves during the spin-rotation. The gestures are an indicator of *Sp-manipulation* ($SpM_{23.5a}$). *NI-visualization* ($NIV_{23.5a}$) is indicated here by two processes: Perceiving the cross-sections as two-dimensional figural units of the cylinder (*dimensional deconstruction*) and relating the properties of the cylinder with those of its cross-sections (when the cylinder is spun around its symmetry axis, the cross-sections do not change shape).

Therefore, it is here the *synergy* of *NI-visualization* and *Sp-manipulation* that supports Lukas' warrant and contributes to the shaping of his refutation by *Reduction ad absurdum*. More specifically, this *synergy* fulfills a double role. From the perspective of the whole argumentation, it takes place inside a refuting argument, thus playing the role of supporting a refutation (Role 4) At the same time, its function in the argument is that of a backing, since it supports the warrant $W_{23.4}$ that is used (Role 5, see next section).

8.4.3 *Sp-manipulation* supporting a refutation

Until now I have presented examples of refutations in students' argumentations, in which *NI-visualization* supported the refutation either alone (see 8.4.1) or in *synergy* with *Sp-manipulation* (see 8.4.2). Nevertheless, I met no case in my data in which *Sp-manipulation* operates alone in the role of supporting a simple contradiction or a refutation by *Reductio ad absurdum* (Role 4). I consider this to be a particularly interesting phenomenon, especially if we consider that this does not happen with any other role (Roles 1, 2, 3 or 5 as shown later).

It seems that one might be able to create a hypothesis or a claim (Role 1, see 8.1.3), draw a conclusion (Role 2, see 8.2.3), explain a visual datum that emerges during the explorations (Role 3, see 8.3.3) or back a warrant (Role 5, see 8.5.3), without necessarily referring to the geometric properties of the objects with which one is dealing with. But it is not easy to refute a statement without them. This may be so for several reasons.

My belief is that the main reason lies in the design of the tasks. In the tasks used in the present research the solids that the students have to work with are not visible to them. This means, that in cases in which, for example, a student disagrees with a conclusion, a hypothesis or a claim, they cannot simply refer to the figure of the solid or to its visible characteristics, in order to support their argument. On the contrary, they will need to access the solid through its cross-sections and their properties. And then also project these properties to the solid itself. This is the only way to argue about the solid, its form and its properties. Consequently, being able to construct the mental image of a solid and manipulate it in their minds (*Sp-manipulation*), would not be enough in order to verbally communicate their reasoning to others. For this the students need the geometric properties of the solid, as well as properties of its lower dimensional figural units, which means that they need to be able to visualize the geometric objects in a non-iconic way (*NI-visualization*).

8.4.4 Epilogue

In this section I have presented examples in which *Sp-manipulation* and *NI-visualization* support the refutation of a statement. Table 8.9, at the beginning of the section, shows the *modes* in which *Sp-manipulation* and *NI-visualization* appear (first column). As I also mention in 8.4.3, in the examples of students' work presented in this section, we have seen only two *modes* (out of the three different *modes* in which *Sp-manipulation* and *NI-visualization* appear in other *roles*): *NI-visualization* alone (subsection 8.4.1) or both together in *synergy* (8.4.2). The *mode* of *Sp-manipulation* alone (8.4.3) does not appear in this *role*. I argue that the task-design may have had a significant impact on this phenomenon. The fact that the solids under exploration are invisible in these tasks, calls for the use of properties in order to build an argument, especially when this argument is a refutation that aims at rejecting an already established a statement, such as a conclusion. An already established statement in the argumentation cannot be rejected without a coherent explanation. In a mathematics class, this means that the refutation must be based on the geometric properties that underpin it.

Table 8.9, also shows the two types of refutation that appear in both *modes* (second column). Those types are: *simple contradiction* or more complex refutation by *Reductio ad absurdum* (RAA). Furthermore, it shows the functions of *Sp-manipulation* and *NI-visualization* (third column) in each type of refutation for each *mode*. As can be seen in the table, when the refutation is a *simple contradiction* (see 3rd and 5th row in Table 8.9), the function of *NI-visualization* and

Sp-manipulation is "contradiction" in both *modes* (see NIV_{18} in 8.4.1, and *synergy* of $NIV_{15\text{-}17}$ and $SpM_{15\text{-}17}$ in 8.4.2). This happens because in *simple contradictions* a conclusion or a hypothesis is rejected by a single statement, a visual datum or an one-step argument. Therefore, the function of *NI-visualization* and *Sp-manipulation* in the argumentation is that of a contradiction that rejects a previously established statement. In contrast, when the refutation emerges by *Reductio ad absurdum* (RAA), the function of *NI-visualization* and *Sp-manipulation* is that of a "warrant" or a "backing" (see NIV_{26} functioning as a warrant in 8.4.1, and *synergy* of $NIV_{23.5a}$ and $SpM_{23.5a}$ functioning as a backing in 8.4.2). RAA is a multi-step argument. Therefore, the use of *NI-visualization* and/or *Sp-manipulation* appears in a specific place in the argument with a specific function (here, warrant or backing), like every other element of the argument. This results in *NI-visualization* and/or *Sp-manipulation* playing more than one role in the argumentation, when they take place in RAA. Apart from the role of supporting a refutation (Role 4), they may also fulfill the role of supporting the creation of a conclusion (Role 2) by functioning as a warrant (see NIV_{26} in 8.4.1) or the role of supporting a backing (Role 5) by functioning in the RAA argument as a backing (see the *synergy* of $NIV_{23.5a}$ and $SpM_{23.5a}$ in 8.4.2).

The fourth column of Table 8.9, shows the *indicators* of *NI-visualization* and *Sp-manipulation* for each *mode* and type of refutation. As in every other *role* (Roles 1, 2, 3 and 5 in sections 8.1, 8.2, 8.2 and 8.5, respectively), *NI-visualization* and *Sp-manipulation* are indicated by specific actions or processes that the students follow. As shown in Table 8.9, *NI-visualization* is indicated by the performance of *dimensional deconstruction* and relating properties of geometric objects of different dimensions (see NIV_{26} and NIV_{18} in 8.4.1, and $NIV_{15\text{-}17}$ and $NIV_{23.3}$ in 8.5.2), and transitioning from the solid to lower dimension figural units' different dimensions (see NIV_{26} in 8.4.1). The common characteristic of all the aforementioned indicators is that they are all processes, which are based on the use of geometric properties. They also demand that one has (and can use) the knowledge of the structure of geometric objects, as well as its properties.

The indicators for *Sp-manipulation* are also varied, but contrary to those of *NI-visualization* they are not based on properties. Indicators of *Sp-manipulation* include (see also Table 8.9): using gestures to describe the orientation or the movement of the solid (see $SpM_{15\text{-}17}$ and $SpM_{23.5a}$ in 8.4.2), verbally describing the movement and the orientation of the solid in space (see $SpM_{23.5a}$ in 8.4.2), and relating the movement of the slider to the movement or the orientation of the solid (see $SpM_{15\text{-}17}$ and $SpM_{23.5a}$ in 8.4.2).

In the next section, I present the last *role* of *Sp-manipulation* and *NI-visualization* that I identified in my data; that of backing a warrant.

8.5 Role 5: Backing a warrant

Until now, I have presented roles of *spatial manipulation* (*Sp-manipulation*) and *non-iconic visualization* (*NI-visualization*), in which they mostly function in argumentations as warrants (in Roles 1, 2, 3 and 4) or as parts of a refutation (in Role 4). In the fifth role, as its name already reveals, *Sp-manipulation* and *NI-visualization* always function in the arguments the same way as a backing would. As I mention in the introduction of the chapter, this happens when the students state a warrant explicitly, connecting a datum with a conclusion, a hypothesis or a claim. If this warrant is supported by the student's *NI-visualization* or *Sp-manipulation* of the solid, then I consider *NI-visualization* or *Sp-manipulation* to function as backings. The role of *Sp-manipulation* and *NI-visualization* is then to support the warrant, exactly as a mathematical statement-backing (e.g. a geometric property, a theorem etc.) would do.

Table 8.12 shows the different *modes* in which *NI-visualization* and *Sp-manipulation* play the fifth role (each of them alone or in *synergy*) and the indicators that hint to their participation in students' argumentation.

Role 5: Backing a warrant	
Mode	**Indicators**
NI-visualization	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Transitioning from lower dimension figural units of the solid to the solid
Synergy of *NI-visualization* and *Sp-manipulation*	• Performing *dimensional deconstruction* • Referring to relations between properties of objects of different dimensions • Transitioning from lower dimension figural units of the solid to the solid • Describing verbally the solid's orientation • Relating the movement of a slider with the movement of the solid
Sp-manipulation	• Using gestures • Describing verbally the solid's movement • Relating the movement of a slider with the movement of the solid

Table 8.12: Role 5 of *NI-visualization* and *Sp-manipulation* – Backing a warrant

8.5.1 *NI-visualization* backing a warrant

In this subsection I present an example from the classroom discussion about the task of the invisible sphere (Task 2)[20], in which Niko initiates the discussion by arguing that the invisible solid is a sphere (see Transcript 8.14). Below, I start with the description of the episode, moving then to the presentation of the reconstruction of the argumentation and finally offering my interpretation of what has happened.

[20] See the global argumentation structure for Episode CD2 in Appendix E8.

Episode description

The episode starts with Niko presenting his opinion about the form of the invisible solid. He says that he mainly explored the tilt (n-slider) and rotation (d-slider) of the solid. Then he says that in both cases the solid rotated around itself and the cross-section remained a circle (utterance 2). He also adds "a sphere consists of many circles, which together produce a sphere" (utterance 6). Finally, he says that since the cross-section is always the same circle both during the tilting and the rotation of the solid, this means that the solid "was definitely a sphere" (utterance 6).

Utterance	Original German transcript	English translation
2	Niko: Jetzt bei der Kugel (unverständlich)? Em, also ich hab das vor allem mit der Neigung oder der Drehung, also es [*the solid*] hat sich halt immer um sich selbst gedreht, und da der Schnittpunkt immer noch ein Kreis war, war eigentlich klar- #01:16:05-5#	Niko: Now, in the case of the sphere (incomprehensible)? Em, so I mainly did it with the tilt or the rotation, so it [*the solid*] always rotated around itself, and since the point of intersection remained a circle, it was actually clear-
3	Frau Karl: Der Schnittpunkt? #01:16:07-0#	Frau Karl: The point of intersection?
4	Niko: Die Schnittfläche meinte ich #01:16:08-9#	Niko: I mean the cross-section
5	Frau Karl: mhm (bejahend) #01:16:07-4#	Frau Karl: mhm (affirmative)
6	Niko: Ja das [*the cross-section*] ist nur ein Kreis geblieben und eine Kugel besteht ja aus ganz vielen Kreisen, die halt zusammen eine Kugel ergeben. Und weil dann, dann hab ich das halt immer, wenn es um sich dreht immer einen Kreis sieht, war das dann mit der Form, bei der Neigung und Drehung, war das auf jeden Fall eine Kugel #01:16:25-6#	Niko: Yes, that [*the cross-section*] is just a circle and a sphere consists of many circles, which together produce a sphere. And because then, I always have that when it turns around you always see a circle, this was the shape during the tilt and the rotation, it was definitely a sphere

Transcript 8.14: Niko's contribution to the classroom discussion on Task 2 (invisible sphere)

Reconstruction of the argumentation

Figure 8.48 illustrates the reconstruction of Niko's argumentation. In Table 8.13 I have listed all the elements of this argumentation using Niko's words from Transcript 8.14. For a better understanding of the argumentation structure, please read the text that follows, and also use Figure 8.48 and Table 8.13.

The argumentation starts with the visual data VD_2, to which Niko refers when he says that he mainly explored the tilt and the rotation of the solid. From this visual data follow two conclusions:

- C_2: it [*the solid*] always rotated around itself
- $C_{4\text{-}6}$: the cross-section remained a circle

These conclusions are then used as data which combined, lead to Niko's final conclusion C_6, that the solid is "definitely a sphere". The explicit warrant W_6, stated by Niko, says "a sphere consists of many circles, which together produce a sphere" and links the two data (C_2/D and $C_{4\text{-}6}$/D) with the final conclusion C_6.

Codes (Figure 8.48 and Transcript 8.14)	German original	English translation
VD_2	ich hab das vor allem mit der Neigung oder der Drehung	I mainly did it with the tilt or the rotation
C_2/D	es [*the solid*] hat sich halt immer um sich selbst gedreht	it [*the solid*] always rotated around itself
$C_{4\text{-}6}/D$	da die Schnittfläche immer noch ein Kreis war	the cross-section remained a circle
W_6	eine Kugel besteht ja aus ganz vielen Kreisen, die halt zusammen eine Kugel ergeben	a sphere consists of many circles, which together produce a sphere
C_6	war das auf jeden Fall eine Kugel	it was definitely a sphere

Table 8.13: Coding of elements in Niko's argumentation (see Figure 8.48)

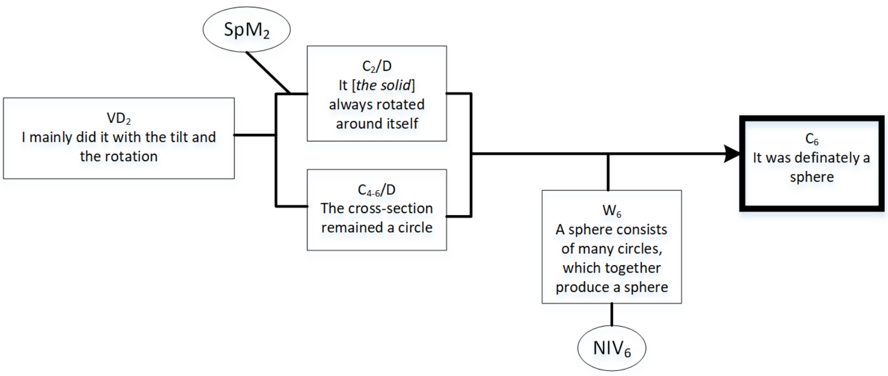


Figure 8.48: Niko's argumentation about the invisible solid being a sphere in Task 2 (see coding in Table 8.13)

Interpretation

In the argumentation structure featured in Figure 8.48, there is a case of *NI-visualization* (NIV_6) that supports the warrant W_6. This will be my focus here. Nevertheless, before I discuss the role and function of NIV_6, I offer my interpretation of the previous parts of Niko's argumentation as well, including the role of *Sp-manipulation* SpM.

As illustrated in Figure 8.48, in his argument, Niko starts from the visual data he perceived during the tilt- and rotation-explorations (VD_2) in his preceding pair-work. Based on those observations he draws two conclusions (C_2 and $C_{4\text{-}6}$). In C_2, Niko says that when he moves the n-slider (tilt) and the d-slider (spin) "it [*the solid*] always turns around itself" [21]. So, in C_2 Niko draws a conclusion about the movement of the solid, based on what happened when he moved the tilt (n) and spin (d) sliders. Since the solid is not visible, Niko's verbal descriptions of the solid's movements indicate that he is able to imagine its movement in space, without seeing it. This is an indicator that Niko employs his *Sp-manipulation*, in order to draw his conclusion. In other words, SpM_2 plays the Role 2 in Niko's argumentation.

Next, Niko also says that the cross-sections of the solid remain always circular during the explorations ($C_{4\text{-}6}$). Using the two previous conclusions as data (C_2 is now used as D_2 and $C_{4\text{-}6}$ is now used as $D_{4\text{-}6}$), he then draws his final conclusion (C_6), that the solid is a sphere. Niko supports this step with a warrant (W_6), according to which "a sphere consists of many circles, which together produce a sphere". This warrant suggests that Niko perceives a sphere, as a geometric object that is constituted by its many two-dimensional circular figural units (an infinite number of circles of different sizes). The ability to apprehend a solid in this way, requires the knowledge of its figural units of lower dimension, as well as their properties. I argue that Niko's warrant is based on such knowledge, which is a process of *dimensional deconstruction* of the sphere to its circular figural units. As I have mentioned before, *dimensional deconstruction* is an indicator of the use of *NI-visualization.* Therefore, I argue that it is Niko's *NI-visualization* that supports his warrant W_6. In other words, NIV_6 functions here as a backing for warrant W_6.

[21] The word for solid in German, is of masculine gender (der Körper), which Niko does not use when he talks about the solid, rather he refers to it with a neutral German article ("es" in utterance 2) instead of the masculine article. Hence, by "it" Niko refers here to the invisible solid. I support my assumption on the fact that the whole time that Niko talks, he uses neutral articles ("das" or "es") to refer to the solid, every time he does not mention the word "solid" (Körper) after the article.

8.5.2 *Synergy* of *NI-visualization* and *Sp-manipulation* backing a warrant

Here, I use another part of the same episode as that in section 8.2.2 to describe how the *synergy* of *NI-visualization* and *Sp-manipulation* can play the role of a backing for a warrant (Role 5) in students' argumentation. I focus on Tom and Lukas' first argumentation stream (see Figure 8.51) from their pair-work on the invisible cone task (Task 3A)[22] and I discuss the roles of the *synergies* of NIV_6 with SpM_6, and NIV_9 with SpM_9 (see Figure 8.51).

Episode description

As I mention above, this excerpt (see Transcript 8.15) takes place at the very beginning of Tom and Lukas' pair-work on the task of the invisible cone. The students start with the height-exploration (h, n=0°, d=0°). Tom drags the height-slider above and below zero. As soon as they start their exploration Lukas says that the solid is a sphere, while Tom says that it is a cone. Tom also provides an explanation for his suggestion (utterance 5). He says that the solid is a cone "because it disappears afterwards-". Lukas accepts Tom's suggestion that the solid is a cone and then the two students discuss the orientation of the cone at its initial position (h=0, n=0°, d=0°). Lukas says that the cone "goes downwards, because it goes to the negative area" (utterance 6). Tom objects to Lukas' statement, saying that the cone is not inverted, rather it stands with its top point upwards (utterances 7-9).

Utterance	Original German transcript	English translation
1	Tom: Ok, n null, d null. h (..) ok. Ich glaub ich weiß. #00:04:22-1#	Tom: Ok, n zero, d zero. h (..) ok. I think I know.
2	Lukas: Kugel. #00:04:22-7#	Lukas: Sphere.
3	Tom: Kegel. #00:04:24-7#	Tom: Cone.
4	Lukas: ach so. #00:04:25-4#	Lukas: Oh so
5	Tom: Weil es [*the cross-section*] dann verschwindet. #00:04:27-3#	Tom: Because it [*the cross-section*] disappears afterwards-
6	Lukas: Ja. Ein Kegel, der nach unten geht [*he means an inverted cone*], aber nur, weil es geht in den negativen Bereich[*under the plane xOy*]. #00:04:31-4#	Lukas: Yes. A cone that goes downwards [*he means an inverted cone*], because it goes to the negative area [*under the plane xOy*].
7	Tom: Nein. Es ist ein Kegel [*he implies an upward cone*]. #00:04:33-0#	Tom: No. It is a cone [*he implies an upward cone*].
8	Lukas: Ach so. #00:04:33-9#	Lukas: Oh so
9	Tom: Guck mal. Der [*the cone*] steht nach oben. Wenn hier oben jetzt die Spitze ist [*Tom points at the z-axis, over the plane xOy, see Figure 8.49*] weißt du, es [*the cone*] geht hier immer weiter nach unten. #00:04:44-3#	Tom: Look. It [*the cone*] stands upwards. If the top is up here [*Tom points at the z-axis, over the plane xOy, see Figure 8.49*], you know, it [*the cone*] always goes further downward.

Transcript 8.15: Tom and Lukas on the height exploration (h, n=0°, d=0°) in Task 3A (invisible cone)

[22] See the global argumentation structure and the worksheet for Episode GR2TL-3A.I in Appendices E7 and G6, respectively.

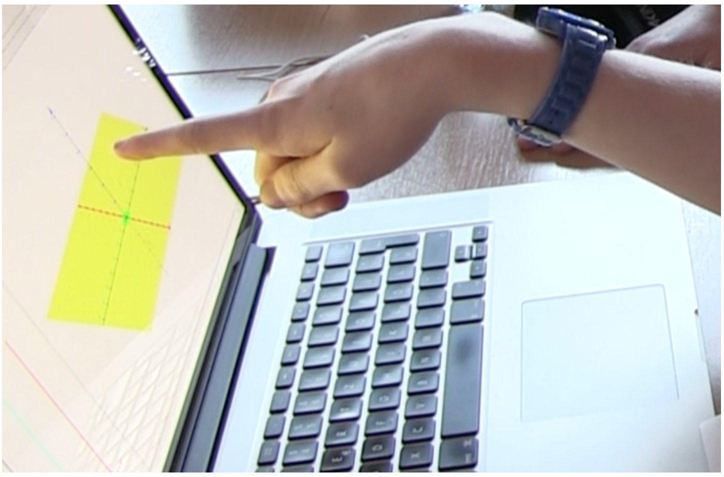


Figure 8.49: Tom points on the screen the initial position of the top of the cone

Reconstruction of the argumentation

Table 8.14 shows the codes of Tom and Lukas' argumentation and Figure 8.51 shows the reconstruction of this argumentation, using the elements of Table 8.14. The argumentation starts with the visual data VD_1 that emerge during the height-exploration of the case (h, n=0°, d=0°); Lukas drags the height-slider above and below h=0, starting from the position (h=0, n=0°, d=0°). What is visible on the screen is a circular cross-section at (h=0, n=0°, d=0°) that disappears as soon as the height-slider is lifted above zero, while when dragged under h=0, the circular cross-section diminishes converging to a single point.

In the argumentation stream (Figure 8.51) the visual data VD_1, are then followed by Lukas' hypothesis H_2 that the solid is a sphere and Tom's claim Cl_3 that the solid is a cone. I elaborate further on the characterization of the students' statements as "hypothesis" and "claim" in the part of the Interpretation of the argumentation stream that follows. Tom draws the claim Cl_3 (cone) from the visual data VD_1 (circular diminishing cross-section under h=0, no cross-sections over h=0), using the warrant W_5, that the solid is a cone "because it [*the cross-section*] disappears afterwards [*for h>0*]". The same warrant also refutes Lukas' sphere hypothesis H_2 (see $R_{1\text{-}5}$).

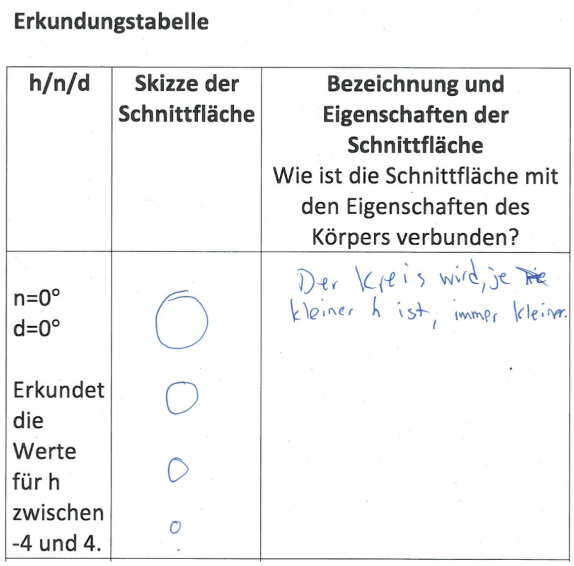
Before I continue with the description of the argumentation stream I would like to present a conclusion that the students draw at this point, and which they do not state verbally but rather note it on the Exploration Matrix of their worksheet (see conclusion C_{W1} in the worksheet of Task 3A, in Appendix G6) only after the completion of their height-exploration (right after utterance 9). I want to state this here, because it is implicitly used as a datum in the next step of the argumentation. Lukas writes down on the Exploration Matrix: "The circle [*circular cross-section*], as h gets smaller, gets smaller and smaller" (see Lukas' original text in German in Figure 8.50 and also the element C_{W1} in Table 8.14).

From the claim Cl_3 that the solid is a cone and the written conclusion C_{W1} that the cross-sections are circles that diminish as h reduces under zero, follows Lukas' conclusion C_6 that the cone "goes downwards" (inverted cone), based on the warrant W_6 that "it goes to the negative area". Then Tom refutes Lukas' conclusion C_6 (inverted cone) (see $R_{6\text{-}9}$). Tom's refuting argument starts with the datum $D_{9.1}$ "If the top [*of the cone*] is up here" (pointing on axis Oz, see Figure 8.49) and the implicit use of the conclusion C_{W1} "The circle [*circular cross-section*], as h gets smaller, gets smaller and smaller" (see Table 8.14 and Figure 8.50). From there follows the conclusion that the cone "stands upwards" (C_9). This conclusion is linked with $D_{9.1}$ and C_{W1}, with the warrant W_9 "it [*the cone*] always goes further downward".

Therefore, the final conclusion C_7 of the stream, is that the solid is an upward cone, which Tom has already stated in utterance 7, before he provides his detailed refutation.

Codes (Figure 8.51)	German original	English translation
VD_1	Ok, n null, d null. h (..)	observations from the exploration of the case (h, n=0°, d=0°)
H_2	Kugel	sphere
Cl_3	Kegel	cone
W_5	Weil es [*the cross-section*] dann verschwindet [*for h>0*]-	Because it [*the cross-section*] disappears afterwards [*for h>0*]-
VD: there are cross-sections only for h<0	-	-
C_{W1} (note in the Exploration Matrix on their worksheet, see Figure 8.50)	Der Kreis [*circular cross-section*] wird, je kleiner h ist, immer kleiner.	The circle [*circular cross-section*], as h gets smaller, gets smaller and smaller.
C_6	Ja. Ein Kegel, der nach unten geht [*Lukas means an inverted cone*]	A cone that goes downwards [*Lukas means an inverted cone*]
W_6	weil es geht in den negativen Bereich [*under xOy*]	because it goes to the negative area [*under xOy*]
$D_{9.1}$	Wenn hier oben jetzt die Spitze ist [*Tom points at the z-axis, over the plane xOy*]	If the top is up here [*Tom points at the z-axis, over the plane xOy*]
$W_{9.1\text{-}2}$	es [*the cone*] geht hier immer weiter nach unten	it [*the cone*] always goes further downward
C_9	Der [*the cone*] steht nach oben.	It [*the cone*] stands upwards.
C_7	Nein. Es ist ein Kegel [*Tom implies an upward cone*].	No. It is a cone [*Tom implies an upward cone*].

Table 8.14: Codes of elements in Tom and Lukas' argumentation stream (see Figure 8.51)



Erkundungstabelle

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Der Kreis wird, je ~~Ke~~ kleiner h ist, immer kleiner.

Figure 8.50: Tom and Lukas' C_{W1} conclusion as notes on their worksheet during the height-exploration (h, n=0°, d=0°)

Translation of Tom and Lukas' note in Figure 8.50:

"*The circle [circular cross-section], as h gets smaller, gets smaller and smaller*"

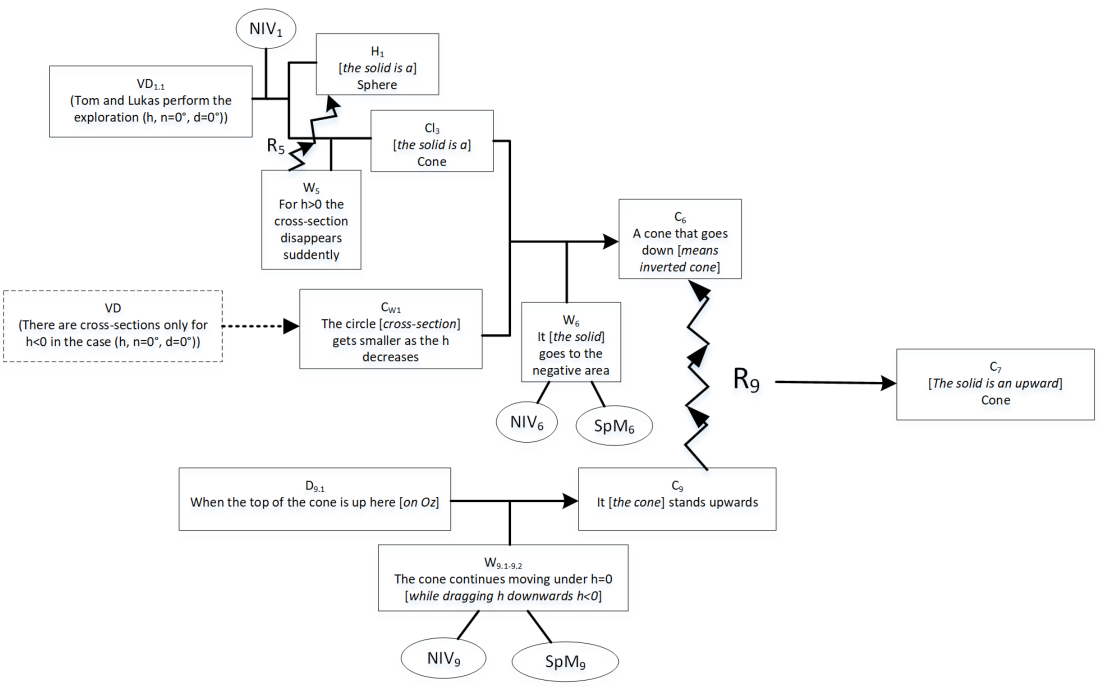


Figure 8.51: Tom and Lukas' argumentation stream

Interpretation

As Figure 8.51 shows, there are two cases in which the *synergy* of *NI-visualization* and *Sp-manipulation* takes place (NIV_6 with SpM_6, and NIV_9 with SpM_9). Here, I focus on the description of these two processes and their role in the argumentation stream. Before this though, I would like to present my interpretation of the previous parts of the stream, including the role of *NI-visualization* NIV_1.

In Chapter 5, I distinguish between a hypothesis and a claim, judging by the epistemic value attributed to them by the student (see subsection Table 5.3 in 5.4.2). If the students treat a supposition as just one possible case, then I call this supposition a "hypothesis". If, on the other hand, the student considers a supposition to be the most possible case, showing the intention to confirm its validity, then I call it a "claim". Based on their observations from the height-exploration (h, n=0°, d=0°), Lukas and Tom create their suppositions. Lukas *hypothesizes* that the solid is a sphere (H_2), without giving specific reasons for it. Tom *claims* that the solid is a cone (Cl_3), because the cross-sections disappear on the one side of the plane xOy (specifically over it). In the present example, Tom (contrary to Lukas) provides a warrant (W_5) for his supposition, indicating that his supposition is a claim. Lukas on the other hand, does not try to provide any explanation about his hypothesis and abandons it the moment he hears Tom's suggestion. Therefore, I consider Lukas' supposition to be merely a hypothesis and not a claim.

In the steps from the visual data VD_1 (circular-cross section that disappears over h=0 and diminishes under h=0) to the hypothesis H_2 (sphere) and the claim Cl_3 (cone), the two students create their suppositions about the form of the invisible solid, solely based on the observation of the cross-sections emerging during the height-exploration. The students perform a transition from the circular cross-sections, as two-dimensional figural units of the invisible solid (VD_1 and W_5), to the form of the solid (H_2 and Cl_3). This process of identifying sub-parts of the solid, which are of lower dimension, is a *dimensional deconstruction* of the invisible solid. *Dimensional deconstruction* constitutes an indicator for the use of *NI-visualization* (NIV_1) in this step. NIV_1 links the visual data VD_1 with hypothesis H_2 and claim Cl_3, thus playing the role of supporting the creation of a hypothesis or a claim (Role 1, see more in Section 8.1).

The warrant W_5 that Tom provides has one more function in the argumentation (besides linking VD_1 and Cl_3). Tom also uses it to refute Lukas' sphere-hypothesis (see refutation $R_{1\text{-}5}$ in Figure 8.51), although he does not provide a detailed explanation of how this statement (W_5) contradicts Lukas' hypothesis.

Another interesting point in this argumentation stream is the role of a written – but not verbally stated – conclusion that Tom and Lukas note on their worksheet (see Figure 8.50 and element C_{W1} in Table 8.14). Although not evident in the transcript, the students add further notes on their worksheet[23], which I use in order to fill possible gaps in their oral argumentation.

After Lukas has accepted Tom's claim that the solid is a cone, he concludes that this cone is inverted (C_6), because "it goes to the negative area" (W_6). It is not possible to say here with absolute certainty what exactly Lukas refers to with the word "it" ("es"

[23] Here I refer to the notes the students keep in the Exploration Matrix provided in their worksheet for each task. See the worksheets in Appendix B and how they are used in the tasks during students' pair-works, in Chapter 5, subsection 5.2.2.1.

in utterance 6). He could be referring to the height-slider that causes the cross-sections to appear when it is dragged towards the negative values, or he could be referring to the orientation or the movement of the cone downwards, under the plane xOy. I believe though that by "it" he refers to the solid, because he uses the same verb ("geht" which is translated as "goes") to describe the orientation of the solid in his previous statement (C_6: A cone that goes downwards [*Lukas means inverted cone*], see Table 8.14). Lukas connects the downward dragging of the height-slider (h<0) and the diminishing cross-sections (C_{W1}), with the orientation of the solid, drawing the conclusion that the cone is inverted (C_6).

Through a similar process, that refutes Lukas' conclusion (C_6) that the cone is inverted, Tom too makes a resembling connection. In refutation $R_{6\text{-}9}$, Tom points at a point on the positive side of the z-Axis (see Figure 8.49) and says that the top of the cone is on it ($D_{9.1}$). He then explains that because the cone always moves downwards, when the h-slider is dragged downwards under zero (he illustrates the dragging while talking) ($W_{9.1\text{-}2}$), this means that the cone stands upwards (with its base on plane xOy) (C_9). With this argument, Tom refutes Lukas' conclusion C_6, that the cone is inverted. Lukas accepts the refutation and together they conclude that the cone stands upwards (with its base on xOy and its top on Oz, as Tom had previously described, see utterance 7 and C_7).

Both Lukas' and Tom's processes towards their conclusions C_6 and C_9, respectively, are supported by the use of *NI-visualization* as well as *Sp-manipulation*. I would now like to explain this in more detail. The two students see the same cross-sections, and eventually that the solid is a cone (see utterances 3-6). So there is no disagreement between them regarding the form of the solid. From their description in C_{W1} I can infer that they both perceive the cone as a solid with circular cross-sections that diminish until they converge to a point (in case (h, n=0°, d=0°)). This suggests that their processes of *dimensional deconstruction* of the cone are the same, which means that it is not a difference in their processes of *NI-visualization* ($NIVz_6$ and $NIVz_9$) that leads to their contradictory conclusions (C_6 and C_9).

Their opinions diverge when it comes to the orientation of the cone. In both cases, the students relate the orientation of the solid with the dragging of the h-slider downwards for h<0 and with the emerging diminishing cross-section. They both agree that when the h-slider is dragged under h=0, then the solid is moved downwards under plane xOy (see W_6 and $W_{9.1\text{-}2}$). They also agree that the circular cross-sections diminish as h reduces under zero (C_{W1}). But, they infer two opposite conclusions (C_6 and C_9). Lukas moves the height-slider under h=0 where the diminishing circular cross-sections appear and from that he infers that the cone "goes downwards, because it goes to the negative area". I believe that here Lukas uses the verb "go" with two different meanings. The first time, "goes" is meant as "oriented" while the second time it means "moves". If my interpretation is correct, then in Utterance 6 Lukas says that the cone is oriented downwards (is inverted) (C_6), because it moves under the plane xOy (W_6) when the height-slider is dragged under h=0 and the circular cross-sections diminish (C_{W1}). On the contrary, Tom uses the datum that the cone has its top on the Oz ($D_{9.1}$) and the statement that when the height-slider is dragged under h=0, the circular cross-sections diminish (C_{W1}), to infer that the cone stands upwards (C_9), because it moves under the plane xOy when the h-slider is dragged under zero ($W_{9.1\text{-}2}$).

This contrast of Tom and Lukas' conclusions about the cone's orientation (C_6 and C_9), despite the use of C_{W1} in both cases and similar warrants (W_6 and $W_{9.1}$), indicates that Tom and Lukas' conclusions are the result of different *Sp-manipulation* processes (SpM_6 and SpM_9) that lead to their different conclusions. Lukas seems to imagine that the diminishing circular cross-sections of the cone, result from the downward movement of an inverted cone through the plane xOy, while Tom imagines the same movement for an upward-standing code.

As shown is this example, when *NI-visualization* and *Sp-manipulation* work in *synergy*, it is not necessary that an invalid conclusion (like Lukas' C_6) is the result of a mistake in both processes. For example, in Lukas' argument for conclusion C_6, we see that his NI-visualization (NIV_6) and the link of properties of the cone and its cross-sections is correct. His misconception regarding the orientation of the solid, arises in the process of *Sp-manipulation*.

In this example, each of the two *synergies* (NIV_6 with SpM_6, and NIV_9 with SpM_9) acts as a backing (support) for the warrant that is used in each case (W_6 and W_9) in order to link specific data with their corresponding conclusions (Cl_3 and C_{W1} (used as data in this argumentation step) to C_6, and $D_{9.1}$ and C_{W1} to C_9, see Figure 8.51). Therefore, their role in the argumentation is to back the warrants that are used, and their function in the argumentation stream is that of a backing (B), exactly as a statement-backing would function here.

8.5.3 *Sp-manipulation* backing a warrant

The example I use here is from Axel and Dave's pair-work on the task of the invisible cube (Task 3C)[24]. *Sp-manipulation* appears in the role of backing for a warrant in one of their arguments. Here, I present this argument (see Figure 8.52) and I show how *Sp-manipulation* functions in it.

Before this argument the students have rejected some of their previous hypotheses (H_3: pyramid, H_{14}: prism, see the argumentation structure of episode GR1AD-3C.I in Appendix E5) and their final claim is that the solid is a cuboid (Cl_{28}). After that, they move on with some further explorations. The argument in Figure 8.52 is a single argument that stands on its own in their argumentation and Axel builds it during the tilt-exploration of the *case* (h=0, n, d=0°).

Episode description

Transcript 8.16 shows the discussion of Axel and Dave. Dave and Axel begin the tilt-exploration from the initial position (h=0, n=0°, d=0°) of the solid. As soon as they move the tilt-slider (n-slider) over zero degrees, Axel says "Yes, of course, then you can see, if you tilt it very slightly, only half of it" (utterance 31, $D_{31.3}$ and $C_{31.2\text{-}3}$ in Figure 8.52). Then Dave wonders if what Axel says is logical (utterance 32), to which Axel responds that it makes sense "Because you lie it in one direction a little high and then the other, the other half disappears" (utterance 33, W_{33} in Figure 8.52). Then Axel and Dave discuss an example of this tilt-exploration. The say that if the tilt-value is n=5° then the left side of the solid sinks 5° under the xOy plane (utterances 34-38).

[24] See the global argumentation structure and the Worksheet for Episode GR1AD-3C.I in Appendices E5 and G4, respectively.

Utterance	Codes	Original German transcript	English translation
31	**$C_{31.2\text{-}3}$** $\underline{D_{31.3}}$	Axel: Ach so, ja. (. . .) Geh, nochmal auf Neigung null. (..) Ah, wenn dann direkt da, ja klar. [*They drag the n-slider above 0°, until 90°*]. Ja klar, **dann sieht man**, wenn man es ganz leicht neigt, **nur die Hälfte** [*he refers to the half part of the solid's base*]. #00:05:36-1#	Axel: Oh yes. (...) Go back to zero tilt. (..) Ah, if then right there, of course. [*They drag the n-slider above 0°, until 90°*]. Yes, of course, then you can see, **if you tilt it very slightly**, only half of it [*he refers to the half part of the solid's base*].
32		Dave: Echt? Ist das logisch? #00:05:37-8#	Dave: Really? Is this logical?
33	**W_{33}** **SpM_{33}**	Axel: Ja. **Weil du liegst es ja in die eine Richtung bisschen hoch und dann verschwindet das andere, die andere Hälfte ja weg** #00:05:44-7#	Axel: Yes. **Because you lie it in one direction a little high and then the other, the other half disappears**.
34		Dave: Ja, weil wir ja ganz unten sind, ne? #00:05:46-1#	Dave: Yes, because we're all down, right?
35		Axel: Ja #00:05:47-3#	Axel: Yes
36		Dave: Und dann geht es ja runter [*under the xOy plane*]. Oder? #00:05:50-2#	Dave: And then it goes down [*under the xOy plane*]. Right?
37		Axel: Ja, jetzt ist das so fünf Grad [*n=5°*] darunter. #00:05:52-4#	Axel: Yes, now it's five degrees [*n = 5°*] below.
38		Dave: Ja. #00:05:55-5#	Dave: Yes

Transcript 8.16: Axel and Dave on Task 3C (invisible cube)

Reconstruction of the argument

Figure 8.52 shows the reconstruction of the argument discussed in Transcript 8.16. I include the claim Cl_{28} as an implicit element (hence the dotted-box) in the argument reconstruction, because from this point on, the students work on the task with the idea in mind, that the solid is a cuboid.

So, the argument starts with the combination of the implicit use of the claim that the solid is a cuboid (Cl_{28} in a dotted-box) and the datum $D_{31.3}$ that the solid tilts when the n-slider is dragged above n=0°. These two data lead to the conclusion that "then you can see (...) only half of it" ($C_{31.2\text{-}3}$). The warrant W_{33} links the data to the conclusion and it says "Because you lie it in one direction a little high and then the other, the other half disappears".

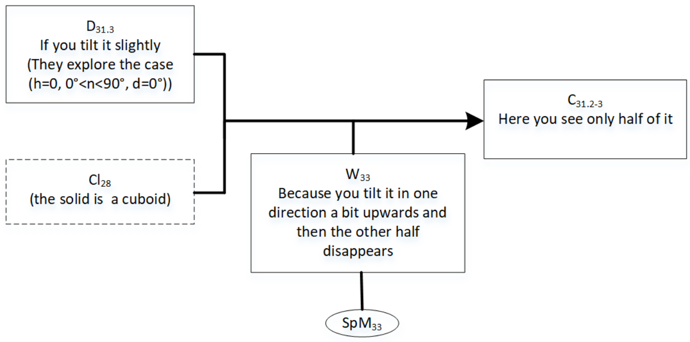


Figure 8.52: Axel's argument for the movement of the solid during the tilt-exploration of the *case* (h=0, n, d=0°)

Interpretation

In the argument presented in Figure 8.52, *Sp-manipulation* takes place. Here, I present my interpretation of what the students say, as well as of the role and function of SpM_{33} in the argument.

Dave and Axel begin the tilt-exploration from the initial position (h=0, n=0°, d=0°) of the solid. As soon as they move the tilt-slider (n-slider) a part of the cross-section disappears from the screen. Axel explains this by saying that if you tilt the solid even slightly (datum $D_{31.3}$) to the side, then you can only see half of the cross-section (conclusion $C_{31.2\text{-}3}$), because when you tilt the solid upwards on the one side, the other half (of the cross-section) disappears (warrant W_{33}). I base my interpretation of the students' ambiguous statements on the extra information provided by the example that follows their argument (see utterances 34-38). As I mention in the "Episode description", the two students discuss an example of the tilt-exploration. The say that if the tilt-value is n=5° then the left side of the solid sinks 5° under the xOy plane (see utterances 34-38).

This example is a verbal description of the relationship between the movement of the slider and the consequent movement of the solid in relation to plane xOy, and it is an indicator of the use of *Sp-manipulation*. SpM_{33} helps the students to make a transition from the movement of the tilt-slider to the movement of the solid in space. I argue that here only *Sp-manipulation* is performed (without *NI-visualization*) since there is no mention of properties regarding the geometric objects of the task (cross-sections or the solid). The role of SpM_{33} here, is to back the warrant W_{33}, and its function is similar to a statement-backing.

8.5.4 Epilogue

In this section I have presented examples in which *Sp-manipulation* and *NI-visualization* back (support) warrants in students' argumentations. Table 8.12, at the beginning of the section, shows the *modes* in which *Sp-manipulation* and *NI-visualization* appear (first column): each alone or both in *synergy*. As the name of this role of *Sp-manipulation* and *NI-visualization* reveals, their function in argumentation is that of a backing (B, see Chapter 2). That means that the two processes function in the argumentation similarly to the way a statement-backing would.

The second column of Table 8.12, shows the *indicators* of *Sp-manipulation* and *NI-visualization* for each *mode*. As in every other *role* (Roles 1 through 4 in sections 8.1, 8.2, 8.3 and 8.4, respectively), *Sp-manipulation* and *NI-visualization* are indicated by specific actions or processes that the students follow. As shown in Table 8.12, *NI-visualization* is indicated by: the performance of *dimensional deconstruction* (see NIV_6 in 8.5.1, and NIV_6 and NIV_9 in 8.5.2), relating properties of geometric objects of different dimensions (see NIV_6 in 8.5.1), and transitioning from lower dimension figural units of the solid to the solid itself (see NIV_6 in 8.5.1, and NIV_6 and NIV_9 in 8.5.2). These indicators of *NI-visualization* are common both when it operates alone, as well as when it operates in *synergy* with *Sp-manipulation*. The common characteristic of all those indicators is that they are all processes that are based on the use of geometric properties. They also demand that one has (and can use) the knowledge of the structure of geometric objects, as well as their properties.

The indicators for *Sp-manipulation* are also varied, but contrary to those of *NI-visualization* they are not based on properties. Indicators of *Sp-manipulation* include (see Table 8.12): verbally describing the movement of the solid in space and relating the movement of the slider to the movement or the orientation of the solid (for both indicators see SpM_6 and SpM_9 in 8.5.2, and SpM_{33} in 8.5.3), as well as using gestures to illustrate the orientation or the movement of the solid (see Tom's gestures in Figure 8.49 in SpM_9 in 8.5.2). The first two indicators appear both when *Sp-manipulation* operates alone as well as when it operates in *synergy* with *NI-visualization*. The last indicator (use of gestures) appears in this role, only when the *Sp-manipulation* operates alone (see fifth row of Table 8.12).

In the next section, I summarize the results of this chapter providing answers to the research questions stated at the beginning of the chapter.

8.6 Summary

As mentioned in the introduction of this chapter, in order to identify the ways in which visualization acts in the context of argumentation, I conducted an argumentation analysis of the collected data, based on Knipping's (2003a, 2003b, 2008) method of reconstructing argumentations[25]. The results of this analysis are argumentation structures that represent the students' oral argumentations (e.g. Figure 8.51). For the reconstruction of students' oral argumentations I used the videos from students' pair-works and from the classroom discussions. I also used the transcripts of these discussions and the notes the students kept on their worksheets, while they worked on the tasks. Through the study and detailed examination of these argumentation structures, I identified different roles of *non-iconic visualization* (*NI-visualization*) and *spatial manipulation* (*Sp-manipulation*) in students' argumentations in exploratory D-transitional tasks designed in a Dynamic Geometry Environment (DGE).

[25] See Chapter 5: Method # 2 in subsection 5.4.2, and Level 3 Analysis in subsection 5.4.4

In this section, I would like to bring together the results that characterize the *roles* of *non-iconic visualization* and *spatial manipulation* in students' argumentations. I do this in three stages, while answering three of my research sub-questions; the ones related to the subject of this chapter. Each of the following subsections is devoted to one of my research questions (RQ):

3.1 How do *non-iconic visualization* and *spatial manipulation* manifest themselves in students' argumentations? (Subsection 8.6.1)

3.2 What are the *roles* of *non-iconic visualization* and *spatial manipulation* in students' argumentations? (Subsection 8.6.2)

3.3 How does the specific *design* of the given tasks influence students' *non-iconic visualization* and *spatial manipulation*? (Subsection 8.6.3)

8.6.1 The indicators of *NI-visualization* and *Sp-manipulation* in students' argumentations

The first research question I would like to discuss, concerns the phenomena or the situations that point to the use of *non-iconic visualization* (*NI-visualization*) and *spatial manipulation* (*Sp-manipulation*) by students in their argumentations (see RQ3.1 above). In other words, I would like to discuss how these two processes manifest themselves in argumentation and how they can be "spotted" through specific *indicators*. These manifestations are the *indicators* of the use of *NI-visualization* and *Sp-manipulation* in argumentation. The *indicators* are like "signs". They are actions I observe in students' discussions that signify the use of either *NI-visualization* or *Sp-manipulation*.

The tables presented at the beginning of each section in this chapter[26] show the *modes* and the *indicators* of *NI-visualization* and *Sp-manipulation* for the corresponding role presented at that section. Bringing all those tables together and comparing their content, I came to the results presented in Table 8.15, which I explain next.

I call *modes*, the ways in which *NI- visualization* and *Sp-manipulation* appear in argumentation. They may appear together (in synergy) or each process may appear alone. As a result, there are three *modes* in which *non-iconic visualization* and *spatial manipulation* play a role in argumentation: *NI-visualization* alone, *Sp-manipulation* alone or *synergy* of *NI-visualization* and *Sp-manipulation* (see left column in Table 8.15). Each of these *modes* has its *indicators* (see right column in Table 8.15). Some of those *indicators* have already been identified and used in previous literature and I use them too when they appear in my data. Such an indicator is, for example, dimensional deconstruction (e.g. Mithalal & Balacheff, 2019). Other *indicators* are specific to the situation in this study and were identified during the argumentation analysis.

[26] See Table 8.2 in Section 8.1, Table 8.3 in Section 8.2, Table 8.7 in Section 8.3, Table 8.9 in Section 8.4, and Table 8.12 in Section 8.5.

Mode	**Indicators**
NI-visualization	• Performing *dimensional deconstruction* (recognizing lower-dimensional figural units) • Referring to relations between properties of objects of different dimensions
Sp-manipulation	• Using gestures • Describing verbally the solid's movement or/and its orientation • Describing verbally the movement of the solid and the consequential change of the cross-sections • Relating the movement of a slider with the movement or the orientation of the solid • Using metaphors
Synergy of NI-manipulation and Sp-manipulation	Combinations of indicators of *NI-visualization* and of *Sp-manipulation*

Table 8.15: Modes and indicators of NI-visualization and Sp-manipulation in students' argumentations

In the case of the *mode* of *NI-visualization* acting alone, one *indicator* is the performance of *dimensional deconstruction* of the invisible solid, or more precisely its "reconstruction" based on its two-dimensional figural units (the visible cross-sections). This is what happens, for example, when the students perceive the cross-sections as lower-dimension figural units of the solid and relate the properties of the cross-sections, with the corresponding properties of the solid itself. As I have mentioned before, *NI-visualization* is a prerequisite for someone to perform *dimensional deconstruction* (Mithalal & Balacheff, 2019). The specific transitions that the students perform in *dimensional deconstructions* are from the cross-section to the solid, and vice versa. An example of a transition from the cross-section to the solid is the recognition of the circular cross-sections as two-dimensional figural units of a sphere, and from that the creation of the claim that the invisible solid can therefore be a sphere (see NIV_{11} in subsection 8.1.1). An example of the reverse process is what Jacob does when he connects the property of the roundness of the side-surface of the cone to the curved line of its parabolic cross-section (see $\text{NIV}_{37.5}$ in subsection 8.3.1). Therefore, students' *verbal descriptions* of the *relations between the properties* of objects of different dimensions are also a valuable *indicator* of *NI-visualization.* During these descriptions, even more transitions are revealed that the students perform between different geometric objects, such as transitioning between figural units of the same dimension, or between figural units of different dimensions (e.g. moving from a property of the two-dimensional cross-section to a property of its edges).

When the students use *Sp-manipulation* exclusively, they do not connect the geometric objects (the solid, its cross-sections and its figural units) with each other based on their properties, at least not explicitly, rather they do so based on the way they imagine the solid moving or being oriented in space. *Sp-manipulation* is indicated by the use of *gestures*[27], the use of *metaphors*[28], or the use of *specific*

[27] e.g. Figure 8.21 (for SpM_{225}) in subsection 8.2.1

[28] e.g. SpM_{20} in subsection 8.2.3

examples[29] about the orientation or the movement of the solid. An *indicator* that is almost always present when *Sp-manipulation* is employed is the *verbal description* of the imagined movement or the orientation of the solid.

Students' use their *Sp-manipulation* in two ways: connecting the movement of the sliders with the movement or the orientation of the solid, or relating the shapes of the cross-section with the orientation (or the movement) of the solid. In the first case, one transitions from the movement of the sliders to the consequent movement of the solid[30]. In the second case, one may go in either direction; from the shape of the cross-section to the movement of the solid[31] or vice versa[32].

There are also times, when the students combine *NI-visualization* with *Sp-manipulation.* This is the *mode* of *synergy.* In *synergy*, the use of *NI-visualization* or *Sp-manipulation* can be spotted through the same indicators as when they are used exclusively. Since the *mode* of *synergy* represents the combined use of both *NI-visualization* and *Sp-manipulation*, it is only logical that the indicators of *synergy* have always been a combination of indicators of the other two *modes.* Each time a *synergy* takes place in argumentation, there is always a dimensional transition taking place, representing the *NI-visualization*, and an action related to the movement of the sliders, representing the *Sp-manipulation.*

8.6.2 The roles of students' *NI-visualization* and *Sp-manipulation* in their argumentations

The results from Level 3 Analysis of my research data, discussed in the previous five sections, show that *non-iconic visualization* (*NI-visualization*) and *spatial manipulation* (*Sp-manipulation*), go hand-in-hand with argumentation, having a significant influence on students' work. Therefore, I would now like to discuss my next research question:

3.2 What are the *roles* of students' *visualization* in their argumentations?

Through my analysis, I have identified five different *roles* that *NI-visualization* and *Sp-manipulation* play in students' argumentations. More specifically, *NI-visualization* and *Sp-manipulation* support students to:

- Create a hypothesis or a claim (Role 1, see Section 8.1)
- Draw a conclusion (Role 2, see Section 8.2)
- Explain visual data (Role 3, see Section 8.3)
- Create a refutation (Role 4, see Section 8.4)
- Back a warrant (Role 5, see Section 8.5)

As discussed previously, I identified three *modes* (ways) in which *NI-visualization* and *Sp-manipulation* may appear in each *role*[33]: each role is performed by *NI-visualization* alone, by *Sp-manipulation* alone, or by their *synergy* (both of them acting together). Another important feature of *NI-visualization and Sp-manipulation*

29 e.g. SpM_{33} in subsection 8.5.3
30 e.g. SpM_{207} in subsection 8.2.2
31 e.g. $SpM_{21.2}$ in subsection 8.1.3
32 e.g. SpM_{51} and SpMW, in subsection 8.1.3
33 See also Table 8.2 in Section 8.1, Table 8.3 in Section 8.2, Table 8.7 in Section 8.3, Table 8.9 in Section 8.4, and Table 8.12 in Section 8.5.

in argumentation is their *function.* By *function*, I mean what Toulmin (1958) means with the same word for the elements of an argument. In the case of my results, *non-iconic visualization* and *spatial manipulation* may *function* as a *warrant* or a *backing*. They may also function as an *element in a refutation-argument* (when the refutation is a Reductio ad absurdum) or just to *support a refutation* (when the refutation is a simple contradiction based on just one element, and not on an entire argument). Table 8.16 shows the *modes*, the *roles* and the *functions* of *NI-visualization and Sp-manipulation* in argumentation.

Roles	Role 1: Creating Hypothesis or Claim	Role 2: Drawing Conclusion	Role 3: Explaining a Visual Datum	Role 4: Creating Refutation	Role 5: Backing a Warrant
Modes	- NI-visualization - Synergy - Sp-manipulation	- NI-visualization - Synergy - Sp-manipulation	- NI-visualization - Synergy - Sp-manipulation	- NI-visualization - Synergy (no Sp-man. here)	- NI-visualization - Synergy - Sp-manipulation
Functions	Warrant	Warrant	Warrant	Warrant, Backing or Refutation	Backing

Table 8.16: The *roles*, *modes* and *functions* of *NI-visualization* and *Sp-manipulation* in students' argumentations

As shown in Table 8.16, in the *roles* of supporting the creation of a hypothesis/claim (Role 1), drawing a conclusion (Role 2) and explaining visual data (Role 3), *NI-visualization* and *Sp-manipulation* always function as *warrants*, while in the *role* of backing a warrant (Role 5) they are only identified to function as a *backing*. On the contrary, the *role* of creating a refutation (Role 4) has been identified in all types of *functions*[34].

<u>What do these *roles* reveal about the contribution of *NI-visualization* and *Sp-manipulation* in students' argumentations?</u>

Table 8.17 below shows how often *NI-visualization* and *Sp-manipulation* have played each *role* in students' argumentations. For example, lets see what the first column means in Role 1. It means that the students have used non-iconic visualization alone (mode NIV) nine times in pair-work episodes and four times in the classroom discussions. Using this table, I will elaborate further on the use of *NI-visualization* and *Sp-manipulation* in students' argumentations.

[34] See details about the functions for Role 4 in Table 8.9, Section 8.4

Roles	Role 1: Creating a Hypothesis or Claim			Role 2: Drawing a Conclusion			Role 3: Explaining a Visual Datum			Role 4: Creating a Refutation			Role 5: Backing a Warrant		
Modes	NIV	Syn	SpM	NIV	Syn	SpM	NIV	Syn	SpM	NIV	Syn	SpM	NIV	Syn	SpM
Pair work	9	1	1	17	3	8	2	1	3	6	0	0	0	2	2
Class. Disc.	4	0	0	13	3	12	2	4	16	0	3	0	5	1	2

Table 8.17: Frequency of non-iconic visualization and spatial manipulation in each role and mode

If we look at the number in the sub-columns of **Role 1** (supporting the creation of a hypothesis or a claim) in Table 8.17, we see that *NI-visualization* acting alone is the dominant *mode* in this *role* (see example in 8.1.1). That means, that when the students create a hypothesis or a claim, they mostly employ their *NI-visualization,* while *Sp-manipulation* seems to have a minimal role. The same phenomenon appears in **Role 4** (creating a refutation) as well. Here too, the students employ mostly their *NI-visualization* in order to refute a statement. Also, this is the only *role*, in which *Sp-manipulation* never appears alone (see subsections 8.4.3 and 8.6.3). That signifies, that the students focus on the geometric properties of the solid and its cross-sections in order to suppose what the form of the invisible solid may be or refute another one. In Role 4, the absence of *Sp-manipulation* acting alone is a shared characteristic in pair-work and in the classroom discussions.

The explanation behind the significantly more extensive use of *NI-visualization* (compared to *Sp-manipulation*) in Roles 1 and 4 lies in the task-design. The students are expected to reason about a three-dimensional solid, which they cannot see. Their only access to the solid is through its two-dimensional cross-sections with the plane xOy. This leads the students to perform connections between the cross-sections and the solid by using their properties and by determining the relationship of a specific cross-section with the solid as a two-dimensional figural unit of it (*dimensional deconstruction* of the solid). As I discussed in the previous subsection, these two processes (using geometric properties and *dimensional deconstruction*) are the two indicators of *NI-visualization.*

Contrary to Role 1 though, in Role 4 (creating a refutation) there is a difference between the *modes* that are used in pair-work and in the classroom discussions. During pair-work the students employ solely their *NI-visualization*[35], while during the classroom discussions they use both *NI-visualization* and *Sp-manipulation* combined (*synergy*)[36]. I would like to provide my explanation for this phenomenon: During their discussions with their classmates and the teacher, in order to explain their refutations, the students accompany their arguments that are based on

[35] See example in 8.4.1.
[36] See example in 8.4.2.

geometric properties[37], with gestures and verbal descriptions about the form and the movement of the solid[38]. As a result, the fact that *NI- visualization* is used every time that a refutation takes place (Role 4), indicates that the use of properties is necessary when a student wants to contradict a statement. The additional use of *Sp-manipulation* during this process in the classroom discussions, suggests that the properties alone are not enough when students want to present and explain their argumentation to others. The use of gestures and descriptions of the solid's movement has aided the students significantly in communicating successfully their arguments to their "audience".

According to my observations from the data analysis[39], *NI-visualization* and *Sp-manipulation* appear most frequently in students' argumentations when they draw a conclusion (**Role 2**). In the two previously presented *roles* (Role 1 and Role 4), the use of *NI-visualization* alone dominated. But in the process of drawing a conclusion, both *NI-visualization* and *Sp-manipulation* have a major involvement in students' argumentation. *NI-visualization* and *Sp-manipulation* may appear separately[40] or they may act in *synergy*[41]. This phenomenon appears both in students' pair-works, as well as in the classroom discussions.

My explanation for the fact that in Role 2 both processes are equally dominant, is that the design of the D-transitional tasks given to the students[42], is such that the three-dimensional geometric object under question is invisible and the students can only see its cross-sections. This means, that their only access to the solid is through its cross-section (two-dimensional subparts of it). To perform transitions from the cross-sections to the invisible solid, reason about it and render arguments that lead to inferential statements is not an easy task. It requires that the students make use of their *NI-visualization* by using the properties of both the solid and its lower dimension figural units in combination (process of *dimensional deconstruction*). At the same time, *Sp-manipulation* is a very valuable and helpful tool. Being able to create a mental image of the hidden solid in one's head and imagine its movement or orientation in respect to the plane of intersection xOy (as a consequence of the movement of the sliders), enhances students' argumentation since they now have an object to operate on when they reason, and to refer to when they articulate an argument. Furthermore, the use of gestures to describe the movement of the solid in space (indicator of *Sp-manipulation*) is a valuable tool in order to communicate one's thought to others (i.e. both in pair-work and in a classroom discussion). Therefore, it is the use of both *NI-visualization* and *Sp-manipulation* that enables the students to draw and justify new conclusions.

The third *role* that *NI-visualization* and *Sp-manipulation* may play in students' argumentation is to support the students in explaining data that they observe on the computer screen (see **Role 3** in Table 8.17). The students use these data in their argumentations as *visual data*[43]. Observing Table 8.17, we see that in most cases in

[37] For this indicator of NI-visualization see 8.6.2.
[38] For these indicators of Sp-manipulation see 8.6.2.
[39] See the numbers in the green-colored cells of Role 2 in Table 8.17.
[40] See examples in 8.2.1 and 8.2.3 respectively.
[41] See example in 8.2.2.
[42] See "Black-box" characteristic in 5.2.2.2, in Chapter 5.
[43] Here the word data is used in the sense of Toulmin, as an element of the argumentation that expresses a fact, see Chapter 2, subsection 2.2.3..

this *role* the students employ their *Sp-manipulation* (19 cases of *Sp-manipulation* as opposed to 5 cases for *synergy* and 4 cases for *NI-visualization* alone). Also, there is a more extensive use of *Sp-manipulation* in the classroom discussions, compared to the pair-works (see the numbers in the columns of *synergy* and *Sp-manipulation* for Role 3). Similarly to Role 4, the use of gestures and the verbal descriptions of the solid's movement and orientation, which are both indicators of *Sp-manipulation,* are two means that help students significantly in communicating their ideas to the rest of the classroom.

The last *role* of *NI-visualization* and *Sp-manipulation* shown in Table 8.17 is that of providing a backing for a warrant (**Role 5**). In this role, *NI-visualization* and *Sp-manipulation* support explicit warrants provided by the students; they "back" the warrants. In this role there is no dominant *mode.* Both *NI-visualization* and *Sp-manipulation* are used, either alone or in *synergy.*

Another interesting result, is a difference between the pair-works and the classroom discussions. In pair-works the students use warrants, which are most frequently based solely on their *Sp-manipulation*[44]. These warrants may, for example, be about the respective movement or orientation of the solid after a specific slider-dragging (change of height, tilt or rotation). There is only one exception of a pair-work episode, where *NI-visualization* is employed in *synergy* with *Sp-manipulation*[45]. On the contrary, when participating in classroom discussions students provide warrants that are robustly based on geometric properties and conditions, therefore on their *NI-visualization*, such as the symmetry of a cylinder in respect to its axis that passes through the centers of its bases.

This difference between pair-work and classroom discussions can be explained based on the social contract of the classroom and what is considered an acceptable and complete argumentation in a specific classroom. In Frau Karl's classroom, an acceptable argumentation had to be expressed and explained clearly and in detail. In cases where the students' argumentations were considered vague or incomplete, Frau Karl would ask questions that would provoke further discussion on the matter[46], until the argumentation was considered complete and the students' answer well justified, something that was a mutual decision of teacher and students. In most cases, this meant that the students would have to enrich their justifications with statements that were based on geometric relationships between the solid and its cross-sections.

In conclusion, what I can infer from the results I just presented is that the students employ mainly – but not exclusively – their *NI-visualization* while creating a hypothesis (Role 1), drawing a conclusion (Role 2), refuting a statement (Role 4), or providing support for a warrant (Role 5). This means that *NI-visualization* is students' first choice in cases of establishing truths (Roles 2 and 4), supporting a statement mathematically (Role 5) and creating a hypothesis regarding an invisible geometric object (Role 1). In the first two cases (establishing truths and supporting a statement), this probably sounds logical and expected, because when establishing truths and supporting statements in geometry, we need to make use of the geometric properties and relationships, which *NI-visualization* includes. But also in the case of

[44] See such a case in 8.5.3.
[45] See example in 8.5.2.
[46] Remember the Q?-elements in the patterns of argumentations in the classroom discussions (see Chapter 7, subsections 7.3.2 and 7.3.3).

creating hypotheses about the form of an invisible solid, the properties of the visible cross-sections are the only "clues" that students have at the beginning of their work on the tasks. As a result, *NI-visualization* is a valuable tool in this process too.

On the other hand, the students employ mostly – but not exclusively – their *Sp-manipulation* when they need to explain a phenomenon they observe on the GeoGebra screen during the explorations (Role 3). This happens also in the process of creating a hypothesis (Role 1) during the classroom discussions. In the first case (Role 3), the students observe what happens with the cross-sections on the screen and they try to explain it. This happens after they have first created one or more hypotheses for the form of the solid. So now that they have an idea of the solid's form – a mental image of it – they have one more tool in their disposal, besides using geometric properties. The use of gestures, the use of metaphors about objects from everyday-life and also relating the slider's movement with the movement of the solid, are all parts of the *spatial manipulation* that helps them to imagine how the solid moves in space, resulting to the creation of those specific cross-sections. All these also help the students in the process of mediating their ideas to their classmates when properties (and consequently NI-visualization) are not enough, or maybe too complicated. This, for example, is what happened when student presented their hypotheses to the rest of the classroom (Sp-manipulation in Role 1 in classroom discussions).

8.6.3 The influence of the design of D-transitional tasks on students' *NI-visualization* and *Sp-manipulation*

In this subsection I discuss the third research question posed in this chapter:

3.3 How does the specific *design* of *D-transitional* tasks influence students' *visualizations*?

Usually, the geometric tasks involving transitions between objects of different dimensions provide students with the higher dimension object asking them a question about a lower-dimensional subpart of it. This could be for example, providing the students with physical models of solids or 3D/2D drawings (two-dimensional drawings) of solids, and asking them to think of possible cross-sections. All the classic paper-and-pencil geometric problems that students are asked to solve, belong in this category too; e.g. to prove that the diagonals of an isosceles trapezium are equal. In these tasks, the main transition is moving from the higher-dimensional object (here the solid or the isosceles trapezium) to its lower-dimensional figural units (here surfaces, bases, cross-sections for the solid, or edges for the trapezium) or equal-dimensional figural units (for example, triangular subparts of the trapezium).

The intention of the tasks used in this research was to engage students in argumentation and challenge their non-iconic visualization. This intention was mirrored in the design of the tasks. I tried to create tasks, which would provoke the students to move beyond iconic visualization. When a student uses iconic visualization, the focus lies on the shape and its perceived features, and the justifications of the student are usually based on arguments such as "it looks like it from the drawing". My aim was to encourage the students to employ their *non-ionic visualization* and use the properties of the geometric objects in order to justify their ideas. In order to do so, I decided to minimize the chances of students relying solely on their visual perception, by making the main object of their "investigation" (the solid) invisible. I also chose to challenge them with tasks that require the opposite process than the one they usually follow. Instead of giving them the solid and asking them to identify its possible cross-sections, I provided them with the opportunity to access any possible cross-section that the solid could have with the plane xOy, and ask them to identify the form of the solid. In other words, the students were asked to argue about a three-dimensional geometric object, based on its lower-dimensional figural units, rather than the other way around.

Judging from the results presented in the previous sections, the characteristics of the tasks (see 5.2.2.2 in Chapter 5), seem to have achieved the main goal. The students engage in argumentations, building arguments that are supported by their *NI-visualization* via *dimensional deconstruction* and transitions that move both from the cross-sections and other lower-dimensional figural units of the solid to the solid itself, and vice versa. On the grounds that the solid is invisible, the students cannot simply rely on their visual perception or iconic visualization in order to justify their ideas. They need to provide arguments that are based on the only thing they have access to: the geometric properties of the objects.

For the same reason, the students could not refer to the consequences of its movements and orientation, if they did not employ their *Sp-manipulation.* The data show that in need of an object upon which to reason, the students create a mental representation of the solid (and its sub-parts), which they then manipulate mentally and argue on. This process supports their argumentation, especially in cases where *NI-visualization* either is not enough (then we have a *synergy*), or is simply too complicated or in any other way not the right tool for the students at that specific point (then *Sp-manipulation* operates alone).

In conclusion, the analysis of the data shows that the design of the tasks has had a significant impact on students' *non-iconic visualization* and *spatial manipulation*, and as a result on their argumentations as well.

8.7 Epilogue

The major contribution of this chapter is that it brings forth the inseparability of visualization from argumentation in geometry. In the literature, visualization has been seen as an important process in learning geometry (see Chapter 3). Nevertheless, I believe that its exact contribution to students' argumentation is a matter that still needs our attention. In my research, *non-iconic visualization* as well as *spatial manipulation* have proven to be not only valuable tools for students' argumentation, but more than that an inseparable part of it. Even in cases, in which students' *NI-visualization* or *Sp-manipulation* was flawed or led them to a wrong statement, this fact did not turn these processes into obstacles, but instead created chances for argumentative negotiation of the outcomes (either between the pair of students, or with the whole classroom). Therefore, I will agree here with Duval (1994, 2005), who claims that in order to learn how to argue mathematically in geometry, one has to engage his/her *NI-visualization* and engage in processes of dimensional deconstruction.

Furthermore, my data analysis methodology and the use of Knipping's (2003a, 2003b, 2008) argumentation analysis method (in Level 3 Analysis) allowed me to analyse not only verbally expressed arguments but also to consider arguments employing visual elements or other types of students' actions (e.g. gestures). In school, arguments are often supported by pictures and manipulatives. The mere verbal focus of traditional argumentation analysis may limit what can be said about students' argumentation. All this also enabled me to introduce *NI-visualization* and *Sp-manipulation* as elements in the reconstructed argumentation structures (elements NIV and SpM, respectively). This has been a decisive step in revealing the *roles* and *functions* of *NI-visualization* and *Sp-manipulation* in argumentation.

9 Conclusion

9.1 Overview

In the first chapter of this work, I began by presenting what I consider a *problematique* regarding the role of visualization in the teaching and learning of geometry. More precisely, I expressed my concern regarding the disconnection of visualization and argumentation in the teaching of geometry within the school context. My intention has been to address this *problematique* in the research field of didactics of mathematics. As I mention throughout the whole dissertation, and already in Chapter 1, visualization is more that visual perception. It is a cognitive process that allows us to gain insight into the structure and properties that define geometric objects. In Duval's (1999) words "visualization makes visible all that is not accessible to vision" (p.6).

The aim of my research has been to analyze students' argumentations in real classroom situations during geometry lesson and to examine the role of visualization in their argumentations. My hypothesis has been that visualization and argumentation in geometry constitute a *unity* of two processes, and should therefore be treated as such, both in teaching as well as in research.

For the purposes of my research, I wanted to create tasks that would be appropriate for the examination of my hypothesis. The objective has been to give students tasks that would "challenge" them to go beyond visual perception and move towards (non-iconic) visualization. Therefore, I designed D-transitional tasks[1] (in a Dynamic Geometry Environment) in which the geometric object under investigation is invisible to the students. In the tasks the students are asked to identify the invisible three-dimensional objects from their visible two-dimensional cross-sections. The rationale behind this design has been that in cases where visual perception fails the students turn towards visualization and towards the use of geometric properties in order to build their arguments regarding the situation they examine. Therefore, with this task-design, I aimed to provide students with situations in which they would have the opportunity to employ their visualization, negotiate their ideas with their peers and engage in argumentation. This was important because it would allow me to examine how students' use visualization in their argumentation and what is the interplay between the two processes.

I also needed to choose the appropriate methodology for the design and the implementation of the research, as well as for the analysis of the data that would be collected. I decided to conduct two experimental lessons in real classroom situations.

[1] Geometric tasks involving transitions from two-dimensional to three-dimensional geometric objects (and vice versa). In such tasks, the correlation of properties between geometric objects of different dimensions is vividly present. The students are asked to recognize a three-dimensional geometric object by having access only to two-dimensional parts of it. See more in Chapter 5, subsection 5.2.2.2.

For the data analysis, I combined two methods of argumentation analysis (Knipping, 2003a, 2003b, 2008; Reid, 2002b) with Duval's (1998, 1999/2002) theory of visualization in geometry. The two methods of argumentation analysis are the method of *patterns of argumentation* (Reid, 2002b) and the method of *argumentation structures* (Knipping, 2003a, 2003b, 2008, see also Knipping & Reid, 2019).

The *patterns of argumentation* offered a coherent overview of the students' actions during their discussions (see Chapter 7). As a result it also provided me with a view of students' overall argumentation and helped me organize their work in step-by-step actions that they took from the beginning of their work until reaching their final conclusions. Furthermore, this method revealed the significance of using various social settings in students' argumentation (see section 7.6) and the influence that the task-design has in students' argumentations (section 7.5).

The *reconstructions of students' argumentation* helped me gain insight into the detailed structure of students' arguments and spot the roles that *non-iconic visualization* and *spatial manipulation* play in their argumentations (see Chapter 8). Duval's (1998, 1999/2002) approach to *visualization* and his description of it in conjunction with the notion of *dimensional deconstruction*, have been useful tools for revealing the *roles* of *non-iconic visualization* in argumentation. Observing the way in which *visualization* works in the frame of argumentation, through the analytical lens of the *argumentation structures* (Knipping, 2003a, 2003b, 2008), I was able to reveal the interconnected relationships and the inseparability of the two processes.

In the following sections, I describe the main results and contributions of my dissertation. Here, I name them briefly:

- I developed a methodology for the analysis of students' visualization in argumentation in geometry (subsection 5.4.2, Chapter 5)
- I created a method for the illustration of students' *non-iconic visualization* and *spatial manipulation* in argumentation structures (Level 3 Analysis (step e) in subsection 5.4.4, Chapter 5)
- I classified three types of *exploration strategies* employed by students while working on the D-transitional tasks (Chapter 6)
- I identified six types of *patterns of argumentation* identified (Chapter 7)
- I identified five different roles that *non-iconic visualization* and *spatial manipulation* play in students' argumentations in D-transitional geometric tasks: 1. Creating a Hypothesis/Claim, 2. Drawing a conclusion, 3. Explaining visual data, 4. Creating a refutation, and 5. Backing a warrant (Chapter 8)
- I found out that *non-iconic visualization* and *spatial manipulation* function in argumentation in three different ways: as "warrant", as "backing", or as an element in a refutation (Chapter 8)
- I showed that *non-iconic visualization* and *spatial manipulation*, are processes inseparable from students' argumentations (Chapter 8)
- I verified that the D-transitional tasks created for this research indeed promote the use of visualization and argumentation in geometry (Chapter 8)

In the next sections, I bring together the results of this work relating them to the already existing literature in the field, and I present the main contributions of my work to the already existing research (Sections 9.2 and 9.3). I also discuss some implications of my work for the teaching of geometry and suggest further steps for future research

(Section 9.4). The chapter closes with the final epilogue for this work (Section 9.5).

9.2 Main contributions of the present work

I consider the most important contribution of this work, to be the results regarding the interconnective relationship between visualization and argumentation in students' work, and the roles of visualization and spatial manipulation in their argumentations. Nevertheless, I do not present those contributions first, because I prefer to follow a structure similar to that of this book. Therefore, I begin with the methodological contribution, moving then to the results that emerged from the data analysis.

Methodological contribution

To find and use suitable methods for data analysis of research is vital in order to be able to answer the research questions one poses as thoroughly as possible. In this work, I needed argumentation analysis methods that would allow me to view students' argumentation as a process of their actions, as well as to be able to observe their detailed structure. The latter was particularly important, in order to be able to identify the contribution of each statement in the argumentation, and reveal which statements may originate from the use of visualization or spatial manipulation.

In the literature, I did not find a unified methodology that would fulfill all the above criteria. Therefore, as mentioned above, I decided to build my own methodology for my data analysis, combining two different argumentation analysis methods (Reid, 2002b; Knipping, 2003a, 2003b, 2008), and Duval's (1998, 1999/2002) theory of visualization in geometry. The tools I used for the analysis of my data are not new, but their combination into one methodology for the analysis of the roles of visualization in argumentation is.

The methodology I created for the analysis of my data has three levels (Chapter 5):

Level 1Analysis – Structure and summary of the episode

Level 2 Analysis – Pattern of argumentation

Level 3 Analysis – Argumentation structures and visualization

This methodology has been used in this work and has fulfilled its designed purpose. Each level of analysis contributed with results that I discuss in more detail in the paragraphs that follow. Here, I would like to focus on the two methodological results of the application of this methodology:

1. The *patterns of argumentation* are the adapted version of Reid's (2002b) *patterns of reasoning*, for the needs of the present research.

   Reid's (2002b) method can be used not only for the reconstruction and illustration of students' *reasoning* (which is an internal mental process), but also for their *argumentations* (which is an externalized process taking place in discussions)[2]. Reid (2002b) uses his method in order to identify and illustrate students' *patterns of reasoning* while working on mainly algebraic tasks. In this work, this method has also been successful in illustrating students' *patterns of argumentations* in the case of geometric tasks. In both cases, the elements of the *patterns* represent students' actions during the course of their work (claiming, drawing a conclusion etc.). With a slight adaptation in the terminology of the elements used for the individual elements of the *patterns*, Reid's method (2002b) can be used to model either the *reasoning* or the *argumentation* of the students. In Chapters 5 (see Table 5.3 in subsection 5.4.2) and 7 (see Table 7.1 in section 7.1) I present these adaptations that are necessary in order to apply the method that Reid (2002b) uses when analyzing reasoning, in order to analyze here students' argumentations.
2. The addition of elements NIV and SpM in *argumentation structures.*

   With Knipping's (2003a, 2003b, 2008) method, argumentations are reconstructed using Toulmin's (1958) functional perspective at the roles of statements within an argument. The result is *argumentation structures* that illustrate entire argumentations. Knipping's (ibid.) method has been used by other researchers as well (e.g. Potari & Psycharis, 2018; Cramer, 2018) for the reconstruction of mathematical argumentation. In my work, I used it for the same purpose, but I wanted to move beyond illustrating students' statements, to include *non-iconic visualization* and *spatial manipulation*, as elements of the argumentation. Therefore, in my analysis I created two more elements that included the argumentation structures: the element NIV represents the use of non-iconic visualization in an argument, and the element SpM represents the use of spatial manipulation in an argument, and consequently in the argumentation too. This later allowed me to identify the exact roles and functions of non-iconic visualization and spatial manipulation in students' argumentations.

Therefore, I would argue that my data analysis methodology is a useful tool for studies that aim to analyze students' argumentations and examine the contribution of visualization in the argumentation. Following, I discuss the contribution of each of the aforementioned analysis levels in more detail.

[2] Note on the difference between reasoning and argumentation in this work: Reasoning refers to a special kind of thinking process that takes place when a person is dealing with a mathematical task. It is considered to be the mental process of human thinking, in a mathematical context, taking place in the mind of the person who is reasoning. Argumentation, in my work, is understood and used as the process of expressing one's reasons to support or reject a statement or an opinion, through verbal articulation, gestures or actions. Argumentation may take place between two or more participants, or a single student may perform it when expressing reasons out loud. See more details in Chapter 2, Section 2.1 for reasoning and subsection 2.2.1 for argumentation.

Exploration strategies

Three types of strategies that students follow during their explorations when working on the tasks, were revealed through Level 1 Analysis of the data. I call them *exploration strategies.* The identification of the tree types of exploration strategies was based on two factors: whether or not there is *initiative*, on the part of students regarding the (h, n, d)-cases and positions they explored, and whether or not there is a specific *intention* that the students have behind the initiative. The three *exploration strategies* are: *free* exploration, *guided* exploration, and *structured* exploration[3].

Patterns of argumentation

Level 2 Analysis of the data revealed six types of patterns of argumentation (three in pair-work and three in classroom discussions)[4]. These patterns show the actions the students take while working on tasks. The observation of these patterns provided me with a view of students' overall argumentation, during each task, and helped me organize their work in the step-by-step actions they took from the beginning of their work until they reached their final conclusions. The results of this analysis have been very valuable in themselves, but also very useful for the next analysis level (Level 3).

With regard to the results themselves, the main contributions have been the following: firstly, looking at the patterns of both the pair-works, as well as the classroom discussions, has revealed the different challenges and opportunities that each of these settings of learning environment bring to the argumentation. In pair-work, students' patterns of argumentation are mainly oriented around exploring the situations (observing data/DO and creating hypotheses and claims/Hyp,Clai), taking initiatives and experimenting (DO and testing/Test), and drawing their conclusions (Conc, 1D). In argumentations taking place in classroom discussions the students collaborated, negotiated, exchanged ideas, made statements (Hyp, Clai, Conc, 1D), argued for or against statements (Conc, 1D, Contra), provided alternative hypotheses or solutions, and they helped each other explain or understand unexpected or surprising phenomena (e.g. how the pentagonal and hexagonal cross-sections of a cube occur[5]). As a result, the six patterns of argumentation show that different social settings provide different opportunities for students' argumentations. Therefore, I consider it very fruitful for students' learning, to provide them with multiple social settings in which they can work.

Level 2 Analysis also reveals the significant influence that the task-design has on students' patterns of argumentation. As discussed in Chapter 7 the use of the Exploration Matrix has been a decisive factor for the way the students organized their work, the exploration strategies they followed (guided exploration based on the Matrix) and consequently their patterns of argumentation as well (see Table 7.8). The patterns show that every time the students performed a guided exploration (aka using the Exploration Matrix) their pattern seems more organized, moving mainly from observing data (DO) to creating hypotheses (Hyp) and then drawing conclusions (Conc, 1D)[6]. On the other hand, when the students take initiative regarding the cases and positions of the solid they want to explore, the patterns are

3 See all the details in Chapter 6.
4 See Chapter 7. For a concise overview refer to Section 7.4.
5 For this, see Episode CD3C-TL in subsection 7.3.1, Chapter 7.
6 See patterns 1PW, 2PW, 1CD and 2CD in Table 7.8.

enriched with testing actions (Test) and longer processes of refuting statements (Contra)[7].

Another interesting point observed in the patterns, is how the classroom discussions differ when all students have worked on the same task before the discussions in comparison to when they have worked on different tasks. When all students had worked on the same task[8], the discussion that followed was based on structured explorations. Contrary to this, the discussions following after the students had worked on different tasks, were built around the presentation of the cases in the Exploration Matrix. In the latter case, the explorations from the Exploration Matrix had been so thoroughly presented by only two students that little to no discussion followed[9]. On the contrary, when the students had all worked on the same task, the discussion was more involving and polyphonic, with many students participating in it and the teacher stimulating the discussion. Consequently, in this setting many different opinions, arguments, and uses of spatial manipulation and non-iconic visualization were presented. I therefore suggest that in order to have richer and more engaging discussions, it may be preferable to let all students work on the same tasks beforehand.

I would now like to comment on the significance of the results of Level 2 Analysis for the next analysis level. Level 3 Analysis is the reconstruction of students' argumentation with Knipping's (2003a, 2003b, 2008) method. This is a more complex argumentation analysis method, which goes into detail in the functional structure of argumentation. Due to the fine details revealed through this method regarding students' argumentation, it has been very valuable to have first obtained a concise overview of students' argumentations through their patterns. Hence, the patterns of argumentation have also acted as a valuable preparatory step before delving into the plurality of information that characterizes argumentation structures.

The interplay between visualization and argumentation

Level 3 Analysis revealed the interconnective relationship between argumentation and the processes of non-iconic visualization (NI-visualization) and spatial manipulation (Sp-manipulation). More precisely, the analysis of the data led to the identification of five *roles* and three *functions* of *NI-visualization* and *Sp-manipulation* in students' argumentation[10]. These roles characterize and differentiate between various ways in which *NI-visualization* and *Sp-manipulation* act in students' argumentation, when the students work on D-transitional tasks designed in a Dynamic Geometry Environment (DGE) (see Chapter 8).

7 See patterns 3PW and 3CD in Table 7.8.

8 See Pattern 3CD – "Reverse debate" pattern in 7.3.3, Chapter 7. This pattern appears during the classroom discussion on Task 2-invisible sphere (episode CD2).

9 See Pattern 1CD in 7.3.1 and Pattern 2CD in 7.3.2, in Chapter 7.

10 See Chapter 8. Concise presentation in 8.6.2.

Those five roles are:

1. Creating a hypothesis or a claim
2. Drawing a conclusion
3. Explaining visual data
4. Creating a refutation
5. Backing a warrant

Furthermore, *NI-visualization* and *Sp-manipulation* have been observed to function in argumentation in three different ways: as *warrants* (in Roles 1, 2, 3, 4), as *backings* for warrants (in Roles 4 and 5) and as *refutations* (Role 5).

NI-visualization operates in argumentation mainly by allowing students to transit between spaces and objects of different dimensions (e.g. from a plane to 3D-space, and from cross-sections to the solid), as well as between their properties. The process of *dimensional deconstruction* has been the main indicator of students' *NI-visualization*. *Sp-manipulation* was indicated by students' verbal descriptions of the movement or the orientation of the solid in space, and by the use of gestures and metaphors. It mainly operated by helping students relate the movement of the sliders with the consequent movement of the solid. In many cases, *NI-visualization* and *Sp-manipulation* acted in *synergy* (collaboration), each fulfilling their purpose in the same *role*.

In my work, *NI-visualization* and *Sp-manipulation* have been facilitators of students' argumentations, and determining factors of the evolution of the argumentations during their work. The better the students could imagine the movement of the solid in space (*Sp-manipulation*), the more precise their mental image of the solid became with time. Also, as more properties were connected between the solid and its sub-parts (*NI-visualization*), the better and more precise their argumentation became as well.

In geometry, *argumentation* and *visualization* are not two processes that run parallel to each other never meeting, neither do they run in opposite directions. On the contrary, as my results have shown, *argumentation* and *visualization* are intertwined, connected and co-dependent processes. Therefore, I suggest that this is how they should also be considered and examined, both in research as well as in teaching: as two collaborating processes in constant interplay.

9.3 Further results – Reconstructing abductive arguments

This research has lead to an interesting result regarding the reconstruction of abductive arguments in the explorative situation, in which the students worked. In the present research, students were not asked to prove their conclusions deductively, rather to justify them as they think necessary based on the type of justification that would be accepted in their class as complete. The students started with the observation of some first data and the creation of their hypotheses and claims. In the argumentations that took place in this study, abduction has not been merely a step of the argumentation, as it happens in other cases in literature (e.g. Knipping, 2003a, 2003b; Cramer, 2018). Here, abduction functions as a process for the *exploration* of an unknown situation. It is a whole part of the argumentation, not merely a "backwards-reasoning" step of it, in order to *explain* a result. The students are exploring a situation, in order to discover something new, which will lead them to their conclusion (i.e. the form of the invisible solid). Therefore, the abductive arguments here are modelled going forwards, rather than backwards (see Chapter 3 for the theoretical details). Although this type of modelling abduction was not the focus of this work, it has nevertheless been a result of it. For more details, please refer to Papadaki, Reid and Knipping (2019).

9.4 Implications for teaching and further research

The research conducted in this work was designed to be implemented in real classroom settings for a very specific reason; I wished my work to be addressed to both the researchers of the field of didactics in mathematics, as well as to the "front-line" protagonists; the mathematics teachers and their students. I believe that research for the sake of research in our field is very interesting, but its real beauty and value lies within its applicability to real teaching situations.

My aim has been to observe the *role* of *visualization* in the process of *argumentation* in geometry, in a real-classroom situation, when students work on D-transitional tasks. Next, I discuss how some of the results of this work could be used in the school practice for the teaching of geometry and I also suggest some points for further research in the teaching and learning of geometry, related to visualization.

For the teaching of geometry

The use of *non-iconic visualization* and *spatial manipulation* in argumentation in the context of geometry can be a valuable tool in lessons, where the focus lies in teaching the properties of geometric objects and the relationships that connect them. As I mention in Chapter 8 (see 8.6.3), in most cases in research as well as in the teaching of geometry, when the students are given geometric tasks that involve transitions between objects of different dimensions, these tasks begin from the higher dimension object (e.g. a solid) asking the students questions about a lower dimension subpart of it (e.g. its possible cross-sections). The intention of the tasks used in this work, was to provide students with a new, out of the ordinary, situation they would need to explore. Through these tasks the students had the opportunity to develop their own *exploration strategies* for their solutions.

Moreover, the characteristics[11] of the tasks had a significant (and intended) influence on students' argumentations, both in the types of arguments they built as well as in the use of non-iconic visualization and spatial manipulation. The *black-box* characteristic, i.e. the fact that the solid in question was invisible, promoted *abductive* and *hypothetico-deductive* argumentation[12]. Also, the fact that the tasks were designed in a Dynamic Geometry Environment (DGE), allowed students to experiment with the situations and explore them to a greater extent. More precisely, students initially created hypotheses or claims about the form of the solid (abduction) and then proceeded to the refutation of some of them through further

[11] See subsection 5.2.2.2 in Chapter 5.

[12] See more details in the results described in Chapter 8 and also in Papadaki, Reid & Knipping (2019).

exploration and data observation (hypothetical deductions). The validation of a specific claim and the final conclusion followed only after the refutation of all the other stated hypotheses. The justification of the final conclusion was either based on the premise that no other hypothesis is possible, or by supporting it with arguments based on the connections of the properties of the cross-sections with those of the solid.

In most cases, the emergence of unexpected or *surprising* cross-sections[13], also made the students turn to the properties of the objects in order to make sense of the visual data they observed. The employment of dimensional deconstruction, an indicator of *non-iconic visualization*, as well as the use of their *spatial manipulation,* constitute vital tools in this process. The design of the tasks seems to have had a significant influence on students' use of *visualization*, and as a result on their argumentation as well.

I therefore believe that the use of tasks with these characteristics in the teaching of geometry can promote students' argumentation and their use of non-iconic visualization, enriching their learning in geometry. In the specific context examined here, which was learning about geometric objects of two and three dimensions, the specific tasks used in this work promote the creation of interconnections between objects of different dimensions (through dimensional deconstruction), thus helping the students create a network of interconnected objects and phenomena in geometry, instead of them being a collection of unrelated geometric objects and properties.

Suggestions for further research

In this study my hypothesis has been that in geometry, argumentation and visualization are two processes in continuous interplay and that therefore this interconnection should be taken into consideration when teaching geometry. This assumption is supported by Mithalal and Balacheff (2019), although their focus does not lie on the illustration of non-iconic visualization in argumentation reconstructions. Mithalal and Balacheff (2019), use Duval's (1998, 1999/2002) theory on visualization to examine how "students' drawing perception has to evolve, from Iconic Visualization to Non-Iconic Visualization" (p. 161). In their research they consider non-iconic visualization to be an essential process for mathematical proving. The results of their research show, that it is possible to design tasks that "provoke the need for intellectual proof" (p. 175). More precisely, when the iconic visualization was no longer reliable in order to solve the task, the students responded to this problem by turning to geometric properties of the three-dimensional figure and relations between its figural units. Therefore, Mithalal and Balacheff (2019) consider the use of non-iconic visualization by the students, a decisive step towards the learning of proving in geometry.

[13] See for example Episode CD3C-TL in subsection 7.3.1, Chapter 7.

As a type of argumentation, proving would also be an interesting context in which the *roles* and *functions* of non-iconic visualization could be examined. Therefore, a future research goal could be to observe students' proving processes and examine if the *roles* and *functions* of non-iconic visualization (and also of spatial manipulation) identified here, are also met in the context of proving.

9.5 Epilogue

The motivation for my research has been to access the core of the relationship between students' visualization and their argumentation, in a way similar to how Duval (1999/2002) describes visualization moving beyond vision and accessing the internal structure of geometric objects. With my study, I intended to make visible what remained unrevealed regarding the contribution of visualization in argumentation.

This study showed that in the context of geometry, argumentation and visualization are two processes that collaborate and are intertwined, with visualization contributing to the evolution of students' argumentation. I conclude this work in the hope that it will contribute to considering and employing visualization and argumentation together, as two inseparable processes in students' learning in geometry, both in the context of teaching and in future research in the field.

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Appendix

In the following pages I provide additional information that can be useful when reading the present work. This is the printed appendix of my dissertation. The list of its contents is provided in the next page. I also include an extended list with the contents of the digital appendix that accompanies the dissertation, where more details are provided, regarding the Levels 2 and 3 of the data analysis.

Contents of the Printed Appendix

E4. Episode GR1AD-3B.II
E5. Episode GR1AD-3C.I
E6. Episode GR2TL-2
E7. Episode GR2TL-3A.I
E8. Episode CD2
E9. Episode CD3A-AD
E10. Episode CD3B-JM
E11. Episode CD3C-TL

F. Table of Roles of non-iconic visualization and spatial manipulation with analysis information

G. Scanned worksheets from the pair-work of video-recorded students
G1. Episode GR1AD-2
G2. Episode GR1AD-3A.III
G3. Episode GR1AD-3B.II
G4. Episode GR1AD-3C.I
G5. Episode GR2TL-2
G6. Episode GR2TL-3A.I

Contents of the Digital Appendix

A. Coding of episodes and transcripts

Episode code	Meaning
GR1AD-2	Group 1 (GR1), students Axel and Dave (AD) work on Task 2 (-2)
GR1AD-3A.III	Group 1 (GR1), students Axel and Dave (AD) work on Task 3A (-3A), their third task (.III) on day 2 of the study
GR1AD-3B.II	Group 1 (GR1), students Axel and Dave (AD) work on Task 3B (-3B), their second task (.II) on day 2 of the study
GR1AD-3C.I	Group 1 (GR1), students Axel and Dave (AD) work on Task 3C (-3C), their first task (.I) on day 2 of the study
GR2TL-2	Group 2 (GR2), students Tom and Lukas (TL) work on Task 2 (-2)
GR2TL-3A.I	Group 2 (GR2), students Tom and Lukas (TL) work on Task 3A (-3A), their first task (.I) on day 2 of the study
CD2	Classroom Discussion (CD) on Task 2 (2)
CD3A-AD	Classroom Discussion (CD) on Task 3A (3A) Axel and Dave present the task (AD)
CD3B-JM	Classroom Discussion (CD) on Task 3B (3B) Jacob and Michael present the task (JM)
CD3C-TL	Classroom Discussion (CD) on Task 3C (3C) Tom and Lukas present the task (TL)

Transcript	Meaning
2	Each utterance is numbered in the transcripts used in the dissertation (e.g. utterance 2).
2.1 2.2	Each line of each utterance is also numbered in the transcripts in the digital appendix (e.g. utterance 2 has two lines, hence line 2.1 and line 2.2).
#00:03:07-6#	Minute in the video recording.
(unverständlich)	(incomprehensible), not possible to understand what the students say
[*Dave explores the case (h=-0,8, n, d=0°)*]	Explanatory/Descriptive notes of the author in square brackets [], written in *italics*.
(..)	2 seconds pause
(...)	3 seconds pause
B1 ist der Punkt,-	A dash is used to denote a disruption of the students' talk.

B. Worksheets of the tasks

B0

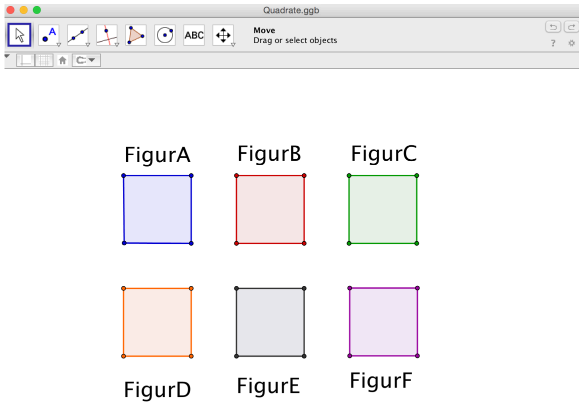
Freitag, den 10. Februar 2017

Arbeitsblatt

Unsere „Undercover“-Namen sind: ....
....

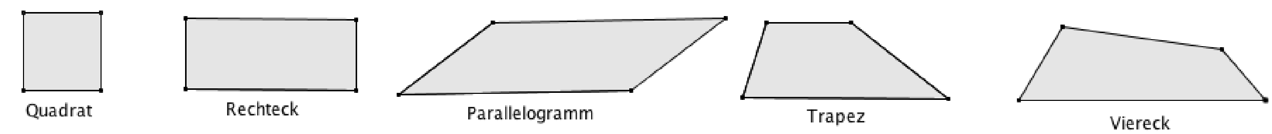
Untersuchung 1. Zeichnung vs Konstruktion

Öffnet die GeoGebra-Datei „Quadrate”. Die Abbildung unten zeigt sechs Quadrate (Figuren A bis F) - oder sehen sie etwa nur aus wie Quadrate?



a. Bewegt die Eckpunkte der einzelnen Quadrate mit der Maus und schreibt eure Beobachtungen auf.

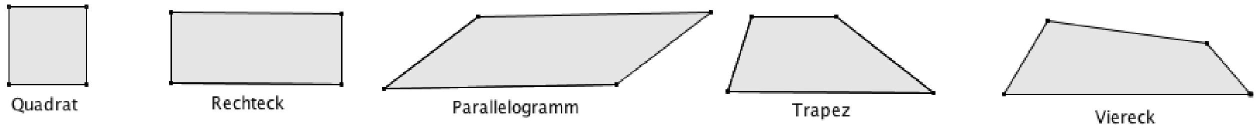
   Welche Eigenschaften der Figuren ändern sich, und welche bleiben unverändert?



b. Öffnet die GeoGebra-Datei “Figur_”, die euch die Lehrerin nennt. Hier könnt ihr eine der Figuren sehen, mit denen ihr zuvor experimentiert habt.
    - Schreibt bitte hier den Namen der Datei auf, die ihr geöffnet habt: Figur……

   Ist es ein echtes Quadrat, oder sieht es nur so aus?

   Aus welchen Eigenschaften der Figur könnt ihr erkennen, um was für ein Viereck es sich handelt?

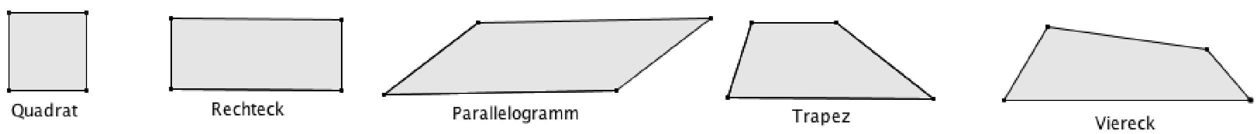


Untersuchung 2. Konstruktion eines gleichseitigen Dreiecks

Öffnet die GeoGebra-Datei „Gleichseitiges Dreieck". Das Ziel dieser Aufgabe besteht in der Konstruktion eines gleichseitigen Dreiecks. Die folgenden Schritte sollen euch dabei helfen.

a. Was sind die Eigenschaften eines gleichseitigen Dreiecks? Fertigt eine Liste dieser Eigenschaften an.

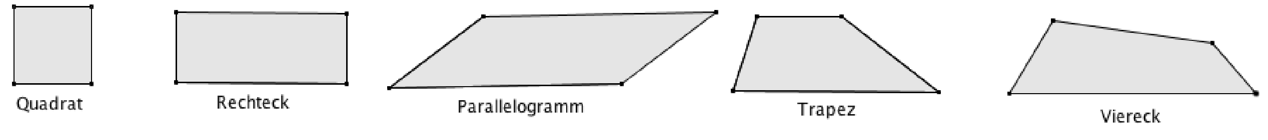
b. Konstruiert eine Strecke AB und dann eine Strecke AC gleicher Länge. Welche „Werkzeuge" in GeoGebra müsst ihr benutzen, um diese gleichen Seiten des gleichseitigen Dreiecks zu konstruieren?
Schreibt auf, wie ihr bei der Konstruktion vorgegangen seid.
Hinweis: Erinnert euch daran, wie wir die Seiten von Figur F in Aufgabe 1 konstruiert haben.



c. Konstruiert Strecke BC. Wo sollte der Punkt C liegen, damit AC, AB und BC gleich lang sind?

d. Wie könnt ihr die Position von Punkt C konstruieren?

e. Konstruiert einen neuen Punkt D dort, wo C sein sollte. Verbindet ihn mit A und B, um zwei neue Strecken zu erschaffen, AD und BD.
Bewegt die Punkte A, B und D. Ist das Dreieck ABD wirklich gleichseitig, oder sieht es nur so aus?
Begründet bitte eure Antwort.



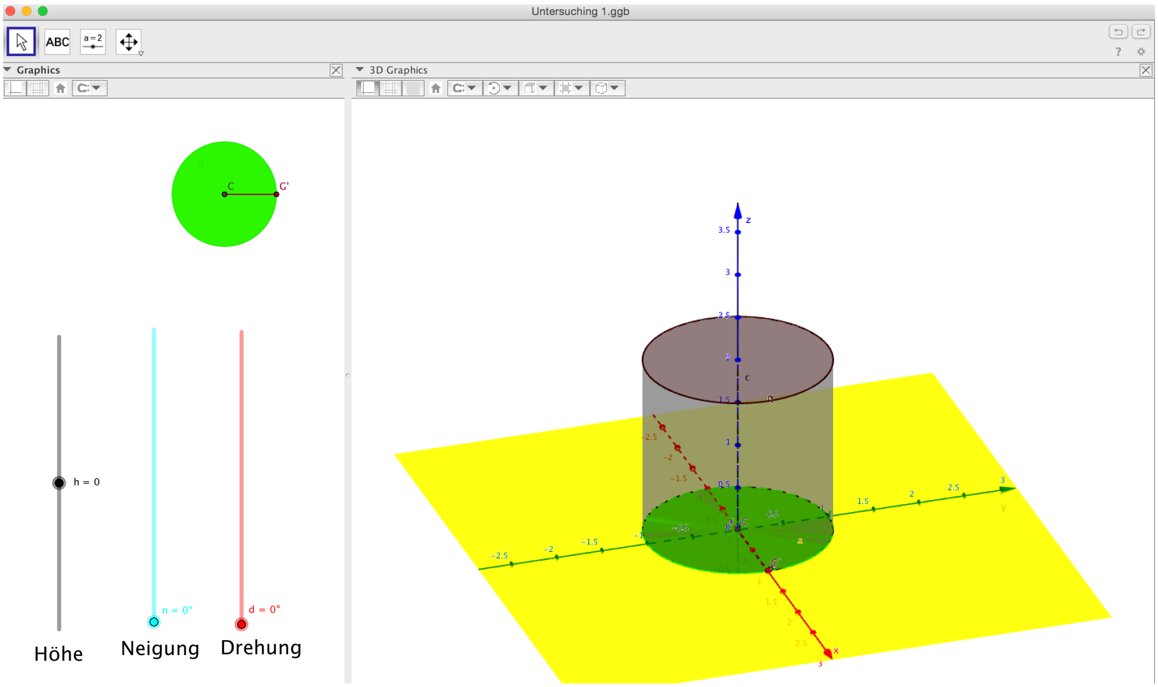
Freitag, den 17. Februar 2017

Arbeitsblatt 1

Unsere „Undercover“-Namen sind: ....
....

Untersuchung 1 – Versinkender Zylinder

Öffne die GeoGebra-Datei „Untersuchung 1”.



In dieser Datei könnt ihr einen Zylinder sehen, der in einer dreidimensionalen Umgebung von GeoGebra 5 (3D Graphics) entworfen ist. Der Zylinder kann mit Hilfe der drei Schieberegler h (Höhe), n (Neigung) und d (Drehung) bewegt werden (siehe auf der linken Seite des Bildschirms).

Neben den Schiebereglern erscheint, in grün, die Schnittfläche des Zylinders mit der gelben Ebene, die von der x- und der y-Achse aufgespannt wird.

a. Was bewirken die Schieberegler h, n und d?
Bewegt die Schieberegler wie ihr möchtet in verschiedenen Kombinationen. Beobachtet, wie die Veränderungen der Schieberegler die Position des Zylinders im Raum beeinflussen.
Wie wirken sich die einzelnen Schieberegler auf die Position des Zylinders im Raum aus?

Schieberegler h	
Schieberegler n	
Schieberegler d	

Kurze Wiederholung

Welcher Körper wird hier untersucht? …………………………………………………………………..

Beschreibt die Eigenschaften des Körpers:

b. Erkundung verschiedener Positionen des Zylinders
   Auf Seite 3 seht ihr eine „Zylinder-Erkundungstabelle“. Jede Zeile bezieht sich auf eine bestimmte Schnittfläche. Diese ist durch die Position bedingt, die der Zylinder in Bezug auf die blaue Ebene einnimmt.

   In der dritten Spalte (mit der Überschrift „Bezeichnung und Eigenschaften der Schnittfläche“) könnt ihr der Schnittfläche eine Bezeichnung geben und ihre Eigenschaften beschreiben. Warum gibt es genau diese Schnittfläche? Was hat sie mit der Form des Zylinders zu tun? Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

   Benutzt die Schieberegler, um die verschiedenen Schnittflächen für die vier gegebenen Positionen zu erzeugen und zu erkunden. Betrachtet außerdem eine weitere Position eurer Wahl und füllt die letzte Zeile der Tabelle aus.

Zylinder-Erkundungstabelle

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=0° d=0°		
h= -0,7 n=45° d=0°		
h=0 n=90° d=0°		
h= -0,75 n=100° d=0°		

B2

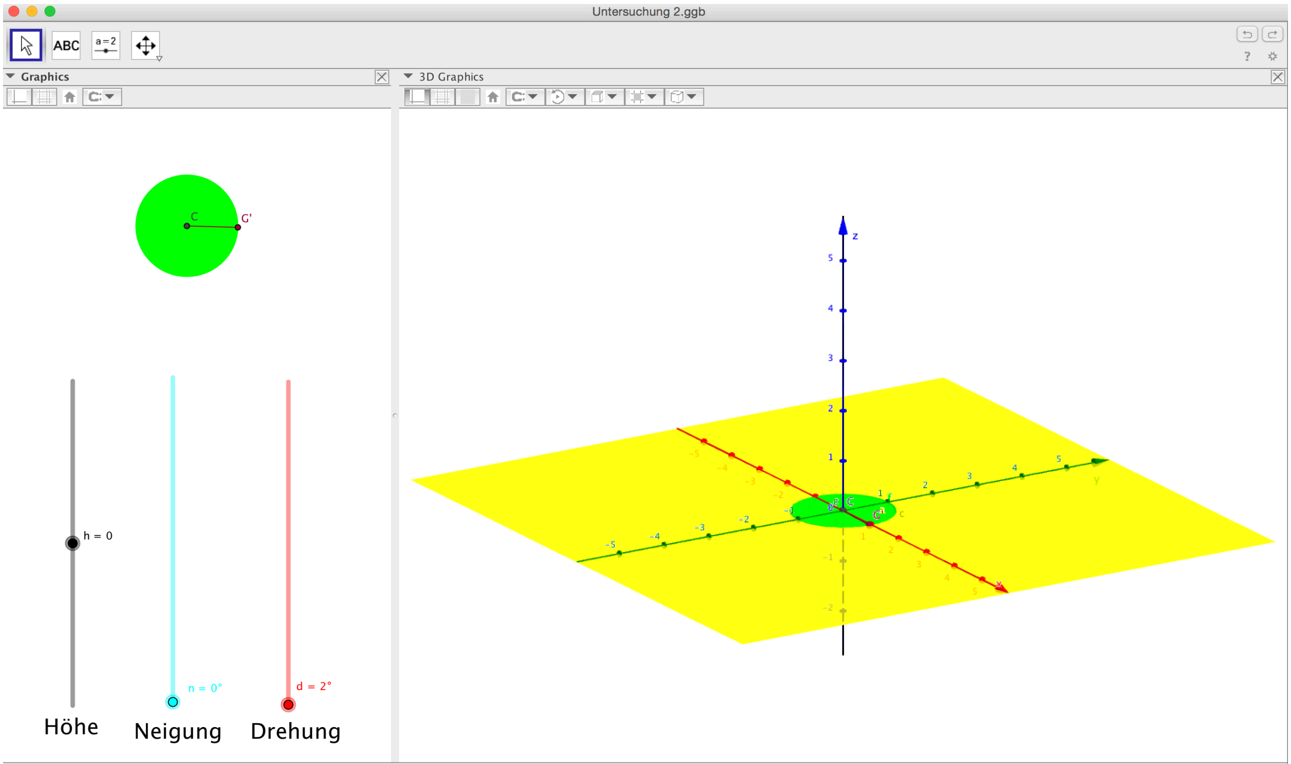
Freitag, den 17. Februar 2017

Arbeitsblatt 2

Unsere „Undercover“-Namen sind: ....
....

Untersuchung 2 – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 2”.



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die "Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2 und 3 soll als Notizbuch für eure Untersuchungen verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h=0,4 d=0° Erkundet die Werte für n zwischen 0° und 360°.		
h=0,4 n=0° Erkundet die Werte für d zwischen 0° und 360°.		

h/n/d	**Skizze der Schnittfläche**	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

B3

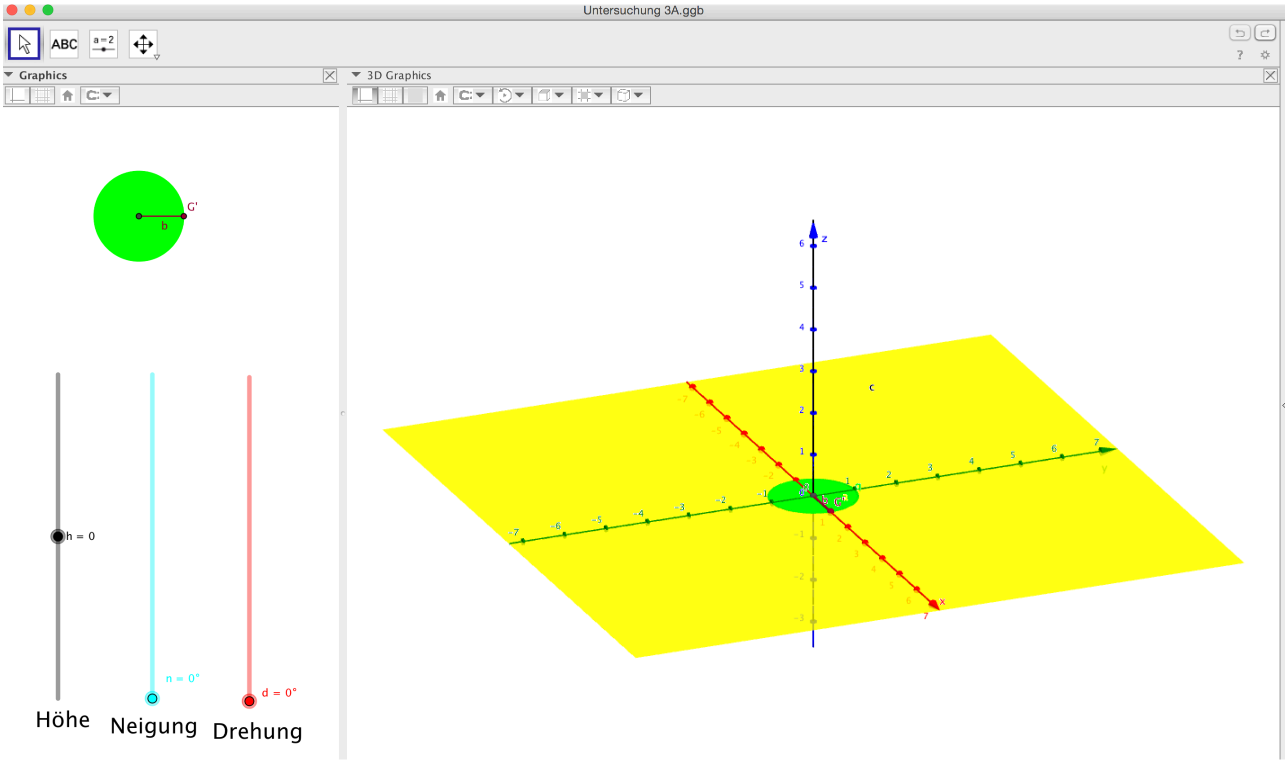
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover“-Namen sind: ....
....

Untersuchung 3A – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffne die GeoGebra-Datei „Untersuchung 3A”.



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren“ zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2, 3 und 4 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h=0 n=90° d=0°		
n=90° d=0° Erkundet die Werte für h zwischen -4 und 4.		

h/n/d	**Skizze der Schnittfläche**	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=45° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h= -0,7 n=45° Erkundet die Werte für d zwischen 0° und 360°.		
h= 0,7 d=0° Erkundet die Werte für n zwischen 0° und 360°.		

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

B4

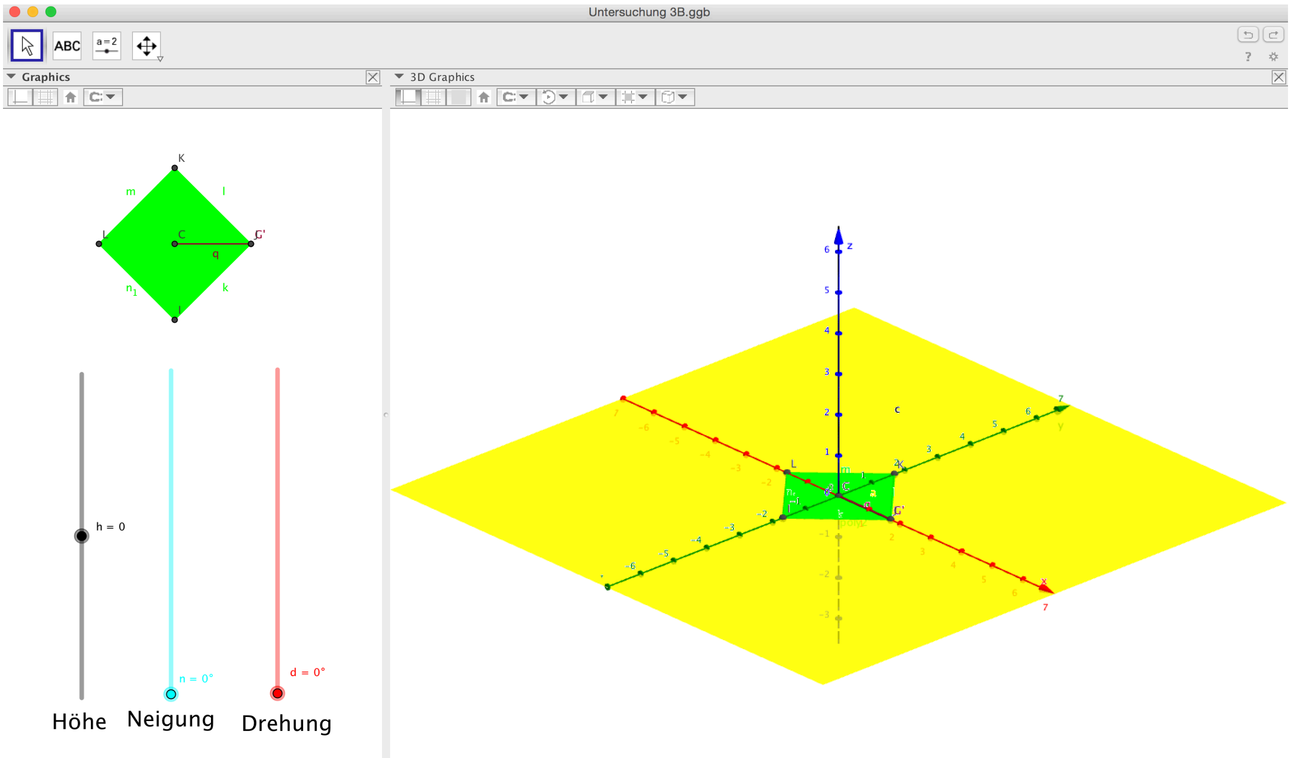
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover“-Namen sind: ....
....

Untersuchung 3B – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 3B”.



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren“ zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2 und 3 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.		
h= -0,5 d=0° Erkundet die Werte für n zwischen 0° und 360°.		

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=90° Erkundet die Werte für d zwischen 0° und 360°.		

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

B5

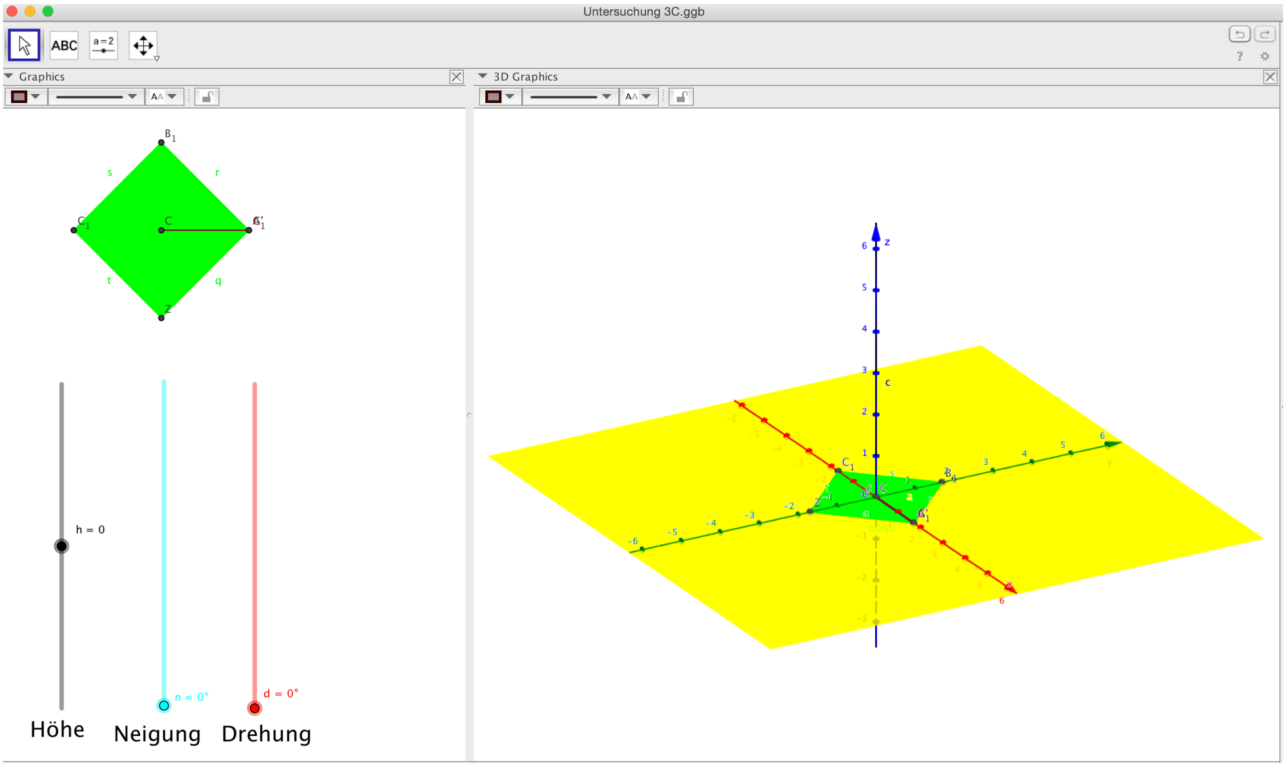
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover"-Namen sind: ....
....

Untersuchung 3C – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 3C".



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren" sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2, 3 und 4 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	**Skizze der Schnittfläche**	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.		
h= -1 d=0° Erkundet die Werte für n zwischen 0° und 360°.		

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=125° d=0° Erkundet die Werte für h zwischen -4 und 4.		
h=0 n=90° d= 0°		
h=0 n=90° d=45°		

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

C. Structure tables of all the episodes

C1

Episode GR1AD-2				
Structure			**Utterance numbers**	**Video Minutes**
1.	Is the solid a cone, a circle or a sphere?		#1-9	00:59:43 – 01:00:08
2.	Observing the cross-section for various values of n, h and d. The solid is a sphere.		#10-12	01:00:09 – 01:00:33
3.	Further exploration with the use of the Exploration Matrix on the worksheet.		#13-29 and #43-61	01:01:07 – 01:02:33 and 01:03:20 – 01:06:03
	3.1	Exploring the case (h, n=0°, d=0°).	#13-20.1	01:01:07 – 01:02:33 and 01:03:20 – 01:06:03
	3.2	Specifying the length of the radius of the sphere.	# 20.1 – 29	01:02:00 – 01:02:33
4.	Discussion with a third student, Jacob, about his conjecture.		#30-42	01:02:33 – 01:03:20
5.	Continuing the exploration using the Exploration Matrix.		#43-61	01:03:20 – 01:06:03
	5.1	Exploring further the case (h=0, n=0°, d=0°)	#43-45	01:03:20 – 01:03:50
	5.2	Exploring the case (h=0,4, n, d=0°)	#46-50	01:04:27 – 01:05:29
	5.3	Exploring the case (h=0,4, n=0°, d)	#51-61	01:05:29 – 01:06:03
6.	The solid is a sphere – Writing their justification on the worksheet.		#63-74	01:07:35 – 01:09:10

C2

Episode GR1AD-3A.III				
Structure			**Utterance numbers**	**Video Minutes**
Exploration using the Exploration Matrix			#3-118	00:43:28 – 00:56:54
1.	Exploration of the case (h, n=0°, d=0°)		#3-29	00:43:28 – 00:45:34
	1.1.	The solid is either a half sphere or a cylinder	#11	00:43:44
	1.2.	Correction of misspoken word: Instead of cylinder, Dave meant cone	#16	00:44:01
	1.3.	The solid is a double cone	#17	00:44:11
	1.4.	The solid is nor a double cone, rather a normal cone	#18	00:44:20
	1.5.	The solid is not a half sphere	#20-28	00:44:54 – 00:45:32
2.	Exploration of the case (h=0, n=90°, d=0º) The cross-section does not change. The cross-section is a vertical one.		#29-44	00:45:32 – 00:47:28
3.	Exploration of the case (h, n=90°, d=0º) The cross-section becomes blunter and then disappears		#45-65	00:47:55 – 00:50:50
4.	Exploration of the case (h, n=45°, d=0º) The cross-sections are oval and semi-circle		#68-91	00:51:09 – 00:53:59
5.	Exploration of the case (h=-0,7, n=45°, d) The cross-section is oval		#92-96.2	00:54:12 – 00:54:53
6.	Exploration of the case (h=0,7, n, d=0º) The cross-sections are from semi-circle to triangle to oval to circular		#96.4-118	00:54:53 – 00:56:54
7.	The solid is a cone – Writing their justification on the worksheet		#119-136	00:56:54 – 00:58:00

<table>
<tr><th colspan="3">Episode GR1AD-3B.II</th><th></th><th></th></tr>
<tr><th colspan="3">Structure</th><th>Utterance numbers</th><th>Video Minutes</th></tr>
<tr><td rowspan="3">1.</td><td colspan="2">Exploration of the case (h, n=0°, d=0°)</td><td>#1-12</td><td>00:33:24 – 00:34:34</td></tr>
<tr><td>1.1.</td><td>The solid is a pyramid</td><td>#3</td><td>00:33:34</td></tr>
<tr><td>1.2.</td><td>The solid is an inverted pyramid</td><td>#10</td><td>00:34:30</td></tr>
<tr><td rowspan="4">2.</td><td colspan="2">Observing the cross-section for various values of n, h and d.</td><td>#13-51</td><td rowspan="4">00:34:36 – 00:40:23</td></tr>
<tr><td>2.1</td><td>Exploration of case (h=-1, n=0°, d)
The pyramid rotates around itself</td><td>#13-15</td></tr>
<tr><td>2.2</td><td>Exploration of case (h=-0,5, n, d=0°)
The cross-sections change from quadrilateral, to pentagon and to triangle</td><td>#16-39</td></tr>
<tr><td>2.3</td><td>Exploration of case (h=0, n=90°, d)
The triangular cross-sections become smaller and bigger.</td><td>#40-51</td></tr>
<tr><td>3.</td><td colspan="2">Back to Task 3C – The solid was a cuboid</td><td>#52-64</td><td>00:40:23 – 00:41:04</td></tr>
<tr><td>4.</td><td colspan="2">The solid is a pyramid with square base – Writing their justification on the worksheet.</td><td>#65-68</td><td>00:41:04 – 00:41:28</td></tr>
</table>

C4

Episode GR1AD-3C.I					
Structure				**Utterance numbers**	**Video Minutes**
1			Based only on the picture on the worksheet: The solid is a pyramid or a cube	#1-8	00:03:00 – 00:03:22
2.			Exploration **without** the use of the Exploration Matrix	#9-76	00:03:22 – 00:08:02
	2.1		Exploration of case (h, n=0º, d=0º) The solid is a cuboid	#9-10	00:03:22
	2.2		Checking the cuboid supposition	#11-13	00:04:02
	2.3		Could the solid be a prism?	#14-16	00:03:43
	2.4		Exploration of the case (h=0, n, d=0°)	#17-40	00:04:29
	2.5		Exploration of the case (h=0, n=35°, d) The cuboid spins	#41-60	00:06:03
	2.6		Exploration of the case (h=0, n, d=0°)	#61-62	00:06:55
	2.7		Hexagonal cross-section at position (h=0,65, n=125º, d=0º)	#64-68	00:07:12
	2.8		Students say: the initial orientation of the solid is positioned with its base on the plane xOy, standing on the positive sup-space xOyOz.	#69-76	00:07:34
3.			Explorations using the Exploration Matrix	#77-260	00:08:01 – 00:27:41
	3.1		Exploration of the case (h, n=0°, d=0°) Observing for which values of h, there are cross-sections	#77-96	00:08:01
	3.2		Exploration of the case (h=-1, n=0°, d) They say, "by varying d, the cross-section does not change"	#97-120	00:10:19
	3.3		Exploration of the case (h=-1, n, d=0°) How can a cuboid have hexagonal and pentagonal cross-sections?	#121-226	00:12:37
		3.3.1	How come hexagons? (#123-143, 00:13:18) Axel says that he cannot understand how can occur the hexagonal cross-section in position (h=-1, n=50°, d=0°) can occur. Dave tries to explain	#123-143	00:13:18
		3.3.2	Pentagonal cross-section While Axel keeps notes, Dave	#144-157	00:16:21

			experiments with further positions. He expresses his surprise to find a pentagon at position (h=-1, n=55º, d=164º)		
	3.4	Exploration of case (h, n=125º, d=0º) Triangular and hexagonal cross-sections		#158-226	00:17:42
		3.4.1	Researcher intervention Discussion with the two students about the hexagonal and pentagonal cross-sections. The students express their difficulty to imagine how a cuboid or a cube could have such cross-sections.	#177-206	00:20:08
		3.4.2	Is the solid a prism? Is it a cuboid? – Hexagonal cross-section at position (h=0,75, n=125º, d=0º) The hexagonal cross-sections give the students the idea that the solid could be a prism. On the other hand, they argue that they are sure that both the top surface and the base of the solid are squares, and so the solid cannot be something else, other than either a cuboid or a cube.	#207-223	00:22:48
	3.5	Exploration of position (h=0, n=90°, d=0°) Students' personal decision: Exploration of position (h=0, n=0º, d=0º) and case (h=0, n, d=0º) Comparing the areas and positions of the different cross-sections.		#227-260	00:24:00
	3.6	Exploration of position (h=0, n=90°, d=45°)			
4.	Rejecting the hypotheses: sphere, cylinder, pyramid, and cone			#261-263	00:27:42 – 00:28:05
5.	Conclusion: The solid is a cuboid, possibly also a cube			#264-286	00:28:06 – 00:29:07
6.	The students write down their justification			#287-308	00:29:07 – 00:31:07

C5

Episode GR2TL-2			
Structure		**Utterance numbers**	**Video Minutes**
1.	Exploration of the case (h, n=0°, d=0°) Circular cross-section getting smaller as h increases over zero or decreases under zero	#7-13	01:00:12 – 01:00:57
2.	Exploration of the position (h=-0,4, n, d=0°) [Misreading of the given case (h=0,4, n, d=0º)] The cross-section is a circle and it does not change	#17-22	01:02:11 – 01:02:40
3.	Exploration of the case (h=-0,4, n=0°, d) [Misreading of the given case (h=0,4, n=0º, d)] The circle (cross-section) remains the same. The solid turns around its axis	#33-38	01:03:48 – 01:04:26
4.	Creation of suppositions: Lukas' hypothesis: The solid is a double truncated cone Tom's claim: The solid is a sphere	#39-64	01:04:28 – 01:05:37
5.	Conclusion: The solid is a sphere	#65-68	01:05:35 – 01:06:05
6.	Exploring case (h=1, d=251º, n)	#72-76	01:06:24 – 01:06:44
7.	Lukas says that if the n-variation changed the shape of the cross-section, it would not fit with the solid being a sphere	#77	01:06:45 – 01:06:49

C6

Episode GR2TL-3A.I			
Structure		**Utterance numbers**	**Video Minutes**
1.	Exploration of the case (h, n=0°, d=0°) Lukas' supposition: sphere Tom's supposition: cone	#1-4	00:04:13 – 00:04:25
1.1	Acceptance of the cone-supposition. Upward cone, not an inverted cone	#5-10	00:04:25 – 00:04:46
1.2	Tom keeps notes on the worksheet for the case (h, n=0º, d=0º)	#11-21	00:04:46 – 00:05:30
2.	Exploration of the position (h=0, n=90°, d=0°) The cross-section is an isosceles triangle	#24-47	00:05:48 – 00:07:06
3.	Exploration of the case (h, n=90°, d=0°) Parabolic cross-sections of different sizes	#51-56	00:07:25 – 00:08:15
4.	Exploration of the case (h, n=45°, d=0°) Lukas describes the orientation and movement of the solid: It lies "crooked"	#82-97	00:10:58 – 00:13:34
5.	Exploration of the case (h=-0,7, n, d=0º) Multiple cross-sections [Misreading of the given case (h=-0,7, n=45º,d)]	#98-131	00:13:36 – 00:16:30
6.	Exploration of the case (h=0,7, n, d=0º) Multiple cross-sections	#138-186	00:17:34 – 00:21:04
7.	Researcher intervention Short discussion on the task. The students justify their conclusion that the solid is a cone	#194-214	00:22:18 – 00:24:00
8.	Exploration of an extra position, of their choice: (h=0, n=80°, d=30°)	#215-227	00:26:30 – 00:27:22
9.	Written justification of the cone conclusion	#236-238	00:28:50 – 00:34:15

C7

Episode CD2				
Structure			**Utterance numbers**	**Video Minutes**
1.	Justifying the sphere-hypothesis		#1-17	01:15:55 – 01:17:43
	1.1	Niko's justification The solid is a sphere because all its cross-sections are circles. Rejecting the idea of the solid being a cuboid via RAA	#1-12	01:15:55 – 01:16:59
	1.2	Jacob's justification The lack of influence of the n-variation to the cross-sections of the solid is a decisive factor, proving that the solid is a sphere.	#14-17	00:17:00 – 01:17:43
2.	Stating and rejecting other hypotheses judging only by the (h, n=0º, d=0º) exploration		#18-54	01:17:44 – 01:23:02
	2.1	The solid could be a cylinder	#23-33	01:17:44 – 01:19:20
	2.2	The solid could be cone	#25 and 34-40	01:18:33 and 01:19:22 – 01:20:03
	2.3	The solid could be a double cone	#41-53	01:20:03 – 01:23:02

C8

Episode CD3A-AD				
Structure			**Utterance numbers**	**Video Minutes**
1.	Axel and Dave's presentation		#1-18	00:58:34 – 01:03:48
	1.1	Going through the positions and cases of the Exploration Matrix	#4-6.7	00:59:46 – 01:02:35
	1.2	The solid is a cone.	#6.8-8	01:02:35 – 01:03:16
	1.3	The solid is an inverted cone.	#10-12	01:03:20 – 01:03:35
	1.4	The solid is not an inverted cone. It is a cone standing with its base on plane xOy.	#13-18	01:03:38 – 01:03:46
2.	How do the curved cross-sections occur?		#19-41	01:03:50 – 01:07:30
	2.1	Jacob's explanation	#20	01:04:19
	2.2	Jacob's explanation with use of gestures	#21-35	01:04:50 – 01:06:28
	2.3	Jacob's explanation with the use of a haptic cone-model	#36-41	01:06:30 – 01:07:28
3.	Presentation of the visible cone in GeoGebra		#42-56	01:07:30 – 01:10:10

C9

Episode CD3B-JM				
Structure			**Utterance numbers**	**Video Minutes**
1.	Jacob and Michael’s presentation		#1-7	01:10:53 – 01:15:43
	1.1	Going through the positions and cases of the Exploration Matrix Supposition: The solid is a pyramid	#1-7.8	01:10:53 – 01:15:00
	1.2	Justification of the conclusion that the solid is a pyramid	#7.9-7.16	01:15:00 – 01:15:43
2.	Ella’s correction of Jacob’s misspeaking mistake: The pyramid has four triangular faces, not three		#8-11	01:15:51 – 01:16:03
3.	Presentation of the visible cone in GeoGebra, by Jacob		#12-18	01:16:06 – 01:19:18

C10

Episode CD3C-TL				
Structure			**Utterance numbers**	**Video Minutes**
1.	Tom and Luka's presentation		#1-24	01:21:09 – 01:26:31
	1.1	Going through the positions and cases of the Exploration Matrix	#1-1.7	
	1.2	Supposing that the solid is a cuboid	#1.9	
	1.3	For the case ($h=-1$, n, $d=0^{\circ}$), the solid tilts, sinking in the plane of intersection xOy	#1.10-12	
	1.4	Continuing the exploration of the cases of the Exploration Matrix	#13-24	
	1.5	Rejection of the idea of the solid being a cylinder	#23	
2.	Theo argues that the solid is not a cuboid, rather a cube, because all its edges are equal		#26-41.2	01:26:33 – 01:28:50
3.	The solid is now visible Discussion on the way hexagonal cross-sections emerge: Michael and researcher		#41-50	01:28:50 – 01:31:05

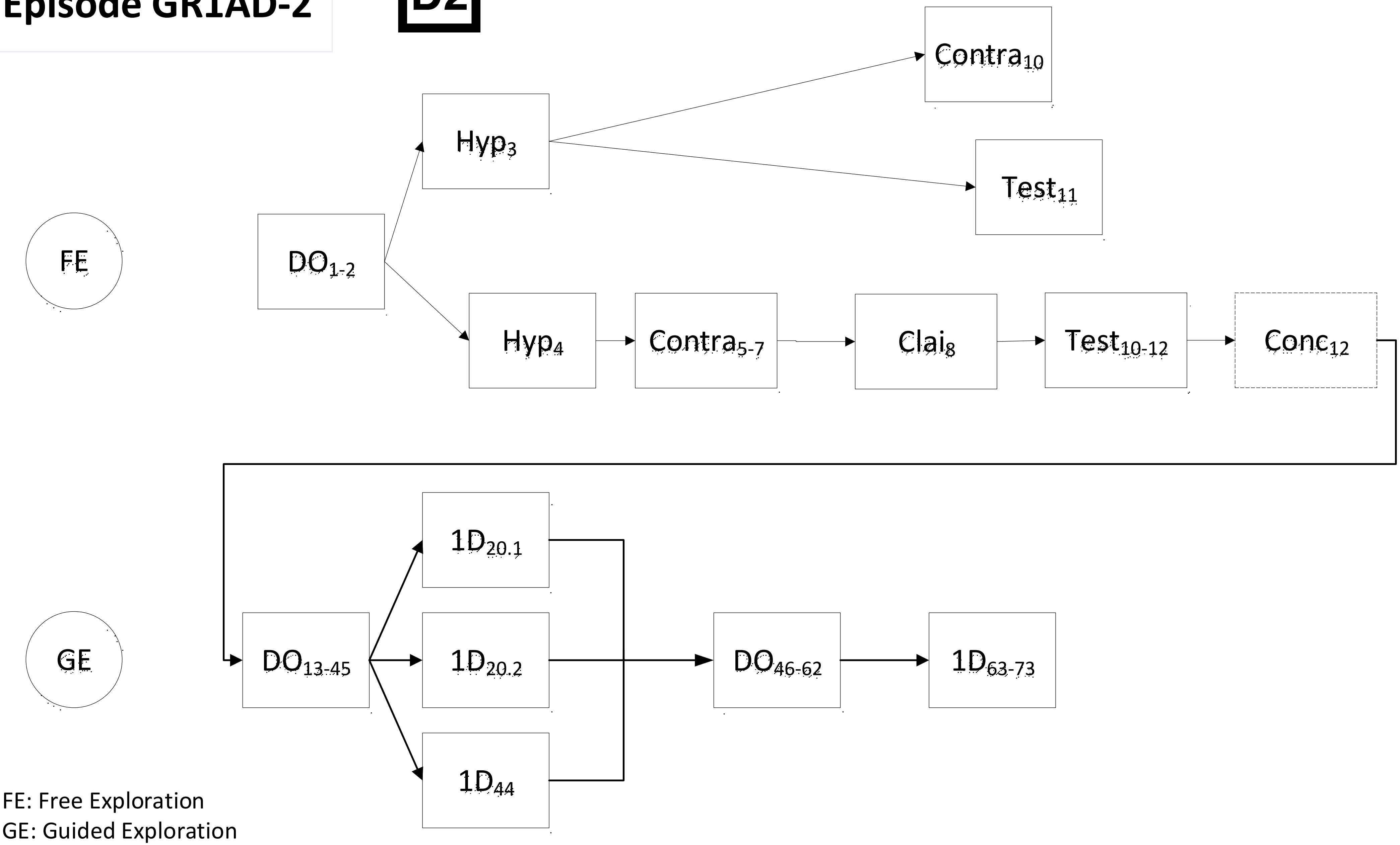
D. Patterns of argumentation

D1 Coding of elements in the patterns of argumentation

DO	Observing data/Data Observation
Hyp	Stating a Hypothesis A *hypothesis* is a supposition created by the students, suggesting a possible case based on the available data. This is a case, which at the moment looks plausible, and whose validity is not yet confirmed.
Clai	Stating a Claim A *claim* is more than a hypothesis; it is more than just a possible case or solution. It is *the* possible case which one considers as the most probable and shows the intention to confirm it or argue in favor of it.
Conc	Stating a Conclusion A *conclusion* is a statement that is accepted by all as true.
Contra	Stating a Contradiction
1D	Drawing a conclusion with simple deduction
Test	Testing
CE	Using a counter example
Q?	The teacher poses a question in order to provoke students' argumentation
$DO_{1\text{-}4}$	The **subscript** next to an element shows the utterances in the episode's transcript in which this element is observed (e.g. Observation of Data in utterances 1 to 4)
Element in dotted box	Implicit element

Episode GR1AD-2

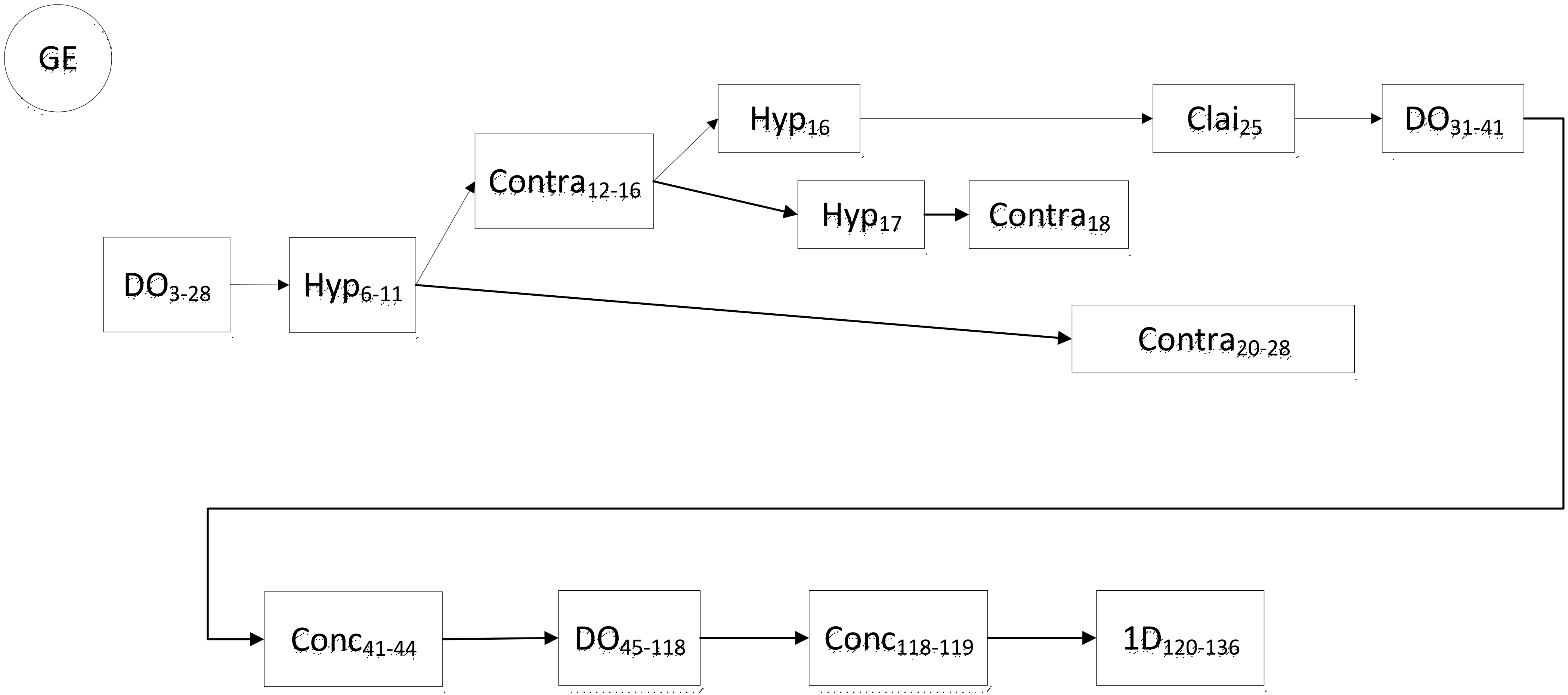
D2



FE: Free Exploration
GE: Guided Exploration

Episode GR1AD-3A.III

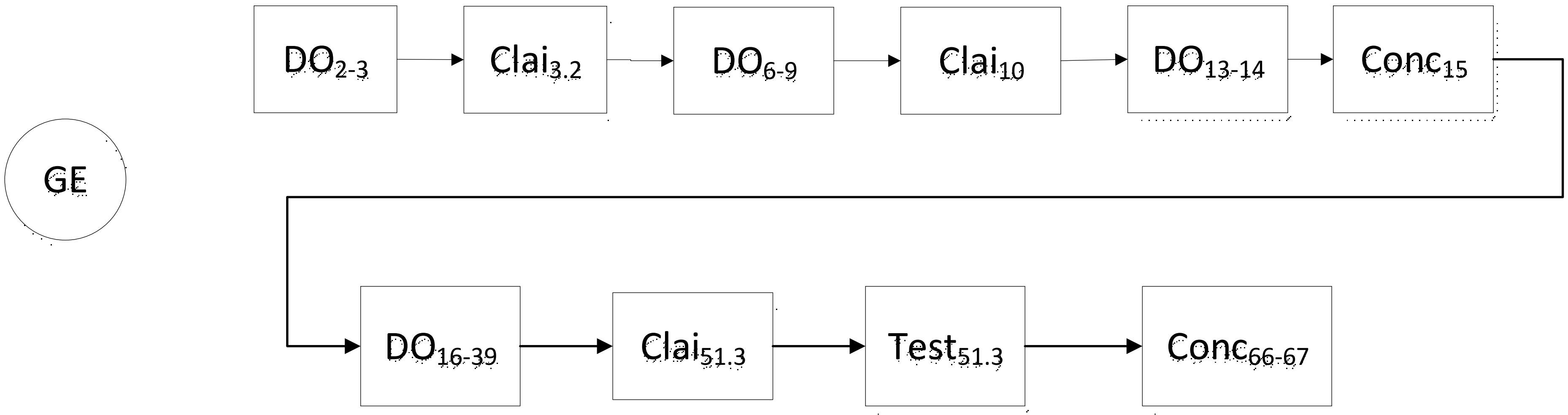
D3



GE: Guided Exploration

Episode GR1AD-3B.II

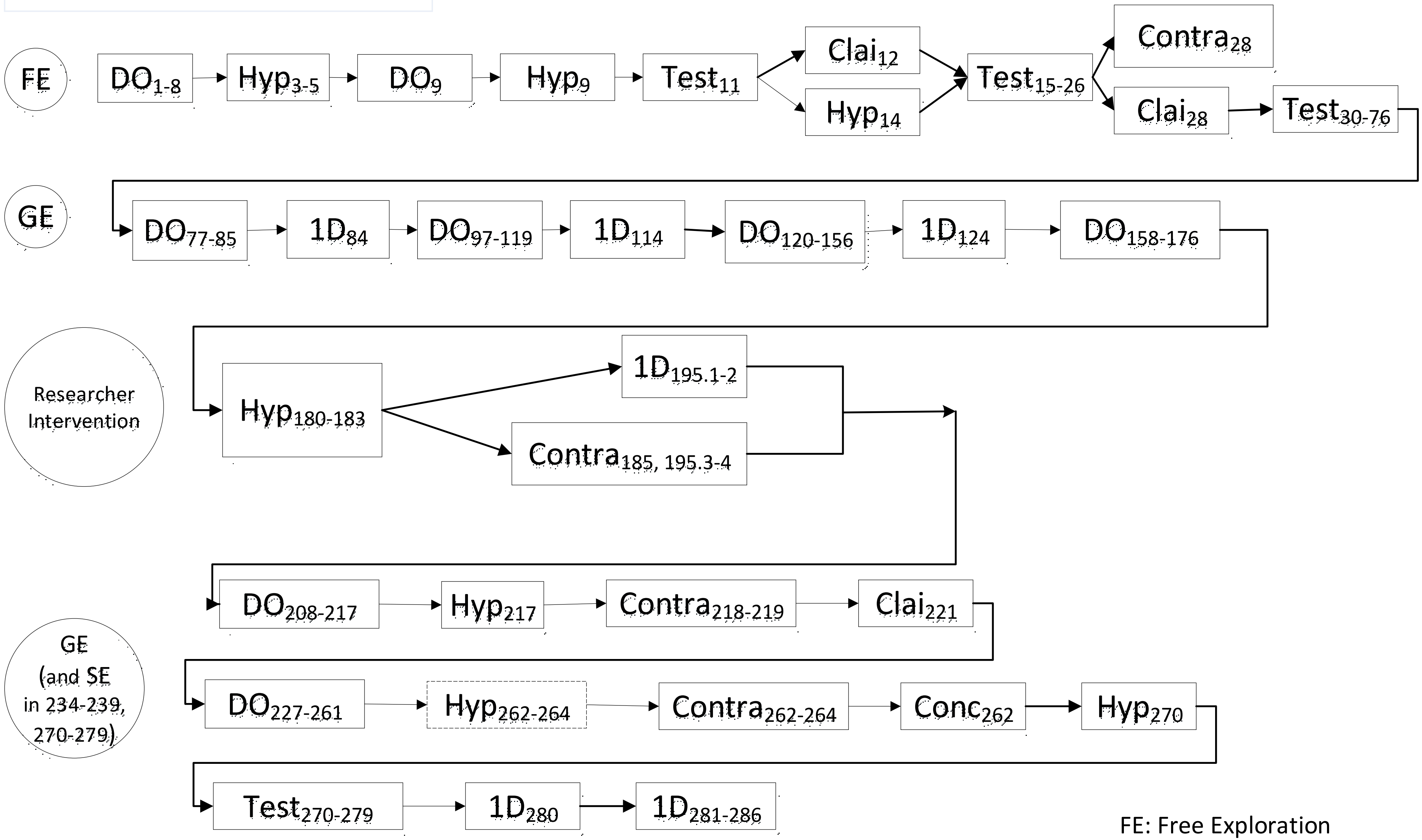
D4



GE: Guided Exploration

Episode GR1AD-3C.I

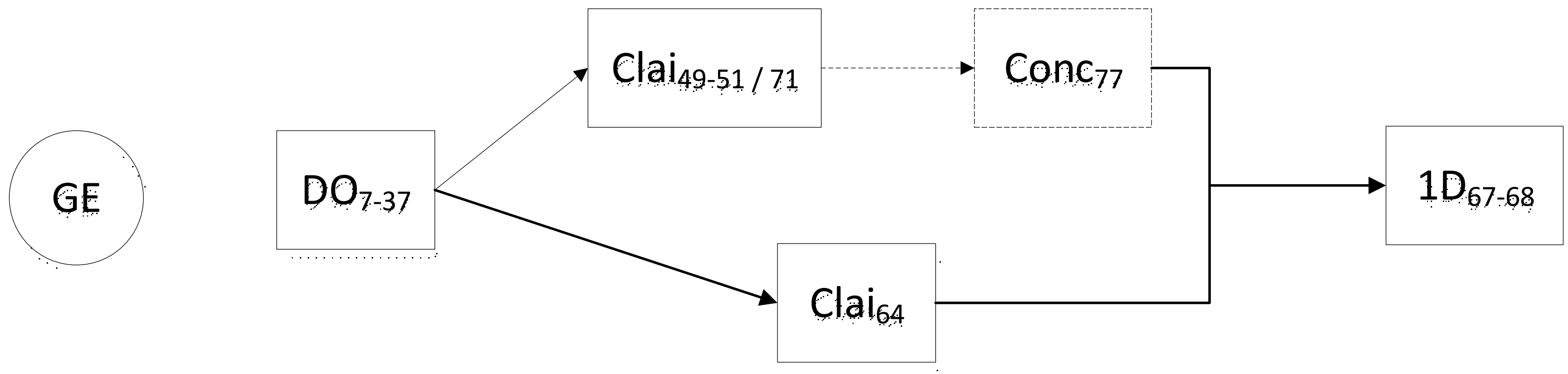
D5



FE: Free Exploration
GE: Guided Exploration
SE: Structured Exploration

Episode GR2TL-2

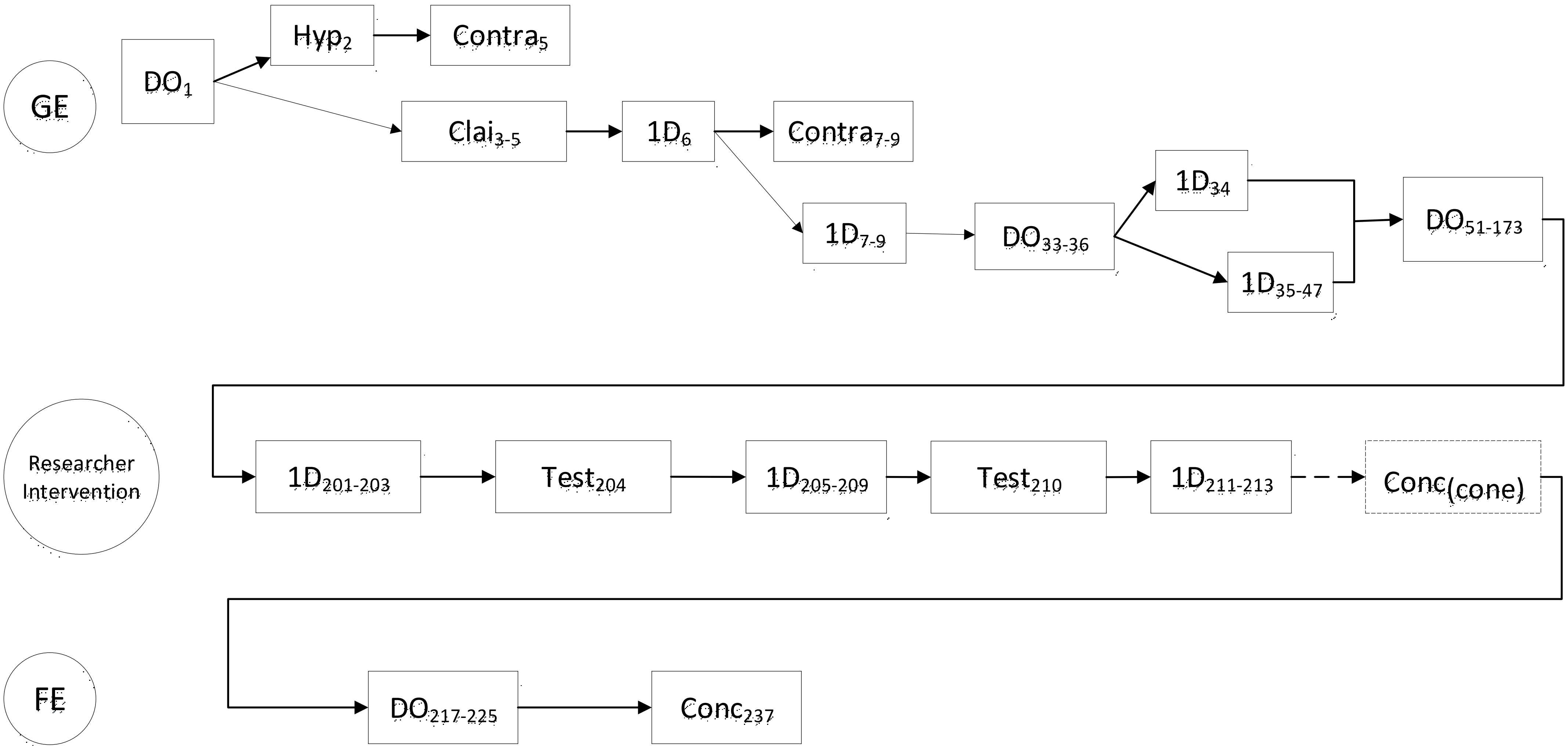
D6



GE: Guided Exploration

Episode GR2TL-3A.I

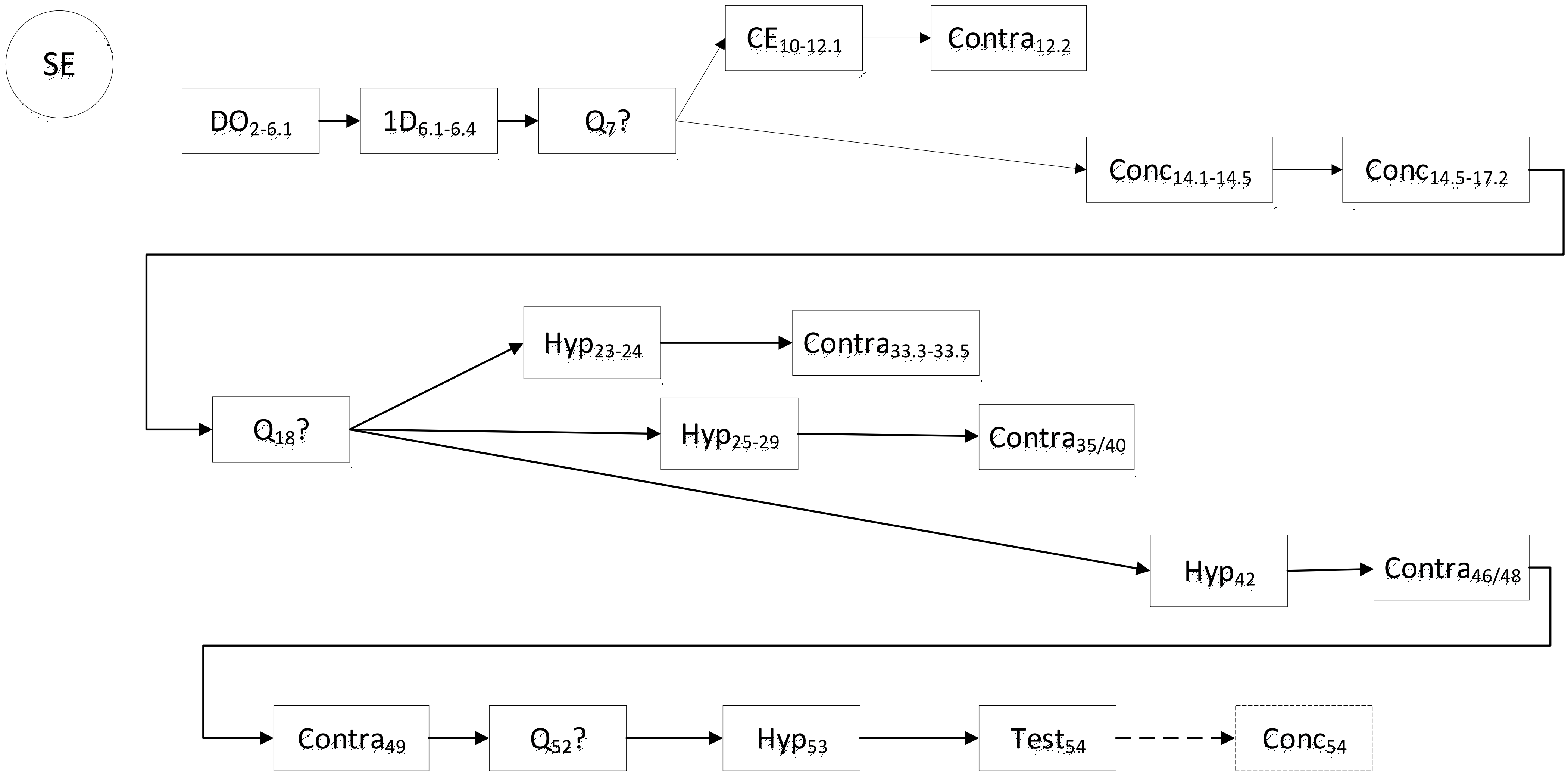
D7



GE: Guided Exploration
FE: Free Exploration

Episode CD2

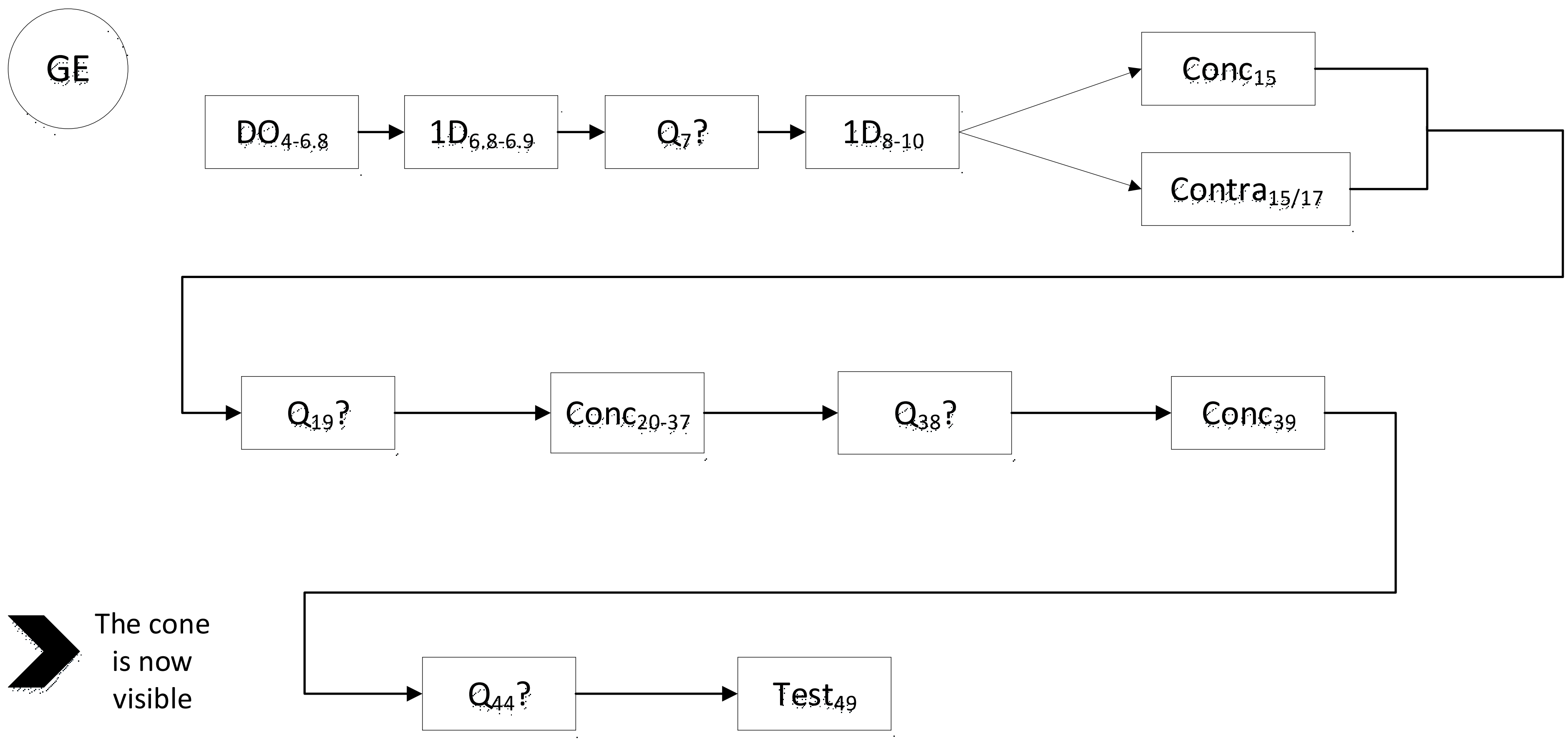
D8



SE: Structured Exploration

Episode CD3A-AD

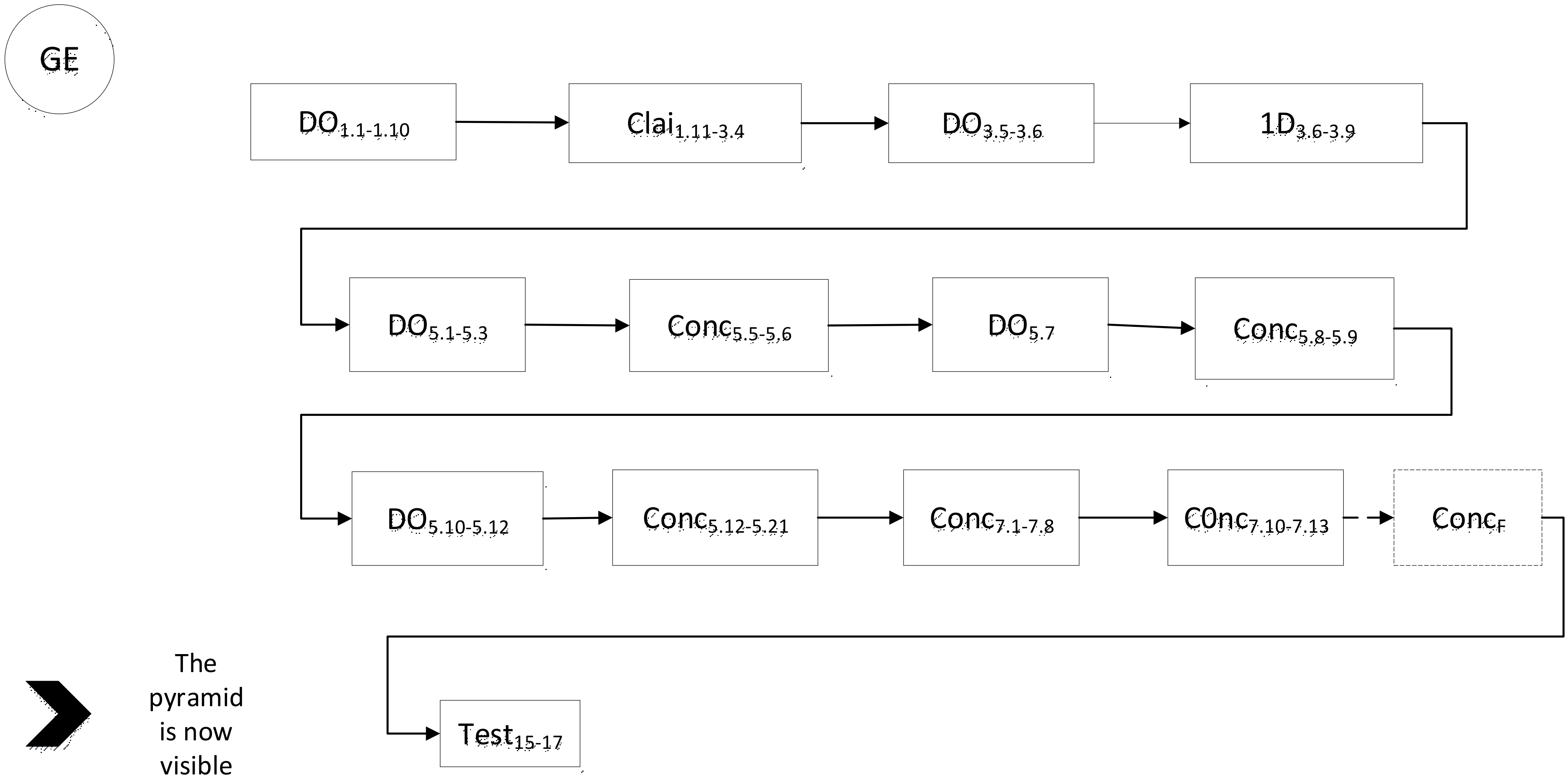
D9



GE: Guided Exploration

Episode CD3B-JM

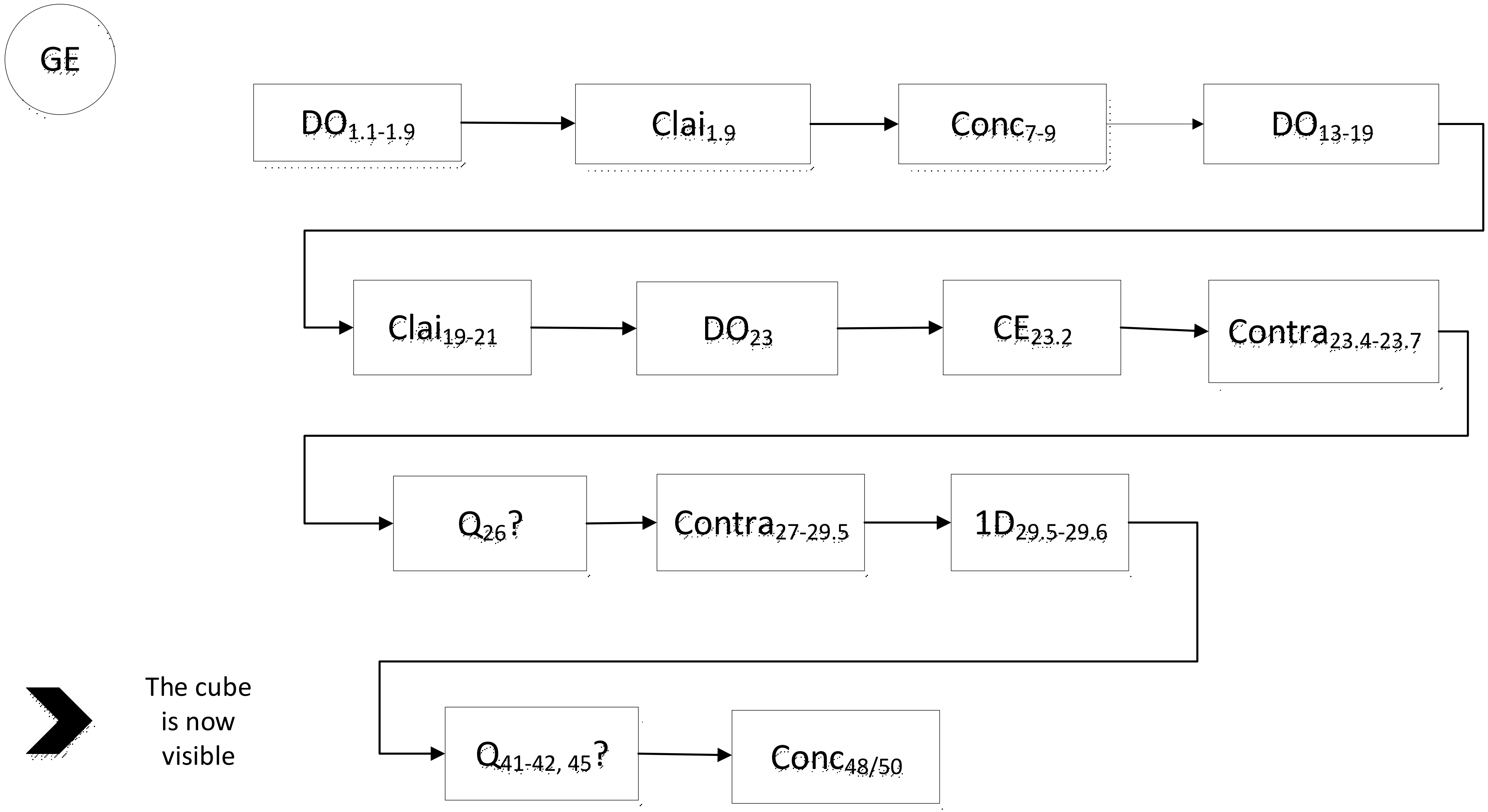
D10



GE: Guided Exploration

Episode CD3C-TL

D11



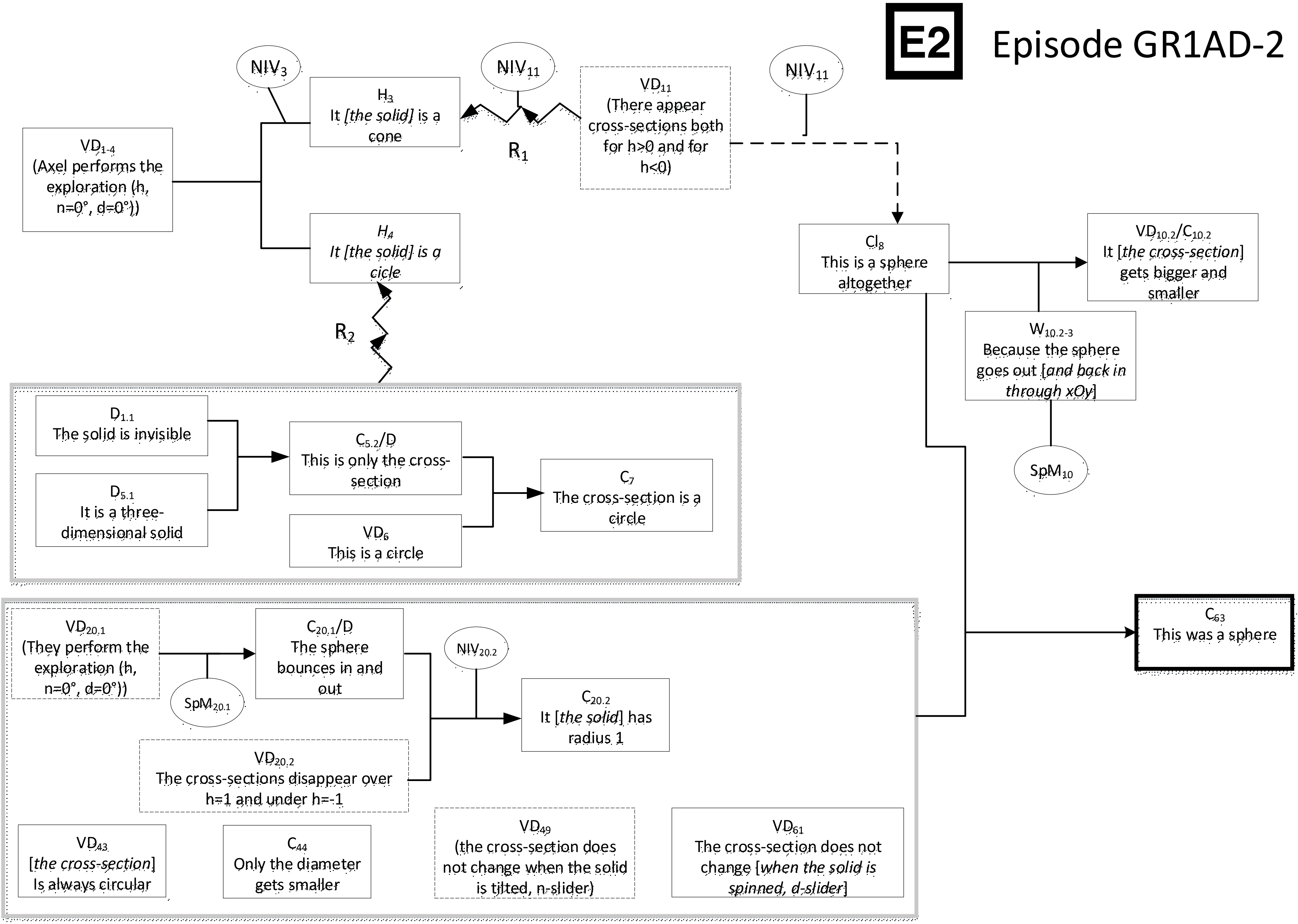
GE: Guided Exploration

E. Global argumentation Structures

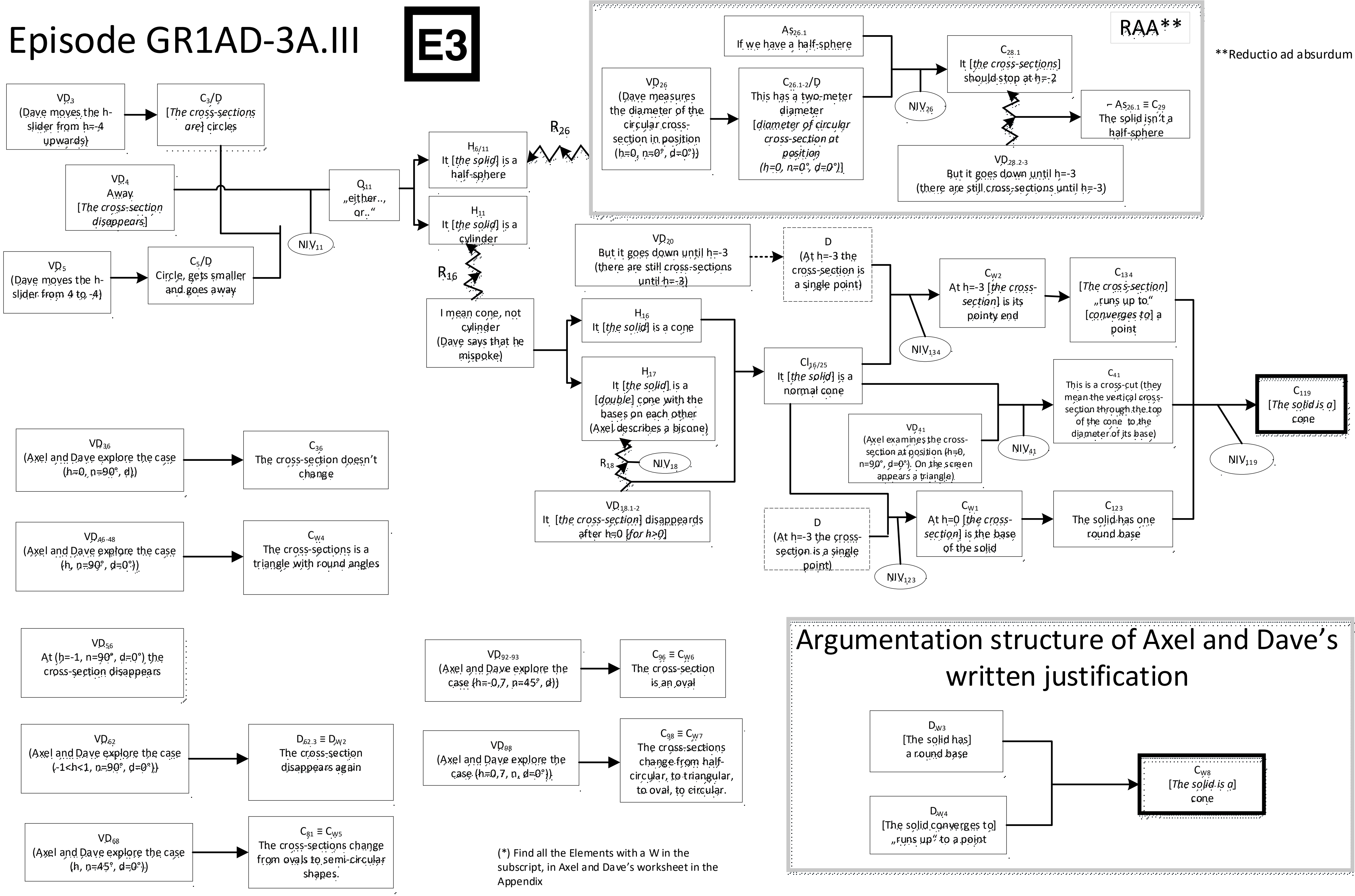
E1 Coding of elements in the global argumentation structures

VD	Visual datum
D	Datum
H	Hypothesis
Cl	Claim
C	Conclusion
Q	Qualifier
W	Warrant
B	Backing
R	Refutation
RAA	Reductio ad absurdum
NIV	Non-iconic visualization
SpM	Spatial manipulation
D_{12}	The **subscript** next to an element shows the utterance(s) in the episode's transcript in which this element is observed (e.g. Datum in utterance 12)
Element in dotted box	Implicit element

E2 Episode GR1AD-2

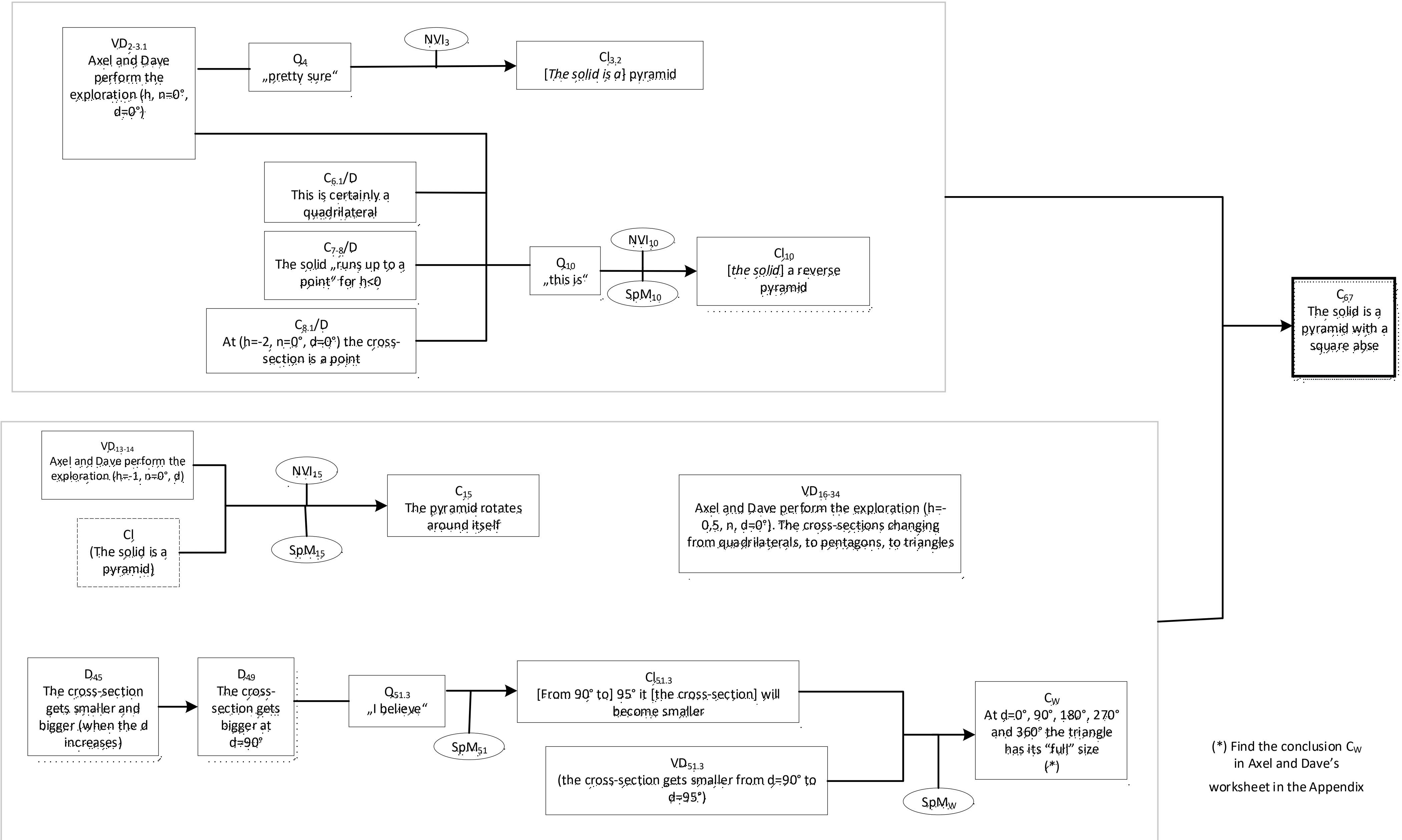


Episode GR1AD-3A.III



Episode GR1AD-3B.II

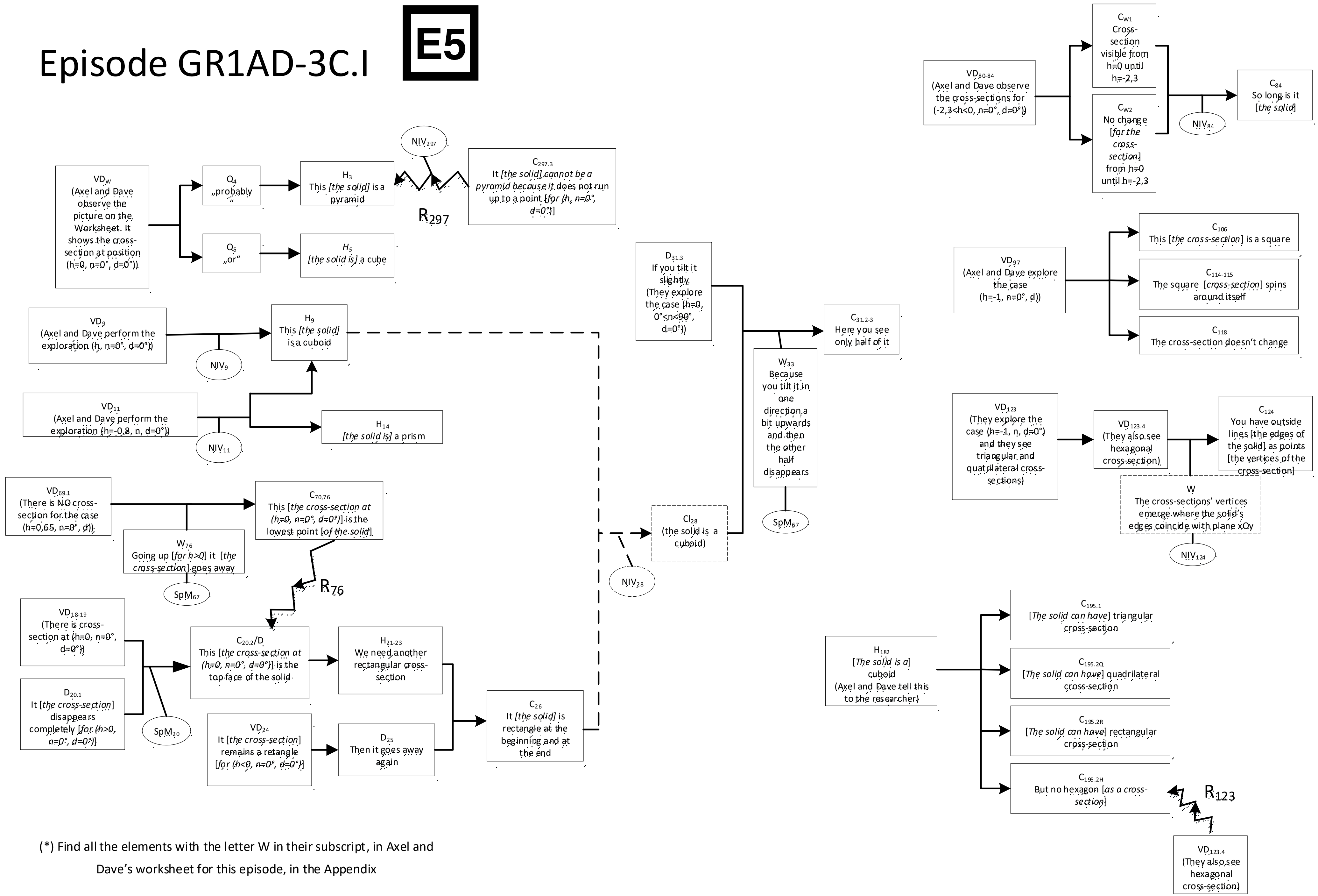
E4



(*) Find the conclusion C_W in Axel and Dave's worksheet in the Appendix

Episode GR1AD-3C.I

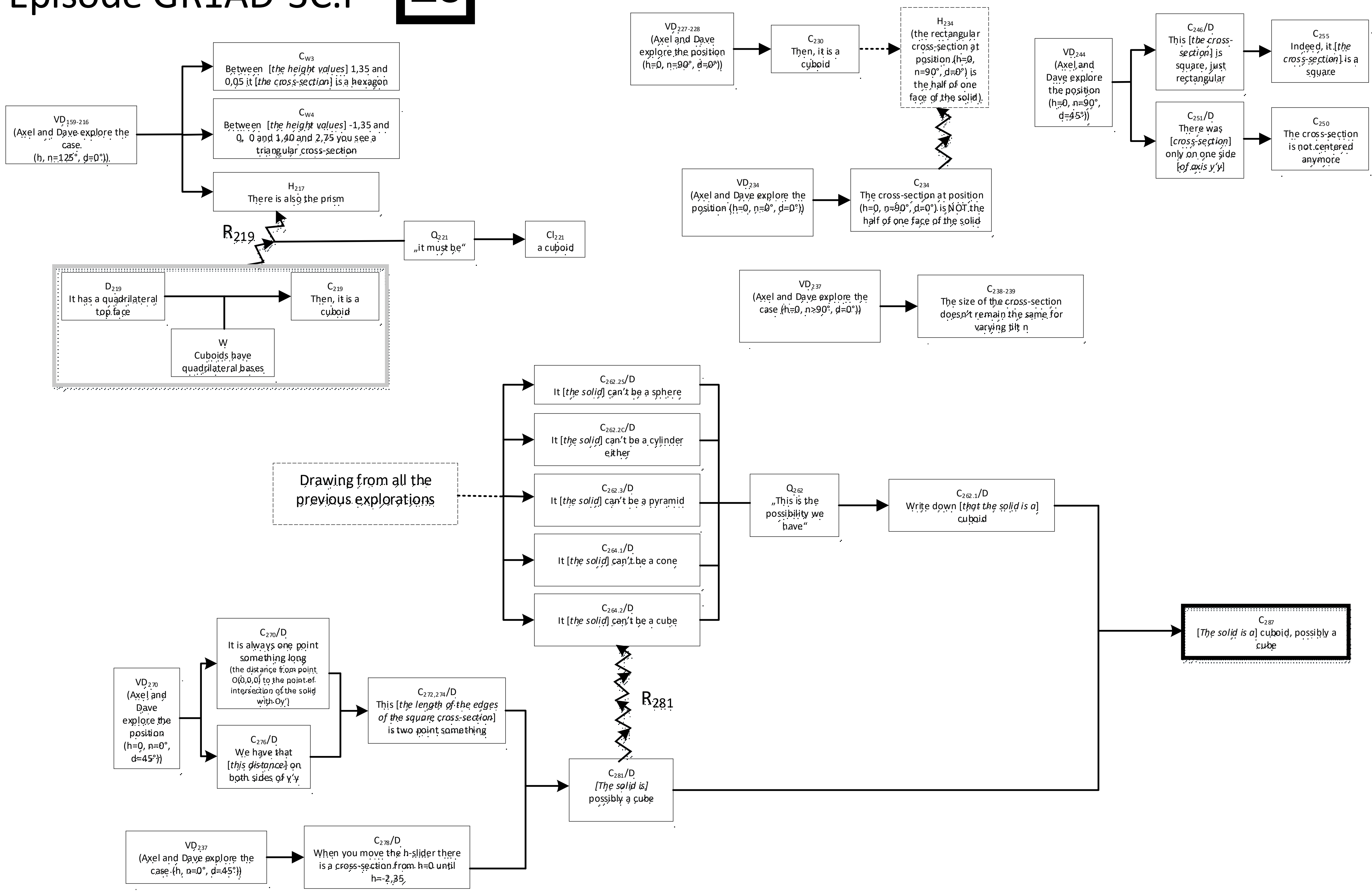
E5



(*) Find all the elements with the letter W in their subscript, in Axel and Dave's worksheet for this episode, in the Appendix

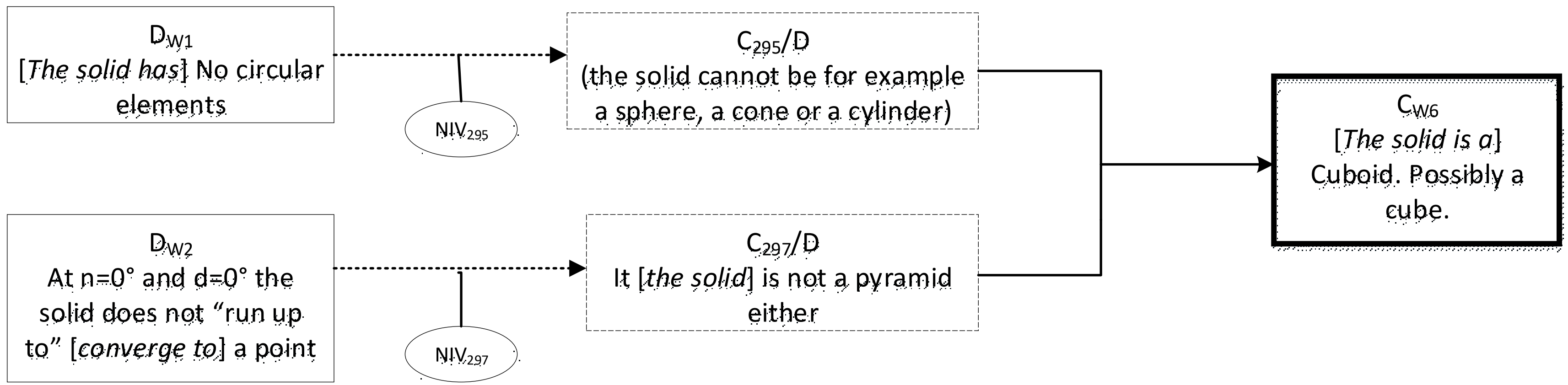
Episode GR1AD-3C.I

E5



Episode GR1AD-3C.I **E5**

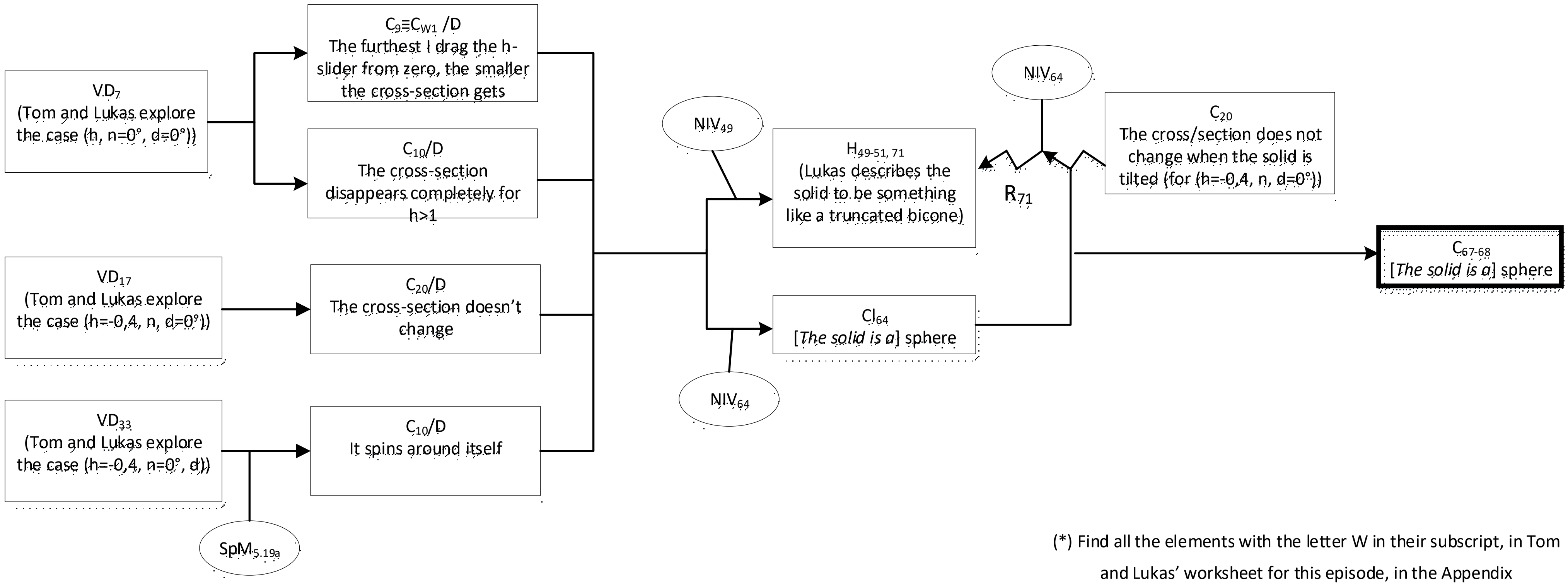
Argumentation structure of Axel and Dave's written justification



(*) Find all the elements with the letter W in their subscript, in Axel and Dave's worksheet for this episode, in the Appendix

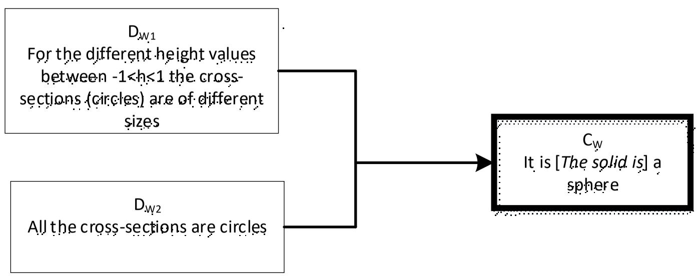
Episode GR2TL-2

E6



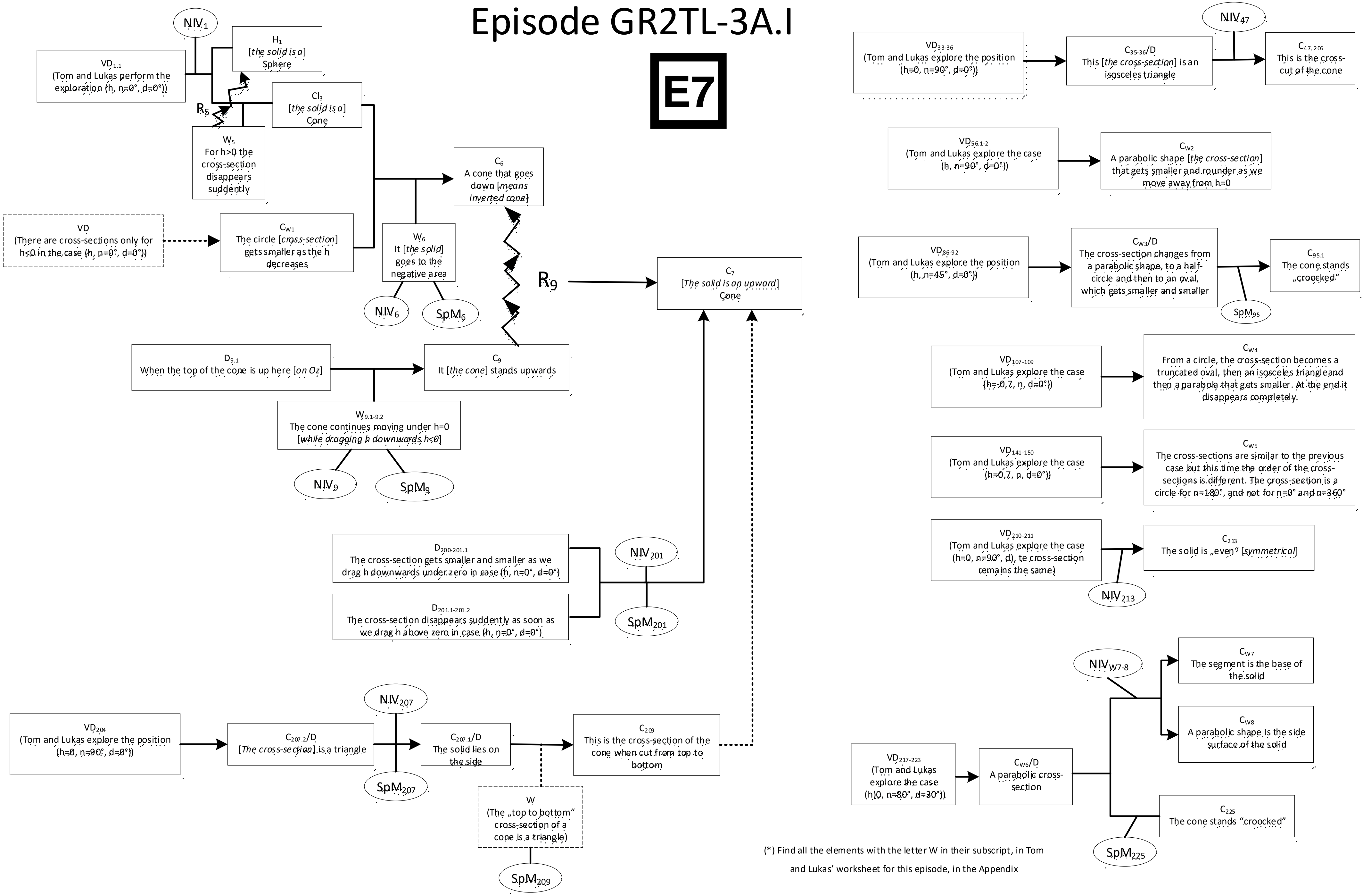
(*) Find all the elements with the letter W in their subscript, in Tom and Lukas' worksheet for this episode, in the Appendix

Argumentation structure of Tom and Lukas' written justification



Episode GR2TL-3A.I

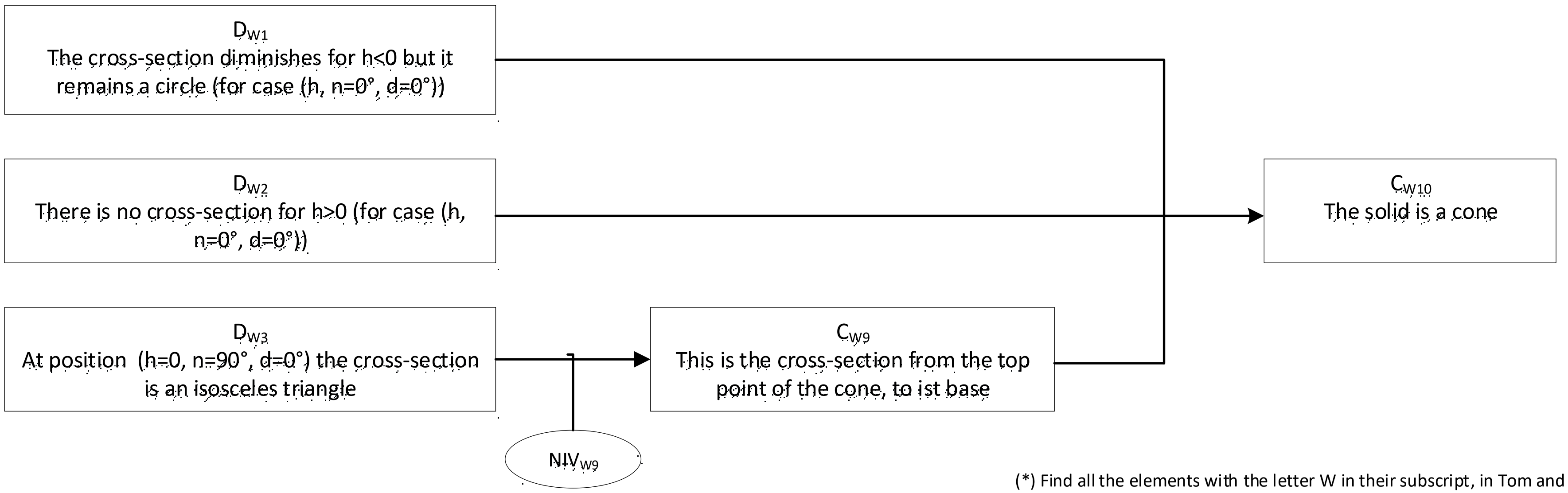
E7



(*) Find all the elements with the letter W in their subscript, in Tom and Lukas' worksheet for this episode, in the Appendix

Episode GR2TL-3A.I E7

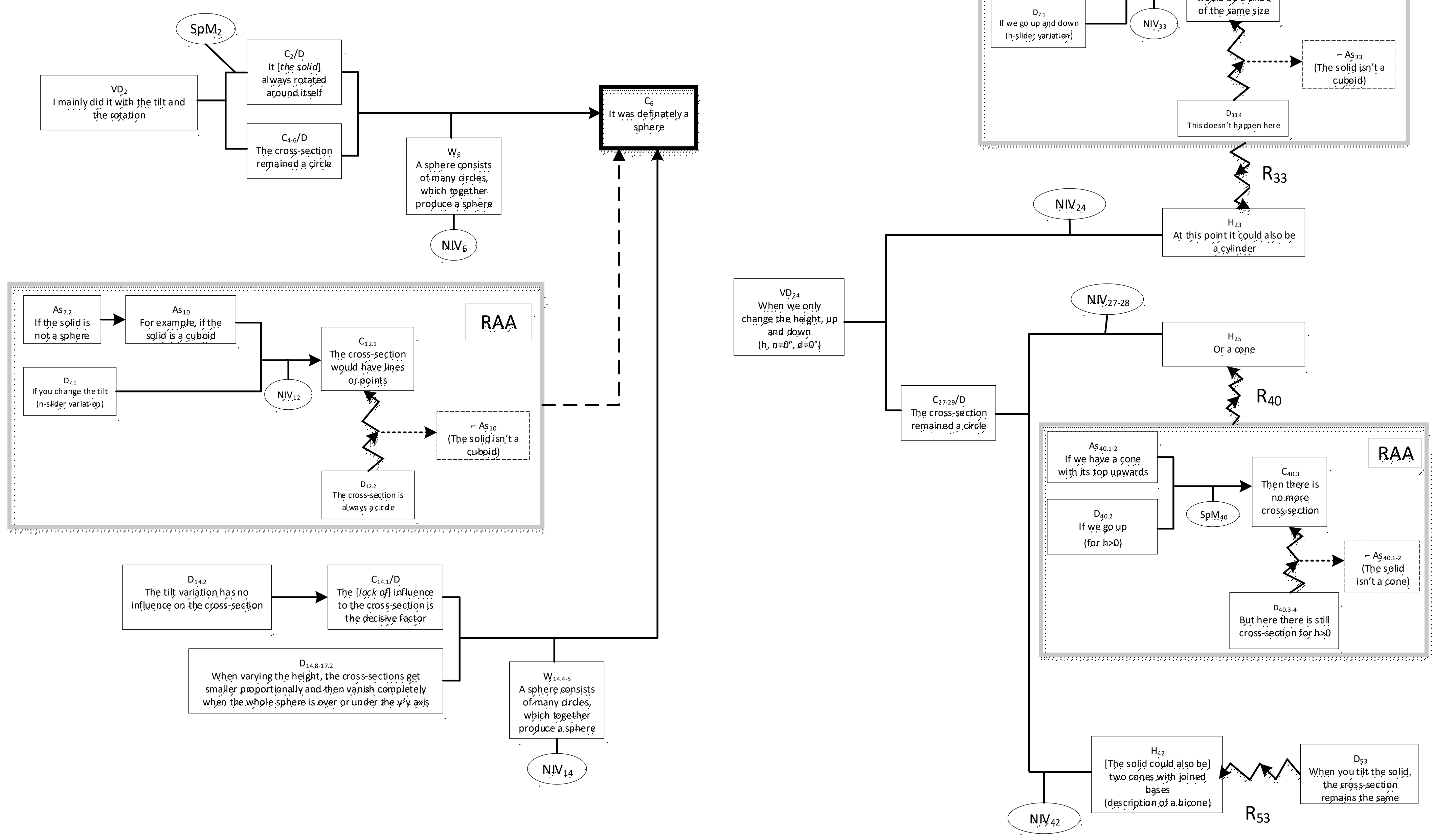
Argumentation structure of Tom and Lukas' written justification



(*) Find all the elements with the letter W in their subscript, in Tom and Lukas' worksheet for this episode, in the Appendix

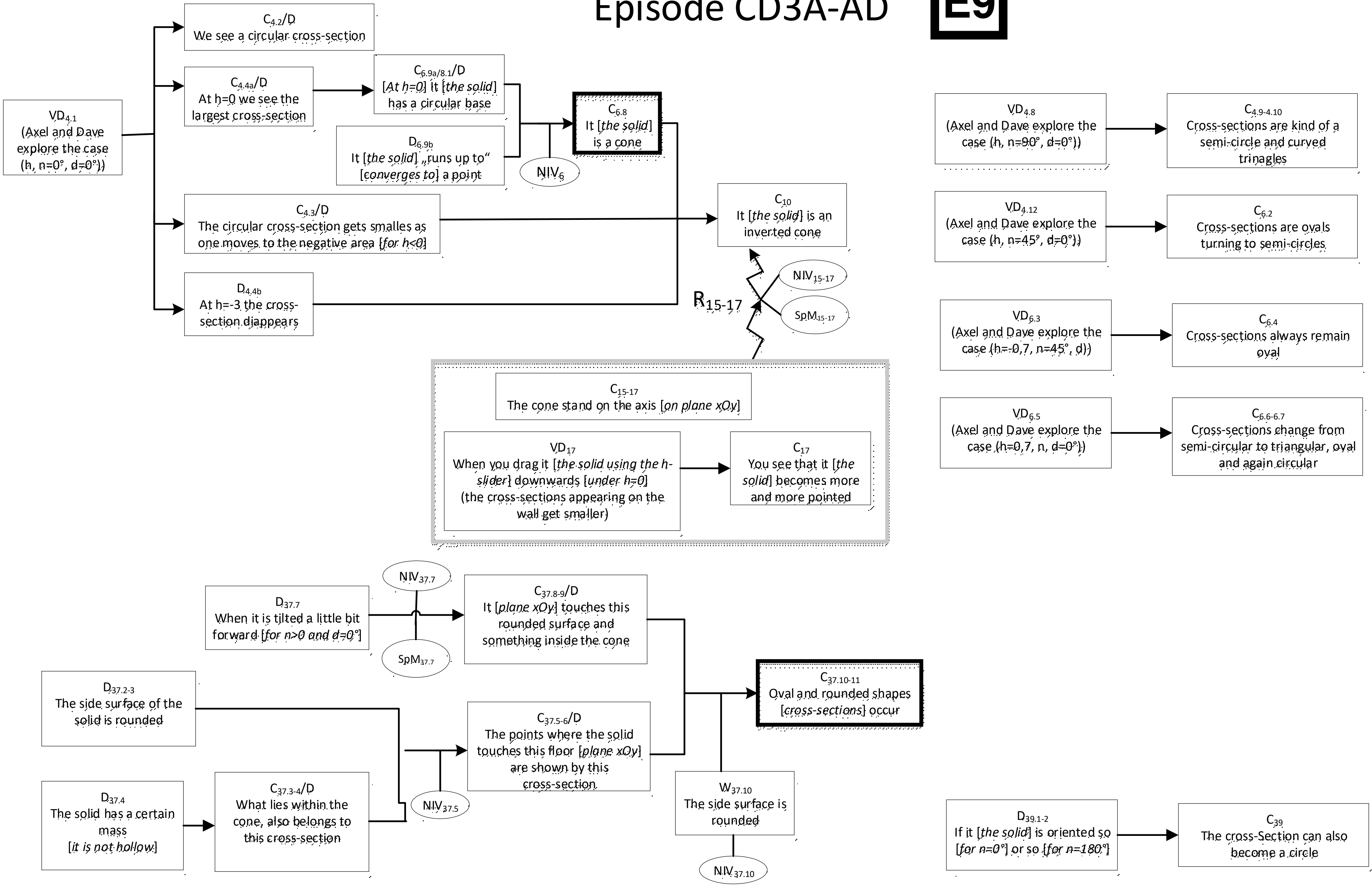
Episode CD2

E8



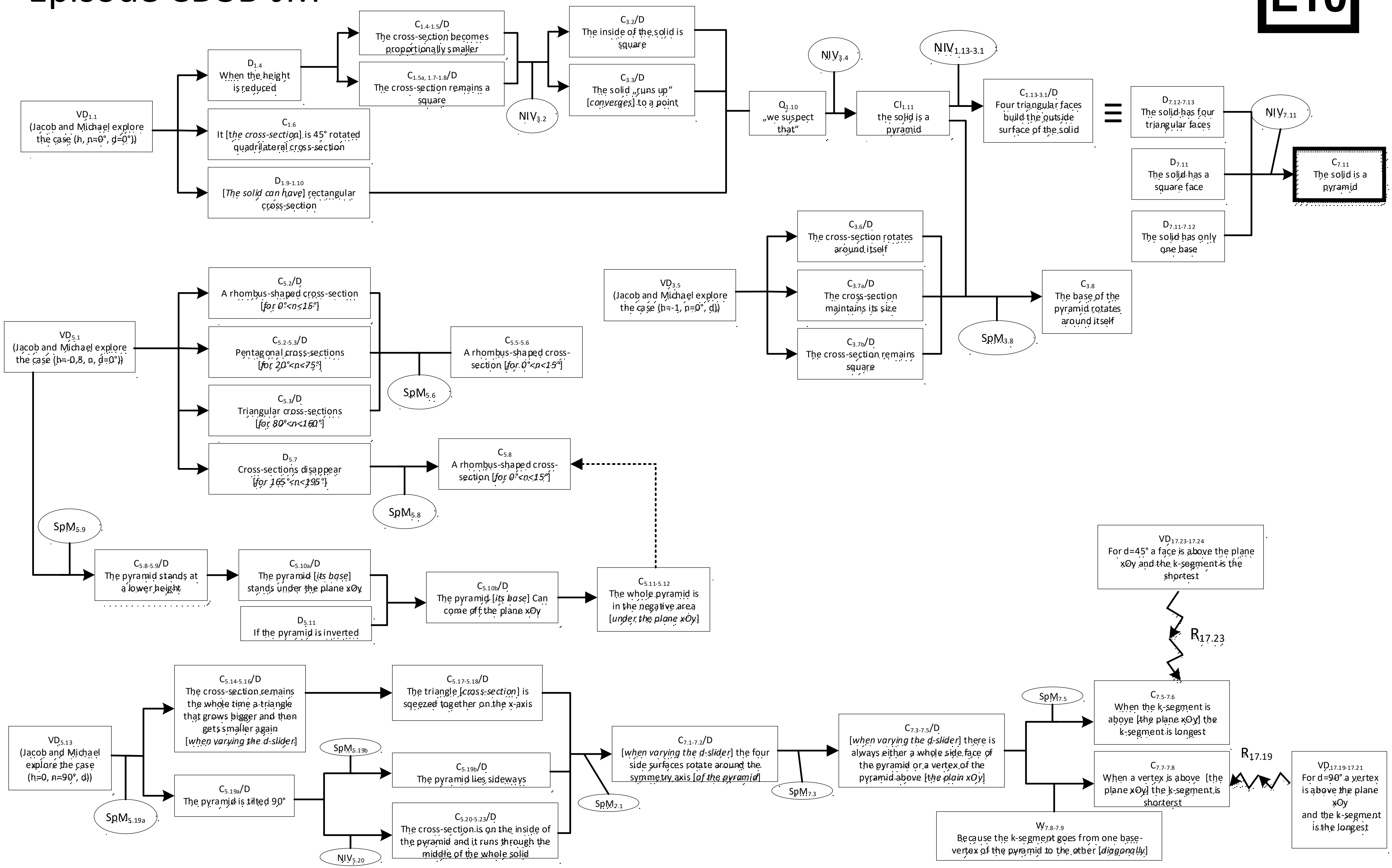
Episode CD3A-AD

E9

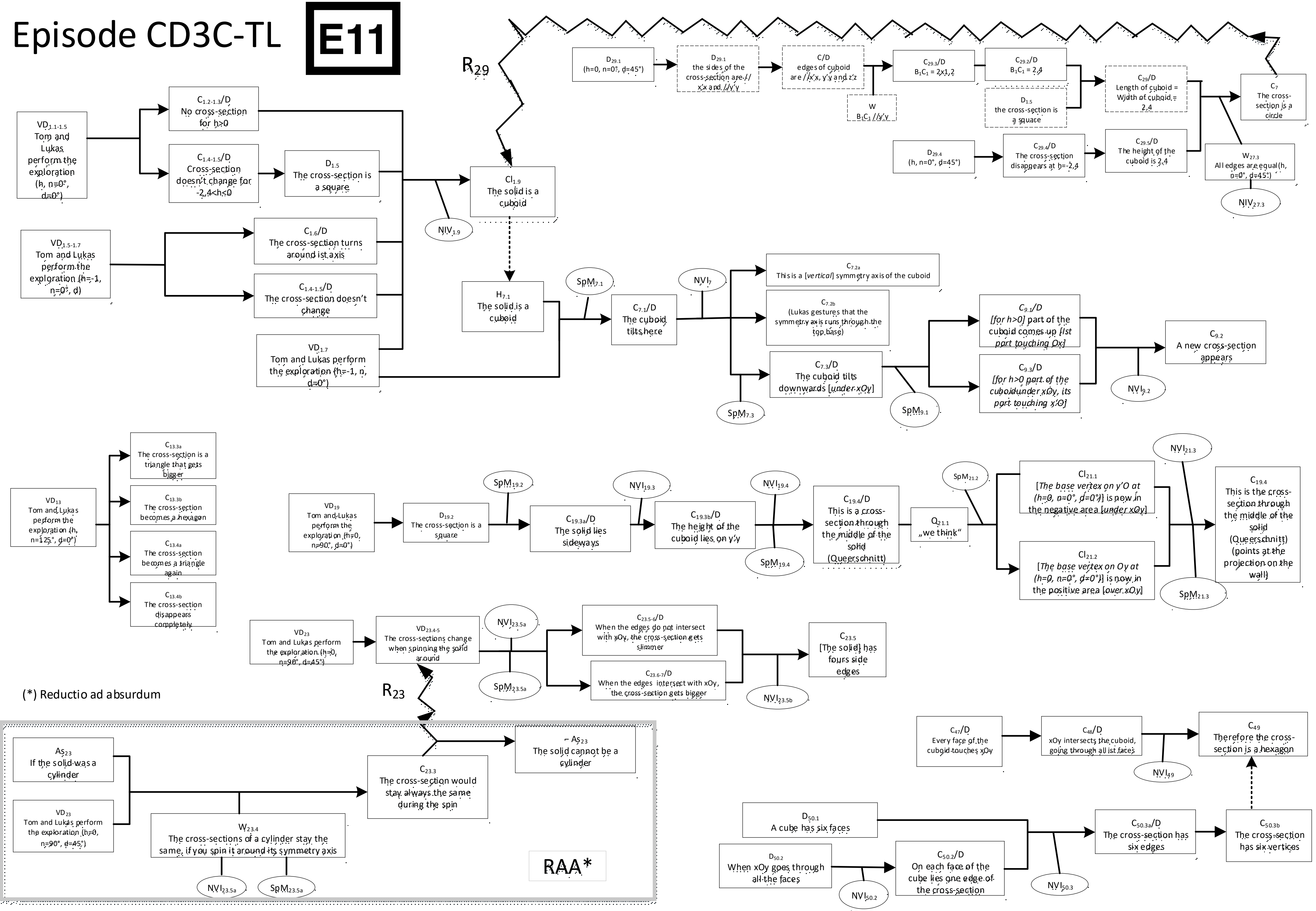


Episode CD3B-JM

E10



Episode CD3C-TL **E11**



(*) Reductio ad absurdum

F. Table of Roles of non-iconic visualization and spatial manipulation with analysis information

		Roles of NI-visualization and Sp-manipulation															
		Role 1: Creating a H/Cl			**Role 2: Drawing a C**			**Role 3: Explaining a VD**			**Role 4: Creating a R**			**Role 5: Backing a W**			
Modes → / **Episodes ↓**		**NIV**	**Syn**	**SpM**	**NIV**	**Syn**	**SpM**	**NIV**	**Syn**	**SpM**	**NIV**	**Syn**	**SpM**	**NIV**	**Syn**	**SpM**	**# of appearances of each case per episode**
Pair-work episodes	**GR1AD-2**	2			1		1			1	1					1	3NIV 0Syn 2SpM
	GR1AD-3A	1			5						1						7NIV 0Syn 0SpM
	GR1AD-3B	1	1	1	1		2	1	1	1							1NIV 2Syn 2SpM
	GR1AD-3C	3			5	1	2	1			3					1	7+2NIV 1Syn 3SpM
	GR2TL-2	1					1			1	1						2NIV 0Syn 1SpM
	GR2TL-3A	1			5	2	2								2		5+1NIV 4Syn 2SpM
Classroom discussions	**CD2**	2			1	1	1			2		1		3			6NIV 1Syn 2SpM
	CD3A-AD				1			1	1			1		1			3NIV 2Syn 0SpM
	CD3B-JM	1			4		9			9							5NIV 0Syn 9SpM
	CD3C-TL	1			7	2	2	1	3	5		1		1	1		9NIV 4Syn 5SpM

G. Scanned worksheets from the pair-work of video-recorded students

G1

Worksheet of Axel and Dave on Task 2
Episode GR1AD-2

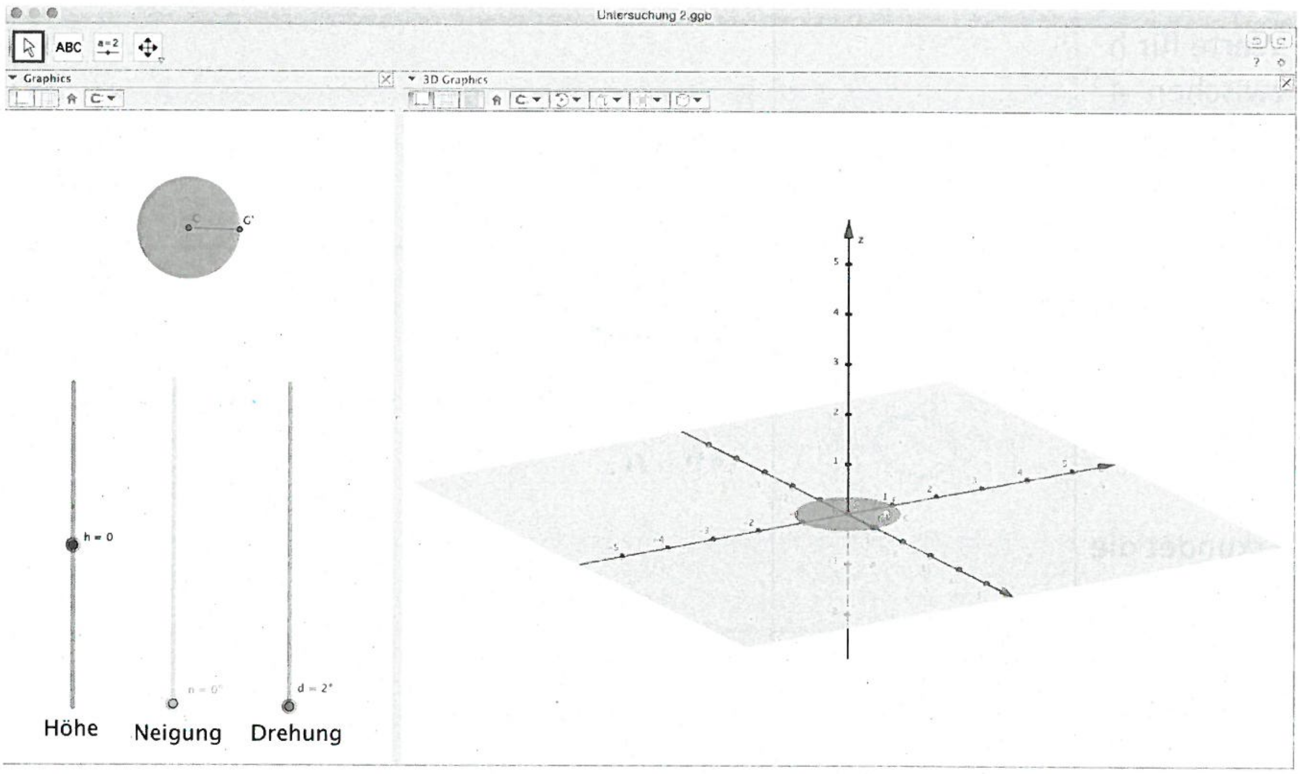
Freitag, den 17. Februar 2017

Arbeitsblatt 2

Unsere „Undercover“-Namen sind: Die Madame
On-Lane

Untersuchung 2 – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 2“.



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die "Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2 und 3 soll als Notizbuch für eure Untersuchungen verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		~~Kreisförmig~~ - immer kreisförmig (1-(-1)) - nur der Durchmesser verändert sich.
h=0,4 d=0° Erkundet die Werte für n zwischen 0° und 360°.		- Kreisförmig - ~~passiert~~ - verändert sich nicht beim Bewegen von n.
h=0,4 n=0° Erkundet die Werte für d zwischen 0° und 360°.		- Kreisförmig - Schnittstelle verändert sich nicht.

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Kugel

-> Schnittstelle, wenn vorhanden, immer kreisförmig. ~~Neig~~

- Neigung hat keine Auswirkung auf Schnittstelle, weil sie eine Kugel die Strecke vom Mittelpunkt zu einer beliebigen Außenstelle immer gleich ist.

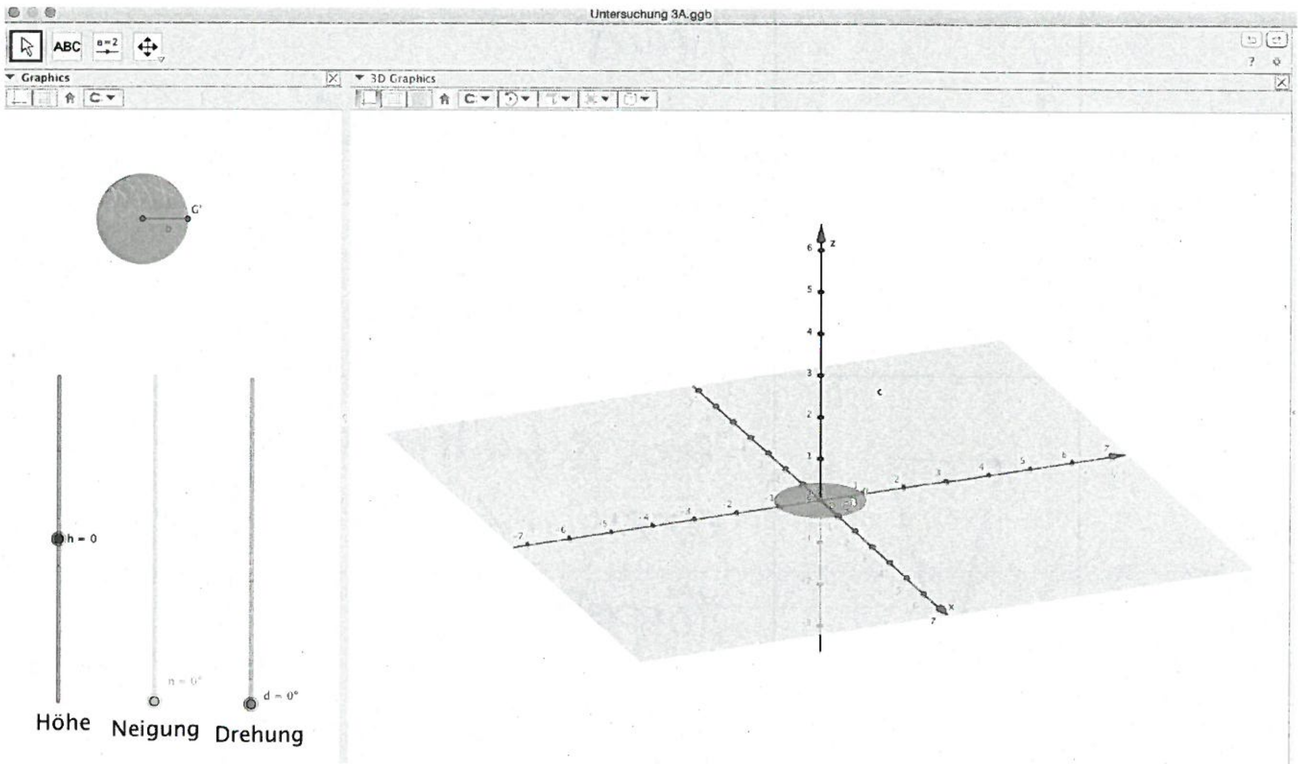
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover"-Namen sind: Oh Leuye
Die Madame

Untersuchung 3A – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffne die GeoGebra-Datei „Untersuchung 3A".

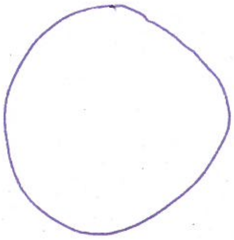


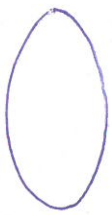
In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren" sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2, 3 und 4 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Datum Dw1 • Die Kreisfläche wird im Negativenbereich kleiner. Conclusion Cw1 Bei h=0° hat es ~~die~~ der Körper seine Grundfläche und bei h=-3° seine "spitzes Ende Conclusion Cw2
h=0 n=90° d=0°		• Die Schnittfläche zeigt einen Längsschnitt des Körpers. Conclusion Cw3
n=90° d=0° Erkundet die Werte für h zwischen -4 und 4.		Conclusion Cw4 • Von h=-1 bis h=0° ~~Wo~~ wird die Schnittfläche größer, man sieht ein „Dreieck" mit runden Ecken. • ~~Dann~~ ab h≥0°+ nimmt die Schnittfläche wieder ab. Datum Dw2

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=45° d=0° Erkundet die Werte für h zwischen -4 und 4.	Von „Oval"-Förmig zu „Halbkreis"-Förmig. **Conclusion Cw5**	
h= -0,7 n=45° Erkundet die Werte für d zwischen 0° und 360°.		• Man sieht ein Oval. **Conclusion Cw6**
h= 0,7 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Die Schnittfläche ist von Halbkreis-, zu Dreieck, zu Oval, zu Kreisförmig. **Conclusion Cw7**	

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

- Kegel

Conclusion Cw8

Datum Dw3 – Eine ~~Rundegrundfläche~~ runde Grundfläche

Datum Dw4 – Läuft spitz zusammen.

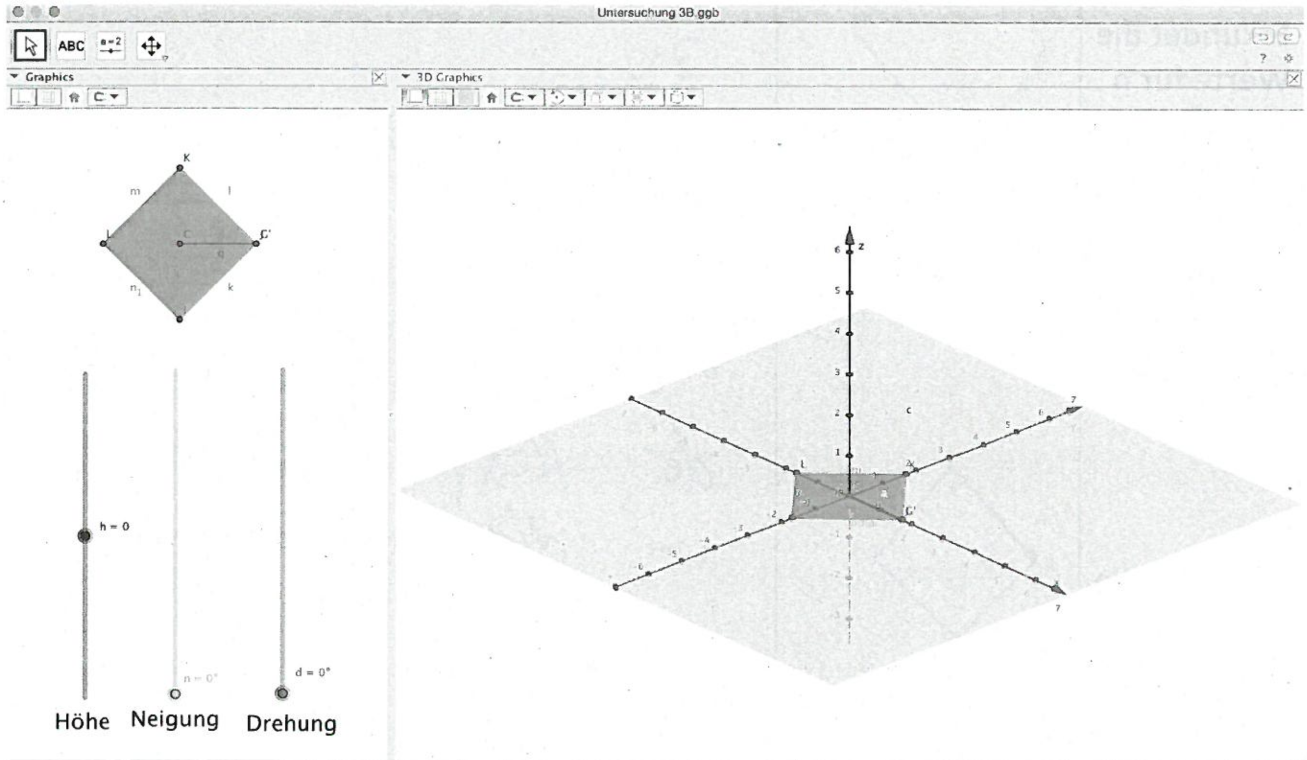
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover"-Namen sind: Oh-Lane,
Die Madame

Untersuchung 3B – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 3B".

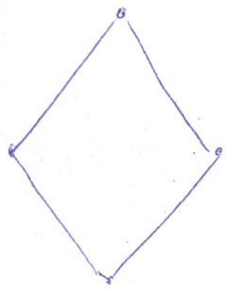
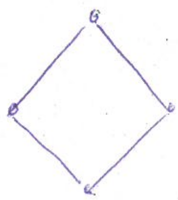


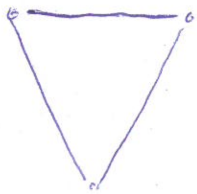
In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren" sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2 und 3 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		• Wenn man in den Negativenbereich geht, läuft der Körper spitz zu.
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.		• Die Schnittfläche und der Körper drehen sich um die eigende Achse.
h= -0,5 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Die Schnittflächen ändert sich von ~~[illegible]~~, 4-,5-, 3-Eck.	

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0 n=90° Erkundet die Werte für d zwischen 0° und 360°.		• Die Schnittfläche wird in 90° intervallen größer und wieder kleiner. Bei 0°, 90°, 180°, 270°, 360° hat das Dreieck die "Volle" größe. conclusion Cw

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

• Pyramide mit Quadratischer Grundfläche.

G4

Worksheet of Axel and Dave on Task 3C
Episode GR1AD-3C.I

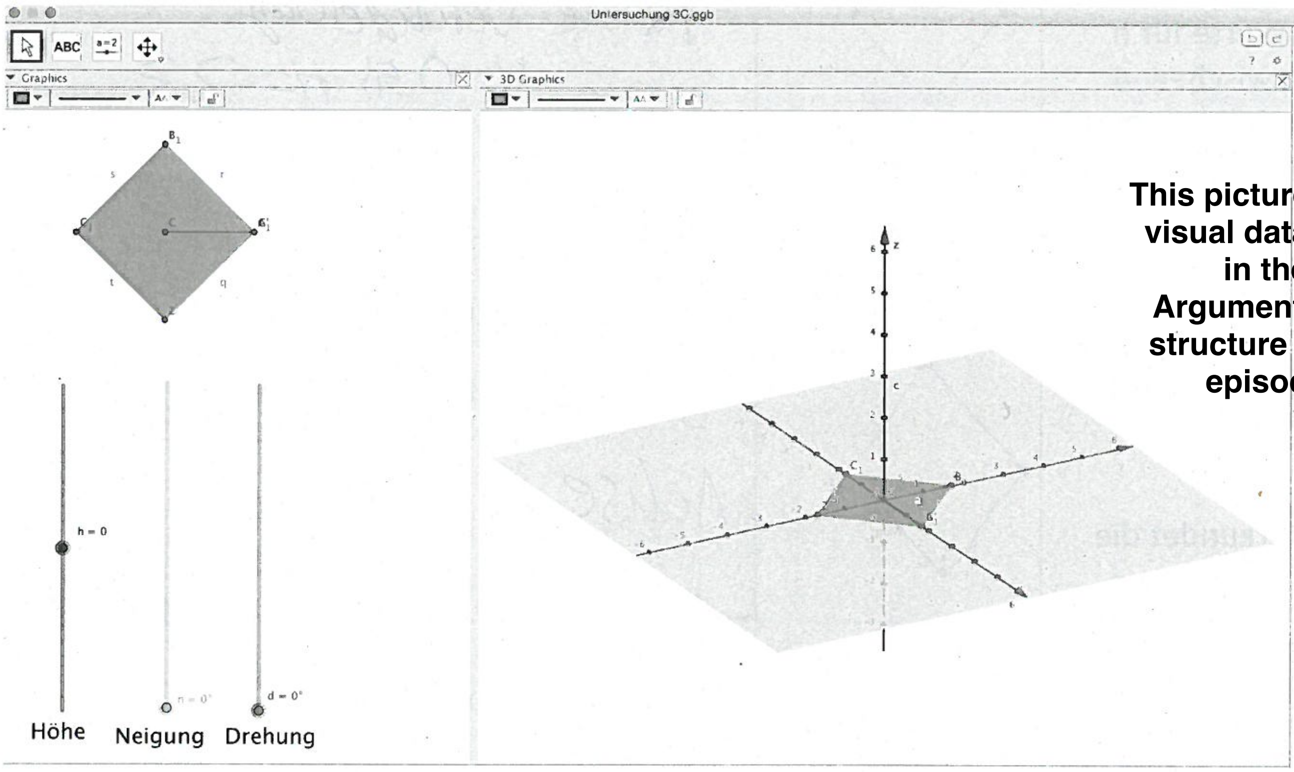
Freitag, den 24. Februar 2017

Arbeitsblatt 3

Unsere „Undercover"-Namen sind: OneLaue
DieMadame

Untersuchung 3C – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 3C".



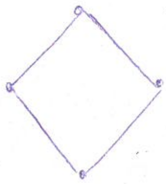
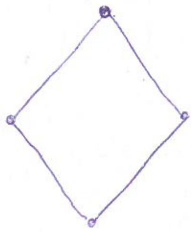
This picture is the visual data VDw in the Argumentation structure of the episode

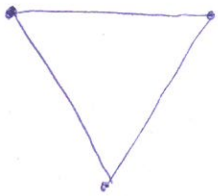
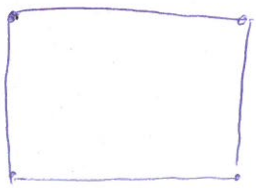
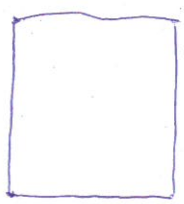
In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht. Eure Mission ist es, die „Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren" sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 2, 3 und 4 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		- ~~sichtb~~ Schnittfläche sichtbar von h=0 bis h=2,3 conclusion Cw1 - Keine Veränderung von H=0 bis H=-2,3 conclusion Cw2
h= -1 n=0° Erkundet die Werte für d zwischen 0° und 360°.	d=0° 	- Das Quadrat dreht sich um die eigende Achse. - Die Schnittfläche verändert sich nicht.
h= -1 d=0° Erkundet die Werte für n zwischen 0° und 360°.	Es gibt 3-, 4-, 6-Ecke.	- Bei der Neigung sieht man zuerst ein Viereck dann ein Sechseck, dann ein Viereck und dann ein Dreieck.

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=125° d=0° Erkundet die Werte für h zwischen -4 und 4.	h=0,15 	- Zwischen 1,35 und 0,05 ist es ein Sechseck. **conclusion Cw3** - zwischen -1,35 und 0,0 und 1,40 und 2,75 sieht man eine Dreieckförmige Schnittfläche. **conclusion Cw4**
h=0 n=90° d= 0°		- Die Schnittfläche ist rechteckig. **conclusion Cw5**
h=0 n=90° d=45°		- Die Schnittfläche ist viereckig. - Quadratisch

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

[redacted]

[redacted]

[redacted]

G5

Worksheet of Tom and Lukas on Task 2
Episode GR2TL-2

Freitag, den 17. Februar 2017

Arbeitsblatt 2

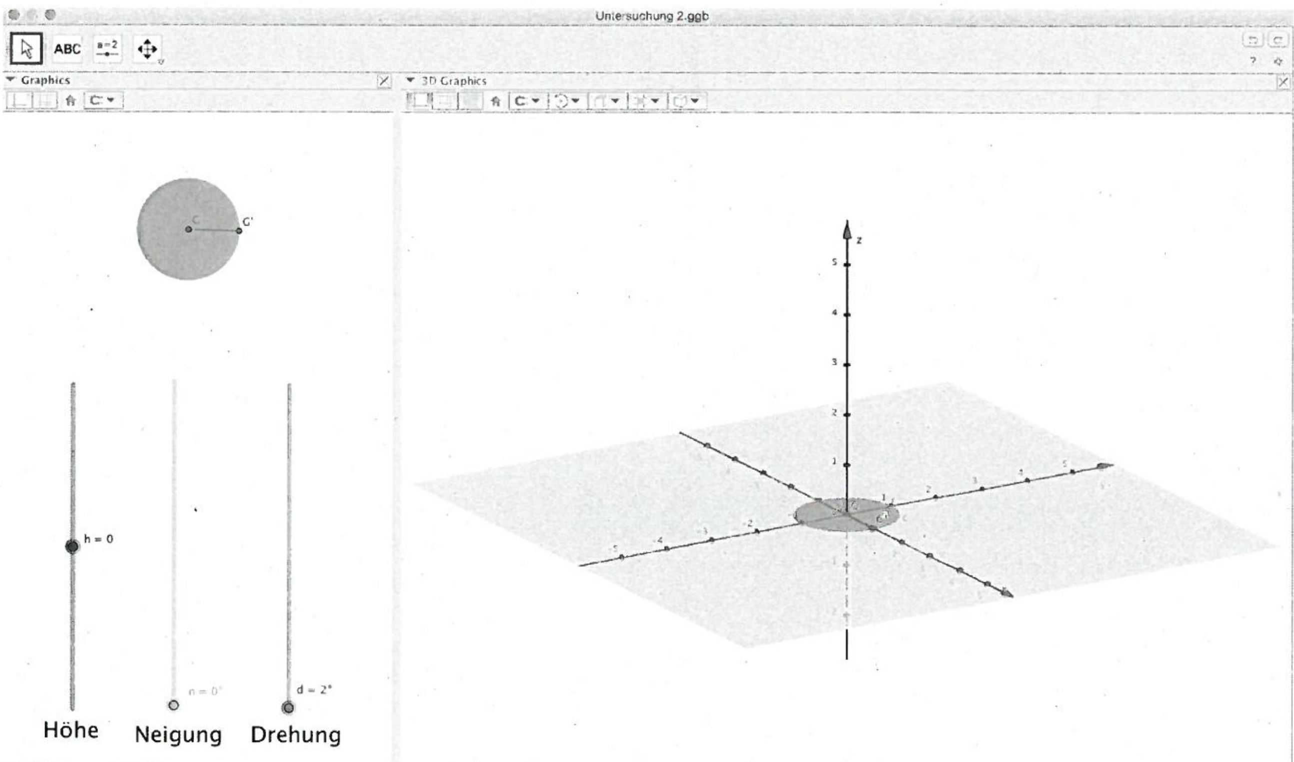
Unsere „Undercover"-Namen sind:

Draculahd ; King Saphiero

....

Untersuchung 2 – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffnet die GeoGebra-Datei „Untersuchung 2".



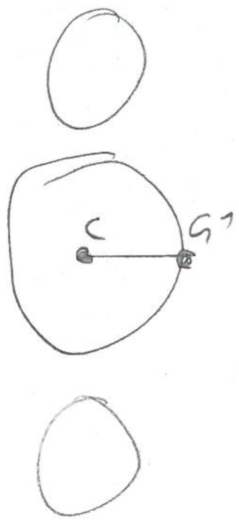
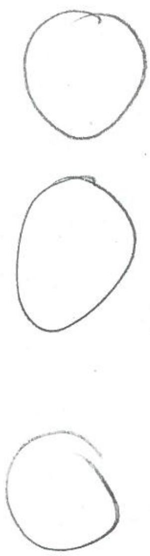
In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem

man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht.
Eure Mission ist es, die "Spuren" zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

a. Die Erkundungstabelle auf den Seiten 3 bis 5 soll als Notizbuch für eure Untersuchungen verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Der Kreis wird ~~größer~~ kleiner, sobald man ~~die~~ h erhöht bzw. ins Negative erniedrigt
h=0,4 d=0° Erkundet die Werte für n zwischen 0° und 360°.		Kreis es passiert nichts mit der Schnittfläche,

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h=0,4 n=0° Erkundet die Werte für d zwischen 0° und 360°.		Der Kreis bleibt gleich, die Figur dreht sich um die eigene Ach-se **Conclusion Cw1**
h=1 d=251° n= 0° bis 360°		Der Punkt verändat sich nicht.

h/n/d	Skizze der Schnittfläche	**Bezeichnung und Eigenschaften der Schnittfläche** Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?

b. Könnt ihr anhand der „Spuren“, die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Es ist eine Kugel. **Conclusion Cw**

Datum Dw1

Bei den verschiedenen ~~Hage~~ Höhen h von 1 bis -1 sind die Schnittflächen (Kreise) unterschiedlich groß. Außerdem besteht die Schnittfläche nur aus Kreisen

Datum Dw2

G6

Worksheet of Tom and Lukas on Task 3A
Episode GR2TL-3A.I

Freitag, den 24. Februar 2017

Arbeitsblatt 3

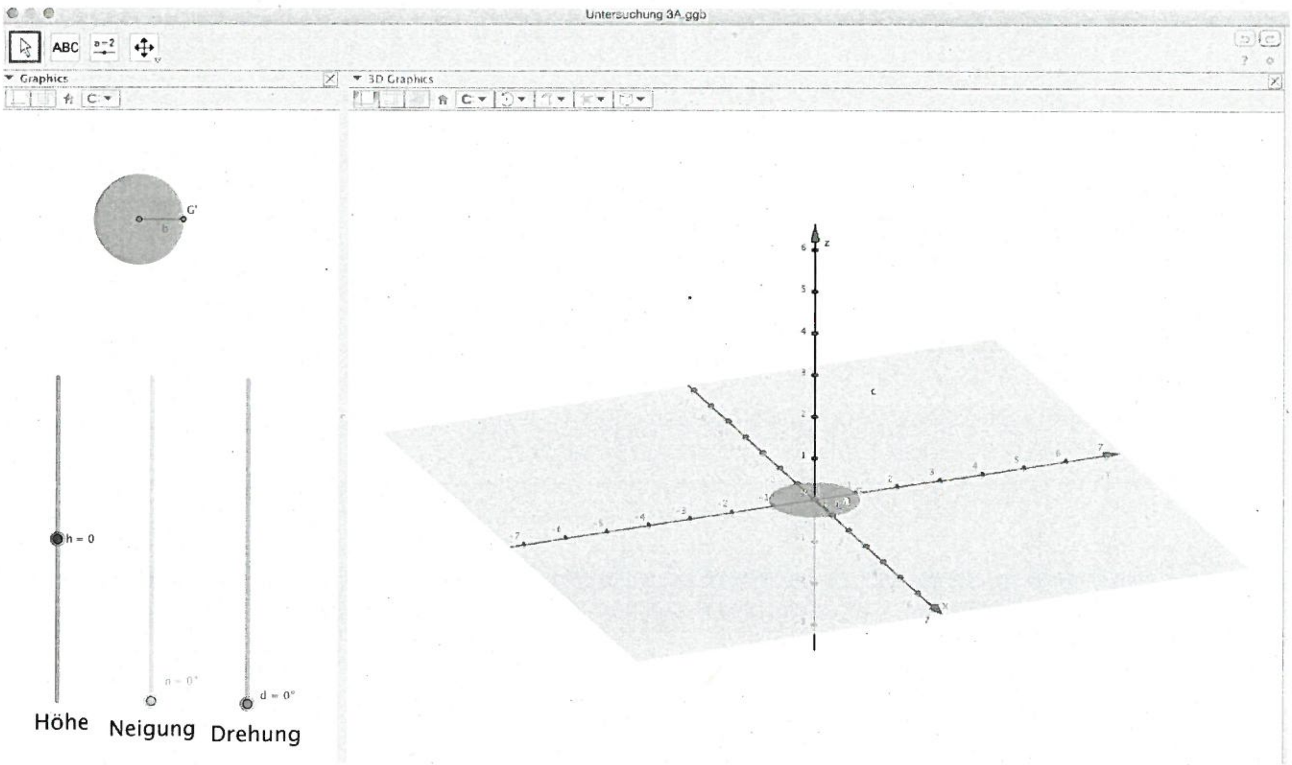
Unsere „Undercover"-Namen sind:

Draculahd

King Saphiero

Untersuchung 3A – Ein unsichtbarer Körper! Kannst Du die Spuren lesen?

Öffne die GeoGebra-Datei „Untersuchung 3A".



In dieser Datei seht ihr ein dreidimensionales Koordinatensystem mit x-, y- und z-Achse, in dem

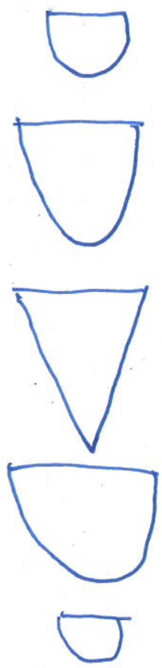
man nur die (grüne) Schnittfläche eines unsichtbaren Körpers sieht.
Eure Mission ist es, die „Spuren“ zu finden, die euch bei der Identifizierung des versteckten Körpers helfen können.

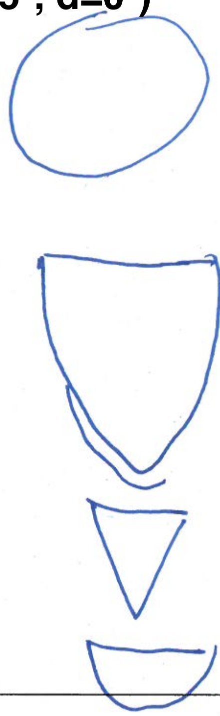
Mit dem Wort „Spuren“ sind die Eigenschaften und Merkmale der verschiedenen Schnittflächen gemeint, die durch die Veränderung der drei Schieberegler sichtbar werden.

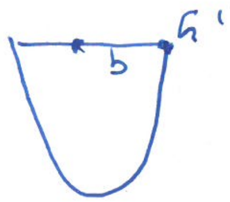
a. Die Erkundungstabelle auf den Seiten 3 bis 6 soll als Notizbuch eurer Untersuchungsverfahren verwendet werden.

Erkundungstabelle

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=0° d=0° Erkundet die Werte für h zwischen -4 und 4.		Der Kreis wird, je ~~Re~~ kleiner h ist, immer kleiner. Conclusion Cw1
h=0 n=90° d=0°		ein gleichschenkliches Dreieck Das ist der Durchmesser der Figur

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
n=90° d=0° Erkundet die Werte für h zwischen -4 und 4.		Eine parabelförmige Form, die, je weiter man sich von h=0 entfernt, immer kleiner und runder wird **Conclusion Cw2**
n=45° d=0° Erkundet die Werte für h zwischen -4 und 4.		Von einer parabelförmigen Schnittfläche verändert sich diese zu einem Halbkreis und weiter zu einem Oval. Dieses wird immer kleiner. **Conclusion Cw3**

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
Here the students actually explored the case (h=-0,7, n=45°, d=0°) ~~h= -0,7~~ ~~n=45°~~ Erkundet die Werte für d zwischen 0° und 360°.		Von einem Kreis wird die Schnittfläche zu einem abgeschnittenen Oval, dann zu einem gleichschenklige Dreieck und dann zu einer immer kleiner werdenen Parabell. Am Ende verschwindet sie komplett. Conclusion Cw4
h= 0,7 d=0° Erkundet die Werte für n zwischen 0° und 360°.		Die Schnittflächen sind ähnlich wie die Schnittflächen von der Aufgabe davor. Nur dieses Mal ist die Abfolge der Schnittflächen fortlaufend, d.h. Der Kreis ist bei n= 180° und nicht bei n= 0°; 360° Conclusion Cw5

h/n/d	Skizze der Schnittfläche	Bezeichnung und Eigenschaften der Schnittfläche Wie ist die Schnittfläche mit den Eigenschaften des Körpers verbunden?
h = 0 n = 80° d = 30°	 **Conclusion Cw7**	Eine parabelförmige Schnittfläche. **Conclusion Cw6** Die Strecke ist die Grundfläche des Körpers. Die parabelförmige Form ist die Fläche des Mantels des Körpers **Conclusion Cw8**
h = -4 h = 1° h = 5		Keine Schnittfläche !!!

b. Könnt ihr anhand der „Spuren", die ihr bis hierhin gesammelt habt, den unsichtbaren Körper identifizieren? Begründet eure Vermutung.

Es ist ein Kegel.
Bei der ersten Einstellung sieht man, dass die Schnittfläche bei $h<0$ immer kleiner wird, aber immer noch ein Kreis bleibt. Bei $h>0$ ist keine Schnittfläche zu sehen. Bei $h=0$, $\alpha=90°$ sieht man, dass die Schnittfläche ein gleichschenkliges Dreieck ist. Das ist der Schnitt von der Grundfläche G zur Spitze des Körpers. Da das ein gleichsch. Dreieck ist, ist der Körper ein Kegel.

Datum Dw1

Datum Dw2

Datum Dw3

Conclusion Cw9

Conclusion Cw10