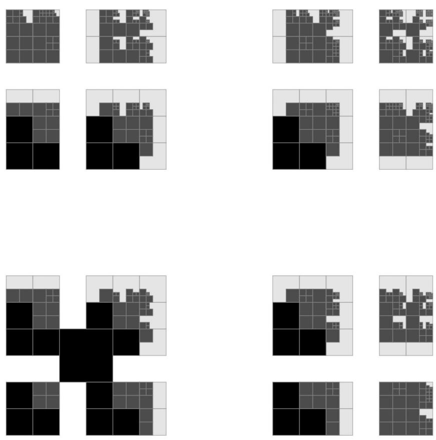
Optimal Partition Problems and Applications to Kreĭn–Feller Operators and Quantization Problems



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Abstract

We study the lower and upper partition entropy and the lower and upper optimized coarse multifractal dimension with respect to certain set functions defined on the set of the dyadic cubes in $\mathbb{R}^d$. For this purpose, we introduce the notion of partition functions, generalizing the well-known L^q-spectrum. We find a formula for the upper partition entropy in terms of the zero of the associated partition function. Further, we establish a connection between the classical works of Solomjak and Birman [BS66; BS74], Borzov [Bor71], and the partition entropy, improving classical results. We give regularity conditions guaranteeing that the lower and upper partition entropy coincide. Based on these general results, we develop a unified framework to tackle both the computation of the upper spectral dimension of Kreĭn–Feller operators with respect to Neumann boundary conditions and the computation of the upper quantization dimension. Furthermore, this enables us to establish regularity conditions, ensuring that the lower and upper spectral dimension, as well as the lower and upper quantization dimension, coincide. The results are illustrated by several examples; in particular, we prove that the spectral dimension and the quantization dimension of self-conformal measures, with or without overlap, exist and can be computed in terms of the L^q-spectrum of the underlying measure. We also determine various lower and upper bounds for the lower and upper spectral dimensions, as well as for the lower and upper quantization dimensions in terms of the associated L^q-spectrum, establishing, in particular, sharp bounds that depend only on the upper Minkowski dimension of the support of the measure. We give first examples in which the lower and upper spectral dimensions do not coincide.

Zusammenfassung

Wir untersuchen die untere und obere Partitionsentropie bezüglich bestimmter Mengenfunktionen, die auf der Menge der d-dimensionalen dyadischen Würfel definiert sind. Zu diesem Zweck führen wir den neuen Begriff der Partitionsfunktion ein, der das bekannte L^q-Spektrum verallgemeinert. Wir finden eine Formel für die obere Partitionsentropie in Form der Nullstelle der zugehörigen Partitionsfunktion. Außerdem stellen wir eine Verbindung zwischen der Partitionsentropie und den klassischen Arbeiten von Solomjak und Birman [BS66; BS74] und Borzov [Bor71] her und verbessern damit klassische Ergebnisse. Darüber hinaus stellen wir Regularitätsbedingungen auf, die garantieren, dass die untere und obere Partitionsentropie übereinstimmen. Aufbauend auf diesen allgemeinen Ergebnissen sind wir in der Lage, einen einheitlichen Rahmen zur Berechnung der oberen Spektraldimension von Kreĭn–Feller-Operatoren unter Berücksichtigung Neumann-Randbedingungen sowie der oberen Quantisierungsdimension zu entwickeln. Weiter können wir so Regularitätsbedingungen aufstellen, die sicherstellen, dass die untere und obere Spektraldimension sowie die untere und obere Quantisierungsdimension übereinstimmen. Die Ergebnisse werden durch eine Reihe von Beispielen veranschaulicht. Insbesondere beweisen wir, dass die Spektraldimension und die Quantisierungsdimension bezüglich selbstkonformer Maße mit und ohne Separierungsbedingungen existieren und mit Hilfe des L^q-Spektrums berechnet werden können. Es werden mehrere untere und obere Schranken für die untere und obere Spektraldimension sowie für die untere und obere Quantisierungsdimension in Abhängigkeit des L^q-Spektrums des zugrunde liegenden Maßes bewiesen, insbesondere erhalten wir scharfe Schranken in Abhängigkeit der oberen Minkowski-Dimension des Trägers des zugrunde liegenden Maßes. Des Weiteren geben wir erste Beispiele an, in denen die obere und untere Spektraldimension nicht übereinstimmen.

Contents

Chapter 1

Introduction

In 1911, H. Weyl [Wey11] studied the following Dirichlet eigenvalue problem

$$\begin{cases} \Delta^D u = -\lambda u, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded domain and Δ^D denotes the classical Laplace operator, i.e. $\Delta^D u = \sum_{i=1}^d \partial u / \partial x_i$ with respect to Dirichlet boundary conditions. Assuming some regularity conditions on Ω, Weyl proved that the associated eigenvalue counting function, denoted by N^D, obeys the following law

$$N^D(x) = \frac{\mathrm{vol}_d(\Omega)\,\omega_d}{(2\pi)^d} x^{d/2} + o\left(x^{d/2}\right), \tag{1}$$

where ω_d is the d-dimensional volume of the unit ball and o denotes the Landau symbol, i.e. $f = o(g)$ if $\limsup_{x\to\infty} |f(x)|/|g(x)| = 0$. Nowadays, the asymptotic expansion (1) is known as *Weyl's law*. This result has been extended to arbitrary bounded domains by Métivier [Mét77]. Weyl's pioneering works have stimulated wide range of activities on this topic, in which many papers are concerned with estimating the remainder term of (1) or investigating generalizations of the Laplace operator in various ways. In the present thesis, we also follow this line of investigation by considering a generalization of the classical Laplacian. The physicist M. Berry [Ber79] conjectured that the remainder term of (1) is driven by the Hausdorff dimension of $\partial\Omega$, i.e.

$$N^D(x) = \frac{\mathrm{vol}_d(\Omega)\,\omega_d}{(2\pi)^d} x^{d/2} + O\left(x^{\dim_H(\partial\Omega)/2}\right).$$

Here, O denotes the Landau symbol, i.e. $g = O(f)$ if $\limsup_{x\to\infty} |g(x)|/|f(x)| < \infty$. Nowadays, this is known as the *Weyl–Berry conjecture*, which turned out to be incorrect, as shown by Brossard and Carmona [BC86]. Moreover, Brossard and Carmona [BC86] suggested replacing the Hausdorff dimension with the upper Minkowski dimension, which is known as the *modified Weyl–Berry conjecture*. A big step forward was made by Lapidus [Lap91], who proved that under the assumption that the upper Minkowski dimension of $\partial\Omega$ lies in $(d-1,d]$ and the Minkowski content of $\partial\Omega$ is finite, the remainder term $o(x^{d/2})$ in (1) can be replaced by $O(x^{\overline{\dim}_M(\partial\Omega)/2})$. However, in the general case, the modified Weyl–Berry conjecture has been disproved by Lapidus and Pomerance [LP96]. Also in this thesis, in the context of Kreĭn–Feller operators, it turns out that the upper Minkowski dimension is also a more appropriate concept for the description of the eigenvalue growth than the Hausdorff dimension (see Theorem 4.10 and Theorem 5.15). It should be noted that in the literature (e.g. see [Fuj87; NX20]) only cases for which the Hausdorff and the upper Minkowski dimensions coincide have been considered so far, obscuring the actual connection.

Another interesting problem is the following famous question by M. Kac [Kac66] "Can one hear the shape of the drum?", which asks whether it is possible to determine the geometry of Ω from the eigenvalues of Δ^D. In general, the answer to this question is "no". In 1964, a first counterexample for the case $d = 16$ was constructed by Milnor [Mil64]. In the following decades, further counterexamples were constructed for $d \geq 4$ by Urakawa [Ura82] and for $d = 2$ by Gordon, Webb, and Wolpert [GWW92]. Consequently, in general, we cannot expect to recover the geometry of Ω. Nevertheless, Weyl's law tells us that some geometric information such as the volume of Ω can be inferred from the eigenvalues of Δ^D. Therefore, it is an interesting and demanding problem to ascertain which information about Ω is encoded by the eigenvalues of Δ^D. We also address Kac's question in the setting of Kreĭn–Feller operators and give partial answers, which can be found in Section 4.3.3 and Section 5.3.3.

1.1 Statement of the problems

We now introduce the problems we study in this thesis.

1.1.1 Spectral problem of Kreĭn–Feller operators

We start to outline the theoretical preliminaries which are necessary to define the Kreĭn–Feller operator $\Delta_\nu^{D/N}$ for a given finite non-zero Borel measure ν on the fixed d-dimensional left-half open unit cube $\mathbf{Q} := (0,1]^d$, $d \in \mathbb{N}$. Let us fix a bounded open set $\Omega \subset \mathbf{Q}$ with Lipschitz boundary (for the definition we refer to Appendix A.2), for which we assume without loss of generality that Ω lies in the open unit cube. The *Sobolev space* $H^1(\Omega)$ is the completion of $C_b^\infty(\overline{\Omega})$ with respect to the metric given by the inner product

$$\langle f, g\rangle_{H^1(\Omega)} := \int_\Omega fg \,\mathrm{d}\Lambda + \int_\Omega \nabla f \nabla g \,\mathrm{d}\Lambda,$$

with $\nabla f := \left(\partial f/\partial_{x_1}, \ldots, \partial f/\partial_{x_d}\right)^T$ (see also Definition A.7 for an equivalent definition). Further, let $H_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ w.r.t. the same metric. Here, Λ denotes the d-dimensional Lebesgue measure, $C_c^\infty(\Omega)$ the vector space of smooth functions with compact support contained in Ω, and $C_b^\infty(\overline{\Omega})$ the vector space of functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f|_\Omega \in C^m(\Omega)$ for all $m \in \mathbb{N}$ with $D^\alpha f|_\Omega$ uniformly continuous on Ω for all $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ (and therefore allowing a unique continuous extension to $\overline{\Omega}$). We will consider the inner product

$$\langle f, g\rangle_{H_0^1(\Omega)} := \int_\Omega \nabla f \nabla g \,\mathrm{d}\Lambda,$$

on $H_0^1(\Omega)$, which gives rise to an equivalent norm as a consequence of the Poincaré inequality (see (PI)). Further, let $L_\nu^2(\overline{\Omega})$ denote the standard Hilbert space (the quotient space of the set of real-valued square-ν-integrable functions with domain $\overline{\Omega}$ with respect to the almost-sure equivalence relation) with inner product

$$\langle f, g\rangle_\nu := \langle f, g\rangle_{L_\nu^2(\Omega)} := \int_{\overline{\Omega}} fg \,\mathrm{d}\nu.$$

Since we mainly focus on the case of Ω being equal to the interior $\mathring{\mathbf{Q}} = (0,1)^d$, we write in this case $H^1 := H^1(\mathring{\mathbf{Q}})$, $H_0^1 := H_0^1(\mathring{\mathbf{Q}})$ and $L_\mu^2 := L_\mu^2(\mathbf{Q})$ with μ being a Borel measure on $\mathbf{Q}$. We will assume that the canonical embedding ι of an appropriate subspace of H^1 into L_ν^2 is continuous and has a dense image. To do so, we first consider the mapping

$$\iota : \left(C_b^\infty\left(\overline{\mathbf{Q}}\right), \langle\cdot,\cdot\rangle_{H^1}\right) \to L_\nu^2, \quad \iota(u) := u,$$

which is continuous if and only if we find a constant $K > 0$ such that

$$\|u\|_{L_\nu^2} \le K \|u\|_{H^1}$$

in which case we can extend the operator ι to H^1. Now, suppose that ι is continuous and note that its image is always dense in L_ν^2 (cf. Proposition 2.11). If ι is not injective, that is, if $\mathcal{N}_\nu := \ker(\iota) = \left\{f \in H^1 : \|\iota(f)\|_{L_\nu^2} = 0\right\}$ is not the null space, then one simply restricts H^1 to

$$\mathcal{N}_\nu^\perp := \left\{f \in H^1 : \forall g \in \mathcal{N}_\nu : \langle f, g\rangle_{H^1} = 0\right\}.$$

For the non-negative quadratic form $(f,g) \mapsto \left\langle \iota^{-1} f, \iota^{-1} g \right\rangle_{H^1}$ restricted to *Neumann boundary conditions* $\iota\left(\mathcal{N}_\nu^\perp\right)$ we write $\mathcal{E}^N$. Replacing H^1 with H_0^1 in the definition of $\mathcal{N}_\nu^\perp$ gives rise to the linear subspace $\mathcal{N}_{0,\nu}^\perp$ of H_0^1 (cf. (2.2.1)) and for the form $(f,g) \mapsto \left\langle \iota^{-1} f, \iota^{-1} g \right\rangle_{H_0^1}$, restricted to the *Dirichlet boundary conditions* $\iota(\mathcal{N}_{0,\nu}^\perp)$ we write $\mathcal{E}^D$. This allows us to define two Kreĭn–Feller operators Δ_ν^D (w.r.t. Dirichlet boundary conditions) and Δ_ν^N (w.r.t. Neumann boundary conditions) with respect to the two different forms via the following characterization

$$f \in \operatorname{dom}\left(\Delta_\nu^{D/N}\right) \iff \forall g \in \operatorname{dom}\left(\mathcal{E}^{D/N}\right) : \mathcal{E}^{D/N}(f,g) = \langle \Delta_\nu^{D/N} f, g \rangle_{L_\nu^2(\overline{\Omega})}.$$

For more details on this form approach, we refer to Section 2.2.5. An important quantity for the investigation of Kreĭn–Feller operators is the *∞-dimension of ν* given by

$$\dim_\infty(\nu) := \liminf_{n\to\infty} \frac{\max_{Q \in \mathcal{D}_n^N} \log(\nu(Q))}{-n\log(2)},$$

where $\mathcal{D}_n^N$ denotes a partition of $\mathbf{Q}$ by cubes of the form $Q := \prod_{i=1}^d I_i$ with

$$I_i := (k_i 2^{-n}, (k_i+1)2^{-n}]$$

for some $k_i \in \{0, \ldots, 2^n - 1\}$. Now, if the *Hu–Lau–Ngai condition* [HLN06]

$$\dim_\infty(\nu) > d - 2 \tag{$\spadesuit$}$$

is fulfilled, then a result of Maz'ya [Maz85] adapted to the dyadic grid (see Lemma 5.3) ensures that the embedding ι is compact and $\Delta_\nu^{D/N}$ admits a countable set of eigenfunctions spanning L_ν^2 with a non-negative and non-decreasing sequence of eigenvalues $\left(\lambda_{n,\nu}^{D/N}\right)_{n\in\mathbb{N}}$ tending to infinity, which correspond to the orthonormal system of eigenfunctions $\left(\varphi_{n,\nu}^{D/N}\right)_{n\in\mathbb{N}}$. As mentioned above, the Hu–Lau–Ngai condition already appeared implicitly in [Tri97, Theorem 30.2 (Isotropic fractal drum)] in the context of Ahlfors–David regular measures, for which we provide more details in Section 5.4.2. The lower and upper exponent of divergence of the eigenvalue counting function $N_\nu^{D/N}(x) := \sup\left\{n \in \mathbb{N} : \lambda_{n,\nu}^{D/N} \le x\right\}$, $x \ge 0$, are given by

$$\underline{s}_\nu^{D/N} := \liminf_{x\to\infty} \frac{\log\left(N_\nu^{D/N}(x)\right)}{\log(x)} \quad \text{and} \quad \overline{s}_\nu^{D/N} := \limsup_{x\to\infty} \frac{\log\left(N_\nu^{D/N}(x)\right)}{\log(x)} \tag{1.1.1}$$

and we refer to these numbers as the *lower* and *upper spectral dimension* of $\mathcal{E}^{D/N}$ (or of $\Delta^{D/N}$ or just of ν), respectively. If the two values coincide, then we denote the common value by $s_\nu^{D/N}$ and call it the *Dirichlet (respect. Neumann) spectral dimension*. There exists a constant C such that for all $k \in \mathbb{N}$ we have $\lambda_{k,\nu}^N \le C\lambda_{k,\nu}^D$

(see Lemma 2.18). This shows that we always have

$$\underline{s}_\nu^D \leq \underline{s}_\nu^N \quad \text{and} \quad \overline{s}_\nu^D \leq \overline{s}_\nu^N.$$

The spectral dimension also provides some essential information on the domains of the associated quadratic form and the Kreĭn–Feller operator, namely via the spectral representation (see for instance [Tri92, Section 4.5.]) given by

- $\operatorname{dom}\left(\mathcal{E}^{D/N}\right) = \left\{\sum_{n\in\mathbb{N}} a_n \varphi_{n,\nu}^{D/N} : \sum_{n\in\mathbb{N}} a_n^2 \lambda_{n,\nu}^{D/N} < \infty\right\}$,
- $\operatorname{dom}\left(\Delta_\nu^{D/N}\right) = \left\{\sum_{n\in\mathbb{N}} a_n \varphi_{n,\nu}^{D/N} : \sum_{n\in\mathbb{N}} a_n^2 \left(\lambda_{n,\nu}^{D/N}\right)^2 < \infty\right\}$.

The knowledge of the growth rate of the eigenvalues leads to many further applications. For instance, it can be used to study heat kernel estimates [GHN20], stochastic heat/wave equations defined by Kreĭn–Feller operators [Ehn19; EH21], the approximation order of Kolmogorov diameters [Eva+09; KN22a], and logarithmic L_2-small ball asymptotics [Naz06].

The Kreĭn–Feller operator for the one-dimensional case was introduced in [Kre51; Fel57] and, since the late 1950's, has been studied by various authors [KK58; Kac59; UH59; MR62; BS70; KW82; Fuj87; SV95; Vol05; Nga11; Fag12; Arz15; DN15; FW17; NTX18; Arz14; FM20; Min20; NX20; PS21; JT22]. For dimensions $d > 1$ however the situation is quite different; in general, it is not even possible to define the Kreĭn–Feller operator for a given Borel measure ν. This is due to the fact that, in general, there is no continuous embedding of the Sobolev space of weakly differentiable functions into the Hilbert space of square-ν-integrable functions L_ν^2 (for example when ν has atoms). For Dirichlet boundary conditions, in [HLN06] a sufficient condition in terms of the maximal asymptotic direction of the L^q-spectrum of ν (the ∞-dimension of ν) has been established, as provided in (♠), which ensures a compact embedding of the relevant Sobolev space into L_ν^2. It is worth pointing out that Triebel already stated this condition implicitly in 1997 in [Tri97]. In 2003 (see [Tri03; Tri04]) he also indicated that there should be a subtle connection between the multifractal concept of the L^q-spectrum and analytic properties of the associated "fractal" operators, a conjecture that we confirm in this thesis.

In contrast to the one-dimensional case, the spectral dimension of Kreĭn–Feller operators is so far known only for very limited number of singular measures. The spectral dimension of Kreĭn–Feller operators for higher dimensions was first computed by Birman and Solomjak [BS70, Theorem 5.1] for absolutely continuous measures, by Naimark and Solomjak [NS95; Sol94] for self-similar measures under the open set condition (OSC), by Triebel [Tri97, Theorem 30.2] in the setting of Ahlfors–David regular measures, and, recently, by Ngai and Xie [NX21] for a class of graph-directed self-similar measures satisfying the graph open set condition. In

[NX21, Sec. 5] Ngai and Xie pointed out that it would also be interesting to study self-similar measures defined by iterated function systems with overlaps on $\mathbb{R}^d$, $d \geq 1$. Indeed, as an application of our general results from Section 5.3, we are able to extend these achievements to self-conformal measures without any restriction on the separation conditions.

In the remainder of this section, we discuss some important results regarding the spectral dimension for the case $d = 1$. Note that we always have $\overline{s}_\nu^{D/N} \leq 1/2$ (see [BS66; BS67]). The case for measures with a non-zero absolutely continuous part was completely solved in an elegant way in [BS70] (see also [MR62]) using a variational approach. In this case, we have for a finite Borel measure ν on $(0,1)$ with absolutely continuous part $\sigma\Lambda$ and singular part η,

$$\lim_{x\to\infty} \frac{N_{\eta+\sigma\Lambda}^{D/N}(x)}{x^{1/2}} = \frac{1}{\pi}\int_{[0,1]} \sqrt{\sigma}\, \mathrm{d}\Lambda,$$

and particularly if σ is non-vanishing, then the spectral dimension exists and equals $1/2$. Besides these estimates, many partial results have been obtained showing that there is a subtle connection between spectral properties and geometric data of ν, which is a major line of investigation since the famous result by H. Weyl [Wey11]. Another important example is the case of self-similar measures ν under the open set condition (OSC) with contractions $r_1, \ldots, r_n \in (0,1)$ and probability weights $p_1, \ldots, p_n \in (0,1)$. It has been shown in [Fuj87; Sol94; UH59] that in this case the spectral dimension $s_\nu^{D/N}$ is given by the unique number $q > 0$ fulfilling

$$\sum_{i=1}^{n} (p_i r_i)^q = 1. \tag{1.1.2}$$

Arzt [Arz14] generalized this result to a class of homogeneous Cantor measures. In Section 4.4.2, we will use a similar construction to find an example for which the spectral dimension does not exist (see Example 4.48 and Example 5.29). Recently, building on the ideas of Arzt [Arz14] and Freiberg, Hambly, and Hutchinson [FHH17], Minorics [Min20; Min17] computed the spectral dimension of random V-variable Cantor measures and random recursive Cantor measures. Another way to generalize the classical self-similar setting under OSC is to drop the assumption of the OSC. Special classes of self-similar measures with overlap have been investigated by Ngai [Nga11], Ngai, Tang, and Xie [NTX18], and Ngai and Xie [NX20]. We will make use of the following notation. For any two functions $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we write $f \ll g$ if there exist positive constants c, x_0 such that $cf(x) \leq g(x)$ for all $x \geq x_0$; we write $f \asymp g$ if both $f \ll g$ and $g \ll f$ hold. The asymptotic behavior of $N_\nu^{D/N}$ strongly depends on the measure ν and it should be noted that the significant difference between the Kreĭn–Feller operator and the classical Laplace operator is that the leading term of the eigenvalue counting function of the Kreĭn–Feller operator may oscillate, as has been pointed out by Triebel [Tri97]. In general, one cannot even expect that $N_\nu^{D/N}$ obeys a power law with a positive exponent $s > 0$, i.e. $N_\nu^{D/N}(x) \asymp x^s$ (for counter examples see, e.g. [Arz14], Example 5.29, and Example 4.51). Therefore, determining the leading term of $N_\nu^{D/N}$ is a challenging problem. However, if we restrict our attention to the spectral dimension, then this problem becomes easier attackable. Surprisingly, we are able to treat arbitrary Borel measures on $(0, 1)$ and determine the upper spectral dimension solely from the data provided by the measure-geometric information carried by the L^q-spectrum of ν. Under mild regularity conditions on the measure we can guarantee the existence of the spectral dimension (see Corollary 4.12). Also, with the help of the L^q-spectrum of ν we are able to construct first examples for which the spectral dimension does not exist (see Section 4.4.2). In this way we give a partial answer to Kac's question in terms of the spectral dimension revealing how the measure theoretic properties of ν and the topological properties of its fractal support determine the spectral dimension (see Theorem 4.10 and Corollary 4.17). This striking connection is the subject of this thesis.

1.1.2 Quantization problem

Quantization refers to the operation of converting input from a continuous or large set of values (e.g. a continuous signal) into a representation space of lower cardinality than the input (e.g. a discrete signal).

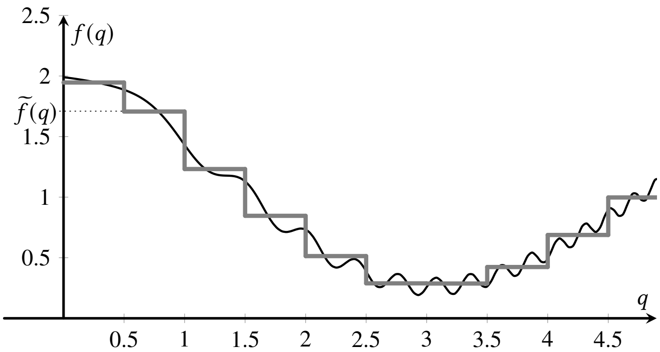

Figure 1.1.1 Simple quantization of the "signal" f by averaging f over the intervals $[i/2, (i+1)/2]$, $i = 0, \ldots, 9$, denoted by $\widetilde{f}$ (gray).

The quantization problem for probability measures originates from information theory, in particular, image compression and data compression. Recently, this theory has attracted increasing attention in applications such as optimal transport problems [JP22], numerical integration [ELP22; Pag15], and mathematical finance [PW12; Hof+14; BFP16; FPS19; BPW10]. From a mathematical point of view, one is concerned with the asymptotics of the errors in approximating a given random variable with a quantized version of that random variable (i.e. taking only finitely many values), in the sense of r-means, $r > 0$.

We start with a stochastic formulation of the quantization problem. Let X be a bounded $\mathbb{R}^d$-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and set $\nu := \mathbb{P} \circ X^{-1}$. For a given $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote the set of all Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}^d$ with $\operatorname{card}\left(f\left(\mathbb{R}^d\right)\right) \le n$. Our goal is to approximate X with a *quantized version* of X, i.e. X will be approximated by elements of the form $f(X)$ with $f \in \mathcal{F}_n$, with respect to the L^r-quasinorm for some $r > 0$, that is,

$$\mathfrak{e}_{n,r}(\nu) := \inf_{f \in \mathcal{F}_n} \left(\int_\Omega |X - f(X)|^r \, \mathrm{d}\mathbb{P} \right)^{1/r} = \inf_{f \in \mathcal{F}_n} \left(\int |x - f(x)|^r \, \mathrm{d}\nu(x) \right)^{1/r}.$$

We call $\mathfrak{e}_{n,r}(\nu)$ the *n-th quantization error* for ν of order $r > 0$.

In the following considerations we assume that ν is a compactly supported Borel probability measure. For every $n \in \mathbb{N}$, we write $\mathcal{A}_n := \{\alpha \subset \mathbb{R}^d : 1 \le \operatorname{card}(\alpha) \le n\}$. In [GL00b, Lemma 3.1] an equivalent formulation of the *n-th quantization error* for ν of order r is given by

$$\mathfrak{e}_{n,r}(\nu) = \inf_{\alpha \in \mathcal{A}_n} \left(\int d(x,\alpha)^r \, \mathrm{d}\nu(x) \right)^{1/r}, \ r > 0, \tag{1.1.3}$$

where $d(x,\alpha) := \min_{y \in \alpha} \|x - y\|$ and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. By [GL00b, Lemma 6.1], we have $\mathfrak{e}_{n,r}(\nu) \to 0$ for $n \to \infty$. In fact, it is well known that $\mathfrak{e}_{n,r}(\nu) = O(n^{-1/d})$ and $\mathfrak{e}_{n,r}(\nu) = o(n^{-1/d})$ for $n \to \infty$ if ν is singular with respect to the Lebesgue measure (see also Proposition 6.1). Hence, it is natural to ask for the "optimal exponent" of convergence. The calculation of this exponent will be one of the main achievement of this thesis. For this purpose we define the *lower* and *upper quantization dimension* for ν of order r by

$$\underline{D}_r(\nu) := \liminf_{n \to \infty} \frac{\log(n)}{-\log \mathfrak{e}_{n,r}(\nu)} \quad \text{and} \quad \overline{D}_r(\nu) := \limsup_{n \to \infty} \frac{\log(n)}{-\log \mathfrak{e}_{n,r}(\nu)}.$$

If $\overline{D}_r(\nu) = \underline{D}_r(\nu)$, then we call the common value the *quantization dimension of ν of order r* and denote it by $D_r(\nu)$. The quantization dimension reflects the exponential rate of this convergence and has been studied by various authors, for example [Del+04; Gra02; LM02; Zhu15a; Zhu15b; ZZS16; KZ16; ZZS17; KZ15; KZ17; Zhu18; Zhu20; ZZ21; Roy13]. A detailed introduction to the mathematical foundations of the quantization problem can be found in [GL00b]. As pointed out for instance in [LM02], "the problem of determining the quantization dimension function for a general probability is open". In this thesis we close this gap for the upper quantization dimension and, under additional regularity conditions, also for the lower quantization dimension. Building on a result of [PS00; Fen07], we confirm the existence of the quantization dimension of self-conformal measures with respect to conformal iterated function systems without any separation conditions.

The following theorem by Zador is a classical result from quantization theory. It was proposed in [Zad82] and then generalized by Bucklew and Wise [BW82]; we refer to [GL00b, Theorem 6.2] for a rigorous proof.

> *Let ν be a Borel probability measure with bounded support and let h denote the density of the absolutely continuous part of ν. Then*
>
> $$\lim_{n\to\infty} n^{-r/d} \mathfrak{e}_{n,r}(\nu)^r = C(r,d)\left(\int h^{\frac{d}{d+r}}(x)\,\mathrm{d}x\right)^{\frac{d+r}{d}}, \; r>0,$$
>
> *where $C(r,d)$ is a constant independent of ν.*

Interestingly, there is a similar result for polyharmonic operators (see for instance [BS70]). While engineers are mainly dealing with absolutely continuous distributions, from a mathematical point of view, the quantization problem is significant for all Borel probability measures with bounded support.

Another important example is the case of self-similar measures ρ under OSC with contractions $r_1, \dots, r_n \in (0,1)$ and probability weights $p_1, \dots, p_n \in (0,1)$. By Graf and Luschgy [GL00a], the quantization dimension exists and is uniquely determined by

$$\sum_{i=1}^{n} \left(p_i r_i^r\right)^{D_r(\nu)/(r+D_r(\nu))} = 1. \tag{1.1.4}$$

Graf and Luschgy's work on the quantization dimension was the starting point of many further generalizations investigating more general classes of fractal measures such as self-affine measures on Bedford-McMullen carpets [KZ16], self-conformal measures [LM02], and inhomogeneous self-similar measures [Zhu08a; Zhu08b]. It is worth mentioning that in the case $r = d = 1$ the formula of the spectral dimension in the self-similar case under OSC is quite similar to the formula of $D_1(\nu)$. In fact, this is no coincidence; we will prove that for general measures ν the spectral dimension and upper quantization dimension are closely related (see Corollary 6.8).

1.1.3 Optimal partition problems and optimized coarse multifractal dimension

Motivated by the study of upper bounds of the spectral dimension of Kreĭn–Feller operators, polyharmonic operators [KN22b; KN22a; KN22c], and quantization dimension [KNZ22], we are interested in the following general combinatorial problem which plays a major role for the investigations in this thesis. Let $\mathfrak{J} : \mathcal{D} \to \mathbb{R}_{\geq 0}$ with $\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n^N$ satisfying the following natural assumptions:

- $\mathfrak{J}$ is monotone, that is, $\mathfrak{J}(Q') \leq \mathfrak{J}(Q)$ for all $Q', Q \in \mathcal{D}$ with $Q' \subset Q$.
- $\mathfrak{J}$ is uniformly vanishing, i.e. $\lim_{n\to\infty} \sup_{Q\in\mathcal{D}_m, n\leq m} \mathfrak{J}(Q) = 0$.
- $\mathfrak{J}$ is locally non-vanishing, i.e. if $\mathfrak{J}(Q) > 0$ for $Q \in \mathcal{D}$, then there exists $Q' \subsetneq Q$, $Q' \in \mathcal{D}$ with $\mathfrak{J}(Q') > 0$.

We are particularly interested in the following class of set functions

$$\mathfrak{J}_{\nu,a,b}(Q) := \begin{cases} \sup_{\widetilde{Q}\in\mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)^b \left|\log\left(\Lambda\left(\widetilde{Q}\right)\right)\right|, & a = 0, \\ \sup_{\widetilde{Q}\in\mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)^b \Lambda\left(\widetilde{Q}\right)^a, & a \neq 0, \end{cases}$$

where $b \geq 0$, $a \in \mathbb{R}$, $\mathcal{D}(Q) := \left\{\widetilde{Q} \in \mathcal{D} : \widetilde{Q} \subset Q\right\}$, and ν is a finite Borel measure on $\mathbf{Q}$, which we call *spectral partition function with parameters a, b*. The spectral partition function arises naturally in the investigation of Kreĭn–Feller operators (for $a = (2-d)/d$, $b = 1$) and the quantization problem (for $a > 0$, $b = 1$). Our goal is to control the asymptotic behavior of

$$\mathcal{M}_{\mathfrak{J}}(x) := \inf\left\{\operatorname{card}(P) : P \in \Pi_{\mathfrak{J}} : \max_{Q\in P} \mathfrak{J}(Q) < 1/x\right\}, \quad x > 1/\mathfrak{J}(\mathbf{Q}),$$

where $\Pi_{\mathfrak{J}}$ denotes the set of finite collections of dyadic cubes such that for all $P \in \Pi_{\mathfrak{J}}$ there exists a partition $\widetilde{P}$ of $\mathbf{Q}$ by dyadic cubes from $\mathcal{D}$ with $P = \{Q \in \widetilde{P} : \mathfrak{J}(Q) > 0\}$. An important quantity for measuring the growth rate of $\mathcal{M}_{\mathfrak{J}}$ is the *lower,* resp. *upper* $\mathfrak{J}$*-partition entropy* defined by

$$\underline{h}_{\mathfrak{J}} := \liminf_{x\to\infty} \frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)}, \quad \overline{h}_{\mathfrak{J}} := \limsup_{x\to\infty} \frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)}. \tag{1.1.5}$$

Under mild conditions on $\mathfrak{J}$, it is closely related to its dual problem (see Proposition 3.11), which is concerned with the control of the asymptotic behavior of

$$\gamma_{\mathfrak{J},n} := \inf_{\substack{P\in\Pi_{\mathfrak{J}},\\ \operatorname{card}(P)\leq n}} \max_{Q\in P} \mathfrak{J}(Q).$$

For the special choice $\mathfrak{J} := \mathfrak{J}_{\nu,a,1}$ with $a > 0$ and ν being a finite Borel measure on $\mathbf{Q}$ (or, more generally, a superadditive function, see Section 3.3), the dual problem has attracted much attention in numerous papers by Birman and Solomjak [BS67; BS70], Borzov [Bor71], and more recently by Davydov, Kozynenko, and Skorokhodov [DKS20] and by Hu, Kopotun, and Yu [HKY00]. The study of $\gamma_{\mathfrak{J},n}$ in [BS67] was motivated by the study of integral operators (see for instance [Bor71]). The classical result by Birman and Solomjak [BS67, Theorem 2.1] states that

$$\gamma_{\mathfrak{J},n} = O\left(n^{-(1+a)}\right),$$

which under the additional assumption that ν is singular with respect to the Lebesgue measure was improved by Borzov [Bor71] to $\gamma_{\mathfrak{J},n} = o\left(n^{-(1+a)}\right)$. It should be noted that in the early 1970s a first attempt was made to find the "right exponent" in the case $d = 1$; however, the estimate obtained in [Bor71, p. 41] depends only on the support of ν. Consequently, this approach ignores important information about the involved measure ν, resulting in an inaccurate estimate of the exponent. In this thesis, we close this gap by giving the exact exponent (see Corollary 3.21).

Motivated by the study of lower estimates of the spectral dimension and quantization dimension (see Section 4.1, Section 5.2 and Section 6.2) we borrow ideas from the coarse multifractal analysis (see [Nga97; Fal14; Rie95] and [Fal97, Chapter 11]), which, roughly speaking, is concerned with the study of global (coarse) properties of compactly supported Borel measures on small dyadic cubes. In contrast to the coarse multifractal analysis in which only bounded Borel measures are considered, we generalize this idea to set functions $\mathfrak{J}$ under the assumptions formulated above. To be more precise, for all $n \in \mathbb{N}$ and $\alpha > 0$ we define

$$\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n) := \operatorname{card}\left(M_n^{D/N}(\alpha)\right), \quad M_{\mathfrak{J},n}^{D/N}(\alpha) := \left\{Q \in \mathcal{D}_n^{D/N} : \mathfrak{J}(Q) \geq 2^{-\alpha n}\right\},$$

with $\mathcal{D}_n^D := \left\{Q \in \mathcal{D}_n^N : \partial\mathbf{Q} \cap \overline{Q} = \varnothing\right\}$, an object motivated by the study of Kreĭn–Feller operators with respect to Dirichlet boundary conditions (see also Section 2.1). We set

$$\overline{F}_{\mathfrak{J}}^{D/N}(\alpha) := \limsup_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n)\right)}{\log(2^n)} \text{ and } \underline{F}_{\mathfrak{J}}^{D/N}(\alpha) := \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n)\right)}{\log(2^n)},$$

with $x \geq 0$ and $\log^+(x) := \max\{0, \log(x)\}$ (where we use the convention that $\log(0) := -\infty$), and refer to the quantities

$$\overline{F}_{\mathfrak{J}}^{D/N} := \sup_{\alpha>0} \frac{\overline{F}_{\mathfrak{J}}^{D/N}(\alpha)}{\alpha} \quad \text{and} \quad \underline{F}_{\mathfrak{J}}^{D/N} := \sup_{\alpha>0} \frac{\underline{F}_{\mathfrak{J}}^{D/N}(\alpha)}{\alpha}$$

as the *upper*, resp. *lower optimized (Dirichlet/Neumann) coarse multifractal dimension* with respect to $\mathfrak{J}$. In Chapter 3, we see that the $\mathfrak{J}$-partition entropy and the optimized coarse multifractal dimension with respect to $\mathfrak{J}$ are strongly linked by ideas from the theory of large deviations.

1.2 Outline and statement of the main results

This thesis is dedicated to the study of general optimal partition problems and their applications for the determination of the spectral dimension of Kreĭn–Feller operators and the quantization dimension. The main achievement of this thesis is the development of a unified framework to tackle both the computation of the upper spectral dimension and the upper quantization dimension.

The thesis is divided into five main parts. In Chapter 2, we provide some preliminary considerations. In Section 2.2, under the Hu–Lau–Ngai condition (♠), we define Kreĭn–Feller operators $\Delta_\nu^{D/N}$ via a form approach with respect to Dirichlet/Neumann boundary conditions. We prove a slight modification of the well-known min-max principle (Proposition 2.17) for the representation of the eigenvalues. The min-max principle is a powerful tool which enables us to reduce the eigenvalue counting problem to the optimal partition problem described in Section 1.1.3. We conclude Section 2.2 by constructing smooth functions via mollifiers. We will use this construction to prove that the condition $\dim_\infty(\nu) < d-2$ implies that there is no continuous embedding of $H^1(\mathbf{Q})$ into $L_\nu^2(\mathbf{Q})$. As a consequence, if $\dim_\infty(\nu) < d-2$, then it is impossible to define the Kreĭn–Feller operator. Section 2.3 is dedicated to the introduction of the new concept of partition functions, which, to a certain extent, is borrowed from the coarse multifractal analysis; for a non-negative, monotone set function $\mathfrak{J} : \mathcal{D} \to \mathbb{R}_{\geq 0}$, the Dirichlet/Neumann partition function of $\mathfrak{J}$ is given by

$$\tau_{\mathfrak{J}}^{D/N}(q) := \limsup_{n\to\infty} \frac{1}{\log(2^n)} \log\left(\sum_{\substack{Q\in\mathcal{D}_n^{D/N},\\ \mathfrak{J}(Q)>0}} \mathfrak{J}(Q)^q \right), \quad q \geq 0.$$

The function $\tau_{\mathfrak{J}}^N$ encodes important information about $\mathfrak{J}$; it provides a quantitative description of the global fluctuation of $\mathfrak{J}$. Furthermore, under mild conditions on $\mathfrak{J}$ (see Lemma 2.25), we have the following important representation of the zero of $\tau_{\mathfrak{J}}^N$ as a critical value:

$$q_{\mathfrak{J}}^N := \inf\{q > 0 : \tau_{\mathfrak{J}}^N(q) < 0\} = \inf\left\{ q > 0 : \sum_{Q\in\mathcal{D}} \mathfrak{J}(Q)^q < \infty \right\}.$$

This representation will be crucial for establishing upper bounds of the $\mathfrak{J}$-partition entropy. An important special case is the Neumann partition function of $\mathfrak{J} = \nu$ which is known as the L^q-spectrum of ν. Throughout this thesis we write $\beta_\nu^{D/N} = \tau_\nu^{D/N}$. The L^q-spectrum of ν provides important information about ν, for example $\beta_\nu^N(0)$ is equal to the upper Minkowski dimension of the support of ν and $\lim_{q\to\infty} \beta_\nu^N(q)/q = -\dim_\infty(\nu)$. In Section 2.4.2, we discuss conditions which guarantee that the Dirichlet partition function and Neumann partition function coincide. We conclude that chapter by computing the (Dirichlet/Neumann) partition function for the spectral partition function for leading examples; we consider absolutely continuous measures, product measures, Ahlfors-David regular measures, and self-conformal measures.

Chapter 3 can be seen as the flagship of this thesis. We develop a machinery which enables us to tackle the problem of the computation of the upper $\mathfrak{J}$-partition entropy under mild assumptions on $\mathfrak{J}$ (see Section 3.4). Section 3.1 is devoted to establishing lower and upper bounds of the lower and upper $\mathfrak{J}$-partition entropy in terms of the zero of $\tau_{\mathfrak{J}}^N$ and the lower and upper optimized coarse multifractal dimension with respect to $\mathfrak{J}$, respectively. This will enable us to use an adaptive approximation algorithm to construct certain partitions of dyadic cubes to estimate the $\mathfrak{J}$-partition entropy from above. A detailed motivation for this approach is given in the beginning of Section 3.1. Based on that, in Section 3.2, we show that the $\mathfrak{J}_{\nu,a,b}$-partition entropy can be bounded from above by

$$q_{\mathfrak{J}_{a,b}}^N = \inf\{q > 0 : \tau_{\mathfrak{J}_{a,b}}^N(q) < 0\}$$

whenever $b\dim_\infty(\nu) + ad > 0$. Section 3.3 is devoted to the study of the corresponding dual problem. We begin with a presentation of an adaptive approximation algorithm by Birman and Solomjak [BS67] to reproduce known results. We then demonstrate how one can use the results of Section 3.1 to improve known upper bounds in terms of $q_{\mathfrak{J}}^N$ (Proposition 3.11). Section 3.4 contains the main results of that chapter. The basic idea of that section is to apply large derivation theory (see Lemma 3.17) to estimate the upper optimized coarse multifractal dimension with respect to $\mathfrak{J}$ from below. The first main result is given by Corollary 3.21. It states that

$$\overline{F}_{\mathfrak{J}}^D = q_{\mathfrak{J}}^D$$

and

$$\overline{F}_{\mathfrak{J}}^N = \overline{h}_{\mathfrak{J}} = q_{\mathfrak{J}}^N.$$

The second main result is concerned with the question under which conditions we can ensure that $\overline{F}_{\mathfrak{J}}^N$ and $\overline{h}_{\mathfrak{J}}$ exist as limits (i.e. $\overline{h}_{\mathfrak{J}} = \underline{h}_{\mathfrak{J}}$ and $\overline{F}_{\mathfrak{J}}^N = \underline{F}_{\mathfrak{J}}^N$). In Corollary 3.23, we impose a sufficient condition (see Definition 3.22); if $\mathfrak{J}$ is Dirichlet/Neumann partition function regular, that is, if $\tau_{\mathfrak{J}}^{D/N}$ is differentiable and exists as a limit in $q_{\mathfrak{J}}^{D/N}$, or $\tau_{\mathfrak{J}}^{D/N}$ exists as a limit on a left-sided neighborhood of $q_{\mathfrak{J}}^{D/N}$, then

$$\underline{F}_{\mathfrak{J}}^D = \overline{F}_{\mathfrak{J}}^D = q_{\mathfrak{J}}^D$$

and

$$\underline{F}_{\mathfrak{J}}^N = q_{\mathfrak{J}}^N = \overline{h}_{\mathfrak{J}} = \underline{h}_{\mathfrak{J}}.$$

Chapter 4 is dedicated to studying the spectral dimension of Kreĭn–Feller operators for the case $d = 1$ with respect to non-zero Borel measures on $(0,1)$. It turns out that the spectral partition function with parameters $(2-d)/d, 1$, given by

$$\mathfrak{J}_\nu(Q) = \mathfrak{J}_{\nu,1,1}(Q) = \nu(Q)\Lambda(Q), \quad Q \in \mathcal{D},$$

is a central object for the calculation of the upper spectral dimension. Its importance stems from the fact that $\mathfrak{J}_\nu$ appears as an embedding constant (which, in fact, is equivalent to the best constant) for the embedding of $H_0^1(Q)$ into $L_\nu^2(Q)$. Combined with the min-max principle, we can reduce the original problem of the computation of the lower and upper spectral dimension to the combinatorial problems with respect to $\mathfrak{J}_\nu$ considered in Chapter 3. In Section 4.1, we establish lower bounds for the lower and upper spectral dimension in terms of the lower and upper optimized coarse multifractal dimension with respect to $\mathfrak{J}_\nu$, respectively. Section 4.2 is devoted to the study of upper bounds for the lower and upper spectral dimension. We show that the lower and upper spectral dimension is bounded from above by the lower and upper $\mathfrak{J}_\nu$-partition entropy, respectively. Further, we obtain upper bounds for the lower spectral dimension in terms of the lower Minkowski dimension of $\operatorname{supp}(\nu)$, denoted by $\underline{\dim}_M(\nu)$, and the ∞-dimension of ν as follows:

$$\underline{s}_\nu^{D/N} \le \underline{h}_{\mathfrak{J}_\nu} \le \frac{\underline{\dim}_M(\nu)}{1+\dim_\infty(\nu)}.$$

In Section 4.3, we present the main results of that chapter. By combining the results of Section 4.1 and Section 4.2 we are able to compute the upper spectral dimension. The first main result (Theorem 4.10) reads as follows:

$$\underline{F}_{\mathfrak{J}_\nu}^N \le \underline{s}_\nu^{D/N} \le \overline{s}_\nu^{D/N} = \overline{F}_{\mathfrak{J}_\nu}^N = q_{\mathfrak{J}_\nu}^N = \overline{h}_{\mathfrak{J}_\nu}.$$

This reveals an interesting connection between the upper spectral dimension, the optimized coarse multifractal dimension with respect to $\mathfrak{J}_\nu$, the $\mathfrak{J}_\nu$-partition entropy, and the spectral partition function. Since the partition function of $\mathfrak{J}_\nu$ is equal to $q \mapsto \beta_\nu^N(q) - q$, we obtain an interesting geometric interpretation of the upper spectral dimension. If $q_{\mathfrak{J}_\nu}^N > 0$, then the upper spectral dimension is given by the fixed point of the L^q-spectrum of the corresponding measure ν (see Figure 1.2.1).

Section 4.3.2 is concerned with the determination of the conditions that ensure the spectral dimension exists. For this purpose we use the regularity results of Section 3.4 applied to $\mathfrak{J}_\nu$. This leads to the following regularity condition, which ensures the existence of the spectral dimension (see Corollary 4.12):

> *If $\mathfrak{J}_\nu$ is Neumann partition function regular, then the spectral dimension exists and is given by $s_\nu^{D/N} = q_{\mathfrak{J}_\nu}^N$.*

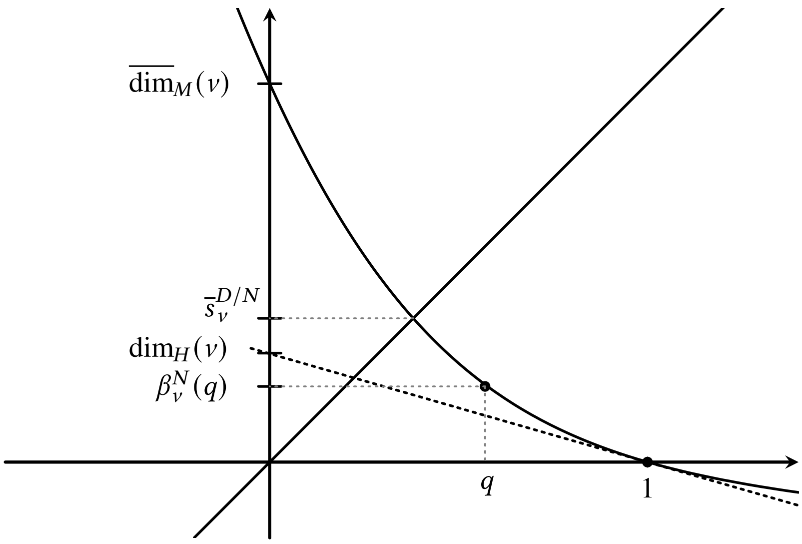

Figure 1.2.1 The intersection point $q_{\mathfrak{J}_\nu}^N$ of the L^q-Spectrum β_ν^N with respect to ν and the identity map. Here ν is chosen to be the $(0.05, 0.95)$-Salem measure with full support $\operatorname{supp}(\nu) = [0,1]$. The intersection of β_ν^N with the y-axis yields the Minkowski dimension of $\operatorname{supp}(\nu)$, namely 1, and the intersection with the (dotted) tangent to β_ν^N in $(0,1)$ yields the Hausdorff dimension $\dim_H(\nu)$ of the measure ν, which equals $(0.05\log(0.05) + 0.95\log(0.95))/\log(2)$.

Fortunately, this regularity condition is usually easy to verify. For instance, we will apply this result for weak Gibbs measures without any separation conditions. In Section 4.3.3, we derive lower and upper bounds for the lower and upper spectral dimension, respectively. For the lower and upper spectral dimension we have the following general bounds depending on the topological support of ν, namely $\overline{\dim}_M(\nu)$, and the right and left derivative $\partial^-\beta_\nu^N$, $\partial^+\beta_\nu^N$ of β_ν^N at 1:

$$\frac{-\partial^+\beta_\nu^N(1)}{1-\partial^-\beta_\nu^N(1)} \le \underline{s}_\nu^{D/N} \le \overline{s}_\nu^{D/N} \le \frac{\overline{\dim}_M(\nu)}{1+\overline{\dim}_M(\nu)} \le \frac{1}{2}$$

and

$$\overline{s}_\nu^{D/N} = \frac{\overline{\dim}_M(\nu)}{1+\overline{\dim}_M(\nu)} \iff -\partial^-\beta_\nu^N(1) = \overline{\dim}_M(\nu).$$

We conclude this chapter with three leading examples in Section 4.4. In the first example we investigate weak Gibbs measures with respect to a C^1-IFS (with or without overlap). Thereby, we generalize the classical result for the self-similar setting under OSC (1.1.2) in three ways:

- In Section 4.4.1.1, we provide a first contribution to the nonlinear setting in a broad sense. More precisely, we consider weak Gibbs measures on fractals which are generated by non-trivial C^1-IFS's under OSC. It turns out that the spectral dimension is given by the zero of the associated pressure function (see (4.4.3)), which constitutes a natural generalization of (1.1.2).

- As a second novelty in Section 4.4.1.3, we drop the assumption of the OSC and allow overlaps. In this situation the computation of the spectral dimension is much more complex compared to (1.1.2). However, by ideas of [Fen07; PS00], we are able to prove the existence of the L^q-spectrum on $(0,1]$ (see Proposition 4.45). This implies that $\mathfrak{J}_\nu$ is Neumann partition function regular. Consequently, Corollary 4.12 yields the existence of the spectral dimension given as the fixed point of the associated L^q-spectrum.

- Our final contribution to the nonlinear setting concerns Gibbs measures on fractals generated by $C^{1+\gamma}$-IFS's under OSC. For this class, using renewal theory in a dynamical context (see for instance [Kom18; KK17]), we are able to prove the spectral asymptotics (see Theorem 4.42)
$$N_\nu^D(t) \asymp t^{z_\nu},$$
where z_ν is the unique zero of the pressure function as defined in (4.4.3).

In Section 4.4.2, we construct a first example with non-converging L^q-spectrum for which the spectral dimension does not exist, with the help of homogeneous Cantor measures (see Example 4.48). We end Chapter 4 with the investigation of purely atomic measures whose spectral dimension exists and attains values in $[0,1/2]$.

In Chapter 5, we discuss the spectral dimension of Kreĭn–Feller operators $\Delta_\nu^{D/N}$ with respect to Dirichlet and Neumann boundary conditions for the case $d > 1$, where ν is a non-zero Borel measure on $\mathbf{Q}$ with $\dim_\infty(\nu) > d-2$. This chapter can be seen as generalization of some results for the one-dimensional case. However, in contrast to the one-dimensional case, there are some difficulties. There is no continuous embedding of the Sobolev space $H^1(\mathbf{Q})$ into $C_b(\overline{\mathbf{Q}})$ and, in general, we cannot guarantee that the spectral dimensions with respect to Dirichlet and Neumann boundary conditions coincide. Thus, many proofs from the one-dimensional case cannot be directly adopted. Again, the main strategy is to use the min-max principle to reduce the problem of the computation of the spectral dimension to the combinatorial problems investigated in Chapter 3, where $\mathfrak{J}$ is chosen to be equal to

$$\mathfrak{J}_\nu(Q) \coloneqq \mathfrak{J}_{\nu,(2-d)/d,1}(Q) = \begin{cases} \sup_{\widetilde{Q}\in\mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)\left|\log\left(\Lambda\left(\widetilde{Q}\right)\right)\right|, & d=2,\\ \sup_{\widetilde{Q}\in\mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)\Lambda\left(\widetilde{Q}\right)^{(2-d)/d}, & d>2,\end{cases}$$

with $Q \in \mathcal{D}$. In Section 5.1, we discuss upper bounds for the lower and upper spectral dimension. Motivated by the ideas of [NS95] (see also Remark 5.2) and the proof of Proposition 4.4, we start with an important observation on the embedding constants on sub-cubes of $\mathbf{Q}$ and the upper spectral dimension in Section 5.1.1. The main result of that section reads as follows:

Suppose there exists a non-negative, uniformly vanishing, monotone set function $\mathfrak{J}$ on $\mathcal{D}$ such that for all $Q \in \mathcal{D}$ and all $u \in C_b^\infty(\overline{Q})$ with $\int_Q u \,\mathrm{d}\Lambda = 0$, we have

$$\|u\|^2_{L^2_\nu(Q)} \leq \mathfrak{J}(Q)\, \|\nabla u\|^2_{L^2_\Lambda(Q)}. \tag{1.2.1}$$

Then we have

$$\overline{s}^D_\nu \leq \overline{s}^N_\nu \leq \overline{h}_{\mathfrak{J}}$$

and

$$\underline{s}^D_\nu \leq \underline{s}^N_\nu \leq \underline{h}_{\mathfrak{J}}.$$

In contrast to the case $d = 1$ (see Lemma 2.2), the best embedding constant on dyadic sub-cubes of $\mathbf{Q}$ from the embedding of $H^1(Q)$ into $L^2_\nu(Q)$ is hard to compute for general measures. Fortunately, the best embedding constant for the embedding of $L^t_{\nu|_Q}(\mathbb{R}^d)$ with $t > 2$ into $H^1(\mathbb{R}^d)$ has been computed by Maz'ya and Preobrazenskii [Maz11, p. 83] for $d = 2$ and by Adam [Ada71; Ada73] (see also [Maz11, p. 67]) for $d > 2$. Using this and the Stein extension established in Lemma 2.8, we show that (1.2.1) is valid for $\mathfrak{J} = \mathfrak{J}_{\nu,(2-d)/d,2/t}$ with $2 < t < 2\dim_\infty(\nu)/(d-2)$ (see Lemma 5.5). By combining the results above, the main results of Section 5.1.2 are the following chains of inequalities:

$$\overline{s}^D_\nu \leq \overline{s}^N_\nu \leq \lim_{t\downarrow 2} \overline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} \leq q^N_{\mathfrak{J}_\nu}$$

and

$$\underline{s}^D_\nu \leq \underline{s}^N_\nu \leq \lim_{t\downarrow 2} \underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}.$$

Section 5.2 is devoted to establishing lower bounds for the lower and upper spectral dimension. Motivated by the proof of Proposition 4.1 for the one-dimensional setting, the lower estimate of the spectral dimension is based on the following general principle which connects the optimized coarse multifractal dimension and the spectral dimension (see Proposition 5.9 for the definition):

Assume there exists a non-negative monotone set function $\mathfrak{J}$ on $\mathcal{D}$ with $\dim_\infty(\mathfrak{J}) > 0$ (see Section 2.3.1) such that for every $Q \in \mathcal{D}$ with $\mathfrak{J}(Q) > 0$ there exists a non-negative and non-zero function $\psi_Q \in C_c^\infty(\mathbb{R}^d)$ with support contained in $\langle \mathring{Q} \rangle_3$ such that

$$\|\psi_Q\|^2_{L^2_\nu} \geq \mathfrak{J}(Q)\, \|\nabla \psi_Q\|^2_{L^2_\Lambda(\mathbb{R}^d)}, \tag{1.2.2}$$

where $\langle \mathring{Q} \rangle_3$ denotes the cube centered and parallel with respect to $\mathring{Q}$ such that $\langle \mathring{Q} \rangle_3 = T(\mathring{Q}) + (1-3)x_0$ with $T(x) = 3x, x \in \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ is the center of Q. Then

$$\underline{F}_{\mathfrak{J}}^N \leq \underline{s}_\nu^N, \quad \overline{F}_{\mathfrak{J}}^N \leq \overline{s}_\nu^N, \quad \underline{F}_{\mathfrak{J}}^D \leq \underline{s}_\nu^D \quad \text{and} \quad \overline{F}_{\mathfrak{J}}^D \leq \overline{s}_\nu^D. \tag{1.2.3}$$

In Section 5.1.2, as an application of the general principle above, we use the results of Section 2.2.6 to construct appropriate functions from $C_c^\infty(\mathbb{R}^d)$ and the min-max principle to demonstrate that (1.2.3) is valid for $\mathfrak{J} = \mathfrak{J}_\nu$.

By combining the lower and upper bounds of the spectral dimension presented in Section 5.2 and Section 5.1, we are able to calculate the upper spectral dimension with respect to Neumann boundary conditions (see Section 5.3). More precisely, we show equality of the optimized coarse multifractal dimension with respect to $\mathfrak{J}_\nu$, the $\mathfrak{J}_\nu$-partition entropy, and the unique zero of $\tau_{\mathfrak{J}_\nu}^N$:

$$\overline{h}_{\mathfrak{J}_\nu} = \overline{s}_\nu^N = q_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^N.$$

Surprisingly, it turns out that in the case $d = 2$, the formula above simplifies to $\overline{s}_\nu^N = 1$, and under the assumption $\nu(\mathring{\mathbf{Q}}) > 0$, we also have $\overline{s}_\nu^D = 1$. Thus, in the case $d = 2$, the upper spectral dimension contains no information about the underlying measure ν. This can be explained by the simple fact that the L^q-spectrum always has a zero at 1.

Another important question is under which conditions we can ensure that the upper spectral dimension with respect to Dirichlet and Neumann boundary conditions coincide. We show that if $\tau_{\mathfrak{J}_\nu}^N(q_{\mathfrak{J}_\nu}^D) = 0$, or equivalently $\overline{F}_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^D$, then the upper Dirichlet and Neumann spectral dimensions fulfill $\overline{s}_\nu^D = \overline{s}_\nu^N = q_{\mathfrak{J}_\nu}^N$. Motivated by this observation, we impose conditions on the Minkowski dimension of $\operatorname{supp}(\nu) \cap \partial\mathbf{Q}$ and on the growth rate of the boundary cubes (see (5.3.2)) to guarantee $\tau_{\mathfrak{J}_\nu}^N(q_{\mathfrak{J}_\nu}^D) = 0$. Further, we address the question of the existence of the spectral dimension. As in the one-dimensional case, we make use of the regularity conditions imposed in Proposition 3.24 to establish the following regularity results.

- *If $\mathfrak{J}_\nu$ is Neumann partition function regular, then the spectral dimension s_ν^N exists.*
- *If $\mathfrak{J}_\nu$ is Dirichlet partition function regular and $\tau_{\mathfrak{J}_\nu}^N(q_{\mathfrak{J}_\nu}^D) = 0$, then both the Dirichlet and Neumann spectral dimension exist and coincide, i.e. $s_\nu^D = s_\nu^N$.*

Additionally, using the formula for $\overline{s}_\nu^N$, we estimate the upper spectral dimension with respect to Neumann boundary conditions in terms of the upper Minkowski dimension of $\operatorname{supp}(\nu)$ and ∞-dimension of ν as follows:

$$\frac{d}{2} \leq \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu) - d + 2} \leq \overline{s}_\nu^N \leq \frac{\dim_\infty(\nu)}{\dim_\infty(\nu) - d + 2}. \tag{1.2.4}$$

We conclude Section 5.3 by discussing Kac's question in view of (1.2.4). Finally, we end Chapter 5 with four examples in Section 5.4. Here, we study absolutely continuous measures, Ahlfors-David regular measures, self-conformal measures without any separation conditions, and we present an example for which the spectral dimension does not exist.

Chapter 6 is dedicated to the study of the lower and upper quantization dimension with respect to a finite Borel measure ν on $\mathbf{Q}$. Here, we use the same strategy as in Chapter 4 and Chapter 5; we reduce the computation of the quantization dimension to the auxiliary combinatorial problems investigated in Chapter 3 for the special choice

$$\mathfrak{J}_{\nu,r/d}(Q) := \mathfrak{J}_{\nu,r/d,1}(Q) = \nu(Q)\Lambda(Q)^{r/d}, \quad Q \in \mathcal{D}.$$

More precisely, we will link the lower and upper quantization dimension to the dual problem studied in Section 3.3 with respect to $\mathfrak{J}_{\nu,r/d}$. Section 6.1 and Section 6.2 are devoted to the study of lower and upper bounds for the lower and upper quantization dimension. Section 6.3 contains the main results of that chapter. By combining the estimates obtained in Section 6.1 and Section 6.2, we are able to compute the upper quantization dimension for the first time. The first main result (see Theorem 6.5) reads as

$$\frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}} \leq \underline{D}_r(\nu) \leq \frac{r\underline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{h}_{\mathfrak{J}_{\nu,r/d}}} \leq \overline{D}_r(\nu) = \frac{r\overline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\overline{h}_{\mathfrak{J}_{\nu,r/d}}} = \frac{rq^N_{\mathfrak{J}_{\nu,r/d}}}{1-q^N_{\mathfrak{J}_{\nu,r/d}}} = \frac{r\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}}. \tag{1.2.5}$$

Interestingly, if $\sup_{x\in(0,1)} \beta_\nu^N(x) > 0$, then the upper quantization dimension coincides with the upper Rényi dimension at $q^N_{\mathfrak{J}_{\nu,r/d}}$, given by

$$\overline{\mathfrak{R}}_\nu(q) := \frac{\beta_\nu^N(q)}{1-q}, \quad q \neq 1.$$

This perspective sheds new light on the connection between the quantization problem and other concepts from fractal geometry in that we obtain a one-to-one correspondence of the upper quantization dimension and the L^q-spectrum restricted to $(0,1)$. Further, as a consequence of the formula for $\overline{D}_r(\nu)$, we derive an alternative representation for $\overline{D}_r(\nu)$ as a critical value:

$$\frac{\overline{D}_r(\nu)}{\overline{D}_r(\nu)+r} = \inf\left\{q>0 : \sum_{Q\in\mathcal{D}} \left(\Lambda(Q)^{r/d}\nu(Q)\right)^q < \infty\right\}.$$

This further results in a surprising connection with the upper spectral dimension for the case $d=r=1$ via

$$\frac{\overline{D}_1(\nu)}{\overline{D}_1(\nu)+1} = \overline{s}_\nu^{D/N}.$$

It is worth mentioning that this formula demonstrates a striking connection between two, at first sight, distinct fields of mathematics (the spectral problem of Kreĭn–Feller Operators and the quantization of compactly supported Borel measures). As a further application we confirm a conjecture of Lindsay [Lin01] which states that

$$r \mapsto \overline{D}_r(\nu), \ \ \textit{is continuous for } r>0 .$$

The second main Theorem 6.5 addresses the existence of the quantization dimension. Based on the regularity conditions imposed in Proposition 3.24 and (1.2.5), we prove the following regularity result:

$$\tau^N_{\mathfrak{J}_{\nu,r/d}} \textit{ is Neumann partition regular} \implies \underline{D}_r(\nu) = \overline{D}_r(\nu) = \frac{r q^N_{\mathfrak{J}_{\nu,r/d}}}{1-q^N_{\mathfrak{J}_{\nu,r/d}}}.$$

We conclude Chapter 6 with a corollary affirming the existence of the quantization dimension for self-conformal measure without any separation conditions.

Certain background information (for instance the relation between self-adjoint operators and quadratic forms) is provided in the Appendix A and referenced throughout this thesis. A list of symbols we use in this thesis is appended, including standard notation (e.g. the set of natural numbers).

This thesis is based on the following publications:

- M. Kesseböhmer and A. Niemann. Spectral dimensions of Kreĭn–Feller operators and L^q-spectra. *Adv. Math.* 399 (2022), Paper No. 108253. DOI: 10.1016/j.aim.2022.108253
- M. Kesseböhmer and A. Niemann. Approximation order of Kolmogorov diameters via L^q-spectra and applications to polyharmonic operators. *J. Funct. Anal.* 283.7 (2022), Paper No. 109598. DOI: 10.1016/j.jfa.2022.109598
- M. Kesseböhmer and A. Niemann. Spectral asymptotics of Kreĭn-Feller operators for weak Gibbs measures on self-conformal fractals with overlaps. *Adv. Math.* 403 (2022), Paper No. 108384. DOI: 10.1016/j.aim.2022.108384

- M. Kesseböhmer and A. Niemann. Spectral dimensions of Kreĭn-Feller operators in higher dimensions. *arXiv: 2202.05247* (2022)
- M. Kesseböhmer, A. Niemann, and S. Zhu. Quantization dimensions of of compactly supported probability measures via Rényi dimensions. *arXiv: 2205.15776* (2022). To appear in: Trans. AMS

Chapter 2

Preliminaries

2.1 Dyadic Partitions

Throughout this thesis we will frequently use dyadic cubes contained in $\mathbf{Q} = (0,1]^d$. Therefore, we list important basic notations from the introduction:

- $\mathcal{D}_n^N = \left\{ \prod_{i=1}^d (k_i 2^{-n}, (k_i+1)2^{-n}] : k_i = 0, \ldots, 2^n - 1 \right\}$,
- $\mathcal{D}_n^D = \left\{ Q \in \mathcal{D}_n^N : \partial \mathbf{Q} \cap \overline{Q} = \varnothing \right\}$,
- $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n^N$,
- $\mathcal{D}(Q) = \left\{ \widetilde{Q} \in \mathcal{D} : \widetilde{Q} \subset Q \right\}$ with $Q \in \mathcal{D}$.

We remark that $\mathcal{D}_{n+1}^N$ is a refinement of $\mathcal{D}_n^N$ for each $n \in \mathbb{N}$, this means that each element of $\mathcal{D}_n^N$ can be decomposed into 2^d disjoint elements of $\mathcal{D}_{n+1}^N$. Hence, it is easy to see that $\mathcal{D}$ is a semiring. The reason why the definition of $\mathcal{D}_n^D$ is appropriate becomes clear in the constructing certain functions with compact support contained in $\mathring{\mathbf{Q}}$ for the proof for the lower bounds in the Dirichlet case (see proof of Proposition 5.9). For this purpose, a certain distance to the boundary of the $\mathbf{Q}$ is needed. The notation $\mathcal{D}_n^N$ is motivated by the proof of the upper estimate of the spectral dimension of the Kreĭn–Feller with respect to Neumann boundary conditions. We end this section with some simple facts about $\mathcal{D}$ and $\mathcal{D}_n^{D/N}$.

Lemma 2.1. *Let $Q_1, Q_2 \in \mathcal{D}$ with $Q_1 \cap Q_2 \neq \varnothing$. Then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Further, for all $n \in \mathbb{N}$, we have*

$$\operatorname{card}\left(\mathcal{D}_n^N \setminus \mathcal{D}_n^D\right) = 2^{dn} - (2^n - 2)^d.$$

2.2 Form approach for Kreĭn–Feller operators

In this section, for fixed $d \in \mathbb{N}$, we define the Kreĭn–Feller operator with respect to a non-zero finite Borel measure ν on $\mathbf{Q} = (0,1]^d$. Note that we allow that $\mathbf{Q} \setminus \mathring{\mathbf{Q}}$ may have positive ν-measure. It should be noted that this assumption is of a technical character; in fact, the following considerations are also valid if we allow positive measure on $[0,1]^d \setminus (0,1)^d$. However, this results in some technical difficulties for the definition of the dyadic cubes of $[0,1]^d$ (for details see [KN22d]). Further, let $\Omega \subset \mathbf{Q}$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Throughout this chapter, we assume $\operatorname{card}(\operatorname{supp}(\nu) \cap \overline{\Omega}) = \infty$, or equivalently that $L^2_\nu(\overline{\Omega})$ is an infinite dimensional vector space.

2.2.1 Sobolev spaces and embeddings

We start by recalling the form approach following ideas in [HLN06]. The space $H^1(\Omega)$ with the bilinear form $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ defines a Hilbert space and $H^1_0(\Omega)$ is a closed subspace. For all $u \in H^1_0(\Omega)$, or $u \in \left\{ f \in H^1(\Omega) : \int_\Omega f \, \mathrm{d}\Lambda = 0 \right\}$, the *Poincaré inequality*, respect. *Poincaré–Wirtinger inequality* (see [Rui12, Lemma 3, p. 500] and Lemma 2.7), reads, for some constant $c > 0$, as follows

$$\|u\|_{L^2_\Lambda(\Omega)} \leq c \, \|\nabla u\|_{L^2_\Lambda(\Omega)} . \tag{PI}$$

Since Ω is a bounded Lipschitz domain, the norm induced by the form $\langle \cdot, \cdot \rangle_{H^1_0(\Omega)}$ is therefore equivalent to the norm induced by $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$ on $H^1_0(\Omega)$. We will consider only those finite Borel measures ν on the closure of Ω for which the following *ν-Poincaré inequality* holds for some $c_1 > 0$:

$$\|u\|_{L^2_\nu(\overline{\Omega})} \leq c_1 \, \|u\|_{H^1(\Omega)} \text{ for all } u \in C^\infty_b\left(\overline{\Omega}\right). \tag{νPI}$$

This then guarantees a continuous embedding of the Sobolev spaces $H^1(\Omega)$ into $L^2_\nu(\overline{\Omega})$ and $H^1_0(\Omega)$ into $L^2_\nu(\Omega)$. In fact, since $C^\infty_b(\overline{\Omega})$ lies dense in $H^1(\Omega)$ (this follows e.g. from Proposition A.1), for every $u \in H^1(\Omega)$ there exists a sequence $(u_n)_n$ of elements of $C^\infty_b(\overline{\Omega})$ such that $u_n \to u$ with respect to the norm of $H^1(\Omega)$. Now, (νPI) implies that $(u_n)_n$ is also a Cauchy sequence in $L^2_\nu(\overline{\Omega})$, hence there exists $\overline{u} \in L^2_\nu(\overline{\Omega})$ such that $u_n \to \overline{u}$ in $L^2_\nu(\overline{\Omega})$. It is easy to see that this limit is independent of the particular choice of $(u_n)_n$ and we therefore obtain in this way a bounded linear operator

$$\iota := \iota_\nu := \iota_{\Omega,\nu} : H^1(\Omega) \to L^2_\nu\left(\overline{\Omega}\right), f \mapsto \overline{f},$$

with $\iota(u) = u$ for all $u \in C_b^\infty(\overline{\Omega})$. If ι is also injective, then we may regard $H^1(\Omega)$ as a subspace of $L_\nu^2(\overline{\Omega})$. In the case the map is not injective we consider the following closed subspace of $H^1(\Omega)$

$$\mathcal{N}_\nu \coloneqq \mathcal{N}_{\nu,\Omega} \coloneqq \ker(\iota) = \left\{ f \in H^1(\Omega) : \|\iota(f)\|_{L_\nu^2(\overline{\Omega})} = 0 \right\}$$

and have the natural embedding

$$\mathcal{N}_\nu^\perp \coloneqq \left\{ f \in H^1(\Omega) : \forall g \in \mathcal{N}_\nu : \langle f, g \rangle_{H^1(\Omega)} = 0 \right\} \hookrightarrow L_\nu^2\left(\overline{\Omega}\right),$$

which is again given by ι. In particular, there exists a sequence $(u_n)_n \in C_b^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1(\Omega)$ and $\iota(u_n) = u_n \to \iota(u)$ in $L_\nu^2(\overline{\Omega})$. The extended version of the Poincaré inequality (νPI) reads as

$$\|\iota(u)\|_{L_\nu^2(\overline{\Omega})} \le c_1 \|u\|_{H^1(\Omega)} \text{ for all } u \in H^1(\Omega).$$

Recall that $C_c^\infty(\Omega) \subset C_b^\infty(\overline{\Omega})$ lies dense in $H_0^1(\Omega)$. Hence, for all $u \in H_0^1(\Omega)$, it follows

$$\|\iota(u)\|_{L_\nu^2(\overline{\Omega})} = \|\iota(u)\|_{L_\nu^2(\Omega)}.$$

Consequently,

$$\mathcal{N}_\nu \cap H_0^1(\Omega) = \left\{ f \in H_0^1(\Omega) : \|\iota(f)\|_{L_\nu^2(\Omega)} = 0 \right\}.$$

Therefore, this embedding carries over to

$$\mathcal{N}_{0,\nu}^\perp \coloneqq \left\{ f \in H_0^1(\Omega) : \forall g \in \mathcal{N}_\nu \cap H_0^1(\Omega) : \langle f, g \rangle_{H_0^1(\Omega)} = 0 \right\} \hookrightarrow L_\nu^2(\Omega) \tag{2.2.1}$$

and by (PI), we have respectively

$$\|\iota(u)\|_{L_\nu^2(\Omega)} \le c_2 \|u\|_{H_0^1(\Omega)} \text{ for all } u \in H_0^1(\Omega),$$

for some $c_2 > 0$.

2.2.2 Sobolev spaces and embeddings in the case $d = 1$

Here, we consider the case $d = 1$ and $\Omega = (a,b)$ with $a, b \in \mathbb{R}$ and $0 \le a < b \le 1$. Due to Lemma A.15 and Proposition A.17 the Sobolev space $H^1(a,b) \coloneqq H^1((a,b))$ is compact embedded into $C([a,b])$, where $C([a,b])$ denotes the vector space of continuous functions on $[a,b]$. Therefore, for elements $f \in H^1(a,b)$, we will always choose the continuous representative of f. Moreover, by Lemma A.15, $H_0^1(a,b) \coloneqq H_0^1((a,b))$ can be identified by $\{f \in H^1(a,b) : f(a) = f(b) = 0\}$. Since $C([a,b]) \subset L_\nu^2([a,b])$ for any finite Borel measure ν on $[a,b]$, the situation in the one-dimensional case becomes much simpler.

Lemma 2.2. *For every $f \in H^1(a,b)$, we have*

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)| \le \left((b-a)^{1/2} + (b-a)^{-1/2}\right)\left(\|\nabla f\|^2_{L^2_\Lambda((a,b))} + \|f\|^2_{L^2_\Lambda((a,b))}\right)^{1/2},$$

and for all $f \in H^1_0(a,b)$ and for every interval I with $\mathring{I} = (a,b)$,

$$\int_I f^2 \, \mathrm{d}\nu \le \nu(I)(b-a)\int_{\mathring{I}} (\nabla f)^2 \, \mathrm{d}\Lambda.$$

Proof. The first inequality was proved in [Kan+09, Lemma 1.4]. Further, for all $f \in H^1_0(a,b)$, an application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_I f(x)^2 \, \mathrm{d}\nu(x) &= \int_I (f(x) - f(a))^2 \, \mathrm{d}\nu(x) \\ &= \int_I \left(\left(\int_{(a,x)} \nabla f \, \mathrm{d}\Lambda\right)^2\right) \mathrm{d}\nu(x) \\ &\le \int_I \left((x-a)\int_{(a,x)} (\nabla f)^2 \, \mathrm{d}\Lambda\right) \mathrm{d}\nu(x) \\ &\le (b-a)\nu(I)\int_{(a,b)} (\nabla f)^2 \, \mathrm{d}\Lambda.\end{aligned}$$

□

Corollary 2.3. *Let $f \in H^1(a,b)$ with $\int_{(a,b)} f \, \mathrm{d}\Lambda = 0$, then we have*

$$\|f\|_{L^2_\nu(I)} \le 5 \cdot \nu(I)\Lambda(I)\|\nabla f\|_{H^1_0(a,b)}$$

with $\mathring{I} = (a,b)$.

Proof. By [Arz15, Lemma 2.3.1], for all $g \in H^1(0,1)$ with $\int_{(0,1)} g \, \mathrm{d}\Lambda = 0$, we have

$$\|g\|^2_{L^2_\Lambda((0,1))} \le \frac{\|\nabla g\|^2_{L^2_\Lambda((0,1))}}{4}$$

and by Lemma 2.2,

$$\sup_{y\in[0,1]} |g(x)|^2 \le 4\|g\|^2_{H^1(0,1)}.$$

Let $h \in (0,1)$ and $b \in \mathbb{R}$ be such that $T((0,1)) = \mathring{I}$ with $T(x) := hx + b$, $x \in [0,1]$.

Hence, for $f \in H^1(a,b)$ with $\int_{(a,b)} f \,\mathrm{d}\Lambda = 0$, we deduce

$$\begin{aligned}\int_I f^2 \,\mathrm{d}\nu &\leq \nu(I) \sup_{x\in I} |f(x)|^2 \\ &= \nu(I) \sup_{x\in[0,1]} |f \circ T(x)|^2 \\ &\leq 4\nu(I) \|f \circ T\|^2_{H^1(0,1)} \\ &\leq 4\nu(I) \|\nabla(f \circ T)\|^2_{L^2_\Lambda(0,1)} \left(1 + \frac{1}{4}\right) \\ &= 5\nu(I)\Lambda(I) \|\nabla(f)\|^2_{L^2_\Lambda(0,1)}.\end{aligned}$$

□

The following lemma shows that (νPI) holds for any finite Borel measure ν on $[a,b]$ and $\iota_{\nu,(a,b)}(u)$ coincides (in the $L^2_\nu([a,b])$ sense) with the natural choice of the continuous representative of $H^1(a,b)$.

Lemma 2.4. *Let ν be a finite Borel measure on $[a,b]$. Then for all $u \in H^1(a,b)$, we have*

$$\int_{[a,b]} u_c^2 \,\mathrm{d}\nu \leq \left((b-a)^{1/2} + (b-a)^{-1/2}\right)^2 \nu([a,b]) \|u\|^2_{H^1(a,b)},$$

and

$$\iota_{\nu,(a,b)}(u) = u_c \ \nu\text{-almost surely,}$$

where u_c denotes the unique continuous representative of u in $H^1(a,b)$.

Proof. By Lemma 2.2, for all $u \in H^1(a,b)$, we obtain

$$\|u_c\|^2_\infty \leq C^2 \|u\|^2_{H^1(a,b)},$$

with $C := (b-a)^{1/2} + (b-a)^{-1/2}$. This leads to

$$\int_{[a,b]} u_c^2 \,\mathrm{d}\nu \leq \nu([a,b]) \|u_c\|^2_\infty \leq C^2 \nu([a,b]) \|u\|^2_{H^1(a,b)},$$

which proves the first claim. Now let $u \in H^1(a,b)$ and $(u_n)_{n\in\mathbb{N}} \in C_b^\infty([a,b])$ such that $u_n \to u$ in the norm of $H^1(a,b)$. Clearly, we have $u_c - u_n \in H^1(a,b) \cap C([a,b])$, which implies

$$\int_{[a,b]} (u_c - u_n)^2 \,\mathrm{d}\nu \leq C \|u_c - u_n\|^2_{H^1(a,b)}.$$

This gives $u_n \to u_c$ for $n \to \infty$ in $L^2_\nu([a,b])$, allowing us to conclude that

$$\iota_{\nu,(a,b)}(u) = u_c. \qquad \square$$

In order to determine $\mathcal{N}_{0,\nu}$, we introduce the following subset of continuous functions on $[a,b]$:

$C_\nu([a,b]) := \{f \in C([a,b]) : f \text{ is affine linear on the components of } [a,b] \setminus \operatorname{supp}(\nu)\}.$

Further, we define the orthogonal complement of $C_\nu([a,b])$ with respect to $H^1_0(a,b)$ by

$\left(C_\nu([a,b]) \cap H^1_0(a,b)\right)^\perp := \left\{f \in H^1_0(a,b) \colon \forall g \in C_\nu([a,b]) \cap H^1_0(a,b) : \langle f,g\rangle_{H^1_0(a,b)} = 0\right\}.$

Proposition 2.5. *Let ν be a Borel measure on (a,b). Then we have*

$$\mathcal{N}_{0,\nu} = \left(C_\nu([a,b]) \cap H^1_0(a,b)\right)^\perp,$$

or equivalently $\mathcal{N}_{0,\nu}^\perp = C_\nu([a,b]) \cap H^1_0(a,b)$.

Proof. Pick $f \in \left(C_\nu([a,b]) \cap H^1_0(a,b)\right)^\perp$. Then we define for $x \in \operatorname{supp}(\nu) \cap (a,b)$

$$g_x(y) := f(x)\frac{(y-a)}{x-a}\mathbb{1}_{[a,x]} + f(x)\frac{b-y}{b-x}\mathbb{1}_{(x,b]}.$$

Hence, using $f(a) = f(b) = 0$, we obtain

$$\begin{aligned}0 = \langle f, g_x\rangle_{H^1_0(a,b)} &= \int_{[a,x]} \nabla f \nabla g_x \,\mathrm{d}\Lambda + \int_{(x,b]} \nabla f \nabla g_x \,\mathrm{d}\Lambda \\ &= \frac{f(x)}{x-a}(f(x)-f(a)) + \frac{-f(x)}{b-x}(f(b)-f(x)) = \frac{f(x)^2}{x-a} + \frac{f(x)^2}{b-x},\end{aligned}$$

and consequently, for all $x \in \operatorname{supp}(\nu) \cap (a,b)$, we have $f(x) = 0$. In particular, we have $\|f\|_{L^2_\nu((a,b))} = 0$. On the other hand, for $f \in \left\{g \in H^1_0(a,b) : \|g\|_{L^2_\nu((a,b))} = 0\right\}$, f vanishes ν-a.e. and, using the continuity of f, we obtain $f = 0$ on $\operatorname{supp}(\nu)$. To simplify notation, assume $a,b \in \operatorname{supp}(\nu)$. Now, with $\biguplus_{i\in I}(a_i,b_i) = [a,b] \setminus \operatorname{supp}(\nu)$, one easily verifies that $\nabla f = \sum_{i\in I} \mathbb{1}_{(a_i,b_i)} \nabla f \in L^2_\Lambda((a,b))$ and we obtain that for all

$g \in C_\nu([a,b]) \cap H_0^1(a,b)$,

$$\begin{aligned}\langle f,g\rangle_{H_0^1(a,b)} &= \int_{[a,b]} \nabla f \nabla g \,\mathrm{d}\Lambda \\ &= \sum_{i\in I} \int_{(a_i,b_i)} \nabla f \nabla g \,\mathrm{d}\Lambda \\ &= \sum_{i\in I} \frac{g(b_i)-g(a_i)}{b_i-a_i}(f(b_i)-f(a_i)) = 0.\end{aligned}$$

This gives $f \in \left(C_\nu([a,b]) \cap H_0^1(a,b)\right)^\perp$. □

2.2.3 Stein extension

We begin with the definition of a Stein extension.

Definition 2.6. We say a bounded domain $A \subset \mathbb{R}^d$ permits a *Stein extension* if there exists a continuous linear operator $\mathfrak{E}_\Omega : H^1(A) \to H^1\left(\mathbb{R}^d\right)$ such that $\mathfrak{E}_\Omega(f)|_A = f$ and

$$\mathfrak{E}_A : C_b^\infty\left(\overline{A}\right) \to C_c^\infty\left(\mathbb{R}^d\right) \text{ with } \mathfrak{E}(f)|_A = f.$$

Necessarily, we then have that $C_b^\infty\left(\overline{A}\right)$ lies dense in $H^1(A)$. The second property above is not standard in the literature but follows from [Ste70, Sec. 3.2 and 3.3] (see Appendix A.2 for a more detailed presentation). Note that every bounded Lipschitz domain permits a Stein extension (see Theorem A.14 in Appendix A.2), in particular $\mathring{\mathbf{Q}}$ as a bounded convex open set, see e.g. [Gri85, Corollary 1.2.2.3] or [Ste70, Example 2, p. 189] is a bounded Lipschitz domain, thus the Stein extension $\mathfrak{E}_{\mathbf{Q}}$ with the above properties is well defined. Note that for any cube $Q \in \mathcal{D}$, by the definition of the weak derivatives, we have $H^1(Q) = H^1(\mathring{Q})$.

Lemma 2.7. *There exists a constant $D_{\boldsymbol{Q}} > 0$ such that for all half-open cubes $Q \subset \boldsymbol{Q}$ with edges parallel to the coordinate axes and $u \in H^1(Q)$,*

$$D_{\boldsymbol{Q}} \|u\|_{H^1(Q)}^2 \le \|\nabla u\|_{L_\Lambda^2(Q)}^2 + \frac{1}{\Lambda(Q)} \left| \int_Q u \,\mathrm{d}\Lambda \right|^2 \le \|u\|_{H^1(Q)}^2 .$$

Proof. Clearly, by the Cauchy-Schwarz inequality, we have for all $u \in H^1(Q)$

$$\frac{1}{\Lambda(Q)} \left| \int_Q u \,\mathrm{d}\Lambda \right|^2 \le \|u\|_{L_\Lambda^2(Q)}^2,$$

proving the second inequality. From [NS01, Lemma 3, p. 500] we obtain that there exists $C_{\mathbf{Q}} > 0$ such that for all $v \in H^1(\mathbf{Q})$

$$C_{\mathbf{Q}}\left(\int_{\mathbf{Q}} v^2 \,\mathrm{d}\Lambda\right) \leq \|\nabla v\|^2_{L^2_\Lambda(\mathbf{Q})} + \left|\int_{\mathbf{Q}} v \,\mathrm{d}\Lambda\right|^2.$$

Let $T : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto x_0 + hx$, with $h \in (0,1)$, $x_0 \in \mathbf{Q}$, such that $Q = T(\mathbf{Q})$. Note that $u \circ T \in H^1(\mathbf{Q})$ and $\|\nabla(u \circ T)\|^2_{L^2_\Lambda(\mathbf{Q})} = h^{2-d}\|\nabla u\|^2_{L^2_\Lambda(Q)}$, for all $u \in H^1(Q)$, leading to

$$\begin{aligned}\frac{C_{\mathbf{Q}}}{h^d}\int_Q u^2 \,\mathrm{d}\Lambda &= C_{\mathbf{Q}}\int_{\mathbf{Q}} u^2 \circ T \,\mathrm{d}\Lambda \\ &\leq \|\nabla(u \circ T)\|^2_{L^2_\Lambda(\mathbf{Q})} + \left|\int_{\mathbf{Q}} u \circ T \,\mathrm{d}\Lambda\right|^2 \\ &= h^{2-d}\|\nabla u\|^2_{L^2_\Lambda(Q)} + h^{-2d}\left|\int_Q u \,\mathrm{d}\Lambda\right|^2.\end{aligned}$$

Hence, using $h < 1$, we obtain

$$\begin{aligned}C_{\mathbf{Q}}\left(\int_Q u^2 \,\mathrm{d}\Lambda + \|\nabla u\|^2_{L^2_\Lambda(Q)}\right) &\leq \left(h^2 + C_{\mathbf{Q}}\right)\|\nabla u\|^2_{L^2_\Lambda(Q)} + h^{-d}\left|\int_Q u \,\mathrm{d}\Lambda\right|^2 \\ &\leq (1 + C_{\mathbf{Q}})\left(\|\nabla u\|^2_{L^2_\Lambda(Q)} + \frac{1}{\Lambda(Q)}\left|\int_Q u \,\mathrm{d}\Lambda\right|^2\right). \qquad \square\end{aligned}$$

Lemma 2.8. *Let $d \geq 2$ and $T : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto x_0 + hx$, with $h \in (0,1)$, $x_0 \in \boldsymbol{Q}$, such that the cube $Q := T(\boldsymbol{Q})$ belongs to $\mathcal{D}$. Then we have:*

1. *$\mathfrak{E}_Q : H^1(Q) \to H^1(\mathbb{R}^d), u \mapsto \mathfrak{E}_{\boldsymbol{Q}}(u \circ T) \circ T^{-1}$ defines a Stein extension with*

$$\|\mathfrak{E}_Q\| \leq \|\mathfrak{E}_{\boldsymbol{Q}}\|/h^2,$$

2. *$\|\mathfrak{E}_Q|_{N_\Lambda(Q)}\| \leq \|\mathfrak{E}_{\boldsymbol{Q}}\|/D_{\boldsymbol{Q}}$ with $N_\Lambda(Q) := \left\{u \in H^1(Q) : \int_Q u \,\mathrm{d}\Lambda = 0\right\}$.*

Proof. We only prove the second claim. Fix $T : x \mapsto x_0 + hx$ such that $T(\mathbf{Q}) = Q$ and assume $\int_Q u \,\mathrm{d}\Lambda = 0$. For a vector space V we write $V^\star := V \setminus \{0\}$. Hence, we obtain

$$\begin{aligned}
\left\|\mathfrak{E}_Q|_{N_\Lambda(Q)}\right\| &= \sup_{u\in N_\Lambda(Q)^\star} \frac{\left\|\mathfrak{E}_Q(u)\right\|_{H^1(\mathbb{R}^d)}}{\|u\|_{H^1(Q)}} = \sup_{u\in N_\Lambda(Q)^\star} \frac{\left\|\mathfrak{E}_{\mathbf{Q}}(u\circ T)\circ T^{-1}\right\|_{H^1(\mathbb{R}^d)}}{\|u\|_{H^1(Q)}} \\
&\le \sup_{u\in N_\Lambda(Q)^\star} \frac{\left(\int \left(\nabla\left(\mathfrak{E}_{\mathbf{Q}}(u\circ T)\circ T^{-1}\right)\right)^2 \,\mathrm{d}\Lambda + \int \left(\mathfrak{E}_{\mathbf{Q}}(u\circ T)\circ T^{-1}\right)^2 \,\mathrm{d}\Lambda\right)^{\frac{1}{2}}}{\left(\int_Q \left(\nabla\left((u\circ T)\circ T^{-1}\right)\right)^2 \,\mathrm{d}\Lambda + \frac{1}{\Lambda(Q)}\left|\int_Q u \,\mathrm{d}\Lambda\right|^2\right)^{\frac{1}{2}}} \\
&= \sup_{u\in N_\Lambda(Q)^\star} \frac{\left(h^{d-2}\int \left(\nabla\left(\mathfrak{E}_{\mathbf{Q}}(u\circ T)\right)\right)^2 \,\mathrm{d}\Lambda + h^d \int \mathfrak{E}(u\circ T)^2 \,\mathrm{d}\Lambda\right)^{\frac{1}{2}}}{\left(h^{d-2}\int_{\mathbf{Q}} \left(\nabla(u\circ T)\right)^2 \,\mathrm{d}\Lambda\right)^{\frac{1}{2}}} \\
&\le \sup_{u\in N_\Lambda(Q)^\star} \frac{\left(h^{d-2}\left(\int \left(\nabla\left(\mathfrak{E}_{\mathbf{Q}}(u\circ T)\right)\right)^2 \,\mathrm{d}\Lambda + \int \mathfrak{E}(u\circ T)^2 \,\mathrm{d}\Lambda\right)\right)^{\frac{1}{2}}}{\left(h^{d-2}\int_{\mathbf{Q}} \left(\nabla(u\circ T)\right)^2 \,\mathrm{d}\Lambda\right)^{\frac{1}{2}}} \\
&\le \sup_{u\in N_\Lambda(Q)^\star} \frac{\left\|\mathfrak{E}_{\mathbf{Q}}(u\circ T)\right\|_{H^1(\mathbb{R}^d)}}{D_{\mathbf{Q}}\|u\circ T\|_{H^1(\mathbf{Q})}} \le \frac{\left\|\mathfrak{E}_{\mathbf{Q}}\right\|}{D_{\mathbf{Q}}},
\end{aligned}$$

where in the last inequality we used the fact that $\int_{\mathbf{Q}} u\circ T \,\mathrm{d}\Lambda = 0$, $h^d < h^{d-2}$, and Lemma 2.7. □

Lemma 2.9. *Assuming that the following Poincaré inequality*

$$\|u\|_{L^2_\nu(\mathbb{R}^d)} \le c_1 \|u\|_{H^1(\mathbb{R}^d)} \text{ for all } u \in C_c^\infty\left(\mathbb{R}^d\right)$$

holds for some $c_1 > 0$. *Let* $\iota_{\mathbb{R}^d} : H^1\left(\mathbb{R}^d\right) \to L^2_\nu\left(\mathbb{R}^d\right)$ *denote the embedding and*

$$\mathfrak{R}_\Omega : L^2_\nu\left(\mathbb{R}^d\right) \to L^2_\nu\left(\overline{\Omega}\right),\ f \mapsto f|_{\overline{\Omega}}$$

the restriction operator. Then we have $\iota_\Omega = \mathfrak{R}_\Omega \circ \iota_{\mathbb{R}^d} \circ \mathfrak{E}_\Omega$.

Proof. First note that $\iota_{\mathbb{R}^d}$ restricted to $C_c^\infty\left(\mathbb{R}^d\right)$ is the identity. Now, using the fact that $\mathfrak{E}_\Omega : C_b^\infty\left(\overline{\Omega}\right) \to C_c^\infty\left(\mathbb{R}^d\right)$ combined with the above observation we find for all $u \in C_b^\infty\left(\overline{\Omega}\right)$,

$$\mathfrak{R}_\Omega\left(\iota_{\mathbb{R}^d} \circ \mathfrak{E}_\Omega(u)\right) = \mathfrak{R}_\Omega\left(\mathfrak{E}_\Omega(u)\right) = u|_{\overline{\Omega}} = \iota_\Omega(u).$$

Since ι_Ω is continuous and $C_b^\infty(\overline{\Omega})$ lies dense in $H^1(\Omega)$, the claim follows. □

2.2.4 Form approach

Since $\iota := \iota_\Omega$ maps $\mathcal{N}_\nu^\perp$ bijectively to $\operatorname{dom}\left(\mathcal{E}^N\right) := \operatorname{dom}\left(\mathcal{E}_\Omega^N\right) := \iota\left(\mathcal{N}_\nu^\perp\right)$ and $\mathcal{N}_{0,\nu}^\perp$ to $\operatorname{dom}\left(\mathcal{E}^D\right) := \operatorname{dom}\left(\mathcal{E}_\Omega^D\right) := \iota\left(\mathcal{N}_{0,\nu}^\perp\right)$, we may define the relevant corresponding forms by the push forward

$$\mathcal{E}^N(u,v) := \mathcal{E}_\Omega^N(u,v) := \left\langle \iota^{-1}u, \iota^{-1}v\right\rangle_{H^1(\Omega)}, \text{ for } u,v \in \operatorname{dom}(\mathcal{E}^N)$$

and

$$\mathcal{E}^D(u,v) := \mathcal{E}_\Omega^D(u,v) := \left\langle \iota^{-1}u, \iota^{-1}v\right\rangle_{H_0^1(\Omega)}, \text{ for } u,v \in \operatorname{dom}(\mathcal{E}^D).$$

In the latter case (Dirichlet case), we always assume that

$$\operatorname{card}\left(\Omega \cap \operatorname{supp}(\nu)\right) = \infty.$$

Lemma 2.10. *$C_c^\infty(\Omega)$ lies dense in $L_\nu^2(\Omega)$ and $C_b^\infty(\overline{\Omega})$ lies dense in $L_\nu^2(\overline{\Omega})$.*

Proof. We start to show that indicator functions $\mathbb{1}_A$ with $A \in \mathfrak{B}(\Omega)$, where $\mathfrak{B}(\Omega)$ denotes the Borel σ-algebra of Ω, can be approximated by functions of $C_c^\infty(\Omega)$. Since ν is a finite Borel measure on $\overline{\Omega}$, by [Els11, 1.16 Satz von Ulam], for fixed $\varepsilon > 0$, there exist a compact set K and an open set $U \subset \Omega$ with $K \subset A \subset U$ such that

$$\nu(U \setminus K) < \varepsilon.$$

Now, using mollifiers (see e.g. Section 2.2.6), we see that there exists $f \in C_c^\infty(U)$ with $f_{|K} = 1$ and $0 \le f \le 1$. Hence,

$$\int |\mathbb{1}_A - f| \,\mathrm{d}\nu \le \nu(U \setminus K) < \varepsilon.$$

Notice, that the simple functions lie dense in $L^2(\Omega)$, proving the first claim. To prove the second claim, consider a bounded open set O such that $\overline{\Omega} \subset O$. Then repeating the previous argument, we obtain that $C_c^\infty(O)$ lies dense in $L_\nu^2(O)$. Furthermore, for each $g \in C_c^\infty(O)$, we have $g|_{\overline{\Omega}} \in C_b^\infty(\overline{\Omega})$. Using $\operatorname{supp}(\nu) \subset \overline{\Omega}$, we deduce that $C_b^\infty(\overline{\Omega})$ lies dense in $L_\nu^2(\overline{\Omega})$. □

Proposition 2.11. *The set $\operatorname{dom}\left(\mathcal{E}^D\right)$ lies dense in $L_\nu^2(\Omega)$ and $\operatorname{dom}\left(\mathcal{E}^N\right)$ lies dense in $L_\nu^2(\overline{\Omega})$.*

Proof. Here we follow the arguments of [HLN06]. By Lemma 2.10, it follows that $C_c^\infty(\Omega)$ lies dense in $L_\nu^2(\Omega)$. This carries over to the orthogonal projection onto $\iota\left(\mathcal{N}_{0,\nu}^\perp\right)$ since $\iota\left(\mathcal{N}_{0,\nu}\right)$ is the zero space in $L_\nu^2(\Omega)$. Similarly, for the Neumann case, by Lemma 2.10, we obtain that $C_b^\infty(\overline{\Omega})$ lies dense in $L_\nu^2(\overline{\Omega})$ which carries over to $\iota\left(\mathcal{N}_\nu^\perp\right)$. □

Proposition 2.12. *Assuming* (νPI)*, we have that* $\operatorname{dom}\left(\mathcal{E}^{D/N}\right)$ *equipped with the inner product* $\langle f,g\rangle_\nu + \mathcal{E}^{D/N}(f,g)$ *defines a Hilbert spaces, i.e.* $\mathcal{E}^{D/N}$ *is a closed form with respect to* $L^2_\nu(\Omega)$ *and* $L^2_\nu(\overline{\Omega})$*, respectively.*

Proof. Since both cases can be treated completely analogously, we only consider the first case: We first observe that $\mathcal{N}_\nu^\perp$ is a closed linear subspace with respect to $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$, which by (νPI) induces a norm that is equivalent to the norm induced by $\langle\cdot,\cdot\rangle_{H^1(\Omega)} + \langle\iota\cdot,\iota\cdot\rangle_\nu$. Therefore, $\left(\mathcal{N}_\nu^\perp, \langle\cdot,\cdot\rangle_{H^1(\Omega)} + \langle\iota\cdot,\iota\cdot\rangle_\nu\right)$ is a Hilbert space and since $\mathcal{E}^N + \langle\cdot,\cdot\rangle_\nu$ is the push-forward of $\langle\cdot,\cdot\rangle_{H^1(\Omega)} + \langle\iota\cdot,\iota\cdot\rangle_\nu$, the claim follows. □

2.2.5 Definition of the Kreĭn–Feller operator

Recall that the Hu–Lau–Ngai condition (♠) is given by

$$\dim_\infty(\nu) = \liminf_{n\to\infty} \frac{\max_{Q\in\mathcal{D}_n^N} \log(\nu(Q))}{-n\log(2)} > d-2.$$

We note that for $Q\in\mathcal{D}$ with $\nu(Q)>0$ we have $\dim_\infty(\nu) \le \dim_\infty\left(\nu|_Q\right)$ and hence the condition (♠) carries over to the restricted measure $\nu|_Q$. We also remark that our definition of $\dim_\infty(\nu)$ is consistent with the usual definition in terms of balls rather than cubes from a uniform lattice (see e.g. [Str93]). Obviously, we always have $\dim_\infty(\nu)\le d$, and the assumption $\dim_\infty(\nu)>0$ excludes the possibility of ν having atoms in higher dimensional case.

Under the Hu–Lau–Ngai condition (♠), from Proposition 2.11 and Proposition 2.12, we deduce that $\left(\mathcal{E}^{D/N}, \operatorname{dom}\left(\mathcal{E}^{D/N}\right)\right)$ is a densely defined closed form on $L^2_\nu(\Omega)$ and $L^2_\nu(\overline{\Omega})$, respectively. Now we are in the position to define the Kreĭn–Feller operator with respect to Dirichlet/Neumann boundary conditions:
For $\left(\mathcal{E}^D, \operatorname{dom}\left(\mathcal{E}^D\right)\right)$, by Lemma A.22 and Theorem A.23, there exists a non-negative self-adjoint operator Δ_ν^D on $L^2_\nu(\Omega)$ such that

$$f\in\operatorname{dom}\left(\Delta_\nu^D\right)\subset\operatorname{dom}\left(\left(\Delta_\nu^D\right)^{1/2}\right)=\operatorname{dom}\left(\mathcal{E}^D\right)$$

if and only if $f\in\operatorname{dom}\left(\mathcal{E}^D\right)$ and there exists $u\in L^2_\nu(\Omega)$ such that

$$\mathcal{E}^D(f,g) = \langle u,g\rangle_{L^2_\nu(\Omega)},\ g\in\operatorname{dom}\left(\mathcal{E}^D\right).$$

Further, for $\left(\mathcal{E}^N, \operatorname{dom}\left(\mathcal{E}^N\right)\right)$, by Theorem A.23 and Lemma A.22, there exists a non-negative self-adjoint operator Δ_ν^N on $L^2_\nu(\overline{\Omega})$ such that

$$f\in\operatorname{dom}\left(\Delta_\nu^N\right)\subset\operatorname{dom}\left(\left(\Delta_\nu^N\right)^{1/2}\right)=\operatorname{dom}\left(\mathcal{E}^N\right)$$

if and only if $f \in \operatorname{dom}(\mathcal{E}^N)$ and there exists $u \in L^2_\nu(\overline{\Omega})$ such that

$$\mathcal{E}^N(f,g) = \langle u,g\rangle_{L^2_\nu(\overline{\Omega})}, g \in \operatorname{dom}\left(\mathcal{E}^N\right).$$

In this case, we have $u = \Delta_\nu^{D/N} f$. Furthermore, we call $\Delta_\nu^{D/N}$ *Kreĭn–Feller operator with respect to Dirichlet/Neumann boundary conditions* and $f \in \operatorname{dom}(\Delta_\nu^D)^\star$ an *(Dirichlet) eigenfunction* with eigenvalue $\lambda \in \mathbb{R}$ if

$$\mathcal{E}^D(f,g) = \lambda \int_\Omega fg \,\mathrm{d}\nu$$

for all $g \in \operatorname{dom}(\mathcal{E}^D)$. Further, we call $f \in \operatorname{dom}(\Delta_\nu^N)^\star$ an *(Neumann) eigenfunction* with eigenvalue $\lambda \in \mathbb{R}$ if

$$\mathcal{E}^N(f,g) = \lambda \int_{\overline{\Omega}} fg \,\mathrm{d}\nu$$

for all $g \in \operatorname{dom}(\mathcal{E}^N)$. Notice in the case $\Omega = \mathring{\mathbf{Q}}$, since ν is Borel measure on $\mathbf{Q}$, we have

$$\mathcal{E}^N(f,g) = \lambda \int_{\mathbf{Q}} fg \,\mathrm{d}\nu.$$

In order to prove that the embeddings

$$\left(\operatorname{dom}\left(\mathcal{E}^D\right), \mathcal{E}^D\right) \hookrightarrow L^2_\nu(\Omega) \text{ and } \left(\operatorname{dom}\left(\mathcal{E}^N\right), \mathcal{E}^N\right) \hookrightarrow L^2_\nu(\overline{\Omega})$$

are compact under the assumption $\dim_\infty(\nu) > d-2$, we need the following result due to Maz'ya [Maz85, Theorem 3, p. 386] and [Maz85, Theorem 4, p. 387].

Theorem 2.13. *For $d > 2$ and $t > 2$, the set $\left\{u \in C_c^\infty(\mathbb{R}^d) : \|u\|_{H^1(\mathbb{R}^d)} \le 1\right\}$ is precompact in $L^t_\nu(\mathbb{R}^d)$ if and only if*

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d, \rho\in(0,r)} \rho^{(1-d/2)} \nu\left(B_\rho(x)\right)^{1/t} = 0,$$

and for $d = 2$, $\left\{u \in C_c^\infty(\mathbb{R}^d) : \|u\|_{H^1(\mathbb{R}^d)} \le 1\right\}$ is precompact in $L^t_\nu(\mathbb{R}^2)$ if and only if

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^2, \rho\in(0,r)} |\log(\rho)|^{1/2} \nu\left(B_\rho(x)\right)^{1/t} = 0,$$

where $B_\rho(x)$ denotes the open unit ball with radius $\rho > 0$ and center x in $\mathbb{R}^d$.

Proposition 2.14. *The assumption $\dim_\infty(\nu) > d-2$ implies (νPI) and the embeddings*

$$\left(\operatorname{dom}\left(\mathcal{E}^D\right), \mathcal{E}^D\right) \hookrightarrow L^2_\nu(\Omega) \textit{ and } \left(\operatorname{dom}\left(\mathcal{E}^N\right), \mathcal{E}^N\right) \hookrightarrow L^2_\nu\left(\overline{\Omega}\right)$$

are compact.

Proof. For the case $d = 1$, the claim follows from Proposition A.17. For $d > 2$ and $t \in (2, 2\dim_\infty(\nu)/(d-2))$, the assumption (νPI) implies

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d, \rho\in(0,r)} \rho^{(1-d/2)} \nu\left(B_\rho(x)\right)^{1/t} = 0,$$

and for $d = 2$,

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^2, \rho\in(0,r)} |\log(\rho)|^{1/2} \nu\left(B_\rho(x)\right)^{1/t} = 0$$

(see [HLN06]). Hence, by [Maz85, Theorem 3 and Theorem 4, p. 583] we know that $\left\{u \in C_c^\infty\left(\mathbb{R}^d\right) : \|u\|_{H^1(\mathbb{R}^d)} \le 1\right\}$ is precompact in $L_\nu^t\left(\mathbb{R}^d\right)$. Thus, there exists $c > 0$ such that for all $u \in C_c^\infty\left(\mathbb{R}^d\right)$, we have

$$\|u\|_{L_\nu^2(\mathbb{R}^d)} \le \|u\|_{L_\nu^t(\mathbb{R}^d)} \le c \cdot \|u\|_{H^1(\mathbb{R}^d)},$$

which implies that the identity map $C_c^\infty\left(\mathbb{R}^d\right) \to L_\nu^2\left(\mathbb{R}^d\right)$ permits a unique continuous continuation $\iota_{\mathbb{R}^d} : H^1\left(\mathbb{R}^d\right) \to L_\nu^2\left(\mathbb{R}^d\right)$. To see that $\iota_{\mathbb{R}^d}$ is compact, fix a bounded sequence $(u_n)_n$ in $H^1\left(\mathbb{R}^d\right)$. We find another sequence $(v_n)_n$ in $C_c^\infty\left(\mathbb{R}^d\right)$ such that $\|v_n - u_n\|_{H^1(\mathbb{R}^d)} \to 0$ for $n \to \infty$ (see for instance [Ada75, Corollary 3.19]). The aforementioned precompactness leads to a subsequence $(v_{n_k})_k$ converging in $L_\nu^2(\mathbb{R}^d)$ to an element v. Then $\left(\iota_{\mathbb{R}^d}(u_{n_k})\right)_k$ also converges to v in $L_\nu^2(\mathbb{R}^d)$, since

$$\begin{aligned}\left\|v - \iota_{\mathbb{R}^d}(u_{n_k})\right\|_{L_\nu^2(\mathbb{R}^d)} &= \left\|v - v_{n_k} + v_{n_k} - \iota_{\mathbb{R}^d}(u_{n_k})\right\|_{L_\nu^2(\mathbb{R}^d)} \\ &\le \left\|v - v_{n_k}\right\|_{L_\nu^2(\mathbb{R}^d)} + \left\|\iota_{\mathbb{R}^d}(v_{n_k} - u_{n_k})\right\|_{L_\nu^2(\mathbb{R}^d)} \\ &\le \left\|v - v_{n_k}\right\|_{L_\nu^2(\mathbb{R}^d)} + c \cdot \left\|v_{n_k} - u_{n_k}\right\|_{H^1(\mathbb{R}^d)} \to 0,\end{aligned}$$

for k tending to infinity. This shows that $\iota_{\mathbb{R}^d}\left(\{u \in H^1(\mathbb{R}^d) : \|u\|_{H^1(\mathbb{R}^d)} \le k\}\right)$ is precompact for every $k > 0$. Since for the Stein extension $\mathfrak{E}_\Omega$ for all $u \in H^1(\Omega)$ we have

$$\|\mathfrak{E}_\Omega(u)\|_{H^1\left(\mathbb{R}^d\right)} \le \|\mathfrak{E}_\Omega\| \, \|u\|_{H^1(\Omega)},$$

it follows that $\iota_{\mathbb{R}^d}(\mathfrak{E}_\Omega(\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \le 1\}))$ is precompact as a subset of

$$\iota_{\mathbb{R}^d}\left(\left\{u \in H^1\left(\mathbb{R}^d\right) : \|u\|_{H^1(\mathbb{R}^d)} \le \|\mathfrak{E}_\Omega\|\right\}\right).$$

Applying the restriction operator $\mathfrak{R}_\Omega$ from Lemma 2.9, we find that

$$\begin{aligned}\left\{u \in \operatorname{dom}\left(\mathcal{E}^N\right) : \left\|\iota^{-1}u\right\|_{H^1(\Omega)} \le 1\right\} &\subset \iota\left(\left\{u \in H^1(\Omega) : \|u\|_{H^1(\Omega)} \le 1\right\}\right) \\ &= \mathfrak{R}_\Omega\left(\iota_{\mathbb{R}^d} \circ \mathfrak{E}_\Omega\left(\left\{u \in H^1 : \|u\|_{H^1(\Omega)} \le 1\right\}\right)\right).\end{aligned}$$

Therefore, $\{u \in \operatorname{dom}\left(\mathcal{E}^N\right) : \left\|\iota^{-1}u\right\|_{H^1(\Omega)} \le 1\}$ is relative compact in $L_\nu^2(\overline{\Omega})$ as a subset of a continuous image of a relatively compact set. In particular, there exists $c_1 > 0$ such that for all $u \in \operatorname{dom}\left(\mathcal{E}^N\right)$

$$\|u\|_{L^2_\nu(\overline{\Omega})} \leq c_1 \left\|\iota^{-1} u\right\|_{H^1(\Omega)}.$$

The Dirichlet case follows by almost the same means without the use of the extension operator (see also [HLN06]). □

As a direct consequence of Theorem A.26 and Proposition 2.14, we obtain the following important corollary.

Corollary 2.15. *Assume* $\dim_\infty(\nu) > d-2$. *Then the operator* $\Delta^{D/N}$ *has compact resolvent and there exists a complete orthonormal basis of eigenvectors* $\left(f_k^{D/N}\right)_{k\in\mathbb{N}}$ *with eigenvalues* $\left(\lambda_{n,\nu}^{D/N}\right)_{n\in\mathbb{N}}$ *with* $\lambda_{n,\nu}^{D/N} \leq \lambda_{n+1,\nu}^{D/N}$ *tending to infinity.*

By Corollary 2.15 we can refer to the lower and upper spectral dimension with respect to Dirichlet and Neumann boundary conditions defined in (1.1.1).

Remark 2.16. Let $\mathcal{E}$ be a closed form with domain $\operatorname{dom}(\mathcal{E})$ densely defined on $\mathcal{H} \in \{L^2_\nu(\Omega), L^2_\nu(\overline{\Omega})\}$, in particular $\operatorname{dom}(\mathcal{E})$ defines a Hilbert space with respect to $(f,g)_{\mathcal{E}} \coloneqq \langle f,g\rangle_{\mathcal{H}} + \mathcal{E}(f,g)$. Moreover, assume that the inclusion from $\left(\operatorname{dom}(\mathcal{E}), \langle\cdot,\cdot\rangle_{\mathcal{E}}\right)$ into $\mathcal{H}$ is compact. Then the *Poincaré–Courant–Fischer–Weyl min-max principle* is applicable, that is for the i-th eigenvalue $\lambda_i(\mathcal{E})$ of $\mathcal{E}$, $i \in \mathbb{N}$, we have (see also Theorem A.27 or [Dav95; KL93])

$$\lambda_i(\mathcal{E}) = \inf\left\{\sup\left\{R(\psi) : \psi \in G^\star\right\} : G <_i \left(\operatorname{dom}(\mathcal{E}), \langle\cdot,\cdot\rangle_{\mathcal{E}}\right)\right\},$$

where we write $G <_i (H, \langle\cdot,\cdot\rangle)$ if G is a linear subspace of the Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and the vector space dimension of G is equal to $i \in \mathbb{N}$; for the *Rayleigh–Ritz quotient* given by $R(\psi) \coloneqq \mathcal{E}(\psi,\psi)/\langle\psi,\psi\rangle_{\mathcal{H}}$.

The following proposition will be crucial for the proof of the upper bound of the spectral dimension as stated in Corollary 5.6.

Proposition 2.17. *We have for all* $i \in \mathbb{N}$,

$$\begin{aligned}\lambda_{i,\nu}^D &= \inf\left\{\sup\left\{R_{H_0^1(\Omega)}(\psi) : \psi \in G^\star\right\} : G <_i \left(\mathcal{N}_{0,\nu}^\perp, \langle\cdot,\cdot\rangle_{H_0^1(\Omega)}\right)\right\}\\ &= \inf\left\{\sup\left\{R_{H_0^1(\Omega)}(\psi) : \psi \in G^\star\right\} : G <_i \left(H_0^1(\Omega), \langle\cdot,\cdot\rangle_{H_0^1(\Omega)}\right)\right\},\end{aligned}$$

where the relevant Rayleigh–Ritz quotient is given by

$$R_{H_0^1(\Omega)}(\psi) \coloneqq \langle\psi,\psi\rangle_{H_0^1(\Omega)} / \langle\iota\psi,\iota\psi\rangle_\nu.$$

The same result holds true for $\mathcal{E}^N$ *with* $\mathcal{N}_{0,\nu}^\perp$ *replaced by* $\mathcal{N}_\nu^\perp$ *and* $H_0^1(\Omega)$ *by* $H^1(\Omega)$.

Proof. The first equality follows by the min-max principle and the fact that

$$\operatorname{dom}\left(\mathcal{E}^D\right) \simeq \mathcal{N}_{0,\nu}^{\perp}.$$

The part '≥' for the second equality follows from the inclusion $\mathcal{N}_{0,\nu}^{\perp} \subset H_0^1(\Omega)$. For the reverse inequality we consider an i-dimensional subspace

$$G = \operatorname{span}(f_1, \ldots, f_i) \subset H_0^1(\Omega).$$

There exists a unique decomposition $f_j = f_{1,j} + f_{2,j}$ with $f_{1,j} \in \mathcal{N}_{0,\nu}^{\perp}$ and $f_{2,j} \in \mathcal{N}_{0,\nu}$, $j = 1, \ldots, i$. Suppose that $\left(f_{j,1}\right)_{j=1,\ldots,i}$ are not linearly independent, then there exists a non-zero element $g \in G \cap \mathcal{N}_{0,\nu}$. To see this, we fix $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$ with $\lambda_1 f_{1,1} + \cdots + \lambda_i f_{1,i} = 0$. Then

$$\underbrace{\lambda_1\left(f_{1,1}+f_{2,1}\right)+\cdots+\lambda_i\left(f_{1,i}+f_{2,i}\right)}_{\in G^{\star}} = \underbrace{\lambda_1 f_{2,1}+\cdots+\lambda_i f_{2,i}}_{\in \mathcal{N}_{0,\nu}} =: g.$$

Using $\mathcal{E}^D(g,g) > 0$, we get in this case

$$\sup\left\{R_{H_0^1(\Omega)}(\psi) : \psi \in G^{\star}\right\} = \infty.$$

Otherwise, using the assumption $f_{1,j} \in \mathcal{N}_{0,\nu}^{\perp}$ and $f_{2,j} \in \mathcal{N}_{0,\nu}$ and particularly $\iota(f_{2,j}) = 0$, we have for every vector $\left(a_j\right)_j \in \mathbb{R}^i \setminus \{0\}$

$$\begin{aligned} R_{H_0^1(\Omega)}\left(\sum_j a_j f_{1,j} + \sum_j a_j f_{2,j}\right) &= \frac{\left\langle \sum_j a_j f_{1,j}, \sum_j a_j f_{1,j}\right\rangle_{H_0^1(\Omega)} + \left\langle \sum_j a_j f_{2,j}, \sum_j a_j f_{2,j}\right\rangle_{H_0^1(\Omega)}}{\left\langle \iota\left(\sum_j a_j f_{1,j}\right), \iota\left(\sum_j a_j f_{1,j}\right)\right\rangle_\nu} \\ &\geq R_{H_0^1(\Omega)}\left(\sum_j a_j f_{1,j}\right). \end{aligned}$$

Note that $\operatorname{span}(f_{1,1}, \ldots, f_{1,i}) \subset \mathcal{N}_{0,\nu}^{\perp}$ is also an i-dimensional subspace in $H_0^1(\Omega)$. Hence, in any case the reverse inequality follows. □

Lemma 2.18. *There exists $C > 0$ such that we have for all $i \in \mathbb{N}$*

$$\lambda_{i,\nu}^N \leq C \lambda_{i,\nu}^D.$$

Proof. Using (PI) and $c > 0$ as defined therein, we obtain for all $u \in H_0^1(\Omega)$ that

$$\langle u, u\rangle_{H^1(\Omega)} \leq (c+1)\langle u, u\rangle_{H_0^1(\Omega)}.$$

Since $H_0^1(\Omega) \subset H^1(\Omega)$, the claim follows from Proposition 2.17 with $C := c+1$. □

The leading idea to obtain lower bounds on $N_\nu^{D/N}$ is to construct appropriate finite dimensional subspaces of $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively. This will be subject of the following lemma.

Lemma 2.19. *Let $\{f_1,\dots,f_n\} \subset H_0^1(\Omega)^\star$ such that $\{f_1,\dots,f_n\}$ is orthogonal in $H_0^1(\Omega)$ and $\{\iota(f_1),\dots,\iota(f_n)\}$ is orthogonal in $L_\nu^2(\Omega)$. Further, assume there exists $C>0$ such that for all $i=1,\dots,n$, we have*

$$\frac{\langle f_i, f_i\rangle_{H_0^1(\Omega)}}{\langle \iota(f_i), \iota(f_i)\rangle_\nu} \le C.$$

Then, $N_\nu^D(C) \ge n$. The same result holds if we replace $H_0^1(\Omega)$ and $L_\nu^2(\Omega)$ by $H^1(\Omega)$ and $L_\nu^2(\overline{\Omega})$.

Proof. For every $(c_1,\dots,c_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n c_i f_i \in H_0^1(\Omega)^\star$, we have

$$\frac{\left\langle \sum_{i=1}^n c_i f_i, \sum_{i=1}^n c_i f_i\right\rangle_{H_0^1(\Omega)}}{\left\langle \sum_{i=1}^n c_i \iota(f_i), \sum_{i=1}^n c_i \iota(f_i)\right\rangle_\nu} = \frac{\sum_{i=1}^n c_i^2 \langle f_i, f_i\rangle_{H_0^1(\Omega)}}{\sum_{i=1}^n c_i^2 \langle \iota(f_i), \iota(f_i)\rangle_\nu} \le C \frac{\sum_{i=1}^n c_i^2 \langle \iota(f_i), \iota(f_i)\rangle_\nu}{\sum_{i=1}^n c_i^2 \langle \iota(f_i), \iota(f_i)\rangle_\nu} = C$$

Thus, by Proposition 2.17, we have $\lambda_{n,\nu}^{D/N} \le C$. □

2.2.6 Smoothing methods

To obtain lower estimates of the lower and upper spectral dimension, it is crucial to construct appropriate finitely dimensional subspaces of $C_c^\infty(\mathbb{R}^d)$ (see Proposition 5.9). In this section, we address this demand via mollifiers.

In the following, we assume that each cube has edges parallel to the coordinate axes. For $s>0$ let $\langle Q\rangle_s$ denote the cube centered and parallel with respect to the cube $Q \subset \mathbb{R}^d$ such that $\Lambda\left(\langle Q\rangle_s\right) = \Lambda(Q)s^d$, $s>0$ (i.e. $\langle Q\rangle_s = T(Q) + (1-s)x_0$ where $T(x) = sx, x \in \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ is the center of Q). Note that we have

$$\left\langle \langle Q\rangle_{1/s}\right\rangle_s = \left\langle s^{-1}Q + (1-s^{-1})x_0\right\rangle_s = Q$$

and if $Q = \prod_{i=1}^d (a_i, b_i]$, then

$$\langle Q\rangle_s = \prod_{i=1}^d \left(-s\frac{b_i - a_i}{2} + \frac{a_i + b_i}{2}, \frac{a_i+b_i}{2} + s\frac{b_i - a_i}{2}\right].$$

Lemma 2.20. *For $m>1$ and $r>0$, let $Q \subset \mathbb{R}^d$ be a cube with side length mr and $Q' := \langle Q\rangle_{1/m}$ the centered and parallel sub-cube with side length r. Then there exists $\varphi_{Q,m} \in C_c^\infty(\mathbb{R}^d)$ which satisfies the following properties:*

1. $0 \leq \varphi_{Q,m}(x) \leq 1$ *for all* $x \in \mathbb{R}^d$,
2. $\mathrm{supp}(\varphi_{Q,m}) \subset \mathring{Q}$,
3. $\varphi_{Q,m}(x) = 1$ *for all* $x \in Q'$,
4. *there exists a constant* $C > 0$ *(depending on d) such that*

$$\left|(\partial/\partial x_i)\, \varphi_{Q,m}(x)\right| \leq C\,(m-1)^{-1}\, r^{-1}$$

for all $i = 1, \ldots, d$ *and* $x \in Q$.

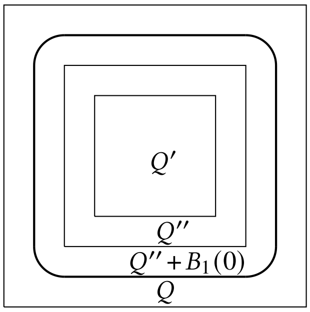

Figure 2.2.1 Illustration of the construction of $\varphi_{Q,m}$ for the case $d = 2$, $m = 5/2$ and $r = 4$.

Proof. Let $\psi : x \mapsto \mathbb{1}_{B_1(0)}(x) c_2 \exp(1/(\|x\|^2 - 1))$ be the normalized *Friedrichs' mollifier* with $c_2 := 1/\int_{B_1(0)} \exp(1/(\|x\|^2 - 1))\, \mathrm{d}x$ and $\psi_\epsilon(x) := \psi(x/\epsilon)/\epsilon^d$ be the mollifier with radius of mollification $\epsilon := (m-1)\, r/6$. For the centered and parallel sub-cube $Q'' := \langle Q \rangle_{(m+2)/(3m)} \subset Q$ with side length $(m+2)\, r/3$ we define $\varphi_{Q,m}$ as the convolution

$$\varphi_{Q,m}(x) := \mathbb{1}_{Q''} \star \psi_\epsilon(x) := \int_{\mathbb{R}^d} \mathbb{1}_{Q''}(y) \psi_\epsilon(x-y)\, \mathrm{d}\Lambda(y),\ x \in \mathbb{R}^d.$$

Since we assume that each cube has edges parallel to the coordinate axes, we can write $Q = \prod_{i=1}^d I_i$ and $Q'' = \prod_{i=1}^d I_i'$ with $\mathring{I}_i = (a_i, b_i)$ and $\mathring{I}_i' = (c_i, d_i)$. Thus, we have

$$\overline{Q'' + B_\varepsilon(0)} \subset \prod_{i=1}^d [c_i - \varepsilon, d_i + \varepsilon].$$

Moreover, note that by the definition of Q'' (recall Q'' is centered with respect to

Q), we have $(a_i+b_i)/2=(c_i+d_i)/2$. Therefore, we find that

$$\begin{aligned}
[c_i-\varepsilon,d_i+\varepsilon] &= \left[\frac{c_i+d_i}{2}-\frac{d_i-c_i}{2}-\frac{(m-1)r}{6},\frac{c_i+d_i}{2}+\frac{d_i-c_i}{2}+\frac{(m-1)r}{6}\right] \\
&= \left[\frac{a_i+b_i}{2}-\frac{(m+2)r}{6}-\frac{(m-1)r}{6},\frac{a_i+b_i}{2}+\frac{(m+2)r}{6}+\frac{(m-1)r}{6}\right] \\
&= \left[\frac{a_i+b_i}{2}-\frac{(2m+1)r}{6},\frac{a_i+b_i}{2}+\frac{(2m+1)r}{6}\right] \\
&\subset \left(\frac{a_i+b_i}{2}-\frac{rm}{2},\frac{a_i+b_i}{2}+\frac{rm}{2}\right) \\
&= \left(\frac{a_i+b_i}{2}-\frac{b_i-a_i}{2},\frac{a_i+b_i}{2}+\frac{b_i-a_i}{2}\right) \\
&= (a_i,b_i).
\end{aligned}$$

It follows that $\operatorname{supp}\left(\varphi_{Q,m}\right)\subset\overline{Q''+B_\varepsilon(0)}\subset\mathring{Q}$. Since

$$\frac{r}{2}+\varepsilon=r\frac{m+2}{6},$$

it follows analogously as above that for each $x\in Q'$, we have

$$x-B_\varepsilon(0)\subset Q'+B_\varepsilon(0)\subset Q''.$$

Hence, for each $x\in Q'$,

$$\begin{aligned}
\varphi_{Q,m}(x) &= \int_{\mathbb{R}^d}\mathbb{1}_{Q''}(y)\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&= \int_{x-B_\varepsilon(0)}\mathbb{1}_{Q''}(y)\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&= \int_{x-B_\varepsilon(0)}\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&= \int_{B_\varepsilon(0)}\psi_\epsilon(y)\,\mathrm{d}\Lambda(y)=1.
\end{aligned}$$

Further, for $x \in Q'' + B_\varepsilon(0)$, we have

$$\begin{aligned}
\left|(\partial/\partial x_i)\,\varphi_{Q,m}(x)\right| &= \left|\int_{Q''} \mathbb{1}_{Q''}(y)\,(\partial/\partial x_i)\,\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y)\right| \\
&\le \int_{Q''} \mathbb{1}_{Q''}(y)\frac{2\,|x_i-y_i|}{\varepsilon^2\left(\left\|\frac{x-y}{\varepsilon}\right\|^2-1\right)^2}\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&\le \frac{1}{\varepsilon}\int_{Q''\cap B_\varepsilon(x)} \mathbb{1}_{Q''}(y)\frac{2}{\left(\left\|\frac{x-y}{\varepsilon}\right\|^2-1\right)^2}\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&\le \frac{1}{\varepsilon}\int_{B_\varepsilon(0)+x} \frac{2}{\left(\left\|\frac{x-y}{\varepsilon}\right\|^2-1\right)^2}\psi_\epsilon(x-y)\,\mathrm{d}\Lambda(y) \\
&= \frac{1}{\varepsilon}\int_{B_\varepsilon(0)} \frac{2}{\left(\left\|\frac{y}{\varepsilon}\right\|^2-1\right)^2}\psi_\epsilon(y)\,\mathrm{d}\Lambda(y) \\
&= \frac{1}{\varepsilon}\int_{B_1(0)} \frac{2}{\left(\|y\|^2-1\right)^2}\psi(y)\,\mathrm{d}\Lambda(y).
\end{aligned}$$

Observing $\int_{B_1(0)}\left(\|y\|^2-1\right)^{-2}\psi(y)\,\mathrm{d}\Lambda(y) < \infty$, the claim follows. □

Lemma 2.21. *Let Q be a cube with side length $mr > 0$, $m > 1$. Then there exists a constant $C > 0$ depending on $m > 1$ and d such that for $\varphi_{Q,m}$ as defined in Lemma 2.20 we have*

$$\frac{\int\left|\nabla\varphi_{Q,m}\right|^2\,\mathrm{d}\Lambda}{\int\left|\varphi_{Q,m}\right|^2\,\mathrm{d}\nu} \le C(m-1)^{-2}m^{1-2/d}\frac{\Lambda\left(\langle Q\rangle_{1/m}\right)^{1-2/d}}{\nu\left(\langle Q\rangle_{1/m}\right)}.$$

Proof. Using Lemma 2.20, we find

$$\begin{aligned}
\frac{\int\left|\nabla\varphi_{Q,m}\right|^2\,\mathrm{d}\Lambda}{\int\varphi_{Q,m}^2\,\mathrm{d}\nu} &\le dC^2\frac{\Lambda(Q)}{(m-1)^2r^2\nu\left(\langle Q\rangle_{1/m}\right)} \le \frac{dC^2}{(m-1)^2m^2}\frac{\Lambda(Q)^{1-2/d}}{\nu\left(\langle Q\rangle_{1/m}\right)} \\
&= dC^2(m-1)^{-2}m^{4-2/d}\frac{\Lambda\left(\langle Q\rangle_{1/m}\right)^{1-2/d}}{\nu\left(\langle Q\rangle_{1/m}\right)}.
\end{aligned}$$
□

The following proposition applies only in the case $d > 2$.

Proposition 2.22. *If $\dim_\infty(\nu) < d-2$ and $d > 2$, then the identity operator*

$$\left(C_b^\infty\left(\overline{\boldsymbol{Q}}\right),\langle\cdot,\cdot\rangle_{H^1}\right)\to L_\nu^2$$

is not continuous.

Proof. First note that

$$\dim_\infty(\nu) = \liminf_{n\to\infty} \frac{\max_{Q\in\mathcal{D}_n^N} \log(\nu(Q))}{-n\log(2)} < d-2$$

implies that there exists a sequence of cubes $(Q_n) \in \mathcal{D}^{\mathbb{N}}$ with strictly decreasing diameters such that $\nu(Q_n) \geq \Lambda(Q_n)^{a/d}$, $n \in \mathbb{N}$, for some $a \in (\dim_\infty(\nu), d-2)$. Now we have for $u_n := \Lambda(\langle Q_n\rangle_2)^{1/d-1/2} \varphi_{\langle Q_n\rangle_2,2}$ with $C > 0$ given in Lemma 2.20

$$\begin{aligned}
\|u_n\|_{H^1}^2 &= \Lambda(\langle Q_n\rangle_2)^{2/d-1}\left(\int_{\mathbf{Q}} \left|\nabla\varphi_{\langle Q_n\rangle_2,2}\right|^2 \,\mathrm{d}\Lambda + \int_{\mathbf{Q}} \left|\varphi_{\langle Q_n\rangle_2,2}\right|^2 \,\mathrm{d}\Lambda\right)\\
&\leq \Lambda(\langle Q_n\rangle_2)^{2/d-1}\left(\frac{2C\Lambda(\langle Q_n\rangle_2)}{\Lambda(Q_n)^{2/d}} + \int_{\mathbf{Q}} \left|\varphi_{\langle Q_n\rangle_2,2}\right|^2 \,\mathrm{d}\Lambda\right)\\
&\leq \Lambda(\langle Q_n\rangle_2)^{2/d-1}\left(4C\Lambda(\langle Q_n\rangle_2)^{-2/d+1} + \Lambda(\langle Q_n\rangle_2)\right) \leq 4(C+1).
\end{aligned}$$

On the other hand we have for n tending to infinity

$$\begin{aligned}
\|u_n\|_{L_\nu^2}^2 &\geq \Lambda(\langle Q_n\rangle_2)^{2/d-1}\,\nu(Q_n) = 2^{2-d}\Lambda(Q_n)^{2/d-1}\,\nu(Q_n)\\
&\geq 2^{2-d}\Lambda(Q_n)^{(a+2-d)/d} \to \infty.
\end{aligned}$$

This proves the claim. □

2.3 Partition functions and L^q-spectra

In this chapter, we investigate the new notion of partition functions with respect to a non-negative monotone function $\mathfrak{J}$ defined on the set of dyadic cubes $\mathcal{D}$. Further, we assume $\mathfrak{J}$ is *locally non-vanishing*, that is, if $\mathfrak{J}(Q) > 0$ for $Q \in \mathcal{D}$, then there exists $Q' \subsetneq Q$, $Q' \in \mathcal{D}$ with $\mathfrak{J}(Q') > 0$. Note that this assumption is satisfied for each specific choice for $\mathfrak{J}$ that we consider. Of particular interests is the case when $\mathfrak{J}$ is chosen to be $\mathfrak{J}(Q) := \nu(Q)$, $Q \in \mathcal{D}$, where ν is a finite Borel measure on $\mathbf{Q}$; in this case we obtain the well-known *L^q-spectrum of ν* . The L^q-spectrum has been studied by various authors, for example Ngai [Nga97], Ngai and Lau [LN98], Heurteaux [Heu07], Hochman [Hoc14], Shmerkin [Shm19], and Ngai and Xie [NX19].

2.3.1 The partition function

We start with recalling the definition of the partition function from the introduction. The Dirichlet/Neumann partition function of $\mathfrak{J}$ is given by

$$\tau_{\mathfrak{J}}^{D/N}(q) = \limsup_{n\to\infty} \tau_{\mathfrak{J},n}^{D/N}(q),\ \tau_{\mathfrak{J},n}^{D/N}(q) = \frac{\log\left(\sum_{C\in\mathcal{D}_n^{D/N}} \mathfrak{J}(C)^q\right)}{\log(2^n)}.$$

Here, we use the convention that $0^0 = 0$, that is for $q = 0$ we neglect the summands with $\mathfrak{J}(Q) = 0$. Further, recall from the introduction

$$q_{\mathfrak{J}}^{D/N} = \inf\left\{q \geq 0 : \tau_{\mathfrak{J}}^{D/N}(q) < 0\right\}$$

and

$$\kappa_{\mathfrak{J}} = \inf\left\{q \geq 0 : \sum_{Q\in\mathcal{D}} \mathfrak{J}(Q)^q < \infty\right\}.$$

Let us begin with some general observations for which we need the following objects:

$$\operatorname{supp}(\mathfrak{J}) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} \left\{\overline{Q} : Q \in \mathcal{D}_n^N, \mathfrak{J}(Q) > 0\right\}$$

and

$$\dim_\infty(\mathfrak{J}) := \liminf_{n\to\infty} \frac{\max_{Q\in\mathcal{D}_n^N} \log(\mathfrak{J}(Q))}{-\log(2^n)}.$$

We call $\dim_\infty(\mathfrak{J})$ the *∞-dimension of* $\mathfrak{J}$ which generalizes the ∞-dimension for ν. Obviously, the following holds.

Lemma 2.23. *If* $\dim_\infty(\mathfrak{J}) > 0$, *then* $\mathfrak{J}$ *is uniformly vanishing, i.e. we have*

$$\lim_{n\to\infty} \max_{C\in\mathcal{D}_n^N} \mathfrak{J}(C) = 0.$$

Remark 2.24. Note that due to the monotonicity of $\mathfrak{J}$ the assumption

$$\lim_{n\to\infty} \max_{C\in\mathcal{D}_n^N} \mathfrak{J}(C) = 0$$

is equivalent to $\lim_{n\to\infty} \sup_{C\in\bigcup_{k\geq n}\mathcal{D}_k^N} \mathfrak{J}(C) = 0$.

In the following lemma we use the convention $-\infty \cdot 0 = 0$.

Lemma 2.25. *For $q \geq 0$, we have*

$$-\dim_\infty(\mathfrak{J})q \leq \tau_{\mathfrak{J}}^N(q) \leq \overline{\dim}_M(\operatorname{supp}(\mathfrak{J})) - \dim_\infty(\mathfrak{J})q. \tag{2.3.1}$$

In particular,

$$q_{\mathfrak{J}}^N \leq \overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))/\dim_\infty(\mathfrak{J}).$$

Further, we have

$$\dim_\infty(\mathfrak{J}) > 0 \iff q_{\mathfrak{J}}^N < \infty.$$

and

$$q_{\mathfrak{J}}^N < \infty \implies \kappa_{\mathfrak{J}} = q_{\mathfrak{J}}^N.$$

Proof. The first claim follows from the following simple inequalities

$$\begin{aligned} q \cdot \max_{Q\in\mathcal{D}_n^N} \log\left(\mathfrak{J}(Q)\right) &\leq \log\left(\sum_{C\in\mathcal{D}_n^N} \mathfrak{J}(C)^q\right) \\ &\leq \log\left(\sum_{C\in\mathcal{D}_n^N,\, \mathfrak{J}(C)>0} 1\right) + q \max_{Q\in\mathcal{D}_n^N} \log(\mathfrak{J}(Q)). \end{aligned}$$

Now, assume $q_{\mathfrak{J}}^N < \infty$. It follows there exists $q > 0$ such that $\tau_{\mathfrak{J}}^N(q) < 0$. Consequently, from (2.3.1) we obtain $-\dim_\infty(\mathfrak{J})q \leq \tau_{\mathfrak{J}}^N(q) < 0$, which yields $\dim_\infty(\mathfrak{J}) > 0$. Reversely, suppose $\dim_\infty(\mathfrak{J}) > 0$. In the case $\dim_\infty(\mathfrak{J}) = \infty$, using (2.3.1), we have $q_{\mathfrak{J}}^N = 0$ due to $\tau_{\mathfrak{J}}^N(q) = -\infty$ for $q > 0$. It is left to consider the case $0 < \dim_\infty(\mathfrak{J}) < \infty$. Then it follows from (2.3.1) that

$$\tau_{\mathfrak{J}}^N(q) < 0 \text{ for all } q > \frac{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))}{\dim_\infty(\mathfrak{J})},$$

proving the implication. Now, assume $q_{\mathfrak{J}}^N < \infty$. Thus, we have $\tau_{\mathfrak{J}}^N(q) < 0$ for all $q > q_{\mathfrak{J}}^N$, and therefore, for every $\varepsilon > 0$ with $\tau_{\mathfrak{J}}^N(q) < -\varepsilon < 0$ and n large enough, we obtain

$$\sum_{Q\in\mathcal{D}_n^N} \mathfrak{J}(Q)^q \leq 2^{-n\varepsilon},$$

implying $\sum_{Q\in\mathcal{D}} \mathfrak{J}(Q)^q < \infty$. This shows $\inf\left\{q \geq 0 : \sum_{Q\in\mathcal{D}} \mathfrak{J}(Q)^q < \infty\right\} \leq q_{\mathfrak{J}}^N$. For the reversed inequality we note that if $q_{\mathfrak{J}}^N = 0$, then the claimed equality is clear.

If, on the other hand, $q_{\mathfrak{J}}^N > 0$, then we necessarily have $\dim_\infty(\mathfrak{J}) < \infty$. Therefore, $\tau_{\mathfrak{J}}^N$ is decreasing, convex (and therefore continuous), and proper (i.e. $\tau_{\mathfrak{J}}^N(q) > -\infty$ for all $q \geq 0$). Hence, it follows that $q_{\mathfrak{J}}^N$ is a zero of $\tau_{\mathfrak{J}}^N$ and for all $0 < q < q_{\mathfrak{J}}^N$

$$0 < \tau_{\mathfrak{J}}^N(q).$$

This implies that for every $0 < \delta < \tau_{\mathfrak{J}}^N(q)$, there is a subsequence $(n_k)_k$ such that

$$2^{n_k \delta} \leq \sum_{Q \in \mathcal{D}_{n_k}^N} \mathfrak{J}(Q)^q$$

and therefore,

$$\infty = \sum_{k \in \mathbb{N}} \sum_{Q \in \mathcal{D}_{n_k}^N} \mathfrak{J}(Q)^q \leq \sum_{Q \in \mathcal{D}} \mathfrak{J}(Q)^q$$

and consequently $q_{\mathfrak{J}}^N \leq \inf\left\{q \geq 0 : \sum_{Q \in \mathcal{D}} \mathfrak{J}(Q)^q < \infty\right\}$. □

Remark 2.26. Note that in the case $\dim_\infty(\mathfrak{J}) \leq 0$, from Lemma 2.25 we deduce that $\tau_{\mathfrak{J}}^N(q)$ is non-negative for $q \geq 0$, hence $q_{\mathfrak{J}}^N = \infty$. However, it is possible that $\kappa_{\mathfrak{J}} < \infty$. Indeed, consider

$$\mathfrak{J}(Q) := \begin{cases} \frac{1}{n}, & \text{if } Q = \left(0, \frac{1}{2^n}\right]^d, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows $\kappa_{\mathfrak{J}} = 1 < q_{\mathfrak{J}}^N = \infty$ and $\dim_\infty(\mathfrak{J}) = 0$.

Lemma 2.27. *If* $\dim_\infty(\mathfrak{J}) \in (0, \infty)$*, then* $\tau_{\mathfrak{J}}^N$ *is convex and strictly decreasing on* $\mathbb{R}_{\geq 0}$*. In particular, if* $q_{\mathfrak{J}}^N > 0$*, then* $q_{\mathfrak{J}}^N$ *is the only zero of* $\tau_{\mathfrak{J}}^N$*.*

Proof. First, note that Lemma 2.25 implies $\tau_{\mathfrak{J}}^N(q) \in \mathbb{R}$ for all $q \geq 0$ and

$$\lim_{q \to \infty} \tau_{\mathfrak{J}}^N(q) = -\infty.$$

Since $\dim_\infty(\mathfrak{J}) > 0$ it follows from Lemma 2.23 that for n large we have $\mathfrak{J}(Q) < 1$, $Q \in \mathcal{D}_n^N$. Hence, it follows that $\tau_{\mathfrak{J}}^N$ is decreasing and as pointwise limit superior of convex functions again convex.
Now, we show that $\tau_{\mathfrak{J}}^N$ is strictly decreasing. Assume there exist q_1, q_2 with $0 < q_1 < q_2$ such that $\tau_{\mathfrak{J}}^N(q_1) = \tau_{\mathfrak{J}}^N(q_2)$. Since $\tau_{\mathfrak{J}}^N$ is decreasing, we obtain $\tau_{\mathfrak{J}}^N(q_1) = \tau_{\mathfrak{J}}^N(q)$ for all $q \in [q_1, q_2]$. Fix $q'' \in (q_1, q_2)$. Since $\tau_{\mathfrak{J}}^N$ is convex, for all $q' > q''$ Theorem A.5 implies

$$0 = \frac{\tau_{\mathfrak{J}}^N(q'') - \tau_{\mathfrak{J}}^N(q_1)}{q'' - q_1} \leq \frac{\tau_{\mathfrak{J}}^N(q') - \tau_{\mathfrak{J}}^N(q_1)}{q' - q_1} \leq 0,$$

which implies $\tau_{\mathfrak{J}}^{N}(q) = \tau_{\mathfrak{J}}^{N}(q_1)$ for all $q > q_1$ which contradicts $\lim_{q\to\infty} \tau_{\mathfrak{J}}^{N}(q) = -\infty$. For the second claim note that, since $\tau_{\mathfrak{J}}^{N}$ is convex and finite on $\mathbb{R}> 0$, it follows that $\tau_{\mathfrak{J}}^{N}$ is continuous on $\mathbb{R}_{>0}$. Hence, we obtain $\tau_{\mathfrak{J}}^{N}(q_{\mathfrak{J}}^{N}) = 0$. Finally, the uniqueness follows from the fact that $\tau_{\mathfrak{J}}^{N}$ is a finite strictly decreasing function. □

We now summarise the above and mention a few more basic characteristics.

Fact 2.28. We make the following elementary observations under the assumption $\dim_\infty(\mathfrak{J}) \in (0,\infty)$:

1. $\lim_{q\to\infty} \tau_{\mathfrak{J}}^{N}(q)/q = -\dim_\infty(\mathfrak{J})$.
2. $\tau_{\mathfrak{J}}^{N}(q) > -\infty$ for all $q \geq 0$.
3. $\tau_{\mathfrak{J}}^{N}(0) = \overline{\dim}_M(\operatorname{supp}(\mathfrak{J})) \leq d$.
4. If $\tau_{\mathfrak{J}}^{N}(1) \geq 0$ and $q_{\mathfrak{J}}^{N} > 1$ hold, then
$$q_{\mathfrak{J}}^{N} \leq \frac{\dim_\infty(\mathfrak{J}) + \tau_{\mathfrak{J}}^{N}(1)}{\dim_\infty(\mathfrak{J})}.$$
5. If $q_{\mathfrak{J}}^{N} > 1$, then
$$\frac{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))}{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J})) - \tau_{\mathfrak{J}}^{N}(1)} \leq q_{\mathfrak{J}}^{N}.$$
6. If $q_{\mathfrak{J}}^{N} < 1$, then
$$q_{\mathfrak{J}}^{N} \leq \frac{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))}{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J})) - \tau_{\mathfrak{J}}^{N}(1)}.$$
7. If $\operatorname{supp}(\mathfrak{J}) \subset \mathring{\mathbf{Q}}$, then we have $\tau_{\mathfrak{J}}^{D}(q) = \tau_{\mathfrak{J}}^{N}(q)$.
8. The partition function is scale invariant, i.e. for $c > 0$, we have $\tau_{c\mathfrak{J}}^{D/N} = \tau_{\mathfrak{J}}^{D/N}$.

Proof. We only give a proof of the assertion in 3, namely
$$\tau_{\mathfrak{J}}^{N}(0) = \overline{\dim}_M(\operatorname{supp}(\mathfrak{J})).$$

First, we observe that if $Q \in \mathcal{D}_n^N$, $Q \cap \operatorname{supp}(\mathfrak{J}) \neq \varnothing$, then there exists $Q' \in \mathcal{D}_n^N$ with $\overline{Q'} \cap \overline{Q} \neq \varnothing$ and $\mathfrak{J}(Q') > 0$. This can be seen as follows: For $x \in Q \cap \operatorname{supp}(\mathfrak{J})$ there exists a subsequence $(n_k)_k$ such that $x \in \overline{Q}_{n_k}$, $Q_{n_k} \in \mathcal{D}_{n_k}$ and $\mathfrak{J}(Q_{n_k}) > 0$. For $k \in \mathbb{N}$ such that $n_k \geq n$ there exists exactly one $Q' \in \mathcal{D}_n^N$ with $Q_{n_k} \subset Q'$. Further, we have $x \in \overline{Q}_{n_k} \subset \overline{Q'}$ implies $\overline{Q'} \cap \overline{Q} \neq \varnothing$ and since $\mathfrak{J}$ is monotone, we have $\mathfrak{J}(Q') > 0$.

Furthermore, for each $Q \in \mathcal{D}_n^N$, we have $\operatorname{card}\left(\left\{Q'' \in \mathcal{D}_n^N : \overline{Q''} \cap \overline{Q} \neq \varnothing\right\}\right) \leq 3^d$. Combining these two observations, we obtain

$$\begin{aligned}
&\operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : Q \cap \operatorname{supp}(\mathfrak{J}) \neq \varnothing\right\}\right) \\
&\leq \operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : \exists Q' \in \mathcal{D}_n^N, \overline{Q'} \cap \overline{Q} \neq \varnothing, \mathfrak{J}(Q') > 0\right\}\right) \\
&\leq 3^d \operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : \mathfrak{J}(Q) > 0\right\}\right),
\end{aligned}$$

implying $\tau_{\mathfrak{J}}^N(0) \geq \overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))$.

For the reversed inequality, we first show that for $Q \in \mathcal{D}_n^N$ with $\mathfrak{J}(Q) > 0$, we have $\overline{Q} \cap \operatorname{supp}(\mathfrak{J}) \neq \varnothing$. Indeed, since $\mathfrak{J}$ is locally non-vanishing there exists a subsequence $(n_k)_k$ with $Q_{n_k} \in \mathcal{D}_{n_k}^N$, $\mathfrak{J}(Q_{n_k}) > 0$ and $Q_{n_k} \subset Q_{n_{k-1}} \subset Q$. Since $\left(\overline{Q}_{n_k}\right)_k$ is a nested sequence of non-empty compact subsets of $\overline{Q}$, we have

$$\varnothing \neq \bigcap_{k \in \mathbb{N}} \overline{Q}_{n_k} \subset \operatorname{supp}(\mathfrak{J}) \cap \overline{Q}.$$

Therefore, we complete the proof by observing

$$\begin{aligned}
\operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : \mathfrak{J}(Q) > 0\right\}\right) &\leq \operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : \overline{Q} \cap \operatorname{supp}(\mathfrak{J}) \neq \varnothing\right\}\right) \\
&\leq 3^d \operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : Q \cap \operatorname{supp}(\mathfrak{J}) \neq \varnothing\right\}\right). \qquad \square
\end{aligned}$$

2.3.2 The (Dirichlet/Neumann) L^q-spectrum

In this section, we collect some important facts about the Dirichlet/Neumann L^q-spectrum. The *Dirichlet/Neumann* L^q-spectrum of ν is given by

$$\beta_\nu^{D/N}(q) \coloneqq \tau_\nu^{D/N}(q) = \limsup_{n \to \infty} \beta_n^{D/N}(q)$$

with

$$\beta_n^{D/N}(q) \coloneqq \beta_{\nu,n}^{D/N}(q) \coloneqq \frac{1}{\log(2^n)} \log\left(\sum_{Q \in \mathcal{D}_n^{D/N}} \nu(Q)^q\right), \; q \in \mathbb{R}.$$

The Neumann L^q-spectrum we also simply call the *L^q-spectrum of* ν. In the Dirichlet case, we will assume $\nu(\mathring{\mathbf{Q}}) > 0$, implying that there exists a sub-cube $Q \in \mathcal{D}$ with $\overline{Q} \subset \mathring{\mathbf{Q}}$, $\nu(Q) > 0$ and hence $-\infty < \beta_{\nu|_Q}^N \leq \beta^D$. In the following, we list some standard facts about the L^q-spectrum.

Fact 2.29. We make the following elementary observations:

1. $\beta_\nu^N(0) = \overline{\dim}_M(\mathrm{supp}(\nu))$, where $\overline{\dim}_M(A)$ denotes the upper Minkowski dimension of $A \subset \mathbb{R}^d$.

2. $\dim_\infty(\nu) \leq d$.

3. $\beta_\nu^N(1) = 0$ and if $\nu(\mathring{\mathbf{Q}}) > 0$, then also $\beta_\nu^D(1) = 0$.

4. For the Dirichlet L^q-spectrum we have $\beta_\nu^D = \beta_{\nu|_{\mathring{\mathbf{Q}}}}^D$.

5. For all $q \geq 0$, we have $-qd \leq \beta_\nu^N(q)$.

6. If $\mathrm{supp}(\nu) \subset \mathring{\mathbf{Q}}$, then we have $\beta_\nu^D = \beta_\nu^N$.

7. If ν is absolutely continuous with density $h \in L_\Lambda^t(\mathbf{Q})$ for some $t > d/2$, then $\beta_\nu^D(q) = \beta_\nu^N(q)$, for all $q \in [0,t]$.

8. The condition $\dim_\infty(\nu) > d-2$ implies that the upper Minkowski dimension $\overline{\dim}_M(\mathrm{supp}(\nu))$ and the Hausdorff dimension $\dim_H(\nu)$ must also lie in $(d-2,d]$. This in particular rules out the possibility of atomic parts of ν if $d \geq 2$.

Fact 2.30. The function β_ν^N will not alter when we take δ-adic cubes instead of dyadic ones (see, e.g. [Rie95, Proposition 2 and Remarks, p. 466] or [Rie93, Proposition 1.6] and note that the definition in [Rie93, Proposition 1.6] coincides with our definition for $q \geq 0$). More precisely, for fixed $\delta > 0$, set

$$G_{\nu,\delta} := G_\delta := \left\{ \prod_{i=1}^d ((k_i-1)\delta, k_i\delta] : k_i \in \mathbb{Z}, \nu\left(\prod_{i=1}^d ((k_i-1)\delta, k_i\delta]\right) > 0 \right\}$$

and let $(\delta_n)_n$ be an *admissible sequence*, i.e. $\delta_n \in (0,1)^{\mathbb{N}}$, $\delta_n \longrightarrow 0$ and there exists a constant $C > 0$ such that $C\delta_n \leq \delta_{n+1} \leq \delta_n$ for all $n \in \mathbb{N}$. Then for $q \geq 0$ we have

$$\limsup_{\delta \downarrow 0} \frac{1}{-\log(\delta)} \log\left(\sum_{C \in G_\delta} \nu(C)^q\right) = \limsup_{m \to \infty} \frac{1}{-\log(\delta_m)} \log\left(\sum_{C \in G_{\delta_m}} \nu(C)^q\right).$$

In particular, for $\delta_m := 2^{-m}$, the above expression coincides with the definition of $\beta_\nu^N(q)$.

Fact 2.31. The function $\beta_\nu^{D/N}$ as a pointwise limit superior of convex functions is again convex and we have (see also [Rie95, Corollary 11])

$$\beta_\nu^N(0) = \overline{\dim}_M(\mathrm{supp}(\nu)) \leq d \quad \text{and} \quad \beta_\nu^{D/N}(1) = 0.$$

The function β_ν^N is non-increasing and non-negative on $\mathbb{R}_{<1}$ and

$$\liminf_{n \to \infty} \beta_n^N(q) \geq -dq$$

for all $q \geq 0$.

Fact 2.32. If ν has an atomic part, then $\beta_\nu^N(q) = 0$, for all $q \geq 1$. If $\nu|_{\mathring{\mathbf{Q}}}$ has an atomic part, then also $\beta_\nu^D(q) = 0$ for all $q \geq 1$.

Proof. We consider only the first case. Assume that ν has an atom in $x_0 \in \overline{\mathbf{Q}}$ and let $q > 1$. Then, for every $n \in \mathbb{N}$, we have $0 < \nu(\{x_0\})^q \leq \sum_{C \in \mathcal{D}_n^N} \nu(C)^q$, implying $0 \leq \beta_\nu^N(q) \leq \beta_\nu^N(1) = 0$. □

2.4 The spectral partition function

In this section, we study the spectral partition function which will play an important role in the study of the spectral dimension as well as for the quantization dimension.

2.4.1 The spectral partition function and connections to the L^q-spectrum

This section is devoted to the special case $\mathfrak{J} = \mathfrak{J}_{\nu,a,b}$, where for $b \geq 0$, $a \in \mathbb{R}$ and $Q \in \mathcal{D}$,

$$\mathfrak{J}_{\nu,a,b}(Q) = \begin{cases} \sup_{\widetilde{Q} \in \mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)^b \left|\log\left(\Lambda\left(\widetilde{Q}\right)\right)\right|, & a = 0, \\ \sup_{\widetilde{Q} \in \mathcal{D}(Q)} \nu\left(\widetilde{Q}\right)^b \left(\Lambda\left(\widetilde{Q}\right)\right)^a, & a \neq 0. \end{cases}$$

Recall that $\tau_{\mathfrak{J}_\nu}^{D/N} = \tau_{\mathfrak{J}_{\nu,(2/d-1),1}}^{D/N}$. We call $\tau_{\mathfrak{J}_{\nu,a,b}}^{D/N}$ the *(Dirichlet/Neumann) spectral partition function* of ν with parameters a, b. For the Dirichlet case we always assume $\nu(\mathring{\mathbf{Q}}) > 0$.

We now elaborate some connections between the L^q-spectrum and the spectral partition function.

Proposition 2.33. *Fix $a \in \mathbb{R}$, $a > 0$ with $b \dim_\infty(\nu) + ad > 0$.*

1. *If $a > 0$, then $\beta_\nu^{D/N}(bq) - adq = \tau_{\mathfrak{J}_{\nu,a,b}}^{D/N}(q)$ for $q \geq 0$.*
2. *If $a < 0$, then $\beta_\nu^{D/N}(bq) - adq \leq \tau_{\mathfrak{J}_{\nu,a,b}}^{D/N}(q) \leq \beta_\nu^{D/N}(q(b + ad/\dim_\infty(\nu)))$ for $q \geq 0$, and in particular, $\tau_{\mathfrak{J}_{\nu,a,b}}^{D/N}(0) = \beta_\nu^{D/N}(0)$.*

Proof. We only consider the case $a < 0$. We have for every $-ad/b < s < \dim_\infty(\nu)$ and n large enough

$$\nu(C) \leq 2^{-sn},$$

with $C \in \mathcal{D}_n^N$. This leads to $n \leq -\log_2(\nu(C))/s$. Hence, for $q \geq 0$, we obtain

$$\nu(C)^{bq}\Lambda(C)^{qa} = \nu(C)2^{-adqn} \leq \nu(C)^{bq}2^{adq\log_2(\nu(C))/s} = \nu(C)^{q(b+ad/s)}.$$

We get $\nu(C)^{bq}\Lambda(C)^{qa} \le \mathfrak{J}_{\nu,a,b}(C)^q \le \nu(C)^{q(b+ad/s)}$ and

$$\tau^{D/N}_{\mathfrak{J}_{\nu,a,b}}(q) \le \beta^{D/N}_\nu(q(b+ad/s)).$$

Finally, the continuity of $\beta^{D/N}_\nu$ gives

$$\tau^{D/N}_{\mathfrak{J}_{\nu,a,b}}(q) \le \beta^{D/N}_\nu\left(q\left(b+ad/\dim_\infty(\nu)\right)\right). \qquad \square$$

Corollary 2.34. *Let $a \neq 0$. Assume $b\dim_\infty(\nu)+ad > 0$ and β^N_ν is linear on $[0,\infty)$. Then, for all $q \ge 0$, we have*

$$\tau^N_{\mathfrak{J}_{\nu,a,b}}(q) = \beta^N_\nu(bq) - adq = \overline{\dim}_M(\nu) - q(b\overline{\dim}_M(\nu)+ad).$$

Proposition 2.35. *Assume $\dim_\infty(\nu) > 0$. Then for all $b > 0$ and $q \ge 0$, we have*

$$\beta^{D/N}_\nu(bq) = \tau^{D/N}_{\mathfrak{J}_{\nu,0,b}}(q).$$

Furthermore, if $\beta^{D/N}_\nu(bq)$ exists as a limit, then

$$\beta^{D/N}_\nu(bq) = \liminf_{n\to\infty} \tau^{D/N}_{\mathfrak{J}_{\nu,0,b},n}(q).$$

Proof. Let $q > 0$. For $\dim_\infty(\nu) > \varepsilon > 0$, we have

$$\nu(C) \le 2^{-\varepsilon n}$$

for n large enough and all $C \in \mathcal{D}^{D/N}_n$. Hence, for every $0 < \delta < b$ and n large, we obtain

$$|\log(\Lambda(C))|^q = (d\log(2)n)^q \le 2^{nq\delta\varepsilon} \le \nu(C)^{-q\delta}.$$

Recall that we neglect the summands with $\nu(C) = 0$. This leads to

$$\begin{aligned}
\log(2)d \sum_{C\in\mathcal{D}^{D/N}_n} \nu(C)^{bq} &\le \sum_{C\in\mathcal{D}^{D/N}_n} \mathfrak{J}_{\nu,0,b}(C)^q \\
&\le \sum_{C\in\mathcal{D}^{D/N}_n} \sup_{Q\in\mathcal{D}(C)} \nu(Q)^{-q\delta}\nu(Q)^{bq}. \\
&= \sum_{C\in\mathcal{D}^{D/N}_n} \nu(C)^{q(b-\delta)}.
\end{aligned}$$

Hence,

$$\beta^{D/N}_\nu(qb) \le \tau^{D/N}_{\mathfrak{J}_{\nu,0,b}}(q) \le \beta^{D/N}_\nu(q(b-\delta))$$

and for $\delta \searrow 0$, the continuity of $\beta^{D/N}_\nu$ gives $\beta^{D/N}_\nu(qb) = \tau^{D/N}_{\mathfrak{J}_{\nu,0,b}}(q)$. Under the assumption that $\beta_\nu^{D/N}$ exists as a limit, we infer

$$\beta_\nu^{D/N}(bq) \leq \liminf_{n\to\infty} \tau^{D/N}_{\mathfrak{J}_{\nu,0,b},n}(q) \leq \liminf_{n\to\infty} \beta_n^{D/N}(q(b-\delta)) \leq \beta_\nu^{D/N}(q(b-\delta)).$$

Now, for $\delta \searrow 0$, the continuity of $\beta_\nu^{D/N}$ proves the claim. □

Corollary 2.36. *If* $d=2$ *and* $\dim_\infty(\nu)>0$*, then* $\tau^N_{\mathfrak{J}_\nu}(1)=\beta^N_\nu(1)=0$*, or equivalently,* $q^N_{\mathfrak{J}_\nu}=1$*. If additionally* $\nu(\mathring{Q})>0$*, then* $\tau^D_{\mathfrak{J}_\nu}(1)=\beta^D_\nu(1)=0$*, or equivalently,* $q^D_{\mathfrak{J}_\nu}=1$.

By virtue of Proposition 2.33 and Proposition 2.35, we arrive at the following list of facts.

Fact 2.37. Assuming $b\dim_\infty(\nu)+ad>0$, then the following list of properties of the spectral partition function applies:

1. $\operatorname{supp}(\mathfrak{J}_{\nu,a,b})=\operatorname{supp}(\nu)$.
2. $\dim_\infty(\mathfrak{J}_{\nu,a,b})=b\dim_\infty(\nu)+ad>0$.
3. We have that $q^N_{\mathfrak{J}_{\nu,a,b}}$ is the unique zero of $\tau^N_{\mathfrak{J}_{\nu,a,b}}$ and if $a<0$, then by Proposition 2.33, we have
$$q^N_{\mathfrak{J}_{\nu,a,b}} \leq \frac{\dim_\infty(\nu)}{b\dim_\infty(\nu)+ad}.$$
If $a>0$, then
$$q^N_{\mathfrak{J}_{\nu,a,b}} \leq \frac{\overline{\dim}_M(\nu)}{b\overline{\dim}_M(\nu)+ad}.$$
4. We have $\dim_\infty(\nu)\leq\overline{\dim}_M(\operatorname{supp}(\nu))$.
5. If $d>1$, then
$$\frac{d}{2} \leq \frac{\overline{\dim}_M(\operatorname{supp}(\nu))}{\overline{\dim}_M(\operatorname{supp}(\nu))-d+2} \leq q^N_{\nu\Lambda^{(2/d-1)}} \leq q^N_{\mathfrak{J}_\nu}. \tag{5}$$
If additionally $\dim_\infty(\nu)=\overline{\dim}_M(\operatorname{supp}(\nu))$, then
$$q^N_{\nu\Lambda^{(2/d-1)}}=q^N_{\mathfrak{J}_\nu}=\frac{\overline{\dim}_M(\operatorname{supp}(\nu))}{\overline{\dim}_M(\operatorname{supp}(\nu))-d+2}.$$
6. If ν is absolutely continuous with density $h\in L^r_\Lambda(\mathbf{Q})$ for some $r>d/2$, then $\tau^D_{\mathfrak{J}_\nu}(q)=\tau^N_{\mathfrak{J}_\nu}(q)=\beta^N_\nu(q)+(d-2)q$, for all $q\in[0,r]$.
7. For the Dirichlet spectral partition function we have $\tau^D_{\mathfrak{J}_{\nu,a,b}}=\tau^D_{\mathfrak{J}_{\nu|_{\mathring{Q}},a,b}}$.

8. For $c>0$, we have $\tau^{D/N}_{\mathfrak{J}_{c\nu,a,b}}=\tau^{D/N}_{\mathfrak{J}_{\nu,a,b}}$ and without loss of generality we can assume that ν is a probability measure.

Proof. We only need to prove assertion (5). The convexity of β^N_ν implies for all $q>1$

$$d(1-q)+(d-2)q\le\overline{\dim}_M(\nu)(1-q)+(d-2)q\le\beta^N_\nu(q)+(2-d)q\le\tau^N_{\mathfrak{J}_\nu}(q),$$

implying the claim. □

2.4.2 Relations between the Dirichlet and Neumann spectral partition functions

In this section, we investigate under which conditions we can guarantee that for given $q\ge 0$, we have $\tau^D_{\mathfrak{J}_\nu}(q)=\tau^N_{\mathfrak{J}_\nu}(q)$. As auxiliary quantities we need

$$\dim^{N\setminus D}_\infty(\nu):=\liminf_{n\to\infty}\frac{\log\left(\max_{Q\in\mathcal{D}^N_n\setminus\mathcal{D}^D_n}\nu(Q)\right)}{-\log(2^n)}$$

and

$$\dim^D_\infty(\nu):=\liminf_{n\to\infty}\frac{\log\left(\max_{Q\in\mathcal{D}^D_n}\nu(Q)\right)}{-\log(2^n)}.$$

In the following, we assume $\dim_\infty(\nu)>d-2$.

Lemma 2.38. *For any $q\ge 0$ such that*

$$\overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\boldsymbol{Q})-q\left(\dim^{N\setminus D}_\infty(\nu)-d+2\right)<\tau^N_{\mathfrak{J}_\nu}(q),$$

we have

$$\tau^D_{\mathfrak{J}_\nu}(q)=\tau^N_{\mathfrak{J}_\nu}(q).$$

In particular, since $\dim_\infty(\nu)\le\dim^{N\setminus D}_\infty(\nu)$, *if*

$$\overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\boldsymbol{Q})-q(\dim_\infty(\nu)-d+2)<\tau^N_{\mathfrak{J}_\nu}(q),$$

then $\tau^D_{\mathfrak{J}_\nu}(q)=\tau^N_{\mathfrak{J}_\nu}(q)$.

Remark 2.39. Using $\overline{\dim}_M(\nu)/(\overline{\dim}_M(\nu)-d+2)\le q^N_{\mathfrak{J}_\nu}$, we find that

$$\overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\mathbf{Q})<\overline{\dim}_M(\nu)\frac{\dim_\infty(\nu)-d+2}{\overline{\dim}_M(\nu)-d+2}$$

implying $\tau^N_{\mathfrak{J}_\nu}\left(q^N_{\mathfrak{J}_\nu}\right)=\tau^D_{\mathfrak{J}_\nu}\left(q^N_{\mathfrak{J}_\nu}\right)=0$.

Proof. First, we consider the case $d > 2$. Notice that

$$\sum_{Q\in\mathcal{D}_n^D} \mathfrak{J}_\nu(Q)^q \le \sum_{Q\in\mathcal{D}_n^N} \mathfrak{J}_\nu(Q)^q = \sum_{Q\in\mathcal{D}_n^D} \mathfrak{J}_\nu(Q)^q + \sum_{Q\in\mathcal{D}_n^N\setminus\mathcal{D}_n^D} \mathfrak{J}_\nu(Q)^q .$$

Set $\tau^{N\setminus D}(q) := \limsup_{n\to\infty} 1/\log(2^n)\log(\sum_{Q\in\mathcal{D}_n^N\setminus\mathcal{D}_n^D} \mathfrak{J}_\nu(Q)^q)$. Then for $q \ge 0$

$$\tau_{\mathfrak{J}_\nu}^D(q) \le \tau_{\mathfrak{J}_\nu}^N(q) = \max\left\{\tau^{N\setminus D}(q), \tau_{\mathfrak{J}_\nu}^D(q)\right\}.$$

Further, we always have

$$\begin{aligned} 0 < \dim_\infty(\nu) - d + 2 \le A &:= \liminf_{n\to\infty} \frac{\log\left(\max_{Q\in\mathcal{D}_n^N\setminus\mathcal{D}_n^D} \mathfrak{J}_\nu(Q)\right)}{-n\log(2)} \\ &= \lim_{q\to\infty} \frac{\tau^{N\setminus D}(q)}{-q}. \end{aligned}$$

By the definition of $\mathfrak{J}_\nu$ we have

$$\dim_\infty^{N\setminus D}(\nu) - d + 2 \ge A$$

and $\dim_\infty^{N\setminus D}(\nu) - d - 2 \ge \dim_\infty(\nu) - d + 2 > 0$. Fix $d - 2 < s < \dim_\infty^{N\setminus D}(\nu)$, then we obtain for all n large and $Q \in \mathcal{D}_n^N \setminus \mathcal{D}_n^D$,

$$\nu(Q)\Lambda(Q)^{2/d-1} \le 2^{n(d-2-s)}.$$

Therefore, $A \ge s - d + 2$, which yields $A = \dim_\infty^{N\setminus D}(\nu) - d + 2$. By the definition of $\tau^{N\setminus D}$, we have

$$\tau^{N\setminus D}(q) \le \overline{\dim}_M(\operatorname{supp}(\nu) \cap \partial\mathbf{Q}) - qA.$$

Hence, by our assumption $\overline{\dim}_M(\operatorname{supp}(\nu) \cap \partial\mathbf{Q}) - qA < \tau_{\mathfrak{J}_\nu}^N(q)$, we obtain

$$\tau^{N\setminus D}(q) < \tau_{\mathfrak{J}_\nu}^N(q).$$

This gives

$$\tau^{N\setminus D}(q) < \tau_{\mathfrak{J}_\nu}^N(q) = \max\left\{\tau^{N\setminus D}(q), \tau_{\mathfrak{J}_\nu}^D(q)\right\} = \tau_{\mathfrak{J}_\nu}^D(q).$$

For the case $d \le 2$, notice that by Proposition 2.35, we have

$$\tau_{\mathfrak{J}_\nu}^{D/N}(q) = \beta_\nu^{D/N}(q) + (d-2)q$$

for $q \ge 0$. Hence, this case follows in a similar way. □

In the next section we will see that many examples, which have been investigated in the literature (see [NS95; Tri97; NX21]), fulfill $\tau_{\mathfrak{J}_\nu}^N = \tau_{\mathfrak{J}_\nu}^D$. It is worth pointing out that in the one-dimensional case, the situation becomes considerably simpler, which follows from the fact that the boundary only contains $\{0,1\}$. The rest of this section is devoted to the proof of $\beta_\nu^D = \beta_\nu^N$ for the case $d=1$ whenever $\nu(\{0,1\})=0$.

Proposition 2.40. *For $d=1$ and $\nu(\{0,1\})=0$,*

$$\dim_\infty^{N\setminus D}(\nu) \geq \dim_\infty(\nu) = \dim_\infty^D(\nu).$$

Proof. First we consider the case $\dim_\infty^D(\nu) > 0$. Then, for $\dim_\infty^D(\nu) > s > 0$ and n large, we obtain

$$\nu(Q) \leq 2^{-sn},\ Q \in \mathcal{D}_n^D.$$

Hence, using $\nu(\{0,1\})=0$, it follows

$$\begin{aligned}
\nu((0,2^{-n}]) &= \sum_{k=0}^\infty \left(\nu\left(\left(0,2^{-(n+k)}\right]\right) - \nu\left(\left(0,2^{-(n+k+1)}\right]\right)\right) \\
&= \sum_{k=0}^\infty \nu\left(\left(2^{-(n+k+1)},2^{-(n+k)}\right]\right) \\
&\leq \sum_{k=0}^\infty 2^{-s(n+k+1)} \leq 2^{-sn}\sum_{k=0}^\infty 2^{-sk}
\end{aligned}$$

and

$$\begin{aligned}
\nu\left(\left(\frac{2^n-1}{2^n},1\right]\right) &= \sum_{k=0}^\infty \nu\left(\left(\frac{2^{n+k}-1}{2^{n+k}},1\right]\right) - \nu\left(\left(\frac{2^{n+k+1}-1}{2^{n+k+1}},1\right]\right) \\
&= \sum_{k=0}^\infty \nu\left(\left(\frac{2^{n+k}-1}{2^{n+k}},\frac{2^{n+k+1}-1}{2^{n+k+1}}\right]\right) \\
&\leq 2^{-sn}\sum_{k=0}^\infty 2^{-sk}.
\end{aligned}$$

Hence, we obtain

$$\dim_\infty^{N\setminus D}(\nu) \geq \dim_\infty^D(\nu).$$

Now, we observe

$$\begin{aligned}\frac{\log\left(\max_{Q\in\mathcal{D}_n^N}\nu(Q)\right)}{-\log(2^n)} &= \frac{\max_{k\in\{D,N\setminus D\}}\log\left(\max_{Q\in\mathcal{D}_n^k}\nu(Q)\right)}{-\log(2^n)}\\ &= \min_{k\in\{D,N\setminus D\}}\frac{\log\left(\max_{Q\in\mathcal{D}_n^k}\nu(Q)\right)}{-\log(2^n)}\\ &= \min\left\{\frac{\log\left(\max_{Q\in\mathcal{D}_n^{N\setminus D}}\nu(Q)\right)}{-\log(2^n)},\frac{\log\left(\max_{Q\in\mathcal{D}_n^D}\nu(Q)\right)}{-\log(2^n)}\right\}.\end{aligned}$$

Thus, we obtain

$$\dim_\infty(\nu)\geq\min\left\{\dim_\infty^{N\setminus D}(\nu),\dim_\infty^D(\nu)\right\}=\dim_\infty^D(\nu)\geq\dim_\infty(\nu).$$

If $\dim_\infty^D(\nu)=0$, then clearly $\dim_\infty(\nu)=0$. Thus, in any cases, we obtain $\dim_\infty(\nu)=\dim_\infty^D(\nu)$. □

Proposition 2.41. *Let $d=1$ and $\nu(\{0,1\})=0$. Then, for all $q\geq 0$, we have*

$$\beta_\nu^D(q)=\beta_\nu^N(q).$$

Proof. If for some $q>0$ we have

$$-q\dim_\infty^{N\setminus D}(\nu)\leq -q\dim_\infty(\nu)<\beta_\nu^N(q),$$

then $\beta_\nu^D(q)=\beta_\nu^N(q)$ by Lemma 2.38. Hence, for all

$$q<\alpha:=\inf\left\{s>0:\beta_\nu^N(s)>-s\dim_\infty(\nu)\right\},$$

we have

$$\beta_\nu^D(q)=\beta_\nu^N(q).$$

Note that we always have $\beta_\nu^N(q)\geq 0$ for all $q\in[0,1]$, implying $\alpha>1$. If $\alpha=\infty$, then we are finished. Otherwise, the convexity of β_ν^N and $\beta_\nu^N(q)\geq -q\dim_\infty(\nu)$ force that $\beta_\nu^N(q)=-q\dim_\infty(\nu)$ for all $q\geq\alpha$. Indeed, if $\beta_\nu^N(q)=-q\dim_\infty(\nu)$, then by Theorem A.5 and (2.3.1), for all $q'\geq q$, we have

$$\begin{aligned}-q'\dim_\infty(\nu)\leq\beta_\nu^N(q')&=\beta_\nu^N(q)+(q'-q)\frac{\beta_\nu^N(q')-\beta_\nu^N(q)}{q'-q}\\ &\leq\beta_\nu^N(q)-(q'-q)\dim_\infty(\nu)=-q'\dim_\infty(\nu).\end{aligned}$$

Therefore, Proposition 2.40 yields

$$-q\dim_{\infty}^{D}(\nu) = -q\dim_{\infty}(\nu) \le \beta_{\nu}^{D}(q) \le \beta_{\nu}^{N}(q) = -q\dim_{\infty}(\nu). \qquad \square$$

Corollary 2.42. *Let $\nu_1,\ldots,\nu_d$ be non-zero Borel measures on $(0,1)$ and define $\nu := \nu_1 \otimes \ldots \otimes \nu_d$. For $q \ge 0$, we assume that $\beta_{\nu_i}^{D/N}(q)$ exists as limit for each $i = 1,\ldots,d-1$. Then,*

$$\beta_{\nu}^{D}(q) = \beta_{\nu}^{N}(q) = \sum_{i=1}^{d} \beta_{\nu_i}^{D/N}(q).$$

Proof. The second equality follows from the simple fact that, for all $q \ge 0$ and all $n \in \mathbb{N}$, we have

$$\sum_{Q\in\mathcal{D}_n^N} \nu(Q)^q = \sum_{Q\in\mathcal{D}_n^N} \prod_{i=1}^{d} \nu_i\left(\pi_i(Q)\right)^q = \prod_{i=1}^{d} \sum_{Q\in\mathcal{D}_n^N} \nu_i\left(\pi_i(Q)\right)^q,$$

where π_i denotes the projection in the i-th component. Further, set

$$\mathcal{D}_n^{1,D} := \{(k2^{-n},(k+1)2^{-n}] : k = 1, 2^n-2\}, n \in \mathbb{N}.$$

Then for the d-folded product $\mathcal{D}_n^D = \mathcal{D}_n^{1,D} \times \ldots \times \mathcal{D}_n^{1,D}$, we have

$$\sum_{Q\in\mathcal{D}_n^D} \nu(Q)^q = \sum_{(Q_1,\ldots,Q_d)\in\left(\mathcal{D}_n^D\right)^d} \prod_{i=1}^{d} \nu_i(Q_i)^q = \prod_{i=1}^{d} \sum_{Q_i\in\mathcal{D}_n^{D,1}} \nu_i\left(Q_i\right)^q.$$

Thus, the first equality follows from Proposition 2.41 and our assumption that the $\beta_{\nu_i}^{D/N}(q)$'s exist as limit for each $i = 1,\ldots,d-1$. $\square$

2.4.3 Examples

In this section, assuming $\dim_{\infty}(\nu) > d-2$, we show that for some particular cases (absolutely continuous measures, product measures, Ahlfors–David regular measures, and self-conformal measures) the spectral partition function is completely determined by the L^q-spectrum. Furthermore, for these classes of measures we investigate under which conditions the Dirichlet and the Neumann L^q-spectra coincide. Later, we will use these results to calculate the spectral dimension for these classes of measures.

2.4.3.1 Absolutely continuous measures

Lemma 2.43. *Let ν be a non-zero absolutely continuous measure with Lebesgue density $f \in L^r_\Lambda(\boldsymbol{Q})$ for some $r \geq 1$. Then, for all $q \in [0,r]$, we have*

$$\liminf_{n\to\infty} \beta^{D/N}_{\nu,n}(q) = \beta^{D/N}_{\nu}(q) = d(1-q).$$

Proof. First, we remark that, since $\nu(\partial\mathbf{Q}) = 0$, there exists an open set $O \subset \overline{\mathbf{Q}}$ with $\nu(O) > 0$. Moreover, we have $\beta^N_\nu(1) = 0$ and $\beta^N_\nu(0) \leq d$. Hence, the convexity of β^N_ν implies

$$\beta^N_\nu(q) \leq d(1-q) \text{ for all } q \in [0,1].$$

Furthermore, for n large, we have $\beta^D_{n,\nu|_O/\nu(O)}(1) = 0$ and $\beta^D_{n,\nu|_O/\nu(O)}(0) \leq d$. Consequently, for all $q \in (1,\infty)$, the convexity of $\beta^D_{n,\nu|_O/\nu(O)}$ and Theorem A.5 give

$$\frac{\beta^D_{n,\nu|_O/\nu(O)}(q) - \beta^D_{n,\nu|_O/\nu(O)}(1)}{q-1} \geq \frac{\beta^D_{n,\nu|_O/\nu(O)}(1) - \beta^D_{n,\nu|_O/\nu(O)}(0)}{1-0} \geq -d.$$

Hence,

$$\beta^D_{n,\nu|_O/\nu(O)}(q) \geq d(1-q).$$

This implies

$$d(1-q) \leq \liminf_{n\to\infty} \beta^D_{n,\nu|_O/\nu(O)}(q) = \liminf_{n\to\infty} \beta^D_{n,\nu|_O}(q) \leq \liminf_{n\to\infty} \beta^{D/N}_{n,\nu}(q).$$

Moreover, by Jensen's inequality, for all $q \in [0,1]$ and n large, we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}^{D/N}_n} \nu(Q)^q &= \sum_{Q\in\mathcal{D}^D_n} \left(\int_Q f \,\mathrm{d}\Lambda/\Lambda(Q)\right)^q \Lambda(Q)^q \\ &\geq \sum_{Q\in\mathcal{D}^{D/N}_n} \Lambda(Q)^{q-1} \int_Q f^q \,\mathrm{d}\Lambda \\ &\geq \Lambda(Q)^{q-1} \int_O f^q \,\mathrm{d}\Lambda,\end{aligned}$$

implying

$$\liminf_{n\to\infty} \beta^{D/N}_{\nu,n}(q) \geq d(1-q).$$

Further, Jensen's inequality, for all $q \in [1,r]$, yields

$$\begin{aligned}\sum_{Q\in\mathcal{D}_n^{D/N}} \nu(Q)^q &= \sum_{Q\in\mathcal{D}_n^{D/N}} \left(\int_Q f \,\mathrm{d}\Lambda/\Lambda(Q)\right)^q \Lambda(Q)^q \\ &\le \Lambda(Q)^{q-1} \sum_{Q\in\mathcal{D}_n^{D/N}} \int_Q f^q \,\mathrm{d}\Lambda \\ &\le \Lambda(Q)^{q-1} \int_{\mathbf{Q}} f^q \,\mathrm{d}\Lambda.\end{aligned}$$

Hence, we obtain

$$\limsup_{n\to\infty} \beta_{\nu,n}^{D/N}(q) \le d(1-q). \qquad \square$$

Proposition 2.44. *Let $d > 2$ and ν be a non-zero absolutely continuous measure with Lebesgue density $f \in L^r_\Lambda(\boldsymbol{Q})$ for some $r \ge d/2$. Then, for all $q \in [0,r]$,*

$$\liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^{D/N}(q) = \tau_{\mathfrak{J}_\nu}^{D/N}(q) = \beta_\nu^{D/N}(q) - (2-d)q = d - 2q.$$

Proof. By Jensen's Inequality, for $d/2 \le q \le r$ and $Q \in \mathcal{D}^{D/N}$, we have

$$\nu(Q)^q = \left(\int_Q f\Lambda(Q)^{-1} \,\mathrm{d}\Lambda\right)^q \Lambda(Q)^q \le \left(\int_Q f^q \,\mathrm{d}\Lambda\right)\Lambda(Q)^{q-1}.$$

This shows that $\nu(Q)^q \Lambda(Q)^{2q/d-q} \le \left(\int_Q f^q \,\mathrm{d}\Lambda\right)\Lambda(Q)^{2q/d-1}$, and since we have $0 \le 2q/d - 1$, we observe that the right-hand side is monotonic in Q. Therefore we get the following upper bound

$$\begin{aligned}\sum_{\widetilde{Q}\in\mathcal{D}_n^{D/N}} \left(\sup_{Q\in\mathcal{D}_n(\widetilde{Q})} \nu(Q)^q \Lambda(Q)^{2q/d-q}\right) &\le \sum_{\widetilde{Q}\in\mathcal{D}_n^{D/N}} \left(\int_{\widetilde{Q}} f^q \,\mathrm{d}\Lambda\right)\Lambda\left(\widetilde{Q}\right)^{2q/d-1} \\ &\le 2^{-n(2q-d)} \|f\|^q_{L^q_\Lambda(\mathbf{Q})}.\end{aligned}$$

Combining this with Lemma 2.43, we obtain

$$\begin{aligned}d - 2q &= \liminf_{n\to\infty} \beta_{\nu,n}^{D/N}(q) + (2-d)q \\ &\le \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^{D/N}(q) \\ &\le \tau_{\mathfrak{J}_\nu}^{D/N}(q) \le d - 2q.\end{aligned}$$

For the remaining case, we use the convexity of $\tau_{\mathfrak{J}_\nu}^{D/N}$, the lower bound obtained above, and the fact that $\tau_{\mathfrak{J}_\nu}^{D/N}(0) \le d$ and $\tau_{\mathfrak{J}_\nu}^{D/N}(r) = d-2r$, to obtain for all $q \in [0,r]$,

$$d-2q \ge \tau_{\mathfrak{J}_\nu}^{D/N}(q) \ge \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^{D/N}(q) \ge \liminf_{n\to\infty} \beta_{\nu,n}^{D/N}(q) + (2-d)q = d-2q. \qquad \square$$

2.4.3.2 Product measures

The following example will be used to give an example for the non-existence of the spectral dimension (see Section 5.4.4). Let $d \ge 3$ and ν_d denotes a non-zero Borel measure on $(0,1)$ and let Λ^1 denote the one-dimensional Lebesgue measure on $(0,1)$. Here, we consider $\nu := \underbrace{\Lambda^1 \otimes \ldots \otimes \Lambda^1}_{(d-1)\text{-times}} \otimes \nu_d$. Then, for $Q \in \mathcal{D}$ we have

$$\mathfrak{J}_\nu(Q) = \sup_{Q' \in \mathcal{D}(Q)} \nu(Q')\Lambda(Q')^{(2-d)/d} = 2^{-n}\nu_d(\pi_d(Q)).$$

Hence, for all $q \ge 0$, we have

$$\tau_{n,\mathfrak{J}_\nu}^{N}(q) = 2^{(d-1)n}2^{-nq} \sum_{Q \in \pi_d \mathcal{D}_n^N} \nu_d(Q)^q$$

and

$$\tau_{n,\mathfrak{J}_\nu}^{D}(q) = (2^n-2)^{d-1}2^{-nq} \sum_{Q \in \pi_d \mathcal{D}_n^D} \nu_d(Q)^q.$$

It follows from Proposition 2.41 that

$$\tau_{\mathfrak{J}_\nu}^{N}(q) = d-1-q+\beta_{\nu_d}^{N}(q) = d-1-q+\beta_{\nu_d}^{D}(q) = \tau_{\mathfrak{J}_\nu}^{D}(q).$$

2.4.3.3 Ahlfors–David regular measures

In this example we assume that ν is an α-Ahlfors–David regular probability measure on $\mathring{\mathbf{Q}}$ with $\alpha \in (d-2,d]$, that is, there exists a constant $K>0$ such that for every $x \in \operatorname{supp}(\nu)$ and $r \in (0, \operatorname{diam}(\operatorname{supp}(\nu))]$ we have

$$K^{-1}r^\alpha \le \nu(B_r(x)) \le Kr^\alpha,$$

where the diameter of a set $A \subset \mathbb{R}^d$ is defined by

$$\operatorname{diam}(A) := \sup\{|x-y| : x,y \in A\}.$$

Then for appropriate $C>0$ and every $Q \in \mathcal{D}$ with $\nu(Q)>0$ we have

$$C^{-1}\Lambda(Q)^{\alpha/d} \le \nu\left(\langle \mathring{Q} \rangle_2\right) \text{ and } \nu(Q) \le C\Lambda(Q)^{\alpha/d}. \tag{2.4.1}$$

This implies

$$\begin{aligned}\dim_M(\nu) &= \lim_{n\to\infty}\frac{\log\left(\operatorname{card}\left(\left\{Q\in\mathcal{D}_n^N : \nu(Q)>0\right\}\right)\right)}{\log(2^n)}\\ &= \lim_{n\to\infty}\frac{\log\left(\operatorname{card}\left(\left\{Q\in\mathcal{D}_n^D : \nu(Q)>0\right\}\right)\right)}{\log(2^n)} = \alpha = \dim_\infty(\nu).\end{aligned}$$

Indeed, since $\nu(\mathring{\mathbf{Q}})>0$ we find an element $E\in\mathcal{D}$ with $\overline{E}\subset\mathring{\mathbf{Q}}$ and $\nu(E)=\varepsilon>0$. Then, on the one hand, for n large we have

$$\operatorname{card}\left(\left\{Q'\in\mathcal{D}_n^D : \nu(Q')>0\right\}\right)C2^{-n\alpha} \geq \sum_{Q\in\mathcal{D}_n^D}\nu(Q)\geq\varepsilon,$$

showing

$$\liminf_{n\to\infty}\frac{\log\left(\operatorname{card}\left(\left\{Q'\in\mathcal{D}_n^D : \nu(Q')>0\right\}\right)\right)}{\log(2^n)}\geq\alpha.$$

On the other hand,

$$\begin{aligned}\operatorname{card}\left(\left\{Q'\in\mathcal{D}_n^N : \nu(Q')>0\right\}\right)C^{-1}2^{-n\alpha}3^{-d} &\leq 3^{-d}\sum_{Q\in\mathcal{D}_n^N}\nu\left(\langle\mathring{Q}\rangle_2\right)\\ &\leq \sum_{Q\in\mathcal{D}_n^N}\nu(Q)=1,\end{aligned}$$

implying

$$\limsup_{n\to\infty}\frac{\log\left(\operatorname{card}\left(\left\{Q'\in\mathcal{D}_n^N : \nu(Q')>0\right\}\right)\right)}{\log(2^n)}\leq\alpha.$$

Now we prove that for all $q\geq 0$

$$\tau_{\mathfrak{J}_\nu}^{D/N}(q)=\beta_\nu^{D/N}(q)+(2-d)q=(\alpha+2-d)q-\alpha$$

exists as a limit. Indeed,

$$\begin{aligned}\sum_{Q\in\mathcal{D}_n^N}\mathfrak{J}_\nu(Q)^q &\leq C\sum_{Q\in\mathcal{D}_n^N}2^{-n(\alpha+2-d)q}\\ &\leq C\operatorname{card}\left(\left\{Q\in\mathcal{D}_n^N : \nu(Q)>0\right\}\right)2^{-n(\alpha+2-d)q}.\end{aligned}$$

Further, we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}_n^D} \nu\left(\langle\mathring{Q}\rangle_2\right)^q &\le \sum_{Q\in\mathcal{D}_n^D}\Bigg(\sum_{\substack{Q'\in\mathcal{D}_{n+1}^D,\\ \overline{Q}\cap\overline{Q'}\neq\varnothing}} \nu(Q')\Bigg)^q\\ &\le \sum_{Q\in\mathcal{D}_n^D} 4^{dq} \max_{\substack{Q'\in\mathcal{D}_{n+1}^D,\\ \overline{Q}\cap\overline{Q'}\neq\varnothing}} \nu(Q')^q\\ &\le 4^{dq+d}\sum_{Q\in\mathcal{D}_{n+1}^D}\nu(Q)^q,\end{aligned}$$

implying

$$\begin{aligned}\sum_{Q\in\mathcal{D}_{n+1}^D}\mathfrak{J}_\nu(Q)^q \ge \sum_{Q\in\mathcal{D}_{n+1}^D}&\nu(Q)^q 2^{(d-2)nq}\\ &\ge 2^{(d-2)nq}4^{-dq-d}\sum_{Q\in\mathcal{D}_n^D,\nu(Q)>0}\nu\left(\langle\mathring{Q}\rangle_2\right)^q\\ &\ge C^{-1}4^{-dq-d}\,\mathrm{card}\left(\left\{Q\in\mathcal{D}_n^D:\nu(Q)>0\right\}\right)2^{-n(\alpha+2-d)q},\end{aligned}$$

proving the claim.

2.4.3.4 Self-conformal measures

We start with some basic definitions.

Definition 2.45. Let $U\subset\mathbb{R}^d$ be an open set. We say a C^1-map $S:U\to\mathbb{R}^d$ is *conformal* if for every $x\in U$ the matrix $S'(x)$, giving the total derivative of S in x, satisfies $|S'(x)\cdot y| = \|S'(x)\|\,|y|\neq 0$ for all $y\in\mathbb{R}^d\setminus\{0\}$ with $\|S'(x)\| := \sup_{|z|=1}|S(x)\cdot z|$.

Definition 2.46. A family of conformal mappings $\{S_i : X\to X\}_{i\in I}$ on a compact set $X\subset\overline{\mathbf{Q}}$ with $I:=\{1,\dots,\ell\}$, $\ell\ge 2$, is a *C^1-conformal iterated function system (C^1-cIFS)* if

1. Each S_i extends to an injective conformal map $S_i : U\to U$ on an open set $X\subset U$,
2. We have uniform contraction, i.e. $\sup\left\{\left\|S_i'(x)\right\| : x\in U\right\}<1$, $i\in I$.
3. The contractions $(S_i)_{i\in I}$ do not share the same fixed point.

For a conformal iterated function system $\{S_i : X\to X\}_{i\in I}$ there exists a unique compact set $\mathcal{K}\subset X$ such that

$$\mathcal{K}=\bigcup_{i\in I}S_i(\mathcal{K}).$$

Let $(p_i)_{i\in I}$ be a positive probability vector and define $p_u := \prod_{i=1}^{|u|} p_{u_i}$. Then there is a unique Borel probability measure ν with support $\mathcal{K}$ such that

$$\nu(A) = \sum_{i=1}^{\ell} p_i \nu\left(S_i^{-1}(A)\right) \tag{2.4.2}$$

for all $A \in \mathfrak{B}\left(\mathbb{R}^d\right)$ (see [Hut81]). We refer to ν as the *self-conformal measure*.

In following we provide some standard notations. We call $I = \{1, \ldots, \ell\}$ *alphabet* and I^m is the set of words of length $m \in \mathbb{N}$ over I and by $I^* = \bigcup_{m\in\mathbb{N}} I^m \cup \{\varnothing\}$ we refer to the set of all words with finite length including the empty word $\varnothing$. Furthermore, the set of words with infinite length will be denoted by $I^{\mathbb{N}}$ equipped with the metric

$$d(x,y) := \begin{cases} 2^{-\min\{i\in\mathbb{N}: x_i = y_i\}}, & \text{if } x \neq y, \\ 0, & \text{x=y.} \end{cases}$$

The length of a finite word $\omega \in I^*$ will be denoted by $|\omega|$ and for the concatenation of $\omega \in I^*$ with $x \in I^* \cup I^{\mathbb{N}}$ we write ωx. The shift-map $\sigma : I^{\mathbb{N}} \cup I^* \longrightarrow I^{\mathbb{N}} \cup I^*$ is defined by $\sigma(\omega) = \varnothing$ for $\omega \in I \cup \{\varnothing\}$, $\sigma(\omega_1 \ldots \omega_m) = \omega_2 \ldots \omega_m$ for $\omega_1 \ldots \omega_m \in I^m$ with $m > 1$ and $\sigma(\omega_1\omega_2, \ldots) = (\omega_2\omega_3 \ldots)$ for $(\omega_1\omega_2 \ldots) \in I^{\mathbb{N}}$. The cylinder set generated by $\omega \in I^*$ is defined by $[\omega] := \left\{\omega x : x \in I^{\mathbb{N}}\right\} \subset I^{\mathbb{N}}$. Further, for $u = u_1 \ldots u_n \in I^n$, $n \in \mathbb{N}$, we set $u^- = u_1 \ldots u_{n-1}$. We say $P \subset I^*$ is a partition of $I^{\mathbb{N}}$ if

$$\bigcup_{\omega\in P} [\omega] = I^{\mathbb{N}} \text{ and } [\omega] \cap [\omega'] = \varnothing, \text{for all } \omega, \omega' \in P \text{ with } \omega \neq \omega'.$$

Now, we are able to give a coding of the self-conformal set in terms of $I^{\mathbb{N}}$. For $\omega \in I^*$ we put $T_\omega := T_{\omega_1} \circ \ldots \circ T_{\omega_n}$ and define $T_\varnothing := \mathrm{id}_{[0,1]}$ to be the identity map on $[0,1]$. For $(\omega_1\omega_2 \ldots) \in I^{\mathbb{N}}$ and $m \in \mathbb{N}$ we define the initial word by $\omega|_n := \omega_1 \ldots \omega_n$. For every $\omega \in I^{\mathbb{N}}$ the intersection $\bigcap_{n\in\mathbb{N}} T_{\omega|_n}([0,1])$ contains exactly one point $x_\omega \in \mathcal{K}$ and gives rise to a surjection $\pi : I^{\mathbb{N}} \longrightarrow \mathcal{K}$, $\omega \mapsto x_\omega$, which we call the *natural coding map*.

It is worth pointing out that we have the following remarkable bounded distortion property in the case $d \geq 2$.

Proposition 2.47 ([MU03][Theorem 4.1.3])**.** *Assume $d \geq 2$. Then there exists $D \geq 1$ such that for all $n \in \mathbb{N}$ and $u \in I^n$*

$$D^{-1} \leq \frac{\left\|S_u'(x)\right\|}{\left\|S_u'(y)\right\|} \leq D \text{ for all } x, y \in U$$

with $S_u = S_{u_1} \circ \ldots \circ S_{u_{|u|}}$.

Proposition 2.48. *Any self-conformal measure ν is atomless.*

Proof. Fix $x \in X$ such that $\nu(\{x\}) = \max\{\nu(\{y\}) : y \in X\} =: m$. Using

$$\nu(\{x\}) = \sum_{\omega \in I^n} p_\omega \nu\left(S_\omega^{-1}\{x\}\right),\ n \in \mathbb{N},$$

we find $\nu\left(S_\omega^{-1}(\{x\})\right) = m$ for every $\omega \in I^n$ and $n \in \mathbb{N}$. Hence, if $m > 0$, we have $x \in S_\omega(X)$ for all $\omega \in I^n$ and $n \in \mathbb{N}$ and x is the common fixed point of all contractions. This contradicts our assumption that the contractions do not share the same fixed point. □

We need the following result from [PS00, Theorem 1.1] and [Fen07, Corollary 4.5]:

Theorem 2.49. *For a self-conformal measure ν, the L^q-spectrum β_ν^N exists as a limit on $\mathbb{R}_{>0}$.*

Example 2.50. In this example, we additionally assume that the S_i's are contractive similitudes $S_1, \dots, S_\ell$ with corresponding contraction ratios h_i, i.e.

$$|S_i(x) - S_i(y)| = h_i|x - y|,\ x, y \in \mathbb{R}^d.$$

Furthermore, we assume the OSC, i.e. there exists a bounded open set $O \subset \mathbb{R}^d$ such that

$$\forall i \neq j : S_i(O) \cap S_j(O) = \varnothing \text{ and } \bigcup_{i=1}^{\ell} S_i(O) \subset O.$$

Then the L^q-spectrum of ν is implicitly given by

$$\sum_{i=1}^{\ell} p_i^q h_i^{\beta_\nu^N(q)} = 1$$

(see for instance [Rie95]).

Proposition 2.51. *Let ν denote a self-conformal measure on $\overline{\boldsymbol{Q}}$ such that $\nu(\partial\boldsymbol{Q}) = 0$ and $\dim_\infty(\nu) > d - 2$. Then for all $q \geq 0$,*

$$\beta_\nu^N(q) + (d-2)q = \liminf_{n\to\infty} \beta_{\nu,n}^N(q) + (d-2)q = \tau_{\mathfrak{J}_\nu}^N(q) = \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^N(q).$$

Proof. We only have to check the case $d > 2$. Note that $a := 2 - d > -\dim_\infty(\nu)$ implies

$$\sup_{Q\in\mathcal{D}} \nu(Q)\Lambda(Q)^{a/d} =: K < \infty.$$

Let I^* denote the set of all finite words generated by the alphabet I. For $n \in \mathbb{N}$, as in [PS00], we let

$$W_n := \{\omega \in I^* : \operatorname{diam}(S_\omega(\mathcal{K})) \leq 2^{-n} < \operatorname{diam}(S_{\omega^-}(\mathcal{K}))\},$$

which defines a partition of $I^{\mathbb{N}}$. Now fix $Q \in \mathcal{D}_n^N$. For any $Q' \subset \mathcal{D}(Q)$ we set

$$I^{Q'} := \{u \in W_n : S_u(\mathcal{K}) \cap Q' \neq \varnothing\}.$$

If $Q' \in \mathcal{D}_{n+m}^N \cap \mathcal{D}(Q)$, $m \in \mathbb{N}$, and $u \in I^{Q'}$, then we have $\operatorname{diam}\left(S_u^{-1}(Q')\right) \leq L2^{-m}$ for some $L > 0$ (see also [PS00, Lemma 2.4]) and hence it is contained in at most 3^d cubes from $\mathcal{D}_{m-k}^N$ with $k := \lceil \log(L)/\log(2) \rceil$ (this gives $\operatorname{diam}\left(S_u^{-1}(Q')\right) \leq 2^{-m+k}$). Also, by definition of $I^{Q'}$ and W_n, we have

$$\bigcup_{u \in I^{Q'}} S_u(\mathcal{K}) \subset \bigcup_{Q'' \in \mathcal{D}_n^N, \overline{Q''} \cap \overline{Q'} \neq \varnothing} Q'' \subset \bigcup_{Q'' \in \mathcal{D}_n^N, \overline{Q''} \cap \overline{Q} \neq \varnothing} Q'' =: Q_3'.$$

Then we have

$$\begin{aligned}
\nu(Q')\Lambda(Q')^{a/d} &= 2^{-a(n+m)} \sum_{u \in W_n} p_u \nu\left(S_u^{-1}(Q')\right) \\
&= 2^{-an} \sum_{u \in I^{Q'}} p_u 2^{-am} \nu\left(S_u^{-1}(Q')\right) \\
&\leq 2^{-an} \sum_{u \in I^{Q'}} p_u 2^{-ak} \sum_{\substack{C \in \mathcal{D}_{m-k}^N, \\ S_u^{-1}(Q') \cap C \neq \varnothing}} 2^{-a(m-k)} \nu(C) \\
&\leq 2^{-ak} 3^d \max_{C \in \mathcal{D}_{m-k}^N} \nu(C)\Lambda(C)^{a/d} 2^{-an} \sum_{u \in I^{Q'}} p_u \\
&\leq 2^{-ak} 3^d \max_{C \in \mathcal{D}_{m-k}^N} \nu(C)\Lambda(C)^{a/d} 2^{-an} \nu\left(\bigcup_{u \in I^{Q'}} S_u(\mathcal{K})\right) \\
&\leq 2^{-ak} 3^d K \nu\left(Q_3'\right) 2^{-an}.
\end{aligned}$$

Since in the above inequality $Q' \in \mathcal{D}(Q)$ was arbitrary, we deduce for $q > 0$,

$$\begin{aligned}
\sum_{Q \in \mathcal{D}_n^N} \mathfrak{J}_\nu(Q)^q &\leq 2^{(d-2)kq} 3^{dq} K^q 2^{-naq} \sum_{Q \in \mathcal{D}_n^N} \nu\left(Q_3'\right)^q \\
&\leq 2^{(d-2)kq} 3^{dq} K^q 2^{-naq} \sum_{Q \in \mathcal{D}_n^N} \left(\sum_{Q' \in \mathcal{D}_n^N, \overline{Q'} \cap \overline{Q} \neq \varnothing} \nu(Q') \right)^q \\
&\leq 2^{(d-2)kq} 3^{dq} K^q 2^{-naq} 3^{dq} \sum_{Q \in \mathcal{D}_n^N} \max_{Q' \in \mathcal{D}_n^N, \overline{Q'} \cap \overline{Q} \neq \varnothing} \nu(Q')^q \\
&\leq 2^{(d-2)kq} 3^{dq} K^q 2^{-naq} 3^{dq+d} \sum_{Q \in \mathcal{D}_n^N} \nu(Q)^q.
\end{aligned}$$

This gives $\beta_\nu^N(q) - aq \geq \tau_{\mathfrak{J}_\nu}^N(q)$. Furthermore, we observe that

$$\beta_\nu^N(0) = \overline{\dim}_M(\nu) = \tau_{\mathfrak{J}_\nu}^N(0).$$

To complete the proof, notice that

$$\sum_{Q \in \mathcal{D}_n^N} \nu(Q)^q \Lambda(Q)^a \leq \sum_{Q \in \mathcal{D}_n^N} \mathfrak{J}_\nu(Q)^q.$$

Finally, Theorem 2.49 gives $\beta_\nu^N(q) - aq \leq \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^N(q)$ for $q > 0$. □

Proposition 2.52. *Let ν denote a self-conformal measure on $\overline{\boldsymbol{Q}}$ with $\nu(\partial \boldsymbol{Q}) = 0$ and $\dim_\infty(\nu) > d-2$. Then*

$$\beta_\nu^N(q) = \beta_\nu^D(q) = \liminf_{n\to\infty} \beta_{\nu,n}^D(q) = \liminf_{n\to\infty} \beta_{\nu,n}^N(q)$$

for all $q > 0$.

Proof. We use the same notation as in the proof of Proposition 2.51. By our assumption there exists $n \in \mathbb{N}$ such that $S_u(\mathcal{K}) \subset \mathring{\mathbf{Q}}$ for some $u \in W_n$. Indeed assume for all $n \in \mathbb{N}$ and $u \in W_n$, we have

$$S_u(\mathcal{K}) \cap \partial \mathbf{Q} \neq \emptyset.$$

Further, using $\sup_{u \in W_n} \operatorname{diam}(S_u(\mathcal{K})) \leq 2^{-n} \to 0$ for $n \to \infty$ and $\mathcal{K} = \bigcup_{u \in W_n} S_u(\mathcal{K})$, we deduce that $\mathcal{K} \subset \partial \mathbf{Q}$. This gives $\nu(\partial \mathbf{Q}) > 0$ contradicting our assumption.

Let us assume that the distance of $S_u(\mathcal{K})$ to the boundary of $\mathbf{Q}$ is at least $2^{-n-m_0+2}\sqrt{d}$ for some $m_0 \in \mathbb{N}$. Then all cubes $Q \in \mathcal{D}_{n+m}^N$ intersecting $S_u(\mathcal{K})$ lie in $\mathcal{D}_{n+m}^D$ for all $m > m_0$. Therefore, using the self-similarity and [PS00, Lemma 2.2 and Lemma 2.4] (with constant C_1 from there) we have for $q > 0$

$$\begin{aligned}\sum_{Q \in \mathcal{D}_{n+m}^D} \nu(Q)^q &= \sum_{Q \in \mathcal{D}_{n+m}^D} \left(\sum_{v \in W_n} p_v \nu\left(S_v^{-1} Q\right) \right)^q \\ &\geq p_u^q \sum_{Q \in \mathcal{D}_{n+m}^N} \nu\left(S_u^{-1} Q\right)^q \geq C_1^{-1} p_u^q \sum_{Q \in \mathcal{D}_m^N} \nu(Q)^q.\end{aligned}$$

This gives $\beta_\nu^N(q) \leq \beta_\nu^D(q)$ and $\liminf_{n\to\infty} \beta_n^N(q) \leq \liminf_{n\to\infty} \beta_n^D(q)$. The reverse inequalities are obvious. Hence, the claim follows from Theorem 2.49. □

Corollary 2.53. *Let ν denote a self-conformal measure on $\boldsymbol{Q}$ with $\nu(\partial \boldsymbol{Q}) = 0$ and $\dim_\infty(\nu) > d-2$. Then, for all $q > 0$, we have*

$$\beta_\nu^N(q) + (d-2)q = \tau_{\mathfrak{J}_\nu}^N(q) = \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^N(q) = \tau_{\mathfrak{J}_\nu}^D(q) = \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^D(q).$$

Proof. The cases $d = 1, 2$ follow immediately from Proposition 2.35 and Proposition 2.52. For $d > 2$, we obtain from Proposition 2.51 and Proposition 2.52 the following chain of inequalities

$$\begin{aligned}\beta_\nu^N(q) + (d-2)q &= \liminf_{n\to\infty} \beta_n^D(q) + (d-2)q \\ &\le \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^D(q) \le \liminf_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^N(q) \\ &= \tau_{\mathfrak{J}_\nu}^N(q) = \beta_\nu^N(q) + (d-2)q.\end{aligned}$$

□

Chapter 3

Partition entropy and optimized coarse multifractal dimension

Throughout this chapter let $\mathfrak{J}$ be a non-negative set function defined on $\mathcal{D}$ which is monotone, locally non-vanishing and uniformly vanishing (for the definitions we refer to Section 2.3). This chapter is devoted to the study of the lower and upper optimized coarse multifractal dimension with respect to $\mathfrak{J}$ and the lower and upper $\mathfrak{J}$-partition entropy. If $\mathfrak{J}$ is equal to the spectral partition function $\mathfrak{J}_{\nu,a,b}$ for a certain choice of a, b, then these quantities will be important in estimating the lower and upper spectral dimension of Kreĭn–Feller operators and the lower and upper quantization dimension. First, we briefly recall the basic definitions from the introduction. The upper, resp. lower $\mathfrak{J}$-partition entropy is given by

$$\overline{h}_{\mathfrak{J}} = \limsup_{x\to\infty} \frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)}, \quad \underline{h}_{\mathfrak{J}} = \liminf_{x\to\infty} \frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)},$$

with

$$\mathcal{M}_{\mathfrak{J}}(x) = \inf\left\{\operatorname{card}(P) : P \in \Pi_{\mathfrak{J}} \mid \max_{C\in P} \mathfrak{J}(C) < 1/x\right\},\ 1/\mathfrak{J}(\mathbf{Q}) < x.$$

Here $\Pi_{\mathfrak{J}}$ denotes the set of finite collections of dyadic cubes such that for all $P \in \Pi_{\mathfrak{J}}$ there exists a partition $\widetilde{P}$ of $\mathbf{Q}$ by dyadic cubes from $\mathcal{D}$ with $P = \left\{Q \in \widetilde{P} : \mathfrak{J}(Q) > 0\right\}$. The lower and upper optimized (Dirichlet/Neumann) coarse multifractal dimension with respect to $\mathfrak{J}$ is given by

$$\underline{F}_{\mathfrak{J}}^{D/N} = \sup_{\alpha>0} \frac{\underline{F}_{\mathfrak{J}}^{D/N}(\alpha)}{\alpha} \quad\text{and}\quad \overline{F}_{\mathfrak{J}}^{D/N} = \sup_{\alpha>0} \frac{\overline{F}_{\mathfrak{J}}^{D/N}(\alpha)}{\alpha}$$

with

$$\underline{F}_{\mathfrak{J}}^{D/N}(\alpha) = \liminf_{n\to\infty} \frac{\log^{+}\left(\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n)\right)}{\log(2^{n})} \text{ and } \overline{F}_{\mathfrak{J}}^{D/N}(\alpha) = \limsup_{n\to\infty} \frac{\log^{+}\left(\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n)\right)}{\log(2^{n})}$$

and

$$\mathcal{N}_{\mathfrak{J},\alpha}^{D/N}(n) = \operatorname{card}\left(M_{n}^{D/N}(\alpha)\right), \quad M_{\mathfrak{J},n}^{D/N}(\alpha) = \left\{Q \in \mathcal{D}_{n}^{D/N} : \mathfrak{J}(Q) \geq 2^{-\alpha n}\right\}.$$

This chapter is divided into four sections. In Section 3.1, we present an adaptive approximation partition algorithm (see Proposition 3.1) which yields an upper bound for the upper $\mathfrak{J}$-partition entropy in terms of the zero of the $\mathfrak{J}$-partition function. Further, we show that the lower and upper $\mathfrak{J}$-partition entropy is always bounded from below by the lower and upper optimized coarse multifractal dimension. In Section 3.2, we prove that the $\mathfrak{J}_{\nu,a,b}$-partition entropy, under the assumption $b\dim_{\infty}(\nu) + ad > 0$, is bounded from above by

$$q_{\mathfrak{J}_{\nu,a,b}}^{N} = \inf\{q \geq 0 : \tau_{\mathfrak{J}_{\nu,ab}}^{N}(q) < 0\}.$$

In Section 3.3, we establish a connection to the classical works of Solomjak and Birman [BS66; BS74], Borzov [Bor71], and the partition entropy. This enables us to partially improve [Bor71, Theorem 1] for a wide class of singular set functions on $\mathcal{D}$. In the last chapter we study the lower and upper optimized coarse multifractal dimension under mild conditions on $\mathfrak{J}$. One of the main result of this section is the identification of the upper optimized coarse multifractal dimension by $q_{\mathfrak{J}}^{D/N}$. Further, we establish regularity conditions (see Definition 3.22) for which we can guarantee that $\underline{F}_{\mathfrak{J}}^{D/N} = \overline{F}_{\mathfrak{J}}^{D/N}$. Later, we will use this regularity result to give criteria that allow us to ensure, depending on the setting, the existence of the spectral dimension or quantization dimension, respectively.

3.1 Bounds for the partition entropy and optimized coarse multifractal dimension

We begin with a motivation for the upper estimate of $\overline{h}_{\mathfrak{J}}$: For a given threshold $0 < t < \mathfrak{J}(\mathbf{Q})$, we will construct partitions by dyadic cubes of $\mathcal{D}$ as a function of t via an *adaptive approximation algorithm* in the sense of [DeV87] (see also [HKY00]) as follows. We say $Q \in \mathcal{D}$ is *bad* if $\mathfrak{J}(Q) \geq t$, otherwise we call Q *good.* We generate a partition of $\mathbf{Q}$ of elements of $\mathcal{D}$ into good intervals which will be denoted by P_t. By the choice of t, we see that $\mathbf{Q}$ is bad. Hence, we put $\mathcal{B} := \{\mathbf{Q}\}$. Now, we divide each element of $\mathcal{B}$ into 2^d cubes of $\mathcal{D}$ of equal size and check whether they are good, in which case we move these cubes to P_t, or they are bad, in which case they are put into $\mathcal{B}$. We repeat this procedure until the set of bad cubes is empty. The process terminates, which is ensured by the assumption that $\mathfrak{J}$ decreases uniformly. Using the definition of $\mathcal{D}$ and $\mathcal{M}_{\mathfrak{J}}$, it follows that the resulting finite partition P_t is optimal (in the sense of minimizing the cardinality, i.e. $\mathcal{M}_{\mathfrak{J}}(1/t) = \mathrm{card}(P_t)$) among all partitions P by dyadic cubes of $\mathcal{D}$ fulfilling $\max_{Q\in P}\mathfrak{J}(Q) < t$. We provide a two-dimensional illustration (Figure 3.1.1) of these partitions P_t for three different values of $t \in (0,1)$ for the particular choice $\mathfrak{J}(Q) = (\nu\otimes\nu)(Q)\Lambda(Q)^2$, $Q \in \mathcal{D}$, where ν denotes the $(p, 1-p)$-Cantor measure supported on the triadic Cantor set.

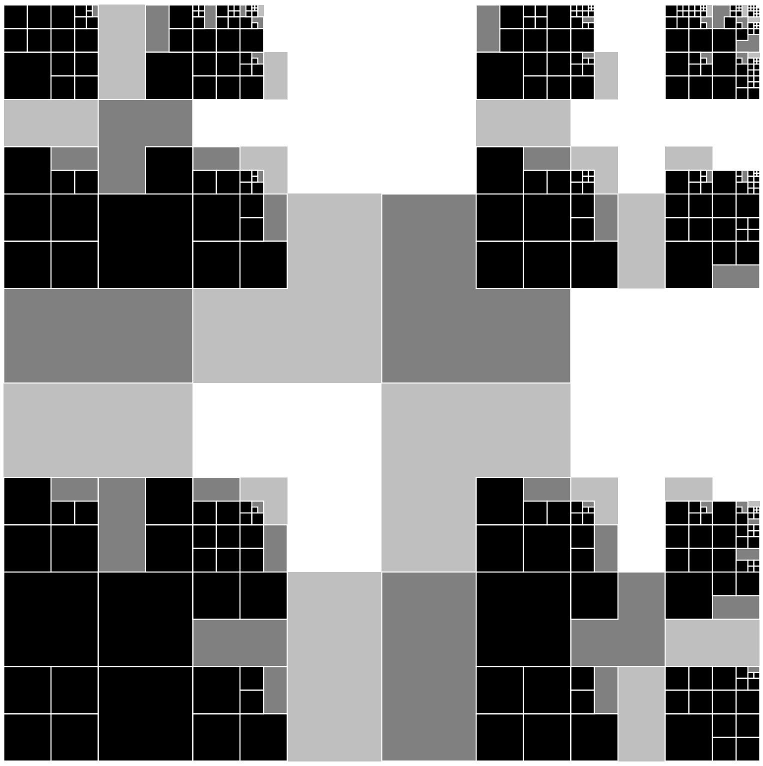

Figure 3.1.1 Illustration of the adaptive approximation algorithm for $\mathfrak{J}(Q) = (\nu\otimes\nu)(Q)\Lambda(Q)^2$, $Q \in \mathcal{D}$, $d = 2$, where ν is the $(0.1, 0.9)$-cantor measure. Here, the light gray cubes belong to $P_{10^{-3}}$, the gray cubes to $P_{10^{-4}}$, and the black cubes to $P_{10^{-7}}$. In this figure we neglected all cubes with ν-measure zero.

Now, the remaining task is to connect the asymptotic behavior of $\mathrm{card}(P_t)$ with the partition function $\tau_{\mathfrak{J}}^N$. Motivated by ideas from large derivation theory and the thermodynamic formalism [Rue04], we are able to bound $\overline{h}_{\mathfrak{J}}$ from above by $q_{\mathfrak{J}}^N$, namely, by comparing the cardinality of P_t and $Q_t := \{Q \in \mathcal{D} : \mathfrak{J}(Q) \geq t\}$. This will be the key idea in the proof of Proposition 3.1.

Proposition 3.1. *For* $0 < t < \mathfrak{J}(\boldsymbol{Q})$*, we have that*

$$P_t \coloneqq \left\{ Q \in \mathcal{D} : \mathfrak{J}(Q) < t \,\&\, \exists Q' \in \mathcal{D}^N_{\left|\log_2(\Lambda(Q))\right|/d-1} : Q' \supset Q \,\&\, \mathfrak{J}(Q') \geq t \right\}$$

is a finite partition of dyadic cubes of $\boldsymbol{Q}$*, and the growth rate of* $\mathrm{card}(P_t)$ *gives rise to the following inequalities:*

$$\overline{F}^N_{\mathfrak{J}} \leq \overline{h}_{\mathfrak{J}} \leq \limsup_{t \downarrow 0} \frac{\log(\mathrm{card}(P_t))}{-\log(t)} \leq \kappa_{\mathfrak{J}} \leq q^N_{\mathfrak{J}}, \tag{3.1.1}$$

$\overline{F}^D_{\mathfrak{J}} \leq q^D_{\mathfrak{J}}$*, and*

$$\underline{F}^N_{\mathfrak{J}} \leq \underline{h}_{\mathfrak{J}} \leq \liminf_{t \downarrow 0} \frac{\log(\mathrm{card}(P_t))}{-\log(t)}.$$

Remark 3.2. At this stage we would like to point out that in the next section (see Proposition 3.20) , we will show equality in the above chain of inequalities (3.1.1) using the coarse multifractal formalism under some mild additional assumptions on $\mathfrak{J}$.

Proof. We only have to consider the case $\kappa_{\mathfrak{J}} < \infty$. The first statement follows from the monotonicity of $\mathfrak{J}$,

$$\lim_{n \to \infty} \sup_{i \geq n, C \in \mathcal{D}^N_i} \mathfrak{J}(C) = 0,$$

the definition of $\mathcal{D}$ and Lemma 2.1. Further, Lemma 2.25 gives $\kappa_{\mathfrak{J}} \leq q^N_{\mathfrak{J}}$ (where equality holds if $\dim_\infty(\mathfrak{J}) > 0$, otherwise $q^N_{\mathfrak{J}} = \infty$). Let $0 < t < \mathfrak{J}(\mathbf{Q})$. Setting

$$R_t \coloneqq \{Q \in \mathcal{D} : \mathfrak{J}(Q) \geq t\},$$

we note that for $Q \in P_t$ there is exactly one $Q' \in R_t \cap \mathcal{D}^N_{\left|\log_2(\Lambda(Q))\right|/d-1}$ with $Q \subset Q'$ and for each $Q' \in R_t \cap \mathcal{D}^N_{\left|\log_2(\Lambda(Q))\right|/d-1}$ there are at most 2^d elements of $P_t \cap \mathcal{D}^N_{\left|\log_2(\Lambda(Q))\right|/d}$ such that they are subsets of Q'. Hence,

$$\mathrm{card}(P_t) \leq 2^d \, \mathrm{card}(R_t).$$

For $q > \kappa_{\mathfrak{J}}$ we obtain

$$\begin{aligned} t^q \operatorname{card}(P_t) &= \sum_{n=1}^{\infty} t^q \sum_{Q \in P_t \cap \mathcal{D}_n^N} 1 \\ &\leq 2^d \sum_{n=0}^{\infty} \sum_{Q \in R_t \cap \mathcal{D}_n^N} t^q \\ &\leq 2^d \sum_{n=0}^{\infty} \sum_{Q \in R_t \cap \mathcal{D}_n^N} \mathfrak{J}(Q)^q \\ &\leq 2^d \sum_{n=0}^{\infty} \sum_{Q \in \mathcal{D}_n^N} \mathfrak{J}(Q)^q < \infty. \end{aligned}$$

This implies

$$\limsup_{t \downarrow 0} \frac{\log(\operatorname{card}(P_t))}{-\log(t)} \leq q.$$

Now, q tending to $\kappa_{\mathfrak{J}}$ proves the third inequality. The second and sixth inequality follow immediately from the observation that $\mathcal{M}_{\mathfrak{J}}(x) \leq \operatorname{card}\left(P_{1/x}\right)$. For $\alpha > 0$, $n \in \mathbb{N}$, and $P \in \Pi_{\mathfrak{J}}$ such that $\max_{C \in P} \mathfrak{J}(C) < 2^{-n\alpha}$, we have

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n) = \operatorname{card}\left(\left\{Q \in \mathcal{D}_n^N : \mathfrak{J}(Q) \geq 2^{-\alpha n}\right\}\right) \leq \operatorname{card}(P),$$

where we used the fact that for each $Q \in \mathcal{D}_n^N$ with $\mathfrak{J}(Q) \geq 2^{-\alpha n}$ there exists at least one $Q' \in \mathcal{D}(Q) \cap P$ and this assignment is injective. Indeed, since $P \in \Pi_{\mathfrak{J}}$, $\mathfrak{J}$ is locally non-vanishing, and $\mathfrak{J}(Q) \geq 1/x$, it follows $Q \cap P \neq \emptyset$. Therefore, using that $\mathfrak{J}$ is monotone, we deduce that there at least one exists $Q' \in \mathcal{D}(Q) \cap P$. Thus, we obtain (for an illustration see Figure 3.1.2)

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n) \leq \mathcal{M}_{\mathfrak{J}}(2^{\alpha n}).$$

To prove $\underline{F}_{\mathfrak{J}}^N \leq \underline{h}_{\mathfrak{J}}$, fix $\alpha > 0$, $x > \max\{1, \mathfrak{J}(\mathbf{Q})\}$, and $P \in \Pi_{\mathfrak{J}}$ such that $\mathcal{M}_{\mathfrak{J}}(x) = \operatorname{card}(P)$. Then there exists $n_x \in \mathbb{N}$ such that

$$2^{-(n_x+1)\alpha} < \frac{1}{x} \leq 2^{-n_x\alpha}.$$

It follows

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n_x) \leq \mathcal{M}_{\mathfrak{J}}(2^{\alpha n_x}) \leq \mathcal{M}_{\mathfrak{J}}(x).$$

Therefore, we obtain

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n_x) \leq \operatorname{card}(P).$$

Hence,

$$\frac{\log\left(\mathcal{N}^{N}_{\alpha,\mathfrak{J}}(n_x)\right)}{\log(2^{n_x\alpha})+\log(2)\alpha} \leq \frac{\log\left(\mathcal{N}^{N}_{\alpha,\mathfrak{J}}(n_x)\right)}{\log(x)} \leq \frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)},$$

implying

$$\liminf_{n\to\infty} \frac{\log\left(\mathcal{N}^{N}_{\alpha,\mathfrak{J}}(n)\right)}{\log(2^{n\alpha})} \leq \liminf_{x\to\infty} \frac{\log\left(\mathcal{N}^{N}_{\alpha,\mathfrak{J}}(n_x)\right)}{\log(2^{n_x\alpha})} \leq \underline{h}_{\mathfrak{J}}.$$

Therefore, taking the supremum over $\alpha > 0$ gives $\underline{F}^{N}_{\mathfrak{J}} \leq \underline{h}_{\mathfrak{J}}$. The last claim follows from the fact that for every $\alpha > 0$ and $q = q^{D}_{\mathfrak{J}}$, we have

$$\mathcal{N}^{D}_{\alpha,\mathfrak{J}}(n) \leq 2^{\alpha n q} \sum_{Q\in\mathcal{D}^{D}_{n}} \mathfrak{J}(Q)^{q}. \qquad \square$$

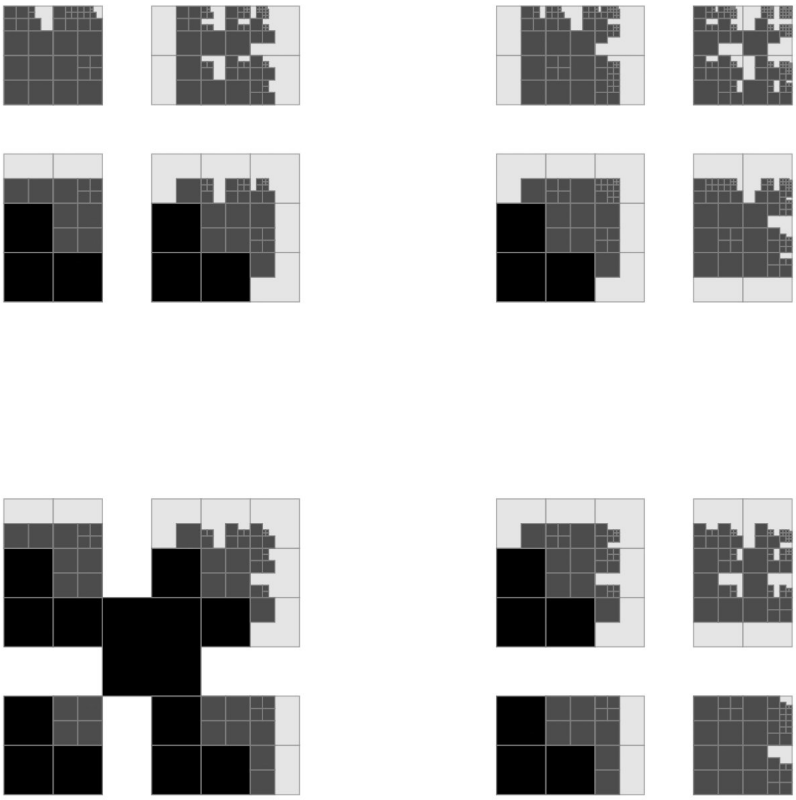

Figure 3.1.2 Illustration of the cubes of $M^{N}_{\alpha,\mathfrak{J}}(n)$ (gray) and $P_{2^{-\alpha n}}$ (underneath of gray cubes in black) for $n = 4$ and $\alpha = 5.734$ with $\mathfrak{J}$ as defined in Figure 3.1.1.

In Section 3.4 (see Proposition 3.20), we will show equality in the above chain of inequalities (3.1.1) using the coarse multifractal formalism under some mild additional assumptions on $\mathfrak{J}$.

Proposition 3.3. *Assume there exists a sequence $(n_k)_k \in \mathbb{N}^{\mathbb{N}}$ and $K > 0$ such that for all $k \in \mathbb{N}$,*

$$\max_{Q\in\mathcal{D}_{n_k}^N} \mathfrak{J}(Q)^{q_{n_k}} \leq \frac{K}{2^{n_k \tau_{\mathfrak{J},n_k}^N(0)}} \sum_{Q\in\mathcal{D}_{n_k}^N} \mathfrak{J}(Q)^{q_{n_k}},$$

where q_{n_k} is the unique zero of $\tau_{n_k}^N$. Further, suppose $\liminf_{k\to\infty} q_{n_k} > 0$. Then we have

$$\underline{h}_{\mathfrak{J}} \leq \liminf_{k\to\infty} q_{n_k}.$$

Proof. First of all, note that we have $\sum_{Q\in\mathcal{D}_{n_k}^N} \mathfrak{J}(Q)^{q_{n_k}} = 1$. Further, since $\mathfrak{J}$ is uniformly decreasing, we choose k large enough such that $\mathfrak{J}(Q) < 1$ for all $Q \in \mathcal{D}_{n_k}^N$. Ensuring that $\tau_{n_k}^N$ has a unique zero. Hence, we obtain

$$\max_{Q\in\mathcal{D}_{n_k}^N} \mathfrak{J}(Q) \leq \frac{K^{1/q_{n_k}}}{2^{n_k \tau_{\mathfrak{J},n_k}^N(0)/q_{n_k}}}.$$

This implies

$$\frac{\log\left(\mathcal{M}_{\mathfrak{J}}\left(\frac{2^{\tau_{\mathfrak{J},n_k}^N(0) n_k/q_{n_k}}}{2K^{1/q_{n_k}}}\right)\right)}{\log\left(\frac{2^{\tau_{\mathfrak{J},n_k}^N(0) n_k/q_{n_k}}}{2K^{1/q_{n_k}}}\right)} \leq \frac{\log\left(2^{n_k \tau_{\mathfrak{J},n_k}^N(0)}\right)}{\log\left(\frac{2^{\tau_{\mathfrak{J},n_k}^N(0) n_k/q_{n_k}}}{2K^{1/q_{n_k}}}\right)},$$

which proves the claim. ◻

3.2 Upper bounds for the $\mathfrak{J}_{\nu,a,b}$-partition entropy

This section is devoted to the study of the $\mathfrak{J}_{\nu,a,b}$-partition entropy, which is ultimately associated with the spectral dimension for a certain choice of parameters $a \in \mathbb{R}$, $b > 0$. Let us introduce the following notation: $\mathcal{M}_{a,b}(x) := \mathcal{M}_{\mathfrak{J}_{\nu,a,b}}(x)$, $x > 0$, as well as

$$\overline{h}_{a,b} := \overline{h}_{\mathfrak{J}_{\nu,a,b}},\ \underline{h}_{a,b} := \underline{h}_{\mathfrak{J}_{\nu,a,b}} \text{ and } \overline{h}_a := \overline{h}_{a,1},\ \underline{h}_a := \underline{h}_{a,1}.$$

The following theorem treats an upper estimate of $\overline{h}_{a,b}$ for the case $a = 0$.

Proposition 3.4. *If $\dim_\infty(\nu) > 0$, then*

$$\overline{h}_{0,b} \leq q_{\mathfrak{J}_{\nu,0,b}}^N = \inf\left\{q \geq 0 : \sum_{C\in\mathcal{D}} \mathfrak{J}_{\nu,0,b}(C)^q < \infty\right\} = \frac{1}{b}.$$

Proof. First, note that Proposition 2.35 implies $\dim_\infty(\mathfrak{J}_{\nu,0,b}) = b\dim_\infty(\nu) > 0$. Therefore, we deduce that $\mathfrak{J}_{\nu,0,b}$ is uniformly vanishing by Lemma 2.23. An application of Proposition 3.1 with $\mathfrak{J} = \mathfrak{J}_{\nu,0,b}$ gives $\overline{h}_{0,b} \le q^N_{\mathfrak{J}_{\nu,0,b}}$. Furthermore, we have by Proposition 2.35 that $\beta^N_\nu(qb) = \tau^N_{\mathfrak{J}_{\nu,0,b}}(q)$. Thus, by Lemma 2.25, we obtain

$$\inf\left\{q \ge 0 : \sum_{C\in\mathcal{D}} \mathfrak{J}_{\nu,0,b}(C)^q < \infty\right\} = q^N_{\mathfrak{J}_{\nu,0,b}} = \frac{1}{b}. \qquad \square$$

The rest of this section deals with the case $a \neq 0$. Recalling the definition of $q^N_{\mathfrak{J}}$, we find $q^N_{\nu\Lambda^a} \le q^N_{\mathfrak{J}_{\nu,a,1}}$ with equality for the case $a > 0$. We need the following elementary lemma.

Lemma 3.5. *For $c, d \in \mathbb{R}$ with $c < d$, let $(f_n : [c,d] \to \mathbb{R})_{n\in\mathbb{N}}$ be a sequence of decreasing functions converging pointwise to a function f. We assume that f_n has a unique zero in x_n, $n \in \mathbb{N}$, and f has a unique zero in x. Then $x = \lim_{n\to\infty} x_n$.*

Proof. Assume that $(x_n)_n$ does not converge to x. Then, because $[c,d]$ is compact, there exists a subsequence $(n_k)_k$ such that $x_{n_k} \to x^* \neq x$ for $k \to \infty$ and $x^* \in [c,d]$. We only consider $x^* < x$, the case $x^* > x$ follows analogously. Then, for k large, we have $x_{n_k} \le (x^* + x)/2$. Thus, for each $y \in ((x^* + x)/2, x)$, we have

$$0 = f_{n_k}(x_{n_k}) > f_{n_k}(y) \ge f_{n_k}(x) \to f(x) = 0, \text{ for } k \to \infty.$$

Consequently, $f(y) = 0$ for all $y \in ((x^* + x)/2, x)$, contradicting the uniqueness of the zero of f. $\square$

Proposition 3.6. *Suppose $b\dim_\infty(\nu) + ad > 0$. If $a < 0$, then*

$$\overline{h}_{a,b} = \frac{\overline{h}_{a/b,1}}{b} \le \frac{q^N_{\mathfrak{J}_{\nu,a/b,1}}}{b} \le \frac{\dim_\infty(\nu)}{b\dim_\infty(\nu) + ad}.$$

If $a > 0$, then

$$\overline{h}_{a,b} \le q^N_{\mathfrak{J}_{\nu,a,b}} = \inf\left\{q > 0 : \beta^N_\nu(bq) < adq\right\} \le \frac{\overline{\dim}_M(\nu)}{b\overline{\dim}_M(\nu) + ad} \le \frac{1}{b+a}.$$

In particular, if $\dim_\infty(\nu) > d-2$ and $d > 2$, then for all $t \in (0, 2\dim_\infty(\nu)/(d-2))$, we have

$$\overline{h}_{2/d-1,2/t} = \frac{t}{2}\overline{h}_{t(2/d-1)/2,1} \le \frac{t}{2} q^N_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} \le \frac{\dim_\infty(\nu)}{2\dim_\infty(\nu)/t + 2 - d}.$$

Moreover, $\lim_{t\downarrow 2} q^N_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} = q^N_{\mathfrak{J}_\nu}$.

Proof. Since $b\dim_\infty(\nu)+ad>0$, we obtain from Fact 2.37 that

$$\dim_\infty(\mathfrak{J}_{\nu,a/b,1})=\dim_\infty(\nu)+ad/b>0.$$

Using the definition of $\mathcal{M}_{a,b}(x)$ and Proposition 3.1 applied to $\mathfrak{J}=\mathfrak{J}_{\nu,a/b,1}$, we obtain

$$\overline{h}_{a,b}=\limsup_{x\to\infty}\frac{\log\left(\mathcal{M}_{a,b}(x)\right)}{\log(x)}=\limsup_{x\to\infty}\frac{\log\left(\mathcal{M}_{a/b,1}\left(x^{1/b}\right)\right)}{b\log(x^{1/b})}=\frac{1}{b}\overline{h}_{a/b,1}\le\frac{q^N_{\mathfrak{J}_{\nu,a/b,1}}}{b},$$

where for the third equality we used the bijectivity of $x\mapsto x^{1/b}$, $x>0$. The estimate of $q^N_{\mathfrak{J}_{\nu,a,b}}$ for the case $a>0$ follows from

$$\beta^N_\nu(bq)\le\overline{\dim}_M(\nu)(1-qb)$$

for all $0\le q\le 1/b$. For the case $a<0$, Fact 2.37 implies

$$q^N_{\mathfrak{J}_{\nu,a/b,1}}\le\dim_\infty(\nu)/(\dim_\infty(\nu)+ad/b).$$

Now, let $\dim_\infty(\nu)>d-2$, $d>2$, and $t\in I:=(0,2\dim_\infty(\nu)/(d-2))$. we obtain

$$\dim_\infty\left(\mathfrak{J}_{\nu,t(2/d-1)/2,1}\right)=\dim_\infty(\nu)+dt(2/d-1)/2>0.$$

Hence, the third claim follows from the first part. The rest of the proof is devoted to the proof of $\lim_{t\downarrow 2}q^N_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}=q^N_{\mathfrak{J}_{\nu,(2/d-1),1}}$. First, observe that, using

$$0<t(d-2)/2<s<\dim_\infty(\nu),$$

we have for n large

$$\nu(C)\le 2^{-sn},\ C\in\mathcal{D}^N_n.$$

Set $a:=2/d-1$ and for fixed $q\ge 0$, consider

$$t\mapsto\tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q)=\limsup_{n\to\infty}\frac{\log\left(\sum_{Q\in\mathcal{D}^N_n}\max_{Q'\in\mathcal{D}(Q)}\nu(Q')^q\left(\Lambda(Q')^{qa}\right)^{t/2}\right)}{\log(2^n)}.$$

Since $f_Q:t\mapsto\nu(Q)^q\left(\Lambda(Q)^{qa}\right)^{t/2}$, $Q\in\mathcal{D}^N_n$ with $\nu(Q)>0$ and $t\in I$, is log-convex, i.e. for all $\theta\in(0,1)$ and $s,t\in\mathbb{R}_{>0}$, we have

$$\log\left(f_Q(\theta t+(1-\theta)s)\right)\le\theta\log\left(f_Q(t)\right)+(1-\theta)\log\left(f_Q(s)\right),$$

it follows that

$$t\mapsto\max_{Q'\in\mathcal{D}(Q)}\nu(Q')^q\left(\Lambda(Q')^{qa}\right)^{t/2}$$

is also log-convex (the existence of the maximum is ensured by

$$\mathfrak{J}_{\nu,at/2,1}(C) \leq 2^{n(-s+(d-2)t/2)}$$

for all $C \in \mathcal{D}_n^N$). Therefore, we get with the Hölder inequality that $t \mapsto \tau^N_{\mathfrak{J}_{\nu,at/2,1},n}(q)$, $t \in I$ is convex, which carries over to the limit superior $t \mapsto \tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q)$, $t \in I$, of convex functions. Further, we have $\tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q) < \infty$ for all $t \in I$. Hence, by Theorem A.5, it follows that $\tau^N_{\mathfrak{J}_{\nu,at/2,1}}$ is continuous on I. In particular, for each $q \geq 0$, we have

$$\lim_{t \to 2} \tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q) = \tau^N_{\mathfrak{J}_{\nu,a,1}}(q)\,.$$

By Lemma 2.27 we deduce that $q^N_{\mathfrak{J}_{\nu,at/2,1}}$ is the unique zero of $q \mapsto \tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q)$. Hence, for fixed $t \in I$, we have that $q \mapsto \tau^N_{\mathfrak{J}_{\nu,at/2,1}}(q)$ is decreasing and has a unique zero given by $q^N_{\mathfrak{J}_{\nu,at/2,1}}$. Further, for all $t \in [2,(2+2\dim_\infty(\nu)/(d-2))/2]$, we have

$$\begin{aligned} 0 \leq q^N_{\mathfrak{J}_{\nu,at/2,1}} &\leq \frac{\dim_\infty(\nu)}{\dim_\infty(\nu)+(2-d)t/2} \\ &\leq \frac{\dim_\infty(\nu)}{\dim_\infty(\nu)+(2-d)(1+\dim_\infty(\nu)/(d-2))/2} =: g. \end{aligned}$$

Now, Lemma 3.5, applied to

$$q \mapsto \tau^N_{\mathfrak{J}_{\nu,at/2,1}}|_{[0,g]},\ t \in [2,(2+\dim_\infty(\nu)/(d-2))/2],$$

implies

$$\lim_{t \downarrow 2} q^N_{\mathfrak{J}_{\nu,at/2,1}} = q^N_{\mathfrak{J}_{\nu,a,1}}\,. \qquad \square$$

3.3 The dual problem

This section is devoted to study the dual problem of $\mathcal{M}_{\mathfrak{J}}$. Recall from Section 1.1.3 that the dual problem is concerned with the control of the asymptotic behavior of

$$\gamma_{\mathfrak{J},n} = \inf_{\substack{P \in \Pi_{\mathfrak{J}},\\ \mathrm{card}(P) \leq n}} \max_{Q \in P} \mathfrak{J}(Q).$$

In particular, we are interested in the special choice $\mathfrak{J}_{J,a}(Q) := J(Q)\Lambda(Q)^a$, $a > 0$, $Q \in \mathcal{D}$, where J is a non-negative, finite, locally non-vanishing, and superadditive function on $\mathcal{D}$, that is, if $Q \in \mathcal{D}$ is decomposed into a finite number of disjoint cubes $(Q_j)_{j=1,\dots,N}$ of $\mathcal{D}$, then $\sum_{j=1}^N J(Q_j) \leq J(Q)$. We are now interested in the growth properties of $\gamma_{\mathfrak{J}_{J,a},n}$. Upper estimates for $\gamma_{\mathfrak{J}_{J,a},n}$ have been first obtained in [BS67; Bor71]. Here, we proceed as follows: First we present an adaptive approximation algorithm going back to Birman/Solomjak [BS67; Bor71] to obtain well-known upper bounds on $\gamma_{\mathfrak{I}_{J,a},n}$. After that we employ the estimates of Proposition 3.1 to partially improve and extend the results in [BS67; Bor71].

In the following we use the terminology as in [DKS20]. Let Ξ_0 be a finite partition of $\mathbf{Q}$ of dyadic cubes from $\mathcal{D}$. We say a partition Ξ' of $\mathbf{Q}$ is an *elementary extension* of Ξ_0 if it can be obtained by uniformly splitting some of its cubes into 2^d equal sized disjoint cubes lying in $\mathcal{D}$ with half side length. We call a partition Ξ *dyadic subdivision* of an initial partition Ξ_0 if it is obtained from the partition Ξ_0 with the help of a finite number of elementary extensions.

Proposition 3.7 ([BS67], [Bor71]). *Let Ξ_0 be a finite partition of $\boldsymbol{Q}$ with dyadic cubes from $\mathcal{D}$ and suppose there exists $\varepsilon > 0$ and a subset $\Xi_0' \subset \Xi_0$ such that*

$$\sum_{Q \in \Xi_0 \setminus \Xi_0'} \Lambda(Q) \leq \varepsilon \text{ and } \sum_{Q \in \Xi_0'} J(Q) \leq \varepsilon.$$

Let $(P_k)_{k \in \mathbb{N}}$ denote a sequence of dyadic partitions obtained recursively as follows: We set $P_0 := \Xi_0$ and, for $k \in \mathbb{N}$, we construct an elementary extension P_k of P_{k-1} by subdividing all cubes $Q \in P_{k-1}$ for which

$$\mathfrak{I}_{J,a}(Q) \geq 2^{-da} G_a(P_{k-1})$$

with $G_a(P_{k-1}) := \max_{Q \in P_{k-1}} \mathfrak{I}_{J,a}(Q)$, into 2^d equal sized cubes. Then, for all $k \in \mathbb{N}$, we have

$$G_a(P_k) = \max_{Q \in P_k} \mathfrak{I}_{J,a}(Q) \leq C \varepsilon^{\min(1,a)} (N_k - N_0)^{-(1+a)} J(\boldsymbol{Q})$$

with $N_k := \operatorname{card}(P_k)$, $k \in \mathbb{N}_0$, and the constant $C > 0$ depends only on a and d. In particular, there exists $C' > 0$ such that for all $n > N_0$,

$$\gamma_{\mathfrak{I}_{J,a},n} \leq C' \varepsilon^{\min(1,a)} n^{-(1+a)} J(\boldsymbol{Q}).$$

Proof. Here, we follow closely [DKS20]. Without loss of generality we may assume $J(\mathbf{Q}) \leq 1$. Fix $k \in \mathbb{N}$ and let S_k denote the set of all cubes from P_{k-1} that are subdivided to obtain P_k. Further, let $S_k^1, S_k^2 \subset S_k$ with $S_k = S_k^1 \cup S_k^2$,

$$\bigcup_{Q \in S_k^1} Q \subset \bigcup_{Q \in \Xi_0 \setminus \Xi_0'} Q,$$

and

$$\bigcup_{Q \in S_k^2} Q \subset \bigcup_{Q \in \Xi_0'} Q.$$

We define $t_k := \operatorname{card}(S_k)$, $t_{1,k} := \operatorname{card}(S_k^1)$, and $t_{2,k} := \operatorname{card}(S_k^2)$. By the definition of

P_k, we have $\min_{Q\in S_k^i} \mathfrak{I}_{J,a}(Q) \geq 2^{-da} G_a(P_{k-1})$ and we obtain

$$\left(2^{-da} G_a(P_{k-1})\right)^{\frac{1}{1+a}} \leq \min_{Q\in S_k^i} \mathfrak{I}_{J,a}(Q)^{\frac{1}{1+a}} \leq \frac{1}{t_{i,k}} \sum_{Q\in S_k^i} \Lambda(Q)^{\frac{a}{1+a}} J(Q)^{\frac{1}{1+a}}.$$

By the Hölder inequality and the superadditivity of J, we obtain for $i = 1,2$,

$$\left(2^{-da} G_a(P_{k-1})\right)^{\frac{1}{1+a}} \leq \frac{1}{t_{i,k}} \left(\sum_{Q\in S_k^i} \Lambda(Q)\right)^{\frac{a}{1+a}} \left(\sum_{Q\in S_k^i} J(Q)\right)^{\frac{1}{1+a}} \leq \frac{1}{t_{i,k}} \varepsilon^{\frac{\min(1,a)}{1+a}}.$$

This is equivalent to

$$t_{i,k} \leq 2^{\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} \left(G_a(P_{k-1})\right)^{-\frac{1}{1+a}}.$$

Since $t_k = t_{1,k} + t_{2,k}$, we have $t_k \leq 2^{1+\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} \left(G_a(P_{k-1})\right)^{-\frac{1}{1+a}}$. By the definition of the dyadic subdivision P_j, $j \in \mathbb{N}$,

$$\begin{aligned} G_a(P_j) &\leq \max\left\{\max_{Q\in P_j\cap P_{j-1}} \mathfrak{I}_{J,a}(Q), \max_{Q\in P_j\setminus P_{j-1}} \mathfrak{I}_{J,a}(Q)\right\} \\ &\leq \max\left\{2^{-ad} G_a(P_{j-1}), \max_{Q\in S_j} 2^{-ad} \mathfrak{I}_{J,a}(Q)\right\} \\ &\leq 2^{-ad} G_a(P_{j-1}). \end{aligned} \tag{3.3.1}$$

Now, applying (3.3.1) recursively, for all integers $j \leq k$, we obtain

$$G_a(P_{k-1}) \leq 2^{-ad(k-j)} G_a(P_{j-1}).$$

Since for all $j \in \mathbb{N}$ we have $N_j - N_{j-1} = (2^d - 1) t_j$. Hence, for all $k \in \mathbb{N}$, we deduce

$$\begin{aligned} N_k - N_0 &= (2^d - 1) \sum_{j=1}^{k} t_j \\ &\leq 2^{1+\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} (2^d - 1) \sum_{j=1}^{k} G_a(P_{j-1})^{-\frac{1}{1+a}} \\ &\leq 2^{1+\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} (2^d - 1) G_a(P_{k-1})^{-\frac{1}{1+a}} \sum_{j=1}^{k} 2^{-\frac{ad}{1+a}(k-j)} \\ &\leq \left(1 - 2^{-\frac{ad}{1+a}}\right)^{-1} 2^{1+\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} (2^d - 1) G_a(P_{k-1})^{-\frac{1}{1+a}} \\ &\leq \left(1 - 2^{-\frac{ad}{1+a}}\right)^{-1} 2^{1+\frac{ad}{1+a}} \varepsilon^{\frac{\min(1,a)}{1+a}} (2^d - 1) 2^{-\frac{ad}{1+a}} G_a(P_k)^{-\frac{1}{1+a}}. \end{aligned}$$

This proves our first claim. For the second claim note that, since $\lim_{k\to\infty} N_k = \infty$, for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $N_{k-1} \le n < N_k$. Furthermore, we always have $N_{k-1} \le N_k \le 2^d N_{k-1}$. Thus, by combining both inequalities, we obtain

$$\begin{aligned} \gamma_{\mathfrak{J}_{J,a},n} &\le \gamma_{\mathfrak{J}_{J,a},N_{k-1}} \\ &\le C\varepsilon^{\min(1,a)} \left(N_{k-1} - N_0\right)^{-(1+a)} \\ &\le C\varepsilon^{\min(1,a)} \left(2^{-d} n - N_0\right)^{-(1+a)}. \end{aligned}$$ □

Definition 3.8. We call J a singular function with respect to Λ if for every $\varepsilon > 0$ there exist two partitions $\Xi_0' \subset \Xi_0 \subset \mathcal{D}$ of $\mathbf{Q}$ such that

$$\sum_{Q \in \Xi_0 \setminus \Xi_0'} \Lambda(Q) \le \varepsilon \text{ and } \sum_{Q \in \Xi_0'} J(Q) \le \varepsilon.$$

Remark 3.9. Since $\mathcal{D}$ is a semiring of sets, it follows that a measure ν which is singular with respect to the Lebesgue measure, is also singular as a function $J = \nu$ in the sense of Definition 3.8.

As an immediate corollary of Proposition 3.7, we obtain the following statement by [Bor71].

Corollary 3.10. *We always have*

$$\gamma_{\mathfrak{J}_{J,a},n} = O\left(n^{-(1+a)}\right) \textit{ and } \mathcal{M}_{\mathfrak{J}_{J,a}}(x) = O\left(x^{1/(1+a)}\right).$$

If additionally J is singular, then

$$\gamma_{\mathfrak{J}_{J,a},n} = o\left(n^{-(1+a)}\right) \textit{ and } \mathcal{M}_{\mathfrak{J}_{J,a}}(x) = o\left(x^{1/(1+a)}\right).$$

Using Proposition 3.1, we are able to extend the class of set functions considered in [BS67, Theorem 2.1] (i.e. we allow set functions $\mathfrak{J}$ for which $\mathfrak{J}$ is only assumed to be non-negative, monotone, and $\dim_\infty(\mathfrak{J}) > 0$). We obtain the following estimate for the upper exponent of divergence of $\gamma_{\mathfrak{J},n}$ given by

$$\overline{\alpha}_{\mathfrak{J}} := \limsup_{n\to\infty} \frac{\log\left(\gamma_{\mathfrak{J},n}\right)}{\log(n)} \text{ and } \underline{\alpha}_{\mathfrak{J}} := \liminf_{n\to\infty} \frac{\log\left(\gamma_{\mathfrak{J},n}\right)}{\log(n)}.$$

Proposition 3.11. *If* $\dim_\infty(\mathfrak{J}) > 0$*, then*

$$-\frac{1}{\overline{h}_{\mathfrak{J}}} = \overline{\alpha}_{\mathfrak{J}} \le -\frac{1}{q_{\mathfrak{J}}^N} \le -\frac{\dim_\infty(\mathfrak{J})}{\overline{\dim}_M(\operatorname{supp}(\mathfrak{J}))} \textit{ and } -\frac{1}{\underline{h}_{\mathfrak{J}}} = \underline{\alpha}_{\mathfrak{J}}.$$

In particular, for $\mathfrak{J} = \mathfrak{J}_{J,a}$*, we have* $\dim_\infty(\mathfrak{J}) = \dim_\infty J + ad > 0$ *and*

$$-\frac{1}{\overline{h}_{\mathfrak{J}_{J,a}}} = \alpha_{\mathfrak{J}_{J,a}} \leq -\frac{1}{q^N_{\mathfrak{J}_{J,a}}} \leq -\frac{\overline{\dim}_M((J)) + ad}{\overline{\dim}_M(\operatorname{supp}(J))} \leq -(1+a).$$

Remark 3.12. If $\tau^N_{\mathfrak{J}_{J,a}}(q) < d(1-q(1+a))$ for some $q \in (0,1)$, then this estimate improves the corresponding results of [Bor71; BS67, Theorem 2.1], where only $\alpha_{\mathfrak{J}_{J,a}} \leq -(1+a)$ has been shown.

Proof. For all $\varepsilon > 0$, we have for n large

$$\mathcal{M}_{\mathfrak{J}}\left(n^{1/(\overline{h}_{\mathfrak{J}}+\varepsilon)}\right) \leq n,$$

this gives $\min_{P\in\Pi_{\mathfrak{J}},\operatorname{card}(P)\leq n} \max_{Q\in P} \mathfrak{J}(Q) \leq n^{-1/(\overline{h}_{\mathfrak{J}}+\varepsilon)}$. Thus, in tandem with Lemma 2.25 and Proposition 3.1, we see that

$$\alpha_{\mathfrak{J}} \leq -1/\overline{h}_{\mathfrak{J}} \leq -1/q^N_{\mathfrak{J}} \leq -\dim_\infty(\mathfrak{J})/\overline{\dim}_M(\operatorname{supp}(\mathfrak{J})).$$

In particular, since $q^N_{\mathfrak{J}} \geq 0$, we have $\alpha_{\mathfrak{J}} < 0$. To prove the equality, it is left to show $\alpha_{\mathfrak{J}} \geq -1/\overline{h}_{\mathfrak{J}}$. First, assume $\alpha_{\mathfrak{J}} > -\infty$, then for $\varepsilon > 0$ with $\alpha_{\mathfrak{J}} + \varepsilon < 0$ and n large, we have

$$\inf_{\substack{P\in\Pi_{\mathfrak{J}},\\ \operatorname{card}(P)\leq n}} \max_{Q\in P} \mathfrak{J}(Q) \leq n^{\alpha_{\mathfrak{J}}+\varepsilon}.$$

By the definition of infimum, there exists $P' \in \Pi_{\mathfrak{J}}$ with $\operatorname{card}(P') \leq n$ such that

$$\max_{Q\in P'} \mathfrak{J}(Q) \leq \left(1+\frac{2}{3}\right) n^{\alpha_{\mathfrak{J}}+\varepsilon} < 2n^{\alpha_{\mathfrak{J}}+\varepsilon},$$

implying $\mathcal{M}_{\mathfrak{J}}\left(n^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2\right) \leq n$. Moreover, for each $x \geq 1$ there exists $m \in \mathbb{N}$ such that

$$m^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2 \leq x < (m+1)^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2.$$

Consequently, for x large, we have

$$\begin{aligned}\frac{\log\left(\mathcal{M}_{\mathfrak{J}}(x)\right)}{\log(x)} &\leq \frac{\log\left(\mathcal{M}_{\mathfrak{J}}\left((m+1)^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2\right)\right)}{\log(m^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2)} \\ &\leq \frac{\log(m+1)}{\log(m^{-(\alpha_{\mathfrak{J}}+\varepsilon)}/2)}.\end{aligned}$$

Therefore, we infer that $\overline{h}_{\mathfrak{J}} \leq -1/\alpha_{\mathfrak{J}}$ or equivalently $\alpha_{\mathfrak{J}} \geq -1/\overline{h}_{\mathfrak{J}}$. In the case $\alpha_{\mathfrak{J}} = -\infty$ it follows in a similar way that $\overline{h}_{\mathfrak{J}} = 0$. Now, we consider the special case $\mathfrak{J} = \mathfrak{J}_{J,a}$. Observe that $\tau^N_{\mathfrak{J}_{J,a}}(q) = \tau^N_J(q) - adq$ for $q \geq 0$ and $\tau^N_J(0) \leq d$. From the fact that J is sub-additive, it follows that $\tau_J^N(1) \le 0$. We only have to consider the case $\tau_J^N(1) > -\infty$. Since τ_J^N is convex, for every $q \in [0,1]$, we deduce

$$\tau_{\mathfrak{J}_{J,a}}^N(q) = \tau_J^N(q) - adq \le \tau_J^N(0)(1-q) - adq \le d(1-q) - adq.$$

This implies $q_{\mathfrak{J}_{J,a}}^N \le \tau_J^N(0)/(\tau_J^N(0)+ad) \le 1/(1+a)$. From Proposition 3.1, we deduce

$$-\frac{1}{\overline{h}_{\mathfrak{J}}} \le -\frac{1}{q_{\mathfrak{J}_{J,a}}^N} \le -\frac{\overline{\dim}_M(\operatorname{supp}(J))+ad}{\overline{\dim}_M(\operatorname{supp}(J))} \le -(1+a). \qquad \square$$

The following proposition establishes an upper bound of $\underline{h}_{\mathfrak{J}_{J,a}}$ in terms of the lower Minkowski dimension of J and the lower ∞-dimension of J.

Proposition 3.13. *If* $\dim_\infty(J) \in [0,\infty)$*, then we have*

$$\underline{h}_{\mathfrak{J}_{J,a}} \le \frac{\underline{\dim}_M(\operatorname{supp}(J))}{ad+\dim_\infty(J)}.$$

Proof. We only consider the case $\dim_\infty(J) > 0$. The case $\dim_\infty(J) = 0$ follows along the same lines. Let $0 < s < \dim_\infty(J)$ and set $a := r/d$. Then, for n large, we have

$$\max_{Q\in\mathcal{D}_n^N} \mathfrak{J}(Q)\Lambda(Q)^a \le 2^{-(s+ad)n} < 2^{-(s+ad)n+1}.$$

This implies

$$\mathcal{M}_{\mathfrak{J}_{J,a}}\left(2^{-(s+ad)n+1}\right) \le 2^{n\tau_{J,n}(0)}.$$

Therefore, we obtain

$$\begin{aligned}
\underline{h}_{\mathfrak{J}_{\nu,a}} &\le \liminf_{n\to\infty} \frac{\log\left(\mathcal{M}_{\mathfrak{J}_{\nu,a}}\left(2^{(s+ad)n-1}\right)\right)}{\log\left(2^{(s+ad)n-1}\right)} \\
&\le \liminf_{n\to\infty} \frac{\log\left(2^{n\beta_{\nu,n}(0)}\right)}{\log\left(2^{(s+ad)n-1}\right)} \\
&= \liminf_{n\to\infty} \frac{\tau_{J,n}(0)}{(s+ad)-1/n} = \frac{\underline{\dim}_M(\operatorname{supp}(J))}{ad+s}.
\end{aligned}$$

Now, $s \uparrow \dim_\infty(J)$ gives $\underline{h}_{\mathfrak{J}_{J,a}} \le \underline{\dim}_M(\operatorname{supp}(J))/(ad+\dim_\infty(J))$. $\square$

3.4 Coarse multifractal analysis

Throughout this section let $\mathfrak{J}$ be a non-negative, monotone and locally non-vanishing set function defined on the set of dyadic cubes $\mathcal{D}$. We additionally assume that

(A1) there exists $a > 0$ and $b \in \mathbb{R}$ such that $\tau_{\mathfrak{J},n}^{D/N}(a) \geq b$ for all n large enough (excluding trivial cases),

(A2) the maximal asymptotic direction of $\tau_{\mathfrak{J}}^{N}$ is negative, i.e. $\dim_\infty(\mathfrak{J}) > 0$ (this generalizes the assumption $\dim_\infty(\nu) - d + 2 > 0$).

Lemma 3.14. *Under the assumptions (A1) and (A2) with a and b as determined there and $L := (b-d)/a < 0$, for all n large enough and $q \geq 0$, we have*

$$b + qL \leq \tau_{\mathfrak{J},n}^{D/N}(q).$$

In particular, $-\infty < \liminf_{n\to\infty} \tau_{\mathfrak{J},n}^{D/N}(q)$ *and* $\dim_\infty(\mathfrak{J}) \leq -L$.

Proof. By our assumptions, we have $\dim_\infty(\mathfrak{J}) > 0$, therefore, for n large, $\tau_{\mathfrak{J},n}^{D/N}$ is monotone decreasing and also $b \leq \tau_{\mathfrak{J},n}^{D/N}(a)$. By the definition of $\tau_{\mathfrak{J},n}^{D/N}$ we have $\tau_{\mathfrak{J},n}^{D/N}(0) \leq d$ for all $n \in \mathbb{N}$ and the convexity of $\tau_{\mathfrak{J},n}^{D/N}$ implies for all $q \in [0, a]$

$$\tau_{\mathfrak{J},n}^{D/N}(q) \leq \tau_{\mathfrak{J},n}^{D/N}(0) + \frac{q\left(\tau_{\mathfrak{J},n}^{D/N}(a) - \tau_{\mathfrak{J},n}^{D/N}(0)\right)}{a}.$$

In particular, by Theorem A.5, the convexity of $\tau_{\mathfrak{J},n}^{D/N}$ implies for $q > a$

$$\frac{\tau_{\mathfrak{J},n}^{D/N}(a) - \tau_{\mathfrak{J},n}^{D/N}(0)}{a} \leq \frac{\tau_{\mathfrak{J},n}^{D/N}(q) - \tau_{\mathfrak{J},n}^{D/N}(0)}{q}.$$

Thus, we obtain

$$\begin{aligned} b + q(b-d)/a &\leq \tau_{\mathfrak{J},n}^{D/N}(0) + \frac{q\left(\tau_{\mathfrak{J},n}^{D/N}(a) - \tau_{\mathfrak{J},n}^{D/N}(0)\right)}{a} \\ &\leq \tau_{\mathfrak{J},n}^{D/N}(0) + \frac{q\left(\tau_{\mathfrak{J},n}^{D/N}(q) - \tau_{\mathfrak{J},n}^{D/N}(0)\right)}{q} = \tau_{\mathfrak{J},n}^{D/N}(q). \end{aligned}$$

Since $\tau_{\mathfrak{J},n}^{D/N}$ is decreasing with $0 \leq \tau_{\mathfrak{J},n}^{D/N}(0) \leq d$ and $\tau_{\mathfrak{J},n}^{D/N}(a) \geq b$, we obtain for all $q \in [0, a]$

$$b + q(b-d)/a \leq b \leq \tau_{\mathfrak{J},n}^{D/N}(a) \leq \tau_{\mathfrak{J},n}^{D/N}(q). \qquad \square$$

Remark 3.15. If $\dim_\infty(\nu) > d-2$ and $\nu(\mathring{\mathbf{Q}}) > 0$, then the assumptions of Lemma 3.14 are satisfied for $\tau_{\mathfrak{J}_\nu,n}^{D/N}$. This follows from

$$\tau_{\mathfrak{J}_\nu,n}^{D/N}(1) \geq d - 2 + \beta_n^{D/N}(1) \geq (d-2) - \delta$$

for $\delta > 0$ and n sufficiently large, where we used $\nu(\mathring{\mathbf{Q}}) > 0$ for the Dirichlet case. Consequently, $\beta_n^D(1) \to 0$, for $n \to \infty$.

Lemma 3.16. *For $\alpha \in (0, \dim_\infty(\mathfrak{J}))$ and n large, we have*

$$\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n) = 0.$$

In particular,

$$\overline{F}_{\mathfrak{J}}^{D/N} = \sup_{\alpha \geq \dim_\infty(\mathfrak{J})} \limsup_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n)\right)}{\alpha \log(2^n)}$$

and

$$\underline{F}_{\mathfrak{J}}^{D/N} = \sup_{\alpha \geq \dim_\infty(\mathfrak{J})} \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n)\right)}{\alpha \log(2^n)}.$$

Proof. For fixed $\alpha > 0$ with $\alpha < \dim_\infty(\mathfrak{J})$, by the definition of $\dim_\infty(\mathfrak{J})$, for n large we have $\max_{Q \in \mathcal{D}_n^{D/N}} \mathfrak{J}(Q) \leq 2^{-\alpha n}$. Hence, for all $Q \in \mathcal{D}_n^N$, we have $\mathfrak{J}(Q) \leq 2^{-n\alpha}$. For every $0 < \alpha' < \alpha$, it follows that $\mathcal{N}_{\alpha',\mathfrak{J}}^{D/N}(n) = 0$. This proves the claim. □

We need the following elementary observation from large deviation theory which seems not to be standard in the relevant literature. For this purpose, we need some standard facts about convex functions, which are summarized in Appendix A.1.

Lemma 3.17. *Suppose $(X_n)_n$ are real-valued random variables on some probability spaces $(\Omega_n, \mathcal{A}_n, \mu_n)_n$ such that the rate function $\mathfrak{c}(t) := \limsup_{n\to\infty} \mathfrak{c}_n(t)$ is a proper convex function with $\mathfrak{c}_n(t) := a_n^{-1} \log\left(\int \exp(tX_n)\, \mathrm{d}\mu_n\right)$, $t \in \mathbb{R}$, $a_n \to \infty$ and such that 0 belongs to the interior of the domain of finiteness $\{t \in \mathbb{R} : \mathfrak{c}(t) < \infty\}$. Let $I = (a, d)$ be an open interval containing the subdifferential $\partial\mathfrak{c}(0) = [b, c]$ of $\mathfrak{c}$ in 0. Then there exists $r > 0$ such that for all n sufficiently large,*

$$\mu_n\left(a_n^{-1} X_n \notin I\right) \leq 2 \exp(-r a_n).$$

Proof. We assume that $\partial\mathfrak{c}(0) = [b, c]$ and $I = (a, d)$ with $a < b \leq c < d$. First note that the assumptions ensure that $-\infty < b \leq c < \infty$. By the Chebychev inequality for all $q > 0$ and $n \in \mathbb{N}$, we have

$$\mu_n\left(a_n^{-1} X_n \geq d\right) = \mu_n(qX_n \geq q a_n d) \leq \exp(-q a_n d) \int \exp(qX_n)\, \mathrm{d}\mu_n,$$

implying

$$\limsup_{n\to\infty} a_n^{-1} \log\left(\mu_n\left(a_n^{-1} X_n \geq d\right)\right) \leq \inf_{q>0} \mathfrak{c}(q) - qd = \inf_{q\in\mathbb{R}} \mathfrak{c}(q) - qd \leq 0,$$

where the equality follows from the assumption $c < d$,

$$\mathfrak{c}(q) - qd \geq \mathfrak{c}(0) + (q-0)c - qd = (c-d)q \geq 0$$

for all $q \leq 0$, $\mathfrak{c}(0) = 0$, and the continuity of $\mathfrak{c}$ at 0. Similarly, we find

$$\limsup_{n\to\infty} a_n^{-1} \log\left(\mu_n\left(a_n^{-1}X_n \leq a\right)\right) \leq \inf_{q<0} \mathfrak{c}(q) - qa = \inf_{q\in\mathbb{R}} \mathfrak{c}(q) - qa.$$

We are left to show that both upper bounds are negative. We show the first case by contradiction – the other case follows in exactly the same way. Assuming $\inf_{q\in\mathbb{R}} \mathfrak{c}(q) - qd \geq 0$, then for all $q \in \mathbb{R}$ we have $\mathfrak{c}(q) - qd \geq 0$, or after rearranging, $\mathfrak{c}(q) - \mathfrak{c}(0) \geq dq$. This means, according to the definition of the sub-differential, that $d \in \partial\mathfrak{c}(0)$, contradicting our assumptions. □

Proposition 3.18. *For a subsequence $(n_k)_k$ define the convex function on $\mathbb{R}_{\geq 0}$ by $B := \limsup_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N}$ and for some $q \geq 0$, we assume $B(q) = \lim_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N}(q)$ and set $[a',b'] := -\partial B(q)$. Then we have $a' \geq \dim_\infty(\mathfrak{J})$ and*

$$\begin{aligned}\frac{a'q + B(q)}{b'} &\leq \sup_{\alpha>b'} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha\log(2^{n_k})} \\ &\leq \sup_{\alpha\geq\dim_\infty(\mathfrak{J})} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha\log(2^{n_k})} = \sup_{\alpha>0} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha\log(2^{n_k})}.\end{aligned}$$

Moreover, if $B(q) = \tau_{\mathfrak{J}}^{D/N}(q)$, then $[a,b] = -\partial\tau_{\mathfrak{J}}^{D/N}(q) \supset -\partial B(q)$. Further, if additionally $0 \leq q \leq q_{\mathfrak{J}}^{D/N}$, then

$$\frac{aq + \tau_{\mathfrak{J}}^{D/N}(q)}{b} \leq \frac{a'q + B(q)}{b'}.$$

Proof. Without loss of generality we can assume $b' < \infty$. Moreover, $\dim_\infty(\mathfrak{J}) > 0$ implies $b' \geq a' \geq \dim_\infty(\mathfrak{J}) > 0$. Indeed, observe that B is again a convex function on $\mathbb{R}$. Thus, by the definition of the sub-differential, we have for all $x > 0$

$$B(q) - a'(x-q) \leq B(x) \leq \tau_{\mathfrak{J}}^{D/N}(x) \leq \tau_{\mathfrak{J}}^{N}(x) \leq -x\dim_\infty(\mathfrak{J}) + d,$$

which gives $a' \geq \dim_\infty(\mathfrak{J}) > 0$. Let $q \geq 0$. Now, for all $k \in \mathbb{N}$ and $s < a' \leq b' < t$, we have

$$\begin{aligned}
\mathcal{N}_{t,\mathfrak{J}}^{D/N}(n_k) &\geq \operatorname{card}(\underbrace{\left\{C \in \mathcal{D}_{n_k}^{D/N} : 2^{-sn_k} > \mathfrak{J}(C) > 2^{-tn_k}\right\}}_{=:L_{n_k}^{s,t}}) \\
&\geq \sum_{C \in L_{n_k}^{s,t}} \mathfrak{J}(C)^q 2^{sn_k q} \\
&= 2^{sn_k q + n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)} \sum_{C \in \mathcal{D}_{n_k}^{D/N}} \mathbb{1}_{L_{n_k}^{s,t}}(C) \mathfrak{J}(C)^q 2^{-n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)} \\
&= 2^{sn_k q + n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)} \left(1 - \sum_{C \in \mathcal{D}_{n_k}^{D/N}} \mathbb{1}_{\left(L_{n_k}^{s,t}\right)^{\complement}}(C) \mathfrak{J}(C)^q 2^{-n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)}\right).
\end{aligned}$$

We use the lower large deviation principle for the process $X_k(C) := \log(\mathfrak{J}(C))$ with probability measure on $\mathcal{D}_{n_k}^{D/N}$ given by $\mu_k(\{C\}) := \mathfrak{J}(C)^q 2^{-n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)}$. We find for the free energy function

$$\begin{aligned}
\mathfrak{c}(x) &:= \limsup_{k\to\infty} \frac{1}{\log(2^{n_k})} \log\left(\mathbb{E}_{\mu_k}(\exp(xX_k))\right) \\
&= \limsup_{k\to\infty} \frac{1}{\log(2^{n_k})} \log\left(\sum_{C \in \mathcal{D}_{n_k}^{D/N}} \mathfrak{J}(C)^{x+q} / 2^{n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)}\right) \\
&= \limsup_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N}(q+x) - B(q) = B(x+q) - B(q),
\end{aligned}$$

with $-\partial\mathfrak{c}(0) = [a', b'] \subset (s, t)$ and hence there exists a constant $r > 0$ depending on s, t, and q such that for k large by Lemma 3.17

$$\sum_{C \in \mathcal{D}_{n_k}^{D/N}} \mathbb{1}_{\left(L_{n_k}^{s,t}\right)^{\complement}}(C) \mathfrak{J}(C)^q / 2^{n_k \tau_{\mathfrak{J},n_k}^{D/N}(q)} = \mu_k\left(\frac{X_k}{\log(2^{n_k})} \notin (-t, -s)\right) \leq 2\exp(-rn_k).$$

Therefore,

$$\liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{t,\mathfrak{J}}^{D/N}(n_k)\right)}{\log(2^{n_k})} \geq sq + B(q),$$

for all $s < a'$ and $t > b'$. Hence, we have

$$\sup_{t>b'} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{t,\mathfrak{J}}^{D/N}(n_k)\right)}{t\log(2^{n_k})} \geq \sup_{t>b'} \frac{a'q + B(q)}{t} = \frac{a'q + B(q)}{b'}.$$

The fact that $-\partial\tau_{\mathfrak{J}}^{D/N}(q) \supset -\partial B(q)$ if $\tau_{\mathfrak{J}}^{D/N}(q) = B(q)$ follows immediately from $\limsup_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N} \leq \tau_{\mathfrak{J}}^{D/N}$. □

Corollary 3.19. *Let* $\mathfrak{J}(Q) := \nu(Q)\Lambda(Q)^\gamma$ *with* $Q \in \mathcal{D}$ *and* $\gamma > 0$. *Then* $\tau_{\mathfrak{J}}^{D/N}(q) = \beta_\nu^{D/N}(q) - \gamma dq$, $q \geq 0$, *and* $\dim_\infty(\mathfrak{J}) = \dim_\infty(\nu) + d\gamma > 0$. *Suppose there exists a subsequence* $(n_k)_k$ *and* $q \in [0,1]$ *such that* $\tau_{\mathfrak{J}}^{D/N}(q) = \lim_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N}(q)$. *Then for* $B := \limsup_{k\to\infty} \tau_{\mathfrak{J},n_k}^{D/N}$, *we have* $-\partial B(q) := [a', b'] \subset -\partial\tau_{\mathfrak{J}}^{D/N}(q) := [a,b]$ *and*

$$\frac{aq + \tau_{\mathfrak{J}}^{D/N}(q)}{b} \leq \frac{a'q + \tau_{\mathfrak{J}}^{D/N}(q)}{b'} \leq \sup_{\alpha \geq \dim_\infty(\nu) + d\gamma} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha \log(2^{n_k})}.$$

Proof. The first claim is obvious since $\gamma > 0$. The second inequality follows immediately from Proposition 3.18 and $\dim_\infty(\mathfrak{J}) = \dim_\infty(\nu) + d\gamma$. To prove the first inequality observe that $-\partial\tau_{\mathfrak{J}}^{D/N}(q) = [a_1 + \gamma d, b_1 + \gamma d]$ with $-\partial\beta_\nu^{D/N}(q) = [a_1, b_1]$. Using $[a', b'] \subset [a_1 + \gamma d, b_1 + \gamma d]$, $\tau_{\mathfrak{J}}^{D/N}(q) = \beta_\nu^{D/N}(q) - d\gamma q$, and $\beta_\nu^{D/N}(q) \geq 0$, we obtain

$$\begin{aligned}\frac{(a_1 + d\gamma)q + \tau_{\mathfrak{J}}^{D/N}(q)}{b_1 + \gamma d} = \frac{a_1 q + \beta_\nu^{D/N}(q)}{b_1 + \gamma d} &\leq \frac{(a_1 + d\gamma)q + \beta_\nu^{D/N}(q) - \gamma d}{b'} \\ &\leq \frac{a'q + \tau_{\mathfrak{J}}^{D/N}(q)}{b'} \\ &\leq \sup_{\alpha \geq \dim_\infty(\nu) + d\gamma} \liminf_{k\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha \log(2^{n_k})}.\end{aligned}$$ □

Proposition 3.20. *We have*

$$\overline{F}_{\mathfrak{J}}^{D/N} = q_{\mathfrak{J}}^{D/N}.$$

Proof. First, note that by Proposition 3.1, we always have

$$\overline{F}_{\mathfrak{J}}^{D/N} \leq q_{\mathfrak{J}}^{D/N}.$$

Hence, we can restrict our attention to the case $q_{\mathfrak{J}}^{D/N} > 0$. Further, by Lemma 3.14, we observe that, for n large, the family of convex functions $\left(\tau_{\mathfrak{J},n}^{D/N}\right)_n$ restricted to $\left[0, q_{\mathfrak{J}}^{D/N} + 1\right]$ only takes values in $\left[\left(q_{\mathfrak{J}}^{D/N} + 1\right)L + b, d\right]$ and on any compact interval $[c,e] \subset \left(0, q_{\mathfrak{J}}^{D/N} + 1\right)$, by Theorem A.5, for all $c \leq x \leq y \leq e$, we have

$$\frac{\tau_{\mathfrak{J},n}^{D/N}(x)-\tau_{\mathfrak{J},n}^{D/N}(0)}{x-0}\leq\frac{\tau_{\mathfrak{J},n}^{D/N}(y)-\tau_{\mathfrak{J},n}^{D/N}(x)}{y-x}\leq\frac{\tau_{\mathfrak{J},n}^{D/N}\left(q_{\mathfrak{J}}^{D/N}+1\right)-\tau_{\mathfrak{J},n}^{D/N}(y)}{q_{\mathfrak{J}}^{D/N}+1-y}.$$

By Lemma 3.14 and the fact $\tau_{\mathfrak{J},n}^{D/N}(0)\leq d$, we obtain

$$\frac{\left(q_{\mathfrak{J}}^{D/N}+1\right)L+b-d}{c}\leq\frac{\tau_{\mathfrak{J},n}^{D/N}(x)-\tau_{\mathfrak{J},n}^{D/N}(0)}{c}\leq\frac{\tau_{\mathfrak{J},n}^{D/N}(x)-\tau_{\mathfrak{J},n}^{D/N}(0)}{x-0}$$

and

$$\frac{\tau_{\mathfrak{J},n}^{D/N}\left(q_{\mathfrak{J}}^{D/N}+1\right)-\tau_{\mathfrak{J},n}^{D/N}(y)}{q_{\mathfrak{J}}^{D/N}+1-y}\leq\frac{d-\left(q_{\mathfrak{J}}^{D/N}+1\right)L-b}{q_{\mathfrak{J}}^{D/N}+1-e},$$

which implies

$$\left|\tau_{\mathfrak{J},n}^{D/N}(y)-\tau_{\mathfrak{J},n}^{D/N}(x)\right|\leq\max\left\{\frac{|b|-\left(q_{\mathfrak{J}}^{D/N}+1\right)L+d}{c},\frac{d-\left(q_{\mathfrak{J}}^{D/N}+1\right)L+|b|}{q_{\mathfrak{J}}^{D/N}+1-e}\right\}|x-y|.$$

Hence, $\left(\tau_{\mathfrak{J},n}^{D/N}|_{[c,e]}\right)_n$ is uniformly bounded and uniformly Lipschitz. Thus, by Arzelà–Ascoli relatively compact. Using this fact, we find a subsequence $(n_k)_k$ such that

$$\lim_{k\to\infty}\tau_{\mathfrak{J},n_k}^{D/N}\left(q_{\mathfrak{J}}^{D/N}\right)=\limsup_{n\to\infty}\tau_{\mathfrak{J},n}^{D/N}\left(q_{\mathfrak{J}}^{D/N}\right)=0$$

and $\tau_{\mathfrak{J},n_k}^{D/N}$ converges uniformly to the proper convex function B on

$$\left[q_{\mathfrak{J}}^{D/N}-\delta,q_{\mathfrak{J}}^{D/N}+\delta\right]\subset\left(0,q_{\mathfrak{J}}^{D/N}+1\right),$$

for δ sufficiently small. We put $[a,b]:=-\partial B\left(q_{\mathfrak{J}}^{D/N}\right)$. Since the points where B is differentiable are dense and since B is convex, we find for every $\delta>\varepsilon>0$ an element $q\in\left(q_{\mathfrak{J}}^{D/N}-\varepsilon,q_{\mathfrak{J}}^{D/N}\right)$ such that B is differentiable at q with $-B'(q)\in[b,b+\varepsilon]$. This follows from the fact that $\tau_{\mathfrak{J}}^{D/N}$ is a decreasing function and that the left-hand derivative of the convex function $\tau_{\mathfrak{J}}^{D/N}$ is left-hand continuous and non-decreasing (see Theorem A.5). Noting $B\leq\tau_{\mathfrak{J}}^{D/N}$, we have $-B'(q)\geq\dim_{\infty}(\mathfrak{J})$. Hence, from

Proposition 3.18 we deduce

$$\sup_{\alpha\geq\dim_{\infty}(\mathfrak{J})}\limsup_{n\to\infty}\frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n)\right)}{\alpha\log(2^{n})}\geq\sup_{\alpha>-B'(q)}\limsup_{k\to\infty}\frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha\log(2^{n_k})}$$
$$\geq\sup_{\alpha>-B'(q)}\liminf_{k\to\infty}\frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n_k)\right)}{\alpha\log(2^{n_k})}$$
$$\geq\frac{-B'(q)\,q+B(q)}{-B'(q)}\geq\frac{b\left(q_{\mathfrak{J}}^{D/N}-\varepsilon\right)}{b+\varepsilon}.$$

Taking the limit $\varepsilon\downarrow 0$ gives the assertion. □

Corollary 3.21. *We have* $\overline{F}_{\mathfrak{J}}^{N}=\overline{h}_{\mathfrak{J}}=-1/\alpha_{\mathfrak{J}}=q_{\mathfrak{J}}^{N}$. *Further, if* $\overline{F}_{\mathfrak{J}}^{D}=\overline{F}_{\mathfrak{J}}^{N}$, *then*

$$\overline{F}_{\mathfrak{J}}^{D}=\overline{h}_{\mathfrak{J}}=q_{\mathfrak{J}}^{D}=q_{\mathfrak{J}}^{N}.$$

By Proposition 3.20, we always have $\overline{F}_{\mathfrak{J}}^{D/N}=q_{\mathfrak{J}}^{D/N}$. It raises the question under which conditions $\overline{F}_{\mathfrak{J}}^{D/N}$ exists as a limit. For this purpose, we establish the following regularity conditions for $\mathfrak{J}$.

Definition 3.22. We define two notions of regularity.

1. We call $\mathfrak{J}$ *Neumann multifractal-regular (N-MF-regular)* if $\underline{F}_{\mathfrak{J}}^{N}=\overline{F}_{\mathfrak{J}}^{N}$ and *Dirichlet multifractal-regular (D-MF-regular)* if $\underline{F}_{\mathfrak{J}}^{D}=\overline{F}_{\mathfrak{J}}^{N}$.
2. We call $\mathfrak{J}$ *Dirichlet/Neumann partition function regular (D/N-PF-regular)* if
   - $q_{\mathfrak{J}}^{D/N}>0$ and $\tau_{\mathfrak{J}}^{D/N}(q)=\liminf_{n\to\infty}\tau_{\mathfrak{J},n}^{D/N}(q)$ for all $q\in\left(q_{\mathfrak{J}}^{D/N}-\varepsilon,q_{\mathfrak{J}}^{D/N}\right)$ for some $\varepsilon>0$, or
   - $q_{\mathfrak{J}}^{D/N}>0$ and $\tau_{\mathfrak{J}}^{D/N}\left(q_{\mathfrak{J}}^{D/N}\right)=\liminf_{n\to\infty}\tau_{\mathfrak{J},n}^{D/N}\left(q_{\mathfrak{J}}^{D/N}\right)$ and $\tau_{\mathfrak{J}}^{D/N}$ is differentiable at $q_{\mathfrak{J}}^{D/N}$.

Corollary 3.23. *If* $\mathfrak{J}$ *is Neumann N-MF-regular, then* $\overline{F}_{\mathfrak{J}}^{N}=\underline{F}_{\mathfrak{J}}^{N}=\overline{h}_{\mathfrak{J}}=\underline{h}_{\mathfrak{J}}=q_{\mathfrak{J}}^{N}$.

Proof. This follows from Proposition 3.1 and Proposition 3.20. □

Proposition 3.24. *If* $\mathfrak{J}$ *is Dirichlet/Neumann PF-regular, then*

$$\underline{F}_{\mathfrak{J}}^{D/N}=q_{\mathfrak{J}}^{D/N}=\overline{F}_{\mathfrak{J}}^{D/N}.$$

In particular, we have

$$\mathfrak{J} \text{ is N-PF-regular} \implies \mathfrak{J} \text{ is N-MF-regular}.$$

Proof. Due to Proposition 3.1, we can restrict our attention to the case $q_{\mathfrak{J}}^{D/N} > 0$. First, we assume there exists $\varepsilon > 0$ such that

$$\tau_{\mathfrak{J}}^{D/N}(q) = \liminf_{n\to\infty} \tau_{\mathfrak{J},n}^{D/N}(q)$$

for all $q \in \left(q_{\mathfrak{J}}^{D/N} - \varepsilon, q_{\mathfrak{J}}^{D/N}\right)$ and set $[a,b] := -\partial\tau_{\mathfrak{J}}^{D/N}\left(q_{\mathfrak{J}}^{D/N}\right)$. Then by the convexity of $\tau_{\mathfrak{J}}^{D/N}$ we find for every $\varepsilon \in \left(0, q_{\mathfrak{J}}^{D/N}\right)$ an element $q \in \left(q_{\mathfrak{J}}^{D/N} - \varepsilon, q_{\mathfrak{J}}^{D/N}\right)$ such that $\tau_{\mathfrak{J}}^{D/N}$ is differentiable at q with $-\left(\tau_{\mathfrak{J}}^{D/N}\right)'(q) \in [b, b+\varepsilon]$ since the points where $\tau_{\mathfrak{J}}^{D/N}$ is differentiable on $(0,\infty)$ lie dense in $(0,\infty)$. This follows from the fact that $\tau_{\mathfrak{J}}^{D/N}$ is a decreasing function and that the left-hand derivative of the convex function $\tau_{\mathfrak{J}}^{D/N}$ is left-hand continuous and non-decreasing (see Theorem A.5). Then by Proposition 3.18 we have

$$\begin{aligned}\sup_{\alpha\geq\dim(\mathfrak{J})} \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n)\right)}{\alpha\log(2^n)} &\geq \sup_{\alpha > -\left(\tau_{\mathfrak{J}}^{D/N}\right)'(q)} \liminf_{n\to\infty} \frac{\log\left(\mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n)\right)}{\alpha\log(2^n)} \\ &\geq \frac{-\left(\tau_{\mathfrak{J}}^{D/N}\right)'(q)\,q + \tau_{\mathfrak{J}}^{D/N}(q)}{-\left(\tau_{\mathfrak{J}}^{D/N}\right)'(q)} \geq \frac{b\left(q_{\mathfrak{J}}^{D/N} - \varepsilon\right)}{b+\varepsilon}.\end{aligned}$$

Taking the limit $\varepsilon \to 0$ proves the claim in this situation. The case that $\tau_{\mathfrak{J}}^{D/N}$ exists as a limit in $q_{\mathfrak{J}}^{D/N}$ and is differentiable at $q_{\mathfrak{J}}^{D/N}$ is covered by Proposition 3.18. □

Corollary 3.25. *If $\mathfrak{J}$ is Neumann PF-regular, then*

$$\underline{F}_{\mathfrak{J}}^{N} = q_{\mathfrak{J}}^{N} = \overline{h}_{\mathfrak{J}} = \underline{h}_{\mathfrak{J}}.$$

Proof. This follows immediately from Proposition 3.1 and Proposition 3.24. □

Chapter 4

Spectral dimension and spectral asymptotic for Kreĭn–Feller operators for the one-dimensional case

Throughout this section, we consider a finite Borel measure ν on $(0,1)$. Further, we assume $\mathrm{card}(\mathrm{supp}(\nu)) = \infty$ to exclude trivial cases. This chapter is devoted to study the spectral dimension and spectral asymptotic of Kreĭn–Feller operators for the case $d = 1$ and $\Omega = (0,1)$. Since the spectral dimension with respect to Dirichlet or Neumann boundary conditions is the same (see Remark 4.5), we will restrict our attention to study Kreĭn–Feller operators with respect to Dirichlet boundary conditions. A big advantage of the case $d = 1$ is that by Proposition A.17 the Sobolev space $H^1((a,b))$ is compactly embedded into $C([a,b])$ with $a < b$. Thus, in this chapter for $f \in H^1 = H^1(0,1)$ we will always pick the continuous representative. Recall that the set of dyadic intervals of $(0,1]$ is given by

$$\mathcal{D} = \{(2^{-n}k, 2^{-n}(k+1)] : k = 0,\dots,2^n - 1 \text{ with } n \in \mathbb{N}\}.$$

Most of the main results of this chapter rely heavily on the theory developed in Chapter 3 applied to $\mathfrak{J}_\nu(Q) = \mathfrak{J}_{\nu,1,1}(Q) = \Lambda(Q)\nu(Q)$, $Q \in \mathcal{D}$, which naturally arises as optimal embedding constant of the embedding of $H_0^1(Q)$ into $L_\nu^2(Q)$:

$$\int_Q f^2 \,\mathrm{d}\nu \le \mathfrak{J}_\nu(Q) \int_Q (\nabla f)^2 \,\mathrm{d}\Lambda \text{ with } f \in H_0^1(Q),\ Q \in \mathcal{D}.$$

In the following, we collect further simplifications that arise in the case $d = 1$:

1. $\beta_\nu^D(q) = \beta_\nu^N(q)$, $q \ge 0$, by Proposition 2.41.

2. $\tau_{n,\mathfrak{J}_\nu}^{D/N}(q) = \beta_n^{D/N}(q) - q$, $q \geq 0$, $n \in \mathbb{N}$.
3. $\tau_{\mathfrak{J}_\nu}^{N}(q) = \beta_\nu^{D/N}(q) - q$, $q \geq 0$. In particular, we have $q_{\mathfrak{J}_\nu}^D = q_{\mathfrak{J}_\nu}^N$.
4. Since, for all $\alpha > 0$, $n \in \mathbb{N}$, we have $\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^D(n) \leq \mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n) \leq \mathcal{N}_{\mathfrak{J}_\nu,\alpha}^D(n) + 2$ and it follows $\overline{F}_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^D$ and $\underline{F}_{\mathfrak{J}_\nu}^N = \underline{F}_{\mathfrak{J}_\nu}^D$.

This justifies the following simplified notation:

$$\beta_\nu := \beta_\nu^{D/N},\ q_{\mathfrak{J}_\nu} := q_{\mathfrak{J}_\nu}^{D/N},\ \overline{F}_{\mathfrak{J}_\nu} := \overline{F}_{\mathfrak{J}_\nu}^{D/N} \text{ and } \underline{F}_{\mathfrak{J}_\nu} := \underline{F}_{\mathfrak{J}_\nu}^{D/N}.$$

The chapter is structured as follows. In Section 4.1, we establish lower bounds for the lower and upper spectral dimension in terms of the lower and upper optimized coarse multifractal dimension with respect to $\mathfrak{J}_\nu$. For this reason, we introduce the new notion of the lower and upper m-reduced partition ν-entropy. In Section 4.2, we present upper bounds of $\overline{s}_\nu^D$ and $\underline{s}_\nu^D$ in terms of $q_{\mathfrak{J}_\nu}$ and $\underline{\dim}_M(\nu)/(1+\dim_\infty(\nu))$, respectively. In Section 4.3, we present our main results – we prove that the upper spectral dimension is given by $q_{\mathfrak{J}_\nu}$ which can be geometrically interpreted as the unique intersection point of the L^q-spectrum and the identity map, provided $q_{\mathfrak{J}_\nu} > 0$. Further, we impose regularity conditions to ensure the existence of the spectral dimension and present general bounds in terms of fractal dimensions. In Section 4.4, we illustrate our general results, developed in Section 4.3, with a number of examples. More precisely, in Section 4.4.1, we compute the spectral dimension of weak Gibbs measures with or without overlap and obtain some refinement estimates under the assumption of the OSC. Moreover, in Section 4.4.2, we discuss an example for which the spectral dimension does not exist. Finally, we conclude by computing the spectral dimension for a class of purely atomic measures in Section 4.4.3.

4.1 Lower bounds for the spectral dimension

We start with the definition of an auxiliary target quantity.

Let Π_0 denote the set of finite disjoint collections of subintervals I of $(0,1]$, and for $m > 1$ and $x > 0$, set

$$\mathcal{N}_m^L(x) := \sup\left\{\operatorname{card}(P) : P \in \Pi_0 \mid \min_{I\in P} \nu\left(\left\langle \overline{I}\right\rangle_{1/m}\right)\Lambda\left(\left\langle \overline{I}\right\rangle_{1/m}\right) \geq \frac{4}{x(m-1)}\right\}.$$

Recall from Section 2.2.6 that $\left\langle \overline{I}\right\rangle_{1/m} \subset I$ denotes the interval of length $\Lambda(I)/m$ centered in $\overline{I}$. Then the *lower and upper m-reduced ν-partition entropy* is given by

$$\underline{h}_\nu^m := \liminf_{x\to\infty} \frac{\log\left(\mathcal{N}_m^L(x)\right)}{\log(x)} \text{ and } \overline{h}_\nu^m := \limsup_{x\to\infty} \frac{\log\left(\mathcal{N}_m^L(x)\right)}{\log(x)}.$$

Proposition 4.1. *For all $x > 0$, we have*

$$\mathcal{N}_m^L(x) \le N_\nu^D(x).$$

Proof. Let $P \in \Pi_0$ such that

$$\min_{I\in P} \nu\left(\left\langle \bar{I}\right\rangle_{1/m}\right)\Lambda\left(\left\langle \bar{I}\right\rangle_{1/m}\right) \ge 4/(x(m-1))$$

and write

$$P := \left\{I_1,\dots,I_{\mathrm{card}(P)}\right\}, \overline{I_i} = [a_i, b_i], \text{ and } \left\langle \overline{I_i}\right\rangle_{1/m} = [c_i, d_i],\ i = 1,\dots,\mathrm{card}(P).$$

For each $i = 1,\dots,\mathrm{card}(P)$, we define

$$f_i(y) := \frac{y-a_i}{c_i-a_i}\mathbb{1}_{[a_i,c_i)}(y) + \mathbb{1}_{[c_i,d_i]}(y) + \frac{b_i-y}{b_i-d_i}\mathbb{1}_{(d_i,b_i]}(y),\ y \in [a,b],$$

which is an element of H_0^1. Notice, by the definition of $\langle\cdot\rangle_{1/m}$, we have

$$c_i - a_i = b_i - d_i = (b_i - a_i)(1-1/m)/2.$$

Hence,

$$\begin{aligned}\frac{\int_{(0,1)} \nabla f_i^2(y)\,\mathrm{d}\Lambda(y)}{\int_{(0,1)} f_i^2(y)\,\mathrm{d}\nu(y)} &\le \frac{\dfrac{1}{c_i-a_i}+\dfrac{1}{b_i-d_i}}{\nu\left(\left\langle \bar{I}\right\rangle_{1/m}\right)}\\ &= \frac{4/(m-1)}{\nu\left(\left\langle \bar{I}\right\rangle_{1/m}\right)\Lambda\left(\left\langle \bar{I}\right\rangle_{1/m}\right)}\\ &\le x.\end{aligned}$$

Since the intervals $(I_i)_i$ are disjoint, the $(f_i)_i$ are mutually orthogonal both in L_ν^2 and in H_0^1, and we obtain that $\mathrm{span}\,(f_i : i = 1,\dots,\mathrm{card}(P))$ is a $\mathrm{card}(P)$-dimensional subspace of H_0^1. Thus, we deduce from Lemma 2.19,

$$\mathrm{card}(P) \le N_\nu^D(x).$$

Taking the supremum over all $P \in \Pi_0$ with

$$\min_{I\in P} \nu\left(\left\langle \bar{I}\right\rangle_{1/m}\right)\Lambda\left(\left\langle \bar{I}\right\rangle_{1/m}\right) \ge 4/(x(m-1))$$

proves the claim. □

Lemma 4.2. *Let $\alpha > 0$. For every $x > 2^{2+\alpha}$, we have*

$$N_\nu^D(x) \geq \mathcal{N}_3^L(x) \geq \frac{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)}{3} - 3$$

with $n_x^\alpha := \lfloor \log\left(\frac{x}{2}\right)/(\log(2)\alpha)\rfloor$, and with $x_n^\alpha := 2^{n\alpha+1}$, for every $n \in \mathbb{N}$, we have

$$N_\nu^D(x_n^\alpha) \geq \frac{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n)}{3} - 3.$$

Proof. For fixed $n \in \mathbb{N}$, $\alpha > 0$, let $(c_1, d_1], \ldots, \left(c_{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n)}, d_{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n)}\right]$ denote the intervals of $M_{\mathfrak{J}_\nu,n}^N(\alpha)$ ordered in the natural way, i.e. $c_i < d_i \leq c_{i+1} < d_{i+1}$. Further, we define

$$D_{n,i} := \left(c_{2+3i} - \frac{1}{2^n}, c_{2+3i}\right] \cup (c_{2+3i}, d_{2+3i}] \cup \left(d_{2+3i}, d_{2+3i} + \frac{1}{2^n}\right)$$

with $i = 0, \ldots, \left\lfloor \mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n)/3 - 2\right\rfloor - 1$. Clearly, we have $D_{n,i} \cap D_{n,j} = \varnothing$ for all $i \neq j$ and

$$\nu\left(\left\langle\overline{D_{n,i}}\right\rangle_{1/3}\right) \Lambda\left(\left\langle\overline{D_{n,i}}\right\rangle_{1/3}\right) \geq \nu\left((c_{2+3i}, d_{2+3i}]\right) \Lambda\left((c_{2+3i}, d_{2+3i}]\right) \geq 2^{-n\alpha}.$$

Hence, for $n_x^\alpha := \lfloor \log\left(\frac{x}{2}\right)/(\log(2)\alpha))\rfloor$ and $x > 2^{2+\alpha}$, we have

$$\nu\left(\left\langle\overline{D_{n_x^\alpha,i}}\right\rangle_{1/3}\right) \Lambda\left(\left\langle\overline{D_{n_x^\alpha,i}}\right\rangle_{1/3}\right) \geq \frac{2}{x}.$$

In tandem with Proposition 4.1, we deduce

$$\begin{aligned} N_\nu^D(x) \geq \mathcal{N}_{3,\nu}^L(x) &= \sup\left\{\operatorname{card}(P) : P \in \Pi_0 : \min_{I \in P} \nu\left(\left\langle \bar{I}\right\rangle_{1/3}\right) \Lambda\left(\left\langle \bar{I}\right\rangle_{1/3}\right) \geq \frac{2}{x}\right\} \\ &\geq \left\lfloor \frac{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)}{3} - 2\right\rfloor \geq \frac{\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)}{3} - 3. \end{aligned}$$ □

Now, we can give a lower bound on $\underline{s}_\nu^D$ and $\overline{s}_\nu^D$ in terms of the lower and upper optimize coarse multifractal dimension as well as the lower and upper m-reduced ν-partition entropy.

Proposition 4.3. *As a general lower (upper) bound for the lower (upper) spectral dimension for all $m \in (1,3]$, we have*

$$\underline{F}_{\mathfrak{J}_\nu} \leq \underline{h}_\nu^m \leq \underline{s}_\nu^D \text{ and } \overline{F}_{\mathfrak{J}_\nu} \leq \overline{h}_\nu^m \leq \overline{s}_\nu^D.$$

Proof. First note that Proposition 4.1 gives $\underline{h}_\nu^m \leq \underline{s}_\nu^D$ and $\overline{h}_\nu^m \leq \overline{s}_\nu^D$ for all $m > 1$. Let

$n_x^\alpha = \left\lfloor \log_2\left(\frac{x}{2}\right)/\alpha \right\rfloor$ and $x > 2^{2+\alpha}$, then by Lemma 4.2 for every $\alpha > 0$, we have

$$\frac{\log\left(3\mathcal{N}_3^L(x)+3\right)}{\log(x)} \geq \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)\right)}{\log(x)}.$$

Hence, for all $1 < m \leq 3$

$$\begin{aligned}
\underline{h}_\nu^m \geq \underline{h}_\nu^3 &= \liminf_{x\to\infty} \frac{\log^+\left(3\mathcal{N}_3^L(x)+3\right)}{\log(x)} \geq \liminf_{x\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)\right)}{\log(x)} \\
&= \liminf_{x\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)\right)}{\alpha\log\left(2^{\log_2\left(\frac{x}{2}\right)/\alpha}\right)+\log(2)} \\
&\geq \liminf_{x\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)\right)}{\alpha\log(2^{n_x^\alpha})+\log(2)(1+\alpha)} \\
&= \liminf_{x\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N\left(n_x^\alpha\right)\right)}{\alpha\log(2^{n_x^\alpha})} \geq \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\alpha}^N(n)\right)}{\alpha\log(2^n)},
\end{aligned}$$

which implies $\underline{h}_\nu^m \geq \underline{F}_{\mathfrak{J}_\nu}$. We also have for $x_m^\eta := 2^{m\eta+1}$ with $m \in \mathbb{N}$ and $\eta > 0$,

$$\begin{aligned}
\overline{h}_\nu^m \geq \overline{h}_\nu^3 \geq \limsup_{m\to\infty} \frac{\log\left(3\mathcal{N}_3^L(x_m^\eta)+3\right)}{\log(x_m^\eta)} &\geq \limsup_{m\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\eta}^N\left(n_{x_m^\eta}^\eta\right)\right)}{\log(x_m^\eta)} \\
\geq \limsup_{m\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\eta}^N\left(n_{x_m^\eta}^\eta\right)\right)}{\eta\log\left(2^{n_{x_m^\eta}^\eta}\right)} &= \limsup_{m\to\infty} \frac{\log^+\left(\mathcal{N}_{\mathfrak{J}_\nu,\eta}^N(m)\right)}{\eta\log(2^m)},
\end{aligned}$$

where we used $n_{x_m^\eta}^\eta = m$. Thus, $\overline{h}_\nu^m \geq \overline{F}_{\mathfrak{J}_\nu}$. □

4.2 Upper bounds for the spectral dimension

We begin with a slight generalization of

$$\mathcal{M}_{\mathfrak{J}_\nu}(x) = \inf\left\{\operatorname{card}(P) : P \in \Pi_\nu \mid \max_{I\in P} \nu(I)\Lambda(I) < 1/x\right\}$$

by allowing left half-open intervals which do not necessarily consist of dyadic intervals. This will be useful for the computation of the spectral dimension and the determination of the spectral asymptotic of specific examples (see Section 4.4.1 and Section 4.4.3).

Let Γ denote the set of left half-open intervals contained in $(0,1]$. We call $P \subset \Gamma$ a *ν-partition of $\mathbf{Q}$* of finitely many left half-open intervals if

- $\operatorname{card}(P) < \infty$,
- $\nu\left(\bigcup_{I\in P} I\right) = \nu(\mathbf{Q})$,
- $I_1 \cap I_2 = \varnothing$ for all $I_1, I_2 \in P$ with $I_1 \neq I_2$,
- $\nu(I) > 0$ for all $I \in P$.

Let Π denote the set of all ν-partitions of left half-open intervals of $\mathbf{Q}$ and

$$\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(x) := \inf\left\{\operatorname{card}(P) : P \in \Pi \mid \max_{I\in P} \nu(I)\Lambda(I) < 1/x\right\}$$

for $x > 1/\nu(\mathbf{Q})$. Before stating our main result of this section, we need some preparations, where we follow [Nga11]. Let $((a_i, a_{i+1}])_{i=0,\dots,n}$ be a partition with $a_i < a_{i+1}$ of $\mathbf{Q}$ and $n \in \mathbb{N}$. Consider the following closed subspace of H_0^1

$$\mathcal{F} := \left\{f \in H_0^1 : f(a_i) = 0,\ i = 1,\dots,n\right\}$$

and define the following equivalence relation on H_0^1, by $x \sim_{H_0^1/\mathcal{F}} y$ if and only if $x - y \in \mathcal{F}$. The associated quotient space is given by

$$H_0^1/\mathcal{F} := \left\{[u]_{H_0^1/\mathcal{F}} : u \in H_0^1\right\},$$

where $[u]_{H_0^1/\mathcal{F}}$ denotes the equivalence class of $u \in H_0^1$ with respect to $\sim$. Further, we define the addition and scalar multiplication on $H_0^1/\mathcal{F}$ in the standard way. For $i = 1,\dots,n$, we define the following elements of H_0^1:

$$f_i : x \mapsto \frac{x - a_{i-1}}{a_i - a_{i-1}} \mathbb{1}_{[a_{i-1},a_i)}(x) + \frac{a_{i+1} - x}{a_{i+1} - a_i} \mathbb{1}_{[a_i,a_{i+1}]}(x).$$

Note that for all $i, j = 1,\dots,n$ we have $f_i(a_j) = \delta_{ij}$, where δ_{ij} is the *Kronecker delta*, i.e.

$$\delta_{ij} := \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{else.} \end{cases}$$

Consequently, for any $u \in H_0^1$ we have

$$u - \sum_{i=1}^{n} u(a_i) f_i \in \mathcal{F}.$$

This implies

$$H_0^1/\mathcal{F} = \operatorname{span}\left([f_i]_{H_0^1/\mathcal{F}} : i = 1,\dots,n\right),$$
$$\dim\left(H_0^1/\mathcal{F}\right) = n.$$

Now we are in the position to establish a link between N_ν^D and $\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}$, allowing us to use the results of Chapter 3.

Proposition 4.4. *For all $x > 0$, we have*

$$N_\nu^D(x) \le 2\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(x) + 1 \le 2\mathcal{M}_{\mathfrak{J}_\nu}(x) + 1$$

and

$$N_\nu^D(x) \le \widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(5x) \le \mathcal{M}_{\mathfrak{J}_\nu}(5x)$$

Proof. Here, we follow the proof of [Kig01, Theorem 4.1.7]. Let $((a_i, a_{i+1}])_{i=0,\dots,n}$ be a partition with $a_i < a_{i+1}$ of $\mathbf{Q}$ and define as above

$$\mathcal{F} = \left\{f \in H_0^1 : f(a_i) = 0,\ i = 1,\dots,n\right\}.$$

Then, $\dim\left(H_0^1/\mathcal{F}\right) = n$ and for any subspace of $L \subset H_0^1$ with $\dim(L) = i$, there exists a linear, injective map $\Phi : L/(L\cap\mathcal{F}) \to H_0^1/\left(H_0^1\cap\mathcal{F}\right) = H_0^1/\mathcal{F}$ with

$$\Phi(\ell + L\cap\mathcal{F}) \coloneqq \ell + \mathcal{F},\ \ell \in L.$$

Thus, the rank–nullity theorem yields

$$\dim(L\cap\mathcal{F}) \ge \dim(L) - N,$$

implying

$$i - N \le \dim(L\cap\mathcal{F}) \le i.$$

Hence, if

$$\mathcal{L}_{i,j} \coloneqq \left\{L :\ L \text{ is a subspace of } H_0^1,\ \dim(L) = i,\ \dim(L\cap\mathcal{F}) = j\right\},$$

then we obtain

$$\left\{L :\ L \text{ is a subspace of } H_0^1,\ \dim(L) = i + N\right\} = \bigcup_{k=0}^{N} \mathcal{L}_{i+N,i+k}.$$

Now with

$$\widetilde{\lambda}_i \coloneqq \inf\left\{\sup\left\{\frac{\langle\psi,\psi\rangle_{H_0^1}}{\langle\psi,\psi\rangle_\nu} : \psi \in G\setminus\{0\}\right\} : G <_i \left(\mathcal{F}, \langle\cdot,\cdot\rangle_{H_0^1}\right)\right\},$$

for every $L \in \mathcal{L}_{i+N,i+k}$ with $k \in \{0, \dots, N\}$, we find that

$$\sup \left\{ \frac{\langle \psi, \psi \rangle_{H_0^1}}{\langle \psi, \psi \rangle_\nu} : \psi \in L \setminus \{0\} \right\} \geq \sup \left\{ \frac{\langle \psi, \psi \rangle_{H_0^1}}{\langle \psi, \psi \rangle_\nu} : \psi \in L \cap \mathcal{F} \setminus \{0\} \right\}$$
$$\geq \widetilde{\lambda}_{i+k} \geq \widetilde{\lambda}_i.$$

We deduce from the min-max principle stated in Proposition 2.17

$$\lambda_{i+N,\nu}^D \geq \widetilde{\lambda}_i,$$

which implies $N_\nu^D(x) \leq \operatorname{card}\left(\{i \in \mathbb{N} : \widetilde{\lambda}_i \leq x\}\right) + N$. Furthermore, using Lemma 2.2, for all $u \in \mathcal{F}$, we have

$$\int u^2 \, \mathrm{d}\nu = \sum_{\substack{i=0,\\ \nu((a_i,a_{i+1}])>0}}^{n} \int_{(a_i,a_{i+1}]} u^2 \, \mathrm{d}\nu$$
$$\leq \max_{i=0,\dots,n} \nu((a_i,a_{i+1}]) \Lambda((a_i,a_{i+1}]) \int_{(0,1)} (\nabla u)^2 \, \mathrm{d}\Lambda.$$

Now, assume $\max_{i=0,\dots,n} \nu((a_i,a_{i+1}]) \Lambda((a_i,a_{i+1}]) < 1/x$. Then,

$$\widetilde{\lambda}_1 \geq \left(\max_{i=0,\dots,n} \nu((a_i,a_{i+1}]) \Lambda((a_i,a_{i+1}]) \right)^{-1} > x.$$

This implies

$$N_\nu^D(x) \leq n.$$

Taking the infimum over all $P \in \Pi$ with $\max_{I \in P} \nu(I)\Lambda(I) < 1/x$ yields

$$N_\nu^D(x) \leq 2\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(x) + 1.$$

The second inequality follows from the fact that $\Pi_{\mathfrak{J}_\nu} \subset \Pi$. Similarly, the second claim follows from Corollary 2.3 by replacing $\mathcal{F}$ with

$$\{f \in H_0^1(0,1) : \int_I f \, \mathrm{d}\Lambda = 0, \ I \in P\}$$

and the fact $\dim(H_0^1/\mathcal{F}) = \operatorname{card}(P)$. □

Remark 4.5. Similarly, one can show that $\dim\left(H^1/H_0^1\right) = 2$. In tandem with the Poincaré inequality (PI) it follows in a similar way as in Proposition 4.4 that $\overline{s}_\nu^D = \overline{s}_\nu^N$ and $\underline{s}_\nu^D = \underline{s}_\nu^N$.

Corollary 4.6. *We have*

$$\overline{s}_{\nu}^{D} \leq \overline{h}_{\mathfrak{J}_{\nu}} = q_{\mathfrak{J}_{\nu}} = \inf\{q \geq 0 : \beta_{\nu}(q) - q \leq 0\} \leq \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu)+1} \text{ and } \underline{s}_{\nu}^{D} \leq \underline{h}_{\mathfrak{J}_{\nu}}.$$

Proof. The first and second inequality follows from Proposition 3.1 and Proposition 4.4 applied to $\mathfrak{J} = \mathfrak{J}_{\nu}$. Moreover, notice that the convexity of β_{ν} implies

$$\beta_{\nu}(q) \leq \overline{\dim}_M(\nu)(1-q), \; q \in [0,1].$$

This yields

$$\inf\{q \geq 0 : \beta_{\nu}(q) - q \leq 0\} \leq \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu)+1}.$$ □

Remark 4.7. The case $\overline{\dim}_M(\nu) = 0$ immediately gives $\overline{s}_{\nu}^{D} = 0$. If we use more information on β_{ν} for the case $\overline{\dim}_M(\nu) > 0$, we find a better upper bound; namely, with $q_1 := \inf\{s : \beta_{\nu}(s) \leq 0\}$, we have

$$\overline{s}_{\nu}^{D} \leq q_{\mathfrak{J}_{\nu}} \leq \frac{q_1 \overline{\dim}_M(\nu)}{q_1 + \overline{\dim}_M(\nu)}.$$

The following proposition complements the connection of the Minkowski dimension by establishing an upper bound of the lower spectral dimension in terms of the lower Minkowski dimension $\underline{\dim}_M(\nu)$ of the support of ν and the ∞-dimension of ν.

Proposition 4.8. *We always have*

$$\underline{s}_{\nu}^{D} \leq \underline{h}_{\mathfrak{J}_{\nu}} \leq \frac{\underline{\dim}_M(\nu)}{1+\dim_{\infty}(\nu)}.$$

Proof. This follows from Proposition 3.13 applied to $J = \nu$ and $a = 1$. □

Proposition 4.9. *Under the assumption that there exists a subsequence $(n_k)_k$ and a constant $K > 0$ such that for all $k \in \mathbb{N}$*

$$\max_{C \in \mathcal{D}_{n_k}} \nu(C)^{q_{n_k}} \leq \frac{K}{2^{\beta_{n_k}(0) n_k}} \sum_{C \in \mathcal{D}_{n_k}} \nu(C)^{q_{n_k}}$$

and $\lim_{k\to\infty} q_{n_k} = \liminf_{n\to\infty} q_n$, *where $q_n \geq 0$ is the unique solution to $\beta_n(q_n) = q_n$, we have*

$$\underline{s}_{\nu}^{D} \leq \underline{h}_{\mathfrak{J}_{\nu}} \leq \liminf_{n\to\infty} q_n \leq \liminf_{n\to\infty} \frac{\beta_n(0)}{1+\beta_n(0)} = \frac{\underline{\dim}_M(\nu)}{1+\underline{\dim}_M(\nu)}.$$

Proof. Due to Proposition 4.8, we only have to consider the case $\underline{\dim}_M(\nu) > 0$ which implies $\liminf_{k\to\infty} q_{n_k} > 0$. Now, Proposition 3.3 applied to $\mathfrak{J} = \mathfrak{J}_\nu$ yields the claim. □

4.3 Main results

In this section, we connect Proposition 4.3, Corollary 4.6, and the general results of Chapter 3 to prove the main results of this chapter.

4.3.1 Upper spectral dimension and lower bounds for the lower spectral dimension

In this , we compute the upper spectral dimension and obtain various lower and upper bounds of the lower and upper spectral dimension.

Theorem 4.10. *For all* $1 < m \leq 3$*, we have*

$$\underline{F}_{\mathfrak{J}_\nu} \leq \underline{h}_\nu^m \leq \underline{s}_\nu^D \leq \underline{h}_{\mathfrak{J}_\nu} \leq \overline{h}_{\mathfrak{J}_\nu} = \overline{h}_\nu^m = \overline{s}_\nu^D = q_{\mathfrak{J}_\nu} = \overline{F}_{\mathfrak{J}_\nu}. \tag{4.3.1}$$

In particular,

$$\overline{s}_\nu^D = q_{\mathfrak{J}_\nu} \leq \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu)+1} \leq 1/2,$$

and the following necessary and sufficient conditions ensuring the existence of spectral dimension:

$$\underline{s}_\nu^D = \overline{s}_\nu^D \implies \underline{h}_{\mathfrak{J}_\nu} = \overline{h}_{\mathfrak{J}_\nu} = s_\nu^D \quad \textit{and} \quad \sup_{m>1} \underline{h}_\nu^m = \overline{h}_{\mathfrak{J}_\nu} \implies \underline{s}_\nu^D = \overline{s}_\nu^D = \overline{h}_{\mathfrak{J}_\nu}.$$

Proof. By Proposition 4.3 and Corollary 4.6, we have

$$\overline{F}_{\mathfrak{J}_\nu} \leq \overline{h}_\nu^m \leq \overline{s}_\nu^D \leq \overline{h}_{\mathfrak{J}_\nu} \leq q_{\mathfrak{J}_\nu}$$

and

$$\underline{F}_{\mathfrak{J}_\nu} \leq \underline{h}_\nu^m \leq \underline{s}_\nu^D \leq \underline{h}_{\mathfrak{J}_\nu}.$$

Moreover, Proposition 3.20 applied to $\mathfrak{J} = \mathfrak{J}_\nu$ yields $\overline{F}_{\mathfrak{J}_\nu} = q_{\mathfrak{J}_\nu}$, and thus, the equalities in (4.3.1) are obtained. □

Remark 4.11. Theorem 4.10 shows that if the spectral dimension exists, then it is given by purely measure-geometric data, which is encoded in the ν-partition entropy $\overline{h}_{\mathfrak{J}_\nu} = \underline{h}_{\mathfrak{J}_\nu}$. We call ν *regular* if $\sup_{m>1} \underline{h}_\nu^m = \overline{h}_{\mathfrak{J}_\nu}$, in which case the spectral dimension exists. If $\mathfrak{J}_\nu$ is Neumann MF-regular (i.e. $\underline{F}_{\mathfrak{J}_\nu} = \overline{F}_{\mathfrak{J}_\nu}$), then in the above chain of inequalities (4.3.1) we have everywhere equality and especially ν is *regular*. Moreover, if for some $m > 1$ we have $\underline{h}^m_{\mathfrak{J}_\nu} \geq 1/2$, then $s^D_\nu = 1/2 = \overline{h}_{\mathfrak{J}_\nu}$.

4.3.2 Regularity results

Here, we investigate the question under which conditions the spectral dimension exists. By combining the results of Section 3.4 and Theorem 4.10, we show that the regularity conditions imposed in Definition 3.22 leads to the following sufficient condition for existence of the spectral dimension.

Corollary 4.12. *If $\mathfrak{J}_\nu$ is Neumann partition function regular, then the spectral dimension exists and is given by $s^D_\nu = q_{\mathfrak{J}_\nu}$.*

Proof. This follows from Corollary 3.23 applied to $\mathfrak{J} = \mathfrak{J}_\nu$ and Theorem 4.10. □

Remark 4.13. Observe that due to $\tau^N_{\mathfrak{J}_\nu,n}(q) = \beta_n(q) - q$ for all $n \in \mathbb{N}$, $q \geq 0$, we infer that $\mathfrak{J}_\nu$ is Dirichlet partition regular if and only if β_ν exists as a limit in $q_{\mathfrak{J}_\nu}$ and β_ν is differentiable at $q_{\mathfrak{J}_\nu}$, or $\beta_\nu(q) = \liminf_{n\to\infty} \beta_n(q)$ for $q \in \left(q_{\mathfrak{J}_\nu} - \varepsilon, q_{\mathfrak{J}_\nu}\right)$ for some $\varepsilon > 0$.

Proposition 4.14. *If for $q \in [0,1]$ we have $\beta_\nu(q) = \lim_{n\to\infty} \beta_n(q)$ and $-\partial\beta_\nu(q) = [a,b]$, then*

$$\frac{aq + \beta_\nu(q)}{1+b} \leq \underline{s}^D_\nu.$$

Proof. By Corollary 3.19 and Proposition 4.3 , we have

$$\frac{aq + \beta_\nu(q)}{1+b} \leq \sup_{t>b} \liminf_{n\to\infty} \frac{\log\left(\mathcal{N}^N_{\mathfrak{J}_\nu,t}(n)\right)}{(1+t)\log(2^n)} \leq \underline{F}_{\mathfrak{J}_\nu} \leq \underline{s}^D_\nu,$$

where we used the fact that $-\partial\tau_{\mathfrak{J}_\nu}(q) = (a+1, b+1]$. □

Remark 4.15. In the case that $\beta_\nu(q) = \lim_{n\to\infty} \beta_n\left(q_{\mathfrak{J}_\nu}\right)$ and β_ν is differentiable at $q_{\mathfrak{J}_\nu}$, we infer $q_{\mathfrak{J}_\nu} \leq \underline{s}^D_\nu$ and hence obtain a direct proof of the regularity statement, namely, $q_{\mathfrak{J}_\nu} = \underline{s}^D_\nu = \overline{s}^D_\nu$.

Also for measures without an absolutely continuous part we have the following rigidity result in terms of reaching the maximum possible value $1/2$ of the spectral dimension.

Corollary 4.16. *We have the following rigidity results:*

1. *If $\overline{s}^D_\nu = 1/2$, then $\beta_\nu(q) = 1 - q$ for all $q \in [0,1]$.*

2. *If* $\beta_\nu(q) = \lim_{n\to\infty} \beta_n(q) = 1 - q$ *for some* $q \in (0,1)$, *then* $\beta_\nu(q) = 1 - q$ *for all* $q \in [0,1]$ *and* $s_\nu^D = 1/2$.

Proof. If $\overline{s}_\nu^D = 1/2$ then it follows from Theorem 4.10 that $1/2 = \overline{s}_\nu = q_{\mathfrak{J}_\nu} \leq 1/2$. The convexity of β_ν and the fact that $\beta_\nu(1) = 0$ and $\beta_\nu(0) \leq 1$ forces $\beta_\nu(q) = 1 - q$ for all $q \in [0,1]$. The second statement is an immediate consequence of Proposition 4.14 by observing that, as in case (1), by convexity we have $\beta_\nu(q) = 1 - q$ for all $q \in [0,1]$. This implies the differentiability of β_ν in the particular point $q \in (0,1)$, where by our assumption $\beta_\nu(q) = \lim_{n\to\infty} \beta_n(q)$. Now, applying Proposition 4.14 gives

$$\frac{1}{2} = \frac{q + 1 - q}{1 + 1} \leq \underline{s}_\nu.$$

Since we always have $\overline{s}_\nu^D \leq 1/2$, our claim follows. □

4.3.3 General bounds in terms of fractal dimensions

As a consequence of Theorem 4.10 and Proposition 4.14 we improve the known general upper bound of the spectral dimension of 1/2 as obtained in [BS66] in terms of the upper Minkowski dimension. Furthermore, we obtain a general lower bound of $\underline{s}_\nu^D$ in terms of the left and right-hand derivative of β_ν.

Corollary 4.17. *For the lower and upper spectral dimension, we have the following general lower and upper bounds depending on the topological support of* ν, *namely* $\overline{\dim}_M(\nu)$, *and left and right derivative of* β_ν *at 1:*

$$\frac{-\partial^+\beta_\nu(1)}{1 - \partial^-\beta_\nu(1)} \leq \underline{s}_\nu^D \leq \overline{s}_\nu^D \leq \frac{\overline{\dim}_M(\nu)}{1 + \overline{\dim}_M(\nu)} \leq \frac{1}{2}$$

and

$$\overline{s}_\nu^D = \frac{\overline{\dim}_M(\nu)}{1 + \overline{\dim}_M(\nu)} \iff -\partial^-\beta_\nu(1) = \overline{\dim}_M(\nu).$$

Proof. The first inequalities follow from Corollary 4.6 and Proposition 4.14, using the fact that β_ν always exists as a limit at 1. The last claim follows from the fact that β_ν is linear on $[0,1]$ if and only if $-\partial^-\beta_\nu(1) = \overline{\dim}_M(\nu)$ and in this case $q_{\mathfrak{J}_\nu} = \overline{\dim}_M(\nu) / \left(1 + \overline{\dim}_M(\nu)\right)$. □

Remark 4.18. It is worth pointing out that these bounds have been first observed in the self-similar case under the open set condition in [SV95, p. 245]. In this case the Minkowski dimension and the Hausdorff dimension of $\mathrm{supp}(\nu)$ coincide as well as β_ν is differentiable at 1 and $\beta_\nu'(1)$ coincides with the Hausdorff dimension of ν (see for instance [Heu07]). Furthermore, note that in the case that ν has an atomic part, we always have $\partial^+\beta_\nu(1) = 0$ (see Fact 2.32). Hence, the lower bound $-\partial^+\beta_\nu(1)/(1-\partial^-\beta_\nu(1))$ is only meaningful in the case of atomless measures.

Regarding Kac's question if β_ν is differentiable at 1, then the spectral dimension is determined by fractal-geometric quantities as follows

$$\dim_H(\nu) = \beta_\nu'(1) \leq \frac{\underline{s}_\nu^D}{1-\underline{s}_\nu^D} \leq \frac{\overline{s}_\nu^D}{1-\overline{s}_\nu^D} \leq \overline{\dim}_M(\nu),$$

where $\dim_H(\nu) := \inf\{\dim_H(A) : \nu(A^\complement) = 0\}$.

4.4 Examples

4.4.1 C^1-cIFS and weak Gibbs measures

In this section, we define weak-Gibbs measures with respect to not necessary linear iterated function systems. We start by pointing out some simplifications of the notion of C^1-cIFS in the one-dimension setting. Let $\Phi := \{T_i : [0,1] \to [0,1] : i = 1,\dots,n\}$, $n \in \mathbb{N}$, be a C^1-cIFS as defined in Definition 2.46. Observe that for every $\varepsilon > 0$ we can extend each T_i to an injective contracting C^1-map $T_i : (-\varepsilon, 1+\varepsilon) \to (-\varepsilon, 1+\varepsilon)$ via

$$\widetilde{T_i}(x) := \begin{cases} T_i(0) + T_i'(0)x, & x \in [-\varepsilon, 0), \\ T_i(x), & x \in [0,1], \\ T_i(1) + T_i'(1)(x-1), & x \in (1, 1+\varepsilon]. \end{cases}$$

Moreover, the notion of conformal maps becomes trivial in the one-dimensional setting. This gives rise to the following equivalent conditions to be a C^1-cIFS:

1. for all $j \in I$ we have $T_j \in C^1([0,1])$ and
$$0 < \inf_{x\in[0,1]} \left|T_j'(x)\right| \leq \sup_{x\in[0,1]} \left|T_j'(x)\right| < 1,$$
2. Φ is non-trivial, i.e. there is more than one contraction and the T_i's do not share a common fixed point.

Definition 4.19. Let $\Phi := \{T_i : [0,1] \to [0,1] : i = 1,\dots,n\}$, $n \in \mathbb{N}$, be a C^1-cIFS. If additionally the $T_1,\dots,T_n$ are $C^{1+\gamma}$-maps with $\gamma \in (0,1)$, we call the system a *$C^{1+\gamma}$-conformal iterated function system* ($C^{1+\gamma}$-cIFS). Here $C^{1+\gamma}$ denotes the set of differentiable maps with γ-Hölder continuous derivative.

In the remainder of this section we fix a C^1-IFS

$$\Phi := \{T_i : [0,1] \to [0,1] : i = 1,\dots,n\}.$$

Recall that the unique non-empty compact invariant set $\mathcal{K} \subset [0,1]$ of a C^1- cIFS Φ is given by

$$\mathcal{K} = \bigcup_{i \in I} T_i(\mathcal{K})$$

with $I = \{1,\dots,n\}$. Let $\mathcal{B}\left(I^{\mathbb{N}}\right)$ denote the Borel σ-algebra of $I^{\mathbb{N}}$. Note that $\mathcal{B}\left(I^{\mathbb{N}}\right)$ is generated by the set of cylinder sets of arbitrary lengths. The set of σ-invariant probability measures on $\mathcal{B}\left(I^{\mathbb{N}}\right)$ is denoted by $\mathcal{M}_\sigma\left(I^{\mathbb{N}}\right)$, where the measure μ is called σ-invariant if $\mu = \mu \circ \sigma^{-1}$.

Definition 4.20. Let $C\left(I^{\mathbb{N}}\right)$ denote the space of continuous real valued functions on $I^{\mathbb{N}}$. For $f \in C\left(I^{\mathbb{N}}\right)$, $\alpha \in (0,1)$, and $n \in \mathbb{N}_0$ define

$$\mathrm{var}_n(f) := \sup\left\{|f(\omega) - f(u)| : \omega, u \in I^{\mathbb{N}} \text{ and } \omega_i = u_i \text{ for all } i \in \{1,\dots,n\}\right\},$$

$$|f|_\alpha := \sup_{n \geq 0} \frac{\mathrm{var}_n(f)}{\alpha^n} \text{ and } \mathcal{F}_\alpha := \left\{f \in C\left(I^{\mathbb{N}}\right) : |f|_\alpha < \infty\right\}.$$

Elements of $\mathcal{F}_\alpha$ are called *α-Hölder continuous* functions on $I^{\mathbb{N}}$. Furthermore, the *Birkhoff sum* of f is defined by $S_n f(x) := \sum_{k=0}^{n-1} f \circ \sigma^k(x)$, $x \in I^{\mathbb{N}}$, $n \in \mathbb{N}$, and $S_0 f := 0$.

Definition 4.21 (Geometric potential function)**.** The *geometric potential function* with respect to Φ is given by

$$\varphi(\omega_1\omega_2\cdots) := \log\left(\left|T'_{\omega_1}(\pi(\omega_2\omega_3\cdots))\right|\right).$$

Remark 4.22. We will make use of the following relation between φ and T'_ω with $\omega = \omega_1 \cdots \omega_n \in I^n$, $n \in \mathbb{N}$. For any $x \in \mathcal{K}$ there exists $\alpha_x \in I^{\mathbb{N}}$ such that $\pi(\alpha_x) = x$. Hence,

$$\left|T'_\omega(x)\right| = \mathrm{e}^{\sum_{i=1}^{|\omega|} \log\left(\left|T'_{\omega_i}\left(T_{\sigma^i(\omega)}(\pi(\alpha_x))\right)\right|\right)} = \mathrm{e}^{\sum_{i=1}^{|\omega|} \log\left(\left|T'_{\omega_i}\left(\sigma^i(\omega\pi(\alpha_x))\right)\right|\right)} = \mathrm{e}^{S_n\varphi(\omega\alpha_x)}.$$

Note that φ is Hölder continuous if the underlying IFS is a $C^{1+\gamma}$-cIFS. Moreover, if all the T_i are affine, then φ depends only on the first coordinate.

Definition 4.23 (Perron-Frobenius operator)**.** Let $\psi \in C(I^{\mathbb{N}})$ (sometimes called potential function). The *Perron-Frobenius operator* (with respect to ψ)

$$L_\psi : C(I^{\mathbb{N}}) \to C(I^{\mathbb{N}})$$

is defined by

$$L_\psi f(x) := \sum_{y\in\sigma^{-1}x} \mathrm{e}^{\psi(y)} f(y),\ x \in I^{\mathbb{N}}.$$

Definition 4.24 (Pressure function)**.** For $f \in C\left(I^{\mathbb{N}}\right)$ the pressure of f is defined by

$$P(f) := \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\omega\in I^n} e^{S_\omega f}\right), \tag{4.4.1}$$

with $S_\omega f := \sup_{x\in[\omega]} S_{|\omega|} f(x)$.

Remark 4.25. The existence of the limit in (4.4.3) follows from the subadditivity of $\log(\sum_{\omega\in I^n} e^{S_\omega f})$ and *Fekete's subadditive lemma.*

Lemma 4.26. *Let $\psi \in C(I^{\mathbb{N}})$ with $L_\psi \mathbb{1} = \mathbb{1}$, $\mathbb{1} := \mathbb{1}_{I^{\mathbb{N}}}$. Then,*

$$P(\psi) = 0.$$

Proof. Using $L_\psi \mathbb{1} = \mathbb{1}$, for all $n \in \mathbb{N}$, $x \in I^{\mathbb{N}}$, we deduce

$$\mathbb{1} = L_\psi^n \mathbb{1} = \sum_{\omega\in I^n} e^{S_n\psi(\omega x)}.$$

This leads to

$$\begin{aligned} -\frac{\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)}{n} + \frac{1}{n}\log\left(\sum_{\omega\in I^n} e^{S_n\psi(\omega x)}\right) &\le \frac{1}{n}\log\left(\sum_{\omega\in I^n} e^{S_\omega\psi}\right) \\ &\le \frac{\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)}{n} + \frac{1}{n}\log\left(\sum_{\omega\in I^n} e^{S_n\psi(\omega x)}\right). \end{aligned}$$

Since ψ is continuous, we have $\lim_{m\to\infty} \mathrm{var}_m(\psi) = 0$ and thus $\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)/n$ tending to zero for sending n to infinity as a Cesàro limit. This gives $P(\psi) = 0$. ◻

Now, we are in the position to define weak Gibbs measures, which is subject of the following proposition.

Proposition 4.27 ([Kes01, Proposition 1])**.** *For any $\psi \in C(I^{\mathbb{N}})$ with $L_\psi \mathbb{1} = \mathbb{1}$ there exists $\mu \in \mathcal{M}_\sigma\left(I^{\mathbb{N}}\right)$ such that*

$$L_\psi^* \mu = \mathrm{e}^{P(\psi)} \mu = \mu,$$

where L_ψ^* denotes the dual operator of L_ψ acting on the set of Borel probability measures supported on $I^{\mathbb{N}}$. We call μ a weak ψ-Gibbs measure *and $\nu := \mu \circ \pi^{-1}$ a* weak ψ-Gibbs measure *with respect to the cIFS Φ.*

Remark 4.28. The existence of the fixed point in Proposition 4.27 follows from the Schauder-Tychonov fixed point theorem (see also [Kes01]) and the σ-invariance of μ follows for $E \in \mathcal{B}\left(I^{\mathbb{N}}\right)$, by

$$\begin{aligned}\mu(\sigma^{-1}(E)) &= \int \sum_{\tau \in I} \mathrm{e}^{S_n\psi(\tau y)} \mathbb{1}_{\sigma^{-1}(E)}(\tau y) \, \mathrm{d}\mu(y) \\ &= \int \sum_{\tau \in I} \mathrm{e}^{S_n\psi(\tau y)} \mathbb{1}_{E}(y) \, \mathrm{d}\mu(y) = \mu(E).\end{aligned}$$

Remark 4.29. The following list of comments proves useful in our context.

1. By [Kes01, Lemma 3], for all $u \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have

$$\mathrm{e}^{-\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)} \leq \frac{\mu([u|_n])}{\mathrm{e}^{S_n\psi(u)}} \leq \mathrm{e}^{\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)}. \tag{4.4.2}$$

   In particular, the measure μ has no atoms, since $\sum_{i=0}^{n} \mathrm{var}_i(\psi) = o(n)$ and $S_n\psi \leq n \max \psi$.

2. The topological support $\mathrm{supp}(\nu)$ of ν is equal to $\mathcal{K}$. To see this, note that $\mathcal{K}$ is covered by the set $\bigcup_{\omega \in I^n} T_\omega([0,1])$, $n \in \mathbb{N}$, and by (4.4.2) each $T_\omega([0,1])$ has positive ν-measure $\nu(T_\omega([0,1])) \geq \exp\left(-\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)\right) \mu([\omega])$.

3. If ψ is additionally Hölder continuous, then μ is the unique invariant ergodic ψ-*Gibbs measure* and the bounds in the above inequality (4.4.2) can be chosen to be positive constants.

4. For an arbitrary Hölder continuous function $\psi : I^{\mathbb{N}} \to \mathbb{R}$ (without assuming $L_\psi \mathbb{1} = \mathbb{1}$) there always exists a σ-invariant ψ-Gibbs measure μ on the symbolic space as a consequence of the general thermodynamic formalism and the Perron-Frobenius theorem for Hölder potentials (see e.g. [Bow08]). Let h denote the only eigenfunction of the Perron-Frobenius operator for the maximal eigenvalue $\lambda > 0$, which is positive and in the same Hölder class. Then $\psi_1 := \psi - \log(\lambda) + \log(h) - \log(h \circ \sigma)$ defines another Hölder continuous function for which $L_{\psi_1} \mathbb{1} = \mathbb{1}$ and μ is the (unique) ψ_1-Gibbs measure, as defined here.

5. If ψ depends only on the first coordinate and is normalized such that we have $p_i := \exp(\psi(i \ldots))$, $i \in I$, defines a probability vector, then μ is in fact a Bernoulli measure and the bounding constants in the above inequalities (4.4.2) can be chosen to be 1. Further, ν coincides with the self-conformal measure as defined in (2.4.2). This can be seen as follows. For all $E \in \mathfrak{B}([0,1])$, we

have

$$\begin{aligned}\nu(E) &= \sum_{u\in I}\mu\left(\pi^{-1}(E)\cap[u]\right)\\ &= \sum_{u\in I}\int_{[u]}\mathbb{1}_E\circ\pi\,\mathrm{d}\mu\\ &= \sum_{u\in I}\int L_\psi^{|u|}(\mathbb{1}_{[u]}(x)\mathbb{1}_E(\pi(x)))\,\mathrm{d}\mu(x)\\ &= \sum_{u\in I}\int\sum_{\tau\in I^{|u|}}\mathrm{e}^{S_{|u|}\psi(\tau x)}\mathbb{1}_{[u]}(\tau x)\mathbb{1}_E(\pi(\tau x))\,\mathrm{d}\mu(x)\\ &= \sum_{u\in I}\int p_u\mathbb{1}_E(\pi(ux))\,\mathrm{d}\mu(x)\\ &= \sum_{u\in I}p_u\nu\left(T_u^{-1}(E)\right).\end{aligned}$$

In particular, if additionally the $(T_i)_i$ are contracting similarities, then ν coincides with the self-similar measure defined in (2.4.2).

For $m\in\mathbb{N}$ we will consider the accelerated shift-space $(I^m)^{\mathbb{N}}$ with natural shift map $\widetilde{\sigma}:(I^m)^{\mathbb{N}}\to(I^m)^{\mathbb{N}}$. Clearly, $(I^m)^{\mathbb{N}}$ can be identified with $I^{\mathbb{N}}$ allowing us to define the *accelerated ergodic sum* for $f\in C\left(I^{\mathbb{N}}\right)$ by

$$\widetilde{S}_n f^m(x) := \sum_{i=0}^{n-1} f^m(\widetilde{\sigma}^i(x)) \text{ with } f^m(x) := \sum_{i=0}^{m-1} f\left(\sigma^i(x)\right).$$

For $\omega\in(I^m)^*$ we let $|\omega|_m$ denote the word length of ω with respect to the alphabet I^m. With this setup we have $\widetilde{S}_n f^m = S_{m\cdot n}f$ and $\widetilde{S}_\omega f^m = \sup_{x\in[\omega]}\widetilde{S}_{|\omega|_m}f^m(x)$ for $\omega\in(I^m)^*$.

Lemma 4.30. *For every $f\in C\left(I^{\mathbb{N}}\right)$ and $m\in\mathbb{N}$, we have*

$$mP(f) = P_{\widetilde{\sigma}}(f^m) := \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\omega\in(I^m)^n}\exp\left(\widetilde{S}_\omega f^m\right)\right).$$

Proof. The assertion follows immediately from the following identity, for all $n\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{n}\log\left(\sum_{\omega\in(I^m)^n}\exp\left(\widetilde{S}_\omega\widetilde{f}\right)\right) &= \frac{1}{n}\log\left(\sum_{\omega\in(I^m)^n}\exp\left(\sup_{x\in[\omega]}\widetilde{S}_{|\omega|_m}f^m(x)\right)\right)\\ &= m\frac{1}{mn}\log\left(\sum_{\omega\in I^{mn}}\exp\left(\sup_{x\in[\omega]}S_{|\omega|}f(x)\right)\right).\end{aligned}$$

□

In the following we show that the *weak bounded distortion property (wBDP)* holds true for the IFS $\Phi = (T_1, \ldots, T_n)$.

Lemma 4.31 (Weak Bounded Distortion Property)**.** *There exists a sequence of non-negative numbers $(b_m)_m$ with $b_m = o(m)$ such that for all $\omega \in I^*$ and $x, y \in [0,1]$*

$$\mathrm{e}^{-b_{|\omega|}} \leq \frac{T'_\omega(x)}{T'_\omega(y)} \leq \mathrm{e}^{b_{|\omega|}}.$$

Proof. Here, we follow the arguments in [KK12, Lemma 3.4]. For all $\omega := \omega_1 \cdots \omega_l \in I^*$, $x, y \in [0,1]$, we have

$$\begin{aligned}\frac{T'_\omega(x)}{T'_\omega(y)} &\leq \exp\left(\sum_{k=1}^{l} \left|\log\left(\left|T'_{\omega_k}\left(T_{\sigma^k\omega}(x)\right)\right|\right) - \log\left(\left|T'_{\omega_k}\left(T_{\sigma^k\omega}(y)\right)\right|\right)\right|\right)\\ &\leq \exp\left(\sum_{k=1}^{l} \underbrace{\max_{x,y\in[0,1]} \max_{i=1,\ldots,n} \left|\log\left(\left|T'_i\left(T_{\sigma^k\omega}(x)\right)\right|\right) - \log\left(\left|T'_i\left(T_{\sigma^k\omega}(y)\right)\right|\right)\right|}_{=:A_{l-k}}\right).\end{aligned}$$

Let $0 < R < 1$ be a common bound for the contraction ratios of the maps $T_1, \ldots, T_n$. Then we have

$$\left|T_{\sigma^k\omega}(x) - T_{\sigma^k\omega}(y)\right| \leq R^{l-k}|x-y| \leq R^{l-k}.$$

Hence, we conclude

$$A_{l-k} \leq \max_{\substack{a,b\in[0,1],\\ |a-b|\leq R^{l-k}}} \left(\max_{i=1,\ldots,n} \left|\log\left(\left|T'_i(a)\right|\right) - \log\left(\left|T'_i(b)\right|\right)\right|\right) =: B_{l-k}.$$

Using that each $T'_1, \ldots, T'_n$ is bounded away from zero and continuous, we obtain $B_k \to 0$ for $k \to \infty$. With $b_m := \sum_{k=0}^{m-1} B_k$ we have $\lim_{m\to\infty} b_m/m$ equals $\lim_{k\to\infty} B_k = 0$ as a Cesàro limit and the second inequality holds. The first inequality follows by interchanging the roles of x and y. □

4.4.1.1 Spectral dimension for weak Gibbs measures under the OSC

Let μ and $\nu = \mu \circ \pi^{-1}$ be weak ψ-Gibbs measures with respect to a C^1-cIFS Φ as defined in Proposition 4.27. In this section we assume the *open set condition* (OSC) with feasible open set $(0,1)$, i.e. $T_i((0,1)) \cap T_j((0,1)) = \varnothing$ and $T_i((0,1)) \subset (0,1)$ for all $i, j \in I$ with $i \neq j$. An important quantity is $\xi := \varphi + \psi$ where φ is the geometric potential with respect to Φ as defined in Definition 4.21. Since

$$p : t \mapsto P(t\xi) \tag{4.4.3}$$

is continuous, strictly monotonically increasing, convex, and

$$\lim_{t\to\pm\infty} p(t) = \mp\infty,$$

there exists a unique number $z_\nu \in \mathbb{R}$ such that $p(z_\nu) = 0$. This section is devoted to identify the spectral dimension of Δ_ν^D with z_ν. We start with some basic observations.

Let $\mathcal{K}^{\text{unique}}$ be the set of points which have a unique preimage of the coding map π. The OSC implies that $\mathcal{K} \setminus \mathcal{K}^{\text{unique}}$ is a countable set. Notice that, due to (4.4.2) μ has no atoms, hence the OSC ensures that ν has also no atoms, implying $\nu\left(\mathcal{K}^{\text{unique}}\right) = 1$.

Lemma 4.32. *For all $\omega \in I^*$ and $f \in H^1$, we have*

$$\int_{I_\omega} (\nabla f)^2 \, \mathrm{d}\Lambda = \int_{[0,1]} (\nabla (f \circ T_\omega))^2 |T_\omega'|^{-1} \, \mathrm{d}\Lambda$$

and

$$\int_{I_\omega} f^2 \, \mathrm{d}\nu = \int_{[0,1]} (f \circ T_\omega)^2 \mathrm{e}^{S_{|\omega|}\psi\circ\pi^{-1}\circ T_\omega} \, \mathrm{d}\nu,$$

with $I_\omega := T_\omega([0,1])$. In particular, we have

$$e^{s_\omega\psi} \min_{x\in[0,1]} |T_\omega'(x)| \le \nu(T_\omega([0,1]))\Lambda(T_\omega([0,1])) \le e^{S_\omega\psi} \max_{x\in[0,1]} |T_\omega'(x)|. \quad (4.4.4)$$

Proof. Clearly, by a change of variables

$$\begin{aligned}\int_{I_\omega} (\nabla f)^2 \, \mathrm{d}\Lambda &= \int_{[0,1]} ((\nabla f) \circ T_\omega)^2 |T_\omega'| \, \mathrm{d}\Lambda \\ &= \int_{[0,1]} (\nabla (f \circ T_\omega))^2 |T_\omega'|^{-1} \, \mathrm{d}\Lambda.\end{aligned}$$

Further using the definition of μ and the OSC, we have

$$\begin{aligned}\int_{I_\omega} f^2 \, \mathrm{d}\nu &= \int_{[\omega]} f^2 \circ \pi \, \mathrm{d}\mu \\ &= \int L_\psi^{|\omega|}(\mathbb{1}_{[\omega]}(x) f^2(\pi(x))) \, \mathrm{d}\mu(x) \\ &= \int \sum_{\tau\in I^{|\omega|}} \mathrm{e}^{S_{|\omega|}\psi(\tau x)} \mathbb{1}_{[\omega]}(\tau x) f^2(\pi(\tau x)) \, \mathrm{d}\mu(x) \\ &= \int \mathrm{e}^{S_{|\omega|}\psi(\omega x)} f^2(\pi(\omega x)) \, \mathrm{d}\mu(x) = \int_{[0,1]} (f \circ T_\omega)^2 \mathrm{e}^{S_{|\omega|}\psi\circ\pi^{-1}\circ T_\omega} \, \mathrm{d}\nu,\end{aligned}$$

where we used the fact that $\pi(\omega x) = T_\omega(\pi(x))$. □

Lemma 4.33. *There exists $C > 0$ such that for $m \in \mathbb{N}$ large enough, for all $x > \frac{C}{r_{m,\min}}$, we have*

$$\left(\frac{x r_{m,\min}}{C}\right)^{\underline{u}_m} \le N_\nu^D(x) \le 2\frac{x^{\overline{u}_m}}{R_{m,\min}^{\overline{u}_m}} + 1$$

where, for $\omega \in I^m$, we set $r_\omega := \exp(s_\omega \varphi - b_m + s_\omega \psi)$, $R_\omega := \exp(S_\omega \varphi + b_m + S_\omega \psi)$, $r_{m,\min} := \min_{i \in I^m} r_i$, and $R_{m,\min} := \min_{i \in I^m} R_i$. Here $(b_m)_m$ is the sequence defined in Lemma 4.31, and $\underline{u}_m, \overline{u}_m \in \mathbb{R}_{>0}$ denotes the unique solutions of

$$\sum_{\omega \in I^m} \mathrm{e}^{\overline{u}_m (S_\omega \varphi + S_\omega \psi + b_m)} = \sum_{\omega \in I^m} \mathrm{e}^{\underline{u}_m (s_\omega \varphi + s_\omega \psi - b_m)} = 1.$$

Proof. This proof follows the arguments used in [KL01, Lemma 2.7]. First, note that for $m \in \mathbb{N}$ sufficiently large, for all $\omega \in I^m$ we have $S_\omega \varphi + S_\omega \psi + b_m < 0$ where we used $b_m = o(m)$ and $S_\omega \psi + S_\omega \varphi \le m(\max \psi + \max \varphi)$. Therefore, there exists $\overline{u}_m \in \mathbb{R}_{>0}$ such that

$$\sum_{\omega \in I^m} R_\omega^{\overline{u}_m} = 1.$$

For $\omega := \omega_1 \cdots \omega_n \in (I^m)^n$, $n \in \mathbb{N}$, define $R_\omega := \prod_{i=1}^{|\omega|} R_{\omega_i}$ and $r_\omega := \prod_{i=1}^{|\omega|} r_{\omega_i}$. Let $x > 1$ and define for $m \in \mathbb{N}$ the following partition of $(I^m)^{\mathbb{N}}$

$$P_{m,x} := \left\{ \omega \in (I^m)^* : R_\omega < \frac{1}{x} \le R_{\omega^-} \right\}$$

with $R_{\omega^-} := \prod_{i=1}^{|\omega|-1} R_{\omega_i}$. Considering the Bernoulli measure on $(I^m)^{\mathbb{N}}$ given by the probability vector $\left(R_\omega^{\overline{u}_m}\right)_{\omega \in I^m}$ and using the fact that $P_{m,x}$ defines a partition of $(I^m)^{\mathbb{N}}$, we obtain

$$\sum_{\omega \in P_{m,x}} R_\omega^{\overline{u}_m} = 1,$$

which leads to $\mathrm{card}(P_{m.x}) \le x^{\overline{u}_m} / \left(R_{m,\min}\right)^{\overline{u}_m}$. Combining Lemma 4.31, (4.4.4), and the chain rule for differentiation, for all $\omega \in P_{m.x}$, we have

$$\begin{aligned} \Lambda(T_\omega([0,1]))\, \nu(T_\omega([0,1])) &\le \prod_{i=1}^{|\omega|} e^{S_{\omega_i} \psi} \max_{x \in [0,1]} |T'_{\omega_i}(x)| \\ &\le \prod_{i=1}^{|\omega|} e^{S_{\omega_i} \psi + S_{\omega_i} \varphi + b_m} = R_\omega < 1/x. \end{aligned}$$

We conclude from Proposition 4.4 and the OSC

$$N_\nu(x) \leq 2\,\mathrm{card}\left(P_{m,x}\right)+1 \leq 2\frac{x^{\overline{u}_m}}{\left(R_{m,\min}\right)^{\overline{u}_m}}+1.$$

For the estimate from below we define for $x > \frac{1}{r_{m,\min}}$ the following partition of $(I^m)^{\mathbb{N}}$

$$\Xi_{m,x} := \left\{\omega \in (I^m)^* : r_\omega < \frac{1}{x r_{m,\min}} \leq r_{\omega^-}\right\},$$

with $r_{\omega^-} := \prod_{i=1}^{|\omega|-1} r_{\omega_i}$. Again, there exists $\underline{u}_m \in \mathbb{R}_{>0}$ such that $\sum_{\omega\in I^m} r_\omega^{\underline{u}_m} = 1$ and we obtain $\sum_{\omega\in\Xi_{m,x}} r_\omega^{\underline{u}_m} = 1$. Hence, it follows

$$1 = \sum_{\omega\in\Xi_{m,x}} r_\omega^{\underline{u}_m} \leq \left(\frac{1}{x r_{m,\min}}\right)^{\underline{u}_m} \mathrm{card}\left(\Xi_{m,x}\right). \tag{4.4.5}$$

Fix $a \in \mathcal{K}\setminus\{0,1\}$ and choose $u_0 \in C_c^\infty((0,1))$ such that $u_0(a) > 0$. For $\omega \in \Xi_{m,x}$, we define

$$u_\omega(x) := \begin{cases} u_0\left(T_\omega^{-1}(x)\right), & x \in T_\omega((0,1)),\\ 0, & x \in [0,1]\setminus T_\omega((0,1)).\end{cases}$$

Clearly, we then have $u_\omega \in C_c^\infty(T_\omega((0,1))) \subset H_0^1$. Using Lemma 4.31, Lemma 4.32, and the chain rule for differentiation, we obtain

$$\begin{aligned}\frac{\int_{[0,1]} (\nabla u_\omega)^2 \,\mathrm{d}\Lambda}{\int_{[0,1]} u_\omega^2 \,\mathrm{d}\mu} &= \frac{\int_{[0,1]} (\nabla u_0)^2 \,|T_\omega'|^{-1} \,\mathrm{d}\Lambda}{\int_{[0,1]} u_0^2 \mathrm{e}^{S_{|\omega|}\psi\circ\pi^{-1}\circ T_\omega} \,\mathrm{d}\nu}\\ &\leq \underbrace{\frac{\int_{[0,1]} (\nabla u_0)^2 \,\mathrm{d}\Lambda}{\int_{[0,1]} u_0^2 \,\mathrm{d}\nu}}_{=:C} \prod_{i=1}^{|\omega|} \frac{1}{\min_{x\in[0,1]} |T_{\omega_i}'(x)|} e^{-s_{\omega_i}\psi}\\ &\leq C\prod_{i=1}^{|\omega|} e^{-s_{\omega_i}\varphi - s_{\omega_i}\psi + b_m} = \frac{C}{r_\omega} \leq \frac{C}{(r_{\omega^-}) r_{m,\min}} \leq C\cdot x.\end{aligned}$$

Since the supports of $(u_\omega)_{\omega\in\Xi_{m,x}}$ are disjoint, it follows that the $(u_\omega)_{\omega\in\Xi_{m,x}}$ are mutually orthogonal both in L_ν^2 and in H_0^1. Consequently, $\mathrm{span}(u_\omega : \omega \in E_t)$ is a $\mathrm{card}\left(\Xi_{m,x}\right)$-dimensional subspace of H_0^1. Therefore, for $x \geq C/r_{m,\min}$, Lemma 2.19 and (4.4.5) give

$$N_\nu^D(C\cdot x) \geq \mathrm{card}\left(\Xi_{m,x}\right) \geq \left(x r_{m,\min}\right)^{\underline{u}_m}. \qquad \square$$

In the case of self-similar measures, we obtain the following classical result of

[Fuj87].

Corollary 4.34. *Assume $T_i' \equiv \sigma_i > 0$ and $\psi(\omega) = \log(p_{\omega_1})$, for $\omega := (\omega_1\omega_2\ldots) \in I^{\mathbb{N}}$, where $(p_i)_i \in (0,1)^n$ is a given probability vector. Then, for all $x > C\left(\min_{i\in I} p_i\sigma_i\right)^{-1}$, we have*

$$x^u\left(\frac{\min_{i\in I} p_i\sigma_i}{C}\right)^u \le N_\nu^D(x) \le \frac{x^u}{(\min_{i\in I} p_i\sigma_i)^u},$$

where u is the unique solution of $\sum_{i=1}^n (\sigma_i p_i)^u = 1$.

Lemma 4.35. *For fixed $m \in \mathbb{N}$ large enough and $\underline{u}_m, \overline{u}_m \in \mathbb{R}_{>0}$ denoting the unique solutions of*

$$\sum_{\omega\in I^m} \mathrm{e}^{\underline{u}_m(S_\omega\varphi + b_{|\omega|} + S_\omega\psi)} = \sum_{\omega\in I^m} \mathrm{e}^{\overline{u}_m(s_\omega\varphi - b_{|\omega|} + s_\omega\psi)} = 1,$$

we have $\lim_{m\to\infty}\overline{u}_m = \lim_{m\to\infty}\underline{u}_m = z_\nu$.

Proof. Define for $m \in \mathbb{N}$ and $t \ge 0$

$$\begin{aligned}\underline{P}_m(t) &:= \frac{1}{m}\log\left(\sum_{\omega\in I^m}\exp(t(s_\omega\varphi - b_m + s_\omega\psi))\right),\\ \overline{P}_m(t) &:= \frac{1}{m}\log\left(\sum_{\omega\in I^m}\exp(t\,(S_\omega\varphi + b_m + S_\omega\psi))\right),\\ P_m(t) &:= \frac{1}{m}\log\left(\sum_{\omega\in I^m}\exp(tS_\omega\xi)\right).\end{aligned}$$

We obtain

$$\begin{aligned}\underline{P}_m(t) &\le P_m(t)\\ &\le \overline{P}_m(t) - t\frac{b_m}{m}\\ &= \frac{1}{m}\log\left(\sum_{\omega\in I^m}\exp(t(s_\omega\varphi + s_\omega\psi + S_\omega\varphi - s_\omega\varphi + S_\omega\psi - s_\omega\psi))\right) - t\frac{b_m}{m}\\ &\le \frac{1}{m}\log\left(\sum_{\omega\in I^m}\exp\left(t(s_\omega\varphi + s_\omega\psi) + t\left(\sum_{j=0}^{m-1}\operatorname{var}_j\psi + \sum_{j=0}^{m-1}\operatorname{var}_j\varphi\right)\right)\right) - t\frac{b_m}{m}\\ &\le \underline{P}_m(t) + \frac{t}{m}\left(\sum_{j=0}^{m-1}\operatorname{var}_j\varphi + \sum_{j=0}^{m-1}\operatorname{var}_j\psi - b_m\right).\end{aligned}$$

Using the continuity of φ, ψ and $\lim_{m\to\infty} b_m/m = 0$, we deduce

$$\lim_{m\to\infty}\overline{P}_m(t) = \lim_{m\to\infty}\underline{P}_m(t) = P(t\xi).$$

Furthermore, for all $t \geq 0$, we have

$$\begin{aligned}\underline{P}_m(t) \leq \overline{P}_m(t) &\leq t\frac{b_m}{m} + \frac{1}{m}\log\left(\sum_{\omega\in I^m} \exp\left(tm\left(\max\psi + \max\varphi\right)\right)\right)\\ &= \log(n) + t\left(\frac{b_m}{m} + (\max\psi + \max\varphi)\right).\end{aligned}$$

Observe that for m large we have $b_m/m \leq -\max\psi/2$ and each of the maps $t \mapsto \overline{P}_m(t)$, $t \mapsto \underline{P}_m(t)$ and $t \mapsto P(t)$ is decreasing and has a unique zero lying in the interval $[0, -\log(n)/(\max\psi/2 + \max\varphi)]$. Hence the statement follows from Lemma 3.5. □

We are now in the position to state our main result of this section which is an immediate consequence of Lemma 4.33 and Lemma 4.35.

Theorem 4.36. *The spectral dimension of* Δ_ν^D *exists and is equal to the unique zero* z_ν *of the pressure function as defined in* (4.4.3). *In particular* $z_\nu = q_{\mathfrak{J}_\nu}$.

Example 4.37. The natural choice of the potential ψ is given by $s\varphi$, where $s \geq 0$ is to be chosen such that $P(s\varphi) = 0$. We then have $\xi = (1+s)\varphi$ and $P((s/(s+1))\xi) = 0$. Thus, in this case, the spectral dimension can be expressed by the simple formula

$$s_\nu^D = \frac{s}{s+1}.$$

4.4.1.2 Spectral asymptotics for Gibbs measures for $C^{1+\gamma}$-cIFS under the OSC

Let μ and ν be as defined in Section 4.4.1. This section is devoted to improve Theorem 4.36 to $N_\nu^D(x) \asymp x^{z_\nu}$ under the additional assumption that ψ is Hölder continuous and the underlying IFS $\{T_1, \ldots, T_m\}$ is $C^{1+\gamma}$, which implies that the associated geometric potential φ is Hölder continuous. In this situation, the following refined bounded distortion property holds (see [KK12, Lemma 3.4]).

Lemma 4.38 (Strong Bounded Distortion Property)**.** *Assume* $T_1, \ldots, T_n$ *are* $C^{1+\gamma}$*-IFS then we have the following* strong bounded distortion property *(***sBDP***). There exists* $a_0 > 0$ *such that for all* $\omega \in I^*$ *and* $x, y \in [0,1]$ *we have*

$$a_0^{-1} \leq \frac{T'_\omega(x)}{T'_\omega(y)} \leq a_0.$$

Using the sBDP, we can improve Lemma 4.32 in the following way.

Lemma 4.39. *For all $i \in \mathbb{N}$, $\omega \in I^*$, and $x, y \in I^{\mathbb{N}}$, we have*

$$\mathrm{e}^{S_{|\omega|}\varphi(\omega y)-\log(a_0)} \le \Lambda\left(T_\omega\left([0,1]\right)\right) \le \mathrm{e}^{S_{|\omega|}\varphi(\omega y)+\log(a_0)},$$

and

$$\mathrm{e}^{S_{|\omega|}\psi(\omega x)-\sum_{n=0}^{\infty}\mathrm{var}_n(\psi)} \le \nu\left(T_\omega\left([0,1]\right)\right) \le \mathrm{e}^{S_{|\omega|}\psi(\omega x)+\sum_{n=0}^{\infty}\mathrm{var}_n(\psi)},$$

where a_0 is defined as in Lemma 4.38. In particular, we have

$$\mathrm{e}^{S_\omega\xi-d_0} \le \nu\left(T_\omega\left([0,1]\right)\right)\Lambda\left(T_\omega\left([0,1]\right)\right) \le \mathrm{e}^{S_\omega\xi+d_0},$$

with $d_0 := \log(a_0) + \sum_{n=0}^{\infty}\mathrm{var}_n(\psi)$ and $\xi := \varphi + \psi$.

Proof. Note, that we have for all $\omega \in I^*$ and $x, z \in I^{\mathbb{N}}$

$$\left|S_{|\omega|}\psi(\omega x) - S_{|\omega|}\psi(\omega z)\right| \le \sum_{k=0}^{\infty}\mathrm{var}_k(\psi)$$

and by Lemma 4.38, for all $y, v \in [0,1]$, we obtain

$$\left|\log\left(\left|T'_\omega(y)\right|\right) - \log\left(\left|T'_\omega(v)\right|\right)\right| \le \log\left(a_0\right).$$

Further, note that there exists a $y \in \mathcal{K}$ such that $\pi(x) = y$ thus we obtain $\log\left(\left|T'_\omega(y)\right|\right) = S_{|\omega|}\varphi(\omega x)$. Hence, the statement follows from Lemma 4.32. □

Lemma 4.40. *For every $t > c > 0$, we have that*

$$\Gamma_t := \{\omega \in I^* : S_\omega\xi < \log(c/t) \le S_{\omega^-}\xi\}$$

is a partition of $I^{\mathbb{N}}$. In particular, for every $\omega \in \Gamma_{t,c}$ and $x \in I^{\mathbb{N}}$, we have

$$\log(M\mathrm{e}^{d_0}t/c) \ge -S_{|\omega|}\xi(\omega x)$$

with $M := \exp\left(\max\left(-\xi\right)\right)$ and $d_0 := \log(a_0) + \sum_{k=0}^{\infty}\mathrm{var}_k(\psi)$ with a_0 defined as in Lemma 4.38.

Proof. First note, that two cylinder sets are either disjoint or one is contained in the other. From $\omega \in \Gamma_{t,c}$ and all $\eta \in I^*$, we have

$$S_{\omega\eta}\xi \le \sup_{x\in I^{\mathbb{N}}} S_{|\omega|}\xi(\omega\eta x) \le \sup_{x\in I^{\mathbb{N}}} S_{|\omega|}\xi(\omega x) = S_\omega\xi < \log(c/t),$$

where we used $\max\xi < 0$, which shows for $\eta \ne \varnothing$ that $\omega\eta \notin \Gamma_{t,c}$. Moreover, since $\min\xi < 0$, it follows that $S_\omega\xi$ converge to $-\infty$ for $|\omega| \to \infty$. Consequently, the set

$\Gamma_{t,c}$ is finite. In particular, for every $\omega \in I^{\mathbb{N}}$, we have $S_{\omega|_n}\xi \to -\infty$ as n tends to infinity. Therefore, there exists $N \in \mathbb{N}$ such that

$$S_{\omega|_N}\xi < \log(c/t) \leq S_{\omega|_{N-1}}\xi$$

and the first statement follows. For the second claim fix $\omega \in \Gamma_{t,c}$, then

$$\begin{aligned}\log(t/c) &\geq -(S_{\omega^-}\xi) = -S_{|\omega-1|}\xi(\omega x) - \left(S_{\omega^-}\xi - S_{|\omega-1|}\xi(\omega x)\right)\\ &\geq -S_{|\omega-1|}\xi(\omega x) - d_0\\ &= -S_{|\omega-1|}\xi(\omega x) - \xi\left(\sigma^{|\omega|-1}(\omega)\,x\right) + \xi\left(\sigma^{|\omega|-1}(\omega)\,x\right) - d_0\\ &\geq -S_{|\omega|}\xi(\omega x) - \log(M) - d_0,\end{aligned}$$

and hence we obtain $\log\left(Me^{d_0}t/c\right) \geq -S_{|\omega|}\xi(\omega x)$. □

Recall for $m \in \mathbb{N}$ and $x \in (I^m)^{\mathbb{N}}$

$$\xi^m(x) = \sum_{i=0}^{m-1} \xi\left(\sigma^i(x)\right).$$

Lemma 4.41. *Set* $d_0 := \log(a_0) + \sum_{k=0}^{\infty} \mathrm{var}_k\,\psi$ *where* a_0 *is defined as in Lemma 4.38. Then for* $t > c > 0$ *and* $m \in \mathbb{N}$ *such that* $-m\max\xi - d_0 > 0$, *and* $x \in (I^m)^{\mathbb{N}}$, *we have that*

$$\Gamma_{t,m}^L := \left\{\omega \in (I^m)^* : -\widetilde{S}_{|\omega|_m}\xi^m(\omega x) \leq \log(t/c) < \min_{v\in I^m} -\widetilde{S}_{|\omega v|_m}\xi^m(\omega v x)\right\}$$

defines a disjoint family, meaning $\omega \neq \omega'$ *implies* $[\omega] \cap [\omega'] = \emptyset$. *With* $k_m := \exp(-m\max\xi)$ *for every* $\omega \in (I^m)^*$

$$\log\left(te^{d_0}/(k_m c)\right) < -\widetilde{S}_{|\omega|_m}\xi^m(\omega x) \leq \log(t/c),$$

we have $\omega \in \Gamma_{t,m}^L$.

Proof. For every $\omega \in \Gamma_{t,m}^L$ and every $v \in I^m$ we have $\log(t/c) < -\widetilde{S}_{|\omega v|_m}\xi^m(\omega v x)$, implying $\omega v \notin \Gamma_{t,m}^L$. Further, using $-m\max\xi - d_0 > 0$ and the BDP, for every

$\eta \in (I^m)^* \setminus \{\varnothing\}$, we have

$$\begin{aligned}
\log(c/t) &> \widetilde{S}_{|\omega v|_m}\xi^m(\omega v x) \\
&\geq \widetilde{S}_{|\omega v|_m}\xi^m(\omega v \eta x) - d_0 \\
&= \widetilde{S}_{|\omega v|_m}\xi^m(\omega v \eta x) + \sum_{i=0}^{|\eta|_m - 1} \left(\xi^m\left(\widetilde{\sigma}^i(\eta)x\right) - \xi^m\left(\widetilde{\sigma}^i(\eta)x\right)\right) - d_0 \\
&\geq \widetilde{S}_{|\omega v \eta|_m}\xi^m(\omega v \eta x) - m \cdot |\eta|_m \max \xi - d_0 \\
&\geq \widetilde{S}_{|\omega v \eta|_m}\xi^m(\omega v \eta x) - m \cdot \max \xi - d_0 \\
&> \widetilde{S}_{|\omega v \eta|_m}\xi^m(\omega v \eta x).
\end{aligned}$$

Thus, for every $\omega \in \Gamma^L_{t,m}$ and $\eta' \in (I^m)^* \setminus \{\varnothing\}$ it follows $\omega\eta' \notin \Gamma^L_{t,m}$.

For the second assertion fix $x \in (I^m)^{\mathbb{N}}$, $\omega \in (I^m)^*$ and assume

$$\log\left(t\mathrm{e}^{d_0}/(k_m c)\right) < -\widetilde{S}_{|\omega|_m}\xi^m(\omega x) \leq \log(t/c).$$

Using the BDP, for all $v \in I^m$, $\omega \in (I^m)^*$, we obtain

$$\left|\widetilde{S}_{|\omega|_m}\xi^m(\omega x) - \widetilde{S}_{|\omega|_m}\xi^m(\omega v x)\right| \leq d_0,$$

and consequently,

$$\begin{aligned}
\log(t/c) &< \log(k_m) - d_0 - \widetilde{S}_{|\omega|_m}\xi^m(\omega x) \\
&\leq \log(k_m) + \xi^m(vx) - \xi^m(vx) - \widetilde{S}_{|\omega|_m}\xi^m(\omega v x) \\
&\leq -\widetilde{S}_{|\omega v|_m}\xi^m(\omega v x).
\end{aligned}$$

Since $-\widetilde{S}_{|\omega|_m}\xi^m(\omega x) \leq \log(t/c)$, we conclude $\omega \in \Gamma^L_{t,m}$. □

Now we are in the position to prove the main theorem of this section.

Theorem 4.42. *We have*

$$N^D_\nu(t) \asymp t^{z_\nu},$$

where z_ν is the unique zero of the pressure function as defined in (4.4.3).

Proof. Let $t > e^{-d_0}$ and $\omega \in \Gamma_{t,e^{-d_0}}$. Then, by Lemma 4.39 and the definition of $\omega \in \Gamma_{t,e^{-d_0}}$, it follows

$$\nu\left(T_\omega([0,1])\right)\Lambda\left(T_\omega([0,1])\right) \leq \mathrm{e}^{S_\omega \xi + d_0} < 1/t.$$

Then from Lemma 4.40 and Proposition 4.4 we infer

$$N_\nu^D(t) \le 2\,\mathrm{card}\left(\Gamma_{t,e^{-d_0}}\right) + 1.$$

Hence, for the upper bound, we are left to show that $\mathrm{card}\left(\Gamma_{t,c}\right) \ll t^{z_\nu}$. For this we use [Kom18, Theorem 3.2] adapted to our situation, i.e.

$$Z(x,t) := \sum_{n=0}^{\infty} \sum_{\sigma^n y = x} \mathbb{1}_{\{-S_n \xi(y) \le \log(t)\}} \sim G(x, \log(t)) t^{z_\nu},$$

where $(x,s) \mapsto G(x,s)$, defined on $I^{\mathbb{N}} \times \mathbb{R}_{>0}$, is bounded from above by inspecting the corresponding function G in [Kom18, Theorem 3.2], and $s \mapsto G(x,s)$ is a constant function in the aperiodic case and a periodic function in the periodic case. Therefore, for fixed $y \in I^{\mathbb{N}}$, with $M := \exp(\max -\xi)$, we have by the second assertion of Lemma 4.40

$$\mathrm{card}\left(\Gamma_t^R\right) \le Z\left(y, M e^{d_0} t/\lambda\right) \ll t^{z_\nu}.$$

For the lower estimate we use an approximation argument involving the strong bounded distortion property. Let $a \in \mathcal{K} \setminus \{0,1\}$. Fix $u_0 \in C_c^\infty((0,1))$ such that $u_0(a) > 0$ and define

$$c_0 := e^{d_0} \frac{\int_{[0,1]} (\nabla u_0)^2 \, \mathrm{d}\Lambda}{\int_{[0,1]} u_0^2 \, \mathrm{d}\nu},$$

where d_0 is defined as in Lemma 4.40. Applying Lemma 4.41 with $x \in (I^m)^{\mathbb{N}}$ and $m \in \mathbb{N}$ such that $\log(k_m) - d_0 > 0$ with $k_m = \exp(-\max \xi^m)$ yields

$$\left\{\omega \in (I^m)^* : \log(t e^{d_0}/(k_m c_0)) < -\widetilde{S}_{|\omega|_m} \xi^m(\omega x) \le \log(t/c_0)\right\} \subset \Gamma_{t,m}^L$$

with

$$\Gamma_{t,m}^L = \left\{\omega \in (I^m)^* : \; -\widetilde{S}_{|\omega|_m} \xi^m(\omega x) \le \log(t/c_0) < \min_{v \in I^m} -\widetilde{S}_{|\omega v|_m} \xi^m(\omega v x)\right\}.$$

For any $\omega \in \Gamma_{t,m}^L$, we define

$$u_\omega(x) := \begin{cases} u_0 \circ T_\omega^{-1}(x), & x \in T_\omega((0,1)), \\ 0, & x \in [0,1] \setminus T_\omega((0,1)), \end{cases}$$

which is an element of $C_c^\infty(T_\omega(0,1))$. Then, by Lemma 4.32, we have

$$\frac{\int_{I_\omega}(\nabla u_\omega)^2\,\mathrm{d}\Lambda}{\int_{I_\omega}u_\omega^2\,\mathrm{d}\nu}=\frac{\int_{[0,1]}(\nabla(u_\omega\circ T_\omega))^2\left|T_\omega'\right|^{-1}\,\mathrm{d}\Lambda}{\int_{[0,1]}(u_\omega\circ T_\omega)^2\mathrm{e}^{S_{|\omega|}\psi\circ\pi^{-1}\circ T_\omega}\,\mathrm{d}\nu}=\frac{\int_{[0,1]}(\nabla(u_0))^2\left|T_\omega'\right|^{-1}\,\mathrm{d}\Lambda}{\int_{[0,1]}u_0^2\mathrm{e}^{S_{|\omega|}\psi\circ\pi^{-1}\circ T_\omega}\,\mathrm{d}\nu}$$
$$\leq c_0e^{-\widetilde{S}_{|\omega|_m}\xi^m(\omega x)}\leq t.$$

Since the supports of $(u_\omega)_{\omega\in\Gamma_{t,m}^L}$ are disjoint, it follows that the $(u_\omega)_{\omega\in\Gamma_{t,m}^L}$ are mutually orthogonal both in L_ν^2 and in H_0^1. Hence, $\operatorname{span}\left(u_\omega:\omega\in\Gamma_{t,m}^L\right)$ is a $\operatorname{card}\left(\Gamma_{t,m}^L\right)$-dimensional subspace of H_0^1. Hence, Lemma 2.19 yields

$$\begin{aligned}N_\nu^D(t)&\geq\operatorname{card}\left(\Gamma_{t,m}^L\right)\\&\geq\operatorname{card}\left(\left\{\omega\in(I^m)^*:\log(t\mathrm{e}^{d_0}/(k_mc))<-\widetilde{S}_{|\omega|_m}\xi^m(\omega x)\leq\log(t/c)\right\}\right).\end{aligned}$$

We conclude

$$N_\nu^D(t)$$
$$\geq\sum_{n=0}^\infty\sum_{\omega\in(I^m)^n}\mathbb{1}_{\left\{-\widetilde{S}_{|\omega|_m}\xi^m(\omega x)\leq\log(t/c)\right\}}-\sum_{n=0}^\infty\sum_{\omega\in(I^m)^n}\mathbb{1}_{\left\{-\widetilde{S}_{|\omega|_m}\xi^m(\omega x)\leq\log(t\mathrm{e}^{d_0}/(k_mc))\right\}}.$$

Moreover, by Lemma 4.30 we have $0=P(z_\nu\xi)=P_{\widetilde{\sigma}}(z_\nu\xi^m)$ as defined in Lemma 4.30. Again, [Kom18, Theorem 3.2] applied to ξ^m gives that there exists a function $(x,s)\mapsto\widetilde{G}(x,s)$ defined on $(I^m)^{\mathbb{N}}\times\mathbb{R}_{>0}$, which is bounded away from zero by inspecting the corresponding function G in [Kom18, Theorem 3.2], such that

$$\widetilde{Z}(x,t):=\sum_{n=0}^\infty\sum_{\omega\in(I^m)^n}\mathbb{1}_{\left\{-\widetilde{S}_{|\omega|_m}\xi^m(\omega x)\leq\log(t)\right\}}\sim\widetilde{G}(x,\log(t))t^{z_\nu}.$$

In the aperiodic case $s\mapsto\widetilde{G}(x,s)$ is a constant function and hence in this case we immediately get $t^{z_\nu}\ll N_{\nu,\Lambda}(t)$. In the periodic case, $s\mapsto\widetilde{G}(x,s)$ is periodic with minimal period $a>0$. To end this, we choose m large enough such that

$\lceil a/(\log(k_m) - d_0) \rceil = 1$ and $\exp(d_0)/k_m < 1$. We finally then have

$$\begin{aligned}
N_\nu^D(t) &\geq \widetilde{Z}(x, t/c) - \widetilde{Z}\left(x, \left(\mathrm{e}^{d_0}/k_m\right)(t/c)\right) \\
&\geq \widetilde{Z}(x, t/c) - \widetilde{Z}\left(x, \left(\mathrm{e}^{d_0}/k_m\right)^{a/(\log(k_m)-d_0)}(t/c)\right) \\
&\sim \widetilde{G}(x, \log(t/c))\left(\frac{t}{c}\right)^{z_\nu} - \widetilde{G}(x, \log(t/c))\left(\frac{t}{c}\right)^{z_\nu}\left(\left(\mathrm{e}^{d_0}/k_m\right)^{a/(\log(k_m)-d_0)}\right)^{z_\nu} \\
&= t^{z_\nu}\frac{\widetilde{G}(x, \log(t/c))}{c^{z_\nu}}\left(1 - \left(\mathrm{e}^{d_0}/k_m\right)^{z_\nu a/(\log(k_m)-d_0)}\right) \\
&\gg t^{z_\nu},
\end{aligned}$$

where we used $\log(t/c) - \log\left(\left(\mathrm{e}^{d_0}/k_m\right)^{a/(\log(k_m)-d_0)} t/c\right) = a$. □

4.4.1.3 Spectral dimension of weak Gibbs measures with overlap

This section relies on results from [PS00; Fen07; BF20] on the L^q-spectrum together with the regularity result stated in Section 4.3.2. Let ν and μ be defined as in Section 4.4.1 such that $\nu(\{0,1\}) = 0$. Here we do not assume any separation conditions for the C^1-cIFS Φ.

Recall that Φ is non-trivial, i.e. there is more than one contraction and the T_i's do not share a common fixed point. Hence, by Proposition 2.48, it follows that self-similar measures with or without OSC are atomless as long as Φ is non-trivial. However, it is an open question under which condition the same applies to weak Gibbs measures without OSC.

First, we will prove that the L^q-spectrum of ν exists as a limit on $(0,1]$. Combining this with Corollary 4.12 we conclude that the spectral dimension exists and is given by $q_{\mathfrak{J}_\nu}$. To this end we need the following lemmas.

Lemma 4.43. *We have for any $G \subset I^*$ with $\biguplus_{u \in G}[u] = I^{\mathbb{N}}$ and $E \in \mathfrak{B}([0,1])$ that*

$$\nu(E) \geq \sum_{u \in G} c_{|u|}\mu([u])\,\nu\left(T_u^{-1}(E)\right)$$

with $c_n := \mathrm{e}^{-\sum_{i=0}^{n-1}\mathrm{var}_i(\psi)}$ (and therefore $\log(c_n) = o(n)$).

Proof. For all $E \in \mathfrak{B}([0,1])$ and $u \in I^*$, we have

$$\begin{aligned}\mu\left(\pi^{-1}(E) \cap [u]\right) &= \int_{[u]} \mathbb{1}_E \circ \pi \, \mathrm{d}\mu = \int L_\psi^{|u|}(\mathbb{1}_{[u]}(x)\mathbb{1}_E(\pi(x))) \, \mathrm{d}\mu(x) \\ &= \int \sum_{\tau \in I^{|u|}} \mathrm{e}^{S_{|u|}\psi(\tau x)} \mathbb{1}_{[u]}(\tau x) \mathbb{1}_E(\pi(\tau x)) \, \mathrm{d}\mu(x) \\ &= \int \mathrm{e}^{S_{|u|}\psi(ux)} \mathbb{1}_E(\pi(ux)) \, \mathrm{d}\mu(x) \\ &\geq \mathrm{e}^{-\sum_{i=0}^{|u|-1} \mathrm{var}_i(\psi)} \nu\left(T_u^{-1}(E)\right) \mu([u]) .\end{aligned}$$

Setting $c_n := \mathrm{e}^{-\sum_{i=0}^{n-1} \mathrm{var}_i(\psi)}$ and summing over $u \in G$, we obtain

$$\nu(E) = \sum_{u \in G} \mu\left(\pi^{-1}(E) \cap [u]\right) \geq \sum_{u \in G} c_{|u|} \mu([u]) \, \nu\left(T_u^{-1}(E)\right).$$

Also, the continuity of the potential ψ implies $\log(c_n) = o(n)$. □

For $u \in I^*$ let us define $\mathcal{K}_u := T_u(\mathcal{K})$. Then, for $n \geq 2$, the set

$$W_n := \{u \in I^* : \mathrm{diam}(\mathcal{K}_u) \leq 2^{-n} < \mathrm{diam}(\mathcal{K}_{u^-})\}$$

defines a partition of $I^{\mathbb{N}}$.

Lemma 4.44. *For any $0 < q < 1$ there exists a sequence $(s_n)_n \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\log(s_n) = o(n)$ and for every $n, m \in \mathbb{N}$ and $\widetilde{Q} \in \mathcal{D}_n^N$, we have*

$$\sum_{\substack{B \in \mathcal{D}_n^N \\ B \sim \widetilde{Q}}} \sum_{\substack{Q \in \mathcal{D}_{m+n}^N, \\ Q \subset B}} \nu(Q)^q \geq s_n \nu(\widetilde{Q})^q \min_{u \in W_n} \sum_{Q \in \mathcal{D}_{m+n}^N} \nu\left(T_u^{-1}(Q)\right)^q$$

where $B \sim \widetilde{Q}$ means that the closures of B and $\widetilde{Q}$ intersect.

Proof. As in [PS00] for $n, m \in \mathbb{N}$, $u \in W_n$, and $A \in \mathcal{D}_n^N$, let us define

$$w(u, A) := \sum_{Q \in \mathcal{D}_{n+m}^N : Q \subset A} \nu\left(T_u^{-1}(Q)\right)^q .$$

The interval $A \in \mathcal{D}_n^N$ on which $w(u, A)$ attains its maximum will be called q-heavy for $u \in W_n$. We will denote the q-heavy box by $H(u)$ (if there are more than one interval which maximizes $w(u, \cdot)$, we choose one of them arbitrarily). Note that every $\mathcal{K}_u$ with $u \in W_n$ intersects at most 3 intervals in $\mathcal{D}_n^N$. Hence, we obtain for all

$u \in W_n$

$$\sum_{Q \in \mathcal{D}_{n+m}^N} \nu\left(T_u^{-1}(Q)\right)^q = \sum_{B \in \mathcal{D}_n^N} \sum_{Q \in \mathcal{D}_{n+m}^N : Q \subset B} \nu\left(T_u^{-1}(Q)\right)^q$$
$$\leq 3 \sum_{Q \in \mathcal{D}_{n+m}^N : Q \subset H(u)} \nu\left(T_u^{-1}(Q)\right)^q .$$

This leads to

$$\sum_{\substack{Q \in \mathcal{D}_{n+m}^N : \\ Q \subset H(u)}} \nu\left(T_u^{-1}(Q)\right)^q \geq \sum_{Q \in \mathcal{D}_{n+m}^N} \frac{\nu\left(T_u^{-1}(Q)\right)^q}{3} \geq \min_{v \in W_n} \sum_{Q \in \mathcal{D}_{n+m}^N} \frac{\nu\left(T_v^{-1}(Q)\right)^q}{3}. \quad (4.4.6)$$

Further, for every $Q \in \mathcal{D}_{n+m}^N$ and $B \in \mathcal{D}_n^N$, by Lemma 4.43, we have

$$\nu(Q) \geq \sum_{u \in W_n} c_{|u|} \mu([u]) \nu\left(T_u^{-1}(Q)\right) \geq \sum_{u \in W_n : B = H(u)} c_{|u|} \mu([u]) \nu\left(T_u^{-1}(Q)\right)$$
$$\geq \left(\min_{u \in W_n} c_{|u|}\right) \sum_{u \in W_n : B = H(u)} \mu([u]) \nu\left(T_u^{-1}(Q)\right).$$

Setting

$$p_-(B) \coloneqq \sum_{u \in W_n : B = H(u)} \mu([u]),$$

and if $p_-(B) > 0$, using the concavity of the function $x \mapsto x^q$ for $0 < q < 1$, we obtain

$$\nu(Q)^q \geq p_-(B)^q \left(\min_{u \in W_n} c_{|u|}\right)^q \left(\sum_{u \in W_n : B = H(u)} \frac{\mu([u]) \nu\left(T_u^{-1}(Q)\right)}{p_-(B)}\right)^q$$
$$\geq p_-(B)^{q-1} \left(\min_{u \in W_n} c_{|u|}\right)^q \sum_{u \in W_n : B = H(u)} \mu([u]) \nu\left(T_u^{-1}(Q)\right)^q .$$

Summing over $Q \in \mathcal{D}_{n+m}^N$ with $Q \subset B$, and using (4.4.6), we infer

$$\sum_{\substack{Q \subset B, \\ Q \in \mathcal{D}_{n+m}^N}} \nu(Q)^q \geq p_-(B)^{q-1} \left(\min_{u \in W_n} c_{|u|}\right)^q \sum_{u \in W_n : B = H(u)} \mu([u]) \sum_{\substack{Q \subset B, \\ Q \in \mathcal{D}_{n+m}^N}} \nu\left(T_u^{-1}(Q)\right)^q$$
$$\geq \frac{p_-(B)^q}{3} \left(\min_{u \in W_n} c_{|u|}\right)^q \min_{v \in W_n} \sum_{Q \in \mathcal{D}_{n+m}^N} \nu\left(T_v^{-1}(Q)\right)^q,$$

which is also valid in the case $p_-(B) = 0$. For $\widetilde{Q} \in \mathcal{D}_n^N$ and $u \in W_n$ with $\mathcal{K}_u \cap \widetilde{Q} \neq \emptyset$, we have $\mathcal{K}_u \subset \bigcup_{B \sim \widetilde{Q}, B \in \mathcal{D}_n^N} B$, as a consequence of $\operatorname{diam}(\mathcal{K}_u) \leq 2^{-n}$. In particular, every $\mathcal{K}_u$ that intersects $\widetilde{Q}$ must have an interval $B \in \mathcal{D}_n^N$ with $B \sim \widetilde{Q}$ which is q-heavy for u. Hence, we obtain

$$\nu\left(\widetilde{Q}\right) \leq \sum_{u \in W_n : \mathcal{K}_u \cap \widetilde{Q} \neq \emptyset} \mu([u]) \leq \sum_{B \sim \widetilde{Q} : B \in \mathcal{D}_n^N} \sum_{u \in W_n : B = H(u)} \mu([u]) = \sum_{B \sim \widetilde{Q} : B \in \mathcal{D}_n^N} p_-(B).$$

Using $0 < q < 1$, we conclude

$$\nu\left(\widetilde{Q}\right)^q \leq \left(\sum_{B \sim \widetilde{Q} : B \in \mathcal{D}_n^N} p_-(B)\right)^q \leq \sum_{B \sim \widetilde{Q} : B \in \mathcal{D}_n^N} p_-(B)^q.$$

Summing over all $B \in \mathcal{D}_n^N$ with $B \sim \widetilde{Q}$ gives

$$\begin{aligned}\sum_{\substack{B \sim \widetilde{Q}: \\ B \in \mathcal{D}_n^N}} \sum_{\substack{Q \subset B: \\ Q \in \mathcal{D}_{n+m}^N}} \nu(Q)^q &\geq \sum_{B \sim \widetilde{Q} : B \in \mathcal{D}_n^N} \frac{p_-(B)^q}{3} \left(\min_{u \in W_n} c_{|u|}\right)^q \min_{v \in W_n} \sum_{Q \in \mathcal{D}_{n+m}^N} \nu\left(T_v^{-1}(Q)\right)^q \\ &\geq \frac{\nu\left(\widetilde{Q}\right)^q}{3} \left(\min_{u \in W_n} c_{|u|}\right)^q \min_{v \in W_n} \sum_{Q \in \mathcal{D}_{n+m}^N} \nu\left(T_v^{-1}(Q)\right)^q.\end{aligned}$$

Note that for every $u \in W_n$, by the definition of W_n, we have

$$|u| < \frac{n \log(2) - \log(\alpha_{\max})}{-\log(\alpha_{\max})}$$

with $\alpha_{\max} := \max_{i=1,\ldots,n} \max_{x \in [0,1]} \left|T_i'(x)\right|$. Thus, setting $s_n := 3^{-1} \min_{u \in W_n} c_{|u|}^q$, we find that $\lim_{n \to \infty} n^{-1} \log(s_n) = 0$, where we used the elementary fact that for any two sequences $(x_n)_n \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $(y_n)_n \in \mathbb{N}^{\mathbb{N}}$ with $x_n = \mathrm{o}(n)$, $y_n \ll n$, we have $x_{y_n} = \mathrm{o}(n)$. □

Proposition 4.45. *The L^q-spectrum β_ν of ν exists on $(0,1]$ as limit.*

Proof. Let $0 < q < 1$. From [Fen07, Proposition 3.3] (which holds true for all Borel probability measures with support $\mathcal{K}$, see remark after Proposition 3.3 in [Fen07]) it follows that there exists a sequence $\left(b_{q,n}\right)_n$ of positive numbers with $\log(b_{q,n}) = o(n)$, such that for all $m, n \in \mathbb{N}$ and $u \in W_n$

$$b_{q,n} \sum_{Q \in \mathcal{D}_m^N} \nu(Q)^q \leq \sum_{C \in \mathcal{D}_{m+n}^N} \nu\left(T_u^{-1}(Q)\right)^q.$$

In tandem with Lemma 4.44, for every $\widetilde{Q} \in \mathcal{D}_n^N$, we obtain

$$\sum_{B\in\mathcal{D}_n^N, B\sim\widetilde{Q}} \sum_{Q\in\mathcal{D}_{m+n}^N : Q\subset B} \nu(Q)^q \geq s_n \nu(\widetilde{Q})^q \min_{u\in W_n} \sum_{Q\in\mathcal{D}_{m+n}^N} \nu\left(T_u^{-1}(Q)\right)^q$$
$$\geq \left(b_{q,n}s_n\right)\nu(\widetilde{Q})^q \sum_{Q\in\mathcal{D}_m^N} \nu(Q)^q.$$

Clearly, $\log\left(b_{q,n}s_n\right) = o(n)$. Hence, we can apply [Fen07, Proposition 4.4], which shows that β_ν exists as a limit on $(0,1]$. □

With this knowledge, we obtain the following theorem.

Theorem 4.46. *The spectral dimension of Δ_ν^D exists and equals $q_{\mathfrak{J}_\nu}$.*

Proof. The proof follows from Corollary 4.12 and Proposition 4.45. □

Corollary 4.47. *Let ν be a self-conformal measure on $[0,1]$. Then the spectral dimension of Δ_ν^D exists and equals $q_{\mathfrak{J}_\nu}$.*

4.4.2 Homogeneous Cantor measures

Let us recall the construction of general homogeneous Cantor measures as in [Arz14; Min20; BH97], allowing us to construct examples for which the spectral dimension does not exist. Let J be finite or countably infinite subset of $\mathbb{N}$. For every $j \in J$ we define an iterated function system $\mathcal{S}^{(j)}$. For $i = 1,2$ let $S_i^{(j)} : [a,b] \to [a,b]$ be defined by

$$S_i^{(j)}(x) = r_i^{(j)}x + c_i^{(j)}$$

with $r_i^{(j)} \in (0,1)$ and $c_i^{(j)} \in \mathbb{R}$ are chosen such that

$$a = S_1^{(j)}(a) < S_1^{(j)}(b) \leq S_2^{(j)}(a) < S_2^{(j)}(b) = b.$$

This ensures the open set condition. We define $\mathcal{S}^{(j)} = \left(S_1^{(j)}, S_2^{(j)}\right)$. Moreover let us define an *environment sequence* $\xi := \left(\xi_i\right)_i \in J^{\mathbb{N}}$. Each ξ_i represents an iterated function system $\mathcal{S}^{(\xi_i)}$. To give a suitable coding, we define the following word space $W_n := \{1,2\}^n$ of words with length n. For $n \in \mathbb{N}$ and $\omega := (\omega_1 \dots \omega_n) \in W_n$, we set

$$S_\omega^{(\xi)} := S_{\omega_1}^{(\xi_1)} \circ S_{\omega_2}^{(\xi_2)} \circ \cdots \circ S_{\omega_n}^{(\xi_n)}.$$

Now, for any environment sequence ξ, we construct a probability measure $\nu^{(\xi)}$ on $[a,b]$ with support $K^{(\xi)}$ defined by

$$K^{(\xi)} := \bigcap_{n=1}^{\infty} \bigcup_{\omega\in W_n} \left(S_{\omega_1}^{(\xi_1)} \circ S_{\omega_2}^{(\xi_2)} \circ \cdots \circ S_{\omega_n}^{(\xi_n)}\right)([a,b]).$$

For any $j \in J$, let $\left(p_1^j, p_2^j\right) \in (0,1)^2$ with $\sum_{i=1}^{2} p_i^j = 1$. Moreover, for $n \in \mathbb{N}$ and $\omega := (\omega_1 \dots \omega_n) \in W_n$, we define $p_\omega^{(\xi)} := \prod_{i=1}^{n} p_{\omega_i}^{(\xi_i)}$. Then define the following sequence of probability measures: $\nu_0 := \frac{1}{b-a}\Lambda|_{[a,b]}$ and for $n \in \mathbb{N}$ and $A \in \mathfrak{B}([a,b])$

$$\nu_n(A) := \sum_{\omega \in W_n} p_\omega^{(\xi)} \nu_0\left(\left(S_\omega^{(\xi)}\right)^{-1}(A)\right).$$

Then we define $\nu^{(\xi)}$ by

$$\nu^{(\xi)}(A) = \lim_{n\to\infty} \nu_n(A).$$

The theorem of Vitali-Hahn-Saks ensures that $\nu^{(\xi)}$ is a probability measure on $\mathfrak{B}([a,b])$ (see [Arz14, Lemma 3.1.2]) and for every $n \in \mathbb{N}$ and $\omega \in W_n$, we have

$$\nu^{(\xi)}\left(S_\omega^{(\xi)}([a,b])\right) = p_\omega^{(\xi)}.$$

Example 4.48 (Homogeneous Cantor measure with non-converging L^q-spectrum)**.** Now, let us consider the following environment

$$\xi_i := \begin{cases} 1, & \exists \ell \in \mathbb{N}_0 : 2^{2\ell} < i \le 2^{2\ell+1}, \\ 2, & \exists \ell \in \mathbb{N}_0 : 2^{2\ell+1} < i \le 2^{2\ell+2}, \end{cases}$$

$S_1^{(1)}(x) := \frac{x}{4}$, $S_2^{(1)}(x) := \frac{x}{4} + \frac{3}{4}$, $S_1^{(2)}(x) := \frac{x}{16}$, and $S_2^{(2)}(x) := \frac{x}{16} + \frac{15}{16}$ for $x \in [0,1]$. Furthermore, let $p_1, p_2 \in (0,1)$ be with $p_1 + p_2 = 1$. Then we define for every $j \in \{1,2\} : p_1^j := p_1$ and $p_2^j := p_2$. First, observe that for $n \in \mathbb{N}$ and $\omega \in W_n$,

$$\Lambda\left(S_\omega^{(\xi)}([0,1])\right) = \begin{cases} 2^{-(8/3)\cdot 2^{2l+1} + 10/3 - 4\left(n - 2^{2l+1}\right)}, & 2^{2l+1} < n \le 2^{2l+2}, \\ 2^{-10/3\cdot 2^{2l+2} + 10/3 - 2\left(n - 2^{2l+2}\right)}, & 2^{2l+2} < n \le 2^{2l+3}. \end{cases}$$

Define for $n \in \mathbb{N}$

$$R(n) := \begin{cases} -(8/3)\cdot 2^{2\ell+1} + 10/3 - 4\left(n - 2^{2\ell+1}\right), & 2^{2\ell+1} < n \le 2^{2\ell+2}, \\ -(10/3)\cdot 2^{2\ell+2} + 10/3 - 2\left(n - 2^{2\ell+2}\right), & 2^{2\ell+2} < n \le 2^{2\ell+3}. \end{cases}$$

Hence, for all $n\in\mathbb{N}$ with $2^{2l+1}<n\leq 2^{2l+2}$, $q\geq 0$, and $\omega\in W_n$, we obtain

$$\begin{aligned}
&\frac{1}{-\log\left(\Lambda\left(S_\omega^{(\xi)}\left([0,1]\right)\right)\right)}\log\left(\sum_{\substack{l=0,\\ \nu\left(\left(l2^{R(n)},(l+1)2^{R(n)}\right]\right)>0}}^{2^{-R(n)}-1}\nu^{(\xi)}\left(\left(l2^{R(n)},(l+1)2^{R(n)}\right]\right)^q\right)\\
&=\frac{1}{-\log\left(\Lambda\left(S_\omega^{(\xi)}\left([0,1]\right)\right)\right)}\log\left(\sum_{l\in W_n}\nu^{(\xi)}\left(S_l^{(\xi)}\left([a,b]\right)\right)^q\right)\\
&=\frac{\log_2\left(p_1^q+p_2^q\right)}{-4\cdot\dfrac{2^{2l+1}}{3n}+4-10/(3n)}
\end{aligned}$$

and, for $2^{2l+2}<n\leq 2^{2l+3}$,

$$\begin{aligned}
\frac{1}{-\log\left(\Lambda\left(S_\omega^{(\xi)}\left([0,1]\right)\right)\right)}\log\left(\sum_{l\in W_n}\nu^{(\xi)}\left(S_l^{(\xi)}\left([a,b]\right)\right)^q\right)&=\frac{\log\left(p_1^q+p_2^q\right)}{-\log\left(\Lambda\left(S_\omega^{(\xi)}\left([0,1]\right)\right)\right)}\\
&=\frac{\log_2\left(p_1^q+p_2^q\right)}{4\frac{2^{2l+2}}{3n}+2-10/(3n)}.
\end{aligned}$$

Therefore, by Fact 2.30,

$$\beta_{\nu^{(\xi)}}(q)=\begin{cases}\frac{3}{8}\log_2\left(p_1^q+p_2^q\right) & \text{for } 0\leq q\leq 1,\\ \frac{3}{10}\log_2\left(p_1^q+p_2^q\right) & \text{for } q>1\end{cases}$$

and

$$\underline{\beta}_{\nu^{(\xi)}}(q):=\liminf_{n\to\infty}\beta_n(q)=\begin{cases}\frac{3}{10}\log_2\left(p_1^q+p_2^q\right) & \text{for } 0\leq q\leq 1\\ \frac{3}{8}\log_2\left(p_1^q+p_2^q\right) & \text{for } q>1.\end{cases}$$

Hence, by Theorem 4.10, the upper spectral dimension of $\Delta^D_{\nu^{(\xi)}}$ is given by the unique solution of

$$p_1^{q_{\mathfrak{I}_{\nu^{(\xi)}}}}+p_2^{q_{\mathfrak{I}_{\nu^{(\xi)}}}}=2^{\left(8\cdot q_{\mathfrak{I}_{\nu^{(\xi)}}}\right)/3}.$$

Furthermore, the unique fixed point of $\underline{\beta}_{\nu^{(\xi)}}$, denoted by $\underline{q}_{\mathfrak{I}_{\nu^{(\xi)}}}$, is the unique solution of

$$\left(2^{-\frac{10}{3}}p_1\right)^{\underline{q}_{\mathfrak{I}_{\nu^{(\xi)}}}}+\left(2^{-\frac{10}{3}}p_2\right)^{\underline{q}_{\mathfrak{I}_{\nu^{(\xi)}}}}=1.$$

See Figure 4.4.1 for the two graphs.

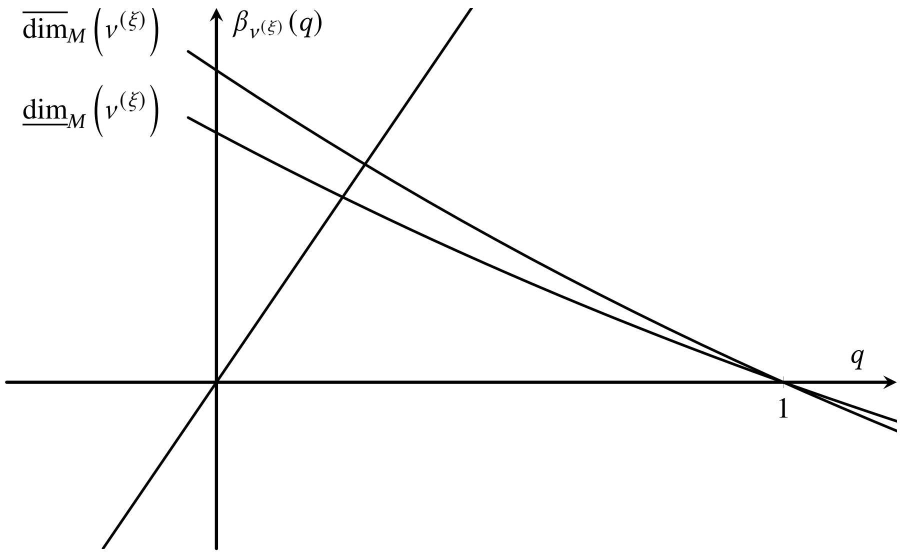

Figure 4.4.1 Illustration of $\beta_{\nu^{(\xi)}}$ and $\liminf_{n\to\infty} \beta_n^{\nu^{(\xi)}}$ with $p_1 = 0.25$.

For the special case $p_1 = 1/2$, we obtain

$$\beta_{\nu^{(\xi)}}(q) = \begin{cases} \frac{3}{8}(1-q) & \text{for } 0 \le q \le 1, \\ \frac{3}{10}(1-q) & \text{for } q > 1. \end{cases}$$

and

$$\underline{\beta}_{\nu^{(\xi)}}(q) := \liminf_{n\to\infty} \beta_n^{\nu^{(\xi)}}(q) = \begin{cases} \frac{3}{10}(1-q) & \text{for } 0 \le q \le 1, \\ \frac{3}{8}(1-q) & \text{for } q > 1. \end{cases}$$

In this special case, we have, $\underline{\dim}_M(\nu) = 3/10$ and $\overline{\dim}_M(\nu) = 3/8$. Now, applying Proposition 4.9 in tandem with Theorem 4.10, we conclude

$$\underline{s}^D_{\nu^{(\xi)}} \le \frac{\underline{\dim}_M(\nu)}{1 + \underline{\dim}_M(\nu)} = \frac{3}{13} < \frac{3}{11} = \frac{\overline{\dim}_M(\nu)}{1 + \overline{\dim}_M(\nu)} = \overline{s}^D_{\nu^{(\xi)}}.$$

Now, let us prove that $\underline{s}^D_{\nu^{(\xi)}} = \frac{3}{13}$. Note that for $m = 1 + \frac{1}{2^7}$ and all $\omega \in W_n$, there exists $\eta \in W_2$ such that

$$S^{(\xi)}_{\omega\eta}([0,1]) \subset \left\langle S^{(\xi)}_{\omega}([0,1]) \right\rangle_{1/m}.$$

Hence, we obtain

$$\nu\left(\left\langle S^{(\xi)}_{\omega}([0,1]) \right\rangle_{1/m}\right) \Lambda\left(\left\langle S^{(\xi)}_{\omega}([0,1]) \right\rangle_{1/m}\right) \ge \frac{2^{-n+R(n)-2}}{m}. \tag{4.4.7}$$

Recall that by Proposition 4.1, for $x>0$,

$$\mathcal{N}_m^L(x)=\sup\left\{\operatorname{card}(P): P\in\Pi_0:\min_{C\in P}\nu\left(\langle I\rangle_{1/m}\right)\Lambda\left(\langle I\rangle_{1/m}\right)\geq\frac{4}{x(m-1)}\right\}\leq N_\nu(x).$$

Now, with $x_n=m\frac{2^{n-R(n)+4}}{m-1}$, by (4.4.7), we have that $\mathcal{N}_m^L(x_n)\geq 2^n$, implying

$$\begin{aligned}
&\frac{\log\left(N_\nu^D(x_n)\right)}{\log(x_n)}\\
&\geq\left(1-\frac{R(n)}{n}+\frac{4}{n}+\frac{\log\left(m/(m-1)\right)}{n\log(2)}\right)^{-1}\\
&=\begin{cases}\left(1+(8/3)\cdot\frac{2^{2l+1}-10/3+4\left(n-2^{2l+1}\right)+4}{n}-\frac{\log(m/(m-1))}{n\log(2)}\right)^{-1}, & 2^{2l+1}<n\leq 2^{2l+2},\\ \left(1+\frac{(10/3)\cdot 2^{2l+2}-10/3+2\left(n-2^{2l+2}\right)+4}{n}-\frac{\log(m/(m-1))}{n\log(2)}\right)^{-1}, & 2^{2l+2}<n\leq 2^{2l+3},\end{cases}\\
&\geq\begin{cases}\left(1+\frac{8}{3}+(4-10/3)/n-\frac{\log(m/(m-1))}{n\log(2)}\right)^{-1}, & 2^{2l+1}<n\leq 2^{2l+2},\\ \left(1+\frac{10}{3}+(4-10/3)/n-\frac{\log(m/(m-1))}{n\log(2)}\right)^{-1}, & 2^{2l+2}<n\leq 2^{2l+3}.\end{cases}
\end{aligned}$$

This shows

$$\liminf_{n\to\infty}\frac{\log\left(N_\nu^D(x_n)\right)}{\log(x_n)}\geq\frac{1}{1+10/3}.$$

Since $x_n\leq x_{n+1}\leq 2^4x_n$, we infer

$$\liminf_{x\to\infty}\frac{\log\left(N_\nu^D(x)\right)}{\log(x)}\geq\liminf_{n\to\infty}\frac{\log\left(N_\nu^D(x_n)\right)}{\log(x_n)}\geq\frac{1}{1+10/3}.$$

Indeed, for $x>0$, choose $n\in\mathbb{N}$ such that $x_n<x\leq x_{n+1}$ and therefore

$$\liminf_{x\to\infty}\frac{\log\left(N_\nu^D(x)\right)}{\log(x)}\geq\liminf_{n\to\infty}\frac{\log\left(N_\nu^D(x_n)\right)}{\log(x_n)+\log(x_{n+1}/x_n)}\geq\frac{1}{1+10/3}.$$

4.4.3 Purely atomic case

In this section we give examples of singular measures η on $(0,1)$ of pure point type such that the spectral dimension attains any value in $[0,1/2]$. To fix notation, throughout this section, we write $\eta:=\sum_k p_k\delta_{x_k}$ with $(p_k)_k\in(\mathbb{R}_{>0})^{\mathbb{N}}$, $\sum_k p_k<\infty$, and $(x_k)_k\in(0,1)^{\mathbb{N}}$.

The first example shows that it is possible for the spectral dimension to be 0 even though the Minkowski dimension is 1.

Example 4.49. In this example we consider purely atomic measures η with $(x_n)_n\in(\mathbb{Q}\cap(0,1))^{\mathbb{N}}$ such that $x_n\neq x_m$ for $m\neq n$, and there exists $C_1>0$ such that $p_n\leq$

$C_1 \mathrm{e}^{-n}$ for all $n \in \mathbb{N}$. We will show that the spectral dimension exists and equals $s_\eta^D = 0$. We define $I_k^n := \left(x_k - \frac{b_n}{\mathrm{e}^n}, x_k + \frac{b_n}{\mathrm{e}^n}\right] \cap [0,1]$ with

$$b_n := \min\left\{\left|x_{l_1} - x_{l_2}\right| : l_1 \neq l_2,\ l_1, l_2 \leq n\right\}$$

for all $k = 1, \ldots, n$. Then,

$$\max_{k=1,\ldots,n} \Lambda\left(I_k^n\right) \nu\left(I_k^n\right) \leq \sum_{l=1}^{\infty} p_l \max_{k=1,\ldots,n} \Lambda(I_k^n) \leq \frac{C_1}{\mathrm{e}-1} \frac{2}{\mathrm{e}^n}.$$

Let A_k^n denote the disjoint half open intervals such that $(0,1] \setminus \bigcup_{k=1}^n I_k^n = \bigcup_{k=1}^{m(n)} A_k^n$ with $m(n) \leq n+1$. Then we conclude

$$\max_{k=1,\ldots,m(n)} \Lambda(A_k^n)\nu(A_k^n) \leq \sum_{l=n+1}^{\infty} p_l \leq C_1 \frac{\mathrm{e}^{-n}}{\mathrm{e}-1}.$$

Now, Proposition 4.4 yields

$$N_\nu^D\left(\frac{e^n(e-1)}{2C_1}\right) \leq 2\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}\left(\frac{e^n(e-1)}{2C_1}\right) \leq 2n+1.$$

Hence, for all $x \geq e(e-1)/(2C_1)$, we have

$$N_\nu^D(x) \leq \frac{4C_1}{e-1} \log(x) + 1.$$

Thus, we obtain $s_\nu^D = 0$.

If $(x_k)_k \in (0,1)^{\mathbb{N}}$ is strictly decreasing, then in [FW17] Δ_η^D is called Kreĭn–Feller operator of Stieltjes type. We start with a general observation which is a consequence of Proposition 4.4.

Lemma 4.50. *Assume that $(x_k)_k \in (0,1)^{\mathbb{N}}$ is strictly decreasing such that for an increasing function $f : \mathbb{N}_{>1} \to \mathbb{R}_+$ and all $k \in \mathbb{N}_{>1}$,*

$$\frac{x_k + x_{k-1}}{2} \sum_{l=k}^{\infty} p_l \leq 1/f(k).$$

Then, for all $x \geq 0$, we have $N_\eta^D(x) \leq \min\{2\check{f}^{-1}(2x)+1, \check{f}^{-1}(6x)\}$ with $\check{f}^{-1}(x) := \inf\{n \in \mathbb{N}_{>1} : f(n) \geq x\}$.

Proof. By our assumption, for all $k \in \mathbb{N}$, we have

$$\nu\left(\left[0, \frac{x_k + x_{k-1}}{2}\right]\right) \Lambda\left(\left[0, \frac{x_k + x_{k-1}}{2}\right]\right) = \frac{x_k + x_{k-1}}{2} \sum_{l=k}^{\infty} p_l.$$

Observe that for $x > 0$

$$\frac{1}{\frac{x_k + x_{k-1}}{2}\sum_{l=k}^{\infty} p_l} \geq f(k) \geq 2x \implies k \geq \check{f}^{-1}(2x).$$

For fixed $k \in \mathbb{N}$ define the following ν-partition

$$I_1 := \left(x_1 - \frac{\min\{x_1 - x_2, 1/x\}}{4}, x_1 + \frac{\min\{1 - x_1, 1/x\})}{4}\right]$$

and

$$I_j := \left(x_j - \frac{\min\left\{x_j - x_{j+1}, 1/x\right\}}{4}, x_j + \frac{\min\left\{x_j - x_{j-1}, 1/x\right\}}{4}\right]$$

for $j = 2, \ldots, k-1$ and $I_k := (0, (x_k + x_{k-1})/2]$. Hence, for $k = \check{f}^{-1}(2x)$, we see that

$$\max_{i=1,\ldots,k} \nu(I_i)\Lambda(I_i) < 1/x.$$

Now, Proposition 4.4 yields

$$N_\eta^D(x) \leq 2\widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(x) + 1 \leq 2\check{f}^{-1}(2x) + 1.$$

Analogously, we obtain

$$N_\eta^D(x) \leq \widetilde{\mathcal{M}}_{\mathfrak{J}_\nu}(5x) \leq \check{f}^{-1}(6x). \qquad \square$$

Example 4.51 (Dirac comb with exponential decay)**.** Observe that if $(x_n)_n \in (0,1)^{\mathbb{N}}$ or $\left(\sum_{k \geq m} p_k\right)_m$ decays exponentially, then by Lemma 4.50, we have $N_\eta^D(x) \ll \log(x)$, hence the spectral dimension s_η^D equals 0. In particular, by Theorem 4.10, we have $q_{\mathfrak{J}_\nu} = 0$. Consequently, the (Neumann) L^q-spectrum is given by

$$\beta_\eta(q) = \begin{cases} \overline{\dim}_M(\eta), & q = 0, \\ 0, & q > 0. \end{cases}$$

Example 4.52 (Dirac comb with at most power law decay)**.** Assume that $(x_k)_k \in (0,1)^{\mathbb{N}}$ is strictly decreasing and

$$p_n \gg n^{-u_1} f_1(n),\ (x_n - x_{n+1}) \gg n^{-u_2} f_2(n),$$

with $u_1, u_2 \geq 1$ and $\lim_{n\to\infty} \log(f_i(n))/\log(n) = 0$ for $i = 1, 2$. Then we have

$$\frac{1}{u_1 + u_2} \leq \underline{s}_\eta^D$$

and, in particular if $u_1 + u_2 = 2$, we have $s_\eta^D = 1/2$. To see this, define

$$I_k := \left[x_k - \frac{\min\{x_k - x_{k+1}, x_{k-1} - x_k\}}{2}, x_k + \frac{\min\{x_k - x_{k+1}, x_{k-1} - x_k\}}{2} \right]$$

for $k = 1, \ldots, n$. We then have $\nu(\langle I_k \rangle_{1/2}) = \nu(I_k)$ and for fixed $\varepsilon > 0$, we have for n large enough

$$Cn^{-(u_1+u_2)-\varepsilon} \leq Cn^{-(u_1+u_2)} f_1(n) f_2(n) \leq \min_{k=1,\ldots,n} \nu\left(\langle I_k \rangle_{1/2}\right) \Lambda\left(\langle I_k \rangle_{1/2}\right)$$

with $C > 0$ suitable, which implies for every $\varepsilon > 0$

$$\mathcal{N}_2^L(x) \gg x^{\frac{1}{u_1+u_2+\varepsilon}}.$$

This proves the claim. Since, in the case $u_1 + u_2 = 2$, we always have $\overline{s}_\eta^D \leq 1/2$. It readily follows that $s_\eta^D = 1/2$.

Example 4.53 (Dirac comb with at most geometric decay and full dimension)**.** Assume that $(x_k)_k \in (0,1)^{\mathbb{N}}$ is strictly decreasing and

$$\frac{f_1(n)}{n} \ll p_n, \frac{f_2(n)}{n} \ll (x_{n-1} - x_n)$$

with $\lim_{n\to\infty} \log(f_i(n))/\log(n) = 0$ for $i = 1, 2$. Then the spectral dimension exists and equals

$$s_\eta^D = 1/2$$

and, for $q \in [0,1]$,

$$\beta_\eta(q) = \begin{cases} 1 - q, & q \in [0,1], \\ 0, & q > 1. \end{cases}$$

Indeed, observe that Example 4.52 implies $1/2 \leq \underline{s}_\eta^D$ and by and Corollary 4.17 Theorem 4.10 we also have $\underline{s}_\eta^D \leq \overline{s}_\eta^D = q_{\mathfrak{J}_\eta} \leq 1/2$, which shows $s_\eta^D = 1/2$. The second statement is then a direct consequence of the first part of Corollary 4.16.

The following example shows that the spectral dimension attains every value in $(0, 1/2)$.

Example 4.54 (Dirac comb with power law decay)**.** If

$$\lim_{n\to\infty} -\log(p_n)/\log(n) = u_1 > 1 \text{ and } x_k := (k+1)^{-u_2},\ k \in \mathbb{N},\ u_2 > 0,$$

then the Neumann L^q-spectrum exists as a limit on the positive half-line and we have

$$\beta_\eta(q) = \begin{cases} \frac{1}{u_2+1} - q\frac{u_1}{u_2+1} & \text{for } q \in [0, 1/u_1], \\ 0 & \text{for } q > 1/u_1. \end{cases}$$

Consequently, η is Neumann $\mathfrak{J}_\eta$-regular and the spectral dimension exists and equals

$$s_\eta^D = \frac{1}{u_1+u_2+1}.$$

In particular, for $u_1 = u_2+1$, we have

$$s_\eta^D = \frac{\dim_M(\eta)}{2}.$$

This can be seen as follows. For every $\varepsilon > 0$, we have uniformly in $n \in \mathbb{N}$

$$n^{-(u_1+\varepsilon)} \ll p_n \ll n^{-u_1+\varepsilon}.$$

For suitable $C > 0$,

$$x_m - x_{m+1} = \frac{(m+1)^{u_2} - m^{u_2}}{m^{u_2}(m+1)^{u_2}} = \frac{1}{(m+1)^{u_2} m} \frac{\left(\frac{m+1}{m}\right)^{u_2} - 1}{1/m} \geq \frac{C}{m^{u_2+1}}.$$

If $2^{-n} < C(m+1)^{-(u_2+1)}$, then $m < (2^n C)^{\frac{1}{u_2+1}}$. Combining these observations, we obtain

$$\begin{aligned}\sum_{C \in \mathcal{D}_n^N} \eta(C)^q &\geq \sum_{k=1}^{C^{1/(u_2+1)}2^{n/(u_2+1)}} p_k^q \\ &\gg \sum_{k=1}^{C^{1/(u_2+1)}2^{n/(u_2+1)}} k^{-(u_1+\epsilon)q} \\ &\asymp 2^{n(-(u_1+\epsilon)q+1)/(u_2+1)}.\end{aligned}$$

For $q \in [0, 1/(u_1+\varepsilon))$, this gives

$$\beta_\eta(q) \geq \liminf_{n\to\infty} \beta_n^\eta(q) \geq \frac{1}{u_2+1} - q\frac{u_1+\varepsilon}{u_2+1}.$$

Letting $\varepsilon \to 0$, showing for $q \in [0, 1/u_1]$

$$\beta_\eta(q) \geq \liminf_{n\to\infty} \beta_n^\eta(q) \geq \begin{cases} \frac{1}{u_2+1} - q\frac{u_1}{u_2+1}, & \text{for } q \in [0, 1/u_1], \\ 0, & \text{for } q > 1/u_1. \end{cases}$$

Moreover, for $m \geq (2^n C)^{\frac{1}{u_2+1}}$ and $k2^{-n} < x_m$, we have $k < 2^n (2^n C)^{-\frac{u_2}{u_2+1}}$. From this inequality, using the integral test for convergence, we obtain for $q > 1/(u_1 - \varepsilon)$,

$$\sum_{Q \in \mathcal{D}_n^N} \eta(Q)^q = \sum_{k=0}^{2^n-1} \left(\sum_{\substack{m \in \mathbb{N}: \\ k2^{-n} < x_m \leq (k+1)2^{-n}}} p_m \right)^q$$

$$\ll \sum_{m < C^{1/u_2} 2^{n/u_2}} m^{q(-u_1+\varepsilon)} + \sum_{k=0}^{C^{-\frac{u_2}{u_2+1}} 2^{\frac{n}{u_2+1}}} \left(\sum_{\substack{m \in \mathbb{N}: \\ k2^{-n} < x_m \leq (k+1)2^{-n}}} p_m \right)^q$$

$$\ll 1 + 2^{n\left(\frac{1}{u_2+1} + q\left(-\frac{(u_1-\varepsilon-1)}{u_2} + \frac{(u_1-\varepsilon-1)-u_2}{u_2(u_2+1)}\right)\right)} = 1 + 2^{n\left(\frac{1}{u_2+1} - q\frac{u_1-\varepsilon}{u_2+1}\right)} \ll 1.$$

Hence, $\beta_\eta(q) = 0$ for $q \geq 1/u_1$. Since $\beta_\eta(0) = 1/(u_2+1)$, by the convexity of β_η, it follows that for all $q \in [0, 1/u_1]$,

$$\frac{1}{u_2+1} - q\frac{u_1}{u_2+1} \leq \liminf_{n\to\infty} \beta_n^\eta(q) \leq \beta_\eta(q) \leq \frac{1}{u_2+1} - q\frac{u_1}{u_2+1}.$$

Corollary 4.12 then gives $s_\eta^D = q_{\mathfrak{J}_\eta} = 1/(u_1+u_2+1)$.

The last example demonstrates how one can improve Example 4.51 if one knows the exact exponential asymptotics of p_n and x_n, which in turn forces an logarithmic asymptotic for the eigenvalue counting function. The following simple lemma is provided for preparation.

Lemma 4.55. *Let $\nu := p\delta_z$ be with $z \in (0,1)$, $p > 0$, $0 < a < z$, and $0 < b < 1 - z$. Then,*

$$\frac{\langle f_{a,b,z}, f_{a,b,z} \rangle_{H_0^1}}{\langle f_{a,b,z}, f_{a,b,z} \rangle_\nu} = \frac{a+b}{pab}$$

with $f_{a,b,z}(x) := \frac{(x-(z-a))}{a} \mathbb{1}_{[z-a,z]} + \frac{(z+b-x)}{b} \mathbb{1}_{(z,z+b]}$.

Example 4.56 (Dirac comb with exponential decay – precise asymptotics)**.** Let us consider the case $\sum_{k=1}^n p_k = 1 - \mathrm{e}^{-\alpha n}$ and $x_k := \mathrm{e}^{-\gamma k}$ for some $\alpha, \gamma > 0$. Then

$$\lim_{x\to\infty} \frac{N_\eta^D(x)}{\log(x)} = \frac{1}{\alpha+\gamma}.$$

To see this, we consider the intervals $A_n := [x_n - (x_n - x_{n+1})/2, x_n + (x_{n-1} - x_n)/2]$ for $n > 1$ and set

$$f_n(y) := f_{(x_n - x_{n+1})/2, (x_{n-1}-x_n)/2, x_n}(y) \text{ for } y \in [0,1].$$

Then, $f_n \in H_0^1$ and by Lemma 4.55,

$$\frac{\langle f_n, f_n \rangle_{H_0^1}}{\langle f_n, f_n \rangle_\eta} = \frac{\mathrm{e}^{(\alpha+\gamma)n} 2\left(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma}\right)}{\left(1-\mathrm{e}^{-\gamma}\right)\left(\mathrm{e}^{\alpha}-1\right)\left(\mathrm{e}^{\gamma}-1\right)}.$$

Notice that for $x > 0$, we have that

$$\frac{\mathrm{e}^{(\alpha+\gamma)n} 2\left(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma}\right)}{\left(1-\mathrm{e}^{-\gamma}\right)\left(\mathrm{e}^{\alpha}-1\right)\left(\mathrm{e}^{\gamma}-1\right)} \leq x$$

implies

$$n \leq \underbrace{\left\lfloor \frac{1}{(\alpha+\gamma)} \log\left(x \frac{\left(1-\mathrm{e}^{-\gamma}\right)\left(\mathrm{e}^{\alpha}-1\right)\left(\mathrm{e}^{\gamma}-1\right)}{2\left(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma}\right)} \right) \right\rfloor}_{=: n_x}.$$

Now, observe that $\operatorname{span}(f_i : i = 1, \ldots, n_x)$ is a n_x-dimensional subspace of H_0^1 and the $(f_i)_i$ are mutually orthogonal both in L_ν^2 and in H_0^1. Hence, by Lemma 2.19, we obtain

$$N_\eta^D(x) \geq \left\lfloor \frac{1}{(\alpha+\gamma)} \log\left(x \frac{\left(1-\mathrm{e}^{-\gamma}\right)\left(\mathrm{e}^{\alpha}-1\right)\left(\mathrm{e}^{\gamma}-1\right)}{2\left(\mathrm{e}^{\gamma} - \mathrm{e}^{-\gamma}\right)} \right) \right\rfloor.$$

For the upper bound, we observe that for $k > 1$,

$$f(k) := \frac{1}{\frac{x_k + x_{k-1}}{2} \sum_{\ell=k}^{\infty} p_\ell} = \frac{2}{\mathrm{e}^{-\gamma k + (k-1)\alpha}\left(1+\mathrm{e}^{-\gamma}\right)} = 2\frac{\mathrm{e}^{(k-1)\alpha + k\gamma}}{1+\mathrm{e}^{-\gamma}}.$$

Then,

$$f(k) \geq x \implies k \geq m_x := \left\lceil \frac{1}{(\alpha+\gamma)} \log\left(\left(1+\mathrm{e}^{-\gamma}\right) \frac{\mathrm{e}^{\alpha} x}{2} \right) \right\rceil.$$

Therefore, Lemma 4.50 applied to f yields for x large,

$$N_\eta^D(x) \leq \check{f}^{-1}(6x) = m_{6x} \leq \frac{1}{(\alpha+\gamma)} \log\left(\left(1+\mathrm{e}^{-\gamma}\right)\mathrm{e}^{\alpha} 6x\right) + 1$$

and the claim follows.

Chapter 5

Spectral dimension for Kreĭn–Feller operators in higher dimensions

Throughout this chapter let ν denote a finite Borel measure on $\mathbf{Q}$ with $\nu(\mathbf{Q}) > 0$, $d > 1$, and $\dim_\infty(\nu) > d-2$. In Chapter 4, we studied the spectral dimension for Kreĭn–Feller operators for the case $d = 1$. This chapter is dedicated to study the spectral dimension of Kreĭn–Feller operators with respect to ν and $\Omega = (0,1)^d$ by extending the ideas for the case $d = 1$ presented in Chapter 4. Similar to the one-dimensional case, we use the results developed in Chapter 3 applied to the spectral partition function $\mathfrak{J}_{\nu,2/d-1,2/t}$ with $t > 2$, which is crucial in our analysis of the spectral dimension. The reason for the importance of $\mathfrak{J}_{\nu,2/d-1,2/t}$ becomes apparent in Section 5.1.2 and Section 5.2.2.

This chapter is organized as follows. In Section 5.1, we establish a connection between the scaling behavior on the embedding constants for the embedding $C_c^\infty(\overline{Q})$ into $L^2_\nu(Q)$, $Q \in \mathcal{D}$ and the lower and upper spectral dimension (see Proposition 5.1). As an application of this general principle, we obtain upper bounds of the lower and upper spectral dimension by using results of Adams (see [Maz11, p. 67] and the references given there) and Maz'ya and Preobrazenskii [MP84]. Section 5.2 is devoted to obtain lower bounds of the lower and upper spectral dimension. Similar to Section 5.1, we first establish a relation to lower bounds on the embedding constants for the embedding $C_c^\infty(\overline{Q})$ into $L^2_\nu(Q)$, $Q \in \mathcal{D}$ and lower bounds of the lower and upper spectral dimension (see Proposition 5.9). In Section 5.2.2, based on this general observation, we obtain lower bounds of the lower and upper spectral dimension by $\underline{F}^{D/N}_{\mathfrak{J}_\nu}$ and $\overline{F}^{D/N}_{\mathfrak{J}_\nu}$, respectively. In Section 5.3, based on the results of Section 5.1 and Section 5.2.2, we present our main results of this chapter. We first prove that the upper Neumann spectral dimension equals $q^N_{\mathfrak{J}_\nu}$. Further, we present conditions under which we can ensure that $\overline{s}_\nu^D = \overline{s}_\nu^N$. Moreover, in the case $d = 2$ and $\nu(\mathring{\mathbf{Q}}) > 0$, we show that $\overline{s}_\nu^D = \overline{s}_\nu^N = 1$. In Section 5.3.2, we impose regularity conditions on $\mathfrak{J}_\nu$ that guarantee the existence of the spectral dimension. We end this chapter with a sequence of examples as an application of our general results of Section 5.3. As a highlight, we confirm the existence of the spectral dimension of self-conformal measures without any separation conditions (see Theorem 5.27). Finally, using Example 4.48, we construct an example for which the spectral dimension does not exist for the case $d = 3$.

5.1 Upper bounds

In this section, we obtain upper bounds for the spectral dimension.

5.1.1 Embedding constants and upper bounds for the spectral dimension

This section establishes a relation between embedding constants on sub-cubes and the lower and upper spectral dimension.

Proposition 5.1. *Suppose there exists a non-negative, uniformly vanishing, monotone set function $\mathfrak{J}$ on $\mathcal{D}$ such that for all $Q \in \mathcal{D}$ and all $u \in C_b^\infty(\overline{Q})$ with $\int_Q u \, \mathrm{d}\Lambda = 0$, we have*

$$\|u\|_{L_\nu^2(Q)}^2 \le \mathfrak{J}(Q) \, \|\nabla u\|_{L_\Lambda^2(Q)}^2 .$$

Then we have

$$\overline{s}_\nu^D \le \overline{s}_\nu^N \le \overline{h}_{\mathfrak{J}} \quad \text{and} \quad \underline{s}_\nu^D \le \underline{s}_\nu^N \le \underline{h}_{\mathfrak{J}}.$$

Proof. For a partition $\Xi \in \Pi_{\mathfrak{J}}$ of $\mathbf{Q}$, let us define the following closed linear subspace of H^1

$$\mathcal{F}_\Xi := \left\{ u \in H^1 : \int_Q u \, \mathrm{d}\Lambda = 0, Q \in \Xi \right\}.$$

We define an equivalence relation $\sim$ on H^1 induced by $\mathcal{F}_\Xi$ as follows: $u \sim v$ if and only if $u - v \in \mathcal{F}_\Xi$. Note that we have $\dim\left(H^1/\mathcal{F}_\Xi\right) = \mathrm{card}(\Xi)$. Further, by our assumption, for all $u \in C_b^\infty(\overline{\mathbf{Q}}) \cap \mathcal{F}_\Xi$ we obtain

$$\begin{aligned} \int u^2 \, \mathrm{d}\nu &= \sum_{Q \in \Xi} \int_Q u^2 \, \mathrm{d}\nu \le \sum_{Q \in \Xi} \mathfrak{J}(Q) \, \|\nabla u\|_{L_\Lambda^2(Q)}^2 \\ &\le \max_{Q \in \Xi} \mathfrak{J}(Q) \sum_{Q \in \Xi} \|\nabla u\|_{L_\Lambda^2(Q)}^2 \le \max_{Q \in \Xi} \mathfrak{J}(Q) \, \|\nabla u\|_{L_\Lambda^2(\mathbf{Q})}^2 . \end{aligned}$$

Next, we show that $C_b^\infty(\overline{\mathbf{Q}}) \cap \mathcal{F}_\Xi$ lies dense in $\mathcal{F}_\Xi$ with respect to H^1. Recall that $C_b^\infty(\overline{\mathbf{Q}})$ lies dense in H^1. Hence, for every $u \in \mathcal{F}_\Xi$, there exists a sequence u_n in $C_b^\infty(\overline{\mathbf{Q}})$ such that $u_n \to u$ in H^1. The Cauchy-Schwarz inequality for all $Q \in \Xi$ gives

$$\left|\int_Q u_n \,\mathrm{d}\Lambda\right| = \left|\int_Q u_n - u \,\mathrm{d}\Lambda\right| \leq \int (u_n - u)^2 \,\mathrm{d}\Lambda \to 0.$$

It follows that $\int_Q u_n \,\mathrm{d}\Lambda \to 0$. Furthermore, for every $Q \in \Xi$ there exists $u_Q \in C_c^\infty(\mathbf{Q})$ such that $u_Q|_{Q^\complement} = 0$ and $\int_Q u_Q \,\mathrm{d}\Lambda = 1$. Then for

$$u_n' := u_n - \sum_{Q \in \Xi} \mathbb{1}_Q \varepsilon_{Q,n} u_Q \in C_b^\infty(\overline{\mathbf{Q}}) \cap \mathcal{F}_\Xi$$

with $\varepsilon_{Q,n} := \int_Q u_n \,\mathrm{d}\Lambda$ we have $u_n' \to u$ in H^1. Thus, for $u \in \mathcal{F}_\Xi$, we obtain

$$\int \iota(u)^2 \,\mathrm{d}\nu \leq \max_{Q \in \Xi} \mathfrak{J}(Q) \left\|\nabla u\right\|_{L^2_\Lambda(\mathbf{Q})}^2.$$

For $i \in \mathbb{N}$ define

$$\lambda^i_{\nu,\mathcal{F}_\Xi} := \inf\left\{\sup\left\{R_{H^1}(\psi) : \psi \in G^\star\right\} : G <_i (\mathcal{F}_\Xi, \langle\cdot,\cdot\rangle_{H^1})\right\}$$

with $R_{H^1}(\psi) := \langle\psi,\psi\rangle_{H^1} / \langle\iota\psi,\iota\psi\rangle_\nu$ and $N_\nu^N(y, \mathcal{F}_\Xi) := \operatorname{card}\left(\left\{i \in \mathbb{N} : \lambda^i_{\nu,\mathcal{F}_\Xi} \leq y\right\}\right)$ with $y > 0$. Thus, $\max_{Q\in\Xi} \mathfrak{J}(Q) < 1/x$, implies

$$\lambda^1_{\nu,\mathcal{F}_\Xi} > x.$$

In view of the min-max principle as stated in Proposition 2.17, we deduce analogously as in the proof of Proposition 4.4

$$N_\nu^N(x) \leq N_\nu^N(x, \mathcal{F}_\Xi) + \operatorname{card}(\Xi) = \operatorname{card}(\Xi),$$

implying $N_\nu^N(x) \leq \mathcal{M}_\mathfrak{J}(x)$, and hence $\overline{s}_\nu^N \leq \overline{h}_\mathfrak{J}$ and $\underline{s}_\nu^N \leq \underline{h}_\mathfrak{J}$. □

Remark 5.2. The ideas underlying in Proposition 5.1 correspond to some extent to those developed in [NS95; Sol94], [NS01, Chapter 5], that is, reducing the problem of estimating the spectral dimension to an auxiliary counting problem. To illustrate the parallel, we present an alternative proof of the upper estimate of the eigenvalue counting function for self-similar measures under OSC (see [Sol94, Theorem 1]). As in the setting in [Sol94], we let ν denote a self-similar measure under OSC with contractive similitudes $S_1, \ldots, S_m$ and corresponding contraction ratios $h_i \in (0,1)$ and probability weights $p_i \in (0,1)$ with $i = 1, \ldots, m$. We assume $\nu(\partial\mathbf{Q}) = 0$ and $\dim_\infty(\nu) > d - 2$, which is in this case equivalent to $\max_i p_i h_i^{2-d} < 1$. For simplicity we assume that the feasible set is given by $\mathring{\mathbf{Q}}$, i.e. $S_j(\mathring{\mathbf{Q}}) \subset \mathring{\mathbf{Q}}$. Instead of $\mathcal{D}$, we will consider a symbolic partition by the cylinder sets $\widetilde{\mathcal{D}} := \{S_\omega(\mathring{\mathbf{Q}}) : \omega \in I^*\}$ with $I := \{1, \dots, m\}$. Then $\mathfrak{J}$ will be replaced by $\widetilde{\mathfrak{J}} : \widetilde{\mathcal{D}} \to \mathbb{R}_{\geq 0}$ with $\widetilde{\mathfrak{J}}(S_\omega(\mathring{\mathbf{Q}})) := p_\omega h_\omega^{2-d}$, $\omega \in I^*$. Now, observe that for $0 < t < \min_{i=1,\dots,m} p_i h_i^{2-d}$, we have

$$\widetilde{P}_t := \left\{\omega \in I^* : p_\omega h_\omega^{2-d} < t \leq p_{\omega^-} h_{\omega^-}^{2-d}\right\}$$

is a partition of $I^{\mathbb{N}}$. Further, let δ denote the unique solution of $\sum_{i=1}^m \left(p_i h_i^{(2-d)}\right)^\delta = 1$. Thus, it follows that

$$\sum_{\omega \in \widetilde{P}_t} \left(p_\omega h_\omega^{(2-d)}\right)^\delta = 1.$$

Furthermore, there exists $K > 0$ such that for all $u \in H^1$ with $\int_{S_\omega(\mathring{\mathbf{Q}})} u \, \mathrm{d}\Lambda = 0$, $\omega \in \widetilde{P}_t$, we have

$$\int \iota(u)^2 \, \mathrm{d}\nu \leq K \max_{\omega \in \widetilde{P}_t} \widetilde{\mathfrak{J}}\left(S_\omega\left(\mathring{\mathbf{Q}}\right)\right) \int_{\mathbf{Q}} |\nabla u|^2 \, \mathrm{d}\Lambda < tK \int_{\mathbf{Q}} |\nabla u|^2 \, \mathrm{d}\Lambda$$

(see [NS01, p. 502]). A similar computation as in the proof of Lemma 4.33 yields the two-sided estimate

$$t^{-\delta} \leq \operatorname{card}\left(\widetilde{P}_t\right) \leq \frac{t^{-\delta}}{\min_{i=1,\dots,m} p_i h_i^{2-d}}.$$

The min-max principle gives

$$N_\nu^N\left((tK)^{-1}\right) \leq \operatorname{card}\left(\widetilde{P}_t\right) \leq \frac{t^{-\delta}}{\min_{i=1,\dots,m} p_i h_i^{2-d}},$$

hence the results of [NS01; NS95, Theorem 1] follow from this simple counting argument without any use of renewal theory. The drawback of the ideas [NS95; Sol94; NS01, Chapter 5] is that they rely heavily on the specific structure of the self-similar measures, whereas our approach via dyadic cubes avoids the use of specific properties of the underlying measure.

5.1.2 Upper bounds on the embedding constants and upper bounds for the spectral dimension

In this section, up to multiplicative uniform constants, we make use of best embedding constants for the embedding $C_c^\infty\left(\mathbb{R}^d\right)$ into $L_\nu^t\left(\mathbb{R}^d\right)$, $t > 2$, to estimate the spectral dimension from above. More precisely, for $d > 2$, the best constant C in

$$\|u\|_{L^t_{\nu|_Q}(\mathbb{R}^d)} \leq C\|u\|_{H^1(\mathbb{R}^d)},\ u \in C_c^\infty\left(\mathbb{R}^d\right),\ Q \in \mathcal{D}, \tag{5.1.1}$$

is equivalent to $\sup_{x\in\mathbb{R}^d,\rho>0} \rho^{(2-d)/2}\nu\left(Q \cap B_\rho(x)\right)^{1/t}$ in the sense that there exist $c_1, c_2 > 0$ only depending on d and t such that

$$c_1 C \leq \sup_{x\in\mathbb{R}^d,\rho>0} \rho^{(2-d)/2}\nu\left(Q \cap B_\rho(x)\right)^{1/t} \leq c_2 C.$$

For $d = 2$, the best constant C in (5.1.1) is equivalent to

$$\sup_{x\in\mathbb{R}^d,0<\rho<1/2} |\log(\rho)|^{1/2}\,\nu\left(Q \cap B_\rho(x)\right)^{1/t}.$$

The result for the case $d > 2$ is a corollary of Adams' Theorem on Riesz potentials (see e.g. [Maz11, p. 67]) and the case $d = 2$ is due to Maz'ya and Preobrazenskii and can be found in [Maz11, p. 83] or [MP84]. The following lemma establishes an alternative representation of the best equivalent constant in terms of dyadic cubes.

Lemma 5.3. *Let $Q \in \mathcal{D}$. Then, for $a < 0$, $b > 0$, $C_1 := \left(2\sqrt{d}\right)^a$, and $C_2 := \left(3\sqrt{d}\right)^{db} 2^{-a}$,*

$$C_1 \mathfrak{I}_{\nu,a/d,b}(Q) \leq \sup_{x\in\mathbb{R}^d,\rho>0} \rho^a \nu\left(Q \cap B_\rho(x)\right)^b \leq C_2 \mathfrak{I}_{\nu,a/d,b}(Q).$$

For $a = 0$, $C_3 := d^{-1}$, and $C_4 := 3^{bd}$,

$$C_3 \mathfrak{I}_{\nu,0,b}(Q) \leq \sup_{x\in\mathbb{R}^d,0<\rho<1/2} |\log(\rho)|\,\nu(Q \cap B_\rho(x))^b \leq C_4 \mathfrak{I}_{\nu,0,b}(Q).$$

Proof. Let $Q \in \mathcal{D}_n^N$. Since $a < 0$, we have

$$\sup_{x\in\mathbb{R}^d,\rho>0} \rho^a \nu\left(Q \cap B_\rho(x)\right)^b = \sup_{x\in\mathbb{R}^d,\rho\leq\sqrt{d}2^{-n+1}} \rho^a \nu\left(Q \cap B_\rho(x)\right)^b.$$

Thus, we assume without loss of generality, that $0 < \rho \leq \sqrt{d}2^{-n+1}$. Then for

$m \geq n-1$ with $\sqrt{d}2^{-(m+1)} < \rho \leq \sqrt{d}2^{-m}$, and $x \in \mathbb{R}^d$, we obtain

$$\begin{aligned}
\rho^a \nu\left(Q \cap B_\rho(x)\right)^b &\leq 2^{-a} \left(\sum_{\substack{Q' \in \mathcal{D}_m^N, \\ Q' \cap Q \cap B_\rho(x) \neq \emptyset}} \nu(Q \cap Q') \right)^b 2^{-ma} \\
&\leq \left(3\sqrt{d}\right)^{db} 2^{-a} \max_{Q' \in \mathcal{D}_m^N} \nu(Q \cap Q')^b \Lambda(Q')^{a/d} \\
&\leq C_2 \sup_{Q' \in \mathcal{D}(Q)} \nu(Q')^b \Lambda(Q')^{a/d} = C_2 \mathfrak{I}_{\nu,a/d,b}(Q),
\end{aligned}$$

where we used the facts that $B_\rho(x)$ can be covered by at most $\left(3\sqrt{d}\right)^d$ elements of $\mathcal{D}_m^N$ and if $Q' \cap Q \neq \emptyset$, then $Q' \subset Q$ for $m \geq n$, as well as

$$\max_{Q' \in \mathcal{D}_{n-1}^N} \nu(Q \cap Q')^b \Lambda(Q')^{a/d} \leq \nu(Q)^b \Lambda(Q)^{a/d} = \max_{Q' \in \mathcal{D}_n^N} \nu(Q \cap Q')^b \Lambda(Q')^{a/d}.$$

Since $x \in \mathbb{R}^d$ and $\rho > 0$ were arbitrary, the second inequality follows.

On the other hand, for $Q' \in \mathcal{D}_m^N$ with $Q' \subset Q$ and $\rho := \sqrt{d}2^{-m+1}$ we find $x \in \mathbb{R}^d$ such that $Q' \subset B_\rho(x)$. Then

$$\begin{aligned}
\nu(Q')^b \Lambda(Q')^{a/d} &\leq \nu\left(Q \cap B_\rho(x)\right)^b 2^{-ma} \\
&\leq \left(\sqrt{d}2\right)^{-a} \nu\left(Q \cap B_\rho(x)\right)^b \rho^a \\
&\leq C_1^{-1} \sup_{x \in \mathbb{R}^d, \rho > 0} \rho^a \nu\left(Q \cap B_\rho(x)\right)^b.
\end{aligned}$$

For the case $a = 0$, for any $2^{-(m+1)} \leq \rho < 2^{-m}$, $m \in \mathbb{N}$, and $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
|\log(\rho)|\, \nu(Q \cap B_\rho(x))^b &\leq |\log(2)\,(m+1)|\, \nu(Q \cap B_{2^{-m}}(x))^b \\
&\leq \left|\log\left(2^{-dm}\right)\right| \left(\sum_{Q' \in \mathcal{D}_m^N, Q' \cap Q \cap B_{2^{-m}}(x) \neq \emptyset} \nu(Q \cap Q') \right)^b \\
&\leq 3^{db} \max_{Q' \in \mathcal{D}_m^N} \nu(Q \cap Q')^b \left|\log\left(\Lambda(Q')\right)\right| \\
&\leq 3^{db} \max_{Q' \in \mathcal{D}(Q)} \nu(Q \cap Q')^b \left|\log\left(\Lambda(Q')\right)\right|,
\end{aligned}$$

where we used that

$$\max_{Q' \in \mathcal{D}_m^N} \nu(Q \cap Q')^b \left|\log\left(\Lambda(Q')\right)\right| \leq \nu(Q)^b \log(2) dm \leq \max_{Q' \in \mathcal{D}_n^N} \nu(Q \cap Q')^b \left|\log\left(\Lambda(Q')\right)\right|$$

for $m \leq n$. On the other hand, for $Q' \in \mathcal{D}_m^N$ with $Q' \subset Q$ and $\rho := \sqrt{d}2^{-m+1}$ we find $x \in \mathbb{R}^d$ such that $Q' \subset B_\rho(x)$. Then

$$\begin{aligned}\nu(Q')^b \left|\log\left(\Lambda(Q')\right)\right| &\leq \nu\left(Q \cap B_\rho(x)\right)^b dm \log(2)\\ &\leq d\nu\left(Q \cap B_\rho(x)\right)^b \left((m+1)\log(2) + \log\left(\sqrt{d}\right)\right)\\ &= d\nu\left(Q \cap B_\rho(x)\right)^b \left|\log(\rho)\right|\\ &\leq d \sup_{x\in\mathbb{R}^d,\rho>0} \left|\log(\rho)\right| \nu\left(Q \cap B_\rho(x)\right)^b. \qquad \square\end{aligned}$$

Corollary 5.4. *For $t > 2$ there exists a constant $C_{t,d} > 0$ such that for all $u \in C_c^\infty\left(\mathbb{R}^d\right)$ and $Q \in \mathcal{D}$, we have*

$$\|u\|_{L^2_{\nu|_Q}(\mathbb{R}^d)} \leq C_{t,d}\mathfrak{J}^{1/2}_{\nu,2/d-1,2/t}(Q)\,\|u\|_{H^1(\mathbb{R}^d)}.$$

Proof. Using [Maz85, Corollary, p. 54] or [Maz85, Theorem, p. 381–382] for $d > 2$ and [Maz85, Corollary 1, p. 382] for $d = 2$, for fixed $t > 2$, we find constants $c_1, c_2 > 0$ independent of $Q \in \mathcal{D}$ and ν such that for all $u \in C_c^\infty\left(\mathbb{R}^d\right)$

$$\begin{aligned}\|u\|_{L^2_{\nu|_Q}(\mathbb{R}^d)} &\leq \|u\|_{L^t_{\nu|_Q}(\mathbb{R}^d)}\\ &\leq c_1\left(\sup_{x\in\mathbb{R}^d,0<r<1/2} \left|\log(r)\right| \nu\left(Q \cap B_r(x)\right)^{2/t}\right)^{1/2} \|u\|_{H^1(\mathbb{R}^d)},\end{aligned}$$

for $d = 2$ (note there is a typo, the constant C_5 has to be replaced by $C_5^{1/p}$, see also for the correct version in [Maz11, p. 83]) and

$$\begin{aligned}\|u\|_{L^2_{\nu|_Q}(\mathbb{R}^d)} &\leq \|u\|_{L^t_{\nu|_Q}(\mathbb{R}^d)}\\ &\leq c_2\left(\sup_{x\in\mathbb{R}^d,\rho>0} \rho^{(2-d)} \nu\left(Q \cap B_\rho(x)\right)^{2/t}\right)^{1/2} \|u\|_{H^1(\mathbb{R}^d)},\end{aligned}$$

for $d > 2$. Therefore, Lemma 5.3 (with $a = 2 - d$ and $b = 2/t$) proves the claim. $\square$

Lemma 5.5. *Then for every $t > 2$ there exists $T_{d,t} > 0$ such that for all $Q \in \mathcal{D}$ and $u \in C_b^\infty\left(\overline{Q}\right)$ with $\int_Q u \,\mathrm{d}\Lambda = 0$, we have*

$$\|u\|_{L^2_{\nu|_Q}(Q)} \leq T_{d,t}\mathfrak{J}_{\nu,2/d-1,2/t}(Q)^{1/2}\,\|\nabla u\|_{L^2_\Lambda(Q)}.$$

Proof. Combining Lemma 2.7 and Corollary 5.4, we have for all $u \in C_b^\infty(\overline{Q})$

$$\begin{aligned}\|u\|_{L^2_{\nu|_Q}(Q)} &= \left\|\mathfrak{E}_Q(u)\right\|_{L^2_{\nu|Q}(\mathbb{R}^d)} \\ &\le C_{t,d}\mathfrak{J}_{\nu,2/d-1,2/t}(Q)^{1/2}\left\|\mathfrak{E}_Q(u)\right\|_{H^1(\mathbb{R}^d)} \\ &\le \underbrace{\frac{C_{t,d}\left\|\mathfrak{E}_\mathbf{Q}\right\|}{D_\mathbf{Q}}}_{=:T_{d,t}}\mathfrak{J}_{\nu,2/d-1,2/t}(Q)^{1/2}\left(\|\nabla u\|^2_{L^2_\Lambda(Q)}+\frac{1}{\Lambda(Q)}\left|\int_Q u\,\mathrm{d}\Lambda\right|^2\right)^{1/2} \\ &= T_{d,t}\mathfrak{J}_{\nu,2/d-1,2/t}(Q)^{1/2}\,\|\nabla u\|^2_{L^2_\Lambda(Q)}.\end{aligned}$$

□

Corollary 5.6. *If* $\dim_\infty(\nu) > d-2$*, then*

$$\overline{s}_\nu^D \le \overline{s}_\nu^N \le \lim_{t\downarrow 2}\overline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} \le q^N_{\mathfrak{J}_\nu} \text{ and } \underline{s}_\nu^D \le \underline{s}_\nu^N \le \lim_{t\downarrow 2}\underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}.$$

In particular, in the case $d = 2$*, we have* $\overline{s}_\nu^N \le 1$.

Proof. Note that $\dim_\infty(\nu) > d-2$ implies that for all $t \in (2, 2\dim_\infty(\nu)/(d-2))$, $\mathfrak{J}_{\nu,2/d-1,2/t}$ is non-negative, monotone and uniformly vanishing on $\mathcal{D}$. Combining Proposition 3.6, Proposition 5.1, and Lemma 5.5, we obtain

$$\overline{s}_\nu^N \le \overline{h}_{\mathfrak{J}_{\nu,(2/d-1),2/t}} = (t/2)\overline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} \le (t/2)q^N_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}$$

and $\underline{s}_\nu^N \le \underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}$ for all $t \in (2, \dim_\infty(\nu)/(d-2))$. The claim follows by letting $t \searrow 2$ and Proposition 3.6. □

5.2 Lower bounds

5.2.1 Lower bound on the spectral dimension

Recall from Section 1.1.3, for $n \in \mathbb{N}$ and $\alpha > 0$,

$$\mathcal{N}^{D/N}_{\alpha,\mathfrak{J}}(n) = \mathrm{card}\left(M^{D/N}_{\alpha,\mathfrak{J}}(n)\right) \text{ with } M^{D/N}_{\alpha,\mathfrak{J}}(n) = \left\{C \in \mathcal{D}_n^{D/N} : \mathfrak{J}(C) \ge 2^{-\alpha n}\right\}.$$

As before, for $s > 0$, we let $\langle Q\rangle_s$ denote the cube centered and parallel with respect to Q such that $\Lambda(Q) = s^{-d}\Lambda\left(\langle Q\rangle_s\right)$, $s > 0$. Recall that we always assume $\dim_\infty(\nu) > d-2$. We start with the following simple geometric lemma.

Lemma 5.7. *Let* $Q, Q' \in \mathcal{D}_n^N$, $n \in \mathbb{N}$*, then*

(1) $\langle \mathring{Q}\rangle_5 \cap \mathring{Q}' = \varnothing$ *implies* $\langle \mathring{Q}\rangle_3 \cap \left\langle \mathring{Q}'\right\rangle_3 = \varnothing$,

(2) $\langle \mathring{Q} \rangle_5 \cap \mathring{Q}' = \varnothing$ *implies* $\left\langle \mathring{Q}' \right\rangle_5 \cap \mathring{Q} = \varnothing$.

Proof. Let us write

$$\mathring{Q} = \prod_{i=1}^{d} (a_i, b_i) \text{ and } \mathring{Q}' = \prod_{i=1}^{d} (c_i, d_i)$$

and note that

$$\langle \mathring{Q} \rangle_5 = \prod_{i=1}^{d} (3a_i - 2b_i, 3b_i - 2a_i) \text{ and } \langle \mathring{Q} \rangle_3 = \prod_{i=1}^{d} (2a_i - b_i, 2b_i - a_i).$$

Now, if $\mathring{Q} \cap \left\langle \mathring{Q}' \right\rangle_5 = \varnothing$, then there exists $j \in \{1, \ldots, d\}$ such that

$$(c_j, d_j) \cap (3a_j - 2b_j, 3b_j - 2a_j) = \varnothing.$$

We only consider the case $d_j < 3a_j - 2b_j$, the case $c_j > 3b_j - 2a_j$ follows similarly. Using $d_j - c_j = b_j - a_j = 2^{-n}$ yields

$$2d_j - c_j = d_j + b_j - a_j < 3a_j - 2b_j + b_j - a_j = 2a_j - b_j,$$

implying

$$\langle \mathring{Q} \rangle_3 \cap \langle \mathring{Q}' \rangle_3 = \prod_{i=1}^{d} (2a_i - b_i, 2b_i - a_i) \cap \prod_{i=1}^{d} (2c_i - d_i, 2d_i - c_i) = \varnothing.$$

Further, we have

$$3d_j - 2c_j = d_j + 2(b_j - a_j) < 3a_j - 2b_j = a_j,$$

allowing us to infer that

$$\langle \mathring{Q}' \rangle_5 \cap \mathring{Q} = \prod_{i=1}^{d} (3c_i - 2d_i, 3d_i - 2c_i) \cap \prod_{i=1}^{d} (a_i, b_i) = \varnothing. \qquad \square$$

We need also the following simple combinatorial lemma.

Lemma 5.8. *For fixed* $\alpha > 0$ *there exists a sequence* $\left(E_{\alpha,n}\right)_n$ *with* $E_{\alpha,n} \subset M_{\alpha,\mathfrak{J}}^{D/N}(n)$,

$$e_{\alpha,n} := \operatorname{card}\left(E_{\alpha,n}\right) \geq \left\lfloor \mathcal{N}_{\alpha,\mathfrak{J}}^{D/N}(n) / 5^d \right\rfloor$$

and for all cubes $Q, Q' \in E_{\alpha,n}$ *with* $Q \neq Q'$ *we have* $\left\langle \mathring{Q} \right\rangle_3 \cap \left\langle \mathring{Q}' \right\rangle_3 = \varnothing$.

Proof. We assume $M^{D/N}_{\alpha,\mathfrak{J}}(n) \neq \varnothing$. We construct inductively a subset $E_{\alpha,n}$ of $M^{D/N}_{\alpha,\mathfrak{J}}(n)$ of cardinality $\operatorname{card}(E_{\alpha,n}) \geq \left\lfloor \mathcal{N}^{D/N}_{\alpha,\mathfrak{J}}(n)/5^d \right\rfloor$ such that for all cubes $Q, Q' \in E_{\alpha,n}$ with $Q \neq Q'$, we have $\langle \mathring{Q} \rangle_3 \cap \left\langle \mathring{Q}' \right\rangle_3 = \varnothing$. At the beginning of the induction we set $D^{(0)} := M^{D/N}_{\alpha,\mathfrak{J}}(n)$. Assume we have constructed

$$D^{(0)} \supset D^{(1)} \supset \ldots \supset D^{(j-1)}$$

such that the following condition holds $\langle \mathring{Q}_j \rangle_5 \cap \mathring{Q} \neq \varnothing$ for some $Q, Q_j \in D^{(j-1)}$ with $Q \neq Q_j$. Then we set

$$D^{(j)} := \left\{ C \in D^{(j-1)} : \mathring{C} \cap \langle \mathring{Q}_j \rangle_5 = \varnothing \right\} \cup \{Q_j\}.$$

By this construction, we have $\operatorname{card}(D^{(j)}) < \operatorname{card}(D^{(j-1)})$, since $\mathring{Q} \cap \langle \mathring{Q}_j \rangle_5 \neq \varnothing$. Further, by Lemma 5.7, for all $Q \in D^{(j)} \setminus \{Q_j\}$, we have $\mathring{Q}_j \cap \langle \mathring{Q} \rangle_5 = \varnothing$, showing $Q_j \in D^{(k)}$ for all $k \geq j$. If otherwise $\langle \mathring{Q} \rangle_5 \cap \mathring{Q}' = \varnothing$ for all $Q, Q' \in D^{(j-1)}$ with $Q \neq Q'$, then we set $E_{\alpha,n} = D^{(j-1)}$. In each inductive step, we remove at most $5^d - 1$ elements of $D^{(j-1)}$, while one element, namely Q_j, is kept. Moreover, by the construction of $E_{\alpha,n}$, for each $Q' \in M^{D/N}_{\alpha,\mathfrak{J}}(n)$ there exists $Q \in E_{\alpha,n}$ such that $\langle \mathring{Q} \rangle_5 \cap \mathring{Q}' \neq \varnothing$. This implies

$$\begin{aligned}
\operatorname{card}(E_{\alpha,n}) &= \sum_{Q \in E_{\alpha,n}} \frac{\operatorname{card}\left(\{Q' \in M^{D/N}_{\alpha,\mathfrak{J}}(n) : \langle \mathring{Q} \rangle_5 \cap \mathring{Q}' \neq \varnothing\}\right)}{\operatorname{card}\left(\{Q' \in M^{D/N}_{\alpha,\mathfrak{J}}(n) : \langle \mathring{Q} \rangle_5 \cap \mathring{Q}' \neq \varnothing\}\right)} \\
&\geq \sum_{Q \in E_{\alpha,n}} \frac{1}{5^d} \operatorname{card}\left(\{Q' \in M^{D/N}_{\alpha,\mathfrak{J}}(n) : \langle \mathring{Q} \rangle_5 \cap \mathring{Q}' \neq \varnothing\}\right) \\
&\geq \frac{\mathcal{N}^{D/N}_{\alpha,\mathfrak{J}}(n)}{5^d}.
\end{aligned}$$

Finally, by Lemma 5.7, we obtain that if $\mathring{Q} \cap \left\langle \mathring{Q}' \right\rangle_5 = \varnothing$ for $Q, Q' \in E_{\alpha,n}$, then $\langle \mathring{Q} \rangle_3 \cap \left\langle \mathring{Q}' \right\rangle_3 = \varnothing$. □

The lower estimate of the spectral dimension is based on the following abstract observation, connecting the optimized coarse multifractal dimension and the spectral dimension.

Proposition 5.9. *Assume there exists a non-negative monotone set function $\mathfrak{J}$ on $\mathcal{D}$ with $\dim_\infty(\mathfrak{J}) > 0$ such that for every $Q \in \mathcal{D}$ with $\mathfrak{J}(Q) > 0$ there exists a non-negative and non-zero function $\psi_Q \in C_c^\infty(\mathbb{R}^d)$ with support contained in $\langle \mathring{Q} \rangle_3$ such that*

$$\left\|\psi_Q\right\|^2_{L^2_\nu} \geq \mathfrak{J}(Q) \left\|\nabla \psi_Q\right\|^2_{L^2_\Lambda(\mathbb{R}^d)}.$$

Then for fixed $\alpha > 0$ and for $x > 0$ large, we have

$$\frac{\mathcal{N}^D_{\alpha,\mathfrak{J}}\left(n_{\alpha,x}\right)}{5^d} - 1 \le N^D_\nu\left(x\right) \text{ and } \frac{\mathcal{N}^N_{\alpha,\mathfrak{J}}\left(n_{\alpha,x}\right)}{2\cdot 5^d} - 1 \le N^N_\nu\left(\frac{x}{D_{\mathbf{Q}}}\right),$$

with $n_{\alpha,x} \coloneqq \left\lfloor \log_2\left(x\right)/\alpha\right\rfloor$. In particular, we have

$$\underline{F}^N_{\mathfrak{J}} \le \underline{s}^N_\nu \text{ and } \overline{F}^N_{\mathfrak{J}} \le \overline{s}^N_\nu, \quad \underline{F}^D_{\mathfrak{J}} \le \underline{s}^D_\nu \text{ and } \overline{F}^D_{\mathfrak{J}} \le \overline{s}^D_\nu.$$

Proof. Fix $\alpha > 0$ and let $E_{\alpha,n}$, $n \in \mathbb{N}$, be given as in Lemma 5.8. Let us first consider the Dirichlet case. Since for each $Q \in \mathcal{D}^D_n$, we have $\partial\mathbf{Q} \cap \overline{Q} = \varnothing$, it follows that $\langle \mathring{Q} \rangle_3 \subset \mathbf{Q}$ and therefore $\psi_Q \in C^\infty_c(\mathring{\mathbf{Q}})$. Now, for $x > 2^\alpha$ we define $n_{\alpha,x} \coloneqq \left\lfloor \log_2\left(x\right)/\alpha\right\rfloor$. Then, for each $Q \in E_{n_{\alpha,x}}$, we have

$$\frac{\int_{\mathbf{Q}} \left|\nabla\psi_Q\right|^2 \,\mathrm{d}\Lambda}{\int \psi_Q^2 \,\mathrm{d}\nu} \le \frac{1}{\mathfrak{J}\left(Q\right)} \le 2^{\alpha n_{\alpha,x}} \le x.$$

Hence, the $\left(\psi_Q : Q \in E_{n_{\alpha,x}}\right) \eqqcolon \left(f_i : i = 1,\dots,e_{n_{\alpha,x}}\right)$ are mutually orthogonal both in L^2_ν and in H^1_0- Thus, we obtain that span $\left(f_i : i = 1,\dots,e_{\alpha,x}\right)$ is an $e_{n_{\alpha,x}}$-dimensional subspace of H^1_0. Hence, we deduce from Lemma 2.19

$$\mathcal{N}^D_{\alpha,\mathfrak{J}}\left(n_{\alpha,x}\right)/5^d - 1 \le e_{n_{\alpha,x}} \le N^D_\nu\left(x\right).$$

In the Neumann case, we proceed similarly. For fixed $\alpha > 0$ again set $n_{\alpha,x} = \left\lfloor \log_2\left(x\right)/\alpha\right\rfloor$ and write $E_{n_{\alpha,x}} = \left\{Q_1,\dots,Q_{\mathrm{card}\left(E_{n_{\alpha,x}}\right)}\right\}$. For each $i = 1,\dots,\left\lfloor e_{n_{\alpha,x}}/2\right\rfloor \eqqcolon N_{\alpha,x}$ we define

$$f_i \coloneqq \left(a_{2i-1}\psi_{Q_{2i-1}} + a_{2i}\psi_{Q_{2i}}\right)|_{\overline{\mathbf{Q}}} \in C^\infty_b\left(\overline{\mathbf{Q}}\right),$$

where we choose $(a_{2i-1}, a_{2i}) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $\int_{\mathbf{Q}} f_i \,\mathrm{d}\Lambda = 0$. Then, by Lemma 2.7, we have

$$\langle f_i, f_i\rangle_{H^1} \le \frac{\int_{\mathbf{Q}} \left|\nabla f_i\right|^2 \,\mathrm{d}\Lambda}{D_{\mathbf{Q}}}.$$

Since we have $\langle \mathring{Q}_j \rangle_3 \cap \langle \mathring{Q}_k \rangle_3 = \varnothing$ for $j \neq k$, combined with the properties of mediants, i.e.

$$\frac{a+b}{c+d} = \frac{a}{c}\frac{c}{c+d} + \frac{b}{d}\frac{d}{c+d} \le \max\left\{\frac{a}{c}, \frac{b}{d}\right\} \text{ for all } a,b,c,d > 0,$$

we obtain

$$\begin{aligned}
\frac{\langle f_i, f_i\rangle_{H^1}}{\int f_i^2 \,\mathrm{d}\nu} &\le \frac{\int_{\mathbf{Q}} |\nabla f_i|^2 \,\mathrm{d}\Lambda}{D_{\mathbf{Q}} \int f_i^2 \,\mathrm{d}\nu} \le \frac{\int |\nabla f_i|^2 \,\mathrm{d}\Lambda}{D_{\mathbf{Q}} \int f_i^2 \,\mathrm{d}\nu} \\
&\le \frac{1}{D_{\mathbf{Q}}} \frac{a_1^2 \int \left(\nabla \psi_{Q_{2i-1}}\right)^2 \,\mathrm{d}\Lambda + a_2^2 \int \left(\nabla \psi_{Q_{2i}}\right)^2 \,\mathrm{d}\Lambda}{a_1^2 \int \psi_{Q_{2i-1}}^2 \,\mathrm{d}\nu + a_2^2 \int \psi_{Q_{2i}}^2 \,\mathrm{d}\nu} \\
&\le \frac{1}{D_{\mathbf{Q}}} \max\left\{ \frac{\int \left(\nabla \psi_{Q_{2i}}\right)^2 \,\mathrm{d}\Lambda}{\int \psi_{Q_{2i}}^2 \,\mathrm{d}\nu}, \frac{\int \left(\nabla \psi_{Q_{2i-1}}\right)^2 \,\mathrm{d}\Lambda}{\int \psi_{Q_{2i-1}}^2 \,\mathrm{d}\nu} \right\} \\
&\le \frac{1}{D_{\mathbf{Q}}} \max\left\{ \frac{1}{\mathfrak{J}(Q_{2i-1})}, \frac{1}{\mathfrak{J}(Q_{2i})} \right\} \le \frac{x}{D_{\mathbf{Q}}}.
\end{aligned}$$

Hence, the f_i are mutually orthogonal in H^1 and also in L_ν^2, we obtain span $\left(f_1, \dots, f_{N_{\alpha,x}}\right)$ is a $N_{\alpha,x}$- dimensional subspace of H^1. Again, an application of Lemma 2.19 gives

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n_{\alpha,x}) / \left(2 \cdot 5^d\right) - 1 \le N_\nu^N \left(x / D_{\mathbf{Q}}\right).$$

Consequently, analogous to the proof of Proposition 4.3, we conclude

$$\overline{s}_\nu^D = \liminf_{x\to\infty} \frac{\log\left(N_\nu^D(x)\right)}{\log(x)} \ge \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\alpha,\mathfrak{J}}^D(n)\right)}{\alpha \log(2^n)} = \frac{\underline{F}_{\mathfrak{J}}^D(\alpha)}{\alpha}.$$

Taking the supremum over all $\alpha > 0$ gives $\underline{F}_{\mathfrak{J}}^D \le \underline{s}_\nu^D$. Furthermore, for $x_{\alpha,n} := 2^{\alpha n}$ with $n \in \mathbb{N}$, we see that

$$\overline{s}_\nu^D \ge \limsup_{n\to\infty} \frac{\log\left(N_\nu^D(x_{\alpha,n})\right)}{\log(x_{\alpha,n})} \ge \limsup_{n\to\infty} \frac{\log^+\left(\mathcal{N}_{\alpha.\mathfrak{J}}^D(n)\right)}{\log(2^n)\alpha} = \frac{\overline{F}_{\mathfrak{J}}^D(\alpha)}{\alpha},$$

implying $\overline{s}_\nu^D \ge \overline{F}_{\mathfrak{J}}^D$. In the Neumann case, using

$$\mathcal{N}_{\alpha,\mathfrak{J}}^N(n_{\alpha,x}) / \left(2 \cdot 5^d\right) - 1 \le N_\nu^N \left(x / D_{\mathbf{Q}}\right),$$

we obtain in the same ways as in the Dirichlet case that $\underline{F}_{\mathfrak{J}}^N \le \underline{s}_\nu^N$ and $\overline{F}_{\mathfrak{J}}^N \le \overline{s}_\nu^N$. □

5.2.2 Lower bound on the embedding constant

We need a slight modification of $\mathfrak{J}_\nu$ for the case $d = 2$. We define

$$\underline{\mathfrak{J}}_\nu(Q) = \sup_{Q' \in \mathcal{D}(Q)} \nu(Q') \Lambda(Q')^{2/d-1}$$

for $Q \in \mathcal{D}$. Hence, in the case $d = 2$, we have $\underline{\mathfrak{J}}_\nu(Q) = \nu(Q)$. Clearly, we again have $\dim_\infty(\underline{\mathfrak{J}}_\nu) > d-2$, $\tau^{D/N}_{\mathfrak{J}_\nu} = \tau^{D/N}_{\underline{\mathfrak{J}}_\nu}$ by Proposition 2.35 and for $d > 2$, $\underline{F}^{D/N}_{\underline{\mathfrak{J}}_\nu} = \underline{F}^{D/N}_{\mathfrak{J}_\nu}$ and $\overline{F}^{D/N}_{\underline{\mathfrak{J}}_\nu} = \overline{F}^{D/N}_{\mathfrak{J}_\nu}$. The case $d = 2$ is covered by the following lemma.

Lemma 5.10. *In the case $d = 2$, we have*

$$\underline{F}^{D/N}_{\underline{\mathfrak{J}}_\nu} = \underline{F}^{D/N}_{\mathfrak{J}_\nu} \textit{ and } \overline{F}^{D/N}_{\underline{\mathfrak{J}}_\nu} = \overline{F}^{D/N}_{\mathfrak{J}_\nu}.$$

Proof. We have always

$$\left\{C \in \mathcal{D}_n^{D/N} : \sup_{Q' \in \mathcal{D}(C)} \nu(Q') \left|\log(\Lambda(Q'))\right| \geq 2^{-\alpha n}\right\} \supset \left\{C \in \mathcal{D}_n^{D/N} : \nu(C) \geq 2^{-\alpha n}\right\},$$

and using $\dim_\infty(\nu) > d-2$, we obtain for every $\delta > 1$ and $n \in \mathbb{N}$ large enough

$$\nu(Q)\left|\log(\Lambda(Q))\right| \leq \nu(Q)^{1/\delta},\ Q \in \mathcal{D}_n^{D/N}.$$

Indeed, for $d-2 < s < \dim_\infty(\nu)$, we have for all n large and $Q \in \mathcal{D}_n^{D/N}$ with $\nu(Q) > 0$

$$\nu(Q) \leq 2^{-sn}.$$

Further, for fixed $0 < \varepsilon < 1$, we have for n large

$$dn\log(2) \leq 2^{\varepsilon sn} \leq \nu(Q)^{-\varepsilon}.$$

Hence, for all $Q \in \mathcal{D}_n^{D/N}$, we obtain

$$\nu(Q)\left|\log(\Lambda(Q))\right| \leq \nu(Q)^{1-\varepsilon}.$$

This leads to

$$\left\{C \in \mathcal{D}_n^{D/N} : \sup_{Q' \in \mathcal{D}(C)} \nu(Q')\left|\log(\Lambda(Q'))\right| \geq 2^{-\alpha n}\right\} \subset \left\{C \in \mathcal{D}_n^{D/N} : \nu(C) \geq 2^{-\alpha\delta n}\right\}.$$

Thus,

$$\frac{\log^+\left(\mathcal{N}^{D/N}_{\alpha,\mathfrak{J}_\nu}(n)\right)}{\alpha n\log(2)} \leq \delta\frac{\log^+\left(\mathcal{N}^{D/N}_{\alpha\delta,\underline{\mathfrak{J}}_\nu}(n)\right)}{\alpha\delta n\log(2)}.$$

Hence, the claim follows. □

Proposition 5.11. *There exists a constant $K > 0$ such that for every $Q \in \mathcal{D}$ with $\underline{\mathfrak{J}}_\nu(Q) > 0$ there exists a function $\psi_Q \in C_c^\infty(\mathbb{R}^d)$ with support contained in $\langle \mathring{Q} \rangle_3$*

and $\|\varphi_Q\|_{L^2_\nu} > 0$ *such that*

$$\|\psi_Q\|^2_{L^2_\nu} \geq K \underline{\mathfrak{J}}_\nu(Q) \|\nabla \psi_Q\|^2_{L^2_\Lambda(\mathbb{R}^d)}.$$

Proof. Since $\dim_\infty(\underline{\mathfrak{J}}_\nu) > 0$, it follows that for each $Q \in \mathcal{D}$ there exists $C_Q \in \mathcal{D}(Q)$ such that $\underline{\mathfrak{J}}_\nu(Q) = \nu(C_Q)\Lambda(C_Q)^{2/d-1}$. Now, choose $\psi_Q := \varphi_{\langle C_Q \rangle_3, 3}$ as in Lemma 2.21. Then $\psi_Q \cdot \mathbb{1}_{C_Q} = \mathbb{1}_{C_Q}$, $\operatorname{supp}(\psi_Q) \subset \langle \mathring{C}_Q \rangle_3 \subset \langle \mathring{Q} \rangle_3$, and

$$\begin{aligned}
\frac{\int |\nabla \psi_Q|^2 \, \mathrm{d}\Lambda}{\int |\psi_Q|^2 \, \mathrm{d}\nu} &\leq C 2^{-2} 3^{1-2/d} \frac{\Lambda\left(\langle \langle C_Q \rangle_3 \rangle_{1/3}\right)^{1-2/d}}{\nu\left(\langle \langle C_Q \rangle_3 \rangle_{1/3}\right)} \\
&= C 2^{-2} 3^{1-2/d} \frac{\Lambda\left(C_Q\right)^{1-2/d}}{\nu\left(C_Q\right)} = C 2^{-2} 3^{1-2/d} \underline{\mathfrak{J}}_\nu(Q)^{-1}.
\end{aligned}$$

□

Proposition 5.12. *Assume* $\nu(\mathring{Q}) > 0$. *Then for fixed* $\alpha > 0$ *and for* $x > 0$ *large, we have*

$$\frac{\mathcal{N}^D_{\alpha, \underline{\mathfrak{J}}_\nu}(n_{\alpha,x})}{5^d} - 1 \leq N^D_\nu(xK)$$

with $n_{\alpha,x} := \lfloor \log_2(x)/\alpha \rfloor$. *In particular,* $\underline{F}^D_{\mathfrak{J}_\nu} \leq \underline{s}^D_\nu$ *and* $\overline{F}^D_{\mathfrak{J}_\nu} \leq \overline{s}^D_\nu$.

Proof. This follows from Proposition 5.9, Lemma 5.10, and Proposition 5.11. □

In the same way we obtain the following proposition for the Neumann case.

Proposition 5.13. *For fixed* $\alpha > 0$, *we have for* $x > 0$ *large*

$$\frac{\mathcal{N}^N_{\alpha, \underline{\mathfrak{J}}_\nu}(n_{\alpha,x})}{2 \cdot 5^d} - 1 \leq N^N_\nu\left(xK/D_{\boldsymbol{Q}}\right)$$

with $n_{\alpha,x} := \lfloor \log_2(x)/\alpha \rfloor$. *In particular,* $\underline{F}^N_{\mathfrak{J}_\nu} \leq \underline{s}^N_\nu$ *and* $\overline{F}^N_{\mathfrak{J}_\nu} \leq \overline{s}^N_\nu$.

5.3 Main results

In this section, we combine the results of Section 5.1 and Section 5.2.2 to prove the main results of this chapter.

5.3.1 Upper spectral dimension, and lower and upper bounds for the lower spectral dimension

To break up the main result (Theorem 5.15) of this section, we start with the following proposition.

Proposition 5.14. *We have*

$$q_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^N = \overline{h}_{\mathfrak{J}_\nu} = \overline{s}_\nu^N \text{ and } \underline{s}_\nu^N \leq \lim_{t\downarrow 2} \underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}.$$

Proof. From Proposition 3.20 and Proposition 5.13 applied to $\mathfrak{J} = \mathfrak{J}_\nu$, we obtain

$$q_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^N \leq \overline{s}_\nu^N.$$

Moreover, Corollary 3.21 and Corollary 5.6 yield

$$\underline{s}_\nu^N \leq \overline{s}_\nu^N \leq \lim_{t\downarrow 2} \overline{h}_{\mathfrak{J}_{\nu,t(1/d-1)/2,1}} \leq q_{\mathfrak{J}_\nu}^N \text{ and } \overline{F}_{\mathfrak{J}_\nu}^N = \overline{h}_{\mathfrak{J}_\nu} = q_{\mathfrak{J}_\nu}^N$$

which proves the claimed equalities. □

Theorem 5.15. *Let ν be a Borel probability measure on $\boldsymbol{Q}$ such that* $\dim_\infty(\nu) > d-2$.

1. *Under Neumann boundary conditions we have*

$$\underline{F}_{\mathfrak{J}_\nu}^N \leq \underline{s}_\nu^N \leq \lim_{t\downarrow 2} \underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}} \leq \overline{h}_{\mathfrak{J}_\nu} = \overline{s}_\nu^N = q_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^N. \tag{5.3.1}$$

2. *Under Dirichlet boundary conditions and $\nu(\mathring{\boldsymbol{Q}}) > 0$ we have*

$$\underline{F}_{\mathfrak{J}_\nu}^D \leq \underline{s}_\nu^D \quad \text{and} \quad \overline{F}_{\mathfrak{J}_\nu}^D = q_{\mathfrak{J}_\nu}^D \leq \overline{s}_\nu^D \leq q_{\mathfrak{J}_\nu}^N.$$

3. *In particular, if $d = 2$, then $\overline{s}_\nu^N = 1$, and under the assumption $\nu(\mathring{\boldsymbol{Q}}) > 0$, we also have $\overline{s}_\nu^D = 1$.*

4. *If $\tau_{\mathfrak{J}_\nu}^N\left(q_{\mathfrak{J}_\nu}^D\right) = 0$, or equivalently $\overline{F}_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^D$, then the upper Dirichlet and Neumann spectral dimensions have the common value $\overline{s}_\nu^D = \overline{s}_\nu^N = q_{\mathfrak{J}_\nu}^N$. This assumption is particularly fulfilled if*

$$\frac{\overline{\dim}_M(\operatorname{supp}(\nu) \cap \partial \boldsymbol{Q})}{\dim_\infty^{N\setminus D}(\nu) - d + 2} < q_{\mathfrak{J}_\nu}^N. \tag{5.3.2}$$

Proof. The first and second claim follow from Proposition 5.12, Proposition 5.13, and Proposition 5.14. Furthermore, by Proposition 2.35, we always have in the case

$d=2$ that $q^N_{\mathfrak{J}_\nu}=1$. For the third claim note that $\nu(\mathring{\mathbf{Q}})>0$ implies that there exists an open cube Q such that $\overline{Q}\subset\mathring{\mathbf{Q}}$, $\nu(Q)>0$, and $\dim_\infty(\nu|_Q)>d-2=0$. Hence, we obtain

$$1=q^N_{\mathfrak{J}_{\nu|_Q}}\le\overline{F}^N_{\mathfrak{J}_{\nu|_Q}}=\overline{F}^D_{\mathfrak{J}_{\nu|_Q}}\le\overline{F}^D_{\mathfrak{J}_\nu}=q^D_{\mathfrak{J}_\nu}\le\overline{s}^D_\nu\le\overline{s}^N_\nu=1.$$

To prove the last claim, we note that (5.3.2) fulfills the assumption of Lemma 2.38, which implies $q^D_{\mathfrak{J}_\nu}=q^N_{\mathfrak{J}_\nu}$. □

Remark 5.16. By Corollary 5.22 we have $q^N_{\mathfrak{J}_\nu}\ge\overline{\dim}_M(\nu)/(\overline{\dim}_M(\nu)-d+2)$. Hence, we can replaces $q^N_{\mathfrak{J}_\nu}$ by $\overline{\dim}_M(\nu)/(\overline{\dim}_M(\nu)-d+2)$ on the right hand side in (5.3.2) making this condition independent of $q^N_{\mathfrak{J}_\nu}$. Moreover, (5.3.2) can easily be verified for particular measures ν such that

1. $\overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\mathbf{Q})<\overline{\dim}_M(\nu)\frac{\dim_\infty(\nu)-d+2}{\overline{\dim}_M(\nu)-d+2}$, in particular, for ν with

$$\dim_\infty(\nu)>d-1 \text{ and } \overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\mathbf{Q})\le\overline{\dim}_M(\nu)/2.$$

2. $\overline{\dim}_M(\operatorname{supp}(\nu)\cap\partial\mathbf{Q})=0$, particularly for $\operatorname{supp}(\nu)\subset\mathring{\mathbf{Q}}$,
3. ν is given by the d-dimensional Lebesgue measure $\Lambda|_{\mathbf{Q}}$ restricted to $\mathbf{Q}$ (then the left-hand side in (5.3.2) is equal to $(d-1)/2$).

Let us also remark that in Section 5.4.4 we present an example for which $\underline{s}^N_\nu<\overline{s}^N_\nu$.

Remark 5.17. The above theorem and the notion of regularity give rise to the following list of observations for measures ν with $\dim_\infty(\nu)>d-2$:

1. If the Neumann spectral dimension with respect to ν exists, then it is given by purely measure-geometric data encoded in the ν-partition entropy, namely we have $\overline{h}_{\mathfrak{J}_\nu}=\lim_{t\downarrow 2}\underline{h}_{\mathfrak{J}_{\nu,t(2/d-1)/2,1}}$ and this value coincides with the spectral dimension.
2. N-MF-regularity implies equality everywhere in the chain of inequalities (5.3.1) and in particular the Neumann spectral dimension exists. If $\mathfrak{J}_\nu$ is D-MF-regular, then we have equality everywhere in all chains of inequalities above and in particular both Neumann and Dirichlet spectral dimensions exist.

5.3.2 Regularity results

The following theorem shows that the spectral partition function is a valuable auxiliary concept to determine the spectral behavior for a given measure ν.

Theorem 5.18. *Under the assumption* $\dim_\infty(\nu) > d-2$ *we have the following regularity result:*

1. *If* $\mathfrak{J}_\nu$ *is N-PF-regular, then it is N-MF-regular and the Neumann spectral dimension* s_ν^N *exists.*
2. *If* $\mathfrak{J}_\nu$ *is D-PF-regular and* $\tau_{\mathfrak{J}_\nu}^N\left(q_{\mathfrak{J}_\nu}^D\right) = 0$*, then both the Dirichlet and Neumann spectral dimension exist and coincide, i.e.* $s_\nu^D = s_\nu^N$*.*

Proof. Under the assumption that $\mathfrak{J}_\nu$ is D/N-PF-regular, we obtain from Proposition 3.24 applied to $\mathfrak{J} = \mathfrak{J}_\nu$, Lemma 2.18, and Theorem 5.15

$$q_{\mathfrak{J}_\nu}^N = \overline{F}_{\mathfrak{J}_\nu}^N = \overline{s}_\nu^N \geq \underline{s}_\nu^{D/N} \geq \underline{F}_{\mathfrak{J}_\nu}^{D/N} = q_{\mathfrak{J}_\nu}^{D/N} \quad \text{and} \quad \overline{s}_\nu^D \leq \overline{s}_\nu^N,$$

proving the claims. □

We will see in Section 5.4.4 that the result is optimal in the sense that there is an example derived from a similar example for $d=1$ in ν which is not $\mathfrak{J}_\nu$ N-PF-regular and for which $\overline{s}_\nu^N > \underline{s}_\nu^N$. It should be noted that PF-regularity is easy accessible if the spectral partition function is essentially given by the L^q-spectrum of ν .

In the following proposition we present lower bounds of the lower spectral dimension in terms of the subdifferential of $\tau_{\mathfrak{J}_\nu}^{D/N}$ at q.

Proposition 5.19. *If* $\dim_\infty(\nu) > d-2$ *and if for* $q \in \left[0, q^{D/N}\right]$ *we have*

$$\tau_{\mathfrak{J}_\nu}^{D/N}(q) = \lim_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^{D/N}(q) \text{ and } -\partial\tau_{\mathfrak{J}_\nu}^{D/N}(q) = [a,b],$$

then

$$\frac{aq + \tau_{\mathfrak{J}_\nu}^{D/N}(q)}{b} \leq \underline{s}_\nu^{D/N}.$$

Proof. This follows from Proposition 3.18, Proposition 5.12, and Proposition 5.13. □

Remark 5.20. In the case that $\tau_{\mathfrak{J}_\nu}^N(q_{\mathfrak{J}_\nu}^N) = \lim_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^N(q_{\mathfrak{J}_\nu}^N)$ and $\tau_{\mathfrak{J}_\nu}^N$ is differentiable at $q_{\mathfrak{J}_\nu}^N$, we infer $q_{\mathfrak{J}_\nu}^N \leq \underline{s}_\nu^N$ and hence we obtain a direct proof of the regularity statement, namely, $q_{\mathfrak{J}_\nu}^N = \underline{s}_\nu^N = \overline{s}_\nu^N$.

Also, if $\tau_{\mathfrak{J}_\nu}^{D/N}(1) = \lim_{n\to\infty} \tau_{\mathfrak{J}_\nu,n}^{D/N}(1) = d-2$, we have the lower bound in terms of left-sided, respectively right-sided, derivative of β_ν^N given by

$$\frac{-\partial^+\tau_{\mathfrak{J}_\nu}^{D/N}(1) - d + 2}{-\partial^-\tau_{\mathfrak{J}_\nu}^{D/N}(1)} \leq \underline{s}_\nu^{D/N}.$$

Corollary 5.21. *For $d = 2$ we have:*

1. *It holds*
$$\frac{-\partial^+ \beta_\nu^N(1)}{-\partial^- \beta_\nu^N(1)} \le \underline{s}_\nu^N \le \overline{s}_\nu^N = 1.$$

2. *If $\beta_\nu^{D/N}$ is differentiable at 1, then $s_\nu^{D/N} = 1$.*

3. *If $\nu(\mathring{Q}) > 0$ and β_ν^N is differentiable at 1, then also β_ν^D is differentiable at 1. In particular, $s_\nu^D = s_\nu^N = 1$.*

5.3.3 General bounds in terms of fractal dimensions

We obtain general bounds for $\overline{s}_\nu^N$ in terms of the upper Minkowski dimension $\overline{\dim}_M(\nu)$ and the possibly smaller lower ∞-dimension $\dim_\infty(\nu)$ of ν (see also Figure 5.3.1).

Corollary 5.22. *Assume $\dim_\infty(\nu) > d - 2$. Then for the Neumann upper spectral dimension we have*

$$\frac{d}{2} \le \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu) - d + 2} \le \overline{s}_\nu^N \le \frac{\dim_\infty(\nu)}{\dim_\infty(\nu) - d + 2}.$$

Proof. Theorem 5.15 gives $\overline{s}_\nu^N = q_{\mathfrak{J}_\nu}^N$, hence the claim follows from the estimates of $q_{\mathfrak{J}_\nu}^N$ obtained in Fact 2.37. □

Remark 5.23. Note that by choosing measures with $\overline{\dim}_M(\nu)$ close to $d - 2$ we can easily find examples where $\overline{s}_\nu^N$ becomes arbitrarily large.

It is also worth mentioning that the analogous situation in the dimension $d = 1$ is quite different (cf. Section 4.3.3), namely the lower bound becomes an upper bound,

$$\overline{s}_\nu^{D/N} \le \frac{\overline{\dim}_M(\nu)}{\overline{\dim}_M(\nu) + 1} \le \frac{1}{2}.$$

This inequalities in Corollary 5.22 naturally links to the famous question by M. Kac [Kac66], *"Can one hear the shape of a drum?"* This question has been modified by various authors e.g. in [Ber79; Ber80; BC86; Lap91], and closer to our context by Triebel in [Tri97]. In the plane, the spectral dimension does not encode any information about the fractal-geometric nature of the underlying measure as we always have $\overline{s}_\nu^{D/N} = 1$ for any bounded Borel measure with $\nu(\mathring{\mathbf{Q}}) > 0$. This has been observed in [Tri97] for the special case of α-Ahlfors–David regular measures. For all other dimensions, our results show that the upper spectral dimension $\overline{s}_\nu^N$ is uniquely

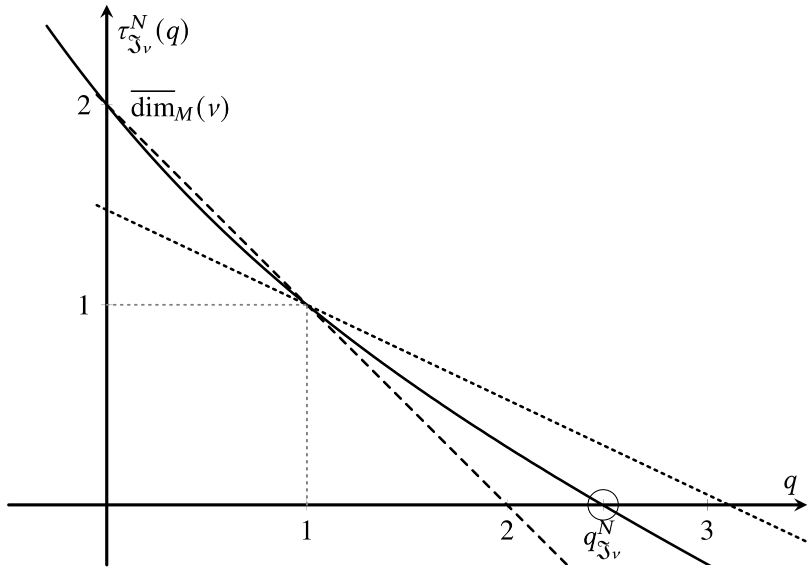

Figure 5.3.1 Partition function $\tau^N_{\mathfrak{J}_\nu}$ in dimension $d = 3$ for the self-similar measure ν supported on the *Sierpiński tetraeder* with all four contraction ratios equal $1/2$ and with probability vector $(0.36, 0.36, 0.2, 0.08)$. Natural bounds for $\overline{s}^N_\nu = q^N_{\mathfrak{J}_\nu}$ in this setting are the zeros of the dashed line $x \mapsto -x\left(\tau^N_{\mathfrak{J}_\nu}(0) - 1\right) + \tau^N_{\mathfrak{J}_\nu}(0)$ and the dotted line $x \mapsto (1-x)\dim_\infty(\nu) + x$ as given in Corollary 5.22. In this case $\tau^N_{\mathfrak{J}_\nu}(0) = \overline{\dim}_M(\nu) = 2$ and $\dim_\infty(\nu) = -\log(0.36)/\log(2) \approx 1.47$

determined by the spectral partition function $\tau^N_{\mathfrak{J}_\nu}$, which in turn reflects many important fractal-geometric properties of ν. For the case $d > 2$, this common ground provides interesting bounds on the upper Minkowski dimension of the support of ν and the lower ∞-dimension of ν in terms of the upper spectral dimension given by

$$\overline{\dim}_M(\nu) \geq \frac{\overline{s}^N_\nu (d-2)}{\overline{s}^N_\nu - 1} \geq \dim_\infty(\nu).$$

So the answer to Kac's question is "partially yes". If additionally the L^q-spectrum β^N_ν is an affine function, we obtain $\beta^N_\nu(q) = \overline{\dim}_M(\nu) + \overline{\dim}_M(\nu)(1-q)$ and with Corollary 5.22

$$\dim_\infty(\nu) = \overline{\dim}_M(\nu) = \frac{\overline{s}^N_\nu (d-2)}{\overline{s}^N_\nu - 1}.$$

In this case, Kac's question regarding dimensional quantities must be answered in the affirmative.

5.4 Examples

Finally, we give some leading examples where the spectral partition function is essentially given by the L^q-spectrum of ν (see Section 2.4.3.1 and Section 2.4.3.4) and in this case we are able to provide the following complete picture.

5.4.1 Absolutely continuous measures

As a first application of our results, we present the case of absolutely continuous measures.

Proposition 5.24. *Let ν be absolutely continuous with respect to Λ with r-integrable density for some $r \geq d/2$. Then the Dirichlet and Neumann spectral partition function exists as a limit with*

$$\tau^N_{\mathfrak{J}_\nu}(q) = \tau^D_{\mathfrak{J}_\nu}(q) = d - 2q, \quad \text{for } q \in [0, r],$$

ν is D/N-PF-regular, and the Dirichlet and Neumann spectral dimension exist and equal $s^D_\nu = s^N_\nu = d/2$.

Proof. We immediately obtain from Proposition 2.44 and Theorem 5.18 that

$$s^D_\nu = s^N_\nu = d/2. \qquad \square$$

Also for absolutely continuous measures we have the following rigidity result in terms of reaching the minimal possible value $d/2$ of the spectral dimension.

Proposition 5.25. *Let ν be an absolutely continuous measure. If* $\dim_\infty(\nu) > d-2$, *then the following rigidity result holds:*

1. *If $\overline{s}_\nu^N = d/2$, then $\tau_{\mathfrak{J}_\nu}^N(q) = d-2q$ for all $q \in [0, d/2]$.*
2. *If $\tau_{\mathfrak{J}_\nu}^N(q) = d-2q$ for some $q > d/2$, then $\tau_{\mathfrak{J}_\nu}^N(q) = d-2q$ for all $q \in [0, d/2]$ and $\overline{s}_\nu^N = d/2$.*

Proof. Suppose $\overline{s}_\nu^N = d/2$. Then by Theorem 5.15, we have $q_{\mathfrak{J}_\nu}^N = d/2$. Moreover, by Lemma 2.43, for all $0 \le q \le 1$, we have

$$d-2q = \beta_\nu^N(q) + (d-2)q \le \tau_{\mathfrak{J}_\nu}^N(q),$$

and the convexity of $\tau_{\mathfrak{J}_\nu}^N$ yields for all $q \in [0, d/2]$

$$\tau_{\mathfrak{J}_\nu}^N(q) \le d-2q.$$

Furthermore, the convexity of β_ν^N for all $q \in [1, d/2]$ yields

$$d-2q \le \beta_\nu^N(q) + (d-2)q \le \tau_{\mathfrak{J}_\nu}^N(q).$$

This proves the first claim. For the second claim assume $\tau_{\mathfrak{J}_\nu}^N(q) = d-2q$ for some $q > d/2$. Again, for all $q' \in [0, q]$, we deduce

$$d-2q' \le \beta_\nu^N(q') + (d-2)q' \le \tau_{\mathfrak{J}_\nu}^N(q') \le d-2q'.$$

In particular, $\tau_{\mathfrak{J}_\nu}^N(d/2) = 0$, which implies $\overline{s}_\nu^N = d/2$. □

5.4.2 Ahlfors–David regular measure

As a second application, we consider a class of measures with linear spectral partition functions, namely we treat *α-Ahlfors–David regular measures ν* on $\mathring{\mathbf{Q}}$ for $\alpha > 0$, i.e. there exist constants $K > 0$ such that for every $x \in \operatorname{supp}(\nu)$ and $r \in (0, \operatorname{diam}(\operatorname{supp}(\nu))]$ we have

$$K^{-1} r^\alpha \le \nu(B_r(x)) \le K r^\alpha.$$

Recall that $B_r(x)$ denotes the open ball with center x and radius $r > 0$. Note that for α-Ahlfors–David regular measures ν we have $\alpha = \dim_M(\nu) = \dim_\infty(\nu)$.

Proposition 5.26. *Assume that ν is α-Ahlfors–David regular with $\alpha \in (d-2, d]$ such that $\nu(\mathring{Q}) > 0$. Then both Neumann and Dirichlet spectral dimensions exist and are given by $s_\nu^D = s_\nu^N = \alpha/(\alpha - d + 2)$.*

Proof. We immediately obtain from Theorem 5.18 and the properties of the partition function provided in Section 2.4.3.3 that $s_\nu^D = s_\nu^N = \alpha/(\alpha - d + 2)$. □

This proposition recovers some of the major achievements on isotropic α-sets Γ (in our terms this means that the α-dimensional Hausdorff measure restricted to Γ is α-Ahlfors–David regular) as investigated by Triebel in his book [Tri97]. This follows in our framework from the fact that the partition function is linear and exists as a limit (see Section 2.4.3.3).

5.4.3 Self-conformal measures

As a third application, we treat self-conformal measures with possible overlaps, following up on a question explicitly posed in [NX21, Sec. 5].

Theorem 5.27. *Let ν be a self-conformal measure as defined in Section 2.4.3.4 with $\nu(\partial \boldsymbol{Q}) = 0$ and $\dim_\infty(\nu) > d-2$. Then the spectral partition function exists as a limit and is given by*

$$\tau_{\mathfrak{J}_\nu}^{D/N}(q) = \beta_\nu^N(q) + (d-2)\,q.$$

Further, ν is D/N-PF-regular and the Dirichlet and Neumann spectral dimension exist and equal $s_\nu^D = s_\nu^N = q_{\mathfrak{J}_\nu}^N$. In particular, in the case $d = 2$, we always have $s_\nu^D = s_\nu^N = 1$.

Proof. Let ν be a self-conformal measure with $\dim_\infty(\nu) > d-2$. Then it follows from Corollary 2.53 that $\mathfrak{J}_\nu$ is D/N-PF-regular and

$$\tau_{\mathfrak{J}_\nu}^D\left(q_{\mathfrak{J}_\nu}^D\right) = \tau_{\mathfrak{J}_\nu}^N\left(q_{\mathfrak{J}_\nu}^D\right) = 0.$$

Now, Theorem 5.15 and Theorem 5.18 give $s_\nu^D = s_\nu^N = q_{\mathfrak{J}_\nu}^N$. □

Remark 5.28. In general, it is difficult to verify the condition $\dim_\infty(\nu) > d-2$, but in the case $d = 2$ a sufficient condition is that the measure ν is invariant with respect to an IFS given by a system of bi-Lipschitz contractions such that the attractor is not a singleton (see [HLN06, Lemma 5.1]). This carries over to self-similar measures, provided that the contractive similitudes do not share the same fixed point, so that $\dim_\infty(\nu) > 0$ and the spectral dimension is given by $s_\nu^D = s_\nu^N = 1$.

5.4.4 Non-existence of the spectral dimension

Here, we present an example for which lower and upper spectral dimension differ.

Example 5.29. Let us consider the homogeneous Cantor measure μ on $(0,1)$ form Example 4.48 with non-converging L^q-spectrum, for which we have

$$\underline{s}_{\mu}^{D/N} = 3/13 < 3/11 = \overline{s}_{\mu}^{D/N},\ \beta_{\mu}^{N}(q) = \begin{cases} \frac{3}{8}(1-q), & q \in [0,1], \\ \frac{3}{10}(1-q), & q > 1, \end{cases}$$

and

$$\underline{\beta}_{\mu}(q) \coloneqq \liminf_{n\to\infty} \beta_{\mu,n}^{N}(q) = \begin{cases} \frac{3}{10}(1-q), & q \in [0,1], \\ \frac{3}{8}(1-q), & q > 1. \end{cases}$$

Take the one-dimensional Lebesgue-measure Λ^1 restricted to $[0,1]$ and define the product measure on $\mathbf{Q}$ by $\nu \coloneqq \mu \otimes \Lambda^1 \otimes \Lambda^1$. Due to the product structure, we have for the L^q-spectrum of ν

$$\beta_{\nu}^{N}(q) = \beta_{\mu}^{N}(q) + \beta_{\Lambda^2}^{N}(q) = \beta_{\mu}^{N}(q) + 2(1-q),\ q \geq 0,$$

and hence $\dim_{\infty}(\nu) = 2 + 3/10 > 1$. Let π_1 denote the projection onto the first coordinate. Then for $t \in [2,4)$, we have

$$\begin{aligned} \tau_{\mathfrak{J}_{\nu,(2/3-1),2/t},n}^{N}(q) &= \frac{1}{\log(2^n)} \log\left(\sum_{Q\in\mathcal{D}_n^N} \sup_{Q'\in\mathcal{D}(Q)} \left(\nu(Q')^{2/t} \Lambda(Q')^{-1/3} \right)^q \right) \\ &= \frac{1}{\log(2^n)} \\ &= \frac{1}{\log(2^n)} \log\left(\sum_{Q\in\pi_1(\mathcal{D}_n^N)} \mu(Q)^{q2/t} 2^{qn(-4/t+1)} 2^{2n} \right) \\ &= \beta_{\mu,n}^{N}(q2/t) - q(4/t-1) + 2 \end{aligned}$$

and the spectral partition function $\tau_{\mathfrak{J}_{\nu,(2/3-1),2/t}}^{N}$ is given by

$$\tau_{\mathfrak{J}_{\nu,(2/3-1),2/t}}^{N}(q) = \beta_{\mu}^{N}(q) - q(t-1) + 2$$

and therefore

$$\tau_{\mathfrak{J}_{\nu,(2/3-1),2/t}}^{N}(q) \neq \liminf_{n\to\infty} \tau_{\mathfrak{J}_{\nu,(2/3-1),2/t},n}^{N}(q) = \underline{\beta}_{\mu}^{N}(q) - q(t-1) + 2$$

for $q \in \mathbb{R}_{\geq 0} \setminus \{1\}$. This gives for the upper spectral dimension $\overline{s}_{\nu}^{N} = q_{\mathfrak{J}_{\nu}}^{N} = 23/13$. Furthermore, by Example 4.48, with $n_k \coloneqq 2^{2k+1}8/3 - 10/3$ and $2 < t < 4$, we have

$$\tau_{\mathfrak{J}_{\nu,(2/3-1),2/t},n_k}^{N}(q) = \beta_{\mu,n_k}^{N}(q2/t) - q(4/t-1) + 2 = \frac{(1-q2/t)}{8/3 - 10/(3n_k)} - q(4/t-1) + 2.$$

Therefore, it follows

$$\begin{aligned} q_{t,n_k} &\coloneqq \inf\{q \geq 0 : \tau^N_{\mathfrak{J}_{\nu,(2/3-1),2/t},n_k}(q) < 0\} \\ &= \left(\frac{1}{8/3 - 10/(3n_k)} + 2\right)\left(\frac{2/t}{8/3 - 10/(3n_k)} + \frac{4}{t} - 1\right)^{-1}. \end{aligned}$$

Moreover, by the construction of μ, we have

$$\begin{aligned} \max_{Q \in \mathcal{D}^N_{n_k}} \mathfrak{J}_{\nu,(2/3-1),2/t}(Q)^{q_{n_k}} &= \max_{Q \in \mathcal{D}^N_{n_k}} \left(\mu(\pi_1(Q))^{q2/t} 2^{qn_k(-4/t+1)} 2^{2n_k}\right)^{q_{n_k}} \\ &= \frac{\sum_{Q \in \mathcal{D}^N_{n_k}} \left(\mu(\pi_1(Q))^{q2/t} 2^{qn_k(-4/t+1)} 2^{2n_k}\right)^{q_{n_k}}}{2^{n_k \tau^N_{\mathfrak{J}_{\nu,(2/3-1),2/t},n_k}(0)}} \\ &= \frac{\sum_{Q \in \mathcal{D}^N_{n_k}} \mathfrak{J}_{\nu,(2/3-1),2/t}(Q)^{q_{n_k}}}{2^{n_k \tau^N_{\mathfrak{J}_{\nu,(2/3-1),2/t},n_k}(0)}}. \end{aligned}$$

Hence, the assumptions of Proposition 3.3 are fulfilled. Thus, combining Proposition 3.3 and Corollary 5.6 yield

$$\underline{s}^N_\nu \leq \lim_{t \downarrow 2} \underline{h}_{\mathfrak{J}_{\nu,(2/3-1),2/t}} \leq \lim_{t \downarrow 2} \lim_{k \to \infty} q_{t,n_k} \leq \lim_{t \downarrow 2} \frac{19}{8}\left(\frac{38}{t8} - 1\right)^{-1} = \frac{19}{11}.$$

Furthermore, by Theorem 5.15 and the result of Section 2.4.3.2, we find that $\overline{s}^D_\nu = \overline{s}^N_\nu$. To summarize, we obtain from the consideration above that

$$\underline{s}^D_\nu \leq \underline{s}^N_\nu \leq 19/11 < \overline{s}^D_\nu = \overline{s}^N_\nu = \frac{23}{13}.$$

It should be remarked that this example can be easily modified to construct an example for non-existing spectral dimension for any $d \geq 3$.

Chapter 6

Quantization Dimension

In this chapter, we study the lower and upper quantization dimension with respect to a compactly supported Borel probability measure ν on $\mathbb{R}^d$. Let us first recall the definition of the lower and upper quantization dimension as given in Section 1.1.2:

$$\underline{D}_r(\nu) = \liminf_{n\to\infty} \frac{\log(n)}{-\log(\mathfrak{e}_{n,r}(\nu))}, \quad \overline{D}_r(\nu) = \limsup_{n\to\infty} \frac{\log(n)}{-\log(\mathfrak{e}_{n,r}(\nu))},$$

with $r > 0$ and

$$\mathfrak{e}_{n,r}(\nu) = \inf_{\alpha\in\mathcal{A}_n} \left(\int \min_{y\in\alpha} \|x - y\|^r \, \mathrm{d}\nu(x) \right)^{1/r} = \inf_{f\in\mathcal{F}_n} \left(\int |x - f(x)|^r \, \mathrm{d}\nu(x) \right)^{1/r},$$

where $\mathcal{A}_n$ is the set of subsets of $\mathbb{R}^d$ with cardinality less then or equal to n and

$$\mathcal{F}_n = \{ f : \mathbb{R} \to A,\ A \in \mathcal{A}_n \}.$$

Note that without loss of generality, we can (and for ease of exposition, we will) assume that the support of ν is contained in $(0,1)^d$. To see this, fix $a \neq 0, b \in \mathbb{R}^d$, and let $\Phi_{a,b}(x) := ax + b$, $x \in \mathbb{R}^d$, such that $\Phi_{a,b}(\mathrm{supp}(\nu)) \subset (0,1)^d$. Then,

$$\begin{aligned}
\mathfrak{e}_{n,r}\left(\nu \circ \Phi_{a,b}^{-1}\right) &= \inf_{f\in\mathcal{F}_n} \left(\int \left| ax + b - f\left(\Phi_{a,b}(x)\right)\right|^r \, \mathrm{d}\nu(x) \right)^{1/r} \\
&= |a| \inf_{f\in\mathcal{F}_n} \left(\int \left| x + \Phi_{-1/a, b/a}\left(f\left(\Phi_{a,b}(x)\right)\right)\right|^r \, \mathrm{d}\nu(x) \right)^{1/r} \\
&= |a| \inf_{f\in\mathcal{F}_n} \left(\int |x - f(x)|^r \, \mathrm{d}\nu(x) \right)^{1/r} = |a| \mathfrak{e}_{n,r}(\nu),
\end{aligned}$$

where we used that $f \mapsto \Phi_{-1/a,b/a} \circ f \circ \Phi_{a,b}$ defines a surjection on $\mathcal{F}_n$. Again, as for the computation of the spectral dimension, the main strategy is to reduce the problem of the determination of the quantization dimension to the combinatorial problems considered in Chapter 3 applied to $\mathfrak{J}_{\nu,a} : Q \mapsto \nu(Q)\Lambda(Q)^{r/d}$ with $Q \in \mathcal{D}$.

This chapter is structured as follows. Section 6.1 is devoted to provide upper bounds of the lower and upper quantization dimension; we obtain bounds in terms of the lower and upper $\mathfrak{J}_{\nu,r/d}$-partition entropy. In Section 6.2, we obtain lower bounds of the lower and upper quantization dimension in terms of the lower and upper optimized coarse multifractal dimension with respect to $\mathfrak{J}_{\nu,r/d}$. The main results of this chapter are presented in Section 6.3. Thereby, we combine the results of Section 6.1 and Section 6.2 to compute the upper quantization dimension and impose regularity conditions that guarantee the existence of the quantization dimension. Finally, we confirm the existence of the quantization dimension for self-conformal measures where no separation conditions are assumed.

6.1 Upper bounds for the quantization dimension

In this section, building on the results of Section 3.3, we establish upper bounds of the quantization dimension in terms of the L^q-spectrum of ν. For this purpose, we recall the notation from Section 3.3: $\mathfrak{J}_{\nu,a}(Q) = \nu(Q)\Lambda(Q)^a$, $a > 0$ and $Q \in \mathcal{D}$. Notice that

$$\tau^N_{\mathfrak{J}_{\nu,a}}(q) = \beta^N_\nu(q) - adq,\ q \geq 0.$$

Further, we define

$$q_r \coloneqq q_{\mathfrak{J}_{\nu,r/d}} = \inf\left\{q > 0 : \beta^N_\nu(q) < rq\right\}.$$

The following proposition establishes an upper bound of the quantization dimension in terms of the L^q-spectrum with respect to ν and the lower $\mathfrak{J}_{\nu,r/d}$-partition entropy.

Proposition 6.1. *For all $n \in \mathbb{N}$, we have*

$$e_{n,r}(\nu)^r \leq \sqrt{d} n \gamma_{\mathfrak{J}_{\nu,r/d},n}.$$

In particular, we have

$$\overline{D}_r(\nu) \leq \frac{r\overline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\overline{h}_{\mathfrak{J}_{\nu,r/d}}} = \frac{rq_r}{1-q_r} \leq \overline{\dim}_M(\nu),$$

and

$$\underline{D}_r(\nu) \leq \frac{r\underline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{h}_{\mathfrak{J}_{\nu,r/d}}}.$$

Proof. We only consider the case $q_r > 0$. The case $q_r = 0$ follows analogously. Note that we always have $q_r < 1$. Let $P \in \Pi_\nu$ with $\operatorname{card}(P) \le n$. Let us write $P = \{Q_1, \ldots, Q_{\operatorname{card}(P)}\}$ and let m_i denote the middle point of the dyadic cube Q_i for $i \le \operatorname{card}(P)$ and set $\alpha_n := (m_1, \ldots, m_{\operatorname{card}(P)})$. Then we have

$$\begin{aligned}
\mathfrak{e}_{n,r}(\nu) &\le \left(\int d(x,\alpha_n)^r \,\mathrm{d}\nu(x)\right)^{1/r} \\
&= \left(\sum_{i=1}^{\operatorname{card}(P)} \int_{Q_i} d(x,\alpha_n)^r \,\mathrm{d}\nu(x)\right)^{1/r} \\
&\le \left(\sum_{i=1}^{\operatorname{card}(P)} \int_{Q_i} d(x,\{m_i\})^r \,\mathrm{d}\nu(x)\right)^{1/r} \\
&\le \sqrt{d}\left(\sum_{i=1}^{\operatorname{card}(P)} \nu(Q_i)\Lambda(Q_i)^{r/d}\right)^{1/r} \\
&\le \sqrt{d}n^{1/r}\left(\max_{Q\in P} \nu(Q)\Lambda(Q)^{r/d}\right)^{1/r}.
\end{aligned}$$

Now, taking the infimum over all $P \in \Pi_\nu$ with $\operatorname{card}(P) \le n$ yields

$$\mathfrak{e}_{n,r}^r(\nu) \le \sqrt{d^r}\, n\gamma_{\mathfrak{J}_{\nu,r/d},n}.$$

Note that by Proposition 3.11, for every $\varepsilon \in (0, 1/q_r - 1)$, we have for n large

$$n\gamma_{\mathfrak{J}_{\nu,r/d},n} \le n^{1-1/q_r+\varepsilon},$$

and, if $\underline{h}_{\mathfrak{J}_{\nu,r/d}} > 0$, there exists a subsequence $(n_k)_k$ such that

$$n_k\gamma_{\mathfrak{J}_{\nu,r/d},n_k} \le n_k^{1-1/\underline{h}_{\mathfrak{J}_{\nu,r/d}}+\varepsilon}.$$

The case $\underline{h}_{\mathfrak{J}_{\nu,r/d}} = 0$ follows again similar. This implies

$$\limsup_{n\to\infty} \frac{-\log(n)}{\log\left(\mathfrak{e}_{n,r}(\nu)\right)} \le \frac{rq_r}{1-q_r} \text{ and } \underline{D}_r(\nu) \le \liminf_{k\to\infty} \frac{-\log(n_k)}{\log\left(\mathfrak{e}_{n_k,r}(\nu)\right)} \le \frac{r\underline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{h}_{\mathfrak{J}_{\nu,r/d}}},$$

where we used that $\lim_{n\to\infty} \mathfrak{e}_{n,r}(\nu) = 0$. Moreover, Proposition 3.11 implies

$$-1/q_r \le -\left(\overline{\dim}_M(\nu) + r\right)/\overline{\dim}_M(\nu),$$

which proves the last inequality. □

Corollary 6.2. *If ν is singular, then*

$$\lim_{n\to\infty} n^{1/d}\mathfrak{e}_{n,r}(\nu) = 0.$$

Proof. Since ν is singular, by Corollary 3.10 and Proposition 6.1, we have

$$\mathfrak{e}_{n,r}(\nu) \le \left(n\gamma_{\mathfrak{J}_{\nu,r/d},n}\right)^{1/r} = o\left(n^{-1/d}\right). \qquad \square$$

By Pötzelberger [Pöt01], for all $r > 0$, we have

$$\underline{D}_r(\nu) \le \underline{\dim}_M(\nu). \tag{6.1.1}$$

The following corollary gives rise to a slight improvement to the estimate in (6.1.1).

Corollary 6.3. *For $r > 0$ such that $\underline{\dim}_M(\nu)/(r+\dim_\infty(\nu)) < 1$, we have*

$$\underline{D}_r(\nu) \le \frac{r\underline{\dim}_M(\nu)}{r+\dim_\infty(\nu)-\underline{\dim}_M(\nu)}.$$

Proof. By Proposition 3.13, we have

$$\underline{h}_{\mathfrak{J}_{\nu,r/d}} \le \frac{\underline{\dim}_M(\nu)}{r+\dim_\infty(\nu)}.$$

Now, the claim follows from Proposition 6.1 and the fact that $x \mapsto x/(1-x)$ is increasing on $(0,1)$. $\square$

6.2 Lower bounds for the quantization dimension

Recall, for $n \in \mathbb{N}$ and $\alpha > 0$,

$$\mathcal{N}^N_{\alpha,\mathfrak{J}}(n) = \operatorname{card}\left(M^N_{\alpha,\mathfrak{J}}(n)\right) \text{ with } M^N_{\alpha,\mathfrak{J}}(n) = \left\{C \in \mathcal{D}^N_n : \mathfrak{J}(C) \ge 2^{-\alpha n}\right\},$$

as well as

$$\overline{F}^N_{\mathfrak{J}}(\alpha) = \limsup_{n\to\infty} \frac{\log^+\left(\mathcal{N}^N_{\alpha,\mathfrak{J}}(n)\right)}{\log(2^n)} \text{ and } \underline{F}^N_{\mathfrak{J}}(\alpha) = \liminf_{n\to\infty} \frac{\log^+\left(\mathcal{N}^N_{\alpha,\mathfrak{J}}(n)\right)}{\log(2^n)},$$

and

$$\overline{F}^N_{\mathfrak{J}} = \sup_{\alpha>0} \frac{\overline{F}^N_{\mathfrak{J}}(\alpha)}{\alpha} \text{ and } \underline{F}^N_{\mathfrak{J}} = \sup_{\alpha>0} \frac{\underline{F}^N_{\mathfrak{J}}(\alpha)}{\alpha}.$$

Recall, for $s > 0$ we let $\langle Q\rangle_s$ denote the cube centered and parallel with respect to Q such that $\Lambda(Q) = s^{-d}\Lambda\left(\langle Q\rangle_s\right)$.

Proposition 6.4. *We have,*

$$\overline{D}_r(\nu) \geq \frac{r q_r}{1-q_r} \quad \textit{and} \quad \underline{D}_r(\nu) \geq \frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}.$$

Proof. Fix $\alpha > 0$ such that $\overline{F}_{\mathfrak{J}_{\nu,r/d}}(\alpha) > 0$. Further, let $(n_k)_k$ be such that

$$\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha) = \lim_{k\to\infty} \frac{\log^+\left(\mathcal{N}^N_{\alpha,\mathfrak{J}_{\nu,r/d}}(n_k)\right)}{\log(2^{n_k})}$$

and let $c_{\alpha,n_k} \coloneqq \operatorname{card}(E_{\alpha,n_k})$ be given as in Lemma 5.8 for $\mathfrak{J}_{\nu,r/d}$. Notice that by our assumption $\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha) > 0$, we infer that $\lim_{k\to\infty} c_{\alpha,n_k} = \infty$. Let $A \subset \mathbb{R}^d$ be of cardinality at most $c_{\alpha,n_k}/2$ and

$$E'_{\alpha,n_k} \coloneqq \left\{Q \in E_{\alpha,n_k} : \min_{a\in A} d(a,Q) \geq 2^{-n_k}\right\}.$$

Since, for all $Q_1, Q_2 \in E_{\alpha,n_k}$ we have $\langle \mathring{Q}_1\rangle_3 \cap \langle \mathring{Q}_2\rangle_3 = \varnothing$. Hence, it follows

$$d(Q_1,Q_2) \geq 2^{-n_k}.$$

Thus, if $d(a,Q) < 2^{-n_k}$ for some $a \in A$ and $Q \in E_{\alpha,n_k}$, then $d(a,Q') \geq 2^{-n_k}$ for all $Q' \in E_{\alpha,n_k} \setminus \{Q\}$ and therefore,

$$\operatorname{card}\left(\left\{Q \in E_{\alpha,n_k} : \min_{a\in A} d(a,Q) < 2^{-n_k}\right\}\right) \leq \operatorname{card}(A).$$

Hence, $\operatorname{card}\left(E'_{\alpha,n_k}\right) \geq c_{\alpha,n_k} - \operatorname{card}(A) \geq c_{\alpha,n_k}/2$ and

$$\begin{aligned}\int d(x,A)^r \,\mathrm{d}\nu(x) &\geq \sum_{Q\in E'_{\alpha,n_k}} \int_Q d(x,A)^r \,\mathrm{d}\nu(x)\\ &\geq \sum_{Q\in E'_{\alpha,n_k}} \nu(Q)\, 2^{-n_k}\\ &\geq \operatorname{card}(E'_{\alpha,n_k}) 2^{-\alpha n_k} \geq c_{\alpha,n_k} 2^{-\alpha n_k - 1}.\end{aligned}$$

Hence, $\mathfrak{e}^r_{\lfloor c_{\alpha,n_k}/2\rfloor,r}(\nu) \geq c_{\alpha,n_k} 2^{-\alpha n_k - 1}$ and we obtain for the first claim

$$\begin{aligned}
\limsup_{k\to\infty} \frac{\log\left(\lfloor c_{\alpha,n_k}/2\rfloor\right)}{-\log\left(\mathfrak{e}_{\lfloor c_{\alpha,n_k}/2\rfloor,r}(\nu)\right)} &\geq \limsup_{k\to\infty} \frac{r\log\left(c_{\alpha,n_k}/4\right)}{-\log\left(e_{\lfloor c_{\alpha,n_k}/2\rfloor,r}(\nu)\right)} \\
&\geq \limsup_{k\to\infty} \frac{r\left(\log\left(\mathcal{N}^N_{\alpha,\mathfrak{J}_{\nu,r/d}}(n_k)/5^d - 1\right) - \log(4)\right)}{-\log\left(\mathcal{N}^N_{\alpha,\mathfrak{J}_{\nu,r/d}}(n_k)/5^d - 1\right) + (\alpha n_k + 1)\log(2)} \\
&= \frac{r\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha}{1 - \overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha}.
\end{aligned}$$

This gives

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log(n)}{-\log\left(\mathfrak{e}_{n,r}(\nu)\right)} \geq \sup_{\alpha>0}\limsup_{k\to\infty} \frac{\log\left(\lfloor c_{\alpha,n_k}/2\rfloor\right)}{-\log\left(\mathfrak{e}_{\lfloor c_{\alpha,n_k}/2\rfloor,r}(\nu)\right)} &\geq \sup_{\alpha>0} \frac{r\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha}{1 - \overline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha} \\
&= \frac{r\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1 - \overline{F}^N_{\mathfrak{J}_{\nu,r/d}}}.
\end{aligned}$$

Therefore, Proposition 3.20 yields

$$\overline{D}_r(\nu) \geq \frac{r\overline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1 - \overline{F}^N_{\mathfrak{J}_{\nu,r/d}}} = \frac{rq_r}{1 - q_r}.$$

For the lower limit assume $\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha) > 0$ and note that for every $\epsilon \in \left(0, \underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)\right)$ and all n large

$$c_{\alpha,n} := \left\lceil 5^{-d}\mathcal{N}^N_{\alpha,\mathfrak{J}_{\nu,r/d}}(n)\right\rceil \geq 2^{n\left(\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha) - \epsilon\right)}.$$

Now, for $k \in \mathbb{N}$, we define

$$n_k := \left\lceil \frac{\log(2k)}{\left(\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha) - \epsilon\right)\log(2)} \right\rceil.$$

Clearly, this gives $c_{\alpha,n_k} \geq 2k$. Then for any subset A with $\mathrm{card}(A) \leq k \leq c_{\alpha,n_k}/2$, we have as above $\operatorname{card}\left(E'_{\alpha,n_k}\right) \geq c_{\alpha,n_k}/2 \geq k$. Then,

$$\begin{aligned}\int d(x,A)^r \,\mathrm{d}\nu(x) &\geq \sum_{Q\in E'_{\alpha,n_k}} \int_Q d(x,A)^r \,\mathrm{d}\nu(x)\\ &\geq \sum_{Q\in E'_{\alpha,n_k}} \nu(Q)\Lambda(Q)^{r/d} \geq k2^{-\alpha n_k}.\end{aligned}$$

Taking the infimum over A with $\operatorname{card}(A) \leq k$, we obtain

$$\mathfrak{e}^r_{k,r}(\nu) \geq k2^{-\alpha n_k}.$$

This gives

$$\begin{aligned}\frac{\log(k)}{-\log\left(\mathfrak{e}_{k,r}(\nu)\right)} &\geq \frac{r\log(k)}{-\log(k)+\alpha n_k \log(2)}\\ &\geq \frac{r\log(k)}{-\log(k)+\alpha\log(2k)/\left(\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)-\epsilon\right)+\alpha\log(2)}.\end{aligned}$$

Taking the lower limit over k and letting ϵ tend to zero, yields

$$\underline{D}_r(\nu) \geq \frac{r}{-1+\alpha/\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)} = \frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}(\alpha)/\alpha}.$$

Finally, taking the supremum for $\alpha > 0$ gives

$$\underline{D}_r(\nu) \geq \frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}. \qquad \square$$

6.3 Main results

Now, we are in the position to prove our main results of this chapter.

Theorem 6.5. *For every $r > 0$, we have*

$$\frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}} \leq \underline{D}_r(\nu) \leq \frac{r\underline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{h}_{\mathfrak{J}_{\nu,r/d}}} \leq \frac{r\overline{h}_{\mathfrak{J}_{\nu,r/d}}}{1-\overline{h}_{\mathfrak{J}_{\nu,r/d}}} = \overline{D}_r(\nu) = \frac{rq_r}{1-q_r}.$$

If in addition $q_r > 0$, then $\overline{\mathfrak{R}}_\nu(q_r) = \beta^N_\nu(q_r)/(1-q) = \overline{D}_r(\nu)$.

Proof. The claim follows by combining Proposition 6.1 and Proposition 6.4. $\square$

Remark 6.6. As illustrated in Figure 6.3.1, due to Theorem 6.5, the upper quantization dimension can be identified by the intersection of the line through $(q_r, \beta_\nu^N(q_r))$ and $(0,1)$ with the y-axis provided $q_r > 0$.

Remark 6.7. By Theorem 6.5 we infer the following one-to-one correspondence between $\overline{D}_r(\nu)$, $r > 0$, and $\beta_\nu^N(q)$, $q \in (0,1)$. For this note that

$$q_r = \frac{\overline{D}_r(\nu)}{r + \overline{D}_r(\nu)}.$$

Hence, if $\overline{D}_r(\nu) > 0$, then

$$\beta_\nu^N\left(\frac{\overline{D}_r(\nu)}{\overline{D}_r(\nu) + r}\right) = \frac{r\overline{D}_r(\nu)}{\overline{D}_r(\nu) + r}.$$

Set $x_0 := \sup\left\{\overline{D}_r(\nu)/\left(r + \overline{D}_r(\nu)\right) : r > 0\right\} < 1$. Then $\beta_\nu^N(q) = 0$ for all $q > x_0$ and for $0 < q < x_0$, we have $\beta_\nu^N(q) > 0$ and

$$\overline{D}_{\beta_\nu^N(q)/q}(\nu) = \frac{\beta_\nu^N(q)}{1-q}. \tag{6.3.1}$$

This result seems to be very promising, since the study of the L^q-spectra is a highly active research area. For instance, the L^q-spectrum for certain classes of self-similar measures with overlaps was computed explicitly by Ngai and Lau [LN98] and Ngai and Xie [NX19]. The formulae derived therein combined with (6.3.1) will lead to a number of interesting formulae for the quantization dimension.

At least for special cases, it has been observed in [KZ15, p. 6] that the upper quantization dimension is often determined by a critical value; we are now in the position to determine this critical value for arbitrary compactly supported probability measures as follows.

Corollary 6.8. *We have*

$$\frac{\overline{D}_r(\nu)}{\overline{D}_r(\nu) + r} = q_r = \kappa_{\mathfrak{J}_{\nu,r/d}} = \inf\left\{q > 0 : \sum_{Q \in \mathcal{D}} \left(\Lambda(Q)^{r/d}\nu(Q)\right)^q < \infty\right\}.$$

In particular, for $d = r = 1$, we have the following connection to the upper spectral dimension of $\Delta_\nu^{D/N}$

$$\frac{\overline{D}_r(\nu)}{\overline{D}_r(\nu) + r} = \overline{s}_\nu^{D/N}.$$

Proof. This follows from Lemma 2.25 applied to $\mathfrak{J} = \mathfrak{J}_{\nu,a}$ and Theorem 6.5. □

Further, using $\overline{D}_r(\nu) = rq_r/(1-q_r)$, we can affirm a conjecture of Lindsay stated in his PhD thesis [Lin01].

Corollary 6.9. *The map $r \mapsto \overline{D}_r(\nu)$ is continuous on $(0,\infty)$.*

Proof. This follows from the fact that $r \mapsto q_r$ is continuous. Indeed, if $q_r = 0$ for some $r > 0$, then $0 \le \beta_\nu^N(q) < rq$ for all $q \in (0,1)$. Consequently, $\lim_{q\downarrow 0} \beta_\nu^N(q) = 0$ and combined with the convexity of β_ν^N and $\beta_\nu^N(1) = 0$, we infer $\beta_\nu^N(q) = 0$ for $q > 0$. Therefore, $q_r = 0$ for all $r > 0$. The case $q_r > 0$ follows from the fact that β_ν^N is continuous and decreasing on $(0,1]$ with $\beta_\nu^N(1) = 0$. □

Theorem 6.10. *The following regularity implication holds:*

$$\tau^N_{\mathfrak{J}_{\nu,r/d}} \text{ is N-PF-regular } \implies \underline{D}_r(\nu) = \overline{D}_r(\nu) = \frac{rq_r}{1-q_r}.$$

Proof. By Proposition 3.24 applied to $\mathfrak{J} = \mathfrak{J}_{\nu,r/d}$, we have $\underline{F}^N_{\mathfrak{J}_{\nu,r/d}} = q_r$. Hence, we can infer from Theorem 6.5

$$\frac{rq_r}{1-q_r} = \frac{r\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}}{1-\underline{F}^N_{\mathfrak{J}_{\nu,r/d}}} \le \underline{D}_r(\nu) \le \overline{D}_r(\nu) \le \frac{rq_r}{1-q_r}.$$ □

As a corollary, we are able to confirm the existence of the quantization dimension for any self-conformal measures.

Corollary 6.11. *Let ν be a self-conformal measure with respect to a C^1-cIFS on $\boldsymbol{Q}$ with no assumptions on the separation conditions. Then,*

$$D_r(\nu) = \frac{rq_r}{1-q_r}.$$

Proof. By Theorem 2.49, we know that β_ν^N exists as a limit on $(0,\infty)$. Since for all $q > 0$ and $n \in \mathbb{N}$, we have $\tau^N_{\mathfrak{J}_{\nu,a},n}(q) = \beta^N_{\nu,n}(q) - qr$. Therefore, it follows that $\tau^N_{\mathfrak{J}_{\nu,a}}$ also exists as a limit for all $q > 0$. We infer that $\mathfrak{J}_{\nu,a}$ is Neumann partition regular. Now, the claim follows from Theorem 6.10. □

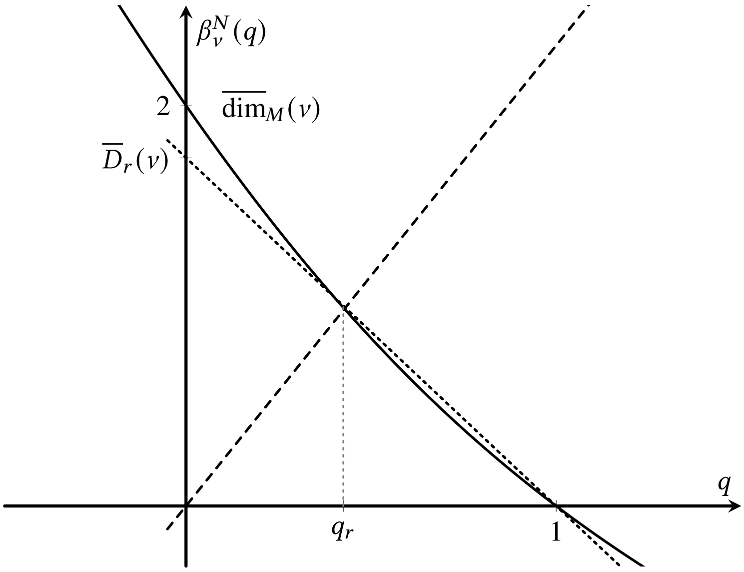

Figure 6.3.1 The L^q-spectrum β_ν^N for the self-similar measure ν supported on the *Sierpiński tetraeder* in $\mathbb{R}^3$ with all four contraction ratios equal $1/2$ and with probability vector $(0.66, 0.2, 0.08, 0.06)$. We have $\beta_\nu^N(0) = \overline{\dim}_M(\nu) = 2$. For $r = 2.3$ (slope of the dotted line) the intersection of the spectrum and the dashed line determines q_r. The dotted line through $\left(q_r, \beta_\nu^N(q_r)\right)$ and $(0,1)$ intersects the y-axis in $\overline{D}_r(\nu)$.

Chapter 7

Open problems and conjectures

This chapter discusses some open problems and conjectures regarding this thesis.

7.1 Dirichlet/Neumann spectral partition function and L^q-spectrum

In Section 2.4, we introduced the Dirichlet/Neumann spectral partition functions. Of special interest is the investigation of the Dirichlet/Neumann spectral partition function $\mathfrak{J}_{\nu,(2-d)/d,1} = \mathfrak{J}_\nu$ which is important for the study of upper spectral dimension with respect to Dirichlet and Neumann boundary conditions. We showed that in the one-dimensional case the Dirichlet and Neumann spectral partition functions $\tau^{D/N}_{\mathfrak{J}_\nu}$ coincide whenever $\nu(\partial\mathbf{Q}) = 0$ due to the fact that the boundary of $\partial\mathbf{Q}$ consists of only two points. In contrast, the situation for the higher dimensional case is much more challenging since the number of the dyadic cubes intersecting the boundary of $\mathbf{Q}$ of level $n \in \mathbb{N}$ tends to infinity as n approaches to infinity (because of $\operatorname{card}\left(\mathcal{D}_n^N \setminus \mathcal{D}_n^D\right) = 2^{dn} - (2^n - 2)^d$ by Lemma 2.1). However, in Section 2.4.3, we proved that for self-conformal measures, product measures, and Ahlfors-David regular measures the Neumann spectral partition function coincides with the Dirichlet spectral partition function whenever $\nu(\partial\mathbf{Q}) = 0$ and $\dim_\infty(\nu) > d-2$. This leads to the question in which situations we can expect that $\tau^D_{\mathfrak{J}_\nu} = \tau^N_{\mathfrak{J}_\nu}$. In general, we could not find any reasons to rule out the possibility that the boundary cubes can dominate the inner cubes. This motivates the following conjecture.

Conjecture 7.1. *There exists a Borel probability measure on* $(0,1)^d$ *with* $d \geq 2$ *such that* $\dim_\infty(\nu) > d-2$ *and*

$$\tau^D_{\mathfrak{J}_\nu}(q) < \tau^N_{\mathfrak{J}_\nu}(q)$$

for some $q \geq 0$.

Another important open problem is the relation between the L^q-spectrum β_ν^N and the spectral partition function $\tau_{\mathfrak{J}_\nu}^N$ under the standard assumption $\dim_\infty(\nu) > d-2$. Obviously, we always have

$$\beta_\nu^N(q) - (2-d)q \le \tau_{\mathfrak{J}_\nu}^N(q),\ q \ge 0.$$

If $d = 1, 2$, then by Proposition 2.35 we have

$$\tau_{\mathfrak{J}_\nu}^N(q) = \beta_\nu^N(q) - (2-d)q,\ q \ge 0. \tag{7.1.1}$$

Unfortunately, in the case $d > 2$, the situation becomes more challenging; this is due to the reason that the level of dyadic cubes, in which the term

$$\mathfrak{J}_\nu(Q) = \sup_{Q \in \mathcal{D}(Q)} \nu(Q)\Lambda(Q)^{(2-d)/d}$$

attains its maximum, is difficult to control. However, we demonstrated in Section 2.4.3 that for self-conformal measures, Ahlfors-David regular measures, and measures whose L^q-spectrum is linear, the equality in (7.1.1) is valid. Nevertheless, the examples considered in Section 2.4.3 are very specific and in some sense very regular. Therefore, in the general case, we conjecture that the equality (7.1.1) fails .

Conjecture 7.2. *There exists a Borel probability measure on* $(0,1)^d$ *with* $d \ge 3$ *such that* $\dim_\infty(\nu) > d-2$ *and*

$$\beta_\nu^N(q) + (d-2)q < \tau_{\mathfrak{J}_\nu}^N(q)$$

for some $q \ge 0$.

7.2 Lower optimized coarse multifractal dimension

In Chapter 3, we saw that by Proposition 3.20,

$$\overline{F}_{\mathfrak{J}}^{D/N} = q_{\mathfrak{J}}^{D/N}.$$

This equality has been crucial for the computation of the upper quantization dimension as well as for the upper spectral dimension. Thus, it is natural to ask whether there is a similar result for $\underline{F}_{\mathfrak{J}}^{D/N}$. The importance of this question becomes apparent when considering the lower spectral dimension and the lower quantization dimension, which are bounded from below in terms of $\underline{F}_{\mathfrak{J}}^{D/N}$ for appropriate choices of $\mathfrak{J}$. Thus, the better understanding of $\underline{F}_{\mathfrak{J}}^{D/N}$ could lead to some progress in understanding of the lower spectral dimension as well as of the lower quantization dimension. An obvious candidate for the value of $\underline{F}_{\mathfrak{J}}^{D/N}$ could be $q_{\mathfrak{J}}^{D/N}$ when replacing the limit superior of the definition of $q_{\mathfrak{J}}^{D/N}$ with the limit inferior. To be more precise, define

$$\underline{\tau}_{\mathfrak{J}}^{D/N}(q) := \liminf_{n\to\infty} \frac{\log\left(\sum_{Q\in\mathcal{D}_n^{D/N}} \mathfrak{J}(Q)^q\right)}{\log(2^n)}, \ q \geq 0.$$

Then a simple computation leads to the following estimate of $\underline{F}_{\mathfrak{J}}^{D/N}$:

$$\underline{F}_{\mathfrak{J}}^{D/N}(\alpha) \leq \frac{\underline{\tau}_{\mathfrak{J}}^{D/N}(q)}{\alpha} + q$$

for any $q, \alpha > 0$. Therefore, one could conjecture that

$$\underline{F}_{\mathfrak{J}}^{D/N} = \inf\left\{q \geq 0 : \underline{\tau}_{\mathfrak{J}}^{D/N}(q) < 0\right\}. \tag{7.2.1}$$

The regularity result stated in Proposition 3.24 tells us that the above inequality turns into an equality if $\mathfrak{J}$ is Dirichlet/Neumann PF-regular. Unfortunately, the method used in the proof of Proposition 3.20 makes heavily use of large derivation theory which requires convexity of $\tau_{\mathfrak{J}}^{D/N}$. This causes many problems since $\underline{\tau}_{\mathfrak{J}}^{D/N}$ does not enjoy convexity in general. Therefore, this makes the situation considerably more challenging and we believe that in general we cannot expect equality in (7.2.1).

7.3 Spectral dimension in the critical case $d = 2$

As a consequence of Theorem 4.10 and Theorem 5.15, under the assumptions

$$\dim_\infty(\nu) > d-2 \text{ and } \nu(\mathring{\mathbf{Q}}) > 0,$$

we obtain the following complete picture of the range of the upper spectral dimension with respect to Dirichlet/Neumann boundary conditions:

$$\begin{aligned} \overline{s}_\nu^{D/N} &\in [0,1/2], && d=1,\\ \overline{s}_\nu^{D/N} &= 1, && d=2,\\ \overline{s}_\nu^{N} &\in (d/2,\infty), && d\geq 3. \end{aligned}$$

Therefore, the case $d = 2$ is the only case for which the upper spectral dimension is constant as function of ν. So the case $d = 2$ seems to be of special interest. Furthermore, every attempt to find an example in which the spectral dimension does not exist has failed, leading to the following conjecture.

Conjecture 7.3. *Let $d = 2$ and $\dim_\infty(\nu) > 0$. Then,*

$$\underline{s}_\nu^N = \overline{s}_\nu^N = 1.$$

If additionally $\nu(\mathring{Q}) > 0$, then we also have $\underline{s}_\nu^D = \overline{s}_\nu^D = 1$.

It would be interesting to find a physical interpretation or explanation of this phenomenon. Also from the mathematical point of view it is interesting to understand this mechanism in more detail. This phenomenon also has been discussed in a more general context in the recent publication by Rozenblum [Roz22], where this case is referred to as a 'critical case'. The proof of Theorem 5.15 shows that the appearance of the value 1 can be explained from the simple fact that the L^q-spectrum has a unique zero at 1. Further, a next big step might be the study of the precise asymptotic behavior of $N_\nu^{D/N}$. It is reasonable that we have $N_\nu^{D/N}(x) \asymp x^{-1}$ for broad classes of measures.

Appendix A

Appendix

A.1 Convex functions

In this chapter, we present basic facts about convex functions on $\mathbb{R}$. We mainly follow [FL01, Chapter 5] and [FV17, Appendix B.2].

Definition A.1. Let $I \subset \mathbb{R}$ be an interval (not necessarily bounded). We say a function $f : I \to \mathbb{R} \cup \{\infty\}$ is convex on I if

$$\forall x, y \in I, \forall t \in (0,1) : \; f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

If f is real valued, then f is convex on I if

$$\forall x, y \in I, \forall t \in [0,1] : \; f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Remark A.2. *A convex function $f : I \to \mathbb{R} \cup \{\infty\}$ can always be extended to $\mathbb{R}$ by setting $f(x) := \infty$ for $x \notin \mathrm{dom}(f)$. The* domain *of f is given by*

$$\mathrm{dom}(f) := \{x \in \mathbb{R} : f(x) < \infty\}.$$

The interior of $\mathrm{dom}(f)$ *will be denoted by* $\mathrm{int}(\mathrm{dom}(f))$. *Notice that* $\mathrm{dom}(f)$ *is a convex set.*

Proposition A.3 ([FL01, Corollary 2.5.2])**.** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then f is continuous on* $\mathrm{int}(\mathrm{dom}(f))$.

Definition A.4 (Subdifferential)**.** Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function. An element $x^* \in \mathbb{R}$ is called *subgradient* of f at $x \in \mathbb{R}$ if for all $z \in \mathbb{R}$,

$$f(z) \geq f(x) + x^*(z - x).$$

The set of subgradients of f at x, denoted by $\partial f(x)$, is called *subdifferential*.

Theorem A.5 ([FV17, Theorem B.12] and [FL01, Proposition 6.5.2]). *Let $f:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ be a convex function. Then the following list of properties applies.*

1. *Let $x,y,z\in\operatorname{int}(\operatorname{dom}(f))$ with $x<y<z$, then*
$$\frac{f(y)-f(x)}{y-x}\le\frac{f(z)-f(x)}{z-x}\le\frac{f(z)-f(y)}{z-y}.$$
2. $\partial^- f(x):=\lim_{z\uparrow x}\dfrac{f(z)-f(x)}{z-x}$ *and* $\partial^+ f(x):=\lim_{z\downarrow x}\dfrac{f(z)-f(x)}{z-x}$ *exist for all* $x\in\operatorname{int}(\operatorname{dom}(f))$.
3. $\partial^- f(x)\le\partial^+ f(x)$ *for all* $x\in\operatorname{int}(\operatorname{dom}(f))$.
4. $\partial^- f$ *and* $\partial^+ f$ *are nondecreasing on* $\operatorname{int}(\operatorname{dom}(f))$.
5. $\partial f(x)=[\partial^- f(x),\partial^+ f(x)]$ *for all* $x\in\operatorname{int}(\operatorname{dom}(f))$.
6. $\{x\in\operatorname{int}(\operatorname{dom}(f)):\partial^- f(x)<\partial^+ f(x)\}$ *is at most countable.*
7. $\partial^- f$ *is left-continuous and* $\partial^- f$ *is right-continuous on* $\operatorname{int}(\operatorname{dom}(f))$.

A.2 Sobolev spaces, Lipschitz domains, and Stein's extension operator

In this section, we briefly review the properties of Stein's extension operator which is needed to define Kreĭn–Feller operators as well as for upper estimates of the lower and upper spectral dimension. Here, we closely follow [Ste70] and [LV19].

We start with the definition of the Sobolev space. For this purpose, we need some preparatory definitions. Let $\Omega\subset\mathbb{R}^d$ be a domain (open and connected). The set of locally integrable functions is given by

$$L^1_{loc}(\Omega):=\{f:\Omega\to\mathbb{R}\text{ measurable}:\int_K|f|\,\mathrm{d}\Lambda<\infty,\ K\subset\Omega,\text{ and }K\text{ compact}\}.$$

Definition A.6 (Weak derivatives). Let $u\in L^1_{loc}(\Omega)$ and $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$. We say u_α is a α^{th}-weak derivative of u if

$$\int_\Omega uD^\alpha\varphi\,\mathrm{d}\Lambda=(-1)^{\sum_{i=1}^d\alpha_i}\int_\Omega u_\alpha\varphi\,\mathrm{d}\Lambda$$

for all $\varphi\in C_c^\infty(\Omega)$ with $D^\alpha\varphi:=\dfrac{\partial^{\sum_{j=1}^d\alpha_j}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}\varphi$. In this case we write $D^\alpha u:=u_\alpha$.

Definition A.7. The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) := \left\{ u \in L^2_\Lambda(\Omega) : D^\alpha u \in L^2_\Lambda(\Omega) \text{ for } (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \text{ with } \sum_{i=1}^d \alpha_i = 1 \right\}.$$

Now, we define the notion of special Lipschitz domains.

Definition A.8. We call $\Omega \subset \mathbb{R}^d$ a *special Lipschitz domain* if

$$\Omega = \left\{ (x_1, \ldots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d > \varphi(x_1, \ldots, x_{d-1}) \right\},$$

where $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz continuous function. The Lipschitz constant of φ, denoted by M, is called *Lipschitz bound* of Ω.

Theorem A.9 ([LV19, Theorem 1])**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain and suppose there exist a rotation $R : \mathbb{R}^d \to \mathbb{R}^d$ and a special Lipschitz domain D such that $\Omega = R(D)$. Then there exists a continuous linear operator $\mathfrak{E}_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that*

$$\mathfrak{E}_\Omega(f)|_\Omega = f \ \Lambda\text{-a.e. and } \mathfrak{E}_\Omega : C_b^\infty\left(\overline{\Omega}\right) \to C^\infty\left(\mathbb{R}^d\right) \text{ with } \mathfrak{E}(f)|_{\overline{\Omega}} = f.$$

Definition A.10. Let $\Omega \subset \mathbb{R}^d$ be a domain (open and connected). We call Ω a *Lipschitz domain* if there exist $\varepsilon > 0$, $N \in \mathbb{N}$, $M > 0$, and a sequence $\{U_i\}_{i=1,\ldots,s}$ of open sets with $s \in \mathbb{N} \cup \{\infty\}$ such that the following conditions are fulfilled:

1. For all $x \in \partial\Omega$ there exists i such that $B_\varepsilon(x) \subset U_i$.
2. For each $y \in \mathbb{R}^d$, we have $\text{card}(\{i \in \{1, \ldots, s\} : y \in U_i\}) \leq N$.
3. For each $i \in \{1, \ldots, s\}$ there exists a special Lipschitz domain whose Lipschitz bound does not exceed M and a rotation $R_i : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$U_i \cap \Omega = U_i \cap R_i(D_i).$$

Remark A.11. *Let Ω be a bounded Lipschitz domain, then only finitely many U_i's are required. This can be seen as follows. We have*

$$\partial\Omega \subset \bigcup_{x \in \partial\Omega} B_{\varepsilon/2}(x) \subset \bigcup_{x \in \partial\Omega} \overline{B_{\varepsilon/2}(x)} \subset \bigcup_{i=1}^s U_i.$$

Since $\partial\Omega$ is compact, it also follows that $\bigcup_{x \in \partial\Omega} \overline{B_{\varepsilon/2}(x)}$ is compact. We infer

$$\bigcup_{x \in \partial\Omega} \overline{B_{\varepsilon/2}(x)} \subset \bigcup_{i \in I} U_i$$

for some finite index set $I \subset \{1, \ldots, s\}$.

Further, in this case, the U_i's can be modified to be bounded. Indeed, for $j \in \{1, \ldots, s\}$, we define

$$\widetilde{U_j} := \bigcup_{\substack{x \in \partial\Omega, \\ B_\varepsilon(x) \subset U_j}} B_\varepsilon(x). \tag{A.2.1}$$

Now, let us check that the conditions in Definition A.10 are fulfilled for $\left(\widetilde{U_j}\right)_{j=1,\ldots,s}$. For any $x \in \partial\Omega$ there exists i such that $B_\varepsilon(x) \subset U_i$ which gives

$$B_\varepsilon(x) \subset \bigcup_{\substack{x \in \partial\Omega, \\ B_\varepsilon(x) \subset U_i}} B_\varepsilon(x) = \widetilde{U_i}.$$

Clearly, using $U_i \cap \Omega = U_i \cap R_i(D_i)$, we infer

$$\bigcup_{\substack{x \in \partial\Omega, \\ B_\varepsilon(x) \subset U_i}} B_\varepsilon(x) \cap \Omega = \bigcup_{\substack{x \in \partial\Omega, \\ B_\varepsilon(x) \subset U_i}} B_\varepsilon(x) \cap R_i(D_i).$$

The following proposition gives rise to an equivalent definition of $H^1(\Omega)$ as presented in the introduction for the case that Ω is a Lipschitz domain.

Proposition A.1 ([Ada75][Theorem 3.18])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then $C_b^\infty\left(\overline{\Omega}\right)$ lies dense in $H^1(\Omega)$ with respect to the norm given by the inner product $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$.*

Example A.12. Suppose Ω is an open bounded convex set. Then Ω is a Lipschitz domain and only finitely many U_i's are needed (see for instance [Ste70, Example 1 on p. 189]).

Now, we briefly outline the construction of the Stein operator for bounded Lipschitz domains (for details see [Ste70, p. 191-192] and [LV19]). Let Ω be a bounded Lipschitz domain with parameters ε, N, M, and $\{(U_i, R_i, D_i) : i \in \{1, \ldots, s\}\}$ with $s \in \mathbb{N}$ (here we choose the $U_i's$ to be bounded, which is always possible by (A.2.1)). There exists $\left(g_i : \mathbb{R}^d \to \mathbb{R}\right)_{i=1,\ldots,s} \in C^\infty\left(\mathbb{R}^d\right)^s$ such that for every $i \in \{1, \ldots, s\}$, we have

1. $0 \leq g_i$ and $\operatorname{supp}(g_i) \subset U_i$,
2. $g_i(x) = 1$ for all $x \in U_i^{\varepsilon/2} := \{x \in U_i : B_{\varepsilon/2}(x) \subset U_i\}$,
3. g_i has bounded derivatives of all orders and the bounds of the derivatives can be chosen independent of i.

Further, we can construct $G^-, G^+ \in C_c^\infty\left(\mathbb{R}^d\right)$ fulfilling the following properties:

1. $\operatorname{supp}(G^+) \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < \varepsilon/2\} \subset \bigcup_{i=1}^s U_i^{\varepsilon/2}$ with
$$\operatorname{dist}(x, A) := \inf_{y \in A} |x - y|,$$
for $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

2. $\operatorname{supp}(G^-) \subset \Omega$.

3. $|G^-|, |G^+| \leq 1$, $G^-(x) + G^+(x) = 1$ for all $x \in \overline{\Omega}$, and $G^-(x) + G^+(x) = 0$ for all $x \in \left\{y \in \mathbb{R}^d : \operatorname{dist}(y, \Omega) \leq \varepsilon/2\right\}^{\complement}$.

Let $\mathfrak{E}_i : H^1(R_i(D_i)) \to H^1(\mathbb{R}^d)$ denote the extension operator of $R_i(D_i)$ defined in Theorem A.9. Now, we define the extension operator $\mathfrak{E}$ for Ω as follows. For $f \in H^1(\Omega)$, we set

$$\mathfrak{E}(f)(x) := G^+(x) \frac{\sum_{i=1}^s g_i(x) \mathfrak{E}_i(g_i f)(x)}{\sum_{i=1}^s g_i^2(x)} + G^-(x) f(x). \tag{A.2.2}$$

Remark A.13. *We remark the following important properties of* $\mathfrak{E}$.

1. For all $x \in \Omega$, we have $\mathfrak{E}(f)(x) = f(x)$.

2. The terms $\mathfrak{E}_i(g_i f)$ are well-defined, since the $g_i f$ are given in $R_i(D_i)$. To be more precise, using $\Omega \cap U_i = U_i \cap R_i(D_i)$ and $\operatorname{supp}(g_i) \subset U_i$, we can extend $g_i f$ by zero on $U_i^{\complement} \cap R_i(D_i)$ which gives $g_i f \in H^1(R_i(D_i))$. If we additionally assume that $f \in C_b^\infty\left(\overline{\Omega}\right)$, then $g_i f \in C_b^\infty\left(\overline{R_i(D_i)}\right)$. Thus, by Theorem A.9, we have $\mathfrak{E}_i(g_i f) \in C^\infty(\mathbb{R}^d)$. Indeed, for $f \in C_b^\infty\left(\overline{\Omega}\right)$, we define
$$E(x) := \begin{cases} f(x) g_i(x), & x \in U_i \cap R_i(D_i), \\ 0, & x \in U_i^{\complement} \cap R_i(D_i). \end{cases}$$
Then we have $E \in C_b^\infty\left(\overline{U_i \cap R_i(D_i)}\right)$ and $\operatorname{supp}(E) \subset \operatorname{supp}(g_i) \cap \overline{\Omega} \subset U_i \cap \overline{\Omega}$. We only have to check that E is infinitely often differentiable on $\partial U_i \cap R_i(D_i)$. Since U_i is bounded, the support of g_i is compact and we deduce
$$\operatorname{dist}(\partial U_i, \operatorname{supp}(g_i)) > 0.$$
Hence, there exists $\delta > 0$ such that for any $x \in \partial U_i \cap R_i(D_i)$ we have $B_\delta(x) \cap \operatorname{supp}(g_i) = \emptyset$. In particular, for each $x \in \partial U_i \cap R_i(D_i)$ it follows $E(y) = 0$ for all $y \in B_\delta(x) \cap R_i(D_i)$.

3. $G^-(x)f(x)$ is well-defined, since the support of G^- lies in Ω, implying that we can extend G^-f by zero outside of Ω. In particular, if $f \in C_b^\infty\left(\overline{\Omega}\right)$, then $G^-f \in C_c^\infty(\Omega)$.

4. Since $\operatorname{supp}(G^+) \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < \varepsilon/2\}$, for $x \in \operatorname{supp}(G^+)$, it follows for at least one i we have $x \in U_i^{\varepsilon/2}$, implying $\sum_{i=1}^s g_i^2(x) \geq 1$. Consequently, we have $G^+/\sum_{i=1}^s g_i^2 \in C_c^\infty\left(\mathbb{R}^d\right)$.

5. Observe that the functions $G^+/\sum_{i=1}^s g_i^2$ and G^- have compact support since Ω is bounded. Thus, if $f \in C_b^\infty\left(\overline{\Omega}\right)$, then $\mathfrak{E}(f) \in C_c^\infty\left(\mathbb{R}^d\right)$ as the product of smooth functions with compact support with a finite sum of smooth functions with compact support.

From [LV19, Theorem 3] and Remark A.13, we obtain the following theorem.

Theorem A.14. *Any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ permits a Stein extension (see Definition 2.6) and the extension operator can be chosen as in* (A.2.2).

A.3 Sobolev spaces in the one-dimensional case

In this section, we collect some important facts about the Sobolev space in the one-dimensional case, which can be found in [Dav95]. Fix $a, b \in \mathbb{R}$ with $a < b$.

Lemma A.15. *A function $f : [a,b] \to \mathbb{R}$ lies in $H^1((a,b))$ if and only if there exist a constant c and $g \in L^2_\Lambda((a,b))$ such that*

$$f(x) = c + \int_{[a,x]} g(y) \, \mathrm{d}\Lambda(y) \tag{A.3.1}$$

for all $x \geq a$. All such functions are uniformly continuous on $[a,b]$. Moreover, g is equal to the weak derivative of f. A function f of the form (A.3.1) lies in $H_0^1(a,b)$ if and only if one also has $f(a) = f(b) = 0$.

Remark A.16. *By Lemma A.15, we have*

$$H^1((a,b)) = \left\{ f : [a,b] \to \mathbb{R} : \exists c \in \mathbb{R}, g \in L^2_\Lambda((a,b)) : \; f(x) = c + \int_{[a,x]} g d\Lambda, \; x \geq a \right\}$$

and $H_0^1((a,b)) = \left\{ f \in H^1((a,b)) \mid f(a) = f(b) = 0 \right\}$, where the equalities hold in the sense of equivalence classes with respect to the relation of almost everywhere equality.

In the following, for elements $f \in H^1((a,b))$, we will always choose the continuous representative.

Proposition A.17. *The embedding of the Hilbert space* $\left(H^1((a,b)), ||\cdot||_{H^1((a,b))}\right)$ *into* $(C([a,b]), ||\cdot||_\infty)$ *is compact. In particular, for every finite Borel measure* ν *on* $[a,b]$*, the embedding of* $\left(H^1((a,b)), ||\cdot||_{H^1((a,b))}\right)$ *into* $L^2_\nu([a,b])$ *is compact.*

Proof. Let $U \subset H^1((a,b))$ be a bounded set with respect to $||\cdot||_{H^1((a,b))}$ which means there exists $c > 0$ such that

$$\sup_{f\in U}\left(\int_{(a,b)} (\nabla f)^2 \, \mathrm{d}\Lambda + \int_{(a,b)} f^2 \, \mathrm{d}\Lambda\right)^{1/2} \le c.$$

We have to show U is precompact in $(C([a,b]), ||\cdot||_\infty)$. This will be done by showing that U is bounded in $C([a,b])$ with respect to $\|\cdot\|_\infty$ and equicontinuous. The Arzelà–Ascoli Theorem then guarantees that U is precompact in $C([a,b])$. Using the Cauchy-Schwarz inequality, for all $x, y \in [a,b]$, we have

$$\begin{aligned}|f(x) - f(y)| &\le \int_{(x,y)} |\nabla f(z)| \, \mathrm{d}\Lambda(z)\\ &\le |x-y|^{1/2}\left(\int_{(a,b)} (\nabla f)^2 \, \mathrm{d}\Lambda\right)^{1/2}\\ &\le c\,|x-y|^{1/2}.\end{aligned}$$

Hence, U is equicontinuous. Moreover, Lemma 2.2 implies that U is also bounded in $(C([a,b]), \|\cdot\|_\infty)$. □

A.4 Self-adjoint operators and quadratic forms

In this section, we summarize basic results and definitions on self-adjoint operators and quadratic forms which we need to define Kreĭn–Feller operators. Here, we follow closely the presentation of [Kig01, Appendix B], which is based on [Dav95]. Throughout this section, let $\mathcal{H}$ be an infinite dimensional real separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$.

Definition A.18. We call a linear map $A : \mathrm{dom}(A) \to \mathcal{H}$ with $\mathrm{dom}(A) \subset \mathcal{H}$ a linear operator if $\mathrm{dom}(A)$ is a dense subspace of $\mathcal{H}$.

Definition A.19. Let A be a linear operator on $\mathcal{H}$.

1. We call A *symmetric* if $\langle Hf, g\rangle = \langle f, Hg\rangle$ for all $f, g \in \mathrm{dom}(A)$.

2. The linear operator A is called *self-adjoint* if A is symmetric and

$$\mathrm{dom}(A) = \{g \in \mathcal{H} : \exists h \in \mathcal{H} \forall f \in \mathrm{dom}(A) : \langle Af, g\rangle = \langle f, h\rangle\}.$$

3. A symmetric operator A is said to be *non-negative* if $\langle Af, f\rangle \geq 0$ for all $f \in \mathcal{H}$.

Proposition A.20. *Let A be a non-negative self-adjoint operator. Then there exists a unique non-negative self-adjoint operator* $G : \operatorname{dom}(G) \to \mathcal{H}$ *such that* $A = G^2$,

$$\operatorname{dom}(A) \subset \operatorname{dom}(G), \text{ and } \operatorname{dom}(A) = \{f \in \mathcal{H} : f \in \operatorname{dom}(G), Gf \in \operatorname{dom}(G)\}.$$

We define $A^{1/2} := G$.

Definition A.21. The form $\mathcal{E} : \operatorname{dom}(\mathcal{E}) \times \operatorname{dom}(\mathcal{E})$ is called a non-negative quadratic form on $\mathcal{H}$ if

1. $\operatorname{dom}(\mathcal{E})$ is dense in $\mathcal{H}$.
2. $\mathcal{E}$ is bilinear and symmetric: $\mathcal{E}(af + bg, h) = a\mathcal{E}(h, f) + b\mathcal{E}(g, h)$ for every $f, g \in \operatorname{dom}(\mathcal{E})$ and $a, b \in \mathbb{R}$.
3. $\mathcal{E}$ is non-negative definite if $\mathcal{E}(f, f) \geq 0$ for all $f \in \operatorname{dom}(\mathcal{E})$.

Let A be a self-adjoint operator on $\mathcal{H}$. Then the associated quadratic form is defined by $\mathcal{E}_A(f, g) := \langle A^{1/2}f, A^{1/2}g\rangle$ with $f, g \in \operatorname{dom}(A^{1/2})$. The following lemma establishes an important characterization of self-adjoint operators in terms of the associated quadratic form, which immediately follows from Proposition A.20 and the definition of self-adjoint operators.

Lemma A.22. *Let A be a non-negative self-adjoint operator on* $\mathcal{H}$*. Then, we have*

$$\operatorname{dom}(A) = \left\{g \in \operatorname{dom}\left(A^{1/2}\right) : \exists h \in \mathcal{H} \forall f \in \operatorname{dom}\left(A^{1/2}\right) : \mathcal{E}_A(f, g) = \langle f, h\rangle\right\}$$

In this case we have $h = Ag$.

The following theorem plays a crucial role for the definition of Kreĭn–Feller operators.

Theorem A.23. *Let* $\mathcal{E}$ *be a non-negative quadratic form on* $\mathcal{H}$ *such that its domain* $\operatorname{dom}(\mathcal{E})$ *is dense in* $\mathcal{H}$*. Then the following conditions are equivalent.*

(1) There exists a non-negative self-adjoint operator A such that

$$\operatorname{dom}(\mathcal{E}) = \operatorname{dom}\left(A^{1/2}\right) \text{ and } \mathcal{E} = \mathcal{E}_A.$$

(2) Let $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \langle f, g\rangle$*. Then* $(\operatorname{dom}(\mathcal{E}), \mathcal{E}_1)$ *is a Hilbert space.*

Definition A.24. Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be Hilbert spaces. We call a linear operator $A : H_1 \to H_2$ compact if $A(B_1)$ is relatively compact, where B_1 denotes the unit ball with respect to $(H_1, \langle\cdot,\cdot\rangle_1)$.

Definition A.25. Let A be a non-negative self-adjoint operator on $\mathcal{H}$. We say A has compact resolvent if the *resolvent* $(A+\mathrm{id})^{-1}$ is a compact operator, where id denotes the identity map.

Theorem A.26. *Let A be a non-negative self-adjoint operator on $\mathcal{H}$. Then the following statements are equivalent.*

1. *A has compact resolvent.*
2. *There exists a complete orthonormal basis $(f_n)_n$ of $\mathcal{H}$ such that $Af_n = \lambda_n f_n$ for all $n \in \mathbb{N}$, where $(\lambda_n)_n$ is a non-negative increasing sequence with $\lambda_n \to \infty$ for $n \to \infty$.*
3. *The identity map from $\left(\mathrm{dom}\left(A^{1/2}\right), \mathcal{E}_1\right)$ into $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is compact, where $\mathcal{E}_1$ is defined as in Theorem A.23.*

We end this chapter with the important min-max principle.

Theorem A.27 (Min-max principle)**.** *Let A be a non-negative self-adjoint operator on $\mathcal{H}$ with compact resolvent and let $(\lambda_n)_n$ denote the eigenvalues of A given by Theorem A.26. Then,*

$$\begin{aligned}\lambda_n &= \inf\left\{\sup\left\{\frac{\mathcal{E}_A(f,f)}{\langle f,f\rangle} : f \in L,\ \langle f,f\rangle > 0\right\} : L \subset \mathrm{dom}\left(A^{1/2}\right),\ \dim(L) = n\right\} \\ &= \inf\left\{\sup\left\{\frac{\mathcal{E}_A(f,f)}{\langle f,f\rangle} : f \in L,\ \langle f,f\rangle > 0\right\} : L \subset \mathrm{dom}(A),\ \dim(L) = n\right\}.\end{aligned}$$

A.5 Fractal dimensions

We define the lower and upper Minkowski dimension, as well as the Hausdorff dimension. These definitions can be found in the classical book from Falconer [Fal97].

Definition A.28 (Minkowski dimension)**.** Let A be a bounded subset of $\mathbb{R}^d$. The *lower and upper Minkowski dimension* are defined as

$$\underline{\dim}_M(A) := \liminf_{n\to\infty} \frac{\log\left(\mathrm{card}\left(\{(k_1,\cdots,k_d) : A \cap Q((k_1,\cdots,k_d)) \neq \emptyset : k_i \in \mathbb{Z}\}\right)\right)}{\log(2^n)}$$

and

$$\overline{\dim}_M(A) := \limsup_{n\to\infty} \frac{\log\left(\mathrm{card}\left(\{(k_1,\ldots,k_d) : A \cap Q((k_1,\ldots,k_d)) \neq \emptyset : k_i \in \mathbb{Z}\}\right)\right)}{\log(2^n)},$$

with $Q((k_1,\ldots,k_d)) := \prod_{i=1}^d (k_i 2^{-n}, (k_i+1)2^{-n}]$ for $(k_1,\ldots,k_d) \in \mathbb{Z}^d$, respectively. If $\underline{\dim}_M(A) = \overline{\dim}_M(A)$, then the common value, the *Minkowski dimension*, is denoted by $\dim_M(A)$.

Definition A.29 (Hausdorff measure and dimension)**.** Let $A \subset \mathbb{R}^d$ and $s \geq 0$. Then the *s-dimensional Hausdorff-measure* is given by

$$\mathcal{H}^s(A) \coloneqq \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(A_i)^s : A \subset \bigcup_{i=1}^{\infty} A_i,\ \operatorname{diam}(A_i) < \delta \right\}.$$

The *Hausdorff dimension of A* is defined as

$$\dim_H(A) \coloneqq \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}.$$

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List of symbols

Symbol	Description, page
$\Delta_\nu^{D/N}$	Kreĭn–Feller operator with respect to Dirichlet/Neumann boundary conditions and ν, 33
β_ν^D	the Dirichlet L^q-spectrum of ν, 46
β_ν^N	the (Neumann) L^q-spectrum of ν, 46
Λ	the d-dimensional Lebesgue measure, 3
∇u	gradient of an element $u \in H^1(\Omega)$, 3
Π	the set of all ν-partitions of left half-open intervals of $(0,1]$, 94
Π_0	denote the set of finite disjoint collections of subintervals I of $(0,1]$, 90
$\Pi_{\mathfrak{J}}$	the set of finite collections of dyadic cubes such that there for all $P \in \Pi_{\mathfrak{J}}$ there exists a partition $\widetilde{P}$ of $\mathbf{Q}$ by dyadic cubes from $\mathcal{D}$ with $P = \left\{Q \in \widetilde{P} : \mathfrak{J}(Q) > 0\right\}$, 10
$\tau_{\mathfrak{J}}^{D/N}$	Dirichlet/Neumann partition function with respect to $\mathfrak{J}$, 12
$<_i$	we write $G <_i (H, \langle\cdot,\cdot\rangle)$ if G is a linear subspace of dimension i of the Hilbert space H, 35
$B_r(x)$	the open unit ball with radius $r > 0$ and center x in $\mathbb{R}^d$, 33
$\mathcal{B}(I^{\mathbb{N}})$	Borel σ-algebra of $I^{\mathbb{N}}$, 102
$\mathcal{B}(\mathbb{R}^d)$	Borel σ-algebra of $\mathbb{R}^d$, 61
$C([a,b])$	the vector space of continuous functions $f : [a,b] \to \mathbb{R}$, 24
$C_b^\infty\left(\overline{\Omega}\right)$	the vector space of functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f\|_\Omega \in C^m(\Omega)$ for all $m \in \mathbb{N}$ with $D^\alpha f\|_\Omega$ uniformly continuous on Ω for all $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, 3
$C_c^\infty(\Omega)$	the vector space of smooth function with compact support contained in Ω, 3
d	dimension of the ambient space, 1
$\mathcal{D}_n^N$	$:= \left\{\prod_{i=1}^d (k_i 2^{-n}, (k_i+1)2^{-n}] : k_i = 0, \ldots, 2^n - 1\right\}$, 22
$\mathcal{D}_n^D$	$:= \left\{Q \in \mathcal{D}_n^N : \partial\mathbf{Q} \cap \overline{Q} = \varnothing\right\}$, 22

$\mathcal{D}(Q)$	$:= \left\{\widetilde{Q} \in \mathcal{D} : \widetilde{Q} \subset Q\right\}$ for $Q \in \mathcal{D}$, 10
$\mathcal{D}$	$:= \bigcup_{n\in\mathbb{N}} \mathcal{D}_n^N$, 9
$\operatorname{diam}(A)$	$:= \sup\{\lvert x-y\rvert : x,y \in A\}$ for $A \subset \mathbb{R}^d$, 58
$\dim_\infty(\mathfrak{J})$	∞-dimension of $\mathfrak{J}$, 42
$\dim_H(A)$	Hausdorff dimension of A, 178
$\underline{\dim}_M(A)$	lower Minkowski dimension of A, 178
$\overline{\dim}_M(A)$	upper Minkowski dimension of A, 178
$\overline{\dim}_M(\nu)$	upper Minkowski dimension of $\operatorname{supp}\nu$, 14
$\underline{D}_r(\nu)$	lower quantization dimension ν of order r, 8
$\overline{D}_r(\nu)$	upper quantization dimension ν of order r, 8
$D_r(\nu)$	quantization dimension ν of order r, 8
$\underline{F}_{\mathfrak{J}}^{D/N}$	lower optimized (Dirichlet/Neumann) coarse multifractal dimension w.r.t. $\mathfrak{J}$, 11
$\overline{F}_{\mathfrak{J}}^{D/N}$	upper optimized (Dirichlet/Neumann) coarse multifractal dimension w.r.t. $\mathfrak{J}$, 11
$H^1(\Omega)$	the Sobolev space of $L^2(\Omega)$ functions with weak first order derivatives in $L^2_\Lambda(\Omega)$, 172
$H^1_0(\Omega)$	the closure of $C_c^\infty(\Omega)$ with respect to $\langle\cdot,\cdot\rangle_{H^1}$, 3
H^1	$:= H^1(\mathbf{Q})$, 3
H^1_0	$:= H^1_0(\mathbf{Q})$, 3
$\underline{h}_{\mathfrak{J}}$	lower $\mathfrak{J}$-partition entropy , 10
$\overline{h}_{\mathfrak{J}}$	lower $\mathfrak{J}$-partition entropy, 10
$\mathfrak{J}_{\nu,a,b}$	spectral partition function with parameters a, b ($a \in \mathbb{R}$, $b > 0$), 10
$\mathfrak{J}_\nu$	$:= \mathfrak{J}_{\nu,(2-d)/d,1}$,10
$\mathfrak{J}_{J,a}$	$:= J(Q)\Lambda(Q)^a$ with $Q \in \mathcal{D}$, 75
$L^2_\nu(A)$	the quotient space of the set of real-valued square-ν-integrable functions with domain $A \in \mathfrak{B}(\mathbb{R}^d)$ with respect to the almost-sure equivalence relation, 3
L^2_μ	$:= L^2_\mu(\mathbf{Q})$ with μ being a Borel measure on $\mathbf{Q}$, 3
L_ψ	Perron-Frobenius operator, 103
$N_\nu^{D/N}$	eigenvalue counting function of $\Delta_\nu^{D/N}$, 4
$\mathcal{N}_\nu^\perp$	$:= \left\{f \in H^1(\Omega) : \forall g \in \mathcal{N}_\nu : \langle f,g\rangle_{H^1(\Omega)} = 0\right\}$, 24
$\mathcal{N}_{0,\nu}^\perp$	$:= \left\{f \in H^1_0(\Omega) : \forall g \in \mathcal{N}_\nu \cap H^1_0(\Omega) : \langle f,g\rangle_{H^1_0(\Omega)} = 0\right\}$, 24
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	$:= \mathbb{N} \cup \{0\}$
o	$f = o(g)$ if $\limsup_{x\to\infty} \lvert f(x)\rvert/\lvert g(x)\rvert = 0$, 1
O	$g = O(f)$ if $\limsup_{x\to\infty} \lvert g(x)\rvert/\lvert f(x)\rvert < \infty$, 1
$\mathbf{Q}$	the left half-open d-dimensional unit cube $(0,1]^d$, 2
$\mathbb{Q}$	the set of rational numbers

$\langle Q \rangle_s$	the cube centered and parallel with respect to Q such that $\langle Q \rangle_s = T(Q) + (1-s)x_0$ with $T(x) = sx, x \in \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ is the center of Q, 37
q_r	$:= \inf \left\{ q > 0 : \beta_\nu^N(q) < rq \right\}$, 157
$\mathbb{R}$	the set of real numbers
$\underline{s}_\nu^{D/N}$	lower spectral dimension with respect to Dirichlet/Neumann boundary conditions, 4
$\overline{s}_\nu^{D/N}$	upper spectral dimension with respect to Dirichlet/Neumann boundary conditions, 4
$s_\nu^{D/N}$	spectral dimension with respect to Dirichlet/Neumann boundary conditions, 4
$V^\star$	$:= V \setminus \{0\}$, 29
$\mathbb{Z}$	the set of integers