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ASPECTS AND IMAGES OF COMPLEX PATH INTEGRALS

AN EPISTEMOLOGICAL ANALYSIS AND A RECONSTRUCTION OF EXPERTS' INTERPRETATIONS OF INTEGRATION IN COMPLEX ANALYSIS

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Aspects and images of complex path integrals

An epistemological analysis and a reconstruction of experts' interpretations of integration in complex analysis

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ABSTRACT

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The first research reports from complex analysis education show that not only novices but also mathematical experts have difficulties in interpreting complex path integrals. Therefore, we deal with two sides of experts' complex analysis discourse in this thesis: On the one hand, we present a comprehensive epistemological analysis of complex path integrals. On the other hand, we reconstruct experts' personal interpretations of these mathematical objects in the form of a multi-case study. The thesis has three major contributions, which are theoretically grounded in the *commognitive framework* and *German subject-matter didactics*:

- We suggest a conceptualisation of *discursive mental images* as narratives and *discursive frames* as sets of metarules in *intuitive mathematical discourses* in order to enrich basic research in mathematics education at university level. It complements acquisitionist perspectives on individuals' mental images of mathematical objects and provides a non-subsumptive and non-prescriptive way to study experts' individual, intuitive interpretations of mathematical objects. (Part i)
- A detailed, historically informed epistemological analysis of definitions of complex path integrals, their discursive embedding, and curricular connections to other mathematical discourses enables us to identify four so-called *aspects* and four *partial aspects* of complex path integrals. These are typical ways of defining complex path integrals by relating them to different mathematical constraints on the integrands, paths, or domains. We also provide a new axiomatic definition for complex path integrals of holomorphic functions. (Part ii)
- This conceptualisation from the first part is used for the analysis of experts' intuitive mathematical discourses about complex path integrals. Our study also includes their individual interpretations and substantiations of central integral theorems in complex analysis. (Part iii)

The reconstructed set of discursive images contains an analogy-based saming of complex and real path integrals, the valuation of the complex path integral as a tool, a mean value interpretation, and others. One expert also attempted to transfer area interpretations for real integrals to complex path integrals. In particular, experts' intuitive interpretations of complex path integrals are primarily narrative rather than figurative. The theoretical construct of discursive frame turns out to be especially helpful as it enables us to highlight commonalities and differences between experts' intuitive mathematical discourses about complex path integrals. Consistent with previous literature, this study confirms that experts enrich their intuitive mathematical discourses with connections to other mathematical discourses such as real or vector analysis. We conclude with perspectives for future research on the teaching and learning of complex path integrals.

KURZZUSAMMENFASSUNG

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Die ersten Forschungsberichte aus der Mathematikdidaktik zur Funktionentheorie zeigen, dass nicht nur Noviz*innen sondern auch Expert*innen Schwierigkeiten haben, komplexe Wegintegrale zu interpretieren. Deshalb widmen wir uns in dieser Arbeit zwei Facetten der Funktionentheoriediskursen von Expert*innen: Auf der einen Seite legen wir eine umfassende epistemologische Analyse von komplexen Wegintegralen vor. Auf der anderen Seite rekonstruieren wir die persönlichen Interpretationen dieser mathematischen Objekte seitens Expert*innen in Form einer multiplen Fallstudie. Diese Arbeit leistet drei wichtige Beiträge basierend auf der theoretischen Rahmung durch das *commognitive framework* und die *deutsche Stoffdidaktik*:

- Wir schlagen eine Konzeptualisierung von *diskursiven Vorstellungen* als Narrative sowie *diskursiven Rahmungen* als Mengen metamathematischer Regeln in *intuitiven mathematischen Diskursen* vor, um damit die mathematikdidaktische Grundlagenforschung für den universitären Bereich anzuregen. Sie komplementiert akquisitionistische Perspektiven auf Vorstellungen von mathematischen Objekten von Individuen und stellt eine nicht-subsumtive und nicht-präskriptive Möglichkeit dar, um die individuellen intuitiven Interpretationen von mathematischen Gegenständen seitens Expert*innen zu erforschen. (Part i)

- Eine detaillierte, mit historischen Bezügen angereicherte epistemologische Analyse von Definitionen komplexer Wegintegrale, ihre diskursiven Einbettungen sowie curricularer Zusammenhänge zu anderen mathematischen Diskursen ermöglicht es uns, vier sogenannte *Aspekte* und vier *partielle Aspekte* komplexer Wegintegrale zu identifizieren. Dabei handelt es sich um typische Definitionsmöglichkeiten von komplexen Wegintegralen, die an verschiedene mathematische Voraussetzungen bezüglich der Integranden, Wege und Definitionsbereiche geknüpft sind. Außerdem stellen wir eine axiomatische Definition komplexer Wegintegrale für holomorphe Funktionen vor. (Part ii)

- Die Konzeptualisierung aus dem ersten Teil der Arbeit wird für die Analyse von intuitiven mathematischen Diskursen über komplexe Wegintegrale von Expert*innen genutzt. Dies beinhaltet auch ihre individuellen Interpretationen und Begründungen von zentralen Integralsätze der Funktionentheorie. (Part iii)

Die rekonstruierten diskursiven Vorstellungen beinhalten eine auf Analogien basierte Vereinheitlichung von komplexen und reellen Wegintegralen, die Wertung des komplexen Wegintegrals als ein Werkzeug, eine Mittelwertinterpretation und weitere. Ein Experte versuchte auch, Flächeninhaltsvorstellungen für reelle Integrale auf komplexe Wegintegrale zu übertragen. Inbesondere sind die intuitiven Interpretationen von Expert*innen vorrangig narrativ

statt bildlich. Das theoretische Konstrukt der diskursiven Rahmungen stellt sich als besonders hilfreich dar, weil es uns erlaubt, Gemeinsamkeiten und Unterschiede zwischen den intuitiven mathematischen Diskursen über komplexe Wegintegrale der Expert*innen hervorzuheben. Übereinstimmend mit der bisherigen Literatur bestätigt diese Studie, dass Expert*innen ihre intuitiven Interpretationen von komplexen Wegintegralen durch Querverbindungen zu anderen mathematischen Diskursen wie der reellen Analysis oder der Vektoranalysis anreichern. Wir beschließen diese Arbeit mit Perspektiven für die zukünftige Forschung zum Lehren und Lernen von komplexen Wegintegralen.

Ideas take their energy from the perceptions of others.
—Haruki Murakami, Killing commendatore

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TABLE OF CONTENTS

1 INTRODUCTION

Is there a simple interpretation for the familiar complex contour integral $\int_C f(z)\,dz$, one as accessible as those for the integrals of real-valued functions? My experience is that students are mystified on first exposure to this concept, and working examples by the formula $\int_a^b f(z(t))z'(t)\,dt$ can be a baffling experience; what sense is a beginning student to make of the results

$$\int_{|z=1|} \operatorname{Re} z\,dz = \pi i \quad \textit{or} \quad \int_1^i \frac{i}{z}\,dz = \frac{-\pi}{2}?$$

—Gluchoff (1991, pp. 641–642)

1.1 MOTIVATION

Many mathematics students struggle with making meaning of the mathematical concepts they encounter in their study programmes (cf. e.g., Davis et al., 2012; Pfeffer, 2017; Rasmussen et al., 2014; Tall, 1991/2002; Tall & Vinner, 1981; Wilzek, 2021; Winsløw & Rasmussen, 2020). Whereas mathematics education at university is ultimately responsible to foster students' participation in formal mathematical discourses, it has nevertheless been argued many times that students should be able to work with a variety of resources and that it is equally important to assist learners build suitable mental images about the mathematical objects they learn (e.g., Bender, 1991; Fischbein, 1987; Greefrath et al., 2016a, 2016b; vom Hofe, 1995; Tall, 2013; Tall & Vinner, 1981; Weber, 2007; Weigand, 2015). These mental constructs are part of current research in university mathematics education in one way or another (e.g., Feudel, 2020; Hamza, 2012; Martin, 2013; Roos, 2020; Tall, 1991/2002, 2008, 2013; Wilzek, 2021, and many others). In particular at the beginning of learning a new topic, it is helpful if the mathematical objects are made accessible to students in various ways, perhaps also by leaving aside full mathematical rigour.

In this context, Hochmuth (2021b) acknowledges that

> [f]or the use of concepts and terms, whether in problem solving, applying, or finding a proof, diverse mental images, representations, and sometimes intuitive references between subject-matter content usually play an important role. The aim of teaching is that students do not only know the *concept definition* and can use it knowledgeably, but also that they have a rich *concept image* at their disposal, which can be reflected upon in its various aspects with regard to its affordances and limits as well as its compatibility with the *concept definition*. (Hochmuth, 2021b, p. 12, own transl., emph. orig.)

Additionally, many studies have shown that many mathematicians also do not only rely on formal reasoning but use of a variety of resources such as drawings and sketches of all kinds (e.g., Burton, 2004; Davis et al., 2012; Hadamard, 1945; Heintz, 2000; Hersh, 1998; Kiesow, 2016; Nardi, 2008; Sfard, 1994). For example, Sfard (2008, p. 149) observes that "the majority of mathematicians use visual imagery even in the most advanced and abstract of discourses" (cf. Sfard, 1994). In particular, the hypothesis that "mathematicians gain conviction solely by logical deduction is too simplistic" (Weber et al., 2014, p. 36). For these reasons, it is an important task in mathematics education to look for multiple ways of making accessible the concepts from advanced mathematics at university for oneself or for others (cf. e.g., Greefrath et al., 2016a, 2016b; Tall, 1991/2002, 2008; Weber & Inglis, 2020; Weber et al., 2014; Weigand, 2019). Since mathematical experts are the leading figures in the development of a mathematical domain at research level and in teaching, it is particularly important to find out how experts find access to and work with objects from tertiary mathematics.

For the teaching and learning of calculus and analysis, mathematics education has already examined various ways to teach and interpret the integral of functions of one real variable. Many interpretations of the Riemann integral $\int_a^b f(x)\,\mathrm{d}x$ are well-known, such as that it is the signed area between the graph of f, the horizontal axis, and the vertical axes at $x = a$ and $x = b$, the reconstruction of a quantity from a given rate of change, or the accumulation of a quantity (e.g., Greefrath et al., 2016a, 2016b, 2021a, 2022). In this context, many teaching materials have been developed and many empirical studies have been conducted to find out how mathematics learners adopt these interpretations and incorporate them to their own set of interpretations of integrals (e.g., Griffiths, 2013; Jones, 2013, 2015, 2018; Kirsch, 2014; Kouropatov & Dreyfus, 2013, 2015; Larsen et al., 2017; Pino-Fan et al., 2018; Roh & Lee, 2017; Tall, 2013; Wasserman et al., 2022; cf. Kidron, 2020; Vinner, 2020; Winsløw, 2020). Additionally, some research is available on double integrals (e.g., Martínez-Planell & Trigueros, 2020, 2021) or real path integrals (e.g., Jones, 2020; Ponce Campuzano et al., 2019) as well.

In 2016, the project *Spotlights Lehre* (from 2019 on *Digi-Spotlights*) started at the University of Bremen. It is part of the *Qualitätsoffensive Lehrerbildung*, which was funded by the German Federal Ministry of Education and Research. I became a member of the mathematics part of this project, *Spotlight-Y* (from 2019 on *Spotlight-Y-Digimath*). Its goal was to develop a lecture on *complex analysis* for the needs of future teachers according to the principles of design based research (for more details and results see Hanke et al., 2021; Hanke & Schäfer, 2018; Schäfer & Hanke, 2022). Consequently, we looked for scientific results about the teaching and learning of complex analysis. Working as a teaching assistant for the course on complex analysis for several years, I additionally experienced first hand that this mathematical domain is very demanding for students and that they ask how the abstract concepts of complex analysis could be explained on an intuitive level.

However, little research on complex analysis education was available so far. While several studies in mathematics education already dealt with complex numbers, only a few dealt with more advanced topics such as continuity or differentiability in complex analysis (e.g., Danenhower, 2000; Soto-Johnson et al., 2016; Troup, 2015). However, practically no empirical research on complex path integrals had been published at that time. Nevertheless, teachers of complex analysis were already interested in improving the teaching and learning of complex path integrals for some time. This shows for example the quote at the beginning of this chapter, in which Gluchoff (1991) reported that students were "mystified" by complex path integrals. Henceforth, I intended to address this gap and chose the complex path integral to be the main mathematical object for my doctoral project.

During the course of my work, a few additional studies appeared in the emerging field of complex analysis education (e.g., Howell et al., 2017a; Soto-Johnson & Hancock, 2018; Troup, 2019; Troup et al., 2017). In particular, the first empirical studies about novices' and experts' understanding of complex path integrals showed that these are indeed a demanding topic for students and that even lecturers struggle to explain how they imagine the complex path integral (Hancock, 2018; Oehrtman et al., 2019; Soto & Oehrtman, 2022). For example, Oehrtman et al. (2019) interviewed experts in complex analysis and noticed that all but one had difficulties to provide a geometrical, physical, or otherwise intuitive interpretation, and reasoned more or less on a symbolic level.

The limited amount of research in complex analysis education contrasts with to the relevance of complex analysis for tertiary mathematics education. Complex analysis is widely taught in many mathematics curricula (for prospective pure and applied mathematicians, preservice teachers, or service courses; e.g., KMK, 2008/2019; UB, 2016, 2021) and has many applications in calculus / real analysis, number theory, physics, engineering and others (e.g., Freitag & Busam, 2006; González, 1992; Karatsuba, 1995; Lang, 1995, 1999; Lawrentjew & Schabat, 1967; Mathews & Howell, 2012; Remmert & Schumacher, 2002; Shorey, 2020; Vaughan, 1997). Henceforth, courses on complex analysis are attended by students of great heterogeneity. Kortemeyer and Frühbis-Krüger (2021) even report on a case where future engineers had to use the residue theorem, but whose curricula did not schedule complex analysis. Moreover, many concepts from other mathematical domains (e.g., real analysis, linear algebra, geometry) appear in complex analysis in one way or another. In particular, the concepts of differentiability and integration students first learn "real" reappear in a "complex" guise. That is, integrals are one of the central "cross-curricular concepts" (cf. Kontorovich, 2018b) in mathematics curricula.

Besides these curricular reasons, there are also epistemological arguments for an increase in mathematics education on complex analysis. Since the set of complex numbers can be visualised as a plane, several objects in complex analysis can be visualised (e.g., points, subsets of the complex numbers, or paths for integration). However, functions are already much harder to visualise since their graphs are subsets of $\mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$, which we cannot draw or model with two or three real coordinates. Therefore, it stands to reason that it is difficult to visualise complex functions in general and complex path integrals in particular. Some authors even state that complex path integrals have no geometrical meaning. For example, while Knopp (1945–1947/1996, p. 33) acknowledges that the Riemann integral can be visualised "as the approximation of a plane area by means of a sum of rectangles", he also argues that for complex path integrals

> [a] simple geometrical interpretation as in the case of real integrals is impossible (Knopp, 1945–1947/1996, p. 39).

Similarly, Gathmann (2017) emphasises that complex path integrals

> in contrast to real integrals — have *no visual meaning* as the area or volume of some set below the graph of a function. However, we will see in this lecture that they are an exceedingly useful tool in complex analysis (Gathmann, 2017, p. 18, own transl., emph. orig.).

Henceforth, we may still ask whether there are other visualisations[1] or intuitive interpretations of complex path integrals. In this regard, the investigation of intuitive interpretations of

1 Visualisation is a multi-faceted notion in mathematics education (Arcavi, 2003; Duval, 1999, 2014; Giaquinto, 2020; Presmeg, 2006, 2020). Here, we use it pragmatically to denote the process or the product of making something visible. Later, when we have introduced the commognitive framework (e.g., Lavie et al., 2019; Sfard, 2008, 2020a) as the main theoretical framework for our thesis, we will use the construct "visual mediator".

complex path integrals is exemplary for other topics in tertiary mathematics, which are hard to visualise as well.

In order to address these desiderata and to develop resources for designing lectures on complex analysis path integrals, we need to examine more closely beforehand how experts in complex analysis work with complex path integrals. In this line, Winsløw et al. (2021, p. 74) observe that

> material that identifies fundamental or central ideas, provides insight into learning difficulties or obstacles for the students [...] is available for teaching at school level, for instance to know about different ways to approach and organise the teaching of derivatives or integrals (cf. Greefrath, Oldenburg, Siller, Ulm, & Weigand, 2016),

but also that "[s]imilar expositions are inaccessible or unavailable when it comes to more advanced subjects (e.g. linear algebra [and complex analysis; EH.]) and their teaching at university level" (Winsløw et al., 2021, p. 74).

Since integrals appear at many places in mathematics curricula, it is likely that novices' but also experts' reasoning about them is influenced by their precedents in real analysis. This is especially probable since there are a variety of interpretations available for real integrals as we have described above. However, it is not clear whether and how the interpretations for other integrals relate to complex path integrals. Therefore, a comprehensive epistemological analysis is relevant. Addressing this desideratum, it is also relevant to take various potential definitions of complex path integrals, its historical genesis, curricular connections, and the transferability of interpretations for other integrals into account (cf. Hefendehl-Hebeker et al., 2019; Hochmuth, 2021b; Sträßer, 2020). Similarly, we may ask how integral theorems from complex analysis (e.g., Cauchy's integral theorem) can be substantiated intuitively.[2]

Since the literature is still scarce and before we can develop teaching-learning materials and assess learners' interpretations of complex path integrals in the long run, we first need to clarify how the main protagonists of mathematical research and teaching deal with these mathematical objects. Hence, we will focus on mathematical experts in this thesis more deeply. We address a *complementary perspective*: On the one hand, we study the endorsed approaches to complex path integrals in the mathematical literature. On the other hand, we investigate experts' intuitive interpretations of these objects. Therefore, we will first engage in an epistemological analysis of complex path integrals (cf. e.g., Bergsten, 2020; Greefrath et al., 2016a, 2016b; Hefendehl-Hebeker, 2016; Hochmuth, 2021b; Hußmann & Prediger, 2016; Hußmann et al., 2016; Sträßer, 2020) and second in dialogues with *experts* in complex analysis (cf. Gläser & Laudel, 2010; Meuser & Nagel, 1991; Pfadenhauer, 2009).

1.2 RESEARCH QUESTIONS AND CONTRIBUTIONS OF THE THESIS

The following research questions will be addressed in this thesis:

A: What are the subject-specific approaches to complex path integrals and how are they embedded into different mathematical areas?

B: i) What are experts' individual ways to interpret complex path integrals?

2 Indeed, visual and other intuitive interpretations of complex path integrals and integral theorems in complex analysis seem to occupy many learners as threads in relevant mathematics forums show (e.g., "math.stackexchange.com", 2021)

ii) What do their mental images of these mathematical objects look like?

iii) How do experts explain central integral theorems from complex analysis intuitively?

Answering these questions, this thesis makes three major contributions to the research gaps presented above:

Contribution to theory

Based on the commognitive framework (Lavie et al., 2019; Sfard, 2008, 2020a), we provide a discursive reconceptualisation of individuals' mental images, namely *discursive images*, as narratives in what we will define as *intuitive mathematical discourses*. This allows us to focus on a broader unit of analysis, namely discourse, to account for the complexity of how experts explain their intuitive interpretations and mental images of complex path integrals. (Part i)

Contribution to subject-matter didactics of complex analysis

We provide a comprehensive, historically informed *epistemological analysis* of complex path integrals. We draw on the notion of *aspects* and *partial aspects* (Greefrath et al., 2016a, 2016b; Roos, 2020), which we reconceptualise commognitively, and identify four *aspects* and four *partial aspects of complex path integrals*. Our epistemological analysis also informs about the curricular entanglement of complex path integrals with other integrals from mathematics curricula. It provides guidance for lecturers teaching complex analysis and our results may be used to develop future designs of lectures and learning materials on complex path integrals. (Part ii)

Contribution to empirical research in complex analysis education

We reconstruct experts' intuitive mathematical discourses about complex path integrals. In particular, we reconstruct underlying mechanisms, which we call *discursive frames* (cf. Heyd-Metzuyanim et al., 2018), guiding experts in constructing their discursive images and other intuitive explanations. Additionally, we describe experts' intuitive explanations of central integral theorems of complex analysis, namely Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and the Existence of primitives for holomorphic functions (Theorem A.20).

1.3 OVERVIEW OF THE THESIS

In order to achieve our goals, we combine the two perspectives indicated above, a *commognitive perspective* (e.g., Lavie et al., 2019; Sfard, 2008, 2015, 2020a) and a *German subject-matter didactical perspective* (e.g., Greefrath et al., 2016a, 2016b; Hefendehl-Hebeker et al., 2019; Hußmann et al., 2016; Sträßer, 2020).

Figure 1.1 at the end of this chapter shows an outline of the thesis. In Part i, we review existing conceptualisations of mental images and explain why we opt to reconceptualise them in terms of the commognitive framework. In Part ii, we perform a comprehensive epistemological analysis by reviewing the existing literature in complex analysis education, analysing the historical discursive development of the concept of complex path integral, as well as the endorsed definitions, interpretations, and cross-curricular connections from mathematics literature. Finally, in Part iii, we will examine experts' intuitive mathematical discourses about complex path integrals.

In Part i, we begin by reviewing existing conceptualisations of mental images in mathematics education such as *concept images* (e.g., Tall & Vinner, 1981; Vinner, 1983, 2020; Vinner & Dreyfus, 1989) or *basic ideas* (e.g., vom Hofe, 1995; vom Hofe & Blum, 2016; Kleine et al., 2005; Salle & Clüver, 2021) (Chapter 2). These are grounded in a cognitive and partly in a prescriptive point of view on the learning of mathematics. However, we argue that a discursive conceptualisation is more beneficial for the goals of this thesis. For one thing, cognitively conceptualised forms of mental imagery are not per se recognisable to researchers and they do not take into account the social and communicative dimension of learning and doing mathematics in interaction with others or oneself. Therefore, we intend to broaden the perspective of analysis to mathematical discourses at large.

We proceed to describe the commognitive framework in Chapter 3. This framework regards mathematics as discourses (Sfard, 2008). Doing mathematics means to participate in mathematical discourses and human thinking is conceptualised as an intra-individual version of communication. The commognitive framework endows us with the ontological and epistemological baseline that mathematical objects are discursive products of human communication. In this framework, mathematical discourses are set apart by their keywords, endorsed narratives, visual mediators, and routines (Sfard, 2008). Here, narratives are understood as stories about mathematical objects (i.e., object-level rules) or of the persons doing mathematics (i.e., metarules), which are to be endorsed by a community of endorsers; visual mediators are all kinds of visually perceptible entities used in mathematical communication; and routines are potentially repetitive discursive actions in mathematical discourses such calculating, proving, defining, and substantiating narratives (Lavie et al., 2019; Sfard, 2008, 2020a).

Afterwards, we define the notion of *intuitive mathematical discourse* in Chapter 4. Instead of conceptualising single mental images cognitively and looking for them in mathematicians' minds, we examine the individual ways, in which mathematicians approach complex path integrals and make them intelligible to themselves and potentially to others in terms of the larger unit of analysis, namely discourse. For that purpose, we find that a cognitive and acquisitionist perspective is not necessary (cf. Sfard, 2006, 2009a). *Discursive images* will then be understood as narratives in intuitive mathematical discourses, which are possibly assisted with visual mediators (Hanke, 2020a, 2020b).

In order to answer research question A, we study the endorsed discourse on complex path integrals. For this to be done, we combine the discursive perspective with the subject-matter didactics perspective in Part ii. Hence, our epistemological analysis in this part of the thesis covers the discursive development of the notion of complex path integral in the literature at large. Implicitly, it accounts for the routines of defining and substantiating in relation to complex path integrals as well as for the endorsement of other narratives about complex path integrals. This part starts with a review of previous research and graphical realisations of complex path integrals in Chapter 5. Then, we outline the necessary background and planning of the epistemological analysis, introduce the notions of *aspects* and *partial aspects* (Greefrath et al., 2016a, 2016b; Roos, 2020), and reconceptualise them from a discursive point of view in Chapter 6.

Complex analysis is one of the domains in mathematics curricula, in which many discourses intersect (e.g., real analysis, vector analysis, and topology). Many mathematical objects students encountered previously reappear here. A mathematically deep epistemological analysis is needed to entangle these inter-discursive relationships and to understand the genesis of the complex analysis discourse at large. We investigate various resources such as historical primary and secondary literature (e.g., Bottazzini & Gray, 2013; Cauchy, 1825), articles from mathematics journals from approximately the last 120 years, as well as current textbooks and lecture notes. For this to be done, Chapter 7 contains an informative historical overview of the use of integrals of complex functions, the origin of one of the definitions of complex path integrals by Cauchy (1825), and the developments on their use in complex analysis in the 20$^{\text{th}}$ century. The main body of the epistemological analysis is then presented in Chapter 8. Finally, we summarise our results in terms of our refinement of the notions of *aspects* and *partial aspects* of complex path integrals in Chapter 9. This refinement is particularly suitable for university mathematics education, in which the routine of defining is of great importance (e.g., Sfard, 2014; Viirman, 2021). According to our definition, an *aspect of complex path integrals* is a narrative, which can function as a definition of the definiendum "complex path integral" (i.e., the part to be defined in a definition) and which can be distinguished in terms of the keywords and other signifiers used in their definientia (i.e., the defining part of a definition; Sill, 2019, p. 96). Furthermore, a *partial aspect of complex path integrals* is an aspect, whose endorsement requires additional constraints for the functions to be integrated and the paths to be integrated along compared to the general setting covered by the aspects. We also add an axiomatic characterisation of complex path integrals (Theorem 8.13), which, to our surprise, we did not find in mathematics literature.

In order to answer research question B, we give a detailed account of experts' intuitive mathematical discourses in Part iii. In an instrumental case study (e.g., Grandy, 2010; Stake, 1995, 2006), we collect data in three semi-structured guideline interviews with experts "at the eye-level" (Pfadenhauer, 2009). Methodological background is presented in Chapter 10 and the methodical implementation is described in Chapter 11. We strive for a "*non-deficit*, *non-prescriptive*, *context-specific*, *example-centred* and *mathematically focused*" account of this data (Nardi, 2016, p. 361, emph. orig.). After an overview of the results in Chapter 12, we proceed with one chapter for the case study of each expert (Chapter 13, Chapter 14, Chapter 15). We reconstruct a list of eight *discursive frames* (i.e., metarules governing intuitive mathematical discourses), some of which appear across our interviews. The *nine discursive images about complex path integrals* we found range from valuations of the complex path integral as a mathematical object without intrinsic meaning or as a tool in complex analysis, as certain mean values, up to narratives derived from integral theorems such as the residue theorem. Our empirical study also confirms Oehrtman et al.'s (2019) finding that experts' individual interpretations of complex path integrals are not consensual. Additionally, we also present more general findings on individual word use and substantiation routines in our three interviews.

We discuss our contributions and findings, the limitations of our study, and suggest directions for further research in Chapter 16.

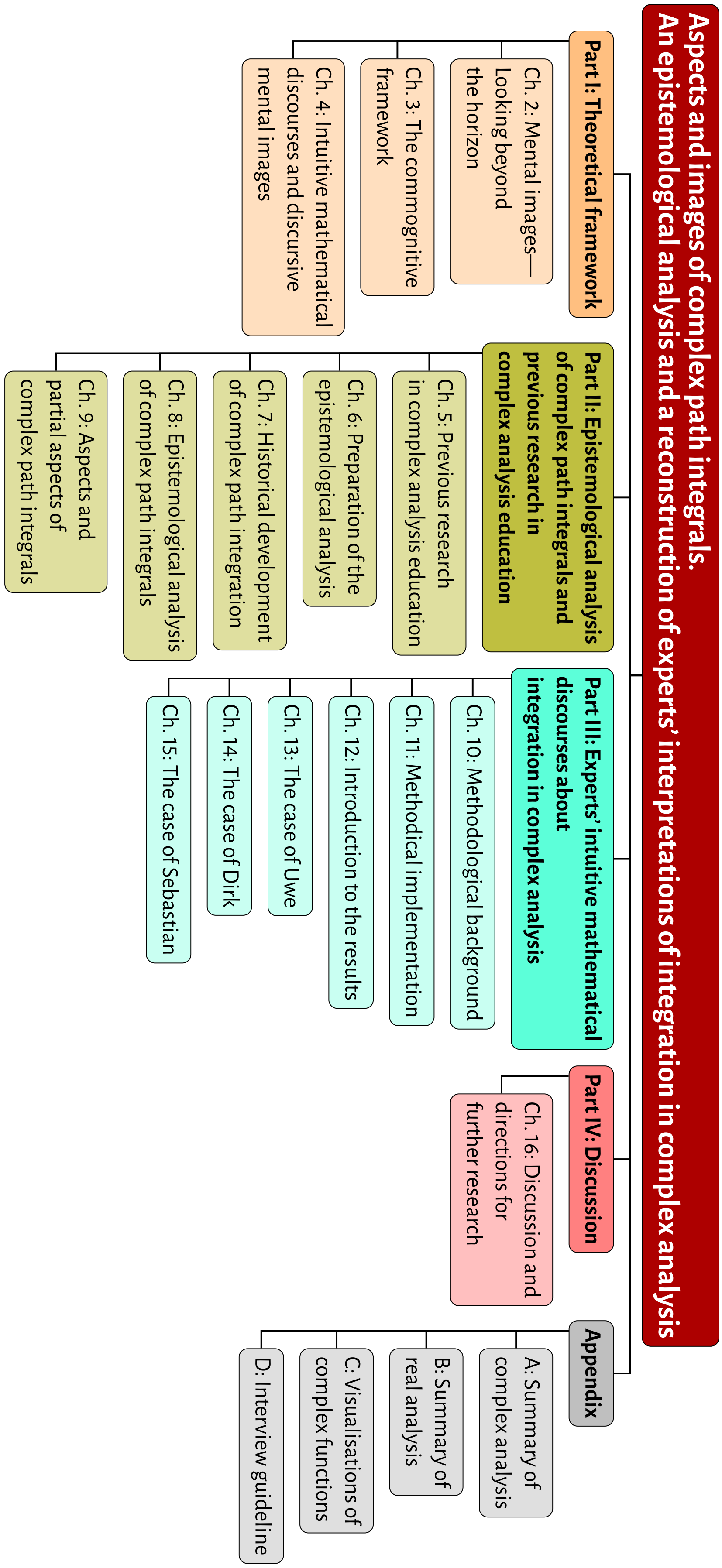

Figure 1.1: Overview of the thesis.

Part I

THEORETICAL FRAMEWORK

OVERVIEW OF PART I

Figure I shows the outline of Part i. We review existing conceptualisations of individuals' mental images of mathematical objects in mathematics education literature and explain why we opt for a discursive reconceptualisation in terms of the commognitive framework for our empirical study on experts' individual interpretations of complex path integrals and integral theorems.

- Chapter 2 contains a literature review of conceptualisations of mental images of mathematical objects in mathematics education literature. Its goal is to describe previous conceptualisations and to explain why a discursive point of view is more appropriate for the study of mathematical experts' individual interpretations of complex path integrals and substantiations of integral theorems in complex analysis. It will be argued why we do not regard mental images about complex path integrals as mental entities or prescriptive categories. Instead, we argue that it is relevant to take a broader unit of analysis, namely discourse, into account for our exploratory research in Part iii.
- In Chapter 3, we explain the discursive framework used in this thesis, namely the *commognitive framework* (e.g., Lavie et al., 2019; Sfard, 2008, 2020a). Here, we clarify what exactly is meant by the word *discourse* in this thesis. In particular, we discuss ontological and epistemological assumptions about mathematical objects for our conceptualisations in the next chapter.
- We present our discursive reconceptualisation of mental images for this thesis in Chapter 4. More precisely, we delineate a certain type of mathematical discourses as *intuitive mathematical discourses*, which will be at the centre of our investigation in Part iii. We describe certain narratives in these discourses as *discursive images*, our replacement of the notion of mental images of mathematical objects. Furthermore, we introduce so-called *discursive frames* with the help of which we will be able to theoretically account for the fabrication of intuitive mathematical discourses. In sum, we replace an acquisitionist point of view on mental images of mathematical objects with a discursive one.

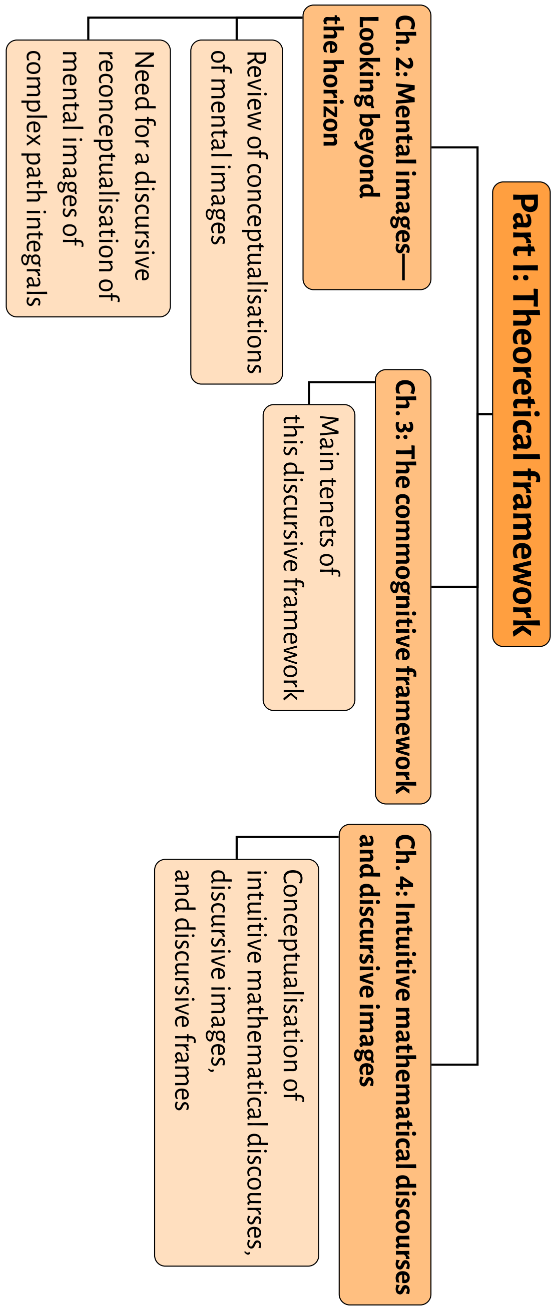

Figure I: Overview of Part i.

2

MENTAL IMAGES—LOOKING BEYOND THE HORIZON

Neither is the imagination a space in which something is imagined, nor are mental images objects of an inner perception over which the imaginer has autonomy. Rather, the fact that one imagines something is a way of characterising one's behaviour in systematic dependence on the attributed content of the mental image, more similar to the characterisation that one presents the content of the mental image.

—von Savigny (1996, p. 54)

In the introduction, we asked which individual interpretations and mental images expert mathematicians have for the complex path integral. More generally, we are interested in how experts access these highly abstract mathematical objects. This immediately raises two ontological questions: What are complex path integrals and what is a mental image of these?

In this chapter, we start with the second question and review literature about mental images in mathematics education. We identify the need for a conceptualisation in terms of a participationist, discursive perspective of learning and doing mathematics (cf. e.g., Cobb, 2006; Sfard, 2006, 2008, 2009a) in order to address research question B (Section 1.2). Then, having clarified what we mean by *discourse* in the *commognitive framework* in Chapter 3, we also answer the first question from the beginning of this chapter and explain our view on the ontology of mathematical objects. Afterwards, we will present our discursive reconceptualisation in Chapter 4, where we finally define *discursive mental images* as narratives in so-called *intuitive mathematical discourses*.

Previous conceptualisations of mental images and similar concepts in mathematics education are often grounded—implicitly or explicitly—in an acquisitionist or psychological vision of learning and doing mathematics (cf. e.g., Schwarz, 2020; Sfard, 1998, 2006, 2008, 2009b), most likely on cognitive or constructivist frameworks (cf. e.g., Cobb, 1994; Confrey & Kazak, 2006; Ernest, 1991, 1994, 2010; Lerman, 2010; Tall, 1991/2002; Thompson, 2020; von Glasersfeld, 2018).[3] In this regard, mental images are usually regarded as mental entities (cf. Schwarz, 2020; Tall & Vinner, 1981; Vinner, 2020; Weber, 2007). Briefly put, one may say that "[w]hat are spoken of as 'visual images', 'mental pictures', 'auditory images' and, in one use, 'ideas' are commonly taken to be entities which are genuinely found existing and found existing elsewhere than in the external world" (Ryle, 1949/2009, p. 222).[4]

In educational research, this has the consequence that many researchers conceptualise ≪mental images≫ as belonging to the minds of individuals. According to Roth (2013),

> ≪[m]ental representation≫ is a researcher construct, whereby what a research participant or learner says is used to make inferences about what might be in their minds. (Roth, 2013, p. 187)

Still others consider it a container word that encompasses various sub-concepts (cf. Manderfeld, 2020), all the way to prescriptive guidelines for the teaching of mathematical concepts (cf. Bergsten, 2020; vom Hofe, 1995; vom Hofe & Blum, 2016; Salle & Clüver, 2021; Sträßer, 2020).

In order to refer to these strands of mathematics education in general, we mimic Roth's (2013) bracket notation, which he used for the words ≪meaning≫ and ≪mental representation≫, and write ≪mental image≫ etc. When we review a particular conceptualisation, we will use the respective authors' wordings and highlight this use as such. When we write mental image (or a similar word) without highlighting, we usually refer to its use as a common sense concept in colloquial communication (e.g., as it may be used by a research participant). Moreover, when we speak of ≪mental images≫, we always understand them as ≪mental images≫ *about* or *of* mathematical concepts or objects.[5]

Outline of this chapter

In Section 2.1, the notion of ≪mental images≫ is broadly embedded into strands of mathematics education research and related areas. Then, we will exemplify our review in Section 2.2, where we discuss *concept images* (e.g., Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989; Vinner & Hershkowitz, 1980) and *basic ideas* (also called *basic mental models*; German: Grundvorstellungen; e.g., Bender, 1991; vom Hofe, 1995; vom Hofe and Blum, 2016. These two notions are frequently used in international and German discourses on mathematics education and have led to important basic and empirical research in mathematics education. In addition to their widespread use, we discuss these two conceptualisations in detail because they illustrate two rather opposing points of view on ≪mental images≫: While the first of these is predominantly descriptive, the second is predominantly prescriptive.[6] Furthermore, we will revisit basic

3 The acquisitionist view regards the learning "as an act of increasing individual possession - as an acquisition of entities such as concepts, knowledge, skills, mental schemas", which "comes to this scholarly discourse [on learning; EH.] directly from everyday expressions, such as acquiring knowledge, forming concepts or constructing meaning" (Sfard, 2006, p. 153; see also Sfard, 1998, 2009a). Thus, everyday life concepts are taken for granted and educational research based on this perspective considers it a major task to conceptualise these concepts for research and to generate potentially useful stories about the acquisition of these concepts.

4 In order to avoid a misunderstanding, one should bear in mind that Ryle (1949/2009) opposed against this point of view that results from the Cartesian dualism of body and mind (see also Wittgenstein, 1953/2009).

5 In Chapter 3, we explain our view on mathematical objects, namely that they are *abstract* and *discursive*, in other words, that they are products of communication. Our own conceptualisation of *discursive mental images* in Chapter 4 will then be in complete accordance with this point of view for they will also be products of communication, namely *narratives*, in *intuitive mathematical discourses*.

6 For example, the notion of basic idea is used to formulate and endorse prescriptions for teaching mathematical concepts (e.g., Bender, 1990, 1991, 1997, 1998; Greefrath et al., 2016a, 2016b; vom Hofe, 1995, 2003; vom Hofe &

ideas of Riemann integrals (Section 2.2.3), which lead to the question whether some of these ideas can be transferred in one way or another to complex path integrals. Also, we give a first hint of what it may look like to regard ≪mental images≫ as narratives, that is, as mathematical stories, which relate mathematical objects to what mathematicians and mathematics educators consider meaningful interpretations of them (Section 2.2.4).

In Section 2.3, we discuss Roth's (2013) and Wittgenstein's (1953/2009) critiques of an acquisitionist view on ≪mental images≫ for research in education and turn towards a discursive, participationist perspective on ≪mental images≫.

The following reasons led to the choice of a discursive perspective on experts' ≪mental images≫ about complex path integrals instead of adopting one of the existing conceptualisations. These reasons will be unfolded in the remainder of this chapter.

(i) Complex path integrals are highly abstract mathematical objects to which we do not have direct graphical access; hence, other means of communication must be used to make them accessible (Chapter 3).

(ii) It is insufficiently clear yet how experts may gain access to these highly abstract objects and how they may reason about them on a level they deem intuitive. In particular, it is not yet clear what a conceptualisation of mental images for tertiary mathematics may look like (cf. Clüver & Salle, 2020; Oehrtman et al., 2019).

(iii) Little research about experts' ≪mental images≫ of complex path integrals is available. Moreover, our initial review of our interview data from Part iii did not allow us to find a consensual set of ≪mental images≫ of complex path integrals. Hence, it proved to be useful to widen the scope of analysis and to not only look for potential ≪mental images≫ but for the fabrication of experts' individual interpretations of complex path integrals at large.

2.1 WHAT IS A ≪MENTAL IMAGE≫?

Mathematics education has taken into account ≪mental images≫ under different names for a long time (e.g., vom Hofe, 1995; Schwarz, 2020; Vinner, 2020; Weigand, 2015). For example, in the context of school, Weigand (2015, p. 261, own transl., emph. orig.) argues that it is important to find out which "*mental objects*" a learner forms when she or he is learning a mathematical concept. In this respect, university mathematics education research has also taken into account students' individual conceptions of mathematical terms for quite some time now (e.g., Feudel, 2020; Feudel & Biehler, 2021a, 2021b; Fischer, 2006; Hamza, 2012; Hamza & O'Shea, 2016; Melhuish et al., 2020; Pfeffer, 2017; Roos, 2020; Tall, 1991/2002; Vinner, 2020; Wilzek, 2021).

In his review article, Thomas (1997/2021) points to the difficulty to describe exactly what is meant by the word ≪mental image≫:

Blum, 2016; Tietze et al., 2000). The notion of concept images serves to contrast learners' individual conceptions of a mathematical concept with their *concept definition* (e.g., Melhuish et al., 2020; Tall & Vinner, 1981; Vinner, 2020; Vinner & Dreyfus, 1989). In particular, both conceptualisations of ≪mental images≫ have been used to compare learners' idiosyncrasies with prescriptive ≪mental images≫ created by mathematics educators (e.g., Feudel, 2020; Feudel & Biehler, 2021a, 2021b; Greefrath et al., 2021a, 2021b, 2022; Hamza, 2012; Hamza & O'Shea, 2016; Roos, 2020, and many others).

> Despite the familiarity of the experience, the precise meaning of the expression 'mental imagery' is remarkably hard to pin down, and differing understandings of it have often added considerably to the confusion of the already complex and fractious debates, amongst philosophers, psychologists, and cognitive scientists, concerning imagery's nature, its psychological functions (if any), and even its very existence. In the philosophical and scientific literature (and a fortiori in everyday discourse), the expression 'mental imagery' (or 'mental images') may be used in any or all of at least three different senses, which are only occasionally explicitly distinguished, and all too often conflated:
>
> {1} quasi-perceptual conscious experience per se;
>
> {2} hypothetical picture-like representations in the mind and/or brain that give rise to {1};
>
> {3} hypothetical inner representations of any sort (picture-like or otherwise) that directly give rise to {1}. (Thomas, 1997/2021, enumeration orig.)

Hence, Thomas (1997/2021) points to the fact that ≪mental images≫ are debated upon in various different scientific disciplines, which range from sociocultural over psychological to neurological points of views (e.g., see also Abraham, 2020). Similarly, ≪visualisation≫ in mathematics comprises internal and external forms or can be regarded as a process or product. For example, Arcavi (2003) describes it as

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Arcavi, 2003, p. 217).

The products of visualisation, a "visual image", can be described as "a mental sign depicting visual or spatial information and inscriptions as symbols, diagrams, information on computer screens, or any external representation with a visual component" (Presmeg, 2020, p. 900; see also Arcavi, 2003; Clemens, 1982a, 1982b; Duval, 1999, 2017; Gutiérrez, 1996; Presmeg, 1986, 1992; Presmeg, 2006, 2020). Gutiérrez (1996) models visualisation furthermore with four elements, namely "mental images", "external representations", "processes", and "abilities of visualization". He defines a mental image as "*any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements*" (Gutiérrez, 1996, p. 9, emph. orig.). On the other hand, imagery need not necessarily be restricted to the visual, but could also refer to any of the five modalities "visual, auditory, tactile, gustatory, and olfactory" (Presmeg, 1992, p. 596).

Although visual media were sometimes accused of having no place in rigorous mathematics, their utility was emphasised conversely, too, for example, for the discovery of propositions and the generation of examples or counterexamples (Giaquinto, 2020). This is also the case for real and complex analysis:

> The utility of visual representations in real and complex analysis is not confined to such simple cases [graphs of functions; EH.]. Visual representations can help us grasp what motivates certain definitions and arguments, and thereby deepen our understanding. Abundant confirmation of this claim can be gathered from working through the text *Visual Complex Analysis* (Needham 1997). (Giaquinto, 2020, emph. in orig.).[7]

7 Needham's (1997) book is an important source for visualising the concepts from complex analysis. We will often refer to it in this thesis, in particular in Chapter 5 and Chapter 8.

Furthermore, a potentially useful distinction between several types of visual images is due to Presmeg (1986): A "visual image" is said to be a "mental scheme depicting visual or spatial information" (Presmeg, 1986, p. 42) and she classifies them into five categories, namely

- "[c]oncrete, pictorial imagery" ("pictures-in-the-mind"),
- "[p]attern imagery" ("pure relationships depicted in a visual spatial scheme"),
- "[m]emory images of formulae" (e.g., memories of formulae with special graphic features such as the root sign),
- "[k]inaesthetic imagery" ("imagery involving muscular activity"), and
- "[d]ynamic imagery" (moving imagery) (Presmeg, 1986, pp. 43–44).

These five types highlight a variety of manifestations of visual images. These can be used to specify the construct of ≪mental image≫ in research in mathematics education, for example, by identifying whether certain individuals prefer or predominantly use one of these types of visual images over another or whether certain mathematical topics give rise to preferences or predominant usages.

According to Weigand (2015), learners of mathematics build their own "mental objects", which go beyond merely pictorial representations. They

> represent not only phenomena producing them, but contain aspects that go beyond the appearance of these phenomena. Mental objects are not images in a photographic sense; rather, they represent a concept on the basis of certain properties and relationships between these properties. They can include linguistic, pictorial, and action-related representations. It is characteristic of *mental objects* that they are general and flexible or dynamic in that mental images refer to a particularly typical representative or to typical representatives – so-called prototypes – but that these can also be transferred to other representatives of the concept. (Weigand, 2015, p. 261–262, own transl.)

One of the important parts in this quotation is that Weigand (2015) does not limit mental objects to "images in a photographic sense", but that they can also pertain *linguistic, pictorial and action-related representations*. Moreover, they are regarded as representatives of mathematical concepts. On the other hand, Griesel et al. (2019, p. 125, own transl., emph. orig.) state very broadly that what they call mental constructs is "*everything that can be meant*", which is fabricated in thoughts. The authors consider it an open question though whether these mental constructs are limited to single persons or whether they could be abstract ideas, which can be potentially thought by multiple individuals (Griesel et al., 2019).

Another important factor for the conceptualisation of ≪mental images≫ and related constructs is language. For example, the German word "Vorstellung" is frequently used in this context (e.g., it is part of the German word "Grundvorstellung" for "basic idea", see Section 2.2.2). Unfortunately, it does not have a perfect translation into English. Rather, several translations are suitable in different contexts and vice versa. Accordingly, just like the word ≪mental image≫ the word ≪Vorstellung≫ leaves room for ambiguity, too, particularly if it is used without a clear conceptualisation (Roth, 2013; Wilzek, 2021). We illustrate this problem with a translation of a part of the German Wikipedia article on "Vorstellung":

> An imagination [Vorstellung] is a content represented in the mind that goes back to a perception or consists in a mental processing of perceived content. An imagination [Vorstellung] can result in something that is not present in reality or could even never be present;

> it is then also called an imagination [Imagination]. In the same way, however, imaginations [Vorstellungen] can also represent realistic future-related expectations, or they can be based on memories. Furthermore, they can be more vivid models of an abstractly given description; in this context, the question has been discussed whether ideas [Vorstellungen] play a role in language comprehension for the meaning of words and sentences (this was denied in Frege's philosophy of language, for example, but is widely assumed today). In contrast to the term or concept, which are structures permanently laid down in the mind (dispositions), ideas [Vorstellungen] (at least in the narrower sense) are concretely occurring phenomena in the mind.
>
> Insofar as the ideas [Vorstellungen] are based on previously experienced perceptions, they can be assigned to certain sensory modalities; a special role is played here by the visual (pictorial) imagination [(bildliche) Vorstellung] (which lends its name to the word "imagination" [Imagination], from the Latin imago "image"). In addition, imaginations [Vorstellungen] of other modalities are possible, such as of smells or tastes, of movement sequences, etc. Imaginations [Vorstellungen] can occur involuntarily, but they are often spoken of as a form of active mental simulation. An example of such a voluntary simulation is the "mental rotation" of an object, which is much studied in psychology and is based on visual and motor ideas [Vorstellungen].
>
> It is characteristic of an imagination [Vorstellungen] that it can exist relatively independently of attitudes to the thing in question: The concept of imagination [Vorstellungen] is neutral to whether the content in question is desired or not. In contrast to what one believes, something imagined does not necessarily have an effect on what one believes to be true or how one will act. (Wikipedia, 2021)[8]

On the basis of this translation, it should have become clear that the use of the German word "Vorstellung" ranges from "ideas" to "imagination" (as a process and product) with varying explanations.

Many other words come (or have already come) into play in the context of ≪mental images≫, for example, "imagery", "mental objects" (Weigand, 2015), "concept image" (Tall & Vinner, 1981; Vinner, 1983), "intuition" (Fischbein, 1987), or "Anschauung" (Buchholtz & Behrens, 2014; Mattheis, 2019; Wilzek, 2021), which is sometimes also translated with "intuition", too (Allmendinger, 2019; Mattheis, 2019). Based on a very broad literature review, Wilzek (2021) characterises words linguistically related to "Anschauung" in a refining way. For example, he describes "Anschaulichkeit" as the "[s]ubjective judgement as to whether suitable conditions for the formation of mental images are present", "Veranschaulichung" as the "[t]ransfer of an inaccessible area to an accessible area as a didactic means in order to increase Anschaulichkeit", or "intuition" as the "[i]mmediate and unquestionable knowledge, preparation for illumination, cognition of objects" (Wilzek, 2021, p. 28, own transl.).[9] In particular, for complex analysis, the second of these characterisations is very important: For reasons of dimension, graphs

8 The translation was done with the online translation tool DeepL (https://www.deepl.com/translator). We have chosen this tool here in order to illustrate the various possible translations of the word "Vorstellung" and its plural form "Vorstellungen" but not to bias the translation ourselves. We include the original German words at important points with the brackets [...].

9 Wilzek (2021) finally suggests the following working definition: "The term 'Anschauung' refers to a cognitive tool that relies on iconic sign processes or visual metaphors. On the continuum between formal and preformal modes of thinking and writing, Anschauung tends to belong to the latter" (Wilzek, 2021, p. 47, own transl.). On the other hand, in relation to Felix Klein's lecture on mathematics from a higher standpoint, Mattheis (2019, p. 97) argues that Felix Klein subsumes "a certain degree of intuition gained through experience" with the word "Anschauung", and Allmendinger (2016) describes that Klein's use of "Anschauung" comprises the use of geometrical representations, prototypical examples, and metaphors (see Allmendinger (2013, 2016, 2019) and Mattheis (2019) for more details).

of complex-valued functions of a complex variable cannot be visualised on a two-dimensional sheet of paper or as a three-dimensional model. Hence, other possibilities have to be found in order to increase their visual accessibility (see Appendix C for an overview of possibilities).

As might be expected, some authors use several of the previously mentioned keywords synonymously, while clearly distinguishing others. For example, Griesel et al. (2019, p. 125) distinguish between "*gedanklichen Konstrukt[en]*" (English: mental / theoretical constructs) and "*mentalen Representation dieses gedanklichen Konstrukts*" (English: mental representation of this mental / theoretical construct) on the one hand, but equate "*Begriff, Idee, Gedanke, Gedankengebilde*" (English: notion / concept, idea, thought, thought-form) with "*gedankliches Konstrukt*" (English: mental / theoretical construct) on the other hand.

Another view on ≪mental images≫ is that of an "umbrella category", which allows to capture a variety of phenomena: "Mental images [Vorstellungen] can be understood in the sense of an umbrella category, whose individual components have to be further differentiated" (Manderfeld, 2020, p. 19, own transl.). From this point of view, ≪mental image≫ is a keyword used to group a set sub-constructs together. For example, Manderfeld (2020, p. 19, own transl.) quotes Thompson (1992), who argues that a "'conception'" (which Manderfeld (2020, p. 19) translates with "Vorstellung" at this point) is a "'more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like'". She finally argues that "Vorstellungen" are a "cluster of internal representations and associations" and an umbrella category that subsumes "beliefs, emotions, and attitudes" (Manderfeld, 2020, p. 26, own transl.).[10]

Last but not least, some authors suggest to use the pair "concept" and "conception" to distinguish mathematical concepts from potential representations learners may acquire (e.g., Sfard, 1991; Simon, 2017; cf. Roth, 2013). For example, Sfard (1991, p. 3) suggests to use the word "'concept' [...] whenever a mathematical idea is concerned in its 'official' form" whereas "'conception'" is "the whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective 'universe of human knowing'". Simon (2017) argues that this distinction is underspecified. Instead, he proposes that a mathematical conception is "an explanatory model generated by mathematics educators to explain behaviours (including verbalizations) of mathematics learners in terms of what they think, know, and understand" (Simon, 2017, pp. 119–120). He goes on to define that "*[a] mathematical conception is an explanatory model used to explain observed abilities and limitations of mathematics learners in terms of their (inferred) ways of knowing*" (Simon, 2017, p. 120, emph. orig.). Thus, according to Simon's (2017) definitions, mathematical conceptions are researchers' constructions to explain learners' behaviour. Complementary, "*mathematical concepts consist of mathematical (mental) objects and the relationships among those objects*" (Simon, 2017, p. 121, emph. orig.), which are built up via the process of reflective abstraction. He also considers a mathematical concept as "*a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship*" (Simon, 2017, p. 123, emph. orig.). This, on the other hand, proposes a view on mathematical concepts which equates them with results from reflective abstraction built up by researchers or teachers, which are then intended to be built up by learners, too (this is comparable to the prescriptive aspect of "basic ideas"; Section 2.2.2).

Summing up, different authors conceptualised ≪mental images≫ (or syntactically related versions such as "mental imagery") in a variety of contexts and scientific disciplines, emphasising different aspects such as picture-like representations, subjective explanations, descrip-

10 To be precise, Manderfeld (2020) does not focus on ≪mental images≫ of mathematical notions here, but rather of the discipline of mathematics education.

tive interpretations, representations of mathematical concepts in individuals' minds, or simply as a keyword to subsume a set of related notions associated with this keyword. In one way or the other, learners' previous encounters of mathematical objects and what they have learned previously is considered to influence students' ≪mental images≫ (see also Bauersfeld, 1983; Biza, 2021; Kontorovich, 2018b; McGowen & Tall, 2010). Generally, however, there seems to be an agreement that picture-like entities belong to what is considered a ≪mental image≫. Nevertheless, this agreement may overshadow that some conceptualisations of ≪mental images≫ often include more than picture-like representations. Moreover, ≪mental images≫ can also contain prescriptive elements developed by teachers and which shall eventually be developed in learners in order to enable them to work with a mathematical concept in a sustainable way.

Now, having raised discourses about ≪mental images≫ in mathematics education from a rather general point of view, let us turn to the two conceptualisations of ≪mental images≫ we have announced at the beginning of this chapter.

2.2 TWO EXEMPLARY CONCEPTUALISATIONS OF ≪MENTAL IMAGE≫ IN MATHEMATICS EDUCATION

Concept images and *basic ideas* are two frequently used conceptualisations of ≪mental images≫. The first stems from the international and the second from the German speaking community of mathematics education, and are part in current research (see the references in the following sections). We will present them exemplary in order to illustrate more concretely the two perspectives addressed before, namely a normative and a descriptive point of view on ≪mental images≫ in mathematics education research.

2.2.1 *Concept images*

Concept images

The idea of *concept image* focuses on individuals' understanding of mathematical notions (Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989; Vinner & Hershkowitz, 1980). As opposed to basic ideas (Section 2.2.2), the notion of concept image is not prescriptive. Rather, research using the notion of concept image focuses on the idiosyncratic associations and ≪mental images≫ of persons doing mathematics:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (Tall & Vinner, 1981, p. 152)

According to this very broad definition, an individual's concept image comprises the "total cognitive structure", "properties", "processes", and "mental pictures" she or he associates with a mathematical notion. "Mental pictures", according to Vinner (1983, p. 293), are "the set of all pictures that have ever been associated" with a mathematical concept in an individuals' mind, and they include "any visual representation" and symbols.

Concept images are not stable and different concept images may be evoked at different times or under different circumstances (*evoked concept image*; Tall & Vinner, 1981, p. 152). They may cause so-called *cognitive conflicts* when two incongruous concept images are evoked simultaneously. Since concept images are evolving over time, single observations only allow to make claims about individuals' "temporary concept image[s]" (Vinner, 1983, p. 297). Stabilised concept images, that is, those that persist over time and can be evoked in different situations, are more difficult to identify and need to be explored beyond isolated observations.

The idea of *concept definition* is in contrast to that of concept image: It is "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). Hence, the notion of concept image is intended to describe individuals' cognitive structures associated to a mathematical notion, regardless of whether it has been previously defined for or by a learner. In particular, mathematics educators using the notion of concept image argue that many learners do not make use of a concept definition when working with a mathematical concept but rather rely on their concept images (Moore, 1994; Vinner, 1983, 1991/2002, 2020).

Similarly to the idiosyncratic nature of concept images, an individual may develop an own definition of a mathematical notion, which is then called a "*personal* concept definition" (Tall & Vinner, 1981, p. 152, emph. orig.). These may comprise definitions encountered in a lecture but do not necessarily have to, nor do they have to be endorsable in any way. In this sense, personal concept definitions are definition attempts, which may or may not be endorsed by others, in particular not by the authorities teaching the concept. It is not unanimous in mathematics education literature though whether an individual's personal concept definition is conceptualised as part of her or his concept image (see the summary by Feudel, 2020, ch. 3.1).

Critical reception of the notion of concept image

Since the conceptualisation of concept images is very broad, it leaves a lot of room for investigating learners' idiosyncratic ≪mental images≫ of mathematical concepts. In this sense, it could in principle be used in mathematical education research at the tertiary level (see e.g., Nardi, 2006; Nardi & Iannone, 2003). For example, it has been used in exploratory studies on mathematical notions, which had not previously been addressed in mathematics education research before or only to a limited extent (e.g., Hamza, 2012; Hamza & O'Shea, 2016; Martin, 2013; Melhuish et al., 2020). Nevertheless, it is not entirely clear from an ontological point of view how broad the defining utterances "total cognitive structure" or "mental pictures" (Tall & Vinner, 1981, p. 152) are to be understood and how they are to be identified. In particular, we wonder what is *not* part of an individual's concept image of a mathematical notion. Since concept images are defined as a cognitive structure, we consider it difficult to conceptualise them for empirical research: All we can potentially observe in empirical research are products of activities of communication and therefore all we can potentially observe is already conceptually different from a cognitive structure. Additionally, mathematics education research focusing on individuals' concept images, omitting socio-cultural circumstances, did not remain without criticism. Bingolbali and Monaghan (2008) therefore introduced a socio-cultural layer to the idea of concept image and showed that students' departmental affiliations may influence mathematics students' concept images of the derivative.

Last but not least, even though research using concept images is mainly descriptive, it is often carried out in a deficit oriented way. That is, concept images, which are not endorsed as sustainable or compliant with a concept definition, are sought to be replaced by "accepted, correct scientific views" (Schwarz, 2020, p. 697). This also presupposes a rather platonistic view on mathematics, according to which mathematically correct statements can be identified and potentially transformed into learners' concept images. Such a point of view on mathematics has been challenged in the literature several times (cf. e.g., Lakatos, 1976/2015; Linnebo, 2017; Sfard, 2008; Wittgenstein, 1964/1967, 1953/2009) and is not used further in this thesis.

2.2.2 *Basic ideas*

Basic ideas

In German mathematics education, the notion of *basic ideas*, also called *basic mental model* (German: Grundvorstellungen) has a long tradition and receives continuous attention (e.g., Bender, 1991; vom Hofe, 1992, 1995, 1998; vom Hofe & Blum, 2016; Kleine et al., 2005). It stems from German subject matter didactics and emphasises applications, the constitution of meaning, as well as intuition (German: Anschauung) (Hefendehl-Hebeker, 2016; Hußmann et al., 2016). One of the main intentions is that learners should develop basic ideas for mathematical concepts in order to deal with these concepts in a meaningful way, such as recognising them in real-life contexts or applications (vom Hofe, 1995, 1998; Salle & Clüver, 2021). Basic ideas are located in a “psychological-constructivist paradigm” (Ullmann, 2015, p. 13, own transl.) and have mainly been used in mathematics education for school mathematics. Basic ideas have been used to research learners’ concept formation of many different mathematical concepts ranging from the basic arithmetic operations, over fractions, the notion of function, continuity, differentiability, and integration in real (school) analysis but also at beginning tertiary level, eventually producing guidelines for teaching, and also for quantitative research (e.g., Bender, 1997; Büchter & Henn, 2010; Feudel, 2020; Feudel & Biehler, 2021a; Greefrath et al., 2016a, 2016b, 2021a, 2021b, 2022; vom Hofe, 1995, 2003; vom Hofe & Blum, 2016; Klinger, 2018, 2019; Padberg & Wartha, 2017; Rembowski, 2015a, 2015b; Roos, 2020; Rütten, 2016; Salle & Clüver, 2021; Schäfer, 2011; Tietze et al., 2000; Wartha, 2007; Wessel, 2015).[11]

Basic ideas have been characterised in various similar but not entirely unanimous ways. On the one hand, they are based on their use and relevance for learners, hence, focusing on a prescriptive facet, and on the other hand, they may be regarded as certain mental entities learners have developed.[12]

Prescriptive facet of basic ideas

Prescriptive facet of basic ideas

Greefrath et al. (2016a, p. 101) describe a basic idea as “a conceptual interpretation that gives it [a mathematical concept; EH.] meaning”. Accordingly, basic ideas function as “mediating elements or as objects of transition between the world of mathematics and the individual conceptual world of the learner” (vom Hofe & Blum, 2016, p. 231). They were also characterised as “an *idea* one simply has to get, to understand what the related mathematical content is essentially about and to make appropriate use of it” (Vohns, 2016, p. 217).

Vom Hofe (1992, 1995, 1998) states three features of basic ideas: They

> can be used to describe *relations between mathematical contents and the phenomenon of the individual generation of concepts*, characterizing three aspects of this phenomenon:

11 For a comparative discussion of basic ideas and concept images see Greefrath et al. (2016a, 2021a), vom Hofe and Blum (2016), and Klinger (2018, 2019). For example, vom Hofe and Blum (2016) claim that normative and descriptive basic ideas are instances of concept images. In contrast, Greefrath et al. (2016a, p. 103) argue that individual basic ideas (see below) are “part of a valid concept image”, a concept image “may contain” individual basic ideas, and a concept definition “is a specific realization” of so-called *aspects* of mathematical concepts (see Section 6.3.1).

12 Complementary to the notion of basic ideas, Greefrath et al. (2016a, 2016b) introduced the notion of *aspects* of mathematical concepts to German subject-matter didactics. Roughly speaking, aspects relate to basic ideas like concept definitions relate to concept images (Greefrath et al., 2016a, pp. 102–103). Since this chapter deals with the discourses on ≪mental images≫, the discussion of aspects will be postponed to Section 6.3 where it functions as a point of departure for the epistemological analysis of complex path integrals (Chapter 8, Chapter 9). Both notions, basic ideas and aspects, will be exemplified for Riemann integrals and used as points of references for the rest of the thesis (Section 2.2.3, Section 6.3.2).

1. *constitution of meaning of mathematical concepts* based on familiar contexts and experiences,
2. *generation of corresponding (visual) representations* which make operative thinking (in the Piagetian sense) possible,
3. *ability to apply a concept to reality* by recognizing corresponding structures in real-life contexts or by modelling a real-life situation with the aid of the mathematical structure. (vom Hofe, 1998, p. 320, emph. orig.)

In this sense, basic ideas of a mathematical notion are prescriptive ideas, which may be used by teachers to structure teaching-learning situations around a mathematical concept. They are "categories, which are constructed by researchers" (Salle & Clüver, 2021, p. 576, own transl. see also Hanke, 2017; Hanke & Schäfer, 2017; Roos, 2020). The general idea is that while learning, learners build their own mental images associated to mathematical concepts, and the teachers' task is to guide them in building up basic ideas (vom Hofe, 1995; vom Hofe & Blum, 2016; Weigand, 2015). Hence, basic ideas may ideally serve as guidelines for teachers or mathematics educators to organise their teaching around meaningful ideas. As such, basic ideas are usually grounded in epistemological analyses of the mathematical notion at hand and are subject to endorsement by a community of mathematics education researchers (Salle & Clüver, 2021). While the characterisation of basic ideas in the quote above mentions (visual) representations, what is considered a visual representation may also include enactive, iconic, or symbolic representations (cf. Salle & Clüver, 2021). Rütten (2016) moreover places basic ideas at the crossroads of cognitive science and linguistics and interprets them as "conceptual metaphors".

It is not implied, however, that basic ideas are necessarily stable and unchangeable. Indeed, they are subject to potential refinement or rejection when mathematical concepts are expanded or abstracted. For example, the basic idea of multiplication as repeated addition may be appropriate in the context of natural numbers but loses its sustainability in the context of rational numbers (Salle & Clüver, 2021, p. 556). In this context, Roos (2020, p. 24, own transl.) refers to basic ideas, which are endorsable only in a restricted context, as "partial basic ideas".

Descriptive facet of basic ideas

Descriptive facet of basic ideas

Complementing the aforementioned prescriptive facet of basic ideas, there is also a *descriptive* aspect of basic ideas. That is, students' individual mental images—"*individual learners' mental images*" (Salle & Clüver, 2021, p. 558, own transl., emph. orig.)—are taken into account, too. Depending on the author(s), these individual mental images are sometimes also called "mental models" (Griesel et al., 2019; vom Hofe & Blum, 2016), which students have actually developed, for example, after having been exposed to the mathematical concept in class. As such, the research discourse on basic ideas aims to integrate the prescriptive view with a descriptive view on mathematical concept formation by learners. As a consequence, individual images are often studied from a comparative, subsuming perspective, which is also occasionally described as deficit oriented (cf. Weber, 2007).

The conceptualisation of the descriptive facet of basic ideas is not entirely consistent though. It is not unanimous what these individuals' mental images are thought to be. Some authors argue that these are "specific manifestations of normative [basic ideas; EH.] in a person" (Greefrath et al., 2021a, p. 650). More precisely, Greefrath et al. (2021b) describe this duality of basic ideas as prescriptive and descriptive at the same time succinctly as follows:

> - *Normative basic ideas* describe what people should generally and ideally imagine with a mathematical concept. This results from subject-didactic analyses of the corresponding concept. The development of such basic concepts in learners is one of the goals of mathematics education; they can provide orientation for the design of teaching-learning processes.
> - *Individual basic ideas* are the actually existing instances of normative basic ideas in an individual. They result from personal learning processes. Different people can therefore differ with respect to which individual basic ideas they have about a mathematical concept. (Greefrath et al., 2021b, p. 5, own transl., emph. and bullet points orig.)

Other authors describe that "[a]nalyzing individual images gives information about the individual explanation models of the students which are integrated into the system of his experiences [...] and which, therefore, can be activated" (vom Hofe, 1998). In this context, a psychologically orientated characterisation of individuals' mental images is that they are

> - *specific mental objects* which show particularly structural and functional aspects of a mathematical subject;
> - *dynamic objects* which can change and develop by new experiences in the course of time;
> - *elements of a cognitive net* in which single ['basic ideas'; EH.] are not isolated but are in correlation to others. (Kleine et al., 2005, p. 229, emph. and bullet points orig.)

Henceforth, there is no agreement as to whether individual (mental) images are individualised versions of normative basic ideas or simply ≪mental images≫ of a mathematical concept, however this may then be conceptualised explicitly.[13]

In any case, this dual perspective on basic ideas is intentional and allows for a constructive interplay between the normative and the descriptive view of learners' ≪mental images≫. Hence, basic ideas address the tension between "theory-free practice" and "practice-free theory" (Lensing, 2021, p. 69, own transl.).[14]

It remains the researcher's task to reconstruct these individual images from empirically gathered data and to compare these reconstructions with prescriptively formulated basic ideas. For example, in quantitative studies, the test subjects answering a questionnaire might choose their answers from a set of possibilities the research team has previously tailored to normative basic ideas (e.g., Greefrath et al., 2021a, 2021b, 2022). This dual view of ≪mental images≫, namely in terms of prescriptions and descriptions, makes it possible to carry out empirical studies on mathematical topics for which there are already enough research results available in order to endorse prescriptive basic ideas (Griesel et al., 2019; Salle & Clüver, 2021; Sträßer, 2020).

13 Yet another view is proposed by Wessel (2015), following Wartha (2007). Wessel (2015) identifies basic ideas with the psychological notion of mental models. As such, individuals' ≪mental images≫ are not the same as individual images (as manifestations of basic ideas) because they possess too concrete properties, which are not important for characterising individuals' manifestations of basic ideas. Also, an (individuals') basic idea is considered as an abstraction of a class of individuals' ≪mental images≫ (German: Vorstellungen), getting rid of superfluous properties; in other words, it is "*the commonality of all mental images*" (Wessel, 2015, p. 7, own transl., emph. in orig.).

14 There is also a third facet of basic ideas, namely the *constructive* aspect. It takes into account how basic ideas can be taught to students and which teaching-learning materials may be appropriate. This facet asks for possible reasons for deviations of individual images from prescriptive basic ideas and aims to provide solutions to such deviations (vom Hofe, 1995, 1998; vom Hofe & Blum, 2016; Salle & Clüver, 2021).

Critical reception of the notion of basic ideas

Despite its popularity and usefulness for school mathematics as we described, the notion of basic ideas has also received some critical reception. It was criticised that the notion of basic ideas does not account for how humans actually learn mathematical concepts, for having little grounding in theories of mathematics learning, and for not taking into account social factors for the learning of mathematics (Ullmann, 2015). Other authors ask to what extent it is justified from an ontological perspective to combine prescriptive and descriptive facets in one construct (Bergsten, 2020; Hanke, 2016, 2017; Hanke & Schäfer, 2017; Rembowski, 2015b; Salle & Clüver, 2021; Vohns, 2016). Particularly, it has been questioned how mental images can be both "mental objects and [..] prescriptive didactical constructs for prototypical metaphorical situations" at the same time (Bergsten, 2020, p. 501). Depending on the authors, it seems that basic ideas are often described primarily in terms of one of the facets (prescriptive or descriptive) and the other is then regarded as complementary.[15]

In Lensing's (2021, ch. 2.3.3) opinion, a weakness of the notion of basic ideas is that its comparative perspective between norm and reality suggests an oversimplification of the complexity of mathematics learning. Moreover, he argues that the notion of basic ideas lacks epistemological means to find out what individual basic ideas actually are and how they can be identified (Lensing, 2021, ch. 2.3.3). This is due to the fact that basic ideas are predominantly characterised with properties they *should* have and the descriptive facet is added for reasons of complementarity, which is in turn little conceptualised.

We have seen that the orientation towards prescriptive basic ideas is a useful approach for teaching practice and mathematics education research. However, such a comparative perspective is not appropriate in the first place when one wants to find out about the potential plethora of individuals' ≪mental images≫ of a mathematical notion (Lensing, 2021, ch. 2.3.3). Whereas comparisons between individuals' ≪mental images≫ may identify commonalities, it is neither guaranteed that these qualify as prescriptive basic ideas, nor do infrequent individuals' ≪mental images≫ disqualify as prescriptive basic ideas per se. We conclude that the application of the notion of basic ideas is less suitable for mathematics education research about a mathematical topic which has not been studied intensively before: On the one hand, individual images have not been elicited before or only very little, and on the other hand, it is not clear yet which potential prescriptive ideas prove to be basic ideas.

Even though the notion of basic ideas is inspiring for this thesis, there are several reasons why we will not use it explicitly in our empirical study. First of all, the characteristic features of basic ideas were mainly developed for school mathematics and it is not clear yet how they relate to important goals in tertiary mathematics education such as the transition to formalisation and deductive reasoning (cf. Clüver & Salle, 2020; Sfard, 2014; Tall, 2008). While we consider one of defining features of basic ideas very important in tertiary mathematics education, namely to constitute meaning of the mathematical notion at hand by connecting it to individuals' previous experiences, we question that the generation of visual representations and applications to reality are appropriate and serve as mandatory features of ≪mental images≫ for tertiary mathematics. In particular, we wonder what "operative thinking" with "(visual) representations" (vom Hofe, 1998, p. 320, emph. omitted) means for tertiary mathematics.[16] For instance, we have described before that graphs of complex functions are contained in $\mathbb{C}^2 \cong \mathbb{R}^4$,

15 We note though that some authors stress that the normative aspect of basic ideas is the historically prevalent and most characteristic (cf. e.g., Hefendehl-Hebeker, 2016; Salle & Clüver, 2021; Sträßer, 2020).

16 In this context, *primary* and *secondary* basic ideas have been distinguished. The former address learners' lives directly while the latter refer to basic ideas which are based on specifically mathematical representations (e.g., Clüver

hence, it is hardly possible to directly visualise complex path integrals.[17] Third, as partly indicated above, little research on complex path integrals in mathematics education is available so far (Section 5.4), which means that we do not have enough evidence in the literature which potential interpretations of complex path integrals are endorsable and pertinent to be taught to students. In particular, the comparative perspective between prescriptive basic ideas and descriptive individuals' basic ideas cannot be put to practice. Last but not least, conceptualising individuals' mental images as mental entities implies a general impossibility to access them (Lensing, 2021, ch. 2.3.3). What we can access are the communicative simulacra of what individuals may consider their mental images of complex path integrals and these seem to be most relevant to us because everything that is hidden and not communicable is hardly useful for potential teaching and learning scenarios (Hanke, 2016, 2017; Hanke & Schäfer, 2017).

2.2.3 *Basic ideas of definite Riemann integrals*

We will now recall the basic ideas for Riemann integrals for further reference.

Let $f\colon [a, b] \longrightarrow \mathbb{R}$ $(a < b)$ be a (bounded integrable) real-valued function of one real variable. Greefrath et al. (2016a, 2016b) identify four basic ideas of definite Riemann integrals (see also Bender, 1990; Büchter & Henn, 2010; Danckwerts & Vogel, 2005, 2006; Tietze et al., 2000). These basic ideas were initially developed for mathematics teaching at school, but they have also been used in undergraduate mathematics education research (Greefrath et al., 2021a). Clearly, these basic ideas overlap with ideas from international research about ≪mental images≫ of Riemann integrals (e.g., Bezuidenhout & Olivier, 2000; Jones, 2013, 2015, 2018; Kouropatov & Dreyfus, 2013, 2015; Mahir, 2009; Orton, 1983; Pino-Fan et al., 2018; Pino-Fan et al., 2017; Rasslan & Tall, 2002; Sealey, 2014; Serhan, 2014; Souto Rubio & Gómez-Chacón, 2011; Thompson, 1994; Thompson & Silverman, 2008; see also the survey by Greefrath et al., 2021a). However, we find that the subdivision into the four basic ideas provides a suitable overview and point for later reference.

The four basic ideas for Riemann integrals described as follows (Greefrath et al., 2016a, pp. 116–121; Greefrath et al., 2016b, pp. 244–254):

Area image of Riemann integrals

(1) The *area image (basic idea of area)* describes the definite integral as the area enclosed by the graph of the function, the x-axis, and the two vertical line segments joining a and b and the graph such that the parts lying above the x-axis are weighted with positive sign and the parts lying below with negative sign (Figure B.1b).

(Re)construction image of Riemann integrals

(2) The *(re)construction image (basic idea of (re)construction)* is the counterpart to the interpretation of the derivative as a rate of change. It asserts that a definite integral can be used to reconstruct an original quantity from its rate of by means of integration.

Average image of Riemann integrals

(3) The *average image (basic idea of average)* asserts that the definite divided by the length of the domain of f,

$$M = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x,$$

& Salle, 2020; vom Hofe, 1995; vom Hofe & Blum, 2016; Salle & Clüver, 2021). However, it is not clear to us how secondary basic ideas fit to tertiary mathematics education.

17 In our epistemological analysis in Chapter 8, we will explore which visualisations are used in relation to complex path integrals more detailed.

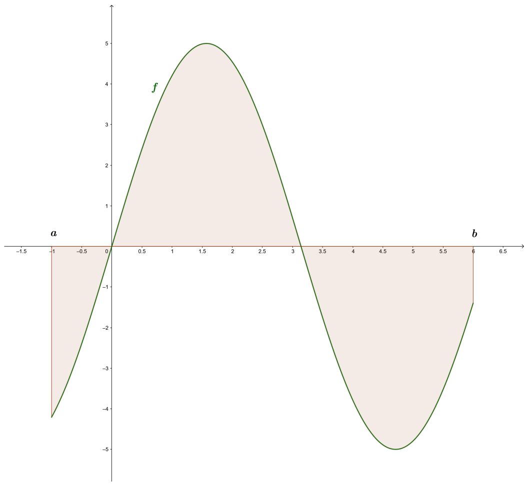

Figure 2.1: A Riemann integral as an oriented area.

is the mean value of the function over its domain. In particular, the Riemann integral of the constant function $[a, b] \longrightarrow \mathbb{R}, x \longmapsto M$, and of f agree:

$$M(b-a) = \int_a^b M \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x.$$

Moreover, in case f is continuous, the mean value theorem guarantees that M is the value of f at some point in $[a, b]$ (Greefrath et al., 2016b, p. 251; Forster, 2016, p. 213).

One can also argue that M is approximated by the mean values of the values $f(\xi_1), f(\xi_2), \ldots, f(\xi_n)$, where each ξ_k is chosen in the kth subinterval of an equidistant partition of $[a, b]$ into n subintervals ($n \in \mathbb{N}$). Letting $n \longrightarrow \infty$, these mean values converge to M, that is, $\frac{1}{n} \sum_{k=1}^{n} f(\xi_k) \longrightarrow M$, which follows basically from the definition of the Riemann integral (Forster, 2016, §18).

Accumulation image of Riemann integrals

(4) In the *accumulation image (basic idea of accumulation)*, integration of f is regarded as a process of accumulating quantities (values of f) in a continuous manner, in the sense of replacing sums when the number of addends gets large, tending to infinity (see also Kouropatov & Dreyfus, 2015). In this sense, "a definite integral is the result of summing up small addends with a product structure" (Greefrath et al., 2021b, p. 6).

The importance of these basic ideas of definite integrals is that they can be used to connect the definite integral to other mathematical topics with which the learners may have (had) experience. For example, the area image may be used to establish a relationship between integrands f and "area functions" $x \longmapsto \int_a^x f(t) \, \mathrm{d}t$ and thus to guide teachers in substantiating the fundamental theorem of calculus (Theorem B.3): In this context, Kirsch (2014) developed a visual substantiation of the portion of this theorem, which asserts that the area function $x \longmapsto \int_a^x f(t) \, \mathrm{d}t$ is a primitive of f (if f is continuous) and Swidan (2020) uses plots of similar "Riemann accumulation functions" in a digital tool to enhance students' learning of the fundamental theorem (see also Swidan & Fried, 2021; Swidan & Naftaliev, 2019; Swidan & Yerushalmy, 2014).

Note that these basic ideas do not explicitly refer to pictures. However, pictures of the graph of a function in which the area below the graph is coloured as in Figure B.1b are suitable for explaining the basic ideas of area. Similarly, the mean value M in the basic idea of average could be presented together with a picture of the graph of f, as just proposed for the area image, together with the set $\{(x, y) \in \mathbb{R}^2 : x \in [a, b], y = M\}$, that is, a horizontal line segment at $y = M$ over the interval $[a, b]$ (e.g., Greefrath et al., 2016b, p. 251). Hence, we see that basic ideas are not simply pictures (understood externally or internally) but that visible entities may be useful to explain basic ideas. Also, the learner is not explicitly part of the description of the basic ideas. Their suitability and relevance of each basic idea for learners has to be inferred from the use of the basic ideas in context.

2.2.4 *A* story-telling view *on basic ideas*

Basic ideas as mathematical stories endorsed by mathematics educators for specific didactical purposes

In this section, we will briefly suggest a point of view on basic ideas, which anticipates how we will conceptualise *discursive mental images* in Chapter 4 for the remainder of the thesis.

The view taken here is that a basic idea neither is a mental entity nor is it necessarily based on a visual representation. Rather, one can regard a basic idea as a special, consistent *mathematical story* about a mathematical notion, which may be taken as a reference for teaching the mathematical notion, which is subsumed under a keyword, and which is endorsed by a community of mathematics educators for the specific didactical purposes we discussed in Section 2.2.2.

For example, for the basic idea of area, the keyword is exactly this, "basic idea of area", and an suitable mathematical story is "The definite integral of the function f is the measure of the area that is enclosed by the graph of f, the horizontal x-axis, and the vertical axes at $x = a$ and at $x = b$, such that the area above the x-axis is taken positively and the area below is taken negatively." A visually perceptible entity (e.g., Figure B.1b) may be used to accompany this story.

The suitability and relevance for a given audience and the endorsability of the story told in a basic idea in a mathematics education community depends on the context and learners and is therefore subject to change when this context changes. For example, in calculus in upper secondary schools, where mostly integrals of continuous functions are covered, other stories would be endorsed than in a lecture on complex analysis, in which the lecturer recalls Riemann integrals only to use them in a definition of complex path integrals. In the first case, the story we just told was already endorsed as a basic idea, while in the second case, this story may be useful as a reminder, but is hardly relevant for the context of complex analysis.

Practitioners may use basic ideas to create mathematical discourses (e.g., in a mathematics classroom), in which the students can construct and endorse similar narratives (Sfard, 2008). For instance, they could substantiate that $\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0$ by drawing conclusions based on the basic idea of area: They may argue that the signed area corresponding to this integral consists of two congruent parts "cancelling each other" because one of these parts is above and the other is below the horizontal axis. In this sense, the story told in the area image provides us with a *rule* for drawing conclusions about Riemann integrals. Again, no picture *on its own* characterises the basic idea and its use (neither Figure B.1b nor any picture showing the graph of the sine function)—it is the specific *story* about Riemann integrals we used to substantiate the equation $\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0$.

We did not need to recur to cognitive entities at all, but we were still able to provide a useful point of view on the basic idea of area.

2.3 TOWARDS A DISCURSIVE PERSPECTIVE ON ≪MENTAL IMAGES≫

All the conceptualisations so far (except for the sketch in Section 2.2.4) relied on an acquisitionist point of view on mathematics learning (see Cobb, 2006; Sfard, 2006, 2009a). They acknowledged that ≪mental images≫ connect mathematics to individuals' previous experiences or real life, are being built up in one way or another, and express individuals' conceptions of a mathematical notion (cf. Simon, 2017). Through the lenses of both, the notion of basic idea and the pair concept image / concept definition, researchers have studied ≪mental images≫ from a *regular* (i.e., an endorsed set of ≪mental images≫ should be taught to or activated in learners) and *singular* perspective (i.e., individuals' idiosyncratic interpretations or representations of mathematical notions) (cf. Weber, 2007).

Critical voices in mathematics education literature have argued that the acquisitionist conceptualisation ≪mental images≫ such as basic ideas and concept images suffer from a poorly developed learning-theoretical foundation as well as from an ontological ambiguity (cf. Lensing, 2021; Roth, 2013; Sfard, 2008; Ullmann, 2015). Although some research extended the research on ≪mental images≫ to groups of individuals and aimed to identify social factors for the formation of individuals' ≪mental images≫, a potential embedding into a participationist learning theory is still due. For example, Ullmann (2015, p. 17, own. transl.) argues that basic ideas are "a rhetorical tool that serves the legitimation and accentuation of the subject in the discourses of teaching and learning mathematics" and suggests to "uncover their embeddedness in socio-cultural practises".

We also addressed before that it has not yet been clarified what might be constitutive features of ≪mental images≫ for tertiary mathematics such as complex path integrals. For example, it is questionable whether visual representation are still relevant or even constitutive for potential interpretations of these integrals. More generally, we wonder whether we could find a set of properties characterising ≪mental images≫ for tertiary mathematics similar to the three properties characterising basic ideas for school mathematics cited in Section 2.2.2 (vom Hofe, 1992, 1995, 1998).

Additionally, even in case we found individuals' ≪mental images≫ of complex path integrals, we would still need to endorse them before we could count either of them as a basic idea. Rather, it seems relevant to us to explore more generally what experts' individual interpretations in complex analysis look like without fixing properties of ≪mental images≫ beforehand in a conceptualisation of ≪mental images≫; in other words, we favour a disobjectification of ≪mental images≫ (see Chapter 3; Sfard, 2008).[18]

Regarding the *empirical ascertainability*, communication is of utmost importance. Previously, ≪mental images≫ were usually conceptualised as invisible cognitive entities and researchers had to derive conclusions about these alleged cognitive constructs by interpreting data (e.g., by comparing data with prescriptions).

Communicative simulacra of mental images

One way to align cognitive conceptualisations of ≪mental images≫ with their empirical ascertainability was proposed by Hanke (2017) and Hanke and Schäfer (2017). They suggest to exclusively study the "communicative simulacra" of ≪mental images≫, that is, the utterances and other perceptible entities individuals make publicly available when they talk about or use their ≪mental images≫. The authors suggest to differentiate between "exclamatory" communicative simulacra as well as the acceptance of other persons' communicative simulacra in terms of "usage" and "acceptance" (Hanke, 2017; Hanke & Schäfer, 2017). Communicative simulacra

18 This is a very general and broad task, which we will not accomplish in one study alone. We will provide an initial step for the case of complex path integrals.

of ≪mental images≫ are called exclamatory if the individual produces them as a response to direct quests for her or his ≪mental images≫ of the mathematical concept. The second and third layer, acceptance of usage and acceptance of attitude, were inspired by the method of "acceptance interviews / surveys" from physics education (Jung, 1992; Wiesner, 1995; Wiesner & Wodzinski, 1996).

All of the following considerations of a discursive perspective on ≪mental images≫ in this thesis can be understood as subsequent refinement of the ideas of communicative simulacra just mentioned. However, in the rest of the thesis, we continue to abandon an acquisitionist perspective on ≪mental images≫ further. Instead, we ground our discursive conceptualisation of ≪mental images≫ in the discursive, participationist framework of commognition (Lavie et al., 2019; Sfard, 2008).

In the next subsections, we review additional criticism of the use of ≪mental images≫ (more precisely, ≪mental representations≫) in mathematics education by Roth (2013) and propose an alternative view on ≪mental images≫ we infer from the philosophy of Wittgenstein (1953/2009).

2.3.1 *Critique of ≪mental representations≫ in mathematics education*

Roth's criticism on mental representation

Roth (2013) considers the use of constructs such as ≪meaning≫ and ≪mental representation≫ of mathematical objects in mathematics education superfluous when these constructs are designed in such a way that they merely point to alleged invisible mental structures. This criticism stems

(1) from the effort to replace conceptualisations of cognitive and invisible entities in favour of directly empirically ascertainable constructs;

(2) from the assumption that doing mathematics is an inherently social activity;[19] and

(3) from the idea to deconstruct (in other words, to "disobjectify"; Sfard, 2008) discourses about ≪mental images≫, ≪meaning≫, or ≪thinking≫ towards discourses, in which researchers pay closer attention to research participants' use of words, which they associated with cognitive concepts in everyday language themselves (cf. Wittgenstein, 1953/2009).[20]

Roth's (2013) suggests to not use everyday concepts like ≪meaning≫ and ≪mental representation≫ unquestioned as constructs in mathematics (and science) education (research) if

19 In Sfard's (2009b, p. 56) words, the participationist perspective on learning due to Vygotsky (1978, 1987) can be summarised by saying that learners "gradually become a competent participant, and eventually a modifier of historically established patterned forms of activity [… . The individualisation of these activities is; EH.] a gradual transition from being an only marginally involved follower of other people's implementation to acting as a competent participant, with full agency over the activity".

20 With this regard, Vossenkuhl (2003) acknowledges Wittgenstein's (1953/2009) philosophical position that

> [w]hat we understand we understand through language alone. We think linguistically and only linguistically, and not logically. And what we think when we think linguistically is shown only by the sentences we use, not by any underlying form. (Vossenkuhl, 2003, p. 138–139, own transl.)
>
> […]
>
> For if the use of a word is, as Wittgenstein now succinctly says, its meaning (PhG 60), any further logical-analytical investigation is superfluous. In any case, it then no longer makes sense to look for structures of meaning below the visible and audible use of words and sentences (Vossenkuhl, 2003, p. 142, own transl.).

this usage suggests that there is *something behind* mathematical concepts, which needs to be recognised in whatever way by people doing mathematics. Likewise, such an approach is not pedagogically expedient, since no implications on the learning of mathematics could be derived from it. (Roth, 2013). Roth's (2013) perspective is influenced by socio-cultural and linguistic philosophical thinkers like Wittgenstein and Vygotsky and draws on ideas from discursive psychology or cultural-historical activity theory (cf. e.g., Barwell, 2013b; Harré & Gillett, 2010; Potter, 2012; Roth & Radford, 2011; te Molder, 2015; Wiggins & Potter, 2017).[21]

Instead of taking "what participants say and do as documents (manifestations) of something else that in itself is inaccessible" (Roth, 2013, p. 188), he proposes to

> retain the concepts of ≪mental representation≫ or ≪conception≫ by qualifying them as a family of expressions or statements that pertain to a particular topic. Thus, rather than thinking about ≪mental representation≫as being something *actually* in the heads of people, it is used to denote the culturally possible ways of talking and making statements about a pertinent phenomenon. (Roth, 2013, p. 188)

Thus, the point of view here has shifted from attributing mental states to individuals to talking about these alleged mental states as a cultural phenomenon. In this regard, an interlocutors' utterances are not exclusively attributed to the speaker but to the discursive situation at whole. The utterances are not categorised according to ≪conceptions≫: "discursive psychologists do not attribute stretches of talk to one underlying ≪conception≫; talk is no longer taken to be a neutral means for reading out and making public what is in the speaker's head" (Roth, 2013, p. 103). In this line of argument, data collection and analysis should not be used for subsumptive reasons. This is especially important when—just as in our case in this thesis—the research participants' utterances cannot be grouped into previously established categories of ≪mental images≫. The underlying idea here can rather be described as "anti-cognitivist, in that it does not seek to develop models of individual cognition (e.g. as a kind of information-processing activity), being more interested in how participants themselves 'model' the cognitive processes of their interlocutors" (Barwell, 2013a, pp. 210–211) and themselves.

People can only come into contact with mathematical objects, or more precisely with what is called mathematical objects, through what they find in communication and artifacts of physical reality. If there really was a ≪meaning≫ etc. behind what is communicated, then it is questionable how it can be conveyed to others or how it could be inferred from an utterance (e.g., we could ask how mathematicians could put a ≪meaning≫ into the mathematical objects in a definition or conversely extract that ≪meaning≫ from an already given definition). Anything one *should* find that one should have of a mathematical concept could therefore only be known at all if one has left them in a form that can be made available to others. Consequently, if there

21 While we will not use discursive psychology explicitly further, we still think that a description of its main tenets fits to the rationale of our thesis. According to Barwell (2013b, p. 599), discursive psychology deals with "the investigation of cognition that started from the everyday, situated production of accounts of mental processes, and that recognized the fundamental role of human interaction in producing these accounts". Te Molder (2015) describes this line of research as

> the study of psychological issues from a participant's perspective. It investigates how people practically manage psychological themes and concepts such as emotion, intent, or agency within talk and text, and to what ends. Rather than revealing psychological phenomena in the laboratory, it looks at how psychology is put to use in everyday lives by people themselves.

We consider our conceptualisation of discursive mental images in Chapter 4 and our empirical study to serve exactly the same idea, namely to study how experts put into practise what they view as their mental images or intuitive interpretations of complex path integrals.

was meaning consisting of more than is conveyed through communication, then every effort to arrive at insights about it would be futile, directly due to its inherent property of not being conveyable through communication. Therefore, the questioning of metaphysical constructs behind mathematical concepts is circular and should better be abandoned.

While Roth (2013) goes on to propose an approach to his deconstructed interpretations of ≪meaning≫, ≪mental representations≫, and ≪conceptions≫ of mathematical objects with a melange from discursive psychology and cultural-historical activity theory, we will look at experts' ≪mental images≫ on complex path integrals and related questions with the commognitive framework. We choose this path because the commognitive framework provides clearly conceptualised aspects of mathematics as a discourse, which we can adapt to our needs in Chapter 4.

2.3.2 *A Wittgensteinian take on ≪mental images≫*

Wittgenstein: ask for the use of the word "mental image"

In this subsection, we recall some of Wittgenstein's (1953/2009, §§363–427) ideas on the ontology of ≪mental images≫. He does not consider ≪mental images≫ as individuals' own images of a concept but rather as what can be expressed about this keyword in language (von Savigny, 2011; Scholz, 1998). These ideas are relevant because the commognitive framework (Sfard, 2008, in particular) contains some of Wittgenstein's ideas alongside those of Vygotsky (see Chapter 3).

According to Wittgenstein (1953/2009),

> [o]ne ought to ask, not what images [Vorstellungen] are or what goes on when one imagines something, but how the word "imagination" [Vorstellung] is used. But that does not mean that I want to talk only about words. For the question of what imagination essentially is, is as much about the "imagination" [Vorstellung] as my question. And I am only saying that this question is not to be clarified – neither for the person who does the imagining, nor for anyone else – by pointing; nor yet by a description of some process. The first question also asks for the clarification of a word; but it makes us expect a wrong kind of answer. (Wittgenstein, 1953/2009, §370)
>
> [...]
>
> the visual world [Vorstellungswelt] is described *completely* by the description of the visual image [Vorstellung]. (Wittgenstein, 1953/2009, §402, emph. in orig.)[22]

These statements underline a significant change in how ≪mental images≫ are conceived: Instead of regarding them as mental entities, Wittgenstein (1953/2009) asks how interlocutors use the keyword ≪mental image≫. In particular, ≪mental images≫ are not images in the sense of an inner picture either. It is moreover insufficient to conceptualise them as representations of things from physical reality. For example, one can imagine a unicorn—in our line of argumentation at this moment, this amounts to describing how one imagines a unicorn—or that a political party will win the next election even though these creatures do not exist or the imagined scenario does not take place (Krüger, 1995). Accordingly, the question what a ≪mental image≫ really is or what happens when one imagines something is not answered and is not answerable at all in case one assumes that such an answer describes a specific mental process, a product resulting from such a process, or an inner event (von Savigny, 1996, p. 62; Hacker, 1990, p. 432).

22 We have again included some instances of the original German wording in brackets [...].

According to Glock (1996, pp. 169–170), Wittgenstein's (1953/2009) ideas on ≪mental images≫and ≪imagination≫ (depending on context, both are translated to "Vorstellung(en)") may be concentrated into four aspects (cf. Flores H., 2001):

(a) One should not ask what mental images are but instead how this keyword is used.

(b) The word "imagination" is used when someone describes her- or himself having a mental image and its use includes the possibility that no mental process (e.g., a process analogous to seeing or perceiving) actually occurs.

(c) Even if images are involved, they are not like physical pictures. Other people's ≪mental images≫ are only accessible if these others make them accessible. What is conceived as the object of imagination is inseparable from the imaginators' description of it (see also Krüger, 1995, p. 76): "The language-game of imagining starts not with a private entity which is then described, but with the expression of what one imagines. A mental image is not a private entity, but the way we imagine something, just as a visual impression is the way we organize what we see" (Glock, 1996, p. 170).

(d) Imagination is no inner version of perception of physical events.

In a nutshell, for Wittgenstein, there is no need to include mental entities to the discourse about ≪mental images≫. A ≪mental image≫ (German: Vorstellung or Vorstellungsbild) is equated with the description of it: "A mental image (German: Vorstellungsbild) is the image which is described when someone describes what he imagines" (Wittgenstein, 1953/2009, §367). In other words,

> that I imagine this or that is not a matter of *what goes on in me* but of the story I tell (Hacker, 1990, p. 397, see also the quote at the beginning of this chapter).

2.4 LOOKING AHEAD

Having raised discourses in mathematics education about ≪mental images≫, we argued exemplary related to *concept images* (Section 2.2.1) and *basic ideas* (Section 2.2.2) that previous conceptualisations of ≪mental images≫ do not fit to the goals of our research, namely to conduct exploratory research about experts' individual interpretations of complex path integrals.

Especially in complex analysis, where visualisation of functions is difficult or impossible for dimensional reasons, and little research is available, we still need to figure out how such processes of making complex path integrals accessible look like. In this context, it is questionable whether having mental images or intuitive explanations can be modelled with a fixed prescription of what these may look like structurally. Therefore, it is appropriate to leave the judgement of what belongs to this sensitising view (cf. Blumer, 1954) on mathematical concepts to the interlocutors engaging in such types of communication. That is, when studying the communication taking place when mathematicians explain mathematical objects on an intuitive level, for themselves or for others, we take into account their assessments of what counts as intuitive or their mental images.

In the next chapter, we will review the main tenets of the commognitive framework, which we use as the central theoretical framework for this thesis (Chapter 3). Afterwards, we will establish our idea of *discursive mental images* as narratives in *intuitive mathematical discourses*, which we use for Part iii of this thesis (Chapter 4). In this empirical part, we will finally examine experts' intuitive mathematical discourses about complex path "at the eye-level" (Pfadenhauer, 2009) in conversations between them as mathematicians and the author of this thesis as university mathematics educator.

3

THE COMMOGNITIVE FRAMEWORK

The word commognition, *a combination of* cognition *and* communication, *was coined to epitomize this claim, that is, to always remind us that human thinking develops, both historically and ontogenetically, through individualization of interpersonal communication. This communication does not have to be verbal or audible. Within commognitive perspective, therefore, cognitive processes and processes of inter-personal communicating are but different manifestations of basically the same phenomenon.*—Ben-Zvi and Sfard (2007, p. 119, emph. orig.)

The aim of this chapter is to describe a discursive perspective on mathematics. To do so, we will describe elements of the commognitive framework (Lavie et al., 2019; Sfard, 2007, 2008, 2015, 2018, 2020a, 2021), especially its ontological and epistemological tenets on human thinking in relation to mathematics. This framework combines a sociocultural perspective on doing mathematics with a perspective on individuals' cognition. Its vision of doing mathematics combines the two perspectives by unifying thinking and communicating as two facets of one phenomenon. Individuals' ways of doing mathematics are seen as individualised versions of collective, historically established patterned ways of doing in mathematical communities, in other words, *mathematical discourses*. The neologism "commognition" resolves this dualism for it combines communication, that is, means of expressing oneself for and with others, and cognition, that is, the individual ways of thinking. In that vein, the commognitive framework conceptualises *thinking as communicating* (Sfard, 2008) and offers various conceptual means for mathematics education research.

In Section 3.1, the commognitive framework is embedded into discursive and participationist approaches in mathematics education. Then, in Section 3.2, the main tenets on communication, discourses, and thinking are summarised. In Section 3.3, we describe its main assumptions about the ontology of mathematical objects and concepts. Section 3.4 contains a summary of the four characterising elements of discourses, namely their use of keywords, visual mediators, endorsed narratives, and routines. Routines will be explained in more depth in Section 3.5. Remarks on the individualisation of mathematical discourses follow in Section 3.6. In Section 3.7, we give an outlook on the next chapter, where we define one of the central notions for the remainder of the thesis, namely *intuitive mathematical discourses*.

3.1 DISCURSIVE AND PARTICIPATIONIST APPROACHES TO MATHEMATICS EDUCATION

Embedding of the commognitive framework

Discursive approaches in mathematics education can be understood as part of what Lerman (2000) calls the "social turn" in mathematics education. This means that "theories that see meaning, thinking, and reasoning as products of social activity" emerge (Lerman, 2000, p. 23). Instead of an acquisitionist view on human activities, the participationist view that underlies the commognitive framework regards

> human capacities as resulting from the fundamental fact that humans are social beings, engaged in collective activities from the day they are born and throughout their lives. [...] it is the collective life that brings about all the other uniquely human characteristics, with the capacity for individualizing the collective – for individual reenactments of collective activities - being one of the most important (Sfard, 2006, p. 157).

Following this basic assumption that collective activities are developmentally prior to those of individuals (Vygotsky, 1978, 1987), an individual's cognition is modelled as an individualised version of participation in collectively patterned ways of communication (Sfard, 2006, 2008; Sfard & Kieran, 2001; see also Lensing, 2021, ch. 2.1). Similar to social constructivist views on doing mathematics, the commognitive framework acknowledges that social circumstances influence how people participate in mathematics.

The word discourse is used in mathematics education with varying meanings (Ryve, 2011). Whilst some of the discursive approaches originate outside mathematics, the commognitive framework is rather focusing on the special case of mathematics, even though the book by Sfard (2008) is also very much self-contained with regards to human thinking in general (Morgan, 2020). The commognitive framework acknowledges "human communication as a topic of research in its own right" rather than "a mere window to something else – to one's thoughts, concepts or mental schemes" (Sfard, 2013, p. 157). Communication is not regarded as subordinate to thinking, that is, as a means of merely expressing thinking, but rather the other way around, that thinking is a type of communication. Even though the commognitive framework is ambitious to conceptualise human thinking more generally than related to mathematics, here, we will describe it more locally in relation to mathematics.[23]

On disobjectification

One of the central assumptions of the commognitive framework is that words are frequently used in metaphorical and objectified ways (Sfard, 2008, 2009a). That is, they seem to refer to

23 For more detailed overviews about discursive approaches in mathematics education see for example Morgan (2020), Ryve (2011), and Sfard (2020b).

self-sustained entities, physically observable, or existent independently from human communicating about them (Sfard, 2020b). Rather, commognition claims that in these cases nouns are used as if they refer to allegedly existing objects. As such they result from turning processes into objects by means of language. While this "objectification" of words into "as-if-objects" is one feature of language that makes communication more effective, it also has drawbacks (Sfard, 2008). On the one hand, several of these alleged objects, even if their existence is taken for granted, remain unobservable, hence inaccessible for research. On the other hand, the objectified use of words in educational research (e.g., knowledge as entity, strong or weak learners, thinking as a mental process, learning disability, etc. to name but a few) gives rise to questioning such entities, which after all have only been baptised as metaphors (Ben-Yehuda et al., 2005; Sfard, 2008, 2018). As such, regarding thinking as communication allows us to conceptualise ideas formerly perceived as cognitive, to disobjectify concepts in mathematics education research discourses, and to ground them in discursive terms. Dualist views portraying communication as a vehicle to transport thinking is not used by participationists: "participationists are likely to view the idea of 'thought-conveyed-in-communication' as but a direct result of an unhelpful objectification" (Sfard, 2006, p. 165, one inverted comma has been corrected, EH.).

This resonates with the critique on the use of ≪mental images≫, ≪meaning≫ or ≪mental representation≫ in Chapter 2. In that chapter, we followed Roth (2013) who suggested to deconstruct ≪meaning≫ or ≪mental representation≫ as alleged objects of mathematics education research and Wittgenstein (1953/2009) who proposed to initiate investigations about ≪mental images≫ by considering interlocutors' usages of this and related words. The conceptualisation of thinking as communication follows a similar motivation as it disobjectifies what has previously been objectified as cognitive constructs in mathematics education research. Hence, making use of conceptual ideas from the commognitive framework allows us to shed different light on ≪mental images≫ in mathematics education research compliant with the efforts of disobjectification.

3.2 THINKING AS COMMUNICATING

> *There is no split between thinking and behaviour; the object of development of change is human activity and not the individual; the focus of analysis is the discourse and not the skill (behaviourism) or the concept (cognitivism) (Sfard, 2008).*
>
> —Ioannou (2018, p. 112)

Thinking

The commognitive view on doing mathematics is that "*thinking mathematically* means participating in a historically developed discourse known as *mathematical*" (Sfard, 2020a, p. 96). Mathematics is an activity that is historically established and newcomers to mathematics learn it by participating in mathematical discourses. The definition of thinking, which the commognitive framework offers, aims to disobjectify the use of the word thinking in educational research (Sfard, 2008, p. 92). It does not aim to explain what thinking really is, but to endow the word thinking with a conceptualisation that allows to "find[] useful ways of talking about the phenomenon" (Sfard, 2008, p. 65; see also Lensing, 2021, ch. 2.1.2). According to Sfard (2008, p. 91),

> [t]hinking, although seemingly inherently private, should not be different [from a collectively patterned activity; EH.]. Cognitive processes may thus be defined as individualized forms of interpersonal communication, whereas communication itself is described as a

> collectively performed rule-driven activity that mediates and coordinates other activities of actors.

Therefore, thinking in commognition is defined as follows:

Thinking

> *Thinking* is an individualized version of (interpersonal) communicating. (Sfard, 2008, p. 81)

This explains the neologism of commognition, which draws communication and cognition together, regarding both as different sides of the same coin. Thinking "can now be thought of as the type of human doing that emerges when individuals become capable of communicating with themselves the way they communicate with others" (Sfard, 2008, p. 91). Therefore, both, thinking and communication are human activities, which are patterned, rule-driven, involve the agency of the participants, use visual mediators, and usually revolve around a selection of specific objects (Sfard, 2008, pp. 84–91).[24]

Discourse

Discourse

A *discourse* is defined as a type of communication "that draw[s] some individuals together while excluding some others" (Sfard, 2008, p. 91).[25] Discourses are "set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). That is, discourses are "means, or sets of tools for telling useful stories, or narratives, about specified types of objects" (Sfard, 2021, pdf p. 4). A discourse about a mathematical object X (or a set of mathematical objects) is accordingly one that deals with this mathematical object. Henceforth, a

> *discourse of X* refers to a *type of communication* that is used in telling stories about X. To define such a type, one does have to specify a set of necessary *keywords with their uses*, [... their] set of *visual mediators* of physical entities, with which participants make clearer what they are talking about, [...their] set of *routines*—of patterned recurrent forms of actions that discourse users perform [..., and] sets of *endorsed narratives* about X [...] that adhere to the truth-defining rules of the discourse. (Sfard, 2021, pdf p. 3)

In the commognitive framework, learning mathematics is understood as the individualisation of mathematical discourses:

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24 Henceforth, there is a strong ontological commitment on what counts as thinking in commognitive research. This is clearly not unquestioned. For example, whilst acknowledging that "interaction and students' epistemic processes (i.e., processes of individual knowledge constitution) are strongly intertwined", some authors, like Erath et al. (2018) in this case, "prefer to *analytically separate* the discourse from the epistemic processes in order to be able to investigate their subtle and strong connections empirically" (Erath et al., 2018, p. 163, emph. orig.).

25 *Communication* is defined as

> a collectively performed patterned activity in which action A of an individual is followed by action B of another individual so that
>
> 1. A belongs to a certain well-defined repertoire of actions known as communicational
> 2. Action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and re-actor. (Sfard, 2008, pp. 86–87)

> The term *individualizing* refers to becoming an agentive participant of this discourse, one who is not just able to act according to the rules of this discourse, but also tends to turn to it on her own accord and to use it in deciding how to proceed. More succinctly, when sufficiently individualized, the discourse becomes the medium of choice for one's thinking, whenever appropriate. (Chan & Sfard, 2020, p. 4, emph. in orig.)

Cognitive, biological, neurological, or psychological processes of human beings are not denied, but are not placed in the centre of attention of commognitive research (Sfard, 2013). Individualisation of mathematical discourses is achieved by participating in these discourses. What changes while this type of learning occurs is the way the individual participates in the discourse: Learning as individualisation is "a gradual transition from being an only marginally involved follower of other people's implementation to acting as a competent participant, with full agency over the activity" (Sfard, 2009b, p. 56).

3.3 OBJECTIFICATION AND MATHEMATICAL OBJECTS

Participants of discourses tell stories about the objects the discourse is about, and this may include that they also tell stories about themselves as human agents of the discourse. These objects in discourse are either entities of the physical world or they are about discursive objects, which means that the talk about them relies on the "metaphor of object" (Sfard, 2008, p. 42, ch. 2).

Mathematical discourses are about mathematical objects. In order to state the commognitive perspective on mathematical objects, we need to look more closely at mechanisms of communication that produce so-called *discursive objects*. These objects are constituted by talking about them, which may enforce the impression of a circular reasoning of what discursive objects are all about. The talk about these discursive objects is very similar to the talk about objects of physical reality; it is as if the discursive objects were objects of the physical world. They are results of processes of *objectification* and in particular *reification* (Sfard, 1994, 2008). In particular, reification is "the birth of a metaphor which brings a mathematical object into existence" (Sfard, 1994, p. 54).

Objectification

Objectification is the process of turning the talk about processes into the talk about objects; in other words, it is the "process in which a noun begins to be used as if it signified an extradiscursive, self-sustained entity (object), independent of human agency; the process consists of two tightly related, but not inseparable subprocesses: reification and alienation" (Sfard, 2008, p. 300, emph. omitted). That is, the objects appear as objectified discursive entities when the human agency is eliminated from the talk about them, in other words, one talks as if these things existed without the involvement of humans.[26] Now, as mentioned in the quote, two subprocesses constitute this phenomenon usually in an interrelated way: *Alienation* is the process of "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of humans beings" (Sfard, 2008, p. 295); and *reification* is the "replacement of talk about processes with talk about objects; usually, reifica-

Alienation

Reification

26 Note that this is an ontological premise about discursive objects. However, we argue that even if someone does not agree that certain objects (e.g., mathematical objects; see below) are results of processes of human communication but instead presupposes their existence somehow or somewhere as self-sustained entities, it is nevertheless accurate to describe the communication about these in the way that is suggested by the ontological premise of the commognitive framework. The important point for this premise is that it prescribes a way of looking at mathematics in terms of communication and discourse rather than in terms of an abstract hidden world to be explored by mathematicians. (cf. Font et al., 2013)

tion requires introduction of a new noun or written symbol" (Sfard, 2008, p. 301). For example, if one talks about somebody having a "'learning disability'", this is a reified version of saying that this person "'cannot cope with even the simplest arithmetical problems in spite of years of instruction'" (Sfard, 2008, p. 301). As such, reification is embedded into the widespread idea of process-object duality of mathematical objects (Sfard, 1991, 1994). It is very special though because it focuses on shifts between processes in communication into objects of communication:

> Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification*, therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. [...] [R]eification is an instantaneous quantum leap: a process solidifies into object, into a static structure. (Sfard, 1991, pp. 19–20)

P-objects and d-objects

The act of reifying is tightly connected to objects of discourses, the discursive objects. In general, Sfard (2008) distinguishes between two classes of objects, *p-objects* (primary objects) and *d-objects* (discursive objects), where the second category is further subdivided. This distinction allows us to grasp more concretely how mathematical objects are understood in the commognitive framework. *P-objects* (primary objects) are objects of physical reality such as materials objects or sounds. The genesis of objects of discourses, *d-objects*, is based on several linguistic processes: *Baptising* is giving something a proper name (i.e., it is the formation of a pair, which consists of the name and a p-object), the result of which is called a *simple d-object*. On the other hand, more interlaced discursive objects, so-called *compound discursive objects* are results of *saming* (i.e., giving a collection of things a name, which are to be considered the same with respect to this assignment of name), *encapsulating* (i.e., using a signifier to talk about a property of multiple objects as a new object), or *reifying* (i.e., using a noun or pronoun to replace the talk about processes with talk about the resulting reified object; as described above). (Sfard, 2008, pp. 167–172)

Baptising

Saming

Encapsulating

Next, we need introduce the ideas of signifiers and their realisations:

Signifiers

Realisation

> *Signifiers* are words or symbols that function as nouns in utterances of discourse participants, whereas the term *realization of a signifier S* refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S. (Sfard, 2008, p. 154, emph. orig.)

Hence, signifiers are words or symbols that function as nouns, while realisations are perceptible entities which may be operated upon while producing or substantiating narratives about a signifier. Both, signifiers and realisations, belong to the same category, namely perceptible entities and often the relation between them is symmetric. That means, a realisation of a signifier may in turn be a signifier of the realisation. Consequently, the pair consisting of a signifier and a realisation of that signifier is ontologically different from the pair consisting of an object and a representation of the object, which would suppose that an object exists independently of the representation and the representation is more or less pointing to the object. (Sfard, 2008, pp. 154–155)

A *realisation tree* of a signifier is the "hierarchically organized set of all the realizations of the given signifier, together with the realization of these realizations, as well as the realizations of these latter realizations, and so forth" (Sfard, 2008, p. 301, emph. omitted). Having defined all these, we can say that a "*(discursive) object signified by S* (or simply *object S*) in a given discourse on *S* is the realization tree of *S* within this discourse" (Sfard, 2008, p. 166).

Discursive object

Mathematical objects

Now, we are ready to define mathematical objects: "*[M]athematical objects are abstract discursive objects with distinctly mathematical signifiers*, that is, signifiers regarded as mathematical" (Sfard, 2008, p. 172). The word "abstract" refers to the fact that they are discursive objects, whose formation involved the process of reification.[27] Even though mathematical objects are used by a community of mathematicians seemingly timelessly, they are actually "personal constructs, and different mathematists may associate different objects with the same signifier" (Sfard, 2008, p. 193). That is because individuals participate in mathematical discourses, and even though they produce endorsed narratives about mathematical objects in processes of endorsement, each of the interlocutors' experience in the respective discourses eventually leads to different realization trees of mathematical signifiers. The definition of mathematical objects turns them "into personalized, contextualized, and discursive constructs" (Kontorovich, 2021b, p. 4). On the other hand, this does not mean that mathematical objects are complete individual idiosyncrasies: The boundary between the individual and the community is abolished as far as the discourses, in which the individuals participate, transform and expand autopoietically through their participants, and the newcomers come into contact with the hitherto established features of the discourse through their participation. Moreover, a *concept* is understood as a "word or other signifier with its discursive use" (Sfard, 2008, p. 296, emph. omitted, see also p. 268).

In conclusion, we follow Ben-Zvi and Sfard (2007, p. 120, emph. orig.) by saying that

> some of the "things" that are investigated by scientists or mathematicians, rather than being found directly in the world, are produced through the discourse itself. Thus, notions such as *velocity* and *energy* in physics, *distribution* and *mean* in statistics, or set and function in mathematics, although clearly related to observable real-world phenomena are, in fact, discursive constructs created for the sake of a better description of reality.

3.4 FOUR ELEMENTS OF DISCOURSES: KEYWORDS, VISUAL MEDIATORS, ENDORSED NARRATIVES, ROUTINES

Four discursive elements delineate different discourses (Figure 3.1; Lavie et al. (2019) and Sfard (2008)):

- Keywords and their use,

27 Abstractness of mathematical objects is not distinctive for commognition. For example, varieties of platonism also regard mathematical objects as abstract (Linnebo, 2017, 2018): "An object is said to be *abstract* [...] if it lacks spatiotemporal location and is causally inefficacious; otherwise it is said to be *concrete*" (Linnebo, 2017, p. 9). But in commognition, they are differently abstract: Their signifiers do not function as pointers to the abstract entities wherever they may be, but they are results of intertwined processes of objectification, particularly reification. According to Quine (1948), there are three possibilities for how abstract objects are conceptualised: as abstractly located in a space of ideas (such as in platonism), as abstractly located in the mind of humans (such as in intuitionism or (radical) constructivism), or as being constituted by language (such as in formalism). With regards to these options, the ontological tenet of commognition would rather fall into the third category, even though not language is the central constituting mechanism but more broadly discourse. Again, as said in footnote 26, this is an ontological premise just like variants of platonism, constructivism, and others, have their premises. In commognition, however, this premise is tightly related to the communication of individuals and groups of individuals. The ontological premise underlying commognition, in my view, is the assumption that mathematical communications are structured as something called discourses and that therefore questions about the ontology of mathematics and its epistemology must be asked according to this assumed constitutivity. Commognition does not divide "between the category of signifiers (mathematical words, symbols etc.) and of signified (the ideal, independently existing abstract objects)" (Sfard, 2021, pdf p. 6–7). According to Morgan and Sfard (2016, p. 103), "[b]y equating mathematics with a form of communication, the discursive approach dispenses with problematic dichotomies and their nebulous ingredients."

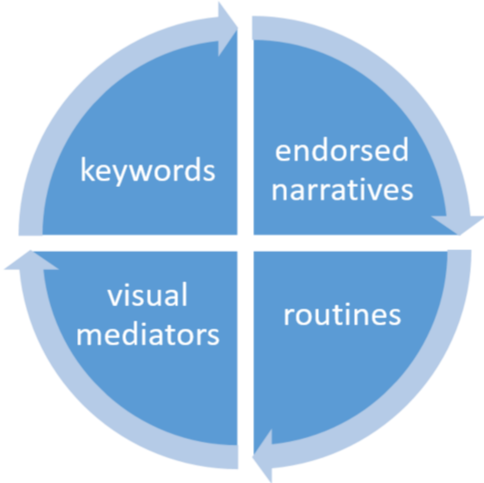


Figure 3.1: The four key elements of discourses

- endorsed narratives,
- visual mediators, and
- routines.

3.4.1 *Keywords*

Keywords in mathematical discourses are for example number words such as "two", "integral", or "holomorphic". Of course, one keyword can appear in multiple discourse. Given the topic of this dissertation, this is evident: "integral" appears in a discourse called complex analysis, one called real analysis, one called measure theory, etc. Sometimes, keywords may be refined with additional keywords. For example, we might use the (compound) keyword "complex path integral". For example, Biza (2021) and Kontorovich (2018b) pay special attention to the fact that certain keywords appear in different mathematical discourses, in the form of "discursive footprints" (Biza, 2021) and "cross-curricular topics" (Kontorovich, 2018b).

Sfard (2008) distinguished three types to describe interlocutors' use of keywords: *passive use* occurs when interlocutors just repeat the word in reaction to more experienced interlocutors (or do not use the word at all); *routine-driven use* occurs when interlocutors use the word "as part of constant discursive sequences" (Sfard, 2008, p. 181); the use is *phrase-driven* if the word is used within certain phrases rather than flexibly; and it is *object-driven* when the use of the word is "guided by the signified object – by the user's awareness of the availability and contextual appropriateness of different realizations of the word" (Sfard, 2008, p. 182).

3.4.2 *Visual mediators*

Visual mediators are any kinds of visible entities, which can be used for the act of communication about mathematical objects. This comprises written sentences, Roman numerals or Greek letters, tables, symbols specially created for use in mathematics such as $\int$ or $\cup$, diagrams, combinations of those such as $f : \Omega \longrightarrow \mathbb{C}$, plots of graphs of functions, but also gestures and much more (see Appendix C for various visual mediators for complex functions). In particular, discourses are much more than language. They can include communication in all kinds of modality such as concrete (physically or embodied) verbal, gestural, iconic, or symbolic (Sfard, 2008, p. 156; see also Ryve et al., 2013).

3.4.3 *Endorsed narratives*

A *narrative* is

Narrative

> a series of utterances, spoken or written, that is framed as a description of objects, or relations between objects, or processes with or by objects, and is subject to endorsement or rejection, that is, to being labeled as "true" or "false[]" (Sfard, 2008, p. 300, emph. omitted).

Here, an utterance is a "communicational act in language", which can be spoken or written (Sfard, 2008, p. 302). Narratives can just be seen as stories about mathematical objects, which are subject to a process of endorsement by a set of endorsers (i.e., a mathematical community).[28] Yet, narratives do not have to respect temporal features (e.g., a differentiation into text genres, such as narrative, description, and argumentation, like in documentary method (Nohl, 2010), or requiring that a narrative is a report of events with a beginning, a development, and end is not included in the definition of narrative in commognition). Narratives count as endorsed when they are "regarded as reflecting the state of affairs in the world and labeled as *true*" (Sfard, 2008, p. 298, emph. orig.). In historically grown discourses, for example complex analysis, the endorsement is consensually attributed to a large community of mathematicians and usually not questioned in general. Endorsement is derived according to rules, that are also accepted by the community of endorsers. (Sfard, 2008)

Endorsed narratives

A mathematical theory is a set of endorsed narratives on X, which has been endorsed relative to the discourse (Sfard, 2021). The narratives have to be consistent to each other according to established discursive rules of consistency (e.g., they should not lead to contradictions). Endorsed narratives take the form of definitions, theorems, lemmas, proofs, etc. For example, definitions and theorems in complex analysis are endorsed narratives. This may also include narratives like "The complex path integral from 1 to 1 along the unit circle traversed anticlockwise of the function $1/z$ is $2\pi i$" and its shortened form making use of symbolic visual mediators "$\int_{|z=1|} \frac{1}{z}\,\mathrm{d}z = 2\pi i$".[29]

3.4.4 *Routines*

Routines are discursants' patterned activities. In order to explain the notion of routine, we need to make a detour about discursive rules and the objects mathematical discourses are all about: mathematical objects.

28 These processes of endorsement can take place very locally in a classroom between pupils and a teacher, scientific communications, but also over a longer period of time. For this reason, one might say that the set of narratives delineating a discourse are "generally *endorsed narratives*, such as theorems, definitions and computational rules. The descriptor 'generally endorsed', used in this last sentence, is to be understood as designating endorsement by the *mathematical community*, with this latter term referring, collectively, to the competent participants of mathematical discourses" (Sfard, 2013, pp. 157–158).

29 This formula may well count as a narrative because it can be read as a story about the signified mathematical objects. It is a story about a complex path integral $\int_{|z=1|} \frac{1}{z}\,\mathrm{d}z$ and the complex number $2\pi i$. The story consists of the utterance that these two are equal.

Before doing so, we would like to remark that it is in general hardly possible to really delineate discourses by giving complete lists of keywords or other descriptions of the four elements of discourses described in this section. etc. This would not only be unfeasible, but also impossible: Mathematical discourses are autopoietic systems, which means that they grow discursively from the inside when new narratives are constructed and endorsed, new keywords are introduced in definitions etc. (Sfard, 2008)

3.5 RULES OF DISCOURSE AND ROUTINES

3.5.1 *Rules of discourse*

Communication was defined as an activity, in which interlocutors act and react to each other based on rules. Two kinds of rules are to be distinguished. One is about the objects of the discourse, the other is about the mathematists, that is, the participants in mathematical discourses. Both kinds of rules are expressed or could at least be expressed in terms of narratives; however, more often than not, many rules are implicit.

Object-level rules

- "*[O]bject-level rules are narratives about regularities in the behavior of objects of the discourse*" (Sfard, 2008, p. 201, emph. orig.).

Metarules

- *Meta-level rules*, or *metarules* for short, "*define patterns in the activity of the discursants trying to produce and substantiate object-level narratives*" (Sfard, 2008, p. 201, emph. orig.).

The object-level rules describe the objects of the discourse and relations between them. The metarules describe the activities of the participants, for example, they may comprise rules that govern the construction of proofs or they may describe what mathematists should do when faced with a certain mathematical task. For instance, one object-level rule on complex path integrals is that whenever f is a holomorphic function on a neighbourhood of the compact closure of the ball $B_r(z_0)$ and z is a point in the ball, then $f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$ (Cauchy's integral formula (Theorem A.22)). In this context, a metarule for the evaluation of an integral of the form $\int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$ is to decide first whether f and z satisfy the previously mentioned requirements and then to use Cauchy's integral formula. Another metarule for the evaluation of the integral is to plug in a parametrisation of the respective path and compute the resulting integral directly.

Discursive rules are variable and evolve over time (e.g., what counts as a function has changed over time). They are often tacit, and when explicated in form of a narrative, these may appear to be "retroactively written into interlocutors' past activities and [are] expected to reappear, possible in a slightly modified version, in these interlocutors' future actions" (Sfard, 2008, p. 203). Discursive rules may appear normative, even though they are assumed to have guided the discourse before they were explicated. Discursive rules are not considered to be fixed and deterministic, they are often tacit and contingent. A commognitivist researcher has the task to retrospectively find out the regulation mechanisms that can be used to describe a discourse and the activities of its participants. She or he can do so by participating in the discourse, but also by observation, taking the stance of an outsider to the discourse. If a person consciously follows a discursive rule, this person can be said to have "*endorsed*" the rule; while if the rule is

Table 3.1: Classification of routines (adapted from Lavie et al., 2019, p. 166).

Routine	Process-oriented	Product-oriented
Practical	Ritual	Deed
Discursive	Ritual	Exploration (construction, substantiation, recall)

described by an observer, it is rather appropriate to say that the observed person "*enacted*" the rule (Sfard, 2008, p. 204, emph. orig.).[30]

It would be wrong to assume that the prescription of the existence of discursive rules means that discourses are rigid and inflexible. This might be true if the rules were algorithmic or deterministic. However, they do not have to be; rather, they may change and develop. According to Sfard (2008), discursive rules are rather flexible, and it is easier to find out when somebody executed a discursive action that is not compliant with tacitly given discursive rules. Discursive rules are not necessarily a matter of necessity but rather, they are "contingent", "variable", "value-laden", and "flexible" (Sfard, 2008, pp. 200–208), which is also reflected in the historical growth and development of mathematical discourses (see e.g., Barnett et al., 2021; Clark, 2019; Güçler, 2016; Kjeldsen & Blomhøj, 2012; Sfard, 2008).

3.5.2 *Routines*

Routines are the fourth discursive element characterising discourses (Lavie et al., 2019; Sfard, 2008). A routine is a "set of metarules defining a discursive pattern that repeats itself in certain types of situations" (Sfard, 2008, p. 301). Routines delineate each other by the *when* they are performed ("applicability conditions") and the *how* ("routine procedure" and "closing conditions") (Sfard, 2008, p. 302, emph. omitted). Routines are like orienting patterns of discourses with which one can describe central discursive activities. They are based on the involvement of the discursant and are "fluid and changeable" rather than rigid and leaving no space for creativity (Sfard, 2008, p. 216). *Routines*

Basically, there are three forms of routine: *explorations*, *deeds*, and *rituals*. Deeds and rituals are process-oriented. This means that they focus on the process of execution, while explorations are oriented towards the creation of endorsable narratives (Table 3.1).

- *Explorations* end with the construction of a new endorsable narrative (where new is relative to the performer of the routine). A narrative in a mathematical discourse is endorsable "if it can be derived according to generally accepted rules from other endorsed narratives" (Sfard, 2008, p. 223). The construction of an endorsable narrative can thus finish with a proposition about a mathematical object, such as a definition, a theorem, an axiom, etc. *Explorations*
  - Explorations can be *constructions*, that is, the process of creation of new endorsable narratives (e.g., by means of mathematical induction, calculation, etc.). (Sfard, 2008, pp. 225–231) *Construction*
  - An exploration can be a *recall*, that is, a memorisation of an endorsed narrative *Recall*

30 Hence, commognitivists should pay attention to how stories are told about discursants. A story told by a discursant about her- or himself is therefore a first-person narrative, but when someone else talks about this person, it is appropriate to keep in mind that these stories actually are third-person narratives (cf. Sfard & Prusak, 2005).

with the help of which a construction process for a new narrative is initiated. (Sfard, 2008, pp. 234–234)

Substantiation

– The third type of exploration is *substantiation*, the "process through which mathematicians become convinced that the narrative can be endorsed" (Sfard, 2008, p. 231). Clearly, producing substantiations is a recursive act: Narratives produced in the course of substantiation often lead to substantiations of these narratives etc. (Sfard, 2008, pp. 231–234)

  A substantiation of a theorem can thus be the construction of a proof, which could be described as the "process of communication with oneself or others that allows for the construction and substantiation of a narrative" (Shinno & Fujita, 2021, p. 4). A substantiation of a definition can be a test for its consistency, the observation that a definition is equivalent to another one, or that the newly constructed definition generalises a previous one, as was illustrated in a case study with an expert mathematician whose research included the proposal of new definitions (Martín-Molina et al., 2018).

  For example, a substantiation of the definition of complex path integrals may be the test whether, when the definition is applied to real-valued functions of a real variable, the complex path integral leads to the Riemann integral of these functions; hence making plausible that the definition for the complex path integral is chosen consistently with regards to Riemann integrals. What is convincing and thus counts as a substantiation may vary among individuals (also for proofs: what counts as proofs may vary between different people, even if experts would probably agree in a lot of cases what counts as a proof for a given mathematical narrative; cf. Hanna and de Villiers, 2012/2021; Harel and Sowder, 2007; Reid and Knipping, 2010).

Deeds

· *Deeds* are practical actions, that is, they "result in a physical change in objects" (Sfard, 2008, p. 236) and are performed when the interlocutor believes that she or he should produce objects rather than narratives. Deeds can be hard to distinguish from explorations, in particular if the change occurs in discursive objects. Whether a routine functions as a deed or exploration depends on what the interlocutor is trying to achieve. (Sfard, 2008, pp. 236–241)

Rituals

· *Rituals* are routines whose closing conditions are "neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people" (Sfard, 2008, p. 241). This means that somebody acting ritualistically does so in order to please an interlocutor. Rituals occur when the performer believes that she or he should act in a very specific manner and cannot or believes to be not allowed to work differently (e.g., performing a strict algorithmic procedure for a certain kind of calculation task in order to please the teacher or to demonstrate that the performer is capable of doing so).

Task situations and precedents

Precedent

Lavie et al. (2019) refined the operationalisation of routine. More specifically, they describe that individuals react to *task situations*, which are "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159), based on so-called *precedents*, that is, "past situations which [a discursant, EH.] interprets as sufficiently similar to the present one to justify repeating what was done then, whether it was done by herself or by another person" (Lavie et al., 2019, p. 160). The collection of those previous situations, with which a current task situation is compared, are called *precedent-search-spaces* (PSS) (Lavie et al., 2019, p. 160). Interlocutors are said to do so with the help of *precedent identifiers*, which are "those features of the current task situation that a person considers as sufficient to view a task situation from the past as a precedent" (Lavie et al., 2019, p. 160). Thus, the task a person is faced with is her or his interpretation of a given task situation, including those features of the situation the person recognises to require an action by comparing it with precedents.

Consequently, Lavie et al. (2019) argue that

> *routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task*. This said, let us immediately add that in certain contexts, we may still indicate some routines by just naming their tasks and without a reference to a particular task situation or a specific performer. Whenever we do this, as would be the case if we said "routine for a quantitative comparison of two sets," we are to be understood as referring to the routine performed by experts in task situations that they interpret as requiring canonically understood quantitative comparison. (Lavie et al., 2019, p. 161)

Hence, whereas their conceptualisation of routine is tightly interwoven with individuals' experiences from participation in previous task situations recognised as similar, the authors nevertheless consider it feasible to refer to certain routines with the help of a name that experienced interlocutors will recognise (Lavie et al., 2019).

The list of routines is not exhaustive and certainly not fine-grained to describe the plethora of how individuals actually participate in mathematical discourses. When studying routines, many facets or subroutines may become apparent (which may in fact appear as researchers' reconstructions). For example, Schüler-Meyer (2020) identifies *classifying* as a facet of defining (i.e., as special case of explorations), and certain features of definitions may be substantiated to establish metarules for defining. Viirman (2014b, p. 523) finds that the routine of defining may consist of definitions "by stipulation, exemplar, contrast, saming, and naming" (cf. Fernández-León et al., 2021; Martín-Molina et al., 2018; Martín-Molina et al., 2020; Nardi et al., 2014; Viirman, 2021).

3.6 COMMOGNITIVE CONFLICTS

Learning mathematics corresponds to a change in discourse. This change can happen on object-level, which corresponds to an enlargement of the discourse with respect to the use of keywords, and construction and endorsement of new routines and narratives ("endogenous expansion of the discourse"; Sfard, 2008, p. 255). Metalevel learning is a change in metarules ("exogenous expansion of the discourse"; Sfard, 2008, p. 256). Especially conflicts with respect to metarules are a vital part of commognitive learning.

Commognitive conflict

These conflicts are called *commognitive conflicts* (Sfard, 2008, 2021). It may be considered as the counterpart to the notion of cognitive conflict in cognitively oriented strands of mathematics education (Barwell, 2009). A commognitive conflict is described as "the encounter between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules" (Sfard, 2008, p. 161) or as "a situation in which different discursants are acting according to different metarules"

(Sfard, 2008, p. 256). With respect to this conceptualisation, commognitive conflicts are different from the idea of cognitive conflict. Sfard (2008, pp. 256–257) describes that cognitive conflicts are conflicts between the world and individuals' beliefs of the world, whereas commognitive conflicts arise from the interaction with discursants.

> Commognitive conflict [...] occurs when seemingly conflicting narratives are originating from different discourses – from discourses that differ in their use of words, in the rules of substantiation, and so forth. (Sfard, 2008, p. 257).

Hence, the conflict is not between an individual's beliefs of a state of affairs in the world and new phenomenon in the world, which causes the conflict as opposed to the held belief. Rather, it is a discursive phenomenon that originates from a different use of elements, according to different rules of discourse, which eventually means between different discourses, which the individuals can be said to follow when a commognitive conflict appeared. The discourses are then "*incommensurable* rather than incompatible" (Sfard, 2008, p. 257). Hence, a commognitive conflict can arise for example when keywords are used according to different rules (e.g., some of which were endorsed in one discourse but not the other).

A very simple example would be the narrative "When I multiply two numbers, the result is bigger than the two numbers I had". For a discourse on natural numbers, this narrative is likely endorsable, but for a discourse on rational numbers, it would no longer be endorsable for simple examples can be given to contradict the given assertion. The origin of the conflict is the use of the keyword "number", when used in one discourse to signify natural numbers, and to rational numbers in another (Sfard, 2021). Hence, interlocutors having endorsed the previously mentioned narrative in a discourse on natural numbers are required to make a change in the metarule that consists of what the signifier "numbers" realises. Similarly, before having entered discourses with complex numbers, participants of real analysis discourse may use the word "number" for elements of $\mathbb{R}$. When complex numbers are introduced a change in metarules has to occur, too, for now the elements of $\mathbb{C}$ are also considered to be numbers (Nachlieli & Elbaum-Cohen, 2021). Depending on the metarules for the realisation of complex numbers, which have been established, commognitive conflicts may arise between different realisations: One possible metarule for the realisation of complex numbers is to use signifiers of the form $a + bi$, where a and b realise real numbers. Another metarule is to use signifiers of the form of pairs of numbers, (a, b) or column vectors $(a, b)^T$. Hence, while participants of one discourse on complex numbers may have endorsed the metarule to meticulously distinguish $a + bi$ from (a, b), participants of another discourse on complex numbers may have endorsed the identity $a + bi = (a, b)$. Both discourses on complex numbers are thus incommensurable.

Commognitive conflicts may be resolved when interlocutors recognise that they were using some words according to different rules, make their uses of these words explicit, identify what the differences between these usages are, and finally agree on one way of usage (likely one that is compliant with the use by experienced interlocutors in a discourse) (Sfard, 2008, 2021; Tabach & Nachlieli, 2016).

Inter- and intra-personal commognitive conflicts

Kontorovich (2021b) suggested a refinement of the notion of commognitive conflicts by pointing out that they can be both, inter- and intrapersonal. A commognitive conflict between different interlocutors are thus interpersonal and a commognitive conflict in narratives of a single discursant are an "intrapersonal commognitive conflict" (or "intra-commognitive conflict[]" for short) (Kontorovich, 2021b, p. 3). In particular, the latter may not be noticed by the discursant her- or himself and hence is a researchers' analytical construct for the analysis of discourses: "Intra-commognitive conflicts draw attention to factual and potential conflicts between mathematical discourses of the same interlocutor" (Kontorovich, 2019, p. 418).

3.7 SUMMARY AND OUTLOOK

The commognitive framework brings forward a theory of mathematical thinking, in which communication and thinking are equated as two sides of the same phenomenon. The participationist point of view allows us to take a symbiotic perspective at discourses at large and their individualisation by mathematists. Doing so, the commognitive framework overcomes the dualism between the social and the individual. In particular, using the commognitive framework, questions about "thinking" are transferred to questions about "communicating".

In order to embed what we will do in the next chapter, we would like to remark that the commognitive framework has been used to describe and do research about many types of mathematical discourses. It is becoming increasingly popular in university mathematics education and is used to explore a wide range of phenomena and topics, ranging from the transition from school to university (e.g., Gavilán-Izquierdo & Gallego-Sánchez, 2021; Schüler-Meyer, 2019, 2020, 2022; Thoma, 2018; Thoma & Nardi, 2018), calculus and real analysis (e.g., Jayakody, 2015; Nardi et al., 2014), group theory (e.g., Ioannou, 2012, 2018), and many other mathematical discourses (see e.g., Antonini et al., 2020; Baccaglini-Frank, 2021; Cooper & Kontorovich, 2021; Heyd-Metzuyanim & Graven, 2019; Kontorovich, 2021b; Nachlieli & Elbaum-Cohen, 2021; Sfard, 2012). The commognitive framework has also been used to conceptualise specific facets pedagogically oriented discourses, too, such as pedagogical discourses of teachers at school (e.g., Heyd-Metzuyanim et al., 2018; Heyd-Metzuyanim & Shabtay, 2019) and university (e.g., Viirman, 2014b, 2015), didactical discourse on proof (Kontorovich, 2021a), or facets of mathematics teachers' knowledge (Cooper, 2016; Papadaki, 2019; Papadaki & Biza, 2020).

In the next chapter, we pursue a similar goal and conceptualise a specific type of mathematical discourses in line with our research goals. Doing so, we will complete our initial task for the study of experts' individual interpretations and potential ≪mental images≫ of complex path integrals by disobjectifying these sensitising concepts and conceptualising *intuitive mathematical discourses*.

4

INTUITIVE MATHEMATICAL DISCOURSES AND DISCURSIVE IMAGES

Even though all my interviewees remarked many times that they frequently resort to visualization [...], they also stressed that pictures, whether mental or in the form of drawings, are only a part of the story.

—Sfard (1994, p. 48)

The goal of this chapter is to describe the promised discursive perspective on ≪mental images≫. As a consequence of the participationist metaphor of learning and doing mathematics, we will not assume that ≪mental images≫ are states of individuals' minds (Sfard, 2006). In line with the commognitive framework, as to which thinking is communicating (Sfard, 2008), we will characterise a version of ≪mental images≫ in terms of discourses. These *discursive images* will be understood as narratives in a special type of mathematical discourses, which we will call *intuitive mathematical discourses*, which were first introduced in Hanke (2020a, 2020b).

In Section 4.1 and Section 4.2, we define intuitive mathematical discourses and discursive images. Then, we describe the four features of discourses (keywords, narratives, visual mediators, and metarules) for these discourses in Section 4.3. In order to account for the mechanisms guiding the construction of discursive images we conceptualised the notion of *discursive frame* in Section 4.4. Lastly, in Section 4.5, we emphasise that intuitive mathematical discourses can be manifested in various ways.

4.1 INTUITIVE MATHEMATICAL DISCOURSES

When mathematicians talk about their ≪mental images≫ or ask about those of others, they construct their narratives based on their own previous discursive actions in similar situations or others they have observed doing so. In commognitive terms, this means that they act according to precedents, in which the word "mental image" or keywords perceived as related were used (Lavie et al., 2019). When this took place in mathematical contexts, the narratives contained keywords for both mathematical objects and what the interlocutor saw as a ≪mental image≫ . Moreover, in these situations, metarules from scholarly mathematical discourses about the use of certain mathematical keywords or the construction of endorsable narratives and their endorsement may not have been at play. Hence, we argue that it is at least likely that mathematists, and especially mathematical experts, have participated in situations in which they described something they counted as a mental image of a mathematical object or as an intuitive or vivid explanation of a mathematical proposition. Even if they have never done this before or have not observed anyone doing it, they could presumably speak of what they consider to be their mental images or intuitive interpretations of mathematical objects or propositions.

In line with the ideas by Wittgenstein (1953/2009) to identify ≪mental images≫ with the talk about them (Section 2.3), we intend not to regard ≪mental images≫ of mathematical objects as something acquired or possessed in any way, which is then translated into language or pictures and communicated to others or oneself. Rather, we look for the discourse that unfolds when interlocutors use keywords from the world field around the keywords "mental images", "imagination" and adjectives like "intuitive" etc. in relation to mathematical objects. Doing so, we disobjectify ≪mental images≫ as objects of mathematics education research in the sense that we do not consider them as fixed, meticulously conceptualised objects located in individuals' minds and then tracked down in empirically gathered data. We intend to stay closer to the actors of mathematical discourses and do not distinguish what interlocutors consider their mental images, or more broadly, their intuitive understanding from their communication about them.

4.1.1 *Intuitive understanding*

In our view, when mathematists use keywords related to ≪mental images≫ , they do this for several interrelated purposes. For instance, they may do so to explain for themselves or for others how they intuitively understand a mathematical object or a mathematical proposition. They may also intend to provide access to the abstract discursive objects we call mathematical objects. The more advanced our mathematical discourses become, the more nested are the processes that have led to their objectification and the more detached are they from the physical world (see Section 3.3). In this case, mathematists, in particular experts who are engaged in teaching, may strive for at least a partial reversion of this process and intend to make these objects accessible to themselves, colleagues, or learners.

Understanding

We caution that we do not use the word "understanding" in an objectified manner (cf. Sfard, 2008, pp. 59–60). The commognitive framework conceptualises *understanding* as follows: It is the

> interpretive term used by **discursants** to assess their own or their interlocutors' ability to follow a given strand or type of **communication** (Sfard, 2008, p. 302, emph. orig.).

This implies that understanding is not an entity that we might observe or a mathematist "has" with respect to a mathematical notion. Rather, this usage of the keyword "understanding" focuses on how mathematists value their own or others' ability to follow certain types of communication. Hence, when we talk about experts' understanding of complex path integrals, we are interested in the narratives they tell about how they perceive their own ways of participating in complex analysis discourse. This includes, implicitly or explicitly, their valuations of what belongs to understanding in their view. Therefore, the use of understanding, in our commognitive interpretation here, is biased by the assessment of the user of this word in relation to an interlocutor (which might be her- or himself) and the mathematical content at hand. In Sfard's (2008, p. 302) words, this means that "the commognitive researcher, rather than assessing participants' understanding, is interested in the interplay of the participants' first- and third-person talk about understanding and their object-level discursive activity".

We consider the talk about potential mental images about mathematical objects embedded in at least two ways of talking about mathematics. First, this talk may arise when interlocutors respond to the task to interpret mathematical objects or propositions and describe how they understand them intuitively. Second, they may value particular descriptions of mathematical objects as particularly valuable for approaching the mathematical objects at hand for themselves or others (e.g., learners).

The case of mathematical experts is of particular relevance here. It is only likely that they have participated in mathematical discourses in teaching the subject, in discussing it with other experts, and in discourses about ≪mental images≫ or ≪intuition≫. For example, we suspect that expert mathematicians are able to describe mathematical objects in relation to how they individually perceive them and may value potential interpretations as appropriate approaches for others.

Excursus to intuition in the literature

In order to motivate our discourse of interest further, we briefly relate our choice of words here, namely the adjective "intuitive", to the literature. We note, though, that we will not explicitly make use of any conceptualisation of intuition from the literature; hence, we include this brief survey to delineate the potential variety of the specific way of talking about mathematics we wish to address when we refer to intuitive mathematical discourses.

Intuition, intuitive knowledge, and similar keywords have been discussed intensively in mathematics and mathematics education literature (e.g., Antonini, 2019; Ben-Zeev & Star, 2001; Burton, 1999, 2004; Davis et al., 2012; Dreyfus & Eisenberg, 1982; Fischbein, 1987; Griffiths, 2013; Hersh, 1998; Mariotti & Pedemonte, 2019; Tieszen, 1989; Tirosh & Tsamir, 2020; Wittmann, 1981). A frequently encountered characteristic is that intuitively accepted statements are those that an individual accepts without a need for further substantiation or at least is convinced of its endorsability. Hence, mathematical metarules for endorsement of narratives may not be followed. However, clearly, it might be the case that someone is able to prove a mathematical proposition but nevertheless does not describe her or his understanding of it as intuitive—or the other way around (Antonini, 2019).

For example, Fischbein (1987) characterises intuitive knowledge as "*a form of cognition which seems to present itself to a person as being self-evident*" (Fischbein, 1987, p. 6, emph. orig.). Additionally, he values it as

> an *idea* which possesses the fundamental properties of concrete, objectively-given reality; *immediacy*—that is to say intrinsic *evidence*—and *certitude* (not formal conventional certitude, but practically meaningful, immanent certitude). (Fischbein, 1987, p. 21, emph. orig.).

Similarly, Mariotti and Pedemonte (2019, p. 760) characterise intuition as "immediate self-evident knowledge" and Antonini (2019, p. 795) describes that it is "associated with a personal feeling of evidence, which is an individual's personal feeling that concepts are obvious and clear, and that properties and statements are intrinsically true". As such, "intuition is a compact and immediate form of knowledge that does not require justification" (Antonini, 2019, p. 795). This intuitive acceptance is described by Ben-Zeev and Star (2001, p. 32) (following Fischbein, Tirosh, and Melamed (1981)) as the "act of accepting a certain solution or interpretation directly without explicit or detailed justification".

Burton (2004, p. 73), referring to Hersh (1998), calls intuition "'an essential part of mathematics'" as "the opposite of rigorous", "visual", "plausible, or convincing in the absence of proof", "incomplete", "based on a physical model or on some special examples", is close to "'heuristic'", or "holistic or integrative as opposed to detailed or analytic". Even though intuition is often seen as a relevant part of mathematical practise, the role of intuitive explanations, which possibly also ground in pictures or related to special instead of general cases, is not unquestioned. Critics may disregard it as superfluous and conflicting with the often heard claim of the infallibility to mathematics (cf. e.g., Davis et al., 2012, p. 441).

We do not want to derive an objective view on intuition from the preceding observations. Instead, we aim at the specific way of describing mathematical objects or propositions, in which the discursants express what they consider to be plausible, visual, vivid, not necessarily requiring further substantiation, in word intuitive, according to their own judgement, in line with the previously mentioned commognitive point of view on understanding. This also agrees with Burton's (2004, pp. 76–79) observation that the word intuition is not used by every mathematician in the same way. Hence, what one person may describe as intuitive may be described by another person with another adjective, and vice versa (Burton, 2004, pp. 76–79) (a phenomenon we have encountered previously when discussing ≪mental images≫; see Chapter 2).

4.1.2 *Defining intuitive mathematical discourses about a mathematical object*

Intuitive understanding

When a mathematist finds that a mathematical statement does not require further reasoning, judges it to be intuitively clear, or explains a mathematical object or proposition with what she or he perceives as a mental image or intuitive explanation, we might say that the mathematist engages in or expresses her or his *intuitive understanding* of the corresponding object or proposition. Hence, we will refer to this type of communication as an *intuitive mathematical discourse* (Hanke, 2020a, 2020b).

Since we will deal with complex path integrals in this thesis, we call the discourses we are interested in *intuitive mathematical discourses about complex path integrals*. Even though a discourse does not only deal with instances of the same class of mathematical objects, we still call it this way here in order to emphasise that we are interested in mathematists' individual interpretations of these objects. In this regard, we define intuitive mathematical discourses about a mathematical object X as follows:

Intuitive mathematical discourse

Intuitive mathematical discourse

An *intuitive mathematical discourse about X* is a type of communication that is shaped by mathematists' descriptions of their intuitive understanding of X. This discourse is about X and other mathematical objects the interlocutors relate to X.

It contains keywords revolving around X but also keywords like "mental image", "intuitive", "vivid", "non-rigorous", "visual" etc.

An intuitive mathematical discourse is shaped in interaction, that is, it depends on the situations that give rise to mathematists' discursive actions in response to one of the various task situations, in which they are asked to describe their intuitive understanding of X or of a proposition about X.

As explained previously, these discourses are forms of communication inseparable from individuals' assessment of their intuitive understanding of a mathematical notion, which is also depending on their personal usage of words like intuition, mental image etc. as explained above. The adjective *intuitive* is used here merely as a pointer to this specific facet of communication, which in cognitively oriented strands of mathematics education are likely to be considered a subfacet of mathematical thinking. This type of communication may occur in various situations and with various goals (see Section 4.5). For example, mathematists engaging in this type of communication may intend to make highly abstract mathematical objects more easily accessible for themselves or others or to explain their personal interpretations of them. Intuitive mathematical discourses may also be directed to less experienced interlocutors for the sake of helping them to participate in a new mathematical discourse. They may also be addressed to experienced interlocutors for discussions on equal footing.

Intuitive mathematical discourses are likely varying from individual to individual. At least for the case of complex path integrals, they are likely not historically established, since no consistent set of interpretations are endorsed in the literature (e.g., as for the Riemann integral in the form of basic ideas Section 2.2.3; see Chapter 5). An individual's intuitive mathematical discourse is idiosyncratic but not completely individual because it is based on discursive elements, in particular keywords, from established mathematical discourses (e.g., complex analysis). In line with Biza's (2021) observation on interlocutors' usage of keywords from discourses they are familiar with in discourses, which are new to them, and Lavie et al.'s (2019) idea of precedent, it may be hypothesised that previous encounters with the use of keywords such as "mental image" or "intuitive" influence mathematists' use related to complex analysis. In Davis et al.'s (2012, p. 433) words, what an individual may consider to be intuitively true depends "on the basic of general experience with similar situations or related subjects". Hence, it is possible that a discursant has dealt with these kinds of precedents previously and has thus constructed well-crafted interpretation of the mathematical object (e.g., this is the case for Ricardo, who interpreted complex path integrals in terms of a metaphor involving a ship for the purpose of teaching it to his students as reported by Oehrtman et al. (2019); Section 5.4.3). On the other hand, two students in the study by Hancock (2018) recalled their lecturers' interpretation of complex path integrals in terms of addition of vectors, hence used a metaphor with which they were educated (Section 5.4.2). However, it is also possible that an interlocutor has never used these keywords in relation to a certain mathematical object such as the complex path integral before. In this case, intuitive mathematical discourses are built up ad hoc during the conversation with other individuals or in soliloquy.

Since experts are fluent participants in mathematical discourses, we are interested in Part iii in their intuitive mathematical discourses about complex path integrals. They have expertise in both, the mathematics and the teaching of the subject, and therefore, it is relevant to understand how they realise mathematical objects and create narratives about them in these discourses (e.g., whether and how they realise them graphically or what they do otherwise). Even though it is not yet clear, which rules govern this kind of communication, we do nevertheless call it intuitive mathematical *discourse* instead of *communication* because the mathematical keywords and the keywords centre around mental images and intuition stem from historically formed discourses.

In order to describe the production of intuitive mathematical discourses in more detail, we introduce two more notions, which will later also be at the heart of our empirical investigation in Part iii. The first is the reconceptualisation of ≪mental images≫ we announced earlier, namely *discursive images* (Section 4.2), and the second are sets of metarules, namely *discursive frames*, governing the construction of discursive images and substantiations in intuitive mathematical discourses (Section 4.4).

4.2 DISCURSIVE MENTAL IMAGES

Discursive (mental) images

We can now redefine mental images of a mathematical object X in terms of intuitive mathematical discourses about X. We define *discursive (mental) images* as certain narratives in intuitive mathematical discourses:

Discursive images

Discursive mental images, or *discursive images* for short, *about the mathematical object X* are narratives in intuitive mathematical discourses about X, which are constructed in response to any task situation to express one's own intuitive understanding about X or to make X accessible for others. Discursive images may be supported with visual mediators.

According to this definition, a visual mediator alone is not enough to count as a discursive image. Rather, discursive images are narratives, that is, stories somebody tells about the mathematical object X in an intuitive mathematical discourse. If a visual mediator is used, it needs to be placed in context to a narrative. For example, as we have suggested in Section 2.2.4, a visual mediator showing the plot of the graph of a function and the area between the graph and the horizontal axis is not a discursive image about Riemann integrals. The discursive image is the story we can tell about Riemann integrals together with this figure.

This conceptualisation also overcomes the duality between the individual and the social, which was lacking in some of the previous conceptualisations of ≪mental images≫ (see Chapter 2). Since a discursive image is a narrative, it is available for potential interpersonal substantiation, endorsement, or rejection. For the same reason, it may be used or modified by others for further constructions of narratives.

The conceptualisation of discursive images as narratives in intuitive mathematical discourses brings another potentially rewarding perspective. The unit of analysis has shifted from single cognitive entities to the surrounding discourses: The commognitive researcher can now investigate the construction of these discursive images in context together with the technical apparatus from the commognitive framework (Lavie et al., 2019; Sfard, 2008, 2018). Accordingly, discursive images are naturally bond to the situations in which they are produced and to the aims the discursants wish to fulfil (e.g., to substantiate a claim, to explain a mathematical object to a learner, etc.; cf. Hanke and Schäfer, 2017).

This shift from a rather narrow unit of analysis (mental images alone) to a broader unit (intuitive mathematical discourse) is particularly valuable for educational research for which there is little subject-matter didactical and empirical research to ground on. This is moreover impor-

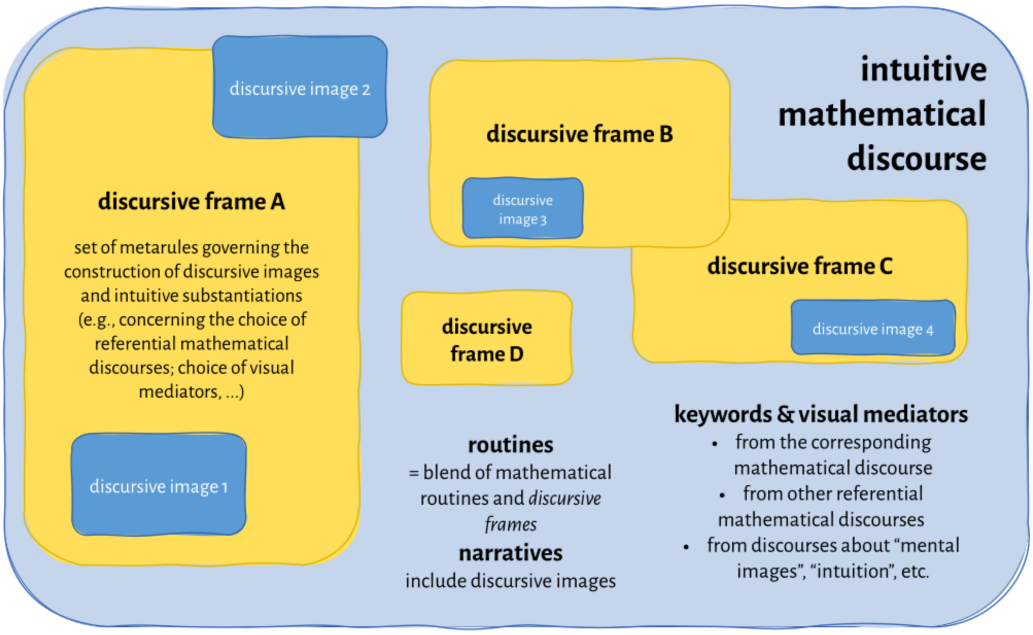


Figure 4.1: Venn diagram illustrating an intuitive mathematical discourse.

tant for educational research on mathematical discourses at university level such as complex analysis, where it is sometimes still possible to realise the mathematical objects directly graphically for dimensional reasons but sometimes not (e.g., one can rather easily draw domains of complex functions but not graphs of complex functions). Hence, mathematists need to find workarounds here in case they intend to produce visual mediators to realise mathematical objects such as complex path integrals. The discursive perspective also allows to find out whether and how mathematists try to transfer known interpretations from other branches of mathematics (e.g., real analysis or measure theory) to the case of complex analysis.

We note further that discursive images are inherently based on what a discursant described to belong to her or his mental images, intuitive explanations etc.[31] Hence, at least in principle, we may differentiate between the narratives a mathematist produces in task situations to express her or his intuitive understanding of a mathematical object or a proposition, which are explicitly valued as a mental images or intuition etc., and those narratives to the same or a similar task situation, which lack this explicit valuation.

4.3 CHARACTERISING INTUITIVE MATHEMATICAL DISCOURSES: KEYWORDS, NARRATIVES, VISUAL MEDIATORS, AND METARULES

In this section, we delineate the features of intuitive mathematical discourses about complex path integration further in terms of their keywords, visual mediators, narratives, and sets of metarules, which we will call *discursive frames*. Broadly speaking, a discursive frame is a set of metarules governing the construction of discursive images or intuitive substantiations. Figure 4.1 illustrates an intuitive mathematical discourse schematically.

31 The idea to regard something as a ≪mental image≫ based on endorsement is not new though. For example, *basic ideas* (Section 2.2.2) are prescriptions about a mathematical object endorsed in a community of mathematics educators (e.g., based on subject-matter didactical analyses; see Salle and Clüver, 2021).

Keywords

Keywords in intuitive mathematical discourses

Keywords in intuitive mathematical discourses about complex path integration come from complex analysis discourse and discourses about ≪mental images≫ and ≪intuition≫ as experienced by its interlocutors previously. They may also use keywords from other referential mathematical discourses, that is, from mathematical discourses, which they consider related in one way or another.

Visual mediators

Visual mediators in intuitive mathematical discourses

Visual mediators in intuitive mathematical discourses are visual mediators from complex analysis and the referential mathematical discourses. Interlocutors may of course create other visual mediators they deem suitable to the task situation.

Narratives

Narratives in intuitive mathematical discourses

Narratives in intuitive mathematical discourses on object-level and meta-level rules from the corresponding and referential mathematical discourses. The most important narratives we are interested in here are

- the discursive images about complex path integrals and
- *intuitive substantiations* for integral theorems from complex analysis discourse.

In these substantiations, the interlocutors aim to substantiate the endorsability or plausibility of a mathematical proposition in a way they value as intuitive, vivid, or especially accessible for others.

Metarules

Metarules in intuitive mathematical discourses

Metarules in intuitive mathematical discourses may be those from mathematical discourses. For instance, metarules from mathematical discourses likely appearing in intuitive mathematical discourses as well are recalling routines, routines for the production of visual mediators, substantiations, and others.

Moreover, it is probable that scholarly metarules (e.g., deductive reasoning, using keywords in the endorsed manner only, etc.; cf. Sfard, 2014; Viirman, 2021) are not followed strictly and are replaced with other metarules for the production of visual mediators or otherwise intuitive explanations. For example, discursants may create narratives in intuitive mathematical discourses, which they know to be non-endorsable in scholarly complex analysis discourse, but reflect on it and aim to reach a certain goal with these narratives nevertheless. Metarules in intuitive mathematical discourses may in particular address whether and how discursants select precedents in order to produce discursive images and intuitive substantiations.

With regards to substantiations, discursants may deviate from substantiations in scholarly discourses. For example, the experts in our empirical study occasionally substantiated an integral theorem with another proposition, which in turn is actually proven with the proposition they wanted to substantiate in the first place. Hence, the logical order of the narratives substantiating each other is reversed. We refer to this type of substantiation as a *retrospective substantiation*.

Pedagogical metarules

Occasionally, mathematicians describing their intuitive interpretations of complex path integrals or integral theorems will explicate metarules about how they (aim to) explain complex path integrals and integral theorems to their students—in other words, *pedagogical metarules* (Gallego-Sánchez et al., 2022; Heyd-Metzuyanim & Shabtay, 2019; Viirman, 2015, see e.g.,): These "shape and orient teachers towards what to teach students, how to teach them, why certain teaching actions are more effective than others [...]" (Heyd-Metzuyanim & Shabtay, 2019, p. 543). Pedagogical metarules may include metarules for teaching in general but also for the usage of visual mediators. With the help of these metarules mathematical experts may describe their own discursive behaviour as lecturers of complex analysis.

A special type of collections of metarules in intuitive mathematical discourses is introduced in the next section.

4.4 DISCURSIVE FRAMES

Since we have turned away from an acquisitionist view on ≪mental images≫, we have to replace questions like "which mental images does an expert have?" to questions like "which narratives does the expert produce during the realisation of an intuitive mathematical discourse?" In particular, the overarching question to be addressed is the "quest for discursive patterns" in this type of communication (Sfard, 2008, p. 200). To this end, we will now introduce the notion of *discursive frame* as a specific set of metarules in intuitive mathematical discourses to theoretically account for mechanisms for the construction of discursive images and intuitive substantiations.

In particular for the case of complex path integrals, we hypothesise that discursants in intuitive mathematical discourses resort to mathematical discourses other than complex analysis, in which they recall a certain interpretation for an integral and question whether it may be transferred to complex path integrals (we discussed this at length from the scholarly perspective in Part ii). Hence, we expect the discursants in intuitive mathematical discourses to borrow narratives or metarules from discourses they relate to complex analysis because they may recall an endorsed or a personal interpretation of an integral from there. These borrowings from other discourses are all the more likely if the discursants lack interpretations or visual mediators or even assume that there are no (endorsed) interpretations for the case of complex path integrals at all.

For example, some experts from our empirical study in Part iii brought up interpretations for real path integrals. In this case, we may infer that vector analysis discourse belongs to their precedents for interpretations of integrals and the experts may have constructed a discursive image that relate complex path integrals to real path integrals. Hence, a particular way to construct a discursive image becomes apparent. The metarules used for this construction may be either made explicit by one of the interlocutors themselves or reconstructed retrospectively by a researcher.

In order to understand these patterns of an interlocutor's discursive actions in her or his intuitive mathematical discourse, it is useful understand how they *frame* their discursive actions (cf. Goffman, 1974/1986; Hammer et al., 2005). Metaphorically speaking, we wish to regard a frame as a "thematic compass" with the help of which discursants respond to the quest to explain a mathematical object or proposition in their intuitive mathematical discourses. In other words, frames are "glasses" through which interlocutors could have looked before carrying out their subsequent discursive actions.

In the following, we briefly embed this point of view to the literature in mathematics education and derive a commognitive conceptualisation. In commognitive terms, we need to look for the metarules governing these choices and actions. In line with the tacitness of metarules as "observer constructs rather than explicit principles that the discursants would follow in a conscious, intentional way" (Sfard, 2008, p. 221), the sets of metarules according to which a discursant constructs a discursive image or an intuitive substantiation may be observer constructs as well.[32]

Similar ideas are used within other frameworks than commognition, too. For example, according to Arzarello et al. (1995, p. 122) a *conceptual frame* is a "structure of the data that is able to get a stereotypical representation of a knowledge" that also entails "specific conceptual aspects of knowledge as an organized set of conceptual notions and operational skills related to some precise pieces of mathematics." Similarly, Weinberg et al. (2014, p. 169) discuss the notion of frame in terms of a "mental structure", which an "individual uses [...] to organize his or her experience, to recognize patterns in new situations and to guide his or her activity". In this context, Barwell (2013a) describes experts' talk in terms of *interpretative repertoires*, that is, "'broadly discernible clusters of terms, descriptions and figures of speech often assembled around metaphors or vivid images'" (Wetherell and Potter (1992), cited in Barwell, 2013b, p. 211).

Heyd-Metzuyanim et al. (2018) proposed a discursive point of view on frames, which we will adopt here. They build on Hammer et al. (2005), who define a frame as

> a set of expectations an individual has about the situation in which she finds herself that affect what she notices and how she thinks to act. An individual's or group's framing of a situation that can have many aspects, including social ("Whom do I expect to interact with here and how?"), affective ("How do I expect to feel about it?"), epistemological ("What do I expect to use to answer questions and build new knowledge?"), and others. (Hammer et al., 2005; cited by Heyd-Metzuyanim et al., 2018, p. 26)

Heyd-Metzuyanim et al. (2018) describe furthermore that discursive metarules governing mathematical and social aspects of mathematical discourses often remain implicit (Sfard, 2008). According to the authors,

> [f]rames lie somewhere in between a specific meta-rule or norm and the much more general set of meta-rules that describe a discourse. They are based on regularities that can be observed in repetitive human interactions, yet most authors describe them in their internalized form, as cognitive structures (Heyd-Metzuyanim et al., 2018, p. 26).

They go on to operationalise the notion of frame commognitively as

> a set of meta-rules, [...] which includes appropriate questions, answers, justifications, and other discursive actions in a situation of solving a mathematical problem or performing a mathematical task. Phenomenologically, frames mean a set of an individual's expectations originating in repetitive interactions. (Heyd-Metzuyanim et al., 2018, p. 26)

These sets of metarules "cohere and reinforce each other" (Heyd-Metzuyanim et al., 2018, p. 29). The authors used the notion of frame in the context of mathematical instruction. It is broader applicable though. For the context of intuitive mathematical discourses, a frame shall

32 In fact, our motivation to look at *discursive frames* arose from our initial analysis of our interview data. It turned out that the experts did not produce a consensual set of discursive images about complex path integrals, but their actions were party similar. For instance, they resorted to other mathematical domains (e.g., vector analysis) multiple times or used similar mechanisms for their explorations of their personal stories about the complex path integral. Hence, we aimed to account for this phenomenon of similarity in discursive actions in intuitive mathematical discourses theoretically.

be understood as a set of metarules governing questioning, answering, and justifying in response to task situation addressing their intuitive understanding of a mathematical object or proposition.

Henceforth, we define *discursive frames*, or *frames* for short, *in intuitive mathematical discourses* as follows: *Discursive frames*

Discursive frames

Discursive frames in intuitive mathematical discourses about X are sets of metarules, possibly in close connection to object-level rules on X, which express "individual's expectations originating in repetitive interactions" (Heyd-Metzuyanim et al., 2018, p. 26) and guide discursants in intuitive mathematical discourses in constructing discursive images and intuitive substantiations. Discursive frames may not be made explicit by the participants themselves and have to be made explicit by the commognitive researcher.

In order to illustrate discursive frames in intuitive mathematical discourses, we again refer to Figure 4.1: Discursive frames are shown as yellow frames. They may overlap, which indicates that a discursant uses more than one frame at a time. A discursive image, which an interlocutor produced following a discursive frame is shown as a blue block within a yellow discursive frame. However, not every discursive image has to be constructed in terms of a discursive frame (after all, discursive frames are researchers' constructions and therefore, it need not be the case that every discursive image can be subordinated to these frames). We also note that not every frame will eventually lead to the construction of a discursive image: In this case, an interlocutor may start to follow certain metarules but does not arrive at a discursive image. In sum, we conclude that routines in intuitive mathematical discourses are blends of mathematical routines and discursive frames.[33]

As we explain in the epistemological analysis (Part ii), there are in fact several mathematical discourses (e.g., on real analysis in one variable, in two variables, measure theory, etc.) with the help of which a mathematist can respond to task situations about complex path integrals or with the help of which a definition for complex path integrals can be substantiated. These potential choices and subsequently the actually performed discursive actions from these potentially to be activated mathematical discourses give rise to mathematists' discursive frames.

For example, when someone tries to argue that a complex path integral also measures some kind of area under the graph of a function (resembling the basic idea of area for Riemann integrals; Section 2.2.3), she or he might be resorting to what could be called the *area frame*. Corresponding metarules are the use of the area interpretation for Riemann integrals or routines for drawing graphs of functions, visually realising areas, and searching an analogy to complex func-

33 At this point, we are also reminded of other commognitive tools for the analysis of discursants' actions. For example, Kontorovich (2021b) defines a *pocket precedent* as a set of precedents which "prescribes distinctive actions that are matched to the precedent identifiers that this person discerned from an assigned task situation" and which are differentiated "in terms of the discursive toolkit that it offers (e.g., words, routines)" (Kontorovich, 2021b, p. 4). Focusing on the use of keywords, Biza (2021) refers to *discursive footprints* as traces of students' experiences in different mathematical discourses where the same keywords are used with respect to different rules. Our notion of discursive frame is insofar different from these two conceptualisations as we focus on sets of metarules for the construction of discursive images and intuitive substantiations, not about their particular use of keywords or about their identification of precedents. Discursive frames however come close to *metanarratives* as introduced by Schüler-Meyer (2019). These account for "patterns in the discursants' activities of producing narratives [...]. In other words, metanarratives are themes behind the individual students' narratives" (Schüler-Meyer, 2019, p. 168). We operationalised these themes here in terms of metarules though.

tions. Hence, it is reasonable to assume that the interlocutor following these metarules aims to take narratives or visual mediators from interpretations of mathematical objects, which are perceived as related to the object at hand (e.g., Riemann integrals), as a starting point to describe a possible interpretation for the complex path integral.

Analysing discursive frames

In order to analyse discursive frames, we need to analyse how the mathematists "describe[] mathematical objects, which words and narratives they use[] and what routines they refer[] to" (Heyd-Metzuyanim et al., 2018, p. 28). From those observation we hypothesise underlying metarules. These are then "group[ed] [...] into frames, that is, sets of meta-rules that cohere and reinforce each other" (Heyd-Metzuyanim et al., 2018, p. 29). Hence, in order to identify a discursive frame, we have to look for possible origins of keywords, narratives, and metarules, which a discursant employs in order to explain a mathematical object or a proposition in a for her or him intuitive way:

- It is possible that a discursant explicates these metarules constituting a discursive frame her- or himself.
- If not, commognitive researchers conjecture a potential underlying set of metarules and examine whether this set reappears at other points in data. In case these metarules can be observed at several points in the data, it is plausible that we have indeed reconstructed a discursive frame.
- Alternatively, the hypothesised metarules are accepted because they are the most sensible explanation for why a discursant produced a specific narrative.

Therefore, descriptions of frames—and intuitive mathematical discourses more generally—are thus to be regarded as "analytical statement[s] with which the observed performance is consistent" (Kontorovich, 2021b, p. 5; see also Kontorovich, 2021a).

4.5 MANIFESTATIONS OF EXPERTS' INTUITIVE MATHEMATICAL DISCOURSES

Manifestations of experts' intuitive mathematical discourses

Mathematical experts may put intuitive mathematical discourses to practice in various situations. The case of experts' intuitive mathematical discourses is particularly important. On the one hand, these intuitive mathematical discourses may contain their personal narratives of the mathematical object in question, including potential idiosyncrasies, which the experts may or may not value as appropriate for others. On the other hand, experts may manifest these discourses with regards to less experienced interlocutors. In this case, the experts have accepted their role as leading figures in the teaching of a mathematical discourse (e.g., complex analysis) and they may develop stories about the highly abstract mathematical objects (e.g., complex path integrals) they find valuable for learners.

Experts' intuitive mathematical discourse may be enacted in various situations, such as

- in teaching: Experts may have developed certain discursive images, which they aim to address in their teaching to assist learners in getting access to the mathematical objects or propositions at hand. The experts may also have developed certain metarules, with the help of which they would like to help students to construct narratives or to enable them see connections between different mathematical discourses, which are based on heuristic observations (e.g., the experts may promote connections between integrals in different mathematical discourses);

- in office hours or informal meetings with students: Students may explicitly ask for a potential meaning or mental images of mathematical objects. Then, the experts need to respond to these quests and they may choose to replace scholarly mathematical metarules with other metarules about making the mathematical objects accessible. It is also possible though that the experts stick to scholarly mathematical discourse, in which the intuitive mathematical discourse is cut off;

- in conversations with colleagues / other experts: In this case, the interlocutors are all familiar with the scholarly mathematical discourse. Intuitive mathematical discourse may satisfy different goals here: Experts may exchange ideas about teaching a certain topic, they may communicate what they regard as the general ideas of their recent research, or tell stories about mathematical objects, in which they omit technical details (e.g., because their interlocutors are not familiar with the needed special technical constraints, because they cannot use paper and pencil or a blackboard, etc.);

- in specific types of communication in which mathematics is addressed for a wider audience (e.g., in popular mathematics texts, podcasts, radio broadcasts, etc.; cf. Barwell, 2013a); or

- in laboratory settings such as an interview with a mathematics educator: In this case, the discourse during the interview is not identical to discourses in authentic situations such as teaching or conversations with colleagues. However, such a laboratory setting is a very useful source of information for mathematics educators: An interviewer may initiate and guide experts in explicating what they consider their intuitive interpretations of mathematical objects and propositions. If the experts developed certain discursive images before, it is likely that these will be uttered in these laboratory settings, too. In these interviews, experts may however also construct discursive images ad hoc and mathematics educators can find out how the experts proceed. Hence, metarules for intuitive mathematical discourses can be identified, too.

We will focus on the last potential manifestation of intuitive mathematical discourses in Part iii: We plan and conduct an instrumental case study (Grandy, 2010; Stake, 1995) for which we collect and analyse data from three semi-guided expert interviews between experts and a mathematics educator "on equal footing" (Pfadenhauer, 2009; cf. Meuser & Nagel, 1991; Nohl, 2017). In this case, the interviewer actively contributes to the unfolding discourse because he specifically asks for the experts' mental images and intuitive explanations and thus, he also stimulates the explication of the metarules of their intuitive mathematical discourses about complex path integration. Expert and interviewer represent a "'commognitive system[]'" (Sfard, 2008, p. 128).

4.6 REFLECTION ON INTUITIVE MATHEMATICAL DISCOURSES

In this chapter, we conceptualised intuitive mathematical discourses as special types of mathematical discourses, in which mathematists either explain mathematical objects or propositions for themselves in a way they would describe as intuitive. This may also be the case when they explain the mathematical objects and propositions to other interlocutors, whose participation in the corresponding mathematical discourse shall be eased. Interlocutors in intuitive mathematical discourses borrow keywords and potentially also narratives from corresponding mathematical discourses. Intuitive mathematical discourses are nevertheless very individual:

> An individual's intuitive mathematical discourse centring on mathematical notions is thus not necessarily about endorsed narratives about mathematical objects per se, like in literate mathematical discourses, but rather about heuristics with which the individual makes sense of these notions. Yet, this can include elements of literate mathematical discourses, e.g., theorems or narratives about related mathematical aspects, of what the individual believes to be endorsable or rejected by other people, or narratives and visual mediators which show the discursants' struggles to express her- or himself. (Hanke, 2020a, p. 107)

Building on this conceptualisation, we reconceptualised ≪mental images≫ in terms of *discursive images*. Thus, discursive images are stories about mathematical objects but not visual mediators alone. Next to discursants' potential inclusion of narratives from scholarly mathematical discourses about the mathematical object in question, other narratives in intuitive mathematical discourses may be intuitive substantiations of mathematical propositions and experts' pedagogical metarules for teaching this mathematical object. Having defined discursive images in terms of narratives, we depend on the discursants' voicing of these narratives. However, these narratives may also be reconstructed close to the utterances of the discursants by commognitive researchers.

The study of intuitive mathematical discourses includes the study of mathematists' self-descriptions of what they perceive as their intuitive understanding of objects and propositions from a mathematical discourse (e.g., complex analysis). Hence, this study depends on the mathematists' willingness to detour from literate mathematical discourses and on their readiness for introspection. For example, it might not be the case that a mathematician judges any kind of a potential mental image to be useful in mathematics and therefore does not explore them further, the opposite, or anything in between. It is also possible that discursants are hesitant to describe their own intuitive understanding or mental images or do not feel able to express them. This uncertainty is then reflected in the realised intuitive mathematical discourses:

> Discursants may be unsure whether their own intuitive narratives are in some sense correct or shared by other discursants, or may be afraid of compromising themselves. Thus, the range of endorsement and the substantiations of the narratives a discursant produces in her or his intuitive discourse may vary notably (either within a discursive community, or with respect to what the single discursant expects as agreement from other discursants). (Hanke, 2020a, p. 107)

In sum, we have broadened the unit of analysis from single ≪mental images≫ to the larger unit of intuitive mathematical discourses. Consequently, discursive images are inseparable from the discourse in which they are produced. This allows us to conduct exploratory research on experts' personal interpretations of complex path integrals and integral theorems from complex analysis from the discursive perspective and to extend the available previous research (see Chapter 5).

Part II

EPISTEMOLOGICAL ANALYSIS OF COMPLEX PATH INTEGRALS AND PREVIOUS RESEARCH IN COMPLEX ANALYSIS EDUCATION

OVERVIEW OF PART II

This part includes our epistemological analysis of complex path integrals. It starts with a comprehensive review of research in complex analysis education and resources for the teaching of complex path integrals. We continue with a comprehensive analysis of the ways in which the complex path integral is introduced in current and historical texts on complex analysis. This analysis includes links to other integrals and other mathematical objects, various possible definitions, and substantiations of these definitions. In particular, we show that many of the central approaches to complex path integrals in recent literature on complex analysis can be embedded into a historical context. We also present an axiomatic characterisation of complex path integrals for holomorphic functions which, to our knowledge, has not yet been discussed in mathematics literature.

Figure II shows the outline of Part ii:

- In Chapter 5, we examine the literature on the education of complex analysis. This includes available resources for the teaching of complex path integrals and the state-of-the-art of empirical educational research in complex analysis.
- Chapter 6 presents background information for our epistemological analyses. Here we introduce in particular the notion of *aspects* and *partial aspects* from German subject-matter didactics (Greefrath et al., 2016a, 2016b; Roos, 2020). We illustrate this theoretical notion with Greefrath et al.'s (2016a, 2022) aspects of Riemann integrals. Moreover, we reconceptualise (partial) aspects in terms of the commognitive framework in order to classify the potential ways to define complex path integrals.
- In Chapter 7, we deal with the historical evolution of the notion of complex path integral. We illustrate how some of our mathematical ancestors had worked with integrals of complex functions before Cauchy (1825) gave the first definition of integrals of complex functions along paths in the complex plane. We also report on developments from the 20$^{\text{th}}$ century regarding the use of complex path integrals for the proofs of central theorems in complex analysis such as that holomorphic functions are analytic.
- The heart of our epistemological analysis is Chapter 8. Here, we explore various approaches to define complex path integrals and how these approaches are substantiated in complex analysis literature. Complex path integrals will also be related to other integrals from real / vector analysis and measure theory, and we discuss whether and which interpretations for other integrals may be transferred to complex path integrals. In this context, we also argue how several important integral theorems in complex analysis and relate to integrals theorems from real / vector analysis.

  Furthermore, having described the use of complex path integrals in complex analysis in terms of the covariation of a mapping with two inputs, namely paths and functions, we prove our axiomatic characterisation for complex path integrals of holomorphic functions (Theorem 8.13).

Affordances and constraints of horizontal (within complex analysis) and vertical (cross-curricular) connections of complex path integrals and other mathematical notions are explored in this chapter, too. This way, we intend to increase the sustainability of mathematical education from the point of view of complex analysis, exemplified with the notion of integral (see also Hanke, 2022b; Hochmuth et al., 2021; Kondratieva & Winsløw, 2018).

- In Chapter 9, we built on our discursive reconceptualisation of aspects and partial aspects from Chapter 6 and present *four aspects* and *four partial aspects of complex path integrals*. These aspects and partial aspects may count as the main result of the part of our epistemological analysis of the possible definitions of complex path integrals. Here, we pay special attention to the restrictions we have to impose on the paths and functions to be integrated.

We believe that Part ii makes a valuable contribution to research on the education of complex analysis by providing lecturers and mathematics educators an in-depth overview of possible approaches to the complex path integral. Our comprehensive epistemological analysis may help lecturers to make reflective decisions for their own teaching and researchers in mathematics education to find starting points for further research.

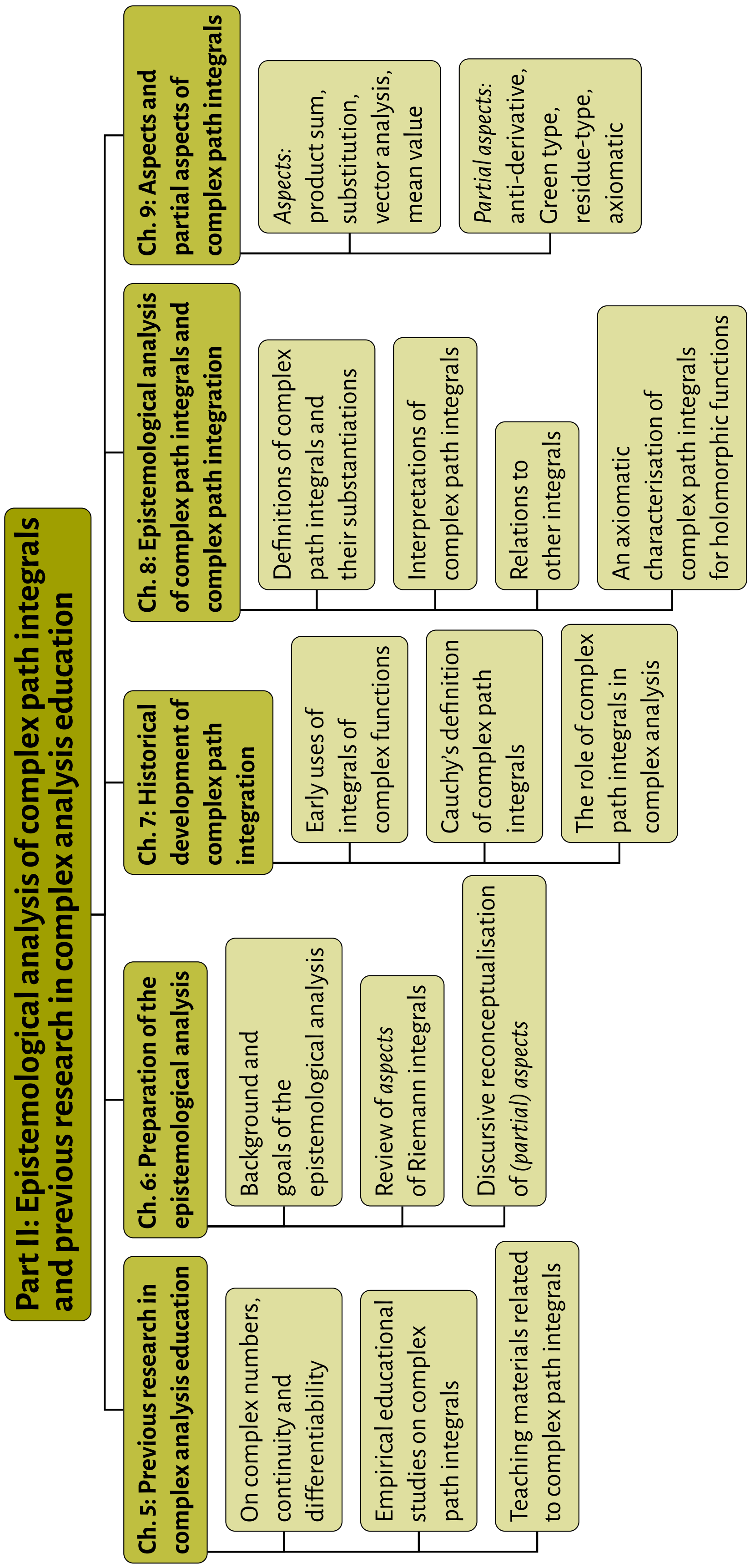


Figure II: Overview of Part ii.

5 LITERATURE REVIEW ON COMPLEX ANALYSIS EDUCATION

Wow, what does that mean? (long pause). Okay, so this is a hard question.
—One of the experts interviewed by Oehrtman et al. (2019, p. 315) on the geometric meaning of the complex path integral

This chapter is a review of literature on educational issues in complex analysis. It covers empirical research but also other literature related to the teaching of complex analysis. Most empirical educational research in this area has addressed the teaching and learning of complex numbers, geometric visualisation (of sets) of complex numbers, complex number arithmetic, or concept images of complex numbers. Recently, drawing on Danenhower's (2000) work, whose thesis is probably the first larger work in mathematics education on complex analysis, some researchers have done pioneering empirical work in mathematics education on more advanced topics in complex analysis like continuity, differentiation, and integration (e.g., Hancock, 2018; Oehrtman et al., 2019; Soto-Johnson et al., 2016; Soto & Oehrtman, 2022; Troup, 2015; Troup et al., 2017). In particular, we will review the studies by Hancock (2018) on students' understanding of integration of complex functions, by Oehrtman et al. (2019) study on experts' "construction of mathematical meaning" of complex path integrals, and by Soto and Oehrtman (2022) on how students realise complex path integrals in an interview without having dealt with integration in their complex analysis course before.

Explicit geometrical or physical interpretations of complex path integrals in textbooks are nevertheless rare. For example, Oehrtman et al. (2019, p. 395) motivate their research in complex analysis education with their observation that "complex analysis textbooks rarely develop foundational geometric interpretations of either differentiation or integration on the complex plane". They also describe that if geometric interpretations are presented in the context of complex path integrals, they mostly represent real path integrals or Riemann integrals of the real or imaginary part of the integrand of a complex path integral rather than the number $\int_\gamma f(z)\,\mathrm{d}z$ itself (see also Soto & Oehrtman, 2017). Some authors of textbooks or lecture notes on complex analysis even claim that it is difficult to find images for complex path integrals compared to their availability in real analysis (e.g., Brown & Churchill, 2009; Gathmann, 2017; Knopp, 1945–1947/1996). Some textbooks (e.g., Polya & Latta, 1974), and specifically Needham's (1997) "Visual complex analysis", the articles by Braden (1987) and Gluchoff (1991), or textbooks emphasising domain colourings for complex functions (e.g., Ponce Campuzano, 2019a; Wegert, 2012) are exceptions. In this context, several teaching proposals on complex analysis have been developed, too (e.g., Brilleslyper et al., 2012; Howell et al., 2017a; Kinney, 2013).[34]

The outline of this chapter is as follows. First, mathematics education literature on the teaching and learning of complex numbers, continuity, and differentiation will be reviewed (Section 5.1). In particular, we include a description of the interpretation of the complex derivative as an "amplitwist" (Needham, 1997) (Section 5.1.3). Then, we concentrate on the visualisation of complex functions (Section 5.2) as well as suggestions for the teaching of integration in complex analysis, mostly from a special issue called "Revitalizing complex analysis" (Howell et al., 2017a) but also in some other journal directed to lecturers of complex analysis (Section 5.3).[35] These three sections are mainly included for the sake of completeness of our review of the state-of-the-art in complex analysis education and to provide some background information we will occasionally refer to again. They may be skipped on first reading. Previous empirical research in mathematics education on integration in complex analysis is mainly based on the research by Danenhower (2000), Hancock (2018), Oehrtman et al. (2019), and Soto and Oehrtman (2022). Even though, only one of them is concerned with experts in complex analysis, we will summarise each of these papers detailed with respect to the research goals of this thesis in Section 5.4, because both, the papers about students and experts in complex analysis share the lack of interpretations for complex path integrals as one of their main results. At the end of this chapter, we will accordingly address the desiderata in complex analysis education, which we tackle in the rest of the thesis (Section 5.5).

34 It seems that the interpretations of complex path integrals, which have been published for the sake of teaching complex analysis (Braden, 1987; Gluchoff, 1991; Needham, 1997; Polya & Latta, 1974, e.g., those by), are not very well disseminated the textbook literature. For instance, none of the German classics by Fischer and Lieb (2003, 2010), Freitag and Busam (2006), Jänich (2004), and Remmert and Schumacher (2002) or the more recent books by Bornemann (2016), Forst and Hoffmann (2012), Müller (2018), and Wegert (2012) refer to them. We do not argue here that textbooks should have referred to them, but this observation underlines that the previously mentioned approaches to complex path integrals do not seem to have entered many textbooks, in line with the observance by Oehrtman et al. (2019), which quoted above. The interpretation of complex path integrals using the so-called *Pólya vector fields* associated to a complex function (Section 8.2.2; Braden, 1987; Needham, 1997, ch. 11; Polya and Latta, 1974, ch. 5.1) has been taken up several times in more specialised literature though (see Section 5.2.2). It remains to be investigated in further research whether these interpretations are either really little known or whether there are other reasons why they rarely appear in textbooks on complex analysis.

35 Consequently, we especially reserve the detailed analysis of the interpretation of complex path integrals for example in Braden (1987), Gluchoff (1991), and Needham (1997) for Chapter 8.

5.1 ON COMPLEX NUMBERS, CONTINUITY, AND COMPLEX DIFFERENTIATION

5.1.1 *On complex numbers*

Much research in mathematics education focused on students', pre-service or in-service teachers', and experts' conceptions of complex numbers with varying theoretical background, from concept image (Chin & Jiew, 2018; Nordlander & Nordlander, 2012) to embodied cognition (Nemirovsky et al., 2012), diagrammatic reasoning (Soto-Johnson & Troup, 2014), or metaphors (Soto-Johnson et al., 2012). One of the earliest investigation in this direction is Danenhower's (2000) doctoral dissertation. Moreover, there are empirical reports or suggestions on the teaching of complex numbers, their algebraic or polar representations, their arithmetic, and complex functions (e.g., Caglayan, 2016; Dittman et al., 2017; Driver & Tarran, 1989; Harel, 2013; Kinney, 2013; Panaoura et al., 2006; Soto-Johnson, 2014). Expert mathematicians' conceptions of complex numbers and functions were addressed, too (Soto-Johnson et al., 2012; Soto-Johnson et al., 2011). For example, Soto-Johnson and Troup (2014) analysed students' reasoning about tasks such as to explain why the identities $z\overline{z} = |z^2|$ or $1/z = 1/r \cdot \mathrm{e}^{-i\theta}$ hold true for any complex number $z = r\mathrm{e}^{i\theta}$ and found strong interplay between speech, inscriptions, and gestures. One of the central findings is that many learners struggle to switch between the algebraic and polar representation of complex numbers or to switch between the symbolic inscription and graphical representation as points in the plane, and that bodily activities can assist learners' reasoning with complex numbers. (cf. Danenhower, 2006; Ghedamsi, 2017; Grønbaek & Winsløw, 2015; Jaworski et al., 2018; Karakok et al., 2013, 2015; Oh et al., 2013; Paoletti et al., 2013; Rosseel & Schneider, 2003; Treffert-Thomas et al., 2019)

In this context, Caglayan (2016) and Nachlieli and Elbaum-Cohen (2021) make use of the commognitive framework (Chapter 3). Caglayan (2016) examines the richness of mathematics teachers' discourses when working in a dynamic geometry environment on the multiplication of complex numbers and roots of unity. Nachlieli and Elbaum-Cohen (2021) highlight that shifts in metarules on numbers need to be carefully emphasised in class when complex numbers are to be introduced. For example, one substantial shift from real number discourse to complex number discourse is that roots can no longer *not* be taken from negative numbers. Furthermore, they describe that it is a pedagogical challenge for teachers to make use of the idea of roots of negative numbers when they initiate a discourse on complex numbers in class (see also Kontorovich, 2018a, 2018b, 2019, 2021b).

5.1.2 *On continuity*

In their case study with five mathematical experts, Soto-Johnson et al. (2016) investigated their reasoning about the continuity of complex functions by making use of a distinction by Schiralli and Sinclair (2003) between "ideational mathematics" (experts' idiosyncratic reasoning) and "conceptual mathematics". The experts were prompted with the interpretation of continuous functions as those whose graph can be traced without lifting the pen from the paper (Branchetti et al., 2020; Hanke, 2018; Hanke & Schäfer, 2017) and then asked to give "geometric representations or explanations [...] to understand continuity of complex-valued functions" (Soto-Johnson et al., 2016, p. 368, emph. omitted). The authors documented four main "IM notions" (ideational mathematics notions), namely "control, physical embodied experiences attempting to capture topological features, preservation of closeness, and paths" (Soto-Johnson et al., 2016, p. 383). Moreover, several examples of "domain-first" conceptions were discovered.

These are descriptions of continuity, in which the domain of the function is taken into account first, as opposed to the definition of continuity of a function at a point using quantifiers (Soto-Johnson et al., 2016).[36]

5.1.3 *On differentiation and the idea of "amplitwist"*

Some research on students' and experts' understanding of the complex derivative and continuity of complex functions is available based on case studies with students (Troup, 2015, 2019; Troup et al., 2017) and experts (Oehrtman et al., 2019) (see also Soto-Johnson and Oehrtman (2011), Soto and Oehrtman (2017), Soto-Johnson et al. (2012), and Soto-Johnson et al. (2011), Troup (2017) for preliminary reports). These works build on Needham's (1997, ch. 4) interpretation of the derivative as an *amplitwist*. If $f\colon \Omega \longrightarrow \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$, there is a function $\varphi\colon \Omega \longrightarrow \mathbb{C}$ such that

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \varphi(z)$$

for $z \in \Omega$ and

$$\lim_{z \to z_0} \frac{\varphi(z)}{z - z_0} = 0$$

(see Proposition A.7). Thus, f is locally approximable by the affine linear function $z \longmapsto f(z_0) + f'(z_0)(z - z_0)$ such that the error function φ is converging to 0 faster than linear when $z \longrightarrow z_0$. Therefore, roughly speaking, one has

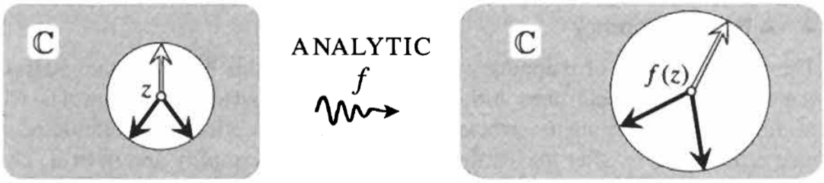
$$f(z) - f(z_0) \approx f'(z_0)(z - z_0)$$

for z close to z_0. Hence, differences of the function values for z close to z_0 can be approximated by the difference of the arguments $z - z_0$ multiplied by $f'(z_0)$. Multiplying a complex number with $f(z_0)$ corresponds to a rotation by $\operatorname{Arg}(f'(z_0))$ and dilation by $|f'(z_0)|$ in the plane. Geometrically, this means that small geometric figures around z_0 are approximately mapped onto rotated and dilated copies of this figure (see Figure 5.1a). In this context, the word *amplitwist* "refers to the act of 'amplifying and twisting' an infinitesimal geometric figure" Needham (1997, p. 198). This is illustrated in Figure 5.1a: An "infinitesimal circle" (Needham, 1997, p. 197) around z is dilated and rotated into another "infinitesimal circle" around $f(z)$. The three arrows shall indicate that the dilation is the same in each direction. An indicator for the function not to be complex differentiable is that the dilations vary with respect to the directions as is shown in Figure 5.1b. According to Needham (1997, p. 197),

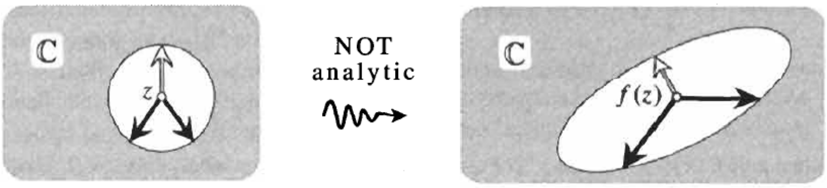
> [holomorphic; EH.] mappings are precisely those whose local effect is an amplitwist: all the infinitesimal complex numbers emanating from a single point are amplified and twisted the same amount.

As such, one might say, borrowing Needham's (1997) words, "infinitesimal circles" are mapped to "infinitesimal circles" under holomorphic functions, but to "infinitesimal ellipses" when the function (more precisely its underlying vector field) is real totally differentiable

36 Recall that the definition for $f\colon D \longrightarrow \mathbb{C}$ to be continuous at $z_0 \in D$ can be written as $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in D : |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$. Accordingly, this definition starts with a quantification about the codomain, namely a prescribed distance for values of the functions in terms of ε.



(a) Illustration of an amplitwist: f is complex differentiable at z (Figure [10] from Needham (1997)).



(b) Illustration of a non-amplitwist: f is totally differentiable as a vector field but not as a complex function) (Figure [9] from Needham (1997)).

Figure 5.1: Illustration of an amplitwist and a non-amplitwist from Needham (1997).[a]

[a] In this figure, the word "analytic" stands for what we call "holomorphic". The wording "not analytic" is used to signify here that the corresponding function is total differentiable as a vector field but not holomorphic as a complex function.

(hence, of course, pictures with geometric figures and their images for a given function are only a guide for detecting complex differentiability).

It was shown that scaffolds with dynamic geometry programmes can assist students to interpret the derivative of a complex function as an amplitwist (Soto-Johnson & Hancock, 2018; Troup, 2015, 2017, 2019; Troup et al., 2017). These studies demonstrate how interpretations of derivatives of real-valued functions of one real variable may interact with students' development of the idea of amplitwist. Troup (2019) summarises that

> moving in different directions between the embodied and symbolic worlds helped the participants better reason in the formal world regarding the derivative of a complex-valued function as an amplitwist, specifically by connecting the symbolic magnitude of the derivative to the embodied dilation of a small circle, and the symbolic argument of the derivative to the embodied rotation of a small circle (Troup, 2019, p. 22).

Most experts' interviewed in the study by Oehrtman et al.'s (2019) explained differentiability in complex analysis by comparing it to differentiability of $\mathbb{R}$- or $\mathbb{R}^2$-valued functions of one or two real variables and related complex differentiability to the Cauchy-Riemann differential equations (A.6). They also took into account the Jacobian of the underlying vector field of complex functions, individual versions of the amplitwist idea, and explained complex differentiability by giving examples. Additionally, some of the experts noted that they prefer to work with equations rather than geometric interpretations (Oehrtman et al., 2019).

5.2 VISUALISING COMPLEX FUNCTIONS AND COMPLEX PATH INTEGRALS

Graphs of real-valued functions of one or two real variables are frequently used to illustrate concepts from real or vector analysis (e.g., Riemann integrals) and accordingly appear in many textbooks on the subjects (cf. Section 2.2.3 and Section 6.3.2). The graph of a complex function $f: \Omega \longrightarrow \mathbb{C}$ is the set

$$\{(z, f(z)) \in \Omega \times \mathbb{C} : z \in \Omega\} \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$$

in 4-space, which is hard to visualise for human beings, and it is not possible to draw them on 2-dimensional paper or to model them in 3-space. Hence, we need other ways to depict the graphs of these functions in general.[37] The following quote underlines this problem:

> *Among the most insightful tools that mathematics has developed is the representation of a function of a real variable by its graph. ... The situation is quite different for a function of a complex variable. The graph is then a surface in four-dimensional space, and not so easily drawn. Many texts in complex analysis are without a single depiction of a function. Nor is it unusual for average students to complete a course in the subject with little idea of what even simple functions, say trigonometric functions, 'look like'.* (Arnold & Rogness, 2008; cit. by Wegert & Semmler, 2011, p. 768, emph. orig.)

However, there are several options to visualise complex functions nevertheless. Basically, these are the following (e.g., Bornemann, 2016, ch. 1.7; Freitag & Busam, 2006, pp. 53–56; Wegert, 2012; Wegert & Semmler, 2011):

(a) separate visualisations of subsets of Ω and their images or preimages with respect to f in $\mathbb{C} \cong \mathbb{R}^2$;

(b) plots of the graphs of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $\mathrm{Arg}(f)$, and the analytic landscape of f (i.e., the graph of $|f|$), or other compositions of functions $r \circ f$, where r is a real-valued function on the image of f, in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$;

(c) vector field plots of the vector field $\mathbf{f}$ or the Pólya vector field $\mathbf{w}_f$ associated to f (e.g., Braden, 1985, 1987; Needham, 1997, chs. 10–11; Polya & Latta, 1974);[38]

(d) one of multiple variants of domain colourings (or phase plots) of f (see e.g., Farris, 1998, 2017; Poelke & Polthier, 2009, 2012; Ponce Campuzano, 2019a, 2019b, 2021; Wegert & Semmler, 2011; and in particular Wegert, 2012); and

(e) coloured analytic landscapes (e.g., Wegert, 2012; Wegert & Semmler, 2011).

Examples for the methods in (a) to (e) can be seen in Appendix C. The methods (a) to (c) appear in several textbooks, lecture notes etc., where they are used to depict complex functions in general (e.g., Bornemann, 2016; Freitag & Busam, 2006; González, 1992; Needham, 1997; Remmert & Schumacher, 2002). For instance, (d) (and accordingly in principle also (e)) can be used to help identifying properties of complex functions such as roots, conformality, differentiability, or singularities and their types singularities (see also Appendix C and Benítez et al., 2013; Farris, 2017; Poelke and Polthier, 2009, 2012; Ponce Campuzano, 2019a, 2019b, 2021; Wegert, 2012; Wegert and Semmler, 2011).

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37 If more properties of the function f are known, for example, that it only takes real values, then, a plot of the graph of the corresponding real-valued function of two real variables in $\mathbb{R}^3$ is possible.

38 Recall that the vector field associated to a complex function $f = u + iv$ of a complex variable is the vector field $\mathbf{f} = (u, v)^T$ and the *Pólya vector field* associated to f is $\mathbf{w}_f = (u, -v)^T$, both in two real variables according to the identification of $\mathbb{C}$ with $\mathbb{R}^2$ (Section A.2; Braden, 1987; Polya and Latta, 1974, ch. 5.1; Needham, 1997, ch. 11).

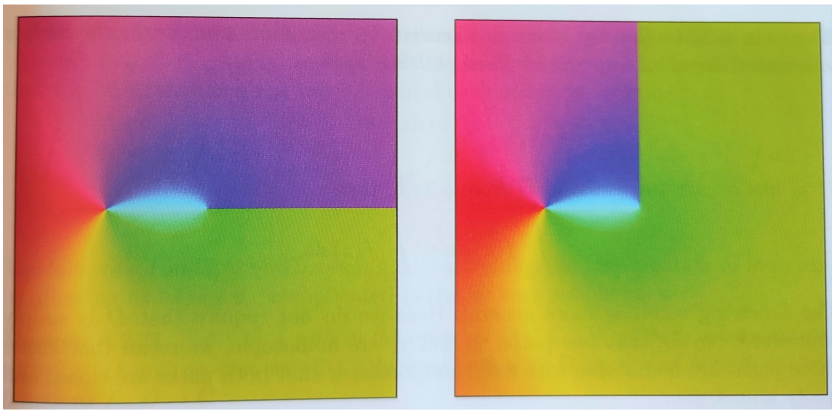

Figure 5.2: Domain colouring of two primitives of $z \longmapsto \frac{1}{z}$ on two sliced squares (Figure 4.13 from Wegert (2012)).[a]

[a] The two primitives are constructed as follows. Let Q denote the eventually to be coloured square in the complex plane shown in both parts of the figure. The centre of Q is the origin. The left part of the figure shows the domain colouring of a primitive function for $z \longmapsto \frac{1}{z}$ on $Q \setminus [0, \infty)$ and the right part shows a primitive function on $Q \setminus \{ti : t \in [0, \infty)\}$. Each primitive function is constructed as $z \longmapsto \int_{\gamma_z} \frac{1}{\zeta} \, d\zeta$, where γ_z is a path from the lower left corner of Q to z (see Existence of primitives for holomorphic functions (Theorem A.20)). The choice of γ_z is different for the left and right part of the figure: One time, γ_z is the path that traverses along a line segment parallel to the real axis first followed by a line segment along the imaginary axis; the other time, γ_z is the path that traverses along a line segment along the imaginary axis first followed by a line segment along the real axis. As can be seen in the figure, the two primitive functions coincide on $Q \setminus ([0, \infty) \cup \{ti : t \in [0, \infty)\})$ except for the top right quadrant.

5.2.1 *Textbooks on visualising complex functions*

Some textbooks and online materials on complex analysis put an emphasis on visualisations (e.g., Forst & Hoffmann, 2012; Kortemeyer, 2020; Needham, 1997; Ponce Campuzano, 2019a; Wegert, 2012). Mostly, these cover complex number arithmetic and geometry of complex numbers (Kortemeyer, 2020), elementary functions of one complex variable (Ponce Campuzano, 2019a, n.d.), or phase portraits of complex functions (Ponce Campuzano, 2019a; Wegert, 2012). In case these books deal with complex integration, the visualisations rarely explicitly signify the complex path integral. Instead, they rather realise paths, integrands, or primitive functions for the integrand.

The book by Forst and Hoffmann (2012) is designed for computation and visualisation of topics in complex analysis. It includes chapters on complex path integration, Cauchy's formula, the residue theorem, and emphasises computation. However, there are no visualisations for complex path integrals. Basically, all visualisations implemented in this book around complex path integrals show traces of paths of integration.

Wegert (2012) makes intensive use of various variants of domain colourings to explore complex functions. A large variety of concepts from complex analysis are illustrated with domain colourings. The figures in the chapter on complex path integrals show domain colourings for selected primitive functions, Cauchy integrals,[39] and illustrates analytic continuation (Wegert, 2012, chs. 4.2–4.3). As an example, Figure 5.2 shows the domain colourings for two primitive functions of $z \longmapsto \frac{1}{z}$ on two subsets of a square in $\mathbb{C}$.

39 A Cauchy integral is a function of the form $z \longmapsto \int_\gamma \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta$, where φ is a continuous function on the trace of a given path γ and z is a point not on the trace of γ (Wegert, 2012, p. 227; cf. Cauchy's integral formula (Theorem A.22)).

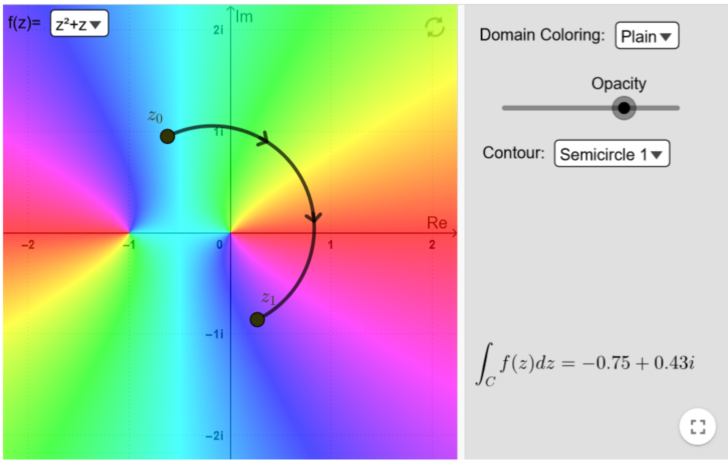

Figure 5.3: Plot of a semi-circular path and domain colouring of $f(z) = z^2 + z$.[a]

[a] The plot was done with the tool provided by Ponce Campuzano (2019a) at https://complex-analysis.com/content/complex_integration.html (retrieved 06/20/2022).

In Ponce Campuzano (2019a), learners can use domain colourings for different functions and plots of paths in these coloured planes to investigate properties of the complex path integrals. An example is shown in Figure 5.3: We see the plot of a semi-circular path in the phase plot of the function $z \longmapsto z^2 + z$; the value of the corresponding integral is shown to the right and is only loosely connected to the colourful plot.[40]

We would like to emphasise that we acknowledge the importance and usefulness of these innovative and carefully designed books and materials very much. Nevertheless, we conclude that explicit visualisations of complex path integrals in the sense of visualisations, which emphasise how the number $\int_\gamma f(z)\,\mathrm{d}z$ is related to the path γ and the function f, are rare. At the appropriate places in Chapter 8, we will continue to embed visual resources on complex path integrals. For instance, we will see an example for the use of domain colourings to illustrate that $\int_{\partial B_1(0)} \frac{1}{z}\,\mathrm{d}z = 2\pi i$ in Section 8.1.1 after we have recalled the definition of complex path integrals in terms of limits of complex Riemann sums.

5.2.2 *Visualising complex functions and complex path integrals using* Pólya vector fields

Complex path integrals can be decomposed into its real and imaginary part,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y,$$

where $f = u + iv$. Kinney (2013) uses different plots for the corresponding vector fields $(u, -v)^T$ (i.e., the Pólya vector field) and $(v, u)^T$, which appear at the right side of the previous equation to illustrate the real and imaginary parts of complex path integrals (see also Section 8.2).

40 Similar plots are used in the teaching proposal by Ponce Campuzano et al. (2019) for the teaching of real path integrals in vector fields.

Similarly, if $\mathbf{w}_f = (u, -v)^T$ denotes the Pólya vector field corresponding to f, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N}.$$

Here, the real part of the complex path integral, $\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T}$, is the real path integral of second kind of $\mathbf{w}_f$ with respect to the tangential component of γ and the imaginary part, $\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N}$ is the real path integral of second kind of $\mathbf{w}_f$ with respect to the normal component of γ. These two integrals measure the flow of $\mathbf{w}_f$ along the trace $\mathrm{tr}(\gamma) = \gamma([a, b])$ and the flux across $\mathrm{tr}(\gamma)$ (with respect to orientation of $\mathrm{tr}(\gamma)$ induced by γ) (see Section 8.1.4 and Section 8.2.2; Braden, 1987; Polya and Latta, 1974, ch. 5.1; Needham, 1997, ch. 11).

Brilleslyper et al. (2012) discuss several applications of complex functions and consequently deal with topics that may appear at the end of an introductory course on complex analysis, subsequent courses, or seminars. They discuss complex path integrals in terms of applications to the study of vector fields and use the Pólya vector fields associated to complex functions (see Section 8.2.2) (Brilleslyper et al., 2012, ch. 3). Other proposals to visualise complex functions and some of their properties with the help of Pólya vector fields date back roughly 30 years. Using technology from that time, Braden (1985, 1987), Gluchoff (1993), Kraines et al. (1990), and Newton and Lofaro (1996) contributed to visualising complex functions in general, power series, singularities, or winding numbers of paths. Braden (1985) explains how the plots of Pólya vector fields can be used to estimate the location of roots, poles, and points of complex differentiability of a complex function f in terms of $\mathbf{w}_f$. For example, complex differentiability of f at some point is equivalent to the vanishing of $\mathrm{div}\,(\mathbf{w}_f)$ and $\mathrm{rot}\,(\mathbf{w}_f)$ (see also Appendix A and Appendix B). Gluchoff (1993) illustrates the convergence of power series using similar plots. The role of Pólya vector fields in complex analysis has also been underlined in more advanced literature (e.g., Ionescu et al., 2021).

Braden (1987) uses Pólya vector fields associated to f for giving a physical interpretation of the complex path integral in terms of two real path integrals of second kind. We will explain these observations detailed in Section 8.1.4 and Section 8.2.2. Kraines et al. (1990) illustrate the formula for the winding number $\mathrm{Ind}_K(0) = \frac{1}{2\pi i}\int_K \frac{\mathrm{d}z}{z}$ for positively oriented circles or limaçons K, some of whose interiors contain the origin and some of whose do not (if the origin is contained in one of these interiors, the formula is a special case of Cauchy's integral formula (Theorem A.22) for $f \equiv 1$, see Equation A.24). For this to be done, they plot K, its images under the mapping $1 \longmapsto 1/z$, and plots of $\frac{1}{2\pi i}\int_K \frac{\mathrm{d}z}{z}$.[41]

Furthermore, Custy (2011), Smith (2008), Wilkinson (2011), and Yoshiaki (2017) designed digital tools to visualise complex path integrals.[42] Generally, these tools show plots of traces of paths and the function to be integrated in terms of vector field plots.

Custy's (2011) tool shows an illustrates the complex path integral of the function $z \longmapsto z^n$ (for variable $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$) along the unit circle starting at 1. The user can adjust $\theta \in [0, 2\pi]$ to parametrise this the circle via $\theta \longmapsto \mathrm{e}^{i\theta}$. According to this variation, the plot shows the circle segment $\{\mathrm{e}^{it} : 0 \leq t \leq \theta\}$, the vector field plot corresponding to $z \longmapsto z^n$ (gray; scaled), the tangential vector to the circle at $\mathrm{e}^{i\theta}$ (red), the complex number $(\mathrm{e}^{i\theta})^n$ displayed as a vector attached to $\mathrm{e}^{i\theta}$ (blue), and the product of these vectors in terms of

41 From the brief description in the paper, we infer that these plots of the integrals are in fact plots of the sets of complex numbers $\left\{\int_a^x \frac{1}{\gamma_K(t)}\gamma_K'(t)\,\mathrm{d}t : x \in [a, b]\right\}$, where $\gamma_K \colon [a, b] \longrightarrow K, t \longmapsto \gamma_K(t)$, is a continuously differentiable parametrisation of the respective contour.

42 These are parts of the *WOLFRAM Demonstrations Project* (https://demonstrations.wolfram.com/; retrieved 04/10/2021).

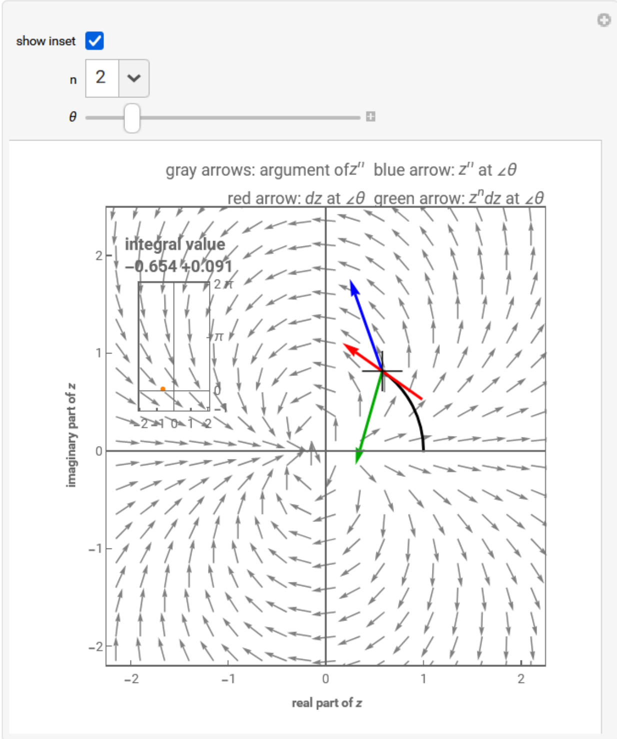

Figure 5.4: Illustration of the tool by Custy (2011).

complex multiplication (green). You can see an example of this tool in Figure 5.4 for the case $n = 2$.

In Wilkinson's (2011) tool, we can select some paths ("curve to integrate over") and functions ("integrand") for integration, and decide whether the normal or tangential component of the Pólya vector field associated to the chosen function shall be drawn (see also Kinney, 2013). The Riemann integral for the function shown in the "tangential component graph" corresponds to the real part of the complex path integral under consideration, and the Riemann integral from the function in the "normal component graph" is the imaginary part of it. In Figure 5.5, we can see an example for the integrand $z \longmapsto \frac{1}{2z+3}$, the circle with radius 2 and centre 0, and the corresponding normal component graph: The corresponding complex path integral is $\int_{\partial B_2(0)} \frac{1}{2z+3}\,\mathrm{d}z = i\pi$, which corresponds to the Riemann integral for the function shown in the normal component graph.[43]

In Smith's (2008) tool, we can draw a path of integration with the mouse and select or enter functions to be integrated along this path (Figure 5.6). The value of the integral is finally shown after clicking on the button "integrate contour". There is however no explanation for how these numerical values relate to the paths or integrands.

Figure 5.7 shows a screenshot of the tool by Yoshiaki (2017). On the top left, we can select an integrand and in the top coordinate system, we can draw a path with the computer mouse. The images of this coordinate system and the drawn path with respect to the chosen integrand are then displayed simultaneously at the top right part of the figure. The corresponding complex path integral is also shown simultaneously as a function of the path being drawn (see also Kraines et al., 1990).

43 To be precise, the graph shown in Figure 5.5 is that of the flow of the Pólya vector field across the circle at $2e^{it}$ as a function of $0 \leq t \leq 2\pi$.

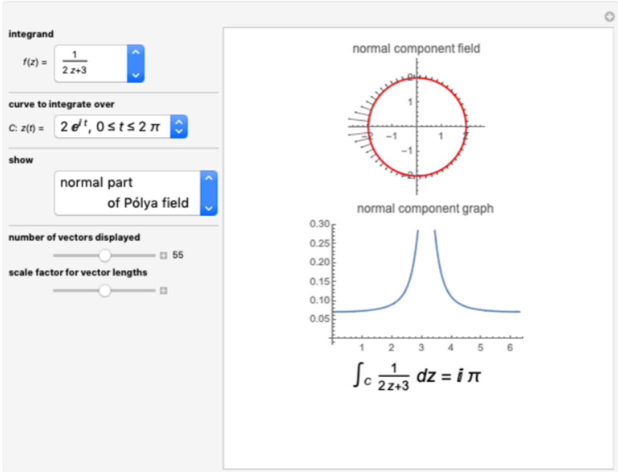

Figure 5.5: Illustration of the tool by Wilkinson (2011).

Finally, some videos demonstrate the relationship between complex path integrals and Pólya vector fields as well (Chen, 2021; Lemmaxiom, 2021; Mathemaniac, 2022). These videos basically repeat the ideas from the sources we described earlier. Their advantage is that a narrator explains what is seen and that these videos include moving pictures rather than static ones. Additionally, the video by zetamath (2022) explains the definition of the complex path integral via complex Riemann sums (Section 8.1.1) and the complex version of the fundamental theorem of calculus (see Equation A.17).

In sum, there are several possibilities to visualise complex functions. At least the plots of vector fields associated to a complex function and phase plots have been used in the context of integration in complex analysis. We will explain the use of these visual mediators further in Chapter 8. Whereas plots of the graphs of the real and imaginary part of a complex function may be useful for the study of the respective complex function, these visual mediators do not seem to have been productively used in the literature to realise complex path integrals.[44] In particular, Pólya vector fields and their real path integrals have been used to realise complex path integrals and therefore, we will explore these connections detailed in our upcoming epistemological analysis, too.

5.3 "REVITALIZING COMPLEX ANALYSIS"

In 2017, a special issue of the mathematics education journal *PRIMUS* was devoted to complex analysis (Howell et al., 2017a). For reasons of complex analysis' beauty, wide range of applications, and the usage of computer technology for visualising complex functions, the editors see benefits to revitalise complex analysis in undergraduate study programmes (Howell et al., 2017b).

The articles in this issue include specialised topics from complex analysis, such as Rouché's theorem (Howell & Schrohe, 2017) or the Gauss-Lucas theorem (Brilleslyper & Schaubroeck, 2017), which may well serve for students' projects. Dittman et al. (2017) and Kinney (2017)

44 In principle, we may plot the graphs of the real and imaginary of the function $[a, b] \longrightarrow \mathbb{C}, t \longmapsto f(\gamma(t))\gamma'(t)$. Their Riemann integrals are the real and imaginary part of $\int_\gamma f(z)\, \mathrm{d}z$. Hence, visual mediators showing the shaded area enclosed by these graphs, the horizontal axis, and the vertical lines at $t = a$ and $t = b$ may be used to realise the complex path integral (see Figure 8.2). However, these mediators do not seem to be used in textbook literature and we conclude that it is hard to relate these plots to the function f itself.

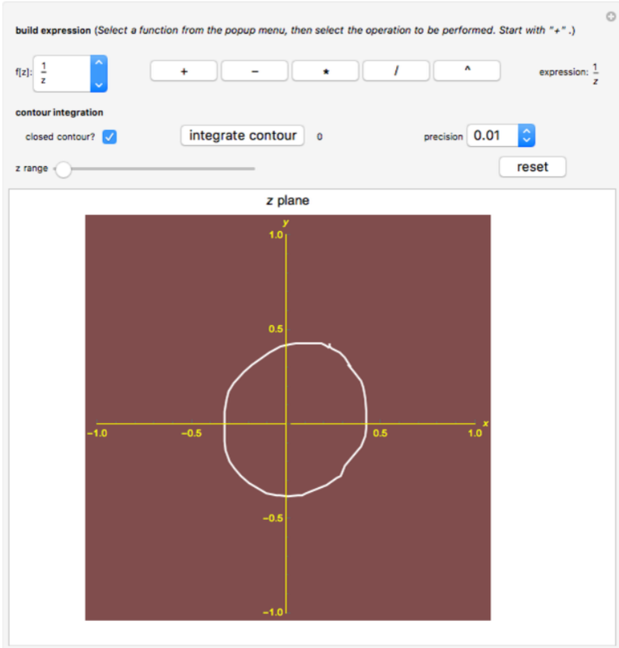

Figure 5.6: Illustration of the tool by Smith (2008).

describe teaching projects with computer algebra or geometry software on complex number arithmetic, properties of functions such as $z \longmapsto z^2$, or infinite products. Newton (2017) sheds light on relations between complex analysis and fluid mechanics, and Bolt (2017) between complex analysis and electrostatics. The authors of these texts emphaise the connections to vector analysis and resulting physical applications (see Section 8.2.2).

Curricular questions for complex analysis are addressed, too. Sachs (2017) advocates a proof-centred course in complex analysis, which seems to reflect an American situation of teaching of complex analysis, where "mathematics majors often graduate without any significant exposure to the depth and beauty of this accessible body of mathematics" (Sachs, 2017, p. 845).[45] The course presented by Sachs (2017) mostly focuses on complex number arithmetic, complex functions, the fundamental theorem of algebra, and non-Euclidean geometry. Similarly, D'Angelo (2017) and Garcia (2017) mainly discuss applications of complex numbers or analogies of complex number arithmetic to special matrices (e.g., unitary or self-adjoint matrices). D'Angelo (2017) also suggests including historical remarks on Cauchy's integral formula to the teaching of complex analysis.

Garcia and Ross (2017) evaluate different proofs of Cauchy's integral theorem (Theorem A.17), namely via Green's theorem (Theorem B.15), Goursat's lemma (Theorem A.19), and Leibniz' rule on the interchangeability of derivative and integral, and a homotopy argument. They value the proof of Goursat's lemma as "beautiful" and suitable for a course on complex analysis because it involves several concepts from the undergraduate curriculum (definition of the derivative, linear approximation, recursion, compactness, and continuity), and does not require the assumption that the holomorphic function to be integrated has a continuous derivative (Garcia & Ross, 2017, p. 761). The other proofs are shown to rely on theorems from real

45 In this sense, the American situation (see also the descriptions of complex analysis courses by Hancock, 2018; Troup, 2015) is not directly comparable to German complex analysis courses for future mathematicians and pre-service teachers, where complex analysis is often taught with emphasis on proof.

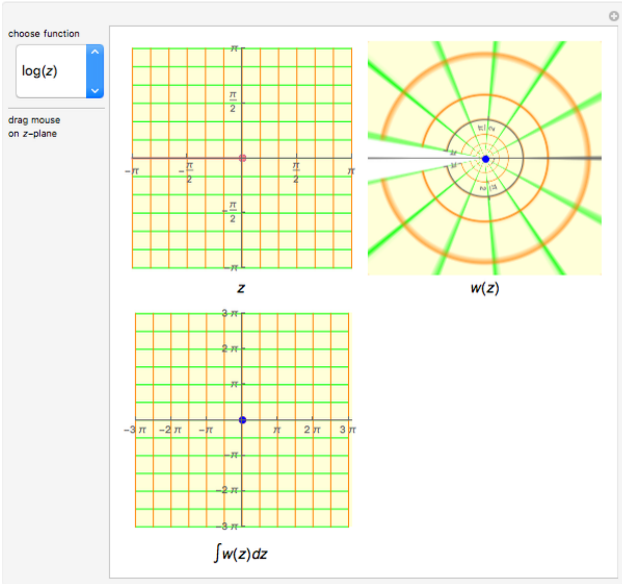

Figure 5.7: Screenshot of the tool by Yoshiaki (2017).

analysis and especially the proof via Green's theorem or the Leibniz rule require the continuity (or integrability) of the derivative of the integrand (Garcia & Ross, 2017).

Farris (2017) uses domain colouring to reason that $\int_{\partial C} z^k \, dz$ equals 0 for $k \in \mathbb{Z} \setminus \{-1\}$ and $2\pi i$ for $k = -1$, and any circle C with centre 0 (see Section 8.2.1). A discussion of a special case of the Residue theorem (Theorem A.29) and the Argument principle (Corollary A.32) follows but is essentially carried out purely formal, whence weakening the geometric argument. Farris (2017, p. 828) also notices that domain colouring cannot explain why the homotopy invariance of complex path integrals holds.

What we can learn from this recent discussion in mathematics education literature is that complex analysis, and in particular the challenges of making complex path integrals and integral theorems accessible to students, is of increasing interest in university mathematics education—from the perspectives of researchers but also lecturers. Lecturers' efforts to increase sustainability of the teaching and learning of complex analysis underlines that there is in fact a need for developing complex analysis education further. In this respect, the thesis fits well into the upcoming branch of university mathematics education.

5.4 INTEGRATION OF COMPLEX FUNCTIONS

In this section, we will summarise findings from the studies by Danenhower (2000), Hancock (2018), Oehrtman et al. (2019), and Soto and Oehrtman (2022) on students' and experts' understanding of integration in complex analysis.[46] We will give as many details as possible with respect to interpretations of complex path integrals found in these studies.[47] Consequently, we

46 We use the word understanding and interpretation here in a rather general way, just to hint at the fact that the studies presented here focus on certain people who engage in doing and talking about integration in complex analysis and possible interpretations about complex path integrals.

47 The authors of these pieces of research use slightly different terminology than is used in this thesis. For example, what we call complex path integral and signify with $\int_\gamma f(z) \, dz$, some of the authors call contour integral and signify with $\int_C f(z) \, dz$. Hence, we occasionally use their significations and word use, too, in particular when quoting from their publications.

may well detour from what the authors themselves considered their main research interests or findings.

In relation to students' and experts' understanding of derivatives and integrals in complex analysis, the phenomenon *thinking real, doing complex*, which originates in Danenhower's (2000) study, is discussed times and again (e.g., Hancock, 2018; Oehrtman et al., 2019; Soto & Oehrtman, 2022). Broadly, this phenomenon means that previous experiences or interpretations of derivatives and integrals from calculus and analysis in one or several variables leave their mark on the interpretation of the complex counterparts.[48]

5.4.1 *Danenhower: Teaching and learning complex analysis*

Danenhower (2000) conducted several interviews with students in different courses on complex analysis in British Colombia (Canada), in which he also covered many questions on integration (Danenhower, 2000, pp. 216–241). Students' answers to two of these questions were analysed in his dissertation:[49]

- Findings on the question "If γ is a circle of radius 2 centered at i, what is $\int_\gamma \frac{|z+i|}{|\bar{z}-i|}\,\mathrm{d}z$?" (Danenhower, 2000, p. 185):

  None of the four students attempting this task recognised that the integrand is constant 1 (because the numerator is the complex conjugate of the denominator and hence their absolute values are the same; except for the removable singularity at $-i$). While Danenhower (2000, p. 186) explains that the question "was intended to test students' knowledge of symbolic methods, so we were not too interested in the details of computing the integrals", we do not get to know much about how the students actually worked on this task. Some students are reported to have correctly plugged in the parametrisation of the path, $\gamma(\theta) = i + 2\mathrm{e}^{i\theta}$, $0 \leq \theta \leq 2\pi$, and obtained the correct result; some incorrectly conjugated the parametrisation of the path to $i - 2\mathrm{e}^{i\theta}$ and got an incorrect result for the integral. One student claimed that the integrand has a singularity at $-i$ on the contour of integration but is analytic (holomorphic) elsewhere in the complex plane, so the integral must be 0 by Cauchy's integral theorem. However, the interview brought up that the student also thought that real-valued functions are analytic. (Danenhower, 2000, pp. 185–187)

- Findings on the question "Find an upper bound for $\left|\int_\gamma z^2 + 2\,\mathrm{d}z\right|$, where the integral is to be evaluated on the contour formed by the segments joining the points $1+i$, $-1+i$, $-1-i$." (Danenhower, 2000, p. 187):

  Two students attempted this task and used Darboux's inequality (Equation A.16), that is, the fact that the modulus of a complex path integral is bounded by the length of the path of integration times the maximum of the modulus of the integrand on the trace of the path. The author also observed that one student struggled to argue that $|z^2| = |z|^2$

48 It has to be acknowledged though that Danenhower (2000) does only hint at this phenomenon. In fact, most of his work focuses on complex number arithmetic and students' switches between algebraic, polar, and vector representation of complex numbers. The phenomenon "thinking real, doing complex [...] was a major theme and had to be eliminated for reasons of space limitations" (Danenhower, 2000, p. 184).

49 He even asked several questions on quite advanced topics. For example, students should explain why the formula for the coefficients in a Laurent series expansion of an analytic (holomorphic) (Equation A.26) is "reasonable" (Danenhower, 2000, p. 217). Unfortunately, answers are not documented.

for any complex number z, which Danenhower (2000) interprets as an instance where "a problem with basic material interfered with later work" (Danenhower, 2000, p. 187).

In view of this last observation and many others, Danenhower (2000, p. 197) asks whether "more time spent on instruction on basic topics would pay significant dividends later in the course". He also suggests that future research should analyse the "interplay between the path of integration and the integrand of a path integral" (Danenhower, 2000, p. 200). Furthermore, Danenhower (2000, p. 201) asks whether it is still timely to motivate the calculus of residues with the evaluation of Riemann integrals, since nowadays Riemann integrals can be approximated numerically with technical aid, and he suggests that the study of conformal mappings could be more appropriate in an introductory course on complex analysis, while "residue theory has become a specialty subject".

5.4.2 *Hancock: Undergraduates' argumentation regarding integration of complex functions*

Hancock (2018) case study on two pairs of students of a complex analysis course focused on their collective argumentation with emphasis on the prerequisites in theorems like Cauchy's integral theorem (Theorem A.17). The interviews were taken after the students completed their final exams. Hancock (2018) made use of the framework of Tall's (2013) three worlds of mathematics (conceptual-embodied, operational-symbolic, and axiomatic-formal) (see also Hancock, 2017, 2019).

Hancock (2018, p. 392) identified three variants of the phenomenon *thinking real, doing complex*:

(1) "purposefully avoiding inappropriate applications of it;

(2) extending real intuition to the complex setting erroneously; and

(3) extending real intuition to the complex setting in productive ways" (Hancock, 2018, p. 392, enum. orig., line breaks EH.).

Since we are primarily interested in the interpretations of the complex path integral, we now focus on the results on the three interview questions, which focus primarily on students' understanding of complex path integrals. However, Hancock (2018) study also includes computational questions and other prompts to gain insight into students' use of prerequisites in integral theorems.

- Findings on the question "If $z = f(t)$ is a parametrized curve described as a complex-valued function of t, how would you provide a geometric interpretation of the identity $\int_a^b \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = f(b) - f(a)$?" (Hancock, 2018, p. 459):

  One pair of students (Dan and Frank) interpreted the integral as "adding up all the derivatives over some interval of time", one of them was "tempted to think of this in terms of real numbers, but the analogy doesn't work" and claimed that the complex path integral is "just like a line integral" (Hancock, 2018, p. 115). One of the students derived the identity $\int_a^b \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \mathrm{d}z$, symbolically reducing the fraction. Moreover, the derivative $\frac{\mathrm{d}z}{\mathrm{d}t}$ was not regarded as a velocity as it was in case of a real valued function of a real variable, and the fact that the integral contained the parameter t sometimes evoked associations

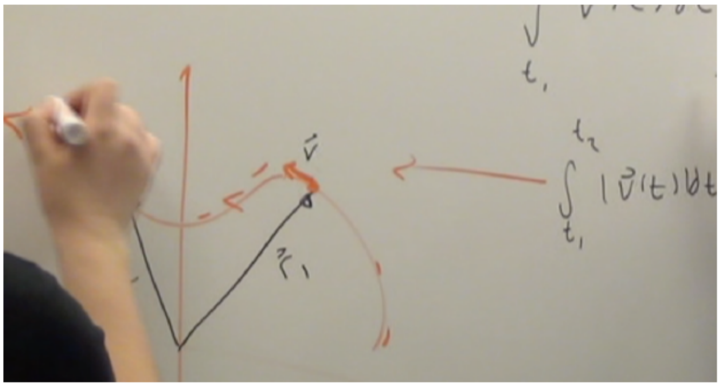

Figure 5.8: Interpretation of an integral of velocities as difference of positions (Figure 35 from Hancock (2018)).

to time rather than an integral along the real axis.[50] Moreover, the students provided a sketch to indicate a path in the complex plane from a to b but were uncertain whether a and b where real numbers or the complex numbers at the initial and terminal of a path parametrised by f. A similar confusion appeared at times in the other interview, too, possibly reinforced by the interviewer, who said the integral was taken along any path from a to b (Hancock, 2018, p. 124), and by using the symbol f, which is often used for an integrand rather than a path.

The equation from the prompt was also labelled as a version of the fundamental theorem of calculus, and the integral on the left hand side was also recognised as a complex path integral of the function 1, which the students then expressed as $\int z(t) \frac{\mathrm{d}z}{\mathrm{d}t} \mathrm{d}t$, where the bounds of integration were implicitly contained in the assumed given parametrisation in $z = f(t)$. (Hancock, 2018, pp. 113–121)

In the other interview (Sean and Riley), the identity from the prompt was similarly recognised as a variant of the fundamental theorem of calculus. One student also emphasised that one has to differentiate between an integral with respect to a complex variable and one with respect to a real variable, which he denoted by $\int f(z) \,\mathrm{d}z$ and $\int f(t) \,\mathrm{d}t$, respectively. The interpretation of real integrals as area was additionally judged to not be applicable here. If the integrand represented a velocity v, the students argue that the integral $\int_{t_1}^{t_2} v(t) \,\mathrm{d}t = \Delta r = r(t_2) - r(t_1)$ may be interpretations as a change in positions ($v = \frac{\mathrm{d}r}{\mathrm{d}t}$) (see also Figure 5.8). Moreover, the students described that an integral of the form $\int_{t_1}^{t_2} |\vec{v}(t)| \,\mathrm{d}t$ yields the arc-length of the curve drawn in Figure 5.8.[51] In relation to the initial task, one student argued added that if $f(t)$ described a velocity, then $\frac{\mathrm{d}z}{\mathrm{d}t}$ would describe an acceleration, and the identity from the task describes a change in velocity. (Hancock, 2018, pp. 122–135)

- Findings on the question "What does the integral of a complex valued function represent? How is this different than the integral of a real valued function? How is it the same?" (Hancock, 2018, p. 460):

50 Interpretations of complex derivatives in terms of velocities are reported in the literature thought. For example, Needham (1997, p. 409) explicitly calls $\nu = \frac{\mathrm{d}z}{\mathrm{d}t}$ "tangential complex velocity" when interpreting the formula "$\int_L f[z] \,\mathrm{d}z = \int_a^b f[z(t)]\nu \,dt$" (in my notation, the left hand side corresponds to $\int_\gamma f(z) \,\mathrm{d}z$ and the right hand side to $\int_a^b f(\gamma(t))\gamma'(t) \,\mathrm{d}t$ (see also Section 8.1.3).

51 The change in notation from v to $\vec{v}$ is original.

This question asked for a "personal characterization of integrals of a complex function" and a comparison with real-valued functions (Hancock, 2018, p. 348). In the first interview, one student questioned whether he is aware of an interpretation and the other student recalled that their professor described integration as the addition of a "bunch of vectors [...] along the curve" (Hancock, 2018, p. 349). This would make sense because complex numbers could be interpreted as vectors. The student explained that he would rather think about formulas and theorems and that there was usually no time to invest in describing what these formulas meant in terms of vectors. The students also recalled interpretation from calculus 1 that real integrals have various meaning such as an area, a travelled distance, or work. (Hancock, 2018, p. 350)

The interviewer continued to ask whether the students could come up with a situation in complex analysis where integration would correspond to an area. The students recalled that double integrals for the density function 1 represent an area, that one could relate a "line integral as a Green's Theorem double integral", which only works in case the path is closed, and which would enable them to "solve the area of something", but this "something" remains unclear (Hancock, 2018, p. 351). While the students seem to be able to relate double integrals to areas and recognised that Green's theorem can be used to equate a line integral with a double integral in case the path is closed, they did not clear relate complex path integrals instead of real path integrals to an area. (Hancock, 2018, pp. 348–352)

The students from the other interview similarly observed that the area interpretation for real integrals does not apply in the complex case and one student also explained that he rather thinks about complex integration in terms of "algebra" (Hancock, 2018, p. 353).

Sean continued to considered a complex function $f(z)$ as an ordered pair of functions $u(x, y)$ and $v(x, y)$ of two real variables and recalled the notion of line integrals (real path integrals of second kind in our terminology) $\int_C \vec{F} \cdot dr$ from calculus 3 (likely, $\vec{F}$ stands for the real vector field corresponding to f, i.e., $\vec{F} = (u, v)^T$). Then, he considered specifically the function $f(z) = z$ and the path to be a semi-circle, and interpreted $\mathrm{d}z$ as an "incremental path" (Hancock, 2018, p. 353) (see Figure 5.9). Then, Sean argued that the dot product of z and the small vector $\mathrm{d}z$ corresponds to the $\vec{F} \cdot \mathrm{d}r$ in $\int_C \vec{F} \cdot \mathrm{d}r$, and the integral adds up these dot products of small vectors (see Figure 5.9). While this is a suitable interpretation for real path integrals of second kind, it does not fit to complex path integrals because it does not take into account the multiplication of complex numbers instead of the dot product of vectors in $\mathbb{R}^2$. (Hancock, 2018, pp. 352–356)

Riley added that if $\vec{F}$ describes a force, the integral measures work. Moreover, the students stated Green's theorem, and similar to the other pair of students, Sean and Riley interpreted integrands in double integrals as a density function. They explained that for the constant density function $f(z) = 1$, the double integral over a region yields the area of this region. Additionally, if $f(x, y)$ is a function of two variable over the region, the students argue that the function may be interpreted as a surface over this region and the double integral corresponds to the volume of this surface. (Hancock, 2018, pp. 358–364)

Sean added that Stokes' theorem generalises Green's theorem for higher dimensions, but that the "third dimension" would make the interpretation in relation to complex quantities "difficult", because applications of complex numbers ("electric fields" or "flow

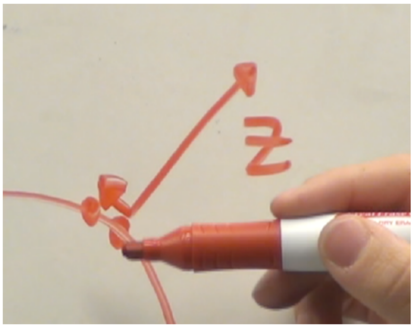

Figure 5.9: Interpretation of the integrand in a line integral from calculus 3 (Figure 197 from Hancock (2018)).

fields") are two-dimensional, not three dimensional as is the surface generated by $f(x, y)$ (Hancock, 2018, p. 361).

Having been prompted by the interviewer as to what makes complex functions difficult to visualise, the students argued that complex functions map from a plane to another plane, hence, the graph would be four-dimensional: "But then you have like $\mathbb{C}^2$ is like four axes and you can't really visualize that at all", but you "still have all the math" (Hancock, 2018, p. 363). (Hancock, 2018, pp. 352–364)

- Findings on the question "When does a complex valued function have an antiderivative? Why would this be useful to know?" (Hancock, 2018, p. 460):

  Frank recalled that analytic functions have anti-derivatives "as long as we're integrating it in some region or on some curve and there is a simply-connected domain that contains the curve, then uh, f has an anti-derivative" and it can be computed using methods from calculus (Hancock, 2018, p. 366). While the existence of primitives for analytic functions is correctly stated, it is interesting that Frank tied the existence of a simply-connected domain containing certain curves to existence of the primitive function. It is not clear though whether Frank is implicitly referring to a method for constructing a primitive in terms of certain integrals.

  The students compared the evaluation of complex path integrals via anti-derivatives or via the parametrisation of the path. They explained that complex path integrals can always be computed by parametrising (by which they likely referred to the formula $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma'(t))\gamma'(t)\,\mathrm{d}t$). Otherwise, the students argue, one would have to "deal with branch cuts and try[] to make a function be analytic on some contour" (Hancock, 2018, p. 367). The students claimed that the computations via parametrisation may get messy, which is no problem for computers. On the other hand, the author noticed that the students used anti-derivatives in explicit calculations of complex path integrals in other interview tasks, whereas the students reported that they would rather apply integral theorems to evaluate complex path integrals instead of computing them explicitly with the parametrisation. (Hancock, 2018, pp. 366–369)

  In the second interview, the existence of primitives was first for analytic (in our terminology: holomorphic) functions without mentioning topological constraints on the domain of the function first, which was later corrected to simply-connected domains. Sean described that $\int_A^B f(z)\,\mathrm{d}z$ may be computed by evaluating the difference of the values of a primitive function for f at A and B, similar to the fundamental theorem of calculus. Also, the approach to use primitive functions for the computation complex path integrals was valued as comfortable in case the path of integration cannot be easily parametrised or needs to be chopped into pieces in order to find parametrisations of each of these. (Hancock, 2018, pp. 371–381)

Sean and Riley illustrated the complex version of the fundamental theorem of calculus with the function $z \longmapsto 1/z$. They argued that this theorem is applicable for a semi-circle from i to $-i$ with an appropriate choice of logarithm, but also that this procedure no longer works for the full circle because the integrand is not holomorphic on any simply-connected domain containing this circle. (Hancock, 2018, pp. 369–383)

Both student pairs interpreted complex path integrals as a process that sums up certain vectors and were well aware that several interpretations from real analysis (e.g., in terms of the area under a graph) cannot be transferred to the complex setting or only with care when interpreting complex quantities as vector quantities. On the other hand, some of the students relied heavily on interpretations of real path integrals of first or second kind, but obscured a precise relationship to complex path integrals or a precise geometric meaning of them.

Regarding the use of the formulas $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t$ and $\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$ (whenever a primitive function F for f exists), both student pairs acknowledged the second for the ease of application. In more complicated settings though, when the functions have branch cuts (Dan and Frank) or the paths are difficult to parametrise (Sean and Riley), either one of the formulas was preferred over the other.

Integral theorems from real or complex analysis (Green's theorem, Cauchy's theorem, Cauchy's integral formula) and formulas like those from the previous paragraph belonged to the repertoire of the students. However, the geometrical and physical interpretations were limited to real path integrals and results from calculus, and it often remained unclear to them whether their reasoning applied to complex or real functions.

The most explicit idiosyncratic interpretation of complex path integration as the sum of a bunch of vectors was borrowed from the students' lecturer. Moreover, some of the students explained that they were not aware whether they even had a personal interpretation of complex path integrals and that they usually employed algebraic or reasoning involving integral theorems or formulas.

In sum, Hancock (2018) showed that the students he interviewed

- were not sure whether they had a geometric, physical, or other kind of interpretation of complex path integrals,
- did not overgeneralise certain interpretations from calculus in one or several variables,
- explained geometrical or physical interpretations of *real* (path) integrals and double integrals even when prompted to describe what the "integral of a complex valued function represent[s]" (Hancock, 2018, p. 460),
- preferred to work with formulas or integral theorems,
- and saw connections between integral theorems in real and complex analysis (e.g., similar versions of the fundamental theorem of calculus).

Thus, Hancock (2018) study indicates that while students may be able to interpret various real integrals with examples from geometry or physics, they tend to avoid their generalisation to complex path integrals (see also Section 5.4.4). Rather, students seem to lack interpretations of integrals in complex analysis. However, since some of the students remembered their lecturer's interpretations of the complex path integral even after their exam, we may infer that lecturer's informal descriptions of complex path integrals may be taken up later by students. As a consequence for this thesis, it is relevant to address lecturers' interpretations of complex path integrals and a natural first step is to consider their own idiosyncratic interpretations, for these are the interpretations potentially to be transformed into teaching.

5.4.3 *Oehrtman et al.: Experts' construction of meaning of the complex path integral*

Besides our own work, Oehrtman et al.'s (2019) study is the first and only available study in this direction so far. The authors investigated "[e]xperts' construction of mathematical meaning for [...] integrals of complex-valued functions". More precisely, the authors aimed to elicit mathematicians' "geometric interpretation of complex differentiation and integration" (Oehrtman et al., 2019, p. 402). Theoretically, Oehrtman et al. (2019) made use of a blend of embodied cognition, a constructivist perspective, and Tall's (2013) "three worlds of mathematics".

Five mathematicians with background in complex analysis (Becky, Judy, Rafael), differential geometry (Luke), and differential equations (Andrew), who had taught introductory complex analysis before, were interviewed. None of them used textbooks with geometrical emphasis on integration (and differentiation) in complex analysis.

In general, the mathematicians explained complex path integrals in different ways. Except for one expert, none of them gave a geometric explanation. Often, the participants recalled and explained definitions or propositions. In particular, complex path integrals were frequently related to integrals of functions of one or two real variables. Also, experts' reasoning did not proceed from concrete instances to generality but was initially rather at the level of generality and formality. (Oehrtman et al., 2019)

More precisely, the following observations were made:

- Andrew responded "he would not think about complex path integrals geometrically" and produced the formula

$$\int_C (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = \int_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_C (v\,\mathrm{d}x + u\,\mathrm{d}y) \qquad (5.1)$$

  (Oehrtman et al., 2019, p. 415). Similarly, Luke multiplied

$$f(z)\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = u\,\mathrm{d}x - v\,\mathrm{d}y + i(v\,\mathrm{d}x + u\,\mathrm{d}y)$$

  (where $f = u + iv$) and obtained the same formula. Luke also claimed it was a hard question to provide a geometrical interpretation of the complex path integral (see the quote at the beginning of this chapter).

  Luke also recalled that the two integrals at the right side of Equation 5.1 "represent things like work and flux, depending on which one you are looking at" (Oehrtman et al., 2019, p. 415). He claims that the first of the two integrals on the right side in Equation 5.1 vanishes if C is a closed curve because no "work" was done over closed loops (Oehrtman et al., 2019, p. 415). The second integral in this equation may be used to obtain "something like Cauchy's integral formula", "but [Luke; EH.] did not 'know what that number means[]'" (Oehrtman et al., 2019, p. 416). However, no constraints on the functions u and

v were made to substantiate the aforementioned claims about the integrals in Equation 5.1 and the precise relationship to Cauchy's integral formula stayed imprecise (or at least were not reported in more detail).

- Some of the other participants also interpreted the real and imaginary part of $\int_C f(z)\,\mathrm{d}z$ physically as certain work and flux, and one of them reasoned that these integrals vanish according to Green's theorem (Lang, 1999, p. 468) if the path is closed. Still, a clear connection to the work and flux of *which* vector field, necessary differentiability conditions, and the relationship to the integrand of a given complex path integral were either not established or not reported in the paper. Additionally, Judy interpreted the integrand f as a force perpendicular to the path of integration (Oehrtman et al., 2019, pp. 415–416; see Section 8.1.4 and Section 8.2.2 for more details on the connections to vector calculus and physics).

- Becky explained the complex path integral as an accumulation similar to Riemann sums in calculus but was not sure *what* is actually being accumulated (Oehrtman et al., 2019, p. 415; see also Soto & Oehrtman, 2022).

- Judy described that she had often thought about complex path integrals as if they were real integrals: "Judy immediately remarked that she teaches integration of complex functions as integrating along the x-axis for the function $f(z) = u(z) + iv(z)$" (Soto & Oehrtman, 2017, p. 1440). For the integration between complex numbers, one "would 'map it to some portion of the curve using alphas and betas and still think of it as a real integral'" (Oehrtman et al., 2019, p. 415).

- Rafael argued that geometrical reasoning and visualisation is "under-utilized" in complex analysis, and that "Gauss and Riemann explicitly relied on such geometrical intuition and geometrical reasoning", hence they "recast Cauchy's formalistic approach to complex analysis" (Oehrtman et al., 2019, p. 414).

  Rafael used the geometric interpretation of multiplication of complex numbers as a dilation and rotation of one of the factors by the modulus and argument of the other. He described integration in the complex plane as a process comparable to a moving ship: He imagines that a ship is sailing on the path of integration and the captain wants to plot the ship's route on a map. When plotting this route, however, the ship's actual path is distorted by inaccuracies of the ship's measuring instruments. In sum, Rafael interpreted the complex path integral as a measure of this distortion.

  More precisely, let us assume the integrand is called f and the path is parametrised as a function $t \longmapsto z(t)$. The path can be imagined to be partitioned into small segments, Δz_k, which correspond to small displacement vectors of the ship. These are rotated and dilated when plotted on the captain's chart. The sum of all these rotated and dilated displacement vectors yields the sketched route on the map.[52] In Rafael's words:

52 Recall that complex path integrals may be defined as a limit of complex Riemann sums, roughly stated as "$\int_K f(z)dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\Delta_i$" (Oehrtman et al., 2019, p. 397). Oehrtman et al. (2019) interpret each Δ_i as a segment of the curve of integration and each z_i as the "midpoint" of the segment; then "each Δ_i may be viewed as rotated and dilated by the argument and magnitude of $f(z_i)$. Finally, these image vectors are summed in the co-domain to obtain the resultant vector" (Oehrtman et al., 2019, p. 397; see Section 8.1.1 for more details).
The notation in the paper is not fully unambiguous. It seems that Δ_i and Δz_k, occasionally also without an index, are used interchangeably. Yet, sometimes, these symbols signify small segments of the curve of integration, differences between points on the curve, or small tangential vectors attached to the path.

> When you do lay these [the displacement vector; EH.] end-to-end, what you're doing— the Riemann sum definition would be a summation of Δw_k's and every Δw_k is an $f(z_k)\Delta z_k$. Or if you want to do a **continuous version of it** (makes a swooping motion with arm), what you're doing is defining $w(t)$ to be the integral of $f(z(t))\frac{dz}{dt}$. And then you integrate from $t = t_0$ to some variable endpoint T. So, this gives you the exact description of your imputed curve. (Oehrtman et al., 2019, 413, emph. orig.)

Here the Δz_ks denote approximations of segments of the original path, and the Δw_ks denote the corresponding segments of the route on the map.[53]

In this context, the function f describes the dilation and rotation of the original route. Since f is not necessarily a constant function, in other words, the measurement errors may not be constant, Rafael argues that these measurements errors depend on the ship's location (Oehrtman et al., 2019, p. 419):

> Interpreting an integrand $f\,(z(t))\,\frac{\mathrm{d}z}{\mathrm{d}t}$ requires providing meaning for multiplying the ship's displacements $\Delta z = \frac{\mathrm{d}z}{\mathrm{d}t}\Delta t$ by the complex number $f(z)$ that depends on the ship's location $z(t)$. (Oehrtman et al., 2019, p. 413)

Additionally, Rafael incorporates his ship metaphor to an explanation of Cauchy's integral theorem (Theorem A.17): He argues that even if the ship's actual path forms a closed loop, the route on the chart may not necessarily be closed, too. This is said to happen in "cases where f is analytic [holomorphic; EH.]" (Oehrtman et al., 2019, p. 414).

Furthermore, Rafael argues that "if it [the complex path integral; EH.] is path independent", then there is a "well-defined mapping" between the real chart and the captain's chart, which might be called "big $F(z)$ ... the complex anti-derivative of little f" (Oehrtman et al., 2019, p. 414). Hence, Rafael obtains a "physical interpretation of the complex anti-derivative as well" (Oehrtman et al., 2019, p. 414).

To summarise, aside from Rafael's idiosyncratic ship metaphor, the experts made frequent use of elements from real analysis in one or two variables, definitions, and theorems. Oehrtman et al. (2019) conclude that their research participants' "found formal concepts [...] to be deeply meaningful in their own right" (Oehrtman et al., 2019, pp. 417–418) and that their "actions within the formal theory served as the source materials for abstraction" (Oehrtman et al., 2019, p. 418). The authors hypothesise that the meanings the experts see in complex path integrals arises from their use as a tool in applications.

It further seems that some of the experts in this study were more confident than the students Hancock (2018) interviewed in stating that they considered complex integration as if it was real integration. We may only hypothesise here why this is the case: This is probably due to the proficiency with formal aspects such as formulas and definitions. However, seeing analogies between Riemann, real path integrals, and complex path integrals also evoked uncertainty: Notwithstanding, the experts' "exhibit[ed] reasoning about mathematics that was primarily formal and symbolic, but most of them initially questioned the possibility of geometric or other concrete interpretations" (Oehrtman et al., 2019, p. 417). It seems that the experts recognise formal similarities between integrals in real and complex analysis, which make them consider these integrals to be the same to some degree. However, except for the case of Rafael, it seems that the experts did not yet explore the boundaries of this sameness more closely.

53 Note that this description mixes differentials $\mathrm{d}z$ with differences Δz, hence, it does not become completely clear which $z(t)$ in $f(z(t))$ determines the dilation and rotation of the respective segment Δz.

Hence, while this study is clearly interesting as it opens the research on experts' intuitive understanding of complex path integrals, we only get to see very little of the discourses on complex path integrals the experts actually enacted. For instance, we only get to see a few of the experts' interpretations. To better understand experts' intuitive interpretations of complex path integrals we need more case studies.

5.4.4 *Soto & Oehrtman: Undergraduates' exploration of contour integration: What is accumulated?*

The last study we review dealt with the question what is accumulated in the case of complex path integrals more closely. Soto and Oehrtman (2022) interviewed three pairs of students from a course on complex analysis, in which integration had not yet been covered, in order to find out how "they can geometrically and visually interpret the inscription for a contour integral, $\int_C f(z)dz$" (Soto & Oehrtman, 2022, p. 1) and how "they could develop a geometric interpretation of integration of complex-valued functions based on their understanding of complex multiplication and real-valued integration" (Soto & Oehrtman, 2022, p. 5). Thus, the authors assume that the interpretation of integrals as an accumulation (Section 2.2.3) applies to complex path integrals as well. Even though we will later study experts' intuitive mathematical discourses about complex path integrals, the study we review in this chapter is nevertheless important because it underlines that students may access complex path integrals by transferring the idea of accumulation they encountered previously during their mathematical studies to complex path integrals, and hence that this idea not be dismissed in our epistemological analysis in this thesis.[54]

Before the interviews, the course had covered Needham's (1997) geometric interpretation of the derivative of a holomorphic function as an amplitwist (Section 5.1.3). The authors "hoped that these embodied experiences would solidify geometric interpretation of multiplication and prepare students to reason geometrically about contour integration" (Soto & Oehrtman, 2022, p. 4). The authors further expected the students to anticipate that the complex path integral is a limit of sums involving addends of the form $f(z_k)\Delta z_k$. Here, z_k is a point on the contour of integration and Δz_k is a tangent vector to the curve (viewed as a complex number; suitable differentiability conditions on C are implied). Furthermore, the students should interpret the product $f(z_k)\Delta z_k$ as a copy of Δz_k rotated by $\operatorname{Arg}(f(z_k))$ and dilated by $|f(z_k)|$ (Soto & Oehrtman, 2022).[55]

During the interviews, the students were first prompted with the interpretation of a Riemann integral as an area under a graph. Then, they had to respond to the following question:

> Suppose that I asked you to explain the geometry behind the integral of the function $f(z)$ over the contour C which is traversed counterclockwise. Based on what you know about

54 We will do so by interpreting complex Riemann sums and their limits geometrically (Section 8.2.1).

55 Writing the complex Riemann sum as $\sum_{k=1}^{n} f(\gamma(t_k))\gamma'(t_k)\Delta t_k$, where $\gamma\colon [a,b] \longrightarrow C$ is parametrising the contour of integration, Soto and Oehrtman (2022, p. 2) geometrically interpret this sum as "the accumulation of tangent vectors $\gamma'(t_k)\Delta t_k$ to the path rotated by $\operatorname{Arg}(f(\gamma(t_k)))$ and stretched by a factor of $|f(\gamma(t_k)|$ [sic!]" (Soto & Oehrtman, 2022, p. 2). Consequently, the "contour integral is the limit of displacements across such a sequence of approximating deformed paths" (Soto & Oehrtman, 2022, p. 2). In this context, the authors use Δz_k to denote the tangent vector $\gamma'(t_k)\Delta t_k$ to the curve of integration (interpreted as a complex number). However, they also use Δz_k for "small-directed segments of the contour" (Soto & Oehrtman, 2022, p. 4). In this respect, the authors use the symbol Δz_k ambiguously.
In the definition of complex path integrals via limits of complex Riemann sums of the form $\sum_{k=1}^{n} f(z_k)\Delta z_k$, the Δz_k usually signify differences of points that partition the contour of integration, which is slightly different from how Soto and Oehrtman (2022) use the notation Δz_k (see Section 8.1.1).

> real-valued integration how might you reason about this? In other words, how might you reason about $\int_C f(z)dz$? (Soto & Oehrtman, 2022, p. 18)

As was expected by the authors, several students transferred the idea of approximating Riemann sums to the complex setting. Moreover, they referred to integrals from real analysis in one or two variables in various different ways when looking for a geometrically realisation of $\int_C f(z)dz$:

- One student interpreted a complex function as a scalar or vector field and the complex path integral accordingly as a real path integral for a scalar or a vector field. He also related it to the area under a curve in 3-space, namely the area of a "slice" (Soto & Oehrtman, 2022, p. 7) as shown in Figure 5.10. After recalling the derivative of a complex function as an amplitwist, he suggested to regard the complex path integrals as the net change of an amplification and twist along the path (Soto & Oehrtman, 2022, pp. 7–8).

- The other groups transferred the idea of a limiting process of sums for the Riemann integral to the case of complex integrals and identified a similar product structure in the addends of such as sum. They realised the addends in these sums as products of function values at points on the contour multiplied with "'little bit[s] of the curve'" (Soto & Oehrtman, 2022, p. 7, emph. omitted).

  Eventually, the students produced idiosyncratic realisations of these limits of product sums such as "$\lim_{n\to\infty} \sum_{i=0}^{\infty} f(z+i\Delta z)\Delta z$" (Soto & Oehrtman, 2022, p. 8) and "$\lim_{\Delta z\to 0} \sum_{i=a}^{b} w_i \Delta z$", where w_i is a corresponding value of the integrand (Soto & Oehrtman, 2022, p. 10).[56] The first limit emphasises the growing number of points involved in a partition of the contour of integration, and the second emphasises that the distance of these points tends to 0. Occasionally, some students assumed that the Δz are always the same and can be factored out of these sums. One student interpreted the factors $f(z+i\Delta z)$ and Δz in the product $f(z+i\Delta z)\Delta z$ as the height and the width of a rectangle in four dimensions. Others interpreted these products as a rotation and dilation of Δz by the argument and modulus of $f(\Delta + i\Delta z)$ (and similarly for the other inscriptions they produced), as was expected by Soto and Oehrtman (2022).

  However, one of the students reversed the geometric interpretation of the products in the summands of the complex Riemann sums: Accordingly, he assumed that $w_i\Delta z$ meant that the function value w_i was rotated by $\mathrm{Arg}(\Delta z)$ and stretched by $|\Delta z|$ (Soto & Oehrtman, 2022, p. 10).

In the last part of the interviews, the students were asked to compute two specific integrals, $\int_C z^2\,\mathrm{d}z$ and $\int_C \frac{1}{z}\,\mathrm{d}z$, for different explicitly given curves C such as a directed line segment on the real axis from 0 to 2, a square with vertices $\pm 2 \pm 2i$ or a circle given by $|z| = 2$ in the complex plane traversed counterclockwise (Soto & Oehrtman, 2022, p. 18).

Soto and Oehrtman (2022) found that these concrete calculations were very challenging for the students and that they always transformed the integrals in question to certain real integrals. Five main observations can be made:

- The students did no longer make use of the product structure of complex Riemann sums, which they explored before. Rather, they jumped back and forth in their view of what is

56 From the context, we infer that the letter i denotes an index variable rather than the imaginary unit here and at similar places.

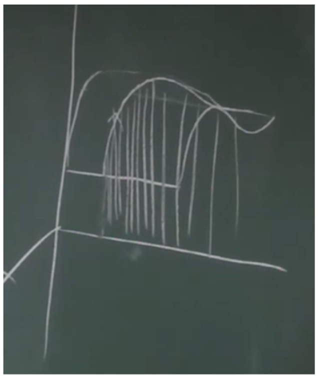

Figure 5.10: Area under a curve in 3-space (Figure 3 from Soto and Oehrtman (2022)).

added up in the complex Riemann sums: Sometimes the students assumed that complex numbers z_k on the contour of integration, function values $f(z_k)$, or what they signified as Δz_k or dz is summed up. (Soto & Oehrtman, 2022, p. 5)

- The students tried to use symmetry of the curves and the integrands, but were uncertain whether these observations allowed them to deduce that the corresponding complex path integrals vanish.

Now let us focus on students' calculations of $\int_C f(z)\,\mathrm{d}z$ for $f(z) = z^2$ and the line segment C from 0 to 2.[57]

- The students assumed that the complex the complex path integral of $f(z) = z^2$ along the given line segment coincides with the real integral $\int_0^2 x^2\,\mathrm{d}x$. Hence, they assumed that the complex path integral restricts to a real integral when the path of integration amounts to an interval on the real number line and the integrand is real on this interval.

- One student decomposed $f(z) = z^2$ into the real and the imaginary part $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ ($z = x + iy$). Assuming further that the complex path integral decomposes into one integral for the real part of the integrand with respect to $\mathrm{d}x$ and one for the imaginary part with respect to $\mathrm{d}y$, he calculated

$$\int_C z^2\,\mathrm{d}z = \int_C u(x,y)\,\mathrm{d}x + i\int_C v(x,y)\,\mathrm{d}y = \int_0^2 (x^2 - y^2)\,\mathrm{d}x + 2i\int_0^0 xy\,\mathrm{d}y$$

for the line segment C from 0 to 2. In particular, the first integral for the real part is taken from 0 to 2, while the second is taken from 0 to 0 (after all, C is the line segment from 0 to 2 on the real axis). Hence, this may be an instance of an overgeneralisation from the procedure of separating a complex numbers into real and imaginary parts to integrals, since this separation is extended to the interval for integration as well.

- Another student interpreted $f(z) = z^2$ as the vector field with defining term $(x^2 - y^2, 2xy)$ ($z = x + iy$) and "'want[ed] to take advantage of Green's theorem' as he wrote $\int_C f(z)dz = \int \overline{f} \cdot d\overline{r} = \iint_D \operatorname{curl} \overline{f} dA$, thus ignoring the complex structure" (Soto & Oehrtman, 2022, p. 11) (here, C denotes the square contour, D likely denotes its interior,

57 For the other function $f(z) = \frac{1}{z}$ and other contours, similar idiosyncratic calculations were observed (Soto & Oehrtman, 2022)).

and dA likely denotes the area element of doubles integrals $\mathrm{d}x\,\mathrm{d}y$; it is not entirely clear though whether $\overline{f}$ denotes the conjugate of f or Pólya vector field).

Overall, the study by Soto and Oehrtman (2022) shows that their students recognised the product structure of complex Riemann sums abstractly while interpreting the expression $\int_C f(z)dz$. The students could also relate the multiplication of complex numbers to a rotation and dilation. However, they did not use this product structure consistently in the calculations of the concretely given examples and did not interpret complex Riemann sums geometrically (e.g., as in Section 8.2.1) as Soto and Oehrtman (2022) expected.

What we can learn for this study is that accumulation is a central idea for integrals, and students may be able to transfer this idea to complex path integrals by mimicking the definition of real integrals i terms of limits of sums. However, similar to Becky from the study by Oehrtman et al. (2019), the students were uncertain about which quantity is accumulated in the case of complex path integrals.

5.5 DESIDERATES TO TACKLE IN THIS THESIS

In sum, research in complex analysis education has so far shown that students and experts resort to interpretations for real integrals of real- or vector-valued functions of one or two real variables when interpreting complex path integrals (Hancock, 2018; Oehrtman et al., 2019; Soto & Oehrtman, 2022). Nevertheless, most of them were unable to reason geometrically or physically about complex path integrals or considered this generally as a hard task. Notwithstanding, both groups of mathematists hesitated to transfer these interpretations to complex path integrals. Only one idiosyncratic interpretation of complex path integrals in terms of the navigation of a ship was reconstructed (Section 5.4.3). Additionally, the interpretation of an integral as an accumulation was reconstructed by the students (Soto & Oehrtman, 2022) and was also addressed by some experts in Oehrtman et al. (2019), but none of the research participants answered what is accumulated in the case of complex path integrals. Additionally, some experts mentioned that they tended to think about complex path integrals as if they were real integrals. It seems that the experts samed real and complex path integrals on a formal level only but not on the level of their interpretations. Moreover, except for one exception, both the novices and the experts were struggling in the task to interpret complex path integrals (geometrically) (Hancock, 2018; Oehrtman et al., 2019; Soto & Oehrtman, 2022).

Last but not least, even though the literature offers some visualisations related to complex path integrals (i.e., plots of vector fields associated to complex functions), visual realisations of complex path integrals beyond symbolic signification do not seem to be widespread and there has not been a systematic review of visualisations for complex path integrals before.

From these observations, we can derive the desiderates, which we will tackle further in this thesis:

(1) We need to explore more detailed how the notion of complex path integral can be introduced and connected to other mathematical notions, in particular other integrals. This pleads for a detailed epistemological analysis of complex path integrals, which shows how complex path integrals are related to their real counterparts and whether certain definitions and interpretations of real integrals may be transferred to their complex counterparts. This includes a review how approaches to complex path integrals are enacted in texts on complex analysis. Potentially, we may identify unifying ways to define and interpret different notions of integral, which occur as a "cross-curricular topic" (Kon-

torovich, 2018b) throughout many mathematics curricula at university level (cf. Hanke, 2022b; Hochmuth et al., 2021; Winsløw et al., 2021). For instance, our epistemological analysis will reveal certain common approaches to define and substantiate definitions of real integrals and complex path integrals. On the other hand, we will also identify certain conflicts between interpretations of real and complex path integrals, particularly for the interpretation of an integral as a certain average (Hanke, 2022c).[58]

(2) More exploratory research on experts' individual interpretations of complex path integrals is needed. This includes the question how they personally think (i.e., communicate) about them intuitively as well as whether and how they relate interpretations for integrals in real and vector analysis. Since the previous studies have shown a rather limited amount of interpretations, we need to broaden our point of view and should not only look for potential interpretations but also how experts organise their discourses when being prompted to discuss their personal interpretations of complex path integrals (e.g., which additional discourses do they refer to). In particular, it remains open which other idiosyncratic interpretations of complex path integrals experts may offer and how they may address these to others.

    In this context, it has also not yet been explored whether and how experienced mathematists may use one of their discursive mental images about complex path integrals to provide an ostensive, vivid substantiations for integral theorems such as Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20), or whether they substantiate these theorems differently.

Consequently, in the course of Chapter 6, Chapter 7, Chapter 8, and Chapter 9 we will perform the epistemological analysis of complex path integrals. In the first of these chapters, we will introduce the notion of *aspects* and *partial aspects* (Greefrath et al., 2016a, 2016b; Roos, 2020) for later us and relate it briefly to the commognitive framework (Chapter 3). In the subsequent two chapters, we will provide a historically informed epistemological analysis of the notion of complex path integral by making use of a huge variety of textbooks, historical sources, and the sources from mathematics and mathematics education literature on visualisations and interpretations of complex path integrals, which we have already introduced in this chapter. Doing so, we will identify endorsed narratives about complex path integrals and metarules for their endorsement. In particular, we will elicit various definitions, substantiations of these definitions, and highlight connections to integrals and their interpretations in real analysis in one and two variables and measure theory. Results of the epistemological analysis will be presented in terms of aspects and partial aspects in Chapter 9. Afterwards, in Part iii, we go on to the empirical investigation of experts' intuitive mathematical discourses about complex path integrals and their intuitive substantiations of the previously mentioned integral theorems.

58 Oehrtman et al. (2019, p. 419) argued that "[f]ormal definitions of derivatives and line integrals were abductively transferred from the real case to explore their viability in the complex" and that the "rich geometry emerged historically and individually only after significant time and experience working with and developing the formal implications of the generalization of the basic structures from real to complex variables". Thus, one part of our epistemological analysis will be a historical overview how this transfer of concepts from real analysis took place.

6 PREPARATION OF THE EPISTEMOLOGICAL ANALYSIS

Epistemological analysis

In this chapter, we prepare our epistemological analysis on complex path integrals. The main objective of the *epistemological analyses* is to systematically and comprehensively reveal the different endorsed approaches to realise complex path integrals and narratives about them in complex analysis literature. We start this introduction by briefly revisiting the relevance and goals of the epistemological analysis (Section 6.1) as well as how we will conduct and present it (Section 6.2). Then, we introduce the notions of *aspects* and *partial aspects* (Greefrath et al., 2016b; Roos, 2020) and illustrate them for the Riemann integral in Section 6.3. This enables us to bundle central results on the main ways to define complex path integrals with respect to different constraints imposed on the integrands and the paths in Chapter 9.

Generally speaking, our epistemological analysis fulfils two objectives: On the one hand, it contributes to mathematics education on complex analysis in general and may serve as a model for a strong subject-matter based epistemological study of mathematical objects at university level. On the other hand, our analysis will be a comprehensive account on the scholarly discourse about complex analysis, including a plethora of narratives and substantiating metarules for their endorsement. Sfard (2014) characterises scholarly mathematical discourses in terms of three main features:

> first, this discourse's extreme objectification; secondly, its reliance on rules of endorsement that privilege analytic thinking and leave little space for empirical evidence; and thirdly, the unprecedented level of rigour [compared to school mathematics; EH.] that is to be attained by following a set of well-defined formal rules (Sfard, 2014, p. 200).

In particular, once keywords have been used in definitions, they should not be used in such a way that they contradict endorsed narratives, narratives should be endorsed by previously established logical-deductive rules (even if only implicitly) etc. (see also Viirman, 2014a). This will enable us to contextualise the empirical data on experts' intuitive mathematical discourses from our interview study in Part iii with the scholarly discourse.[59]

59 However, we would like to point out here as a precaution that we will not analyse our empirical data in a prescriptive way. This would also contradict our goal to elicit experts' individual, intuitive interpretations. The comparative horizon given by the scholarly discourse will rather help us identifying certain features of an experts' intuitive mathematical discourse on complex path integrals (e.g., one of our experts Uwe used the keyword "holomorphic" inter-

6.1 RELEVANCE AND GOALS OF THE EPISTEMOLOGICAL ANALYSIS

Epistemological analyses (depending on the focus and context, other authors also call them subject-matter or subject-matter didactic analyses) question what is taught to mathematics students (Hochmuth, 2021b). Their goal is to reveal "[e]ssential structures and domain-specific ways of thinking", and the "inner network of paths by which the components are connected and possible learning paths throughout the domain" are to be shown (Hefendehl-Hebeker et al., 2019, p. 26). Such an analysis aims to "present the contents in a way that is compatible with the standards of the field, and at the same time appropriate to the learners and the requirements of teaching" (Hefendehl-Hebeker et al., 2019, p. 29). Since mathematists usually get to know complex path integrals after having met integrals outside complex analysis, it is expedient to enable them to see connections but also differences to those integrals.

On the other hand, it is also well-known that mathematists' previous encounters with mathematical notions may have a lasting impact on their further mathematical studies when the notion reappears in one way or another (e.g., Bauersfeld, 1983; Kontorovich, 2021b; McGowen & Tall, 2010; Tall & Vinner, 1981). This influence of previously learned facets of these "cross-curricular topics" (Kontorovich, 2018b) on the way learners approach and work with the reappearing topics has received a great deal of attention so far and is relevant for all kinds of transitions in mathematics education (e.g., Biza et al., 2016; Hochmuth et al., 2021).[60] Based on the commognitive perspective on mathematics, reappearing topics, in particular when signified with similar keywords and eventually causing commognitive conflicts, are part of the growth of mathematical discourses and their individualisations (e.g., Biza, 2021; Lavie et al., 2019; Nachlieli & Elbaum-Cohen, 2021; Schüler-Meyer, 2019; Sfard, 2008, 2014, 2021; Thoma, 2018; Thoma & Nardi, 2018). In particular, it is likely that mathematists activate precedents from their encounters with other kinds of integrals (e.g., certain idiosyncratic or endorsed interpretations they met) to those in complex analysis. In the same vein, experts may equally resort to precedents, too, but their precedent-search-spaces are likely to be larger than those from novices.

Hence, it is important to analyse the various approaches to define and realise the complex path integral in relation to other integrals and mathematical concepts students may have encountered previously, in particular in the sequence from analysis in one variable over analysis in several variables to complex analysis (cf. Hochmuth et al., 2021). Having analysed complex path integrals deeply, vertical coherence in the teaching of the cross-curricular topic of integrals may be enhanced in the sense that common ideas or approaches are carved out and used to relate different integrals and their interpretations to each other (e.g., Soto and Oehrtman (2022) discuss this coherence with respect to the idea of accumulation and Hanke (2022b) discusses it with respect to mean values and measuring). As such, the epistemological analysis can

nally consistent in the respective interview, however not completely consistent with the scholarly discourse, and this influences the discursive images on complex path integrals he produces.

60 A "cross-curricular concept" is a mathematical concept, which reappears within a certain curriculum or in various mathematical discourses and is signified with the same or at least a similar name or symbols. According to Kontorovich (2018b, p. 6), their "domanial shift and the substantial change are potential sources for students' difficulties and mistakes". Kontorovich (2018b) argues that words as signifiers can be "homonymous" if their use in different discourses are barely related (e.g., "graph" in graph theory vs. the graph of a function as a point set), or they can be "polysemous" if they are related. For example, many mathematical objects are baptised "integral" and signified with $\int$ in one way or another and are closely related to each other. Thus, the word "integral" may count as polysemous. It is subject to the investigation in our analysis how different "integrals" from various branches of mathematics are related to the complex path integral.

then be used later for empirical research about the teaching and learning of complex analysis (cf. Hochmuth, 2021b).

Additionally, multiple approaches to define and work with complex path integrals become especially important when students, who did not learn complex analysis previously, have to deal with complex path integrals in applications (e.g., to compute certain real integrals with the help of the residue theorem, in analytical number theory, etc.) or when lecturers face the challenge to teach complex analysis to students with highly varying background (e.g., mixed cohorts of different study programmes) (see e.g., Kondratieva & Winsløw, 2018). For example, Kortemeyer and Frühbis-Krüger (2021) report a case from electrical engineering where students had to use the residue theorem to compute certain integrals but their mathematics courses did not schedule complex analysis. Therefore, an important goal of teaching complex path integrals is "to ensure that the mathematical content is connectable for students to later acquire such content themselves" (Kortemeyer & Frühbis-Krüger, 2021, p. 34).

Moreover, for instance following Barnett et al. (2021), Clark (2019), Güçler (2016), and Kjeldsen and Blomhøj (2012), the reflection on the historical evolution of mathematical concepts informs the development of mathematical discourses and potentially also the learners' individualisation of these discourses. Therefore, it is worthwhile to retrace the historical development of the notion of complex path integral, too. In particular, these historical observations shed light on how preceding generations of mathematicians had worked with integrals of complex functions, when and how they first defined them, which alternatives are possible, how certain theorems or proofs have been developed over time etc. In other words, a historical analysis sheds light on the *phylogenesis* of the nowadays endorsed complex analysis discourse (cf. Sfard, 1995, 2008).

The main question guiding our epistemological analysis is how mathematists can address complex path integrals in the form of definitions and various realisations of the signifiers "complex path integral" and "$\int_\gamma f(z)\,\mathrm{d}z$". Hence, we ask which stories about complex path integrals may function as a definition and how these stories are endorsed in the literature. Additionally, we present an alternative definition of for a specific set of constraints on the class of integrands and paths based on a set of properties of complex path integrals, which we abstracted into axioms (Theorem 8.13). Roughly speaking, the epistemological analysis shall illustrate the plethora of ways complex path integrals may come into being discursively, and also which additional visual mediators are used in this context. It also addresses connections to other mathematical areas, in particular other integrals (Kontorovich, 2018b). Commognitively speaking, we may say that our epistemological analysis deals with the routines of defining and substantiating at large: It analyses object-level narratives about complex path integrals, which may be used as a definition of complex path integrals or which turn out to be equivalent to these under certain constraints, as well as their substantiations, that is, the justifications for why these definitions are chosen and endorsed in certain textbooks or other sources. Additionally, other object-level narratives about the complex path integrals in the form of important theorems are presented in the context of their connections to other integrals and integral theorems.

6.2 CONDUCTION AND PRESENTATION OF THE EPISTEMOLOGICAL ANALYSIS

Rather than analysing the discourse about complex path integrals in one source, the epistemological analysis here will cover realisations of complex path integrals including visual mediators, object-level narratives on complex path integrals and geometrical or physical interpreta-

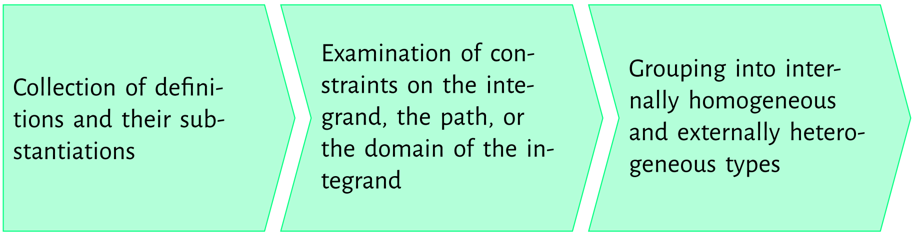

Figure 6.1: Analysis of definitions of complex path integrals (adapted from Hanke, 2022a).

tion as present in the scholarly literature at large (textbooks, lecture notes, historical sources, and articles in mathematics or mathematics education journals).

Figure 6.1 shows an overview of the analysis of the definitions of complex path integrals. We considered approximately 50 textbooks or lecture notes on complex analysis, real analysis, and the history of mathematics, as well as articles in mathematics and mathematics education journals and primary sources from Cauchy's time to present (e.g., Ahlfors, 1979; Bak & Popvassilev, 2017; Beardon, 1979; Bornemann, 2016; Bottazzini, 1986, 2003; Bottazzini & Gray, 2013; Braden, 1987; Burckel, 1979, 2021; Cauchy, 1825; Connell, 1961; Ettlinger, 1922; Fischer & Lieb, 2003, 2010; Forst & Hoffmann, 2012; Forster, 2017b; Freitag & Busam, 2006; Gluchoff, 1991; González, 1992; Grabiner, 1981/2005; Gray, 2015; Heffter, 1960; Heuser, 2008; Isaev, 2017; Klazar, 2018, 2019a, 2019b, 2020; Kneser, 1966; Königsberger, 2004b; Lang, 1999; Lorenz, 1997; Müller, 2018; Neuenschwander, 1996; Polya & Latta, 1974; Ponce Campuzano, 2019a; Porcelli & Connell, 1961; Pringsheim, 1901, 1903; Puiseux, 1850; Remmert & Schumacher, 2002; Riemann, 1851/1867; Smithies, 2005, 1997/2009; Stäckel, 1900; Stewart & Tall, 2018; Trahan, 1965; Wegert, 2012; Whyburn, 1964).

Definitions of complex path integrals were analysed with respect to the constraints on the integrand, the path, and occasionally the domain of the integrand, as well as the signifiers used in the definientia.[61] We also investigated how the authors substantiated their definitions and potential alternatives. The definitions were then organised into internally homogeneous and externally heterogeneous groups. We discuss these groups in terms of the notions of *aspects* and *partial aspects* (Greefrath et al., 2016a, 2016b; Roos, 2020), which we will introduce and refine in the next subsection.

We have explained previously how we place the epistemological analysis in the context of the commognitive framework. However, to simplify the reading and to keep the rest of Part ii self-contained, we will present the epistemological analysis rather classical way, in the sense that we will mainly stay on object-level and tend to hold back on the use of commognitive vocabulary.

6.3 ASPECTS AND PARTIAL ASPECTS

6.3.1 *Aspects, partial aspects, and their discursive reconceptualisation*

Aspect The notion of *aspect* of a mathematical concept was introduced by Greefrath et al. (2016a, 2016b) to complement the idea of *basic ideas* (Section 2.2.2) in German subject matter didactics. An *aspect* is "a subdomain of the concept that can be used to characterize it on the basis of mathematical content" (Greefrath et al., 2016a, p. 101). Moreover, "[a]spects emphasise the con-

61 Recall that the *definiendum* denotes what is defined in a definition and the *definiens* (plural: definientia) is what defines the definiendum (Sill, 2019, p. 96).

nection to the mathematical content, and their identification serve for a finer and thus more detailed structuring of the normative layer" (Roos, 2020, p. 22, own transl.). An aspect thus represents a way of approaching a mathematical concept with the help of which it is possible to define the concept (Greefrath et al., 2016b, p. 17).

The main difference between aspects and (basic) images is that the former eventually lead to a definition of the mathematical object in question, while the latter are ideas that mathematics educators consider helpful for teaching and give meaning to them (however this is conceptualised in the respective publications) (cf. Salle & Clüver, 2021).

Roos (2020) proposed a refinement of the notion of aspect. She observed that several characterisations of a mathematical notion are sufficient in one mathematical context but not in another. For example, this happens when the set of mathematical objects under consideration is restricted to a subset (e.g., Riemann integrals for continuous functions can be characterised in ways, which are not suitable for those the more general set of Riemann integrals for Riemann-integrable functions; Section 6.3.2). For this reason, Roos (2020) introduced the notion *partial aspect*. A *partial aspect* of a mathematical notion is an aspect, which, in comparison with the more general aspects, is only valid under additional constraints. The word "partial" is used to "clarify that [partial aspects; EH.] do not capture the full concept definition" (Roos, 2020, p. 27, own transl.). However, the differentiation between aspects and partial aspects is relative to the constraints chosen as the general. For example, the Riemann integral may be defined for bounded real-valued functions on compact intervals of real numbers. A definition in this context may thus be based on one aspect. However, certain characterisations of Riemann integral are only applicable for a smaller class of integrands (e.g., continuous functions with given anti-derivatives), in other words, they may be based on a partial aspect.

Partial aspect

In the original conceptualisation, Greefrath et al. (2016a, 2016b) distinguish aspects of a mathematical concept in terms of different "subdomain[s] of the concept" (Greefrath et al., 2016a, p. 101). This calls for a reconcepualisation of aspects and partial aspects within the commognitive framework, which we present in this subsection and which we find particularly useful for university mathematics education.

A general scholarly mathematical metarule is to define a mathematical object as clearly as possible, and to be able to refer back to a definition for substantiating further narratives about the mathematical object (Sfard, 2014; Viirman, 2021). We do not argue that definitions should not be motivated, but we think that this metarule justifies Hence, we may say at scholarly level definitions are characterising mathematical objects. Definitions may come from different discourses and we can distinguish them in terms of the keywords or signifiers used in the definientia (cf. Sfard, 2021).

Therefore, one way to look at aspects is to differentiate them according to keywords and other signifiers used in the definiens; hence, aspects of the complex path integral can be described discursively also as follows:

Discursive reconceptualisation of aspects and partial aspects

> An *aspect* of the complex path integral is an object-level narrative about the complex path integral, which may function as a definition and which can be distinguished from other narratives based on the keywords and signifiers used in the definiens. Furthermore, *a partial aspect* is an aspect, which is endorsable with respect to specified additional constraints. What counts as the general situation to which the aspects apply depends on the particular discourse.

Accordingly, a narrative intended as a definition of the definiendum "complex path integral" may contain a keyword or signifier from a previously established discourse (e.g., "Riemann sum" from real analysis). However, this keyword may then be further specified to distinguish it from the realisation of the keyword in the other discourse (e.g., "*complex* Riemann sum"). Figuratively speaking, a previous discourse leaves its mark on a newly formulated aspect.

Moreover, in order to recall an aspect or partial aspects, it is furthermore convenient to baptise them (just as is the case for the basic ideas we have previously explained; Section 2.2.2). For example, in the previous subsection, we have encountered aspects for Riemann integrals such as the "product sum aspect", "anti-derivative aspect", and "measure aspect".

In the next subsection, we illustrate the notion of aspect and partial aspect for Riemann integrals. Finally, we would like to recall again that we will condense our results on the definitions of complex path integrals and their substantiations into aspects and partial aspects in Chapter 9.

6.3.2 *Aspects of Riemann integrals*

Aspects of Riemann integrals

Aspects of the *(definite) Riemann integral* are (1) the *product sum aspect*, (2) the *anti-derivative aspect*, and (3) the *measure aspect* (Greefrath et al., 2016a, pp. 114–116; Greefrath et al., 2016b, pp. 239–244; Weigand et al., 2017). Similar characterisations are discussed at other places (e.g., Bender, 1990; Büchter & Henn, 2015; Danckwerts & Vogel, 2005, 2006; vom Hofe & Blum, 2016; Jones, 2015; Thompson, 1994; Thompson & Silverman, 2008; Tietze et al., 2000). At this point, we will follow the exposition by Greefrath et al. (2016a, 2016b), which we consider concise and clear, and since we consider the idea of aspect as a useful point of reference for our own epistemological on complex path integrals. Since the authors did not use commognitive vocabulary, we keep their terminology here, too.

We will now take a closer look at these aspects in the sense of (Greefrath et al., 2016a, pp. 114–116; Greefrath et al., 2016b, pp. 239–244) and discuss them in terms of their potential applicability in the context of university mathematics (see e.g., Forster, 2016; Ross, 2013, for the mathematical background):

Product sum aspect

(1) The *product sum aspect* characterises the definite integral $\int_a^b f(x)\,\mathrm{d}x$ for a bounded function on $[a, b]$ as the supremum of lower sums and the infimum of upper sums in case these coincide. A lower sum is a sum of the form $\sum_{k=1}^{n} m_k(t_k - t_{k-1})$, where $a = t_0 < t_1 < \cdots < t_n = b$ denotes a partition of the interval $[a, b]$, on which f is defined, m_k denotes the infimum of f on $[t_{k-1}, t_k]$, $n \in \mathbb{N}$, $k = 1, 2, \ldots, n$. Analogously, an upper sum is a sum o the form $\sum_{k=1}^{n} M_k(t_k - t_{k-1})$, where M_k denotes the supremum of f on $[t_{k-1}, t_k]$, $n \in \mathbb{N}$, $k = 1, 2, \ldots, n$. This notion of integral is sometimes called *Darboux integral*. Equivalently, it is possible to choose m_k to be any of the values of f on $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, and let $n \to \infty$. In this case, this notion of integral is the *Riemann integral*. In fact, the Riemann and Darboux approach to integration are equivalent.[62]

The main observation here, which also explains the name of the aspect, is that the Riemann integral is characterised as limits of *product sums*, that is, sums of the form

$$a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n, \quad n \in \mathbb{N}.$$

62 According to Elstrodt (2018, p. 94), at least five mathematicians developed the approach to Riemann integrals via upper and lower sums (Thomae, Ascoli, du Bois-Reymond, Smith, and Darboux).

The first sequence of factors $a_1, \ldots, a_n$, is formed by certain values related to the integrand (e.g., $a_k = m_k$ as above), the other sequence $b_1, \ldots, b_n$ is the sequence of differences of the points in the subdivisions of the interval of the integrand.

(2) In the *anti-derivative aspect*, Greefrath et al. (2016a, pp. 114–115) *define* the integral of a function $f\colon [a, b] \longrightarrow \mathbb{R}$ with an anti-derivative F to be *Anti-derivative aspect*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{6.1}$$

The authors state that this readily enables the calculation of Riemann integrals and they recognise anti-derivatives as a "revision of differentiation" (more likely, they meant "inversion") (Bender, 1990a, cited by Greefrath et al., 2016a, p. 115).

The anti-derivative aspect is further inspired by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x),$$

in case f is continuous (Greefrath et al., 2016b, p. 240). However, having an anti-derivative and being Riemann integrable are two different concepts (Heuser, 2003, p. 452).[63] It is questionable how integrability or the existence of primitive functions can be captured by means of defining the number $\int_a^b f(x)\,\mathrm{d}x$. Notwithstanding, whenever F is a primitive of f and f is Riemann-integrable, then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ is endorsable (Heuser, 2003, pp. 452–453). Greefrath et al. (2016b, p. 243) agree that the definition of the definite integral via the difference of the values of a primitive function of the integrand at the upper and lower bound is problematic, and that then, the definite integral still has to be developed into a meaningful concept. Consequently, it seems very questionable that the anti-derivative aspect is really an aspect for the real definite integral, especially at university where the set of functions considered for integration is not limited to integrable functions with anti-derivatives. Hence, in our view, this aspect is rather a partial aspect.[64]

63 For example, there is an anti-derivative for the function

$$f\colon [-1,1] \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 2x\sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0 \end{cases},$$

but f is not Riemann-integrable; and

$$g\colon [-1,1] \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 0, & x < 0, \\ 1, & x \geq 0 \end{cases}$$

is a Riemann-integrable function without an anti-derivative (Greefrath et al., 2016b, pp. 242–243). Moreover, there are bounded functions with an anti-derivative that are not Riemann-integrable (Goffman, 1977; Gordon, 2016).

64 We note though that there is a version of integral, for which the anti-derivative aspect is a full aspect: namely the so-called *Denjoy* integral (or *Perron integral*, *Henstock-Kurzweil integral*, *gauge integral*) (Rudin, 1987; Shenitzer & Steprāns, 1994; Swartz & Thomson, 1988). One can show that if a function $F\colon [a, b] \longrightarrow \mathbb{R}$ is differentiable possibly except at countably many points and has the derivative $f = F'$ (whenever it exists), then the Denjoy integral of f over $[a, b]$ is defined and is equal $F(b) - F(a)$ (Dieudonné, 1966; Swartz & Thomson, 1988). Hence, the anti-derivative aspect seems to be more appropriate for this integral. In this context, Dieudonné (1966, chs. VII–VIII) studies integrals of *regulated functions*. A regulated function is a function $f\colon [a, b] \longrightarrow \mathbb{R}$, which is the limit of a uniformly convergent sequence of *step functions*. A function $t\colon [a, b] \longrightarrow \mathbb{R}$ is a step function if there is a partition $a = x_0 < \ldots < x_m = b$, $m \in \mathbb{N}$, of $[a, b]$ such that $t(x) \equiv c_k$ for $x \in (x_{k-1}, x_k)$, $c_k \in \mathbb{R}$, $k = 1, \ldots, m$

Measure aspect

(3) The *measure aspect* characterises definite integrals as instruments to measure lengths, areas, volumes, physical quantities etc. The integral of a real-valued function over a real interval $[a, b]$ can be considered as the Lebesgue-measure of the area enclosed by the graph of the function, the x-axis and the two vertical line segments from a and b to the graph. Also, given a non-negative integrable function $f\colon [a, b] \longrightarrow [0, \infty)$, the assignment $A \longmapsto \int_A f(x)\,\mathrm{d}x$ can be seen as a measure of A with weight function f (in case of existence of the integral). In general, the integral of a given integrand f as a function of sets A, $A \longmapsto \int_A f(x)\,\mathrm{d}x$, does not possess the non-negativity property of measures though (Axler, 2020; Greefrath et al., 2016b, p. 244).

One may add another partial aspect to that list, which we may baptise the *step function partial aspect*. We will briefly refer to it in Section 8.4.1. If t is a step function as in footnote 64, we define its *elementary integral* to be $\int_a^b t := \sum_{k=1}^m c_k(x_k - x_{k-1})$. This definition is well-defined since it does not dependent on the choice of subdivision in the construction of t (Amann & Escher, 2006, p. 17). We can phrase the narrative from the step function partial aspect as follows: If $f\colon [a, b] \longrightarrow \mathbb{R}$ is the uniform limit of a sequence of step functions $t_n\colon [a, b] \longrightarrow \mathbb{R}$, then we define the integral of f as the limit of the elementary integrals of these step functions,

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty} \int_a^b t_n$$

(Amann & Escher, 2006, ch. VI.3; Wendland & Steinbach, 2005, ch. 6; see also Behrends, 2007; Forster, 2016).[65]

(Amann & Escher, 2006, p. 4; Dieudonné, 1966, p. 139). He shows that for every regulated function f on $[a, b]$ there is a function $g\colon [a, b] \longrightarrow \mathbb{R}$, such that $g'(x) = f(x)$ except for at most a countable set in $[a, b]$ (Dieudonné, 1966, p. 160). Then, *defines* the (Denjoy) integral of a such a regulated to be $g(b) - g(a)$, which is independent of the choice of g (Dieudonné, 1966, ch. VIII.7).

65 To be precise, we remark that the set of regulated functions is a proper subset of the set of Riemann integrable functions (e.g., Tietze et al., 2000, p. 199). Hence, the step function partial aspect is partial because it is endorsable only for the set of regulated functions. However, if one replaces the fact that f is the uniform limit of a sequence of step functions by the property that there are two sequences of step functions $(t_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ on $[a, b]$ such that $t_n(x) \leq f(x) \leq T_n(x)$ for all $n \in \mathbb{N}$ and $x \in [a, b]$, and $\lim_{n\to\infty} \int_a^b (T_n - t_n) = 0$, then f is Riemann integrable and $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b t_n = \lim_{n\to\infty} \int_a^b T_n$ (see Amann & Escher, 2006; Herfort & Reinhardt, 1980; Tietze et al., 2000, for more details).

7 HISTORICAL DEVELOPMENT OF COMPLEX PATH INTEGRATION

The theory to be erected here did not spring fully-armed from the head of Zeus, but condensed gradually out of the primordial vapors.—Burckel (1979, p. 51)

The purpose of this chapter is to enrich the epistemological analysis with a historical perspective. In particular, we describe elements of the historical development leading to the definition of the complex path integral by Cauchy (1789–1857), which predates and resembles one of the possible definitions used in modern texts on complex analysis. The overview in this chapter shows that much of the discourses on complex path integrals we encounter nowadays goes back to uses of integrals with complex functions even before complex path integrals were first rigorously defined, which is in accordance with the commognitive tenets on the genesis of mathematical objects (Chapter 3). Accordingly, this chapter illustrates how the mathematical object "complex path integral" came into being. Roughly speaking, we see that the historical discourse started off with the discourse on differential forms and anti-derivatives until a Riemann sum definition was first given, which clarified existence claims for integrals of complex functions along paths in the complex plane.

The history of complex analysis has been studied for more than one hundred and twenty years (e.g., Bottazzini, 1986, 2003; Bottazzini & Gray, 2013; Brill & Noether, 1894; Ettlinger, 1922; Grattan-Guinness, 2005a; Gray, 2015; Markusewitsch, 1955; Neuenschwander, 1978, 1981, 1996; Smithies, 1997/2009; Stäckel, 1900). For the historical emergence and uses of complex numbers, the reader is referred to Nahin (2010), Stillwell (2020, ch. 14–15), and Wieleitner (1927). For an introduction to the history of complex analysis, the reader is referred to Bottazzini (2003), Nahin (2010, ch. 7), and Stillwell (2020, ch. 16), as well as the historical remarks by Remmert and Schumacher (2002) and Roy (2013). More detailed and comprehensive treatments on the history of complex analysis and the passage from real to complex analysis are given by Bottazzini (1986), Bottazzini and Gray (2013), Gray (2015), and Smithies (1997/2009).

Additionally, Burckel (2021) provides a lot of details on the history of many theorems in complex analysis. For a primer on the historical development of the Riemann integral and the increase of rigour in real analysis, the reader is advised to consult selected chapters in Grabiner (1981/2005), Grattan-Guinness (2005b), Jahnke (2003), and Stillwell (2020) and the didactically oriented summary in Tietze et al. (2000, ch. 6).

7.1 DEFINITE AND INDEFINITE INTEGRALS AROUND THE 18TH CENTURY

Indefinite integrals are treated like anti-derivatives.

In Euler's (1707–1783) times, it was usual to regard indefinite integrals $\int f(x)\,\mathrm{d}x$ as primitives, that is, any differentiable function F whose derivative is the integrand f; this was the case for real- or complex-valued functions of a real or complex variable (Bottazzini & Gray, 2013, ch. 2; Smithies, 1997/2009, ch. 1; Grabiner, 1981/2005, ch. 6). Similarly, mathematicians in the 18th interpreted the integral of a differential form

$$\int P\,\mathrm{d}x + Q\,\mathrm{d}y$$

as a function whose differential is the integrated form "whereas nowadays the expression $\int Pdx + Qdy$ is understood as a line integral" (Bottazzini & Gray, 2013, p. 86).

One of the earliest usages of complex functions in integrals occurred when Clairaut (1713–1765), Leibniz (1646–1716), Bernoulli (1667–1748), and d'Alembert (1717-1783) amongst others investigated integrals of rational functions or differential equations at the beginning of the 18th century (Bottazzini & Gray, 2013, ch. 2). They performed substitutions involving complex numbers such as $z = ib\frac{t-1}{t+1}$ when integrating the differential $\frac{a\,\mathrm{d}z}{b^2+z^2}$ and obtained the "differential of an *imaginary logarithm*" $\frac{-a\,\mathrm{d}t}{ibt}$ (Stäckel, 1900, p. 110, emph. orig., own transl.).

Method of splitting up integrands and integrals with complex variables and functions into their real and imaginary parts

Similarly, Euler, d'Alembert, Lagrange (1736–1813), or Laplace (1749–1827), to recall some of the prominent names of that time, manipulated differential forms with complex quantities in their treatment of exact differential forms (Smithies, 1997/2009, ch. 1). For example, Euler (1755) concluded that if $u\,\mathrm{d}x + v\,\mathrm{d}y$ and $u\,\mathrm{d}y - v\,\mathrm{d}x$ are exact (i.e., the differential of a function), then $(u - iv)(\mathrm{d}x + i\,\mathrm{d}y)$ and $(u + iv)(\mathrm{d}x - i\,\mathrm{d}y)$ are also exact (Bottazzini & Gray, 2013, pp. 87–88). Euler (1777) also considered the expression $\Delta(z) = \int Z(z)\,\mathrm{d}z$ for a complex variable $z = x + iy$ and the form $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, and split up the integrand Z and the integral Δ into real and imaginary parts, that is, $Z = M + Ni$ and $\Delta = P + Qi$. Formally, he obtains

$$P + Qi = \int Z\,\mathrm{d}z = \int (M + Ni)(\mathrm{d}x + i\,\mathrm{d}y),$$

and thus the two integrals

$$P = \int M\,\mathrm{d}x - N\,\mathrm{d}y$$

and

$$Q = \int N\,\mathrm{d}x + M\,\mathrm{d}y,$$

"so that $M\,\mathrm{d}x - N\,\mathrm{d}y$ and $N\,\mathrm{d}x + M\,\mathrm{d}y$ must be both complete differentials; by Clairaut's conditions,[66] it follows that

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}\text{"}$$

(Smithies, 1997/2009, p. 12; see also Bottazzini and Gray, 2013, ch. 2; Stäckel, 1900). These equations apparently look like what are today known as the Cauchy-Riemann differential equations (Equation A.6). However, here in the historical context, as in the example by Euler, these differential equations were not obtained from an explicit definition of the complex derivative of a complex function but from formal computations with differential forms involving complex functions. As already indicated above, the mathematicians were originally not interested in integrals of complex functions per se. Rather, they aimed to compute specific different real integrals, such as $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,\mathrm{d}x = \frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$ (Bottazzini & Gray, 2013, p. 94; Stäckel, 1900, p. 114). Later, for example in 1818, Cauchy also obtained the aforementioned differential equations, and, exemplary in 1831, he published the version of the Cauchy-Riemann partial differential equations in polar coordinates $\frac{\partial f(\tilde{x})}{\partial X} = \frac{1}{iX}\frac{\partial f(\tilde{x})}{\partial p}$ for $\tilde{x} = X\mathrm{e}^{pi}$ (Bottazzini & Gray, 2013, p. 150).

Preliminary stage of the anti-derivative aspects of definite integrals

Definite integrals between two real numbers were also treated as the differences of the values of primitive functions at the boundaries of integration, which is what we refer to as a version of the fundamental theorem of calculus. According to Smithies (2005, p. 381), in the course of the 18th and beginning of the 19th century, the integral $\int_a^b f(x)\,\mathrm{d}x$ was used to denote the difference $F(b) - F(a)$ whenever F satisfies $F' = f$. On the other hand, the view that an integral is a certain sum of infinitesimal quantities of the type $f(x_j)\,\mathrm{d}x$ was regarded as a theorem rather than a definition (Smithies, 2005, p. 381). In particular the conception of a definite integral as a difference of the values of a primitive at the upper and lower limit of integration led to problems. For example, a naive application of the anti-derivative aspect led to "equations" like

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -2,$$

or

$$\begin{aligned}\int_{-1}^{1} \frac{\mathrm{d}x}{x} &= \log(x)\Big|_{-1}^{1} = -\log(-1) \\ &= \text{the infinity of values } (2n+1)\pi i\end{aligned} \tag{7.1}$$

—the first integral evaluates to a negative number even though the integrand is positive, and the second integral evaluates to infinitely many imaginary values (for the logarithm was known to be multivalued) even though the integrand is real (Smithies, 1997/2009, p. 86; Bottazzini & Gray, 2013, p. 122). This was not regarded as commensurable with an integral being the sum of infinitesimal quantities related to $\frac{\mathrm{d}x}{x}$ or $\frac{\mathrm{d}x}{x^2}$ (after all, $\frac{1}{x^2}$ is positive and $\frac{1}{x}$ is real for all $x \in \mathbb{R} \setminus \{0\}$).

66 Likely, this condition says that the form $U\,\mathrm{d}x + V\,\mathrm{d}y$ is exact if $\frac{\partial}{\partial y}U = \frac{\partial}{\partial x}V$ and vice versa. It is also likely that the partial derivatives were assumed to be continuous. Thus, Bottazzini and Gray (2013) hypothesise that "Clairaut assumed that the integral of the differential form was independent of the path of integration" (Bottazzini & Gray, 2013, p. 85).

In this light, Poisson (1781–1840) writes in 1820 that the "theorem $\int_a^b f(x)dx = F(b) - F(a)$ where $dF(x) = f(x)dx$ ceases to hold if $f(x)$ becomes infinite in the interval of integration" (Bottazzini & Gray, 2013, p. 122). Additionally, for instance for the integral in Equation 7.1, he formally substitutes $x = -(\cos(z) + i\sin(z))$ into $\int_{-1}^{1} \frac{\mathrm{d}x}{x}$, integrates from $z = 0$ to $z = (2n+1)\pi$ ($n \in \mathbb{N}_0$), and gets

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x} = i\int_{0}^{(2n+1)\pi} \mathrm{d}z = (2n+1)\pi i$$

(Bottazzini & Gray, 2013, p. 122)). Thus, instead of using the fundamental theorem of calculus formally, a complex substitution leads to the same result as in Equation 7.1. Poisson's substitution is noteworthy because it transforms the real integral in question into another integral avoiding the singularity at 0. Thus, the multivaluedness of the right integral in Equation 7.1 seems endorsable even if it conflicts with the expectation that $\int_{-1}^{1} \frac{\mathrm{d}x}{x}$ should not be imaginary (and does not exist as an improper Riemann integral as we would say today). Poisson also suggested to treat "imaginary substitutions", as we have just described exemplarily, as a heuristic method to calculate integrals, the result of which should be confirmed without these substitutions afterwards (Smithies, 2005, p. 378).

Poisson was not alone to anticipate integration through regions in the complex plane, as the following remarkable excerpt from a letter (1811) of Gauß (1777–1855) to Bessel (1784–1846) shows:

Gauß anticipates path independence of complex path integrals.

> What should we make of $\int \varphi x \cdot dx$ for $x = a + bi$? Obviously, if we're to proceed from clear concepts, we have to assume that x passes, via infinitely small increments (each of the form $\alpha + i\beta$), from what value at which the integral is supposed to be 0, to $x = a + bi$ and that then all the $\varphi x \cdot dx$ are summed up. In this way the meaning is made precise. [...] The continuous passage from one value of x to another $a + bi$ accordingly occurs along a curve and is consequently possible in infinitely many ways. But I maintain that the integral $\int \varphi x \cdot dx$ computed via two different such passages always gets the same value as long as $\varphi x = \infty$ never occurs in the region of the plane enclosed by the curves describing these two passages. This is a very beautiful theorem, whose not-so-difficult proof I will give when an appropriate occasion comes up. [...] The passage from point to point can always be carried out without ever touching on where $\varphi x = \infty$. However, I demand that those points be avoided lest the original basic conception of $\int \varphi x \cdot dx$ lose its clarity and lead to contradictions. Moreover it is also clear from this how a function generated by $\int \varphi x \cdot dx$ could have several values for the same x, depending on whether a point where $\varphi x = \infty$ is gone around not at all, once, or several times. (translation from German according to Remmert, 1998, pp. 167–168)

Thus, Gauß was aware of what is now attributed to Cauchy as Cauchy's integral theorem (Theorem A.17) and dependence of complex path integrals between two points in the complex plane on the chosen path. Additionally, he explained that integrals along paths winding around singularities (which he refers to as points x with $\varphi x = \infty$) may lead to multiple values of the integral: "One can see that even then Gauss dominated a circle of thought that only decades later forms the adornment of Cauchy's most important works" (Brill & Noether, 1894, p. 159, own transl.). Gauß did not publish or develop these ideas further though (Wußing, 2009, p. 250).

In sum, Bottazzini and Gray (2013, p. 126) conclude that the use of complex substitutions to evaluate real integrals (as done by d'Alembert) was a "skilful, successful practice without a coherent and satisfactory theory", but Cauchy's "research work on definite integrals [...] apparently persuaded him to abandon the prevailing view at the time that the definite integral should be defined via the primitive function and, instead, to define the definite integral as limits of [product sums, EH.]" (Bottazzini & Gray, 2013, p. 122).

7.2 PRECISION OF INTEGRATION OVER A REAL INTERVAL

Cauchy (1823): integrals are limits of product sums.

In 1823, Cauchy (1823, leçon 21) defined the real integral $\int_{x_0}^{X} f(x)\,\mathrm{d}x$ of a continuous real-valued function f as a limit of sums of the form

$$\sum_{k=1}^{n} (x_k - x_{k-1}) f\left(x_{k-1}\right), \tag{7.2}$$

and gave a proof for the convergence of these sums as the number of subdivisions of (equidistant) partitions of the interval of real numbers $[x_0, X]$ tends towards infinity. Hence, Cauchy eventually changed the view that product sums are not used for approximations of integrals only, but instead, this definition made it possible to prove the existence of definite integrals for continuous functions (Lützen, 2003, pp. 171–172). Later, this definition of definite integral was refined to what we nowadays know as the *Riemann* or *Darboux integral*, which identifies a larger class of integrable functions on closed intervals of real numbers than the continuous functions (Hochkirchen, 2003). Overall, Cauchy avoided geometric interpretations and figures, and worked with formulas only (Smithies, 2005). The area image of the definite integral is briefly touched at one point (Cauchy, 1823, leçon 23). Also, Cauchy (1823, leçon 23) formulated the linearity property

$$\int_{x_0}^{X} (au + bv + cw \ldots)\,\mathrm{d}x = a\int_{x_0}^{X} u\,\mathrm{d}x + b\int_{x_0}^{X} v\,\mathrm{d}x + c\int_{x_0}^{X} w\,\mathrm{d}x + \ldots$$

for functions $u, v, w, \ldots$ of the variable x and constants $a, b, c, \ldots$, and extended it to integrals of complex-valued functions:

$$\int_{x_0}^{X} (u + iv)\,\mathrm{d}x = \int_{x_0}^{X} u\,\mathrm{d}x + i\int_{x_0}^{X} v\,\mathrm{d}x.$$

7.3 EARLY VERSIONS OF INTEGRAL THEOREMS IN COMPLEX ANALYSIS

Cauchy's first detailed treatment of integration of complex functions was his *Mémoire* (Cauchy, 1814/1882) though. It is said to be the "first deduction by rigorous methods of the formulæ, hitherto obtained by purely formal processes, for evaluating definite integrals" (Ettlinger, 1922, p. 256), that is, formulas obtained by "imaginary substitutions" to evaluate real integrals as described in the previous section. Cauchy's reasoning depended on for his days typical assumptions that the functions involved are sufficiently smooth and partial derivatives commute (Smithies, 2005; see also Piña Aguirre, 2018).

He starts from the equations $\frac{\partial}{\partial x}\int f(z)\,\mathrm{d}z = f(z)\frac{\partial z}{\partial x}$ and $\frac{\partial}{\partial y}\int f(z)\,\mathrm{d}z = f(z)\frac{\partial z}{\partial y}$, where z is a function of two real variables x and y, and differentiates the first with respect to y and the second with respect to x. He obtains

$$\frac{\partial}{\partial y}\left[f(z)\frac{\partial z}{\partial x}\right] \quad \text{and} \quad \frac{\partial}{\partial x}\left[f(z)\frac{\partial z}{\partial y}\right]$$

(Cauchy, 1814/1882, pp. 336–337). Then, he repeats a similar argument replacing z with $M + Ni$, where M and N are themselves real-valued functions of x and y.

Starting from $f(M+Ni) = P + Qi$, for which he tacitly assumes that $f(M - Ni) = P - Qi$, Cauchy (1814/1882, pp. 336–339) considered the expressions

$$S + Ti = f(M+Ni)\frac{\partial(M+Ni)}{\partial x} \tag{7.3}$$

and

$$U + Vi = f(M+Ni)\frac{\partial(M+Ni)}{\partial y}. \tag{7.4}$$

Differentiating these equations with respect to x, y, and both, and using tacitly that the partial derivatives commute, yielded

$$\frac{\partial S}{\partial y} = \frac{\partial U}{\partial x} \quad \text{and} \quad \frac{\partial T}{\partial y} = \frac{\partial V}{\partial x} \tag{7.5}$$

(Ettlinger, 1922, p. 257; Smithies, 1997/2009, p. 29).[67] Then, he integrated both sides of Equation 7.5 along the sides of a rectangle R, whose bottom left corner is $(0,0)$ and whose top right corner is (a,b) $(a, b > 0)$. Cauchy obtained[68]

$$\int_0^a S\,\mathrm{d}x - \int_0^a s\,\mathrm{d}x = \int_0^b U\,\mathrm{d}y - \int_0^b u\,\mathrm{d}y \tag{7.6}$$

and

$$\int_0^a T\,\mathrm{d}x - \int_0^a t\,\mathrm{d}x = \int_0^b V\,\mathrm{d}y - \int_0^b v\,\mathrm{d}y, \tag{7.7}$$

where $S = S(x,b)$, $s = S(x,0)$, $T = T(x,b)$, and $t = T(x,0)$ are functions of x alone, and $U = U(a,y)$, $u = U(0,y)$, $V = V(a,y)$, and $v = V(0,y)$ are functions of y alone (Ettlinger, 1922, p. 258).

Hence, Cauchy constantly separated all variables and functions into their real and imaginary parts and he assumed that the functions were sufficiently smooth. When he resubmitted the mémoire roughly ten years later in 1825, he added a footnote (Cauchy, 1814/1882, p. 338) stating that similar equations to those in Equation 7.6 and Equation 7.7 "could be replaced by the 'imaginary formula'

67 Note that if $M + Ni$ is simply $z = x + iy$ and $f(z) = u(z) + iv(z)$, then the equations in (7.5) become $\frac{\partial}{\partial y}u + i\frac{\partial}{\partial y}v = i\frac{\partial}{\partial x}u - \frac{\partial}{\partial x}v$, the Cauchy-Riemann differential equation Equation A.7. Cauchy-Riemann differential equations had already been published by d'Alembert in 1752, too (Smithies, 2005, pp. 379–380).

68 The notation and presentation of Cauchy's results as presented here is due to (Ettlinger, 1922).

$$\begin{aligned}&\int_0^x (S(x,z)+iT(x,z))\,\mathrm{d}x - \int_0^x (S(x,0)+iT(x,0))\,\mathrm{d}x\\ &= \int_0^z (U(x,z)+iV(x,z))\,\mathrm{d}z - \int_0^z (U(0,z)+iT(0,z))\,\mathrm{d}z\text{"}\end{aligned} \tag{7.8}$$

(Bottazzini and Gray, 2013, p. 103; the change of the upper boundaries of integration when compared to Equation 7.6 and Equation 7.7 is a result from Ettlinger's (1922) presentation). Remarkably, here, Cauchy explicitly noted an equation of integrals of complex functions.

Equation 7.6 and Equation 7.7 can be interpreted as a precursor of Cauchy's integral formula (Theorem A.22): Multiplying the first of these equations by -1 and the second by $-i$ and adding, we obtain that the complex path integral (in modern terminology) of the function $f(M+Ni)$ along the boundary L of the aforementioned rectangle vanishes:

$$\int_L f(M+Ni)\frac{\mathrm{d}(M+Ni)}{\mathrm{d}(x+yi)}\,\mathrm{d}(x+yi) = \int_L f(M+Ni)\,\mathrm{d}(M+Ni) = \int_L f(z)\,\mathrm{d}z = 0 \tag{7.9}$$

(Ettlinger, 1922, pp. 257–258; Remmert & Schumacher, 2002, pp. 174–176; Smithies, 1997/2009, p. 29). In other words, this is a version of Cauchy's integral theorem for a rectangle. Afterwards in the same paper, Cauchy (1814/1882) applied his results to evaluate integrals of functions of one real variable, which had been his original motivation for the paper. He also deduced a first version of the Residue theorem (Theorem A.29) when considering the case that the integrated function has an "infinity", that is, a singularity, inside the rectangle around which it was integrated (Cauchy, 1814/1882; Ettlinger, 1922; Smithies, 1997/2009).

In the above argument, Cauchy reasoned with the interchangeability of the order of integration of double integrals of the functions in Equation 7.5 over the rectangle in question. In the second part of the *Mémoire* (Cauchy, 1814/1882, pp. 388–394), he investigated the case that the order of integration is not interchangeable. For this, he introduced correction terms, which were made up of the differences of the double integrals with both possible orders of integration, that is, terms of the form $\int_{x_0}^X \int_{y_0}^Y \cdots \mathrm{d}y\,\mathrm{d}x - \int_{y_0}^Y \int_{x_0}^X \cdots \mathrm{d}x\,\mathrm{d}y$. The integrands in these correction terms came from equations like Equation 7.5 (Smithies, 2005, p. 383). According to Smithies (2005, p. 383) and using the modern notation of complex functions $f(z) = P(x,y)+iQ(x,y)$ for $z = x+iy$, Cauchy's correction terms for the real and imaginary part of $f(z)$ amount to

$$A = -\int_{y_0}^Y [Q(X,y)-Q(x_0,y)]\,\mathrm{d}y - \int_{x_0}^X [P(x,Y)-P(x,y_0)]\,\mathrm{d}x$$

for the difference of the double integrals associated to P and

$$B = \int_{y_0}^Y [P(X,y)-P(x_0,y)]\,\mathrm{d}y - \int_{x_0}^X [Q(x,Y)-Q(x,y_0)]\,\mathrm{d}x$$

for the difference of the double integrals associated to Q (for the details see Cauchy, 1814/1882; Ettlinger, 1922; Smithies, 2005, 1997/2009). Putting these together, one obtains

$$\begin{aligned} A + iB &= \int_{y_0}^{Y} [f(X+iy) - f(x_0+iy)]\,\mathrm{d}y - \int_{x_0}^{X} [f(x+iY) - f(x+iy_0)]\,\mathrm{d}x \\ &= \int_{\partial R} f(z)\,\mathrm{d}z, \end{aligned}$$

where for the last integral we made use of today's notion of complex path integral of f along the boundary of the rectangle R counterclockwise. Hence, Cauchy implicitly developed an expression for the complex path integral $\int_{\partial R} f(z)\,\mathrm{d}z$ in terms of correction terms for certain double integrals (Smithies, 2005).

Early appearances of Cauchy's integral formula and the residue theorem

Formulations of Cauchy's integral formula (Theorem A.22) and the Residue theorem (Theorem A.29) date back to Cauchy, too. For example, one early version of Cauchy's integral formula was stated as

$$\frac{1}{2}\int_0^{\pi} \left(\frac{f\left(\mathrm{e}^{pi}\right)}{1 - a\mathrm{e}^{-pi}} + \frac{f\left(\mathrm{e}^{-pi}\right)}{1 - a\mathrm{e}^{pi}} \right) \mathrm{d}p = \pi f(a)$$

(1819) or

$$2\pi f(x) = \int_{-\pi}^{\pi} \frac{\tilde{x} f(\tilde{x})}{\tilde{x} - x}\,\mathrm{d}p$$

for $\tilde{x} = \mathrm{e}^{pi}$ (1831) (Bottazzini & Gray, 2013, pp. 121, 151). In particular, the early versions of this formula are not signified as complex path integrals but as integrals with a complex integrand over a real interval. Cauchy, Laplace, Chebyshev (1821–1894), and Laurent (1813–1854) followed Cauchy's work and refined the expansion of holomorphic functions around a point of its domain into Taylor series or in case of an isolated singularity into a Laurent series (Bottazzini & Gray, 2013, ch. 3.5).

In particular, the integrals in these results were not realised with the help of complex path integrals. Instead, the integrals were notated as ordinary real integrals over intervals of real numbers (or double integrals in the real plane). From what has been said so far, we can see that many mathematicians initially worked quite freely with integrals of complex functions. An explicit definition of complex path integrals had not yet been given. Nevertheless, Cauchy produced preliminary versions of all the important integrals theorems (Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Residue theorem (Theorem A.29)) that we still teach today in virtually every course on complex analysis. Instead of using complex path integrals, he notated all the occurring integrals we presented here as ordinary real integrals over intervals of real numbers (or double integrals in the real plane). Despite all these achievements, conflicts arose when object- and meta-rules for real integrals (such as applying substitutions) were abductively transferred to integrals of complex functions, which astonished our mathematical ancestors and could not be explained with the help of the available discourse on real integrals. Therefore, Cauchy considered it necessary to define integrals of complex functions he was working with.

7.4 HISTORICAL DEFINITIONS OF COMPLEX PATH INTEGRALS

7.4.1 *Cauchy's mémoire (1825)*

Cauchy's "paper on definite integrals between imaginary limits" (1825, own transl. of the title) finally contained a definition of integrals of complex functions. Eventually, the year 1825 may be considered the birth of complex analysis (Stäckel, 1900). Nahin (2010, p. 187) describes this paper as "the very beginnings of modern complex function theory" and Bottazzini and Gray (2013, p. 133) as Cauchy's "masterwork".

Lützen (2003, p. 171) surmises that the inconsistencies of regarding the definite integral as the difference between two values of a primitive and what we nowadays call path dependence of path integrals, led Cauchy to revise the notion of integral and to look for a sounder definition. Grabiner (1981/2005, ch. 6) argues that Cauchy was motivated to define integrals (for both, real and complex functions) in order to have a definition, which is also applicable in case the integrand does not have a primitive function. In particular, she argues that Cauchy's interest in integration of complex functions "was a major impetus to his often expressed dissatisfaction with purely formal analogies in mathematical reasoning" (Grabiner, 1981/2005, p. 143).

Mimicking Equation 7.2, Cauchy provides the following definition:[69]

Cauchy's (1825) definition of complex path integrals

> it is convenient to represent by the notation
>
> $$\int_{x_0+iy_0}^{X+iY} f(z)\,\mathrm{d}z$$
>
> the limit, or one of the limits, to which the sum of the products of the form
>
> $$\begin{cases} [(x_1 - x_0) + (y_1 - y_0)i]f(x_0 + y_0 i), \\ [(x_2 - x_1) + (y_2 - y_1)i]f(x_1 + y_1 i), \\ \text{etc.} \\ [(X - x_{n-1}) + (Y - y_{n-1})i]f(x_{n-1} + y_{n-1} i), \end{cases} \tag{7.10}$$
>
> converges. (Cauchy, 1825, p. 3)

Here, Cauchy (1825, p. 3) additionally describes that the finite sequences of real numbers $x_0, x_1, \ldots, x_{n-1}, X$ and $y_0, y_1, \ldots, y_{n-1}, Y$ are arranged monotonically increasing or decreasing ($n \in \mathbb{N}$). Smithies (2005, p. 386) claims that the required monotony results from the product sum definition of the real definite integral (Cauchy, 1823). From a general perspective nowadays, this is obviously quite a restriction. Such sequences can be obtained, according to Cauchy, with two continuous and monotonic functions φ and χ of a new variable t on an interval of real numbers $[t_0, T]$ such that $x_0 = \phi(t_0)$, $X = \phi(T)$, $y_0 = \chi(t_0)$, $Y = \chi(t)$ as well as $x_n = \phi(t_n)$ and $y_n = \chi(t_n)$ for ascending values $t_0 < t_1 < \ldots t_{n-1} < T$.

Then, taking for granted that φ and χ are continuously differentiable, Cauchy (1825, pp. 4–5) used that one has "very nearly" (Cauchy, 1825, p. 4, own transl.) that $x_1 - x_0 = (t_1 - t_0)\varphi'(t_0)$

69 We changed $\sqrt{-1}$ to i throughout.

etc. and $y_1 - y_0 = (t_1 - t_0)\chi'(t_0)$ etc. Using these approximate equalities, the addends in Equation 7.10 are transformed to

$$\begin{cases} (t_1 - t_0)[\varphi'(t_0) + i\chi'(t_0)]f[\varphi(t_0) + i\chi(t_0)], \\ (t_2 - t_1)[\varphi'(t_1) + i\chi'(t_1)]f[\varphi(t_1) + i\chi(t_1)], \\ \text{etc.} \\ (T - t_{n-1})[\varphi'(t_{n-1}) + i\chi'(t_{n-1})]f[\varphi(t_{n-1}) + i\chi(t_{n-1})]. \end{cases}$$

A parametric formula for the complex path integral

Adding them up and taking the limit yields

$$\int_{t_0}^{T} [\phi'(t) + i\chi'(t)]\, f\, [\phi(t) + i\chi(t)]\, \mathrm{d}t. \tag{7.11}$$

(Cauchy, 1825, pp. 4–5). This formula agrees with what is nowadays written as

$$\int_{t_0}^{T} f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

for the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ of the function f and the path $\gamma = \varphi + i\chi$ on $[t_0, T]$, and which may also be used as a definition (e.g., Lang, 1999, p. 95; Remmert & Schumacher, 2002, p. 155).

Grabiner (1981/2005, pp. 161–162) describes Cauchy's definition of complex path integrals as "purely foundational", but

> it does not enter into the proofs of any of the theorems [of the memoir; EH.] once the parametrized form [Equation 7.11; EH.] has been obtained. Of course, Cauchy could not define the complex integral as an antiderivative; yet it had to be introduced somehow [...] Cauchy, however, wanted a rigorous foundation for his theory of complex integration, and the basis he chose was his definition of real-valued integrals.

From this point of view, Cauchy's definition of complex path integrals has an epistemological function, namely defining it first before using it.

In the rest of the paper, as Grabiner (1981/2005) described, Cauchy (1825) does not use this definition further. He goes on to prove that the integral does not depend on the chosen monotonic functions φ and χ if the function $f(x + yi)$ stays "finite and continuous" (Cauchy, 1825, p. 5) for x between x_0 and X and y between y_0 and Y with the method of calculus of variations (Cauchy, 1825, pp. 5–6).

However, we would like to point out that it is not unanimous, which exact conditions Cauchy required for the integrand f and the path from $x_0 + iy_0$ to $X + iY$ in general. Remmert and Schumacher (2002, p. 175) and Smithies (2005, p. 387) claim that Cauchy tacitly assumed continuous differentiability of f in his proof. However, the reception of Cauchy's work in the literature is not unanimous whether Cauchy implicitly assumed that continuous functions are (continuously) differentiable or additionally satisfy the Cauchy-Riemann differential equations (Bottazzini & Gray, 2013, p. 135). It is also not clear whether or until when Cauchy assumed that all of the functions he considered are differentiable or continuously differentiable (e.g., Smithies, 1997/2009, pp. 28, 158, 173, 201).

Furthermore, we note that Cauchy (1825, p. 6) remarked that "the function under the sign $\int$ reduces [...] to an exact differential", hence the independence of the complex path integral under the above conditions could have been foreseen. Smithies (2005, p. 387) interprets this remark in the sense that

$$f(\mathrm{d}x + i\,\mathrm{d}z) = U\,\mathrm{d}x + V\,\mathrm{d}y$$

is exact because $\frac{\partial}{\partial y}U = \frac{\partial}{\partial x}V$ (in fact, $U = f$ and $V = if$ and $\frac{\partial}{\partial y}f = \frac{\partial}{\partial x}if$ is the Cauchy-Riemann differential equation ((A.7)).

As we may have already anticipated from Cauchy (1814/1882) (Section 7.3), Cauchy was also aware that the integral $\int_{x_0+iy_0}^{X+iY} f(z)\,\mathrm{d}z$ depends on the path between the points (x_0, y_0) and (X, Y) in the plane. He discusses some of the possibilities (Cauchy, 1825, pp. 20–24). For instance, on the one hand, he calls the integral along the diagonal between the two points

$$\int_0^1 [X - x_0 + (Y - y_0)i] f[(x_0 + y_0 i)(1 - t) + (X + Yi)]\,\mathrm{d}t$$

"*la valeur moyenne* [the average value; EH.]" (Cauchy, 1825, p. 23). On the other hand, the two integrals along the line segments given by $y = y_0$ and $x = X$,

$$\int_{x_0}^{X} f(x + y_0 i)\,\mathrm{d}x + i\int_{y_0}^{Y} f(X + yi)\,\mathrm{d}y$$

or the line segments given by $x = x_0$ and $y = Y$,

$$\int_{x_0}^{X} f(x + Yi)\,\mathrm{d}x + i\int_{y_0}^{Y} f(x_0 + yi)\,\mathrm{d}y$$

are called "*valeurs extrêmes* [extreme values; EH.]" (Cauchy, 1825, p. 24).

Last but not least, we would like to point out that Cauchy did rarely use geometrical language in the aforementioned works (Cauchy, 1825, 1814/1882) or changed between algebraic to geometric realisations of what we conceive as paths in today's terminology rather suddenly (Bottazzini & Gray, 2013, p. 137). Later though, he began to use geometrical language more freely (Smithies, 2005). Also, historians of mathematics disagree as to whether and when Cauchy considered complex-valued functions as functions of a complex variable as such or rather as functions depending on two real variables (Bottazzini & Gray, 2013, p. 137; Smithies, 1997/2009, p. 187). In this context, Smithies (2005, p. 389) argues that Cauchy did not realise until about 1851 that a function $P(x, y) + iQ(x, y)$ of two real variables x and y can be understood as a function of the complex variable $z = x + iy$, and that a complex differentiable function has to satisfy the Cauchy-Riemann differential equations (Equation A.6).

7.4.2 *Definitions of the complex path integral in other historical sources*

Let us now turn to some examples of definitions of complex path integrals in other historical sources. The textbook by Casorati (1868, sezione terza, capitolo quatro) contains a definition of the complex path integral "$\int_{z_0}^{\mathbf{z}} f(z)dz$" for a complex path integral between the two complex numbers z_0 and $\mathbf{z}$, which is very similar to Cauchy's definition (Casorati, 1868, p. 358). It is defined as the limit of sums of the form

$$f(z_0)(z_1 - z_0) + f(z_1)(z_2 - z_1) + \cdots + f(z_{n-1})(\mathbf{z} - z_{n-1}),$$

Fig. 20.

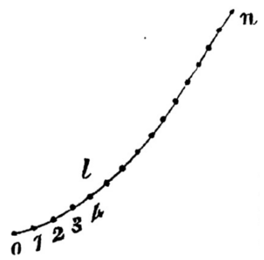

Figure 7.1: Subdivision of a contour (Figure 20 from Casorati (1868)).

where the $z_1, z_2, \ldots, z_{n-1}$ "constitute a continuous, but otherwise complex succession from z_0 and $\mathbf{z}$" for n growing to ∞ (Casorati, 1868, p. 358, own transl.). Casorati also includes a separation into real and imaginary part, namely

$$\text{“}\int_0^l \left[\varphi \frac{dx}{dl} - \psi \frac{dy}{dl}\right] dl + i \int_0^l \left[\varphi \frac{dy}{dl} + \psi \frac{dx}{dl}\right] dl\text{”},$$

where φ and ψ are the real and imaginary part of the integrand f, and x and y describe the real and imaginary part of the path from z_0 to $\mathbf{z}$ as a function on $[0, l]$ (Casorati, 1868, p. 359; the multiple usage of l is original). Moreover, this textbook also includes a picture of the subdivision of a trace of a path (Figure 7.1).[70]

Riemann (1826–1866) also used the product sum aspect to define an integral of the form $\int_a^b f(z)\,\mathrm{d}z$, where a and b are complex numbers and f is a complex function (Neuenschwander, 1996). Riemann used geometric language explicitly. For example, he described that the complex variable z traces a curve between two points a and b in the complex plane, and therefore, in order to define the integral of a function of z, a curve between a and b has to be fixed (Neuenschwander, 1996, p. 29). The terms $\mathrm{d}x$ and $\mathrm{d}y$ are interpreted as changes with respect to x and y, and $\mathrm{d}x + i\,\mathrm{d}y$ is said to be the "change of integration variables along each element of the curve and depicts geometrically the length and direction of the latter" (Neuenschwander, 1996, p. 29, own transl.).

Riemann further considered the case that y is a function of x, $y = \varphi(x)$, and accordingly $f(x + iy)$ is a function depending on x alone, $F(x)$. In doing so, Riemann derived the identity

$$\int_a^b f(x+iy)(\mathrm{d}x + i\,\mathrm{d}y) = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x + i \int_{x_1}^{x_2} F(x)\frac{\partial\varphi(x)}{\partial x}\cdot \mathrm{d}x$$

(Neuenschwander, 1996, p. 29). He used this subdivision of the complex path integral into its real and imaginary part to apply propositions from real analysis (Neuenschwander, 1996, p. 30).

Riemann also discusses that one gets a function $w = \int_a^b f(z)\,\mathrm{d}z$ of b if the complex number $a = p + iq$ is fixed and $b = r + is$ is allowed to vary:[71] He parametrises the path implicitly

70 The points of subdivision are signified with natural numbers from 0 to n instead of the actual points on the contour. To be precise though, this figure appears right after the definition of $\int_{z_0}^{\mathbf{z}} f(z)dz$ in a slightly more general context, when Casorati (1868, p. 360) defines an integral of the form $\int P dQ$, where both P and Q are functions of two real variables x and y, as a limit of sums of the form $P_0(Q_1 - Q_0) + P_1(Q_2 - Q_1) + \ldots + P_{n-1}(Q_n - Q_{n-1})$. The previous case of integral is obtained for $Q = x + iy$ (Casorati, 1868, p. 360).

71 "Since $\int_a^b f(z)dz$ always possesses the same value no matter along which path from a to b the passage may occur, whilst only everywhere *between* these paths the function remains continuous and finite, the value of the integral is only dependent and utterly determined upon the two limits; and it is also easy to demonstrate that this value

by an equation $\varphi(x,y) = 0$, which he specifies to $y = \chi(x)$ and $x = \psi(y)$ for functions χ and ψ, and gets

$$w = \int_p^r f\left(x + i\chi(x)\right) \mathrm{d}x + i \int_q^s f\left(\psi(y) + iy\right) \mathrm{d}y.$$

Then, he differentiated w with respect to r and s and obtained a version of the Cauchy-Riemann differential equations $\frac{\partial w}{\partial s} = i\frac{\partial w}{\partial r}$ (Equation A.7) (Neuenschwander, 1996, p. 36). Accordingly, w is a primitive function for f. Again, just as above, when Cauchy derived a version of this differential equation, Riemann obtains it here in the context of integration for a primitive function. But according to Nahin (2010, p. 193), Riemann was the first to obtain the Cauchy-Riemann differential equations similar to today's textbook version such as in Equation A.5.

7.5 DEVELOPMENT OF THE FOUNDATIONS OF COMPLEX ANALYSIS IN THE 20TH CENTURY AND THE ROLE OF THE COMPLEX PATH INTEGRALS

Early proofs of Cauchy's integral theorem (Theorem A.17) made use of the property that the derivative of a holomorphic function is continuous. Thus, the continuity of the derivative was required as an additional constraint in the prerequisites of the theorem or, occasionally, even included in definitions of holomorphic functions. However, at the turn of the 20th century, it was proven that the derivative of a holomorphic function is automatically continuous, so that it was no longer necessary as an additional constraint. Also, mathematicians discussed the constraints that must be imposed on a complex for it to be *analytic* (see e.g., Heffter, 1960). Surprisingly, until the middle of the 20th, all proofs of this proposition about a differentiability property of complex functions relied on integration:

> These facts have been established by means of Cauchy's integral formula and a natural question has arisen, whether there exists another method of proof, not employing that tool, a formula which is not directly related to the concept of differentiability and whose validity for differentiable functions can be viewed as a fortunate incidence (Shisha, 1989b, p. 117).

Indeed, integration could finally be omitted from the proofs of the analyticity of holomorphic functions. Hence, both these efforts (eliminating continuity of the derivative of a holomorphic function in proofs of Cauchy's integral theorem and eliminating integrals from proofs of the analyticity of holomorphic functions) shed light on the use of complex path integrals in complex analysis. We devote the rest of this chapter to discuss these developments.

7.5.1 *Minimising the prerequisites for Cauchy's integral theorem and for the analyticity of holomorphic functions*

Towards Goursat's lemma: Eliminating the continuity of the derivative of holomorphic functions in proofs of Cauchy's integral theorem

At the turn of the 20th century, surprising results appeared that related complex integration to complex differentiability. Joint efforts by Goursat (1858–1936), Moore (1862–1932), and Pringsheim (1850–1941) showed in particular that the *continuity of the derivative* of a holomorphic function follows automatically and hence does not need to be assumed as an additional requirement on the derivatives of the holomorphic functions in Cauchy's integral theorem (e.g.,

appears as a *function* of one of the limits if one requires the other to be fixed" (Neuenschwander, 1996, p. 36, own transl., emph. orig.).

Goursat, 1884, 1900; Moore, 1900; Pringsheim, 1901, 1903; see Gray, 2000, for an overview). Moreover, according to Morera's theorem (Theorem A.24), a continuous complex function is holomorphic if and only if the complex path integral of it along the boundary of every triangle, which is completely contained in the domain of the function, vanishes. This theorem can therefore, in principle, be used to define holomorphic functions without using the difference quotient but using complex path integrals instead (cf. Macintyre & Wilbur, 1967).

One way to prove Cauchy's integral theorem without assuming that derivatives of holomorphic functions are continuous is to prove and use a variant of Goursat's lemma (Theorem A.19), which asserts that any complex path integral of a holomorphic function along the boundary of a triangle Δ vanishes if the interior of the triangle is completely contained in the domain of the holomorphic function (see Section A.6). In Goursat's lemma, it is only required that the function is holomorphic but not additionally that its derivative is continuous. Hence, continuity of the derivative of a holomorphic function does not have to be assumed. In particular, since holomorphic functions are analytic (Theorem A.25), the continuity of the derivative of a holomorphic function follows as well.

In a proof of Goursat's lemma, we only need a few rather elementary properties of complex path integrals such as $\int_{\partial\Delta} \mathrm{d}z = 0$ and $\int_{\partial\Delta} z\,\mathrm{d}z = 0$, $\mathbb{C}$-linearity, and a careful estimation using nested subtriangles and Darboux's inequality Equation A.16 (e.g., Freitag & Busam, 2006, ch. II.§2). Modern proofs mimic the original methods by Goursat (1884, 1900), Moore (1900), and Pringsheim (1901, 1903) (see also Bottazzini & Gray, 2013, pp. 638–649; Gray, 2000)).[72] Proving a variant of Cauchy's integral theorem with one of Goursat's lemma can be considered to be a “standard” in recent textbook literature (Bottazzini & Gray, 2013, p. 692). We will also mimic this proof for our axiomatic characterisation of complex path integrals of holomorphic functions (Theorem 8.13).

Therefore, we believe it is reasonable to exclude the continuity of the derivative from the definition of holomorphic functions. For example, Garcia and Ross (2017) and Remmert and Schumacher (2002) value the approach to use Goursat's lemma since it leads to a proof of Cauchy's integral theorem without Green's theorem.[73]

Many articles in mathematics journals over the last 100 years discussed different versions of Cauchy's integral theorem. Many of them were explicitly written to refine necessary technical constraints on the classes of integrands or paths, to place the theorem better into curricular circumstances, and aimed at the use in university or college classrooms (e.g., Ahmad, 1955; Arsove, 1955; Bak & Popvassilev, 2017; Beckenbach, 1943; Bôcher, 1896; Dixon, 1971; Flatto & Shisha, 1973; Garcia & Ross, 2017; González-Velasco, 1980; Hanche-Olsen, 2008; Lass, 1953; Lax, 2007; Loeb, 1991, 1993; Minami, 1942; Pollard, 1923; Redheffer, 1969; Výborný, 1979; Widder, 1946; Wyler, 1965). For example, there are also proofs of Cauchy's integral theorem with the help of integral theorems from real analysis such as Green's theorem (Theorem B.15) relating double and path integrals, which in turn require the continuous differentiability of the integrands (e.g., Lang, 1999, p. 468). Hence, these proofs could also be replaced with Goursat's lemma (Garcia & Ross, 2017). Nevertheless, there are also proofs of Cauchy's integral theorem,

72 Pathak's (2019, ch. 3.4.3) proof of Goursat's lemma for more general paths than triangular paths resembles the original versions in Moore (1900) and Pringsheim (1903) closely.

73 What may have counted or still counts as proof of versions of Cauchy's integral theorem is a delicate subject in itself because some errors in early proofs have been unnoticed for quite some time in the history of this theorem. We would like to refer the reader to Bottazzini and Gray (2013, pp. 638–649) and Gray (2000) for more information.

which require the continuity of the derivative but do not use double integrals (e.g., Bôcher, 1896; Mittag-Leffler, 1875, 1923).[74]

In many textbooks, the theorem that holomorphic functions are analytic (Theorem A.25) is proven roughly speaking as follows: In the first step, Cauchy's integral theorem is used to derive Cauchy's integral formula for holomorphic functions. Then in the second step, the integrand in Cauchy's integral formula is developed into a geometric series and integration and summation are swapped, which is possible because the complex path commutes with uniform limits of sequences of functions (e.g., Freitag & Busam, 2006; Lang, 1999; Remmert & Schumacher, 2002). In particular, the proofs relied on complex path integrals.[75]

Heffter (1960) considered the study of which complex functions are analytic to be the *grounds of complex analysis* (German: Begründung der Funktionentheorie). He presented six approaches to this theorem, which are based on the work of Cauchy, Weierstraß (1815–1897), Goursat, Looman (1907–1989), Menchoff, Morera (1856–1907), and himself. In particular, Heffter asked to find and minimise other properties of complex functions, which guarantee that it is analytic (see also Pringsheim, 1920, 1925).

Heffter's own approaches were based on vanishing properties of complex path integrals similar to those in Morera's theorem and he aimed to keep the class of paths, for which the complex path integral has to be defined, in order to prove the aforementioned theorem as minimal as possible. Heffter (1960, ch. B.V) defined complex path integrals for paths made of line segments parallel to the coordinate axes only, and introduced the "axes-parallel unique integrability" (German: achsenparallele eindeutige Integrierbarkeit) condition for a continuous complex function. This condition means that the complex path integral of the given function vanishes along every path along the boundary of a rectangle with sides parallel to the coordinate axes contained in the domain of the function. Heffter (1960, ch. B.V) demonstrated that a continuous complex function is analytic if and only if it is axes-parallel uniquely integrable. In particular, the function was not required to be holomorphic here.

In a further approach, suitable for polar coordinates of complex numbers, Heffter (1951) showed that it is also sufficient to require that the integrals

$$\int_{\partial K_{r_1,r_2,\varphi_1,\varphi_2}} \frac{f(z)}{z-\zeta}\,\mathrm{d}z$$

vanish along the positively oriented boundaries of "arc rectangles" (German: Kreisbogenrechtecke) $K_{r_1,r_2,\varphi_1,\varphi_2} := \left\{\zeta + re^{i\varphi} \in \mathbb{C} :\ r_1 \le r \le r_2,\ \varphi_1 \le \varphi \le \varphi_2\right\}$ ($\zeta \in \mathbb{C}, 0 < r_1 < r_2 < \infty, 0 \le \varphi_1 < \varphi_2 < 2\pi$), which are contained in the domain of f, in order to prove the analyticity of a continuous complex function f on Ω (see also Heffter, 1960, ch. B.IV).

In all these approaches, analyticity of holomorphic functions is proved via complex path integrals (e.g., Freitag & Busam, 2006; Lang, 1999; Remmert & Schumacher, 2002). However, the opposite direction was also subject to mathematics research in the 20th century: Researchers aimed to prove the analyticity of holomorphic function and other theorems without integrals at all. In this context, Beardon (1979) presents Cauchy's integral theorem quite late in his textbook (p. 160 of 234) after having proved a lot of propositions about holomorphic functions,

74 Mittag-Leffler argued that an earlier proof without double integrals is due to Malmsten or Briout and Bouquet, but neither could we nor could Bottazzini and Gray (2013, p. 645) find this paper. The papers by Mittag-Leffler (1875, 1923) also contain preliminary versions of what we nowadays call *star domains* (e.g., Freitag & Busam, 2006, p. 76).

75 We will recall the core elements of this proof in Section 8.4 when we illustrate the properties of complex path integrals as used in many courses and textbooks in complex analysis (Properties 8.12).

which in turn appear quite late in other textbooks. He argues that his late statement and proof of Cauchy's integral theorem

> can only emphasize the fact that integration plays no essential part in the topological consequences of analyticity [...] (Beardon, 1979, p. 160).[76]

We remarked before that the proof that holomorphic functions are analytic relied on the use of Cauchy's integral formula. As such, we may say that a differentiability property of complex functions was proven with integration. It seems natural to ask whether the analyticity of holomorphic functions can also be proven without integration.[77] Let us now briefly summarise these developments, in which complex path integrals were eliminated from the proofs of many important theorems in complex analysis.

7.5.2 *Topological and complex analysis*

Topological methods: Series expansions of holomorphic functions can be proven without integration.

The continuity of the derivative of a holomorphic function, its infinite differentiability, and the local expansion into power and Laurent series proven *without complex path integrals* at all with methods from topological analysis (Whyburn, 1964). This approach "deviates from the mainstream procedures of Classical Analysis" (Shisha, 1989b, p. 117) and is rarely found in textbooks on complex analysis but see González (1992, ch. 6.21–6.22) and Beardon (1979).

Plunkett (1959) proved the continuity of the derivative of a holomorphic function and Connell (1965) that the derivative of a holomorphic function is holomorphic itself without complex path integrals (see also Klazar, 2018). Connell (1961), Porcelli and Connell (1961), and Read (1961) proved the analyticity of holomorphic functions (i.e., its development into Taylor series) and other theorems like Liouville's theorem (i.e., bounded entire functions are constant). Laurent series developments for holomorphic functions with singularities were proven without integration, too (e.g., Beesack, 1972; Leland, 1971a). Pringsheim (1925, § 52) proved Taylor and Laurent series developments, too, but with another method using certain mean values associated to a complex function (see footnote 77).[78]

The topological proofs we mentioned in this section basically rely on the property of holomorphic functions on connected domains to be constant or *open* and *light* (Eggleston & Ursell,

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76 We do not recall at this point, which topological consequences Beardon (1979) refers to. However, the quote fits to "analyticity", too.

77 For example, also Weierstraß did not approve that Cauchy's proof of the power series development of holomorphic functions "was not based on 'on the first elements of analysis' but on the concept of integral. Hence Weierstrass's dislike, for he wanted 'function theory [complex analysis; EH.] to be founded by means [of] elementary theorems on basic operations'" (Bottazzini & Gray, 2013, p. 470). On the contrary, Weierstraß's approach to complex analysis was based on the definition of a holomorphic function "as one that has a local power series representation at each point" (Burckel, 2021, p. 309). Pringsheim favours Weierstraß's approach to complex analysis, too (e.g., Pringsheim, 1896, 1920, 1925). For instance, Pringsheim's (1925) textbook does not contain any complex path integral (except in side remarks). Instead, the author uses a "mean value method" (Pringsheim, 1896, 1920, 1925): "These mean values, by the way, are mentally much simpler arithmetic constructions than the complex integrals and stand in a much more transparent, I would like to say, more speaking relationship to the generating function [the function taken the mean value of; EH.] than the latter" (1925, p. VI, own transl.). These definition of these mean values is a bit technical and will not be given here. It reminds us of Riemann sums though. In this sense, Remmert and Schumacher (2002, p. 315, own transl.) describe that Pringsheim's "mean values 'can always be seen as special cases of certain integrals', [thus,] his proof, which was only outwardly free of integrals, did not catch on".

78 Shisha (1989a, 1989b) also proves the Taylor and Laurent series expansion theorems without Cauchy's integral theorem and integral formula but with a method using a "generalized Riemann integral" (the Denjoy integral; see Section 6.3.2; Shisha, 1989b, p. 119). See also Leland (1964, 1965, 1966, 1971a, 1971b) for more on the development of complex analysis without making use of complex path integrals and Burckel (2021, pp. 382–384) for more references.

1952). Recall that a function is *open* if the image of every open set is open and *light* if the preimage of every point is totally disconnected (Eggleston & Ursell, 1952). Burckel (2021) describes this procedure as the "Stoïlow-Whyburn program", which is

> much broader than merely eliminating integration. It aims to derive the whole local theory of the non-constant holomorphic function f topologically from just two properties: *lightness* (f is not constant on any non-singleton continuum) and *openness*; these in turn are derived using no other analytic hypothesis than the $\mathbb{C}$-differentiability of f (Burckel, 2021, p. 384, emph. orig.).

In particular, since it can be shown that there are local power series developments for holomorphic function, the local existence of primitive functions for holomorphic functions (and global existence if the domain of the function is simply-connected) can also be proved without complex integration. Essentially, this allows us to define the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ locally as the difference $F(\gamma(b)) - F(\gamma(a))$ for a local primitive F of f such that the trace of γ is contained in the domain of F (and globally if the domain of f is simply-connected).

Finally, we would like to point out that power series expansion of holomorphic functions can also be shown *without differentiation* but with complex path integration (Macintyre & Wilbur, 1967). In this sense, both, the notion of complex differentiable function and the notion of complex path integral could in principle be used without the other to prove important properties of holomorphic functions. This shows that both, differentiation and complex path integration, are very closely related (see also the list of equivalent properties of holomorphic functions in the Fundamental theorem of complex function theory (Theorem A.26)).

8 EPISTEMOLOGICAL ANALYSIS OF COMPLEX PATH INTEGRALS AND COMPLEX PATH INTEGRATION

Formal definitions of derivatives and line integrals were abductively transferred from the real case to explore their viability in the complex; they yielded surprising new and powerful results, which compelled new meaning derived from this formal potency.—Oehrtman et al. (2019, p. 419)

This chapter provides the main body of the epistemological analysis. We focus on a plethora of ways to introduce the mathematical object "complex path integral". We highlight interpretations, which were explicitly designed to assist the teaching and learning of complex path integrals, and provide lots of connections to other mathematical objects.

Section 8.1 contains different ways to define complex path integrals based on their relationship to integrals in real and vector analysis, together with substantiations of these approaches as presented in textbooks on complex analysis. Section 8.2 is devoted to a geometrical-physical interpretation of the complex path integral by making use of the *Pólya vector field* associated to a complex function (Braden, 1987; Needham, 1997; Polya & Latta, 1974), which we have already multiply encountered in Chapter 5, and the interpretation of the complex path integral as an *average* (Gluchoff, 1991). We cover relationships between complex integration, complex differentiation, and the existence of primitives in Section 8.3. We also discuss the relationship between primitives for holomorphic functions and path integrals from the point of view of vector analysis. Furthermore, we propose a view on the complex path integral as a certain mapping with two arguments (integrands and paths) in Section 8.4. With this view, we are able to describe the use of complex path integrals in the context of an introductory lecture on complex analysis for the proofs of the main theorems like Goursat's lemma (Theorem A.19), Cauchy's integral theorem (Theorem A.17), or Cauchy's integral formula (Theorem A.22). However, this view is not intended to justify a purely formalistic approach to the teaching of integration in complex analysis. Rather, it serves to point out important aspects to be mastered when learning and to be promoted in teaching accordingly. At this point, as a proof of principle, an axiomatic characterisation of complex path integrals is also presented, which condenses properties of the complex path integrals into defining axioms (Theorem 8.13). Results of the epistemological analysis will be summarised in terms of aspects and partial aspects of complex path integrals in Chapter 9.

The reader is invited to use Appendix A and Appendix B for a reminder of the main definitions and propositions in complex and real analysis. These appendices cover everything we need in our epistemological analysis. Furthermore, they contain many references to the literature and some metatext with pointers to less standardised contents of real and complex analysis, which might be of particular interest to lecturers.

8.1 DEFINITIONS OF THE COMPLEX PATH INTEGRALS AND THEIR SUBSTANTIATIONS

This section deals with different definitions of complex path integrals and their substantiations in the literature. This means that we present different definitions and how these definitions are legitimated by their authors.[79] A substantiation of a mathematical notion is understood as an argument for why it is a useful notion and a substantiation of a definition is understood as an explicit argumentation why the definition is chosen. Sometimes, the definitions of the complex path integral are not distinguishable from their substantiations because construction routines may "act as their own substantiations" (Viirman, 2014b, p. 521). As such, a distinction between a certain definition and its substantiation are often not made or not made clearly in a textbook.

Section 8.1.1 covers the extension of the product sum aspect for Riemann integrals to the complex setting. Additionally, we will apply this definition to substantiate some properties of complex path integrals. Then, Section 8.1.2 and Section 8.1.3 provide a more direct approach for (piecewise) continuously differentiable paths. In Section 8.1.4, connections to real path integrals are established with a focus on integral theorems and the existence of primitive functions. In Section 8.1.6, we reflect on substantiations in the literature on the choice of paths for integration.

79 We refer to different textbooks on complex analysis throughout. This is not supposed to indicate that the particular content appears exclusively in the cited text. Rather, we refer to a particular text when the author(s) put(s) special emphasis on the particular content, even though it may well appear similarly in other texts. In case the cited source is the only one of which we are aware that addresses a specific topic, we highlight it as such.

For the integration of a complex function from one complex number to another, we need to clarify how the first is linked to the second. That is, one has to choose a *path* whose initial and terminal point are the two given complex numbers. In general, the complex path integrals along different paths between the same initial and terminal point will differ. Let us recall that a *path* in a domain $\Omega \subseteq \mathbb{C}$ is a continuous function $\gamma \colon [a, b] \longrightarrow \Omega$ from a non-empty interval of real numbers $[a, b]$ to Ω. Its *trace* is the set $\mathrm{tr}(\gamma) = \gamma([a, b])$. A *curve* is a subset in the complex plane $\mathbb{C}$, which is the trace of a path γ; in this case, γ is called a *parametrisation* of the curve.

Some authors prefer to define complex path integrals along paths and some along curves. In the second case, a parameterisation of the curve has to be given additionally or is implied from the context. Essentially, the notion of complex path integral is sensitive with respect to a direction of the curve. This is reflected in the terminology as well: Some authors call complex path integrals also line integrals, curvilinear integrals etc. In our view, it is therefore preferable define complex path integrals along paths instead of curves because paths already are a parametrisation of their trace.[80]

Convention 8.1. From now on let $\gamma \colon [a, b] \longrightarrow \mathbb{C}$ be a (piecewise) continuously differentiable path and $f \colon \mathrm{tr}(\gamma) \longrightarrow \mathbb{C}$ a continuous function. Whenever we write $f = u + iv$, it is tacitly assumed that $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.[81]

Note that we require f to be continuous. On the one hand, this avoids technical difficulties on the existence of complex path integrals based on one or another definition. On the other hand, this condition is compliant with most parts of complex analysis literature. At some points, we will need additionally constraints on f though (e.g., that f is holomorphic). We will highlight these additional constraints throughout.

8.1.1 *Adapting Riemann sums*

In Section 7.4, we have seen that Cauchy (1825) defined the complex path integral

$$\int_{x_0+iy_0}^{X+iY} f(z)\,\mathrm{d}z$$

between complex numbers $x_0 + iy_0$ and $X + iY$ as the limits of certain sums. In order to link the two points $x_0 + iy_0$ and $X + iY$ in the complex plane, he used monotonous paths linking x_0 to X and y_0 to Y, and the addends in the sums he considered were made of points lying on these linking paths. In modern definitions of complex path integrals using similar sums, Cauchy's assumption that the real and imaginary parts of $x_0 + iy_0$ should vary monotonously to $X + iY$ is dropped. Instead, complex path integrals are defined for continuous complex functions defined on the traces of (piecewise) continuously differentiable paths (or occasionally even more general paths) linking two points in the complex plane.

80 Burckel (2021, p. 60) has a similar opinion: "The difficult task of defining the integral of f over a curve thought of merely as the *set* $\gamma([a, b])$, or of analyzing the invariance of our definition under different parametrizations of the set $\gamma([a, b])$, is not worth pursuing."

81 Recall that for every piecewise continuously differentiable path, there is a continuously differentiable reparametrisation of the original path (Section A.5). Hence, it is in principle sufficient to restrict ourselves to continuously differentiable paths, but it is more convenient to allow piecewise continuously differentiable paths, which also seems to be the prevalent point of view in complex analysis literature.

Complex Riemann sums

Product sum aspect of complex path integrals

Consider a partition $P\colon a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ of length n and set $\ell(P) := \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}$ to be the *norm* of P. Choose a tag vector for P, that is, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ for $\xi_k \in [t_{k-1}, t_k]$, and set $(\Delta\gamma)_k = \gamma(t_k) - \gamma(t_{k-1})$ $(k = 1, 2, \ldots, n)$. We call the sum

$$R\left(f, \gamma, P, \xi\right) = \sum_{k=1}^{n} f\left(\gamma(\xi_k)\right) (\Delta\gamma)_k \tag{8.1}$$

a *complex Riemann sum for f, γ, P, and ξ*. If the limit

$$\lim_{n\to\infty} R\left(f, \gamma, P^{(n)}, \xi^{(n)}\right) \tag{8.2}$$

exists and equals the same $c \in \mathbb{C}$ for every sequence of partitions

$$\left(P^{(n)} : a = t_0^{(n)} < t_1^{(n)} < \ldots < t_{\nu_n}^{(n)} = b\right)_{n\in\mathbb{N}}$$

of $[a, b]$ of length $\nu_n \in \mathbb{N}$ such that $\ell(P^{(n)}) \overset{n\to\infty}{\longrightarrow} 0$ and every sequence of tag vectors

$$\left(\xi^{(n)} = (\xi_1^{(n)}, \ldots, \xi_{\nu_n}^{(n)})\right)_{n\in\mathbb{N}}$$

for these partitions, then the *complex path integral of f along γ* is said to exist and to equal c,

$$\int_\gamma f := \int_\gamma f(z)\,\mathrm{d}z := c.$$

This definition of the complex path integral clearly mimics the definition for Riemann integrals (Definition B.2) and is a full aspect of complex path integrals, which we also call the *product sum aspect* of complex path integrals. It is also clear from the definition that if the path goes from one real number a to another b and the integrand is a real-valued function of one real variable, the usual Riemann integral is obtained. Additionally, note that this definition also works if γ is rectifiable instead of (piecewise) continuously differentiable, because it can be shown that the limit above exists in this case, too (e.g., González, 1992, ch. 7.6).

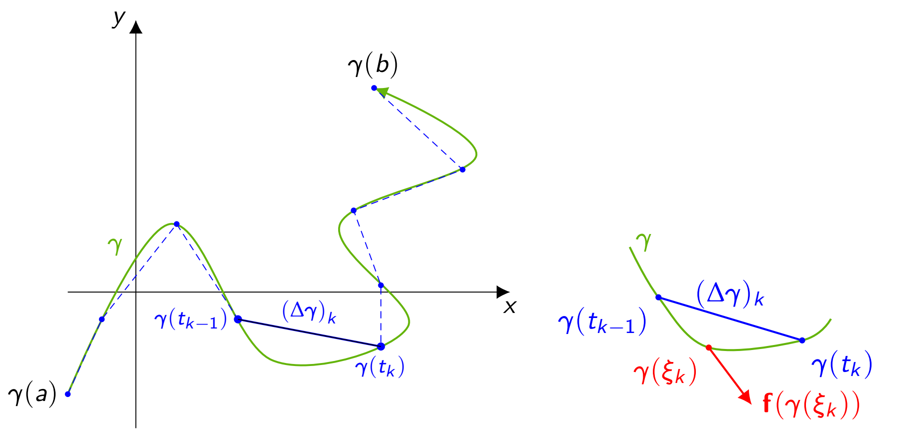
Riemann sums approximate the complex path integral since, by definition of the limit, for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ and a tag vector ξ for P, such that

$$\left|\int_\gamma f(z)\,\mathrm{d}z - R(f, \gamma, P, \xi)\right| < \varepsilon. \tag{8.3}$$

In that sense, complex Riemann sums can be used to approximate complex path integrals, and the approximation can be made arbitrarily good.

Moreover, it can be shown that if f is a continuous function on a domain Ω, γ is rectifiable path in Ω, and $\varepsilon > 0$, then, there is a polygonal path Γ in Ω (i.e., a piecewise continuously differentiable path such that the traces of all its continuously differentiable parts are line segments in Ω) such that

$$\left|\int_\gamma f(z)\,\mathrm{d}z - \int_\Gamma f(z)\,\mathrm{d}z\right| < \varepsilon \tag{8.4}$$



(a) Subdivision of a path.

(b) Quantities for the addend $f(\gamma(\xi_k)) \cdot (\Delta\gamma)_k$.

Figure 8.1: Formation of Riemann sums for the complex path integral.

(Conway, 1973/1978, p. 65). In that sense, complex path integrals along arbitrary (piecewise) continuously differentiable paths can be approximated arbitrarily good by complex path integrals along polygonal paths.

A common visual mediator for the partition $\gamma(P)\colon \{\gamma(a), \gamma(t_1), \ldots, \gamma(b)\}$ of $\operatorname{tr}(\gamma)$ (i.e., the points on $\operatorname{tr}(\gamma)$ given by the values of γ at the elements of the partition of $[a, b]$) is shown in Figure 8.1a. One may regard the polygonal path generated by the points in $\gamma(P)$ in order as a polygonal approximation of γ. Figure 8.1b shows an amplified segment of the trace of γ and one of the complex numbers $f(\gamma(\xi_k))$ as the corresponding vector $\mathbf{f}(\gamma(\xi_k))$ (here, $\mathbf{f} = (u, v)^T$ denotes the vector field associated to $f = u + iv$). The product of complex numbers $f(\gamma(\xi_k)) \cdot (\Delta\gamma)_k$ is the corresponding addend in the complex Riemann sum.

Next to substantiating the Riemann sum definition of complex path integrals of f along γ with the analogy to Riemann integrals for real-valued functions on intervals of real numbers, it may also be substantiated with an analogy to real path integrals of second kind. That is, if the integrand were a vector field, say the vector field $\mathbf{f}$, then the real path integral of second kind $\int_\gamma \mathbf{f}\, \mathrm{d}\mathbf{T}$ of $\mathbf{f}$ along γ (now interpreted as a path in $\mathbb{R}^2 \cong \mathbb{C}$) was the limit of sums of the form

$$\sum_{k=1}^{n} \langle \mathbf{f}(\gamma(\xi_k)), (\Delta\gamma)_k \rangle, \tag{8.5}$$

where $(\Delta\gamma)_k$ is also interpreted as a vector in $\mathbb{R}^2$ (see Section B.2.2). Accordingly, the complex Riemann sums in Equation 8.1 correspond to the sums in Equation 8.5 but with a change in products: In Equation 8.1, the product comes from the multiplication of complex numbers, in Equation 8.5 it comes from the scalar product of vectors in $\mathbb{R}^2$.

Another substantiation of the Riemann sum definition of complex path integrals is more closely related to the existence of the limit in Equation 8.2. Splitting up the integrand and the path of a complex path integral into their real and imaginary parts, the real and imaginary part of complex Riemann sums may be identified as Riemann sums for certain real path integrals of second kind as well. For instance, if we write $f = u + iv$, $\gamma = \gamma_1 + i\gamma_2$,

$(\Delta\gamma_1)_k = \mathrm{Re}\left((\Delta\gamma)_k\right)$ and $(\Delta\gamma_2)_k = \mathrm{Im}\left((\Delta\gamma)_k\right)$ ($k = 1, 2, \ldots, n$), the complex Riemann sums decompose to

$$\sum_{k=1}^{n} f\left(\gamma(\xi_k)\right)(\Delta\gamma)_k = \sum_{k=1}^{n} \left(u(\gamma(\xi_k))(\Delta\gamma_1)_k - v(\gamma(\xi_k))(\Delta\gamma_2)_k\right) + i\sum_{k=1}^{n} \left(v(\gamma(\xi_k))(\Delta\gamma_1)_k + u(\gamma(\xi_k))(\Delta\gamma_2)_k\right)$$

(e.g., Ponnusamy & Silverman, 2006, ch. 7). These two sums at the right hand side are Riemann sums for the real path integrals of second kind of the vector fields $\mathbf{w}_f = (u, -v)^T$ (i.e., the Pólya vector field associated to f; Braden, 1987; Polya and Latta, 1974) and $(v, u)^T$ (see also Section 8.1.4 and Section B.2.2). Now, the complex path integral exists if and only if $\int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y$ and $\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y$ exist (e.g., when u and v are continuous), and then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y. \tag{8.6}$$

Yet another argument for the convergence of the complex Riemann sums is to interpret $R(f, \gamma, P, \xi)$ as Riemann-Stieltjes sums and use their convergence (e.g., Stewart & Tall, 2018, ch. 6.2; Heins, 1968, ch. V.1; Holland, 1980, ch. 5.1; see also Apostol, 1981). Using this approach, complex path integrals are occasionally also signified as Riemann-Stieltjes integrals such as

$$\int_\gamma f = \int_a^b f\left(\gamma(t)\right)\mathrm{d}\gamma(t)$$

(Apostol, 1981, p. 436) or

$$\int_\gamma f = \int_a^b (f \circ \gamma)\,\mathrm{d}\gamma$$

(Heins, 1968, p. 102).

In all approaches presented in this section, the definition of the complex path integral is substantiated in one way or another in terms of an analogy to Riemann or Riemann-Stieltjes integrals. Therefore, we conclude that the definition of complex path integrals in terms of limits of product sums amounts to an *aspect* of complex path integrals. This aspect grounds in discourses on integrals in real analysis, is consistent (even though more a bit more general) with Cauchy's (1825) definition (Section 7.4) and with the product sum aspect for Riemann integrals (Section 6.3.2).

In line with these observations, Jeffrey (1992, p. 295) claims that the definition of the complex path integral via limits of sums is "[t]he fundamental idea underlying the integration of a complex function $f(z)$" and that Equation 8.6 is the "meaning" of the complex contour integral (his name for the complex path integral). Similarly, Needham (1997) motivates the definition of complex path integrals with product sums as the result of a formal transfer of the definition of Riemann integrals:

> In the case of real integration we began with a clear geometric objective ("Find the area!") and then invented the integral as a means to this end. In the complex case we will reverse this process, that is, we will first blindly attempt to generalize real integrals (via Riemann Sums) and only afterwards will we ask ourselves what we have created. (Needham, 1997, p. 383)

While Needham (1997) additionally reminds his readers of the area interpretation of Riemann integrals, the aforementioned introduction to the definition of complex path integrals does not go beyond the formal analogy to Riemann integrals at first. However, he announces that he will take his readers on a tour to find out "what we have created" afterwards (Needham, 1997, p. 383). In particular, he develops a geometric interpretation of complex Riemann sums, which we will review in Section 8.2.1.

8.1.2 *Substitution principle*

Substitution aspect of complex path integrals

Another definition of the complex path integral can be motivated with integration by substitution. Roughly stated, the change of variable $x = \psi(t)$ with a continuously differentiable function ψ in a real integral $\int \varphi(x)\,\mathrm{d}x$ of a continuous function $\varphi\colon [\psi(a), \psi(b)] \longrightarrow \mathbb{R}$ $(\psi(a) < \psi(b))$ yields

$$\int_{\psi(a)}^{\psi(b)} \varphi(x)\,\mathrm{d}x = \int_a^b \varphi(\psi(t))\psi'(t)\,\mathrm{d}t \tag{8.7}$$

(Forster, 2016, p. 223).

Now, if we want to integrate the complex function f on the trace of a continuously differentiable path γ, we need to consider the function values $f(z)$, where z runs through $\mathrm{tr}(\gamma)$. For each $z \in \mathrm{tr}(\gamma)$ there is at least one $t \in [a, b]$ such $z = \gamma(t)$. Now, a heuristic application of the substitution $z = \gamma(t)$ in the expression $\int_\gamma f(z)\,\mathrm{d}z$ suggests that

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \tag{8.8}$$

is a reasonable definition.[82] Changing $\frac{\mathrm{d}z}{\mathrm{d}t} = \gamma'(t)$ symbolically to $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$ makes the formula in Equation 8.8 even more suggestive (Lvovski, 2020, p. 37; Ponnusamy & Silverman, 2006, p. 203). For example, Beardon (1979, p. 139) emphasises that

> [t]his definition is, of course, motivated by the familiar rule for the change of variable $(z = \gamma(t))$ in an integral. We adopt it here in preference to other definitions because it is readily available, easy to use and entirely adequate for our purposes.

In particular, the definition in Equation 8.8 gives rises to an aspect of complex path integrals that we would like to call *substitution aspect*.

According to Beck et al. (2018, p. 54), the definition of the complex path integral via Equation 8.8 "should come as no surprise". If γ is only piecewise continuously differentiable, the formula in Equation 8.8 has to be applied to each continuously differentiable part of γ and added up (technically, one also needs to argue that the definition does not depend on the partition of γ into continuously differentiable parts).

82 We have seen in Section 7.1 the historical routine to perform substitutions involving complex functions in order to evaluate real integrals. This method itself was not strictly endorsed because a solid definition of complex path integrals was absent at first. However, it was partly endorsed since its application led to results that could be checked without it afterwards. Here, we use a formal substitution to motivate the definition in Equation 8.8. In fact, since complex path integrals and real integrals coincide if the path of integration parametrises a real interval and the function is real-valued, we are at least able to endorse some of the substitutions our mathematical ancestors made.

However, we still need to explain what the right side in Equation 8.8 signifies. For this purpose, we may define the integral of a continuous complex function $g\colon [a, b] \longrightarrow \mathbb{C}$ defined on an interval of real numbers termwise for the real and imaginary part, that is,

$$\int_a^b g(t)\,\mathrm{d}t := \int_a^b \mathrm{Re}(g)(t)\,\mathrm{d}t + i\int_a^b \mathrm{Im}(g)(t)\,\mathrm{d}t. \tag{8.9}$$

Since g is continuous, $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$ are continuous, too, and both integrals at the right side exist. Then, the right side of Equation 8.8 is defined with Equation 8.9 for $g = (f\circ\gamma)\cdot\gamma'$, which is a (piecewise) continuous function on $[a, b]$.

Another application of the substitution rule shows that the complex path integral is invariant under a reparametrisation of the path of integration in the following sense. Let $\psi\colon [c, d] \longrightarrow [a, b]$ be a continuously differentiable function such that $a = \psi(c)$ and $b = \psi(d)$. Then, the chain rule for $\sigma = \gamma\circ\psi$ implies

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t\\ &= \int_c^d f(\gamma(\psi(\tau)))\,(\gamma\circ\psi)'(\tau)\,\mathrm{d}\tau = \int_\sigma f(z)\,\mathrm{d}z.\end{aligned} \tag{8.10}$$

Accordingly, one might say that γ and σ both parametrise the same curve and $\int_\gamma f(z)\,\mathrm{d}z$ and $\int_\sigma f(z)\,\mathrm{d}z$ agree (cf. Section A.5). In particular, this way we have another retrospective heuristic substantiation for the appearance of the derivative of the path in the defining equation of the complex path integral. It goes hand in hand with the chain rule for composite functions when reparametrising the original path.

Greene and Krantz (2006, ch. 2.1) follow yet a different way to motivate definition Equation 8.8. First, they prove that $\phi(b) - \phi(a) = \int_a^b \phi'(t)\,\mathrm{d}t$ for a piecewise continuously differentiable function $\phi\colon [a, b] \longrightarrow \mathbb{R}$. Having then defined Equation 8.9, they generalise the previous equation to $\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\,\mathrm{d}t$ for a piecewise continuously differentiable path $\gamma\colon [a, b] \longrightarrow U \subseteq \mathbb{C}$. Replacing γ with $f\circ\gamma$ for a continuously partially differentiable real-valued function $f\colon U \longrightarrow \mathbb{R}$, the authors establish that

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t))\cdot\gamma_1'(t) + \frac{\partial f}{\partial y}(\gamma(t))\cdot\gamma_2'(t)\right)\mathrm{d}t, \tag{8.11}$$

where $\gamma_1 = \mathrm{Re}(\gamma)$ and $\gamma_2 = \mathrm{Im}(\gamma)$ and remark that Equation 8.11 remains valid if f is complex-valued and continuous, since one can apply the formula for real and imaginary part separately and use Equation 8.9.[83] Then, for "motivational purposes", they "extend[] [Equa-

83 In the original, Greene and Krantz (2006) write $\frac{\mathrm{d}\gamma_k}{\mathrm{d}t}$ instead of $\gamma_k'(t)$ for $k = 1, 2$. We changed this notation here because we prefer not to omit the argument t of these derivatives at this point.

tion 8.11; EH.] to complex-valued functions, when $f = u + iv$ is holomorphic" (Greene & Krantz, 2006, p. 31). In this case, the integral on the right-hand side in Equation 8.11 becomes

$$\int_a^b \left(\frac{\partial u}{\partial x} (\gamma(t)) \cdot \gamma_1'(t) + i \frac{\partial v}{\partial x} (\gamma(t)) \cdot \gamma_1'(t) + \right. \\ \left. \frac{\partial u}{\partial y} (\gamma(t)) \cdot \gamma_2'(t) + i \frac{\partial v}{\partial x} (\gamma(t)) \cdot \gamma_2'(t) \right) \mathrm{d}t \\ = \int_a^b \left(\left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] |_{\gamma(t)} \right) \cdot (\gamma_1'(t) + i\gamma_2'(t)) \, \mathrm{d}t.$$

In the last step, the Cauchy-Riemann differential equations Equation A.6 were used. Thus, Greene and Krantz (2006, p. 32) obtain

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial z} (\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t, \tag{8.12}$$

since $\frac{\partial u}{\partial x}(z) + i \frac{\partial v}{\partial x}(z) = \frac{\partial f}{\partial x}(z) = f'(z)$ for $z \in U$. Finally, they *define*

$$\oint_\gamma F(z) \, \mathrm{d}z = \int_a^b F(\gamma(t)) \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}t} \, \mathrm{d}t$$

for a continuous complex function F (Greene & Krantz, 2006, p. 32). Right within the definition, probably to differentiate the integral here from real path integrals, the authors emphasise that the "multiplicative $\cdot$ here is to be interpreted as multiplication of complex numbers."[84]

Clearly, if the approach based on limits of complex Riemann sums is used to define complex path integrals, Equation 8.8 can be proven afterwards (see e.g., González, 1992, ch. 7.6). In this sense, the Riemann sum approach is more general that the approach in this and the next subsection because it works for a larger class of paths.

8.1.3 *The parametric definition*

The parametric formula for complex path integrals

Whereas in the previous subsection, the definition of complex path integrals was substantiated with the substitution rule, it is oftentimes directly given in terms of

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t)) \, \gamma'(t) \, \mathrm{d}t \tag{8.13}$$

without an explanation why one should define it like this (e.g., Ahlfors, 1979; Bornemann, 2016; Fischer & Lieb, 2003, 2010; Freitag & Busam, 2006; Fritzsche, 2019; Müller, 2018). Hence, when we refer to the parametric definition, we intend to emphasise that no substantiation based on the substitution rule is present. Stewart and Tall (2018, pp. 111–112, 122) call the definition in Equation 8.13 the "short cut" when compared with the definition in Section 8.1.1. If the path is piecewise continuously differentiable, then the right side may be interpreted again as the sum of the integrals along the continuously differentiable paths or as a Riemann integral for

84 Note that, Greene and Krantz (2006) use the symbol $\oint$ instead of $\int$ to denote the complex path integral for an arbitrary (piecewise) continuously differentiable path. Usually, the sign $\oint$ is reserved for complex path integrals along closed paths only though.

complex-valued functions, where the finite number of points where γ' does not exist does not affect the integral.

Using a (complex) Riemann integral to define the complex path integral as in Equation 8.13 has the main advantage that questions about the existence of a complex path integral are answered with the existence of (complex) Riemann integrals. For instance, this is how Ahlfors (1979) substantiates the parametric definition of complex path integrals:

> They can be defined as a limit process with mimics the definition of a real definite integral. Actually we shall prefer to define complex definite integrals [complex path integrals, EH.] in terms of real integrals. This will save us from repeating existence proofs which are essentially the same as in the real case. Naturally, the reader must be thoroughly familiar with the theory of definite integrals of real continuous functions. (Ahlfors, 1979, p. 101)

Again, it is easy to see that the complex path integral restricts to the usual Riemann integral if γ is a parametrisation of $[a, b]$ (i.e., $\gamma\colon [a, b] \longrightarrow [a, b], t \longmapsto t$) and f is a real-valued function on this interval, because the right side in Equation 8.13 evaluates to $\int_a^b f(t)\,\mathrm{d}t$.

Example 8.2. Let us consider the function

$$\begin{aligned} f\colon \mathbb{C} &\longrightarrow \mathbb{C}, \\ z &\longmapsto (x+2y)+iy^2, \end{aligned}$$

where $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$. Let $\gamma\colon [0, 2\pi] \longrightarrow \mathbb{C}$, $t \longmapsto e^{it}$ be the standard parametrisation of the unit circle. Then, $f(\gamma(t)) = \cos(t) + 2\sin(t) + i\sin(t)^2$ and $\gamma'(t) = ie^t$ for $t \in [0, 2\pi]$; hence

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_0^{2\pi} \left(\cos(t) + 2\sin(t) + i\sin(t)^2\right) \cdot ie^{it}\,\mathrm{d}t \\ &= \int_0^{2\pi} -\cos(t)\sin(t) - 2\sin(t)^2 - \cos(t)\sin(t)^2\,\mathrm{d}t + \\ &\qquad i\int_0^{2\pi} \cos(t)^2 + 2\cos(t)\sin(t) - \sin(t)^3\,\mathrm{d}t \\ &= -2\pi + i\pi. \end{aligned}$$

Jänich (2001, p. 50) suggests transferring the area interpretation for Riemann integrals to complex path integrals and visualises the corresponding areas enclosed by the graphs of the real and imaginary part of $[a, b] \longrightarrow \mathbb{C}$, $t \longmapsto f(\gamma(t))\gamma'(t)$.

Let us apply this procedure to our example. The corresponding plots of the real part (i.e., $t \longmapsto -\cos(t)\sin(t) - 2\sin(t)^2 - \cos(t)\sin(t)^2$, light blue) and of the imaginary part (i.e., $t \longmapsto \cos(t)^2 + 2\cos(t)\sin(t) - \sin(t)^3$, light green) of $t \longmapsto f(\gamma(t))\gamma'(t)$ can be seen in Figure 8.2. ◇

In order to avoid a separate definition for the right-hand side in Equation 8.13 via Equation 8.9, some authors include the separation of the integral into two integrals for the real and imaginary part directly into the defining equation for the complex path integral. Thus, as a variant of Equation 8.13, one may find

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b \operatorname{Re}\left(f(\gamma(t))\gamma'(t)\right)\mathrm{d}t + i\int_a^b \operatorname{Im}\left(f(\gamma(t))\gamma'(t)\right)\mathrm{d}t$$

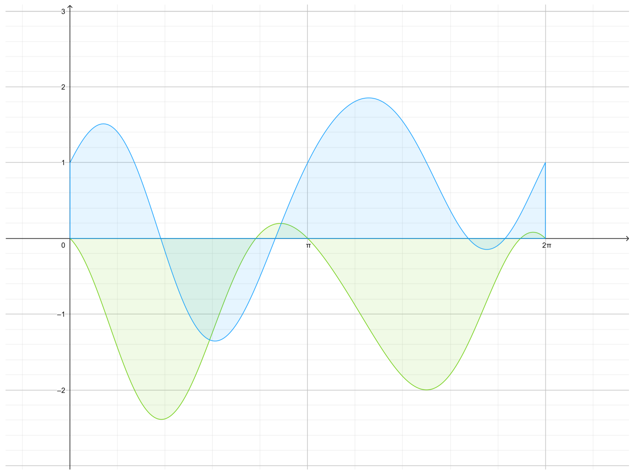


Figure 8.2: Plot of the real and imaginary part of $t \longmapsto f(\gamma(t))\gamma'(t)$ from Example 8.2.

(e.g., Klazar, 2019b) or

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &:= \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b f(\gamma_1(t)+i\gamma_2(t))\,(\gamma_1'(t)+i\gamma_2'(t))\,\mathrm{d}t \\ &= \int_a^b \operatorname{Re}\left(f(\gamma_1(t)+i\gamma_2(t))\,(\gamma_1'(t)+i\gamma_2'(t))\right)\mathrm{d}t \\ &\qquad + i\int_a^b \operatorname{Im}\left(f(\gamma_1(t)+i\gamma_2(t))\,(\gamma_1'(t)+i\gamma_2'(t))\right)\mathrm{d}t,\end{aligned}$$

(Isaev, 2017, p. 44) in the literature, too.

Even if this ad hoc definition with the parametric definition is chosen, we plead for at least some substantiations why the notion of complex path integral is meaningful or how it connects to integrals students may have encountered earlier in their curricula. For example, one may explain that the notion of complex path integrals is an extension of the Riemann integral to the complex setting. One can do so by computing the right side in Equation 8.13 for a path of the form $\gamma\colon [a,b] \longrightarrow [a,b] \subseteq \mathbb{C}, t \longmapsto t$, and a real-valued function $f\colon [a,b] \longrightarrow \mathbb{R}$ explicitly because then, Equation 8.13 immediately becomes $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(t)\,\mathrm{d}t$. Another potential substantiation is to argue that it is useful to have included the derivative of the path in Equation 8.13 because this way, the chain rule implies the invariance of complex path integrals from the parametrisation of the path as in Equation 8.10. Both these substantiations are clearly useful for other potential definitions of complex path integrals, too, but here, we emphasise them because in the preceding two subsections, the textbook authors chose other substantiations for their choices of definition.

8.1.4 *Connections to real path integrals*

In this section, we deal with the relationship between complex path integrals and integral from analysis in one and two real variables. That is, we connect it to path integrals of first and second kind and double integrals. Doing so, we obtain several formulas for complex path integrals, which could in principle be also used as a definition under suitable constraints.

Analogous structure in real and complex path integrals

Let us begin by comparing the complex path integral to real path integrals. For this purpose and depending on the context, we will use γ to signify both, a path in $\mathbb{C}$ or a path in $\mathbb{R}^2$. Let $g\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{R}$ denote a continuous real-valued function and $\mathbf{F} = (P, Q)^T\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{R}^2$ a continuous vector field on the trace of γ.

Analogous structure in real and complex path integrals

For the moment, let us denote multiplication in $\mathbb{R}$ by $\cdot_{\mathbb{R}}$, multiplication in $\mathbb{C}$ by $\cdot_{\mathbb{C}}$, the standard scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^2$ by $*_{\mathbb{R}^2}$, and the Euclidean norm in $\mathbb{R}^2$ by $\|\cdot\|$. Then, the *path integral of first kind* for g along γ is given by

$$\int_\gamma g\,\mathrm{d}s = \int_a^b g\left(\gamma(t)\right) \cdot_{\mathbb{R}} \left\|\gamma'(t)\right\| \mathrm{d}t, \tag{8.14}$$

the *path integral of second kind* for $\mathbf{F}$ is given by

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \int_a^b \mathbf{F}\left(\gamma(t)\right) *_{\mathbb{R}^2} \gamma'(t)\,\mathrm{d}t, \tag{8.15}$$

which we may also signify as

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

and, as we have seen previously, the complex path integral is given by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot_{\mathbb{C}} \gamma'(t)\,\mathrm{d}t.$$

Hence, all of the integrands at the right sides of these equations constitute of products of the integrand composed with γ and γ' (or $|\gamma'|$). With this regard, the only difference is the respective chosen product: In the real path integral of first kind, the product is the multiplication of real numbers; in the real path integral of second kind, it is the standard scalar product on $\mathbb{R}^2$, and in case of the complex path integral, it is the multiplication of complex numbers. Accordingly, we may substantiate the notion of complex path integral in terms of another analogy to real integrals: Its realisation in Equation 8.8 mimics the realisation of path integrals in Equation 8.14 and Equation 8.15 but with multiplication of complex numbers.

Similarly, if γ is only rectifiable, a similar product structure can be observed in the Riemann sums for these three integrals as well. Given a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ and tags $\xi_k \in [t_{k-1}, t_k]$ ($k = 1, \ldots, n$) the Riemann sums for the real path integral of first kind have the form

$$\sum_{k=1}^n g\left(\gamma(\xi_k)\right) \cdot_{\mathbb{R}} \|\gamma(t_k) - \gamma(t_{k-1})\|\,.$$

For the real path integral of second kind they have the form

$$\sum_{k=1}^n \mathbf{F}\left(\gamma(\xi_k)\right) *_{\mathbb{R}^2} \left(\gamma(t_k) - \gamma(t_{k-1})\right),$$

and, as we have seen above, for the complex path integral they have the form

$$\sum_{k=1}^{n} f\left(\gamma(\xi_k)\right) \cdot_{\mathbb{C}} \left(\gamma(t_k) - \gamma(t_{k-1})\right).$$

In all cases, we say that the respective path integral exists if the limits of these Riemann sums exist for $n \to \infty$ and $\max\{t_k - t_{k-t} : k = 1, \ldots, n\} \longrightarrow 0$.

In the real path integral of second kind, the letter $\mathbf{T}$ is used to emphasise that the *tangential* vector field γ' is involved. As a variant, let us introduce the real path integral of second kind with respect to the *normal* component of γ as

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N} = \int_a^b \langle \mathbf{F}(\gamma(t)), \mathbf{n}(t)\rangle\,\mathrm{d}t. \tag{8.16}$$

Here, $\mathbf{n}$ is the vector field obtained by turning γ' clockwise (that is to say that $\mathbf{n}(t)$ points into a direction orthogonal to $\mathrm{tr}(\gamma)$ at $\gamma(t)$ ($t \in [a, b]$). This can be expressed as $\mathbf{n}(t) = (\gamma_2'(t), -\gamma_1'(t))^T$, or in terms of complex numbers as $\mathbf{n}(t) = -i\gamma'(t)$ for $t \in [a, b]$ (the points t where $\gamma'(t)$ and hence $\mathbf{n}(t)$ do not exist, will not contribute to either of the integrals we consider here; we may of course define $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N}$ in terms of Riemann sums similar to $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$). Also note that

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N} = \int_\gamma P\,\mathrm{d}y - Q\,\mathrm{d}x = \int_\gamma \begin{pmatrix} -Q \\ P \end{pmatrix} \mathrm{d}\mathbf{T} = \int_\gamma \mathbf{J}\mathbf{F}\,\mathrm{d}\mathbf{T},$$

where $\mathbf{J}$ is the (2×2)-matrix that describes a rotation by $\pi/2$ counterclockwise.

Realising the complex path integral via real path integrals of second kind

Vector analysis aspect of complex path integrals

Let $\mathbf{w}_f = (u, -v)^T$ be the Pólya vector field associated to f (Braden, 1987; Polya & Latta, 1974). Now, the main observation is that the separation of $\int_\gamma f(z)\,\mathrm{d}z$ for $f = u + iv$ into real and imaginary part yields

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}y + u\,\mathrm{d}x = \int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N}. \tag{8.17}$$

Hence, these formula realise the real part of $\int_\gamma f(z)\,\mathrm{d}z$ as the real path integral of second kind of the Pólya vector field $\mathbf{w}_f$ with respect to the tangential field of γ and the imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$ to the real path integral of second kind of $\mathbf{w}_f$ with respect to the normal vector field of γ. Additionally, the imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$ can also be expressed as the real path integral of second kind of the transformed vector field $\mathbf{J}\mathbf{w}_f$ with respect to the tangential field of γ. Hence, we may use these formulas equally well to define complex path integrals. In other words, we have found an aspect of complex path integrals, the *vector analysis aspect*.

Note that here, both integrands on the right-hand side are *not* the vector field $\mathbf{f}$ associated to f as might have been expected. To state it clearly, in general $\int_\gamma f(z)\,\mathrm{d}z$ is neither equal to

$$\int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{T}, \quad \int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{N}, \quad \int_\gamma u\,\mathrm{d}s + i\int_\gamma v\,\mathrm{d}s,$$

nor to another apparently immediate guess. So, the Pólya vector field of a complex function is a suitable choice in order to realise the real and imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$ as real path integrals.

Since the Pólya vector field associated to $\overline{f}$ is $\mathbf{f}$, we may replace f in Equation 8.17 with $\overline{f}$ and obtain

$$\int_\gamma \overline{f}(z)\,\mathrm{d}z = \int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{N}. \tag{8.18}$$

In case, we realise both of the real path integrals at the right-hand side of Equation 8.17 as real path integral of second kind with respect to the tangential field, we can also write

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{J}\mathbf{w}_f\,\mathrm{d}\mathbf{T}$$

and

$$\int_\gamma \overline{f}(z)\,\mathrm{d}z = \int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{J}\mathbf{f}\,\mathrm{d}\mathbf{T}.$$

As a variant, we may pass from real to complex vector fields and formally plug the complex vector field $(f, if)^T$ into Equation 8.15. We get

$$\int_\gamma \begin{pmatrix} f \\ if \end{pmatrix} \mathrm{d}\mathbf{T} = \int_a^b \left\langle \begin{pmatrix} f(\gamma(t)) \\ if(\gamma(t)) \end{pmatrix}, \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} \right\rangle \mathrm{d}t = \int_\gamma f(z)\,\mathrm{d}z. \tag{8.19}$$

Last but not least, we recall a further variant for the realisation of Equation 8.17 due to Barakat (2014). The separation of the product of complex numbers $z \cdot w$ into real and imaginary part yields

$$z \cdot w = \langle \overline{z}, w\rangle + i\langle \overline{z}, w^\perp\rangle.^{85} \tag{8.20}$$

This decomposition of $z \cdot w$ into real and imaginary part is then transferred to the product of the function $f \circ \gamma$ and γ', which corresponds to the integrand f along γ multiplied with the "direction" γ'. More precisely, we apply Equation 8.20 to $z = \overline{f}(\gamma(t))$ and $w = \gamma'(t)$ with $t \in [a, b]$ and obtain

$$\left\langle \overline{f}(\gamma(t)), \gamma'(t)\right\rangle = \left\langle \overline{f}(\gamma(t)), \gamma'(t)\right\rangle + i\left\langle \overline{f}(\gamma(t)), \gamma'(t)^\perp\right\rangle.$$

Then, integrating termwise (Equation 8.9), we obtain another realisation of the separation of the complex path integral into its real and imaginary part (Barakat, 2014, p. 18):

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b \langle(\overline{f}\circ\gamma)(t), \gamma'(t)\rangle + i\int_a^b \langle(\overline{f}\circ\gamma)(t), \gamma'(t)^\perp\rangle\,\mathrm{d}t.$$

In sum, we may obtain different realisations for the complex path integral as a whole or for the real and imaginary part separately in terms of (real) path integrals of second kind. In particular, once these integrals are established, the complex path integral can be defined accordingly (e.g., with Equation 8.17 or Equation 8.19). In particular, this procedure has the advantage that existence proofs can be imported from real analysis.

85 Here, $\langle\cdot,\cdot\rangle$ is used for the Euclidean scalar product applied to the realisation of complex numbers as vectors in $\mathbb{R}^2$. That is, $\langle z, w\rangle = \mathrm{Re}(z)\,\mathrm{Re}(w) + \mathrm{Im}(z)\,\mathrm{Im}(w)$ for $z, w \in \mathbb{C}$. $w^\perp$ denotes the result of turning w (more precisely, its realisation as a vector) clockwise by $\pi/2$, $w^\perp = -ia$.

We are going to interpret these formulas, in particular Equation 8.17, physically in Section 8.2.2.

Integration of complex differential forms

One may also rephrase some of the observations in the last paragraph in terms of differential forms (for a rigorous discussion of differential forms in complex analysis see Berenstein and Gay (1991); recall that we also described historical precedents in Section 7.1). Symbolically, one may multiply the expressions $f = u + iv$ and $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, which gives

Integration of complex differential forms

$$f\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = (u\,\mathrm{d}x - v\,\mathrm{d}y) + i(u\,\mathrm{d}y + v\,\mathrm{d}x),$$

to formally substantiate the equation

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma u\,\mathrm{d}y + v\,\mathrm{d}x,$$

which we have seen previously. For example, Ponnusamy and Silverman (2006, p. 203) use this formal calculation *after* they have established the last equation. Similarly, one may multiply formally $f\,\mathrm{d}z = f\,\mathrm{d}x + if\,\mathrm{d}y$ to set

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f\,\mathrm{d}x + i\int_\gamma f\,\mathrm{d}y,$$

where the two integrals at the right side denote path integrals with respect to the two coordinates x and y each. More precisely, this means that $\int_\gamma f\,\mathrm{d}x = \int_\gamma f\,\mathrm{d}x + 0\,\mathrm{d}y$ and $\int_\gamma f\,\mathrm{d}y = \int_\gamma 0\,\mathrm{d}x + f\,\mathrm{d}y$ (cf. Isaev, 2017, p. 44).

Ferus (2009, ch. 3.1) obtains the same formulas by transferring the definition of the integral of a real differential form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$,

$$\int_\gamma \omega = \int_\gamma P\left(\gamma(t)\right)\gamma_1'(t) + Q\left(\gamma(t)\right)\gamma_2'(t)\,\mathrm{d}t$$

to the case where P and Q are complex-valued functions ($\gamma = \gamma_1 + i\gamma_2$). Let $P = f$ and $Q = if$. Then, Ferus (2009) defines the complex differential form

$$f\,\mathrm{d}z := P\,\mathrm{d}x + Q\,\mathrm{d}y = f\,\mathrm{d}x + if\,\mathrm{d}y$$

and deduces that

$$\int_\gamma f\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma_1'(t) + if(\gamma(t))\gamma_2'(t)\,\mathrm{d}t = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Last but not least, Königsberger (2004b, ch. 5.1) introduced complex differentials and their integral in another way. The differential of a differentiable function of two real variables g is $\mathrm{d}g = \frac{\partial}{\partial x}g\,\mathrm{d}x + \frac{\partial}{\partial y}g\,\mathrm{d}y$. He transfers this definition to the case, where $g = f$ is a holomorphic function defined on a domain $\Omega \subseteq \mathbb{C}$. Then, according to the Cauchy-Riemann differential equation (A.7) at $a \in \Omega$ we have $f'(a) = \frac{\partial}{\partial x} f(a) = -i\frac{\partial}{\partial y} f(a)$, and thus the differential of the holomorphic function f has the form

$$\begin{aligned} \mathrm{d}f(a) &= \frac{\partial}{\partial x} f(a)\,\mathrm{d}x + \frac{\partial}{\partial y} f(a)\,\mathrm{d}y \\ &= f'(a)\,\mathrm{d}x + i f'(a)\,\mathrm{d}y = f'(a)(\mathrm{d}x + i\,\mathrm{d}y) = f'(a)\,\mathrm{d}z. \end{aligned}$$

Here, z can be also see as the function $(x, y) \longmapsto x + iy$ on $\mathbb{R}^2 \cong \mathbb{C}$. Accordingly, the definition

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma_1'(t) + i f(\gamma(t))\gamma_2'(t)\,\mathrm{d}t = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

is substantiated. Notice that the substantiations by Ferus (2009) and Königsberger (2004b) are very similar to each other, but the differential forms are realised slightly differently. However, Königsberger (2004b) based his derivation on holomorphicity of f was needed.

8.1.5 *Connection to double integrals*

Green-type partial aspect of complex path integrals

Trahan (1965) has developed a “new approach” for deriving integral theorems in complex analysis with “innovations in notation [...] in order to facilitate manipulations” (Trahan, 1965, p. 132). Let $f = u + iv \colon \Omega \longrightarrow \mathbb{C}$ be a function such that u and v are continuously partially differentiable. Trahan (1965, p. 132) considers the difference

$$\delta f := f_x' - f_y'$$

between what he calls the *directional derivatives*

$$f_x' := \partial_1 u + i\partial_1 v \quad \text{and} \quad f_y' := \partial_2 v - i\partial_2 u$$

(Trahan, 1965, p. 132). A computation shows that $\delta f = 2\bar{\partial} f$. Additionally, he sets

$$\mathrm{d}A := \mathrm{d}x(i\,\mathrm{d}y).$$

Then, given a rectifiable simple closed path γ in Ω, whose interior $\mathrm{int}(\gamma)$ is contained in Ω an application of Green’s theorem (Theorem B.15) yields that

$$\int_\gamma f(z)\,\mathrm{d}z = \iint_{\mathrm{int}(\gamma)} \delta f\,\mathrm{d}A. \tag{8.21}$$

This formula realises the complex path integral as a certain double integral. Such a double integral is understood in the following way: If $D \subseteq \Omega$ is a region and h is a complex-valued on D such that the two double integrals $\int_D \mathrm{Re}(h)\,\mathrm{d}\mathcal{A}$ and $\int_D \mathrm{Im}(h)\,\mathrm{d}\mathcal{A}$ exist, then, we set $\int_D h\,\mathrm{d}\mathcal{A} := \int_D \mathrm{Re}(h)\,\mathrm{d}\mathcal{A} + i\int_D \mathrm{Im}(h)\,\mathrm{d}\mathcal{A}$. Therefore, Equation 8.21 yields a partial aspect for complex path integrals for continuously differentiable functions and rectifiable simple closed paths.

We are, however, sceptical about the extent to which it is really appropriate to speak of a new approach in complex analysis, as Trahan (1965) does. Rather, it seems that the particular realisation of complex path integrals in this special case is an application of Green’s integral theorem. The special choice of the differential operator δ resembles a trick in notation, with the help of which the complex path integral may be written particularly succinctly as the double integral above. In the rest of his paper, Trahan (1965) goes on to prove several results with the help of his approach, in particular a version of Cauchy's integral formula and the residue theorem.

Since δf is a multiple of the Wirtinger derivative $\bar{\partial} f$ (Equation A.9), namely $\delta f = 2\bar{\partial} f$, and $\mathrm{d}A$ is a multiple of the area element $\mathrm{d}\mathcal{A} = \mathrm{d}x\,\mathrm{d}y$ for double integrals, namely $\mathrm{d}A = i\,\mathrm{d}\mathcal{A}$, Equation 8.21 can also be written as

$$\int_\gamma f(z)\,\mathrm{d}z = 2i \iint_{\mathrm{int}(\gamma)} \bar{\partial} f\,\mathrm{d}\mathcal{A},$$

which amounts to the complex version of Green's theorem (Theorem B.15) in Equation A.31. In particular, we immediately see that $\int_\gamma f(z)\,\mathrm{d}z$ vanishes under the required constraints on f and γ if f is holomorphic since then the Cauchy-Riemann differential equation (Equation A.7) is satisfied.

8.1.6 *Reflection on the class of paths in complex path integrals*

Since the derivative of γ appears very prominently in Equation 8.13 or in formula related to path integrals of second kind, with the help of which complex path integrals may be defined, learners may question why the class of paths for integration is restricted to (piecewise) continuously differentiable paths. After all, for Riemann integrals, double integrals, or integrals in measure theory, no differentiability conditions are imposed on the "domain of integration". In these cases, the integrals are defined over certain classes of sets. Only if the students have previously dealt with path integrals in vector analysis, they may have encountered a similar restriction to the class of paths before.

Therefore, we take this opportunity to reflect on how the choice of the class of paths for integration in complex analysis is substantiated in the literature (if it is substantiated at all). In fact, some textbook authors deal with this question in both, personal and utilitarian, ways. For example, Burckel (1979, p. 44) describes that his

> attitude toward [paths; EH.] and integrals is frankly utilitarian. There is no need to integrate over curves more general than piecewise smooth ones, so we will not. (see also Burckel, 2021, p. 59)

Nevertheless, it is possible to define complex path integrals for a larger class of paths (e.g., rectifiable paths in case of Section 8.1.1). Therefore, Burckel (1979, p. 45) also acknowledges that

> aesthetic considerations might lead one to want to have $\int_\gamma f$ defined for any curve γ (lying in the domain of the holomorphic function f) [..., which] can easily be done by approximating γ uniformly with piecewise smooth curves (whether γ is rectifiable in the above sense or not). (Burckel, 1979, p. 45; Burckel, 2021, p. 60)

(see Section A.10 for a description how Burckel's statement can be made precise). Similarly, Gathmann (2017, p. 17, own transl.) describes that

> it is possible to define integrals for some functions [holomorphic functions; EH.] along arbitrary paths, but this is endowed with much more work and therefore will not be treated here since almost all paths occurring in practice are continuously differentiable anyhow.

Greenleaf (1972) also substantiates the definition $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ implicitly with the fact that the class of (piecewise) continuously differentiable paths is sufficient for his book, but also with its practical use compared to the definition via limits of complex Riemann sums:

> This definition may seem a bit formal; we have framed it to mimic the change of variable formula (write $dz = d\gamma/dt$ and take an ordinary Riemann integral in place of $\int_\gamma \cdots dz$), but the natural geometric origin of the complex line integral and its true similarity to the Riemann integral are obscured. In more advanced texts the integral $\int_\gamma f(z)\,dz$ is defined by a limit process that is very similar to the construction of Riemann integrals as limits of 'Riemann sums'; formula [(8.13); EH.] is then deduced from the limit definition as a *theorem*, valid when γ is piecewise smooth and the integrand $f(z)$ is continuous. In calculating the values of line integrals we would face serious difficulties if we had to evaluate these integrals as limits of partial sums; [Equation 8.13; EH.] determines the line integral in terms of already familiar objects (Riemann integrals), and all results discussed in this book are derived from this formula, so we will not be hampered by taking it as the definition of the line integrals. (Greenleaf, 1972, pp. 263–264)

However, Greenleaf (1972) also acknowledges that the parametric definition hides the relationship to Riemann integrals and "the natural geometric origin" of the complex path integral. As such, this remark seems to be helpful for learners of complex analysis because it hints at the definition of the Riemann integral students of a complex analysis course are likely familiar with.

Whereas the previous substantiations for the choice of the class of paths are primarily utilitarian, there are also approaches in the literature, which aim at minimal technical requirements or simplicity for teaching. For instance, in one of his approaches to prove the analyticity of holomorphic functions Heffter (1960, own transl.) defines "staircase integrals" (German: Treppenintegrale) (i.e., complex path integrals along polygonal paths with sides parallel to the two coordinate axes) and Klazar (2019a) defines complex path integrals for directed line segments and oriented boundaries of rectangles (see Section 7.5 for variants and reasons for these and similar approaches).

8.2 INTERPRETATIONS OF THE COMPLEX PATH INTEGRAL

8.2.1 *Geometrical interpretation of complex Riemann sums*

Geometrical interpretation of complex Riemann sums as concatenations of vectors

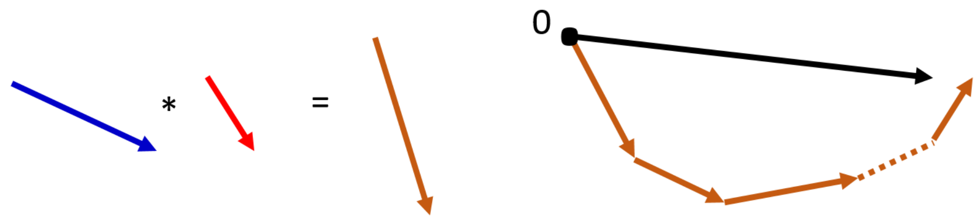
We recall that the arithmetic operations of addition and multiplication of complex numbers can be interpreted as geometric transformations (see Remark A.2): The addition of complex numbers corresponds to the concatenation of the corresponding vectors in $\mathbb{R}^2$ and the multiplication corresponds to a rotation and dilation of one of the corresponding vectors by the argument and length of the other.

The following interpretation is due to Needham (1997, ch. 8). Let us consider a complex Riemann sum

$$R(f,\gamma,P,\xi) = \sum_{k=1}^{n} f(\gamma(\xi_k))(\Delta\gamma)_k.$$

Each of the addends

$$\widetilde{\Delta}_k := f\left(\gamma(\xi_k)\right)\left(\Delta\gamma\right)_k, \quad k = 1,\ldots,n,$$



(a) Pictograph of the multiplication of two complex numbers shown as a blue and red vector and their product as a brown vector..

(b) Pictograph of a complex Riemann sum: The sum of complex numbers is shown as a concatenation of brown vectors; the black vector stands for the exact value of the complex path integral..

Figure 8.3: Geometry of Riemann sums.

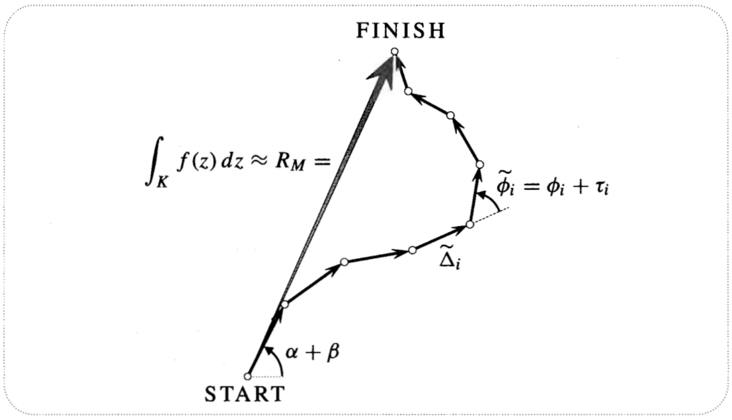


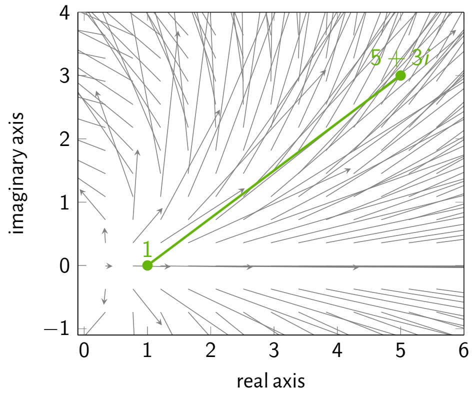
Figure 8.4: Visualisation of the Riemann sum approximation for the complex path integral (Figure [9] from Needham (1997)).

in this sum is the product of $(\Delta\gamma)_k$ and a value of the function $f(\gamma(\xi_k))$ at a point on the trace of γ between $\gamma(t_{k-1})$ and $\gamma(t_k)$ (see Figure 8.1). Geometrically, $\widetilde{\Delta}_k$ is a copy of the number $(\Delta\gamma)_k$ represented as a vector, which was dilated with the modulus and rotated with the argument of $f(\gamma(\xi_k))$ (see Figure 8.1b and Figure 8.3a, where we have drawn the corresponding parts with the same colours). Accordingly, $R(f, \gamma, P, \xi)$ can be interpreted as the concatenation of all the addends $\widetilde{\Delta}_k$ regarded as vectors in the plane (Figure 8.3b). As a consequence of the approximation in Equation 8.3, the endpoint of this concatenation will lie within a small neighbourhood of the actual value $\int_\gamma f(z)\,\mathrm{d}z$ for sufficiently large n.

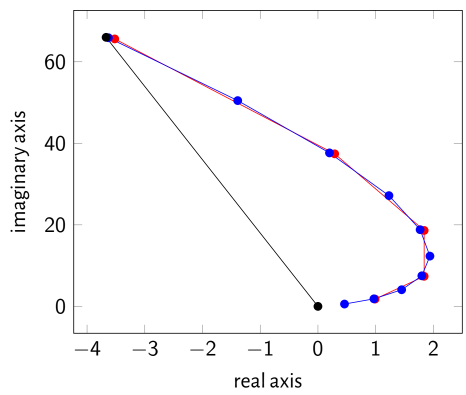
In Figure 8.4, we show one of the many figures from Needham (1997) with complex Riemann sums. The small arrows in this figure realise the addends in a complex Riemann sum, which Needham signifies with R_M and realises with the long vector. The angles ϕ_i and τ_i signify the arguments of $(\Delta\gamma)_{i+1}$ and $f(\gamma(\xi_{i+1}))$ (the $+1$ in the index results from the numbering we have chosen previously).

Example 8.3. Consider the path $\gamma\colon [0, 1] \longrightarrow \mathbb{C}$ given by $\gamma(t) = 1 + t(4 + 3i)$ and the function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) = z^2$. The plot of the vector field associated to f, that is, $\mathbf{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, (x, y) \longmapsto (x^2 - y^2, 2xy)$, and the trace of γ are shown in Figure 8.5a.

Figure 8.5b shows two complex Riemann sums in form of vector concatenations. For the red one, we have chosen partition of $[0, 1]$ into 5 segments of equal lengths and for the blue one we have chosen the partition of $[0, 1]$ into 10 segments of equal lengths. The tag vectors associated



(a) Plot of the vector field $\mathbf{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (gray), $(x, y) \longmapsto (x^2 - y^2, 2xy)$ and the trace of $\gamma\colon [0, 1] \longrightarrow \mathbb{C}, \gamma(t) = 1 + t(4 + 3i)$ (green).



(b) Riemann sums for $f(z) = z^2$ and the path from 1 to $2 + i$ (red: the chosen partition of the path consists of 5 segments, blue: the chosen partition of the path consists of 10 segments; the tag vector consists of the midpoints of the partition entries each).

Figure 8.5: Formation of Riemann sums for the complex path integral.

to each partition consisted of the mid-points of the subdivisions of the intervals. The integral $\int_\gamma z^2 \,\mathrm{d}z = -\frac{11}{3} + 66i$ is shown in black. ◇

Visual reasoning with complex Riemann sums

Needham (1997) uses this geometrical representation of complex Riemann sums to calculate a lot of complex path integrals for special classes of functions, such as rational or exponential functions. But he also uses it for heuristic proofs of many propositions about complex path integrals, ranging from elementary properties like the $\mathbb{C}$-linearity of the complex path integral with respect to the integrand, variants of Cauchy's integral theorem (Theorem A.17) or Existence of primitives for holomorphic functions (Theorem A.20).

Let us review two of his examples.

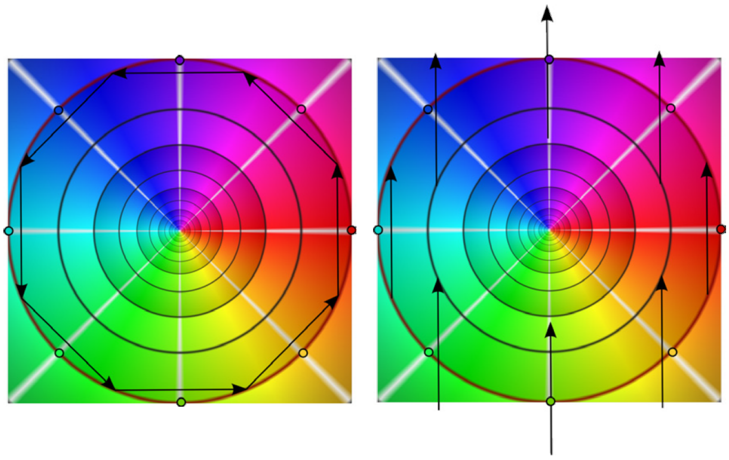
Evaluating $\int_{\partial B_1(0)} \mathrm{d}z/z$

Having evaluated the integral

$$\int_{\partial B_1(0)} \frac{1}{z} \,\mathrm{d}z = 2\pi i,$$

it is quite natural to ask for a geometric explanation for the question why the resulting arrows from the complex Riemann sums (at least approximately) point upwards have length 2π? We will now recapitulate the Needham's (1997, pp. 388–389) explanation and will enhance it with domains colourings as in Poelke and Polthier (2009).

Figure 8.6a shows a domain colouring of the function $z \longmapsto 1/z$ (see Appendix C for domain colourings). On the left, eight black vectors and eight coloured points (the eight roots of unity) on the unit circle are drawn. The arrows correspond to the complex numbers $(\Delta\gamma)_k$ and the points to $f(\gamma(\xi_k))$ ($k = 0, 1, \ldots, 7$; each counted counterclockwise starting with the arrow pointing upwards at the most right position and the red dot at number 1). The colour of the points represent the value of the integrand corresponding to the colour wheel in Figure 8.6b. For example, the point at 1 is coloured red, which represents the number 1, the point at i is coloured violet, which represents the number $-i$, and so on.



(a) Phase plot of $z \longmapsto 1/z$ and a partition of the unit circle; from Poelke and Polthier (2009, p. 739).

(b) Colour wheel; from Poelke and Polthier (2009, p. 736).

Figure 8.6: Domain colouring for the substantiation of $\int_{\partial B_1(0)} = 2\pi i$.

Now, in order to estimate the corresponding complex Riemann sum, $(\Delta\gamma_k)$ has to be multiplied with the function value of the kth point $f(\xi_k)$. For example, the first addend is pointing upwards (product of the arrow point upwards and 1), and the third addend is also pointing upwards (product of the arrow pointing left and $-i$). In total, the complex Riemann sum for the given tagged partition consists of the concatenation of eight arrows pointing upwards and is therefore purely imaginary. Moreover, "[w]hen we have circled around γ, we have gathered a total amount of 2π, pointing into the direction i, and the resulting value $2\pi i$ is understandable" (Poelke & Polthier, 2009, p. 739). Farris (2017, pp. 839–840) also uses domains colourings and complex Riemann sums and extends the argument just shown to the integrands $z \longmapsto 1/z^k$, $k \in \mathbb{Z} \setminus \{1\}$, for which the integrals evaluate to 0.[86]

86 This procedure of computing or estimating complex Riemann sums is applied frequently by Needham (1997). The general procedure is as follows: In order to compute a complex path integral $I = \int_\gamma f(z)\,\mathrm{d}z$, complex Riemann sums R are used as an approximation. Needham always carefully chooses the partitions of γ and corresponding tag vectors, which eventually lead to a manageable computation of R. However, sometimes the $(\Delta\gamma)_k$, which appear in the complex Riemann sums, are also approximated with tangential vectors attached to $\mathrm{tr}(\gamma)$. In this case, a sum S is computed, which approximates R. Accordingly, two approximations $I \approx R$ and $R \approx S$ happen at the same time. Hence, approximations of complex path integrals $I \approx R$ are computed with approximations $R \approx S$ as well, which makes it hard to follow why the two approximation errors do not add up irreparably and how the geometric estimations work in general rather than for simple examples. A detailed analysis of the geometric arguments by Needham (1997) has to be postponed for another study though.

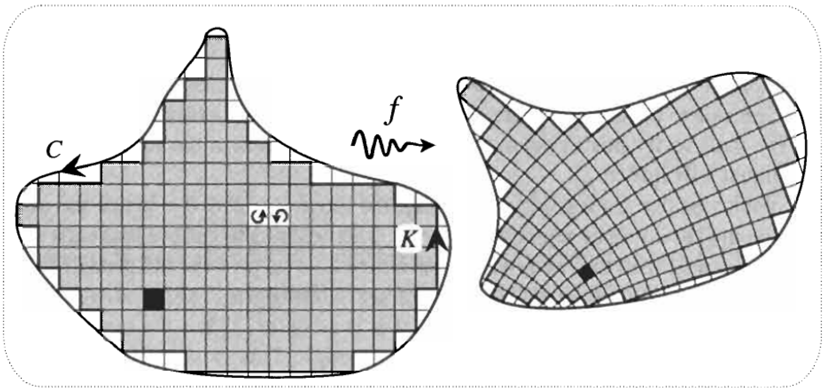


Figure 8.7: Division of the interior of a path into small squares.

In Poelke and Polthier's (2009), it does not become completely clear whether the resulting length of the arrow of 2π comes from the circumference of the unit circle or from the full rotation of the path along the circle. For instance, Farris (2017, p. 840), also using domain colourings, argues that 2π comes from the circumference of the circle. However, as $\int_{\partial B_r(0)} \frac{1}{z}\,\mathrm{d}z = 2\pi i$ is valid for every $r > 0$, at least some more justification is needed, because here circumference of the circle $2\pi r$ is not independent of r. One may argue that multiplying line segments given by points on the boundary of a circle of radius r with values of the function $z \longmapsto 1/z$ normalise the length of the resulting vector back to 2π (Needham, 1997, pp. 388–389).

Cauchy's integral theorem

Geometrically motivated heuristic proof of Cauchy's integral

Recall that Cauchy's integral theorem (Theorem A.17) states that the complex path integral of a holomorphic function along a closed (piecewise continuously differentiable path vanishes. We will now describe Needham's (1997, ch. 8.X) geometric substantiation for a simple closed, simple, (piecewise) continuously differentiable path γ with trace $C = \mathrm{tr}(\gamma)$ as in Figure 8.7.

Let f be a holomorphic function on an open neighbourhood of $\overline{\mathrm{int}(\gamma)}$ and let $\varepsilon > 0$. Then, cover $\overline{\mathrm{int}(\gamma)}$ with a grid of squares of side lengths ε parallel to the coordinate axes such as in the left part of Figure 8.7. Let K denote the boundary of the shaded collection of squares in Figure 8.7 with the indicated orientation. Since f is holomorphic, it can be locally approximated by an affine $\mathbb{C}$-linear mapping. Hence, it is an amplitwist (Section 5.1.3) and thus maps small squares approximately to small squares such as in the right part of Figure 8.7.

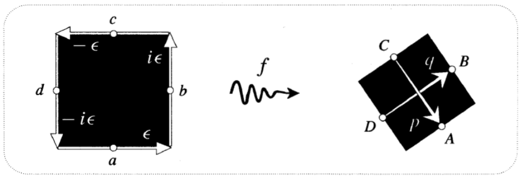
Since the complex path integral is additive in paths and complex path integrals change their sign when the orientation of a path is reversed, Needham (1997, p. 411) concludes that

$$\text{“}\int_K f(z)\,\mathrm{d}z = \sum_{\text{shaded squares}} \int_{\square} f(z)\,\mathrm{d}z\text{”}, \tag{8.22}$$

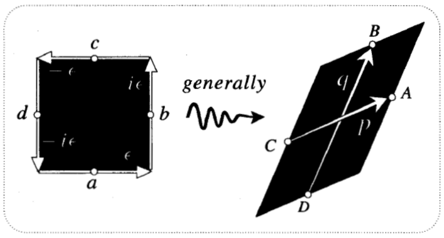
where the sum is taken over shaded squares $\square$ with counterclockwise parametrisation according to Figure 8.7. The degenerated squares between C and K are neglected here and it is argued that the difference between $\int_C f(z)\,\mathrm{d}z$ and $\int_K f(z)\,\mathrm{d}z$ can also be neglected as ε tends to 0.[87]

Now, Needham argues that $\int_\square f(z)\,\mathrm{d}z$ vanishes and hence that $\int_K f(z)\,\mathrm{d}z$ vanishes, too. For this purpose, Needham (1997, pp. 412–413) considers the Riemann sum for $\int_\square f(z)\,\mathrm{d}z$ with

87 Needham (1997, pp. 410–414) estimates the number of squares and the error made more detailed, too.



(a) Transformation of a small square under a holomorphic mapping.



(b) Transformation of a small square under an $\mathbb{R}$-linear mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Figure 8.8: Plots of transformation of a small square and a small parallelogram (adapted from Needham (1997, pp. 411–413)).

four addends of the form $f(\xi)\eta$, where η ranges over the edges of $\square$ and ξ ranges over the corresponding midpoints a, b, c, d (see Figure 8.8a). Let $p = A - C$ and $q = B - D$, where A, B, C, D are the midpoints of $f(\square)$ as in Figure 8.8a. Since p is the same as q rotated by $\pi/2$ clockwise, that is, $p = -iq$, the complex Riemann sum is

$$A\varepsilon + C(-\varepsilon) + Bi\varepsilon + D(-i\epsilon) = p\varepsilon + qi\varepsilon = 0.$$

Hence,

$$\text{“}\int_{\square} f(z)\,\mathrm{d}z = \varepsilon(p + iq) = 0\text{”}$$

and “[w]e conclude [...] that the vanishing of loop integrals for analytic mappings is indeed the nonlocal manifestation of their local amplitwist property” (Needham, 1997, p. 413).[88]

8.2.2 *Physical interpretation of the complex path integral using Pólya vector fields*

In this subsection, we review the physical interpretation of complex path integrals, which is based on the use of the Pólya vector field $\mathbf{w}_f$ associated to a complex function f. It seems that Braden (1987) and Needham (1997) attempted to popularise it at the end of the last century based on the text by Polya and Latta (1974). Nevertheless, applications of complex analysis physics appeared in textbooks on complex analysis as well, including the interpretation we will

88 In particular, Needham equates $\int_{\square} f(z)\,\mathrm{d}z$ with the single complex Riemann sum mentioned previously. While this reasoning seems very plausible at first sight, there are several problematic aspects. First, it relied on the property that $\int_K f(z)\,\mathrm{d}z$ converges to $\int_C f(z)\,\mathrm{d}z$ if $\varepsilon \longrightarrow 0$. This is not endorsable for a general function f (since then, complex path integrals may depend on the chosen path between the same initial and terminal points) and for a holomorphic function f it is essentially equivalent to Cauchy's integral theorem or a variant of it and would have to be proven first (cf. Theorem A.26). Second, it is rather hard to follow why the approximations involved in Needham's argument work out as planned for holomorphic but not for non-holomorphic functions f. For instance, in the argument presented above for holomorphic f, Needham concludes from the vanishing of a single complex Riemann sum that $\int_{\square} f(z)\,\mathrm{d}z$ vanishes, too. However, he also argues that if f is not holomorphic (but $\mathbf{f}$ is real totally differentiable), the corresponding complex Riemann sum for $\int_{\square} f(z)\,\mathrm{d}z$ with the four addends as described above does not vanish in general, since the square is only mapped approximately to a small parallelogram (see Figure 8.8b; Needham, 1997, p. 413). But since complex Riemann sums only approximate complex path integrals in general, we would be in need of a more careful argument to conclude that $\int_{\square} f(z)\,\mathrm{d}z \neq 0$ when f is a general non-holomorphic function, which is however not possible (in fact, there are non-holomorphic functions f and small squares $\square$ such that $\int_{\square} f(z)\,\mathrm{d}z = 0$ (e.g., take $\square$ to be the square with vertices $\varepsilon + \varepsilon i, -\varepsilon + \varepsilon i, -\varepsilon - \varepsilon i$, and $\varepsilon - \varepsilon i$, and $f(z) = 1/z^2$)). The problem when and why a single complex Riemann sum is considered to be the value of a complex path integral or at least as an approximation with sufficient accuracy seems to be require further discussion (see also footnote 86).

present now. For example, Lawrentjew and Schabat (1967, pp. 264–276) and González (1992, ch. 7.29) describe the application to vector fields that model the motion of ideal incompressible fluids very detailed, too. This physical interpretation was then taken up in several recent publications (Brilleslyper et al., 2012, ch. 3; Needham, 1997, ch. 11; Kinney, 2013), computer tools (Custy, 2011; Wilkinson, 2011), and even in videos (e.g., Chen, 2021; Lemmaxiom, 2021; Mathemaniac, 2022). As we have already explained at the beginning of Chapter 5, this interpretation nevertheless seems to be hardly represented in textbooks on complex analysis.

To motivate a physical interpretation of complex path integrals, Braden (1987) cites an earlier edition of Brown and Churchill (2009), who

> express the problem succinctly: "Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane." In 1974, George Pólya suggested a simple solution, but his idea does not seem to be widely appreciated. (Braden, 1987, p. 321)

We recall that a real path integral of second kind with respect to the tangential field models the work of a particle done along the path in case the integrand models a force field (e.g., of an electrical field) or the flow along the path in case the integrand models a fluid's motion; in this case, the real path integral of second kind with respect to the normal component represents the flux of the fluid across the path (Section B.2.2; Braden, 1987; Polya and Latta, 1974).

Work and flux of the Pólya vector field

Therefore, Equation 8.17 and Equation 8.18, that is,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N} \quad \text{and} \quad \int_\gamma \overline{f}(z)\,\mathrm{d}z = \int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{f}\,\mathrm{d}\mathbf{N},$$

enable us to see the following (Braden, 1987; Brilleslyper et al., 2012, ch. 3; Needham, 1997, ch. 11; Polya & Latta, 1974; Spiegel et al., 2009):

- If we interpret $\mathbf{w}_f$ and $(v, u)^T$ as the velocity fields of a fluid, then the real part of $\int_\gamma f(z)\,\mathrm{d}z$ represents the *flow of a fluid modelled by* $\mathbf{w}_f$ *along* γ and the imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$ represents the *flux of* $\mathbf{w}_f$ *across* γ or the *flow of the fluid modelled by* $(v, u)^T$ *along* γ.

- If we interpret $\mathbf{w}_f$ and $(v, u)^T$ as force field, the real part of $\int_\gamma f(z)\,\mathrm{d}z$ represents the *work done by a particle moving along* γ *in the field* $\mathbf{w}_f$ and the imaginary part $\int_\gamma f(z)\,\mathrm{d}z$ the *work done by a particle moving along* γ *in the field* $(v, u)^T$.

- Replacing f with $\overline{f}$ and $\mathbf{w}_f$ with $\mathbf{f}$ we get similar interpretations for real and imaginary part of the complex path integral $\int_\gamma \overline{f}(z)\,\mathrm{d}z$.

In particular, we emphasise once again that neither $\int_\gamma f(z)\,\mathrm{d}z$ nor its real and imaginary part are the work / flow or flux of the vector field $\mathbf{f}$ naturally associated to f. The work / flow or flux of $\mathbf{f}$ does however appear as the real and the imaginary part in the complex path integral of the conjugate f, $\int_\gamma \overline{f}(z)\,\mathrm{d}z$.

Before we illustrate these relationships with a few examples and visualisations from the literature, we go more into detail about the physical interpretations in relation to potential theory (i.e., the mathematics of harmonic functions and conservative vector fields).

Complex path integrals in potential theory

According to Equation 8.17 and, $\mathbf{w}_f$ appears in the real path integrals of second side but not $\mathbf{f}$. While this seems to break the natural correspondence between complex numbers and vectors in $\mathbb{R}^2$ at first (cf. Equation A.1), it also brings a few advantages in interpretation, in particular when γ is a closed path. These interpretations connect complex analysis and potential theory and potential theory following (i.e., roughly speaking, the study of harmonic functions and conservative vector fields). We follow the exposition in Arens et al. (2018, pp. 1220–1228), Bärwolff (2017, pp. 738–743), Brilleslyper et al. (2012, ch. 3), and Lawrentjew and Schabat (1967, pp. 264–276).

We assume that γ is a simple closed (piecewise) continuously differentiable path in a simply-connected domain Ω and that $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2)^T$ is a continuously differentiable stationary planar vector field on Ω. We may imagine that this vector field models the motion of a fluid, which depends on the location (x, y) only. Then, the *flux across a simple closed path* γ is

$$N = \int_\gamma \mathfrak{A}\, \mathrm{d}\mathbf{N} = \int_\gamma -\mathfrak{A}_2\, \mathrm{d}x + \mathfrak{A}_1\, \mathrm{d}y = \iint_{\mathrm{int}(\gamma)} \operatorname{div} \mathfrak{A}\, \mathrm{d}S,$$

and the *circulation along* γ is

$$\Gamma = \int_\gamma \mathfrak{A}\, \mathrm{d}\mathbf{T} = \int_\gamma \mathfrak{A}_1\, \mathrm{d}x + \mathfrak{A}_2\, \mathrm{d}y = \iint_{\mathrm{int}(\gamma)} \operatorname{rot} \mathfrak{A}\, \mathrm{d}S.$$

Each of the last equal signs is justified by Green's theorem (Theorem B.15). Recall furthermore that

$$\operatorname{div} \mathfrak{A} = \partial_1 \mathfrak{A}_1 + \partial_2 \mathfrak{A}_2$$

and

$$\operatorname{rot} \mathfrak{A} = \partial_1 \mathfrak{A}_2 - \partial_2 \mathfrak{A}_1.$$

If $\operatorname{div} \mathfrak{A} \equiv 0$ on Ω, $\mathfrak{A}$ is called source- and sink-free or solenoidal, and then we consequently have $N = 0$. In this case, $(-\mathfrak{A}_2, \mathfrak{A}_1)^T$ is a conservative vector field and hence there is a potential v for it, which means that $\partial_1 u = -\mathfrak{A}_2$ and $\partial_2 u = \mathfrak{A}_1$. If $\operatorname{div} \mathfrak{A} \equiv 0$ on Ω, $\mathfrak{A}$ is called rotation-free or irrotational (or that it has no vortices), and then we consequently have $\Gamma = 0$. In this case, $\mathfrak{A}$ is a conservative vector field and it has a potential u, which means that $\partial_1 u = \mathfrak{A}_1$ and $\partial_2 u = \mathfrak{A}_2$.

In particular, if $\mathfrak{A}$ is solenoidal and irrotational, we have

$$\partial_1 \mathfrak{A}_1 = \partial_2 \mathfrak{A}_2 \quad \text{and} \quad \partial_2 \mathfrak{A}_1 = -\partial_1 \mathfrak{A}_2.$$

In other words, $\mathfrak{A}_1 + i\mathfrak{A}_2$ satisfies the Cauchy-Riemann differential equations (Equation A.6). Moreover, the potentials u and v from above are conjugate harmonic functions, and $f := u + iv$ is holomorphic. In this situation, we call f a *complex potential for* $\mathfrak{A}$. In sum, we obtain

$$\mathfrak{A} = \nabla u = (\partial_1 u, \partial_2 u)^T \cong \partial_1 u + i\partial_2 u = \partial_1 u - i\partial_1 v = \overline{f'}, \tag{8.23}$$

where $\cong$ signifies the identification of a complex number with a vector in $\mathbb{R}^2$ the second to last equal sign follows from the Cauchy-Riemann differential equations (Equation A.6), as well as

$$\Gamma + iN = \int_\gamma f'(z)\,\mathrm{d}z = \int_\gamma \mathfrak{A}_1\,\mathrm{d}x + \mathfrak{A}_2\,\mathrm{d}y + i\int_\gamma \mathfrak{A}_1\,\mathrm{d}y - \mathfrak{A}_2\,\mathrm{d}x.$$

Since Green's theorem (Theorem B.15) implies that

$$N = \iint_{\mathrm{int}(\gamma)} \operatorname{div}\mathfrak{A}\,\mathrm{d}S,$$

and

$$\Gamma = \iint_{\mathrm{int}(\gamma)} \operatorname{rot}\mathfrak{A}\,\mathrm{d}S,$$

we can deduce that $N = 0$ if $\mathfrak{A}$ is solenoidal and $\Gamma = 0$ if $\mathfrak{A}$ is irrotational. In particular, Cauchy's integral theorem (Theorem A.17) is a consequence from the previous observations and Equation 8.17.

The physical interpretation of all we have described here may thus summarised as follows:

> If the region Ω contains no sources and vortices, one can specify an analytic function $f(z) = u(z) + iv(z)$ in this region — the complex potential of the field — such that
>
> $$\mathfrak{V} = \overline{f'(z)}.$$
>
> Conversely, one can interpret any function analytic in the domain Ω as the complex potential of a plane flow of an ideal incompressible fluid free of sources and vortices (277 Lawrentjew & Schabat, 1967, own transl.).

Hence, we have argued why the complex conjugate of complex potential function for a given vector field is a useful object for further study. Reversing the point of view from vector fields to complex functions, we have mutually substantiated why the Pólya vector field associated to a complex function is a suitable vector field to study complex path integrals. Transferring our observation that $N = \Gamma = 0$ if $\mathfrak{A}$ is solenoidal and irrotational to $\mathbf{w}_f$, we see accordingly that $\int_\gamma f(z)\,\mathrm{d}z = 0$ if $\mathbf{w}_f$ is solenoidal and irrotational.[89]

We see at least three benefits of the physical interpretation of complex path integrals with the help of the associated Pólya vector fields:

(1) It establishes a clear connection between real and complex path integrals in terms of Equation 8.17, where the real path integrals of $\mathbf{w}_f$ can be interpreted physically compared to f. However, this interpretation relies on the separate interpretation of the real and imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$. Additionally, the integrals on the right-hand side of Equation 8.17 can be estimated visually (see Braden, 1987, for details).

(2) Propositions about complex path integrals can be rephrased and proven with the help of real analysis. For example, as described above, Cauchy's integral theorem (Theorem

89 Of course, technically both, $\mathbf{f}$ and $\mathbf{w}_f$, contain the same information; after all, only their second component functions differ in terms of sign. However, we aimed to show the physical important of the Pólya vector field. Additionally, we note that there are inner-mathematical applications of our observations in this section, too. For example, Brilleslyper and Schaubroeck (2017) use the Pólya vector field to prove the *Gauß-Lucas theorem*, which asserts that the roots of p' of a complex polynomial p lie in the convex hull of the roots of p: In physical terms, the reason is that the sources of p'/p are at the roots of p.

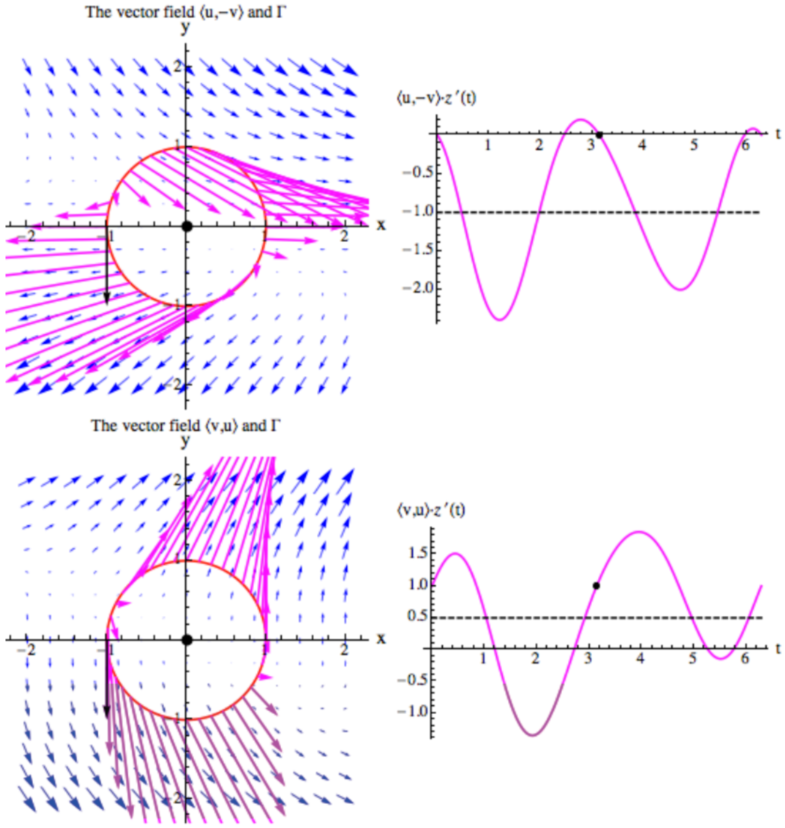


Figure 8.9: Illustration of the identity $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u,-v)^T\,\mathrm{d}\mathbf{T} + \int_\gamma (v,u)^T\,\mathrm{d}\mathbf{T}$ (Figure 12 from Kinney (2013)).

A.17) can be rephrased such that $\int_\gamma f(z)\,\mathrm{d}z$ vanishes if $\mathbf{w}_f$ is a solenoidal and irrotational vector field.

(3) It can be used to derive non-standard properties of the complex path integral: For example, Braden (1987) proves that $\left|\int_\gamma f(z)\,\mathrm{d}z\right| = \int_\gamma |f(z)|\,\mathrm{d}s$ if and only if $\mathbf{w}_f$ and the tangential vector field for γ, enclose a constant angle. It needs to be emphasised however, that Braden (1987) additionally uses the homotopy invariance of the complex path integral for paths relative to initial and terminal points.

Visualising Pólya vector fields and their integrals

There are a few computer tools, teaching materials, and videos on the interpretation of complex path integrals in terms of the physical properties of the Pólya vector field Brilleslyper et al. (2012), Chen (2021), Custy (2011), Kinney (2013), Lemmaxiom (2021), Mathemaniac (2022), and Wilkinson (2011). Since these visualisations are quite alike to each other, we will exemplarily focus on two.

Example 8.4. Let us look again at the function f with the term $f(x+iy) = x + 2y + iy^2$ and the path γ taken as the standard parametrisation of the unit circle from Example 8.2.

Plots of the associated Pólya vector field $\mathbf{w}_f = (u,-v)^T$ and the vector field $(v,u)^T$ are shown in Figure 8.9 on the left. They serve to illustrate the flow of $\mathbf{w}_f$ and $(v,u)^T$ along the unit circle in counterclockwise direction.

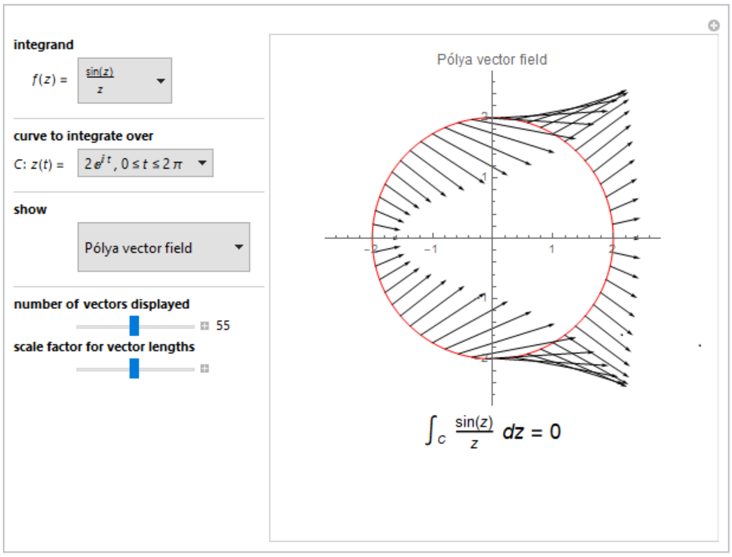


Figure 8.10: Illustration of the complex path integral $\int_{\partial B_2(0)} \frac{\sin(z)}{z}\,\mathrm{d}z$ by the Pólya vector field $\mathbf{w}$ associated to $z \longmapsto \frac{\sin(z)}{z}$ (drawn with the tool from Wilkinson (2011)).

The unit circle is drawn in red. The pink vectors illustrate the vector fields on unit circle (scaled) and the blue vectors illustrate them outside the unit circle (scaled down a lot). The graphs of the functions $t \longmapsto \langle (u(t), -v(t))^T, \gamma'(t)\rangle$ and $t \longmapsto \langle (v(t), u(t))^T, \gamma'(t)\rangle$, which appear in the integrals for the real and imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$ are shown on the right. Additionally, the respective mean values of these functions are shown as a dotted horizontal line (-1 in the first and $1/2$ in the second case). Hence, we have $\int_\gamma f(z)\,\mathrm{d}z = 2\pi(-1 + i/2)$. ◇

Another implementation of Braden's (1987) approach was prepared by Wilkinson (2011). You can select several integrands, paths for integration, and possibilities for the plot on the right side. In the example in Figure 8.10, you can see Pólya vector field associated to the function $z \longmapsto \frac{\sin(z)}{z}$ at the boundary of the circle $C := \partial B_2(0)$ (black vectors). The integral $\int_C \frac{\sin(z)}{z}\,\mathrm{d}z$ is 0. In visual terms, this means that the same amount of fluid represented by the black vectors flows into the circle as flows out.

Evaluating $\int_{\partial B_1(0)} z^n\,\mathrm{d}z$

Let us look at a concrete example for the use of the physical interpretation of Equation 8.17. For this purpose, we look again at the example from Section 8.2.1, where we have argued geometrical why $\int_{\partial B_1(0)} \frac{1}{z}\,\mathrm{d}z = 2\pi i$.

The Pólya vector field for associated to the function $z \longmapsto \frac{1}{z} = \frac{x-iy}{x^2+y^2}$ is given by

$$\mathbf{w}(x,y) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix}, \quad (x,y)^T \in \mathbb{R}^2 \setminus \left\{(0,0)^T\right\}.$$

The vectors of this field are normal to the unit circle (Figure 8.11). Thus, the normal component of $\mathbf{w}$ along the unit circle is constant 1 and the tangential component of it vanishes, and therefore, the real part of $\int_{\partial B_1(0)} \frac{1}{z}\,\mathrm{d}z$ is 0 and the imaginary part is 2π (Braden, 1987, p. 322).

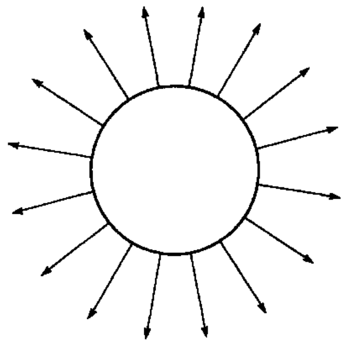


Figure 8.11: The Pólya vector field associated to $z \longmapsto 1/z$ on the unit circle (Figure 1 from Braden (1987)).

A dynamic variant of the plot in Figure 8.11 is implemented by Custy (2011). However, it shows the natural vector field **f** associated to a complex function instead of the Pólya vector field: His tool can plot the vector fields naturally associated to $z \longmapsto z^n$ for adjustable n. The case $n = -1$ is shown in Figure 8.12. The plot realises the vector field given by

$$\mathbf{f}(x, y) = \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{-y}{x^2+y^2} \end{pmatrix}, \quad (x, y)^T \in \mathbb{R}^2 \setminus \left\{(0, 0)^T\right\}$$

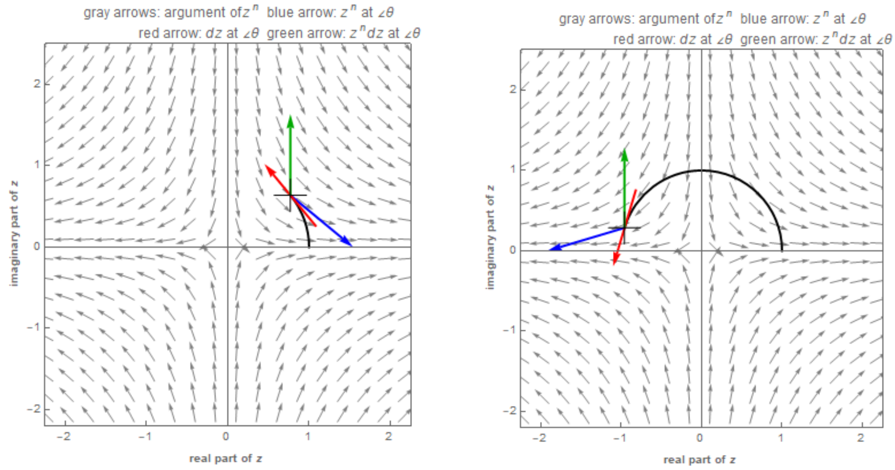
in gray (scaled). A slider allows the user to move a point along the unit circle (indicated with the large $+$-sign). At this point, the corresponding vector of the vector field is drawn in blue, the tangential vector at to the unit circle in red, as well as the tangential component of the vector field at that point in green. These green arrows represent the number $f(\gamma(t))\gamma'(t)$ for $t \in [0, 2\pi]$. We can see clearly that the green vectors all point upwards (Figure 8.12a and Figure 8.12b show two examples). In other words, these green arrows represent purely imaginary numbers, in accordance with our previous observations.

Figure 8.13 shows a similar visualisation for the function $z \longmapsto z^2$. In this case, however, the green vectors rotate when the point on the unit circle varies. For each possible direction of these vectors there is another pointing into the opposite direction, hence, Custy's (2011) tool illustrates dynamically that the corresponding complex path integral $\int_{\partial B_1(0)} z^2 \, \mathrm{d}z$ evaluates to 0.

8.2.3 *Average interpretations of integration of complex functions*

Gluchoff's average interpretation of complex path integrals

The main part of this section is the interpretation of the complex path integral as a certain average by (Gluchoff, 1991). Gluchoff (1991) emphasises that this is an interpretation, as it says in the title of his paper. Nevertheless, it turns out that it is indeed a full aspect of complex path integrals. Hence, it can be used to substantiate a definition of the complex path integral (see Chapter 9). We will now explain this interpretation and relate it to other mean value interpretations for other integrals. This is particularly important because mean value interpretations for other integrals may lead to commognitive conflicts and hence are likely to be challenges for students.



(a) Vectors of vector field of $z \longmapsto 1/z$, tangential vector to unit circle and their product as complex numbers roughly at the point $\mathrm{e}^{\pi i/4}$.

(b) Vectors of vector field of $z \longmapsto 1/z$, tangential vector to unit circle and their product as complex numbers roughly at the point $\mathrm{e}^{9\pi i/10}$.

Figure 8.12: Plot of vector field for the function $z \longmapsto 1/z$ and pieces of the unit circle (black): Vector field is drawn blue, tangential field vector is drawn red, their product in in terms of complex numbers is drawn as a green vector (drawn with the tool by Custy (2011)).

Gluchoff's average interpretation of the complex path integral

Mean value aspect of complex path integrals

Gluchoff (1991, p. 642) claims that the "idea of the integral as an average should be familiar to students from earlier calculus courses". Therefore, he suggests an average interpretation of the complex path integral, which "generally transforms the [complex path; EH.] integral into an object of substance" (Gluchoff, 1991, p. 644).

Let us follow Gluchoff's (1991) terminology here according to which complex path integrals are integrals for complex functions along curves. We assume further that C has a (piecewise) continuously differentiable parametrisation and that C is simple or simple closed and regular.[90] He argues that the complex path integral of a function f along a curve C can be seen as the scaled average

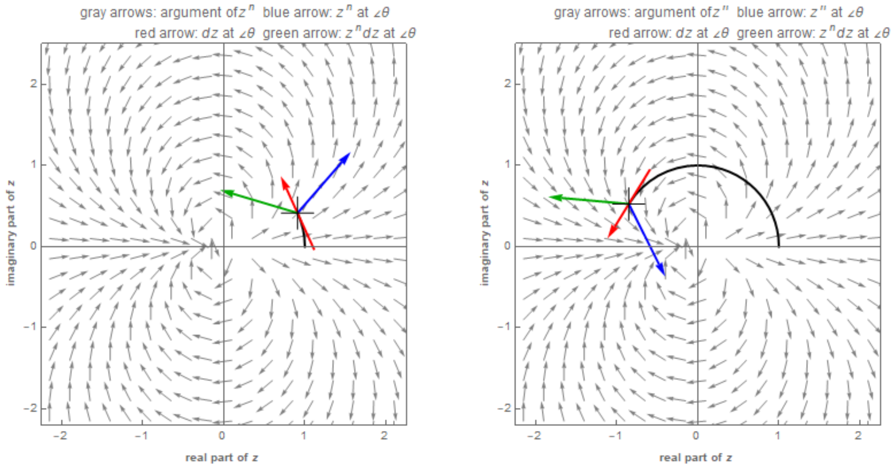
$$L(C) \cdot \operatorname*{av}_{z \in C} (f(z)T(z))$$

where $L(C)$, the length of C, is the scaling factor, $T(z)$ is the unit tangent vector at the point z on the curve (given an orientation of the curve), and $\mathrm{av}_{z\in C}$ stands for the average of values when z runs through the points of C. In other words, we have

$$\frac{1}{L(C)} \int_C f(z)\,\mathrm{d}z = \operatorname*{av}_{z \in C} (f(z)T(z)) . \tag{8.24}$$

We will now describe in more detail how this formula can be derived and how we can precisely define the average used here. Thus, we obtain an aspect for complex path integrals from Equation 8.24, which we call the *mean value aspect*.

90 In fact, Gluchoff (1991) does not specify the constraints on C. The differentiability condition shall ensure that T exist. We assume that the curve is simple or simple closed in order to assure that no points of it are "run through" multiple times except for possibly the initial and terminal point. We consider this to be a sensible prerequisite for speaking of the average of a function along a path. Technically, however, our definition of average also works for non-simple or non-simple closed (piecewise) continuously differentiable paths.



(a) Vectors of vector field of $z \longmapsto 1/z^2$, tangential vector to unit circle and their product as complex numbers roughly at the point $e^{\pi i/6}$.

(b) Vectors of vector field of $z \longmapsto 1/z^2$, tangential vector to unit circle and their product as complex numbers roughly at the point $e^{4\pi i/5}$.

Figure 8.13: Plot of vector field for the function $z \longmapsto 1/z^2$ and pieces of the unit circle (black): Vector field is drawn blue, tangential field vector is drawn red, their product in in terms of complex numbers is drawn as a green vector (drawn with the tool by Custy (2011)).

Gluchoff (1991) derives this formula from the Riemann sum approach to define the complex path integral as described in Section 8.1.1. Gluchoff (1991) starts with the complex Riemann sum

$$\sum_{k=1}^{n} f(z_k^*)(z_k - z_{k-1}),$$

where the points $z_0, z_1, \ldots, z_n$ are chosen on C such that their differences have length $L(C)/n$, z_0 and z_n agree with the initial and terminal point of C, z_k^* are points on the curve between z_{k-1} and z_k ($k = 1, \ldots, n-1$). He then chooses $z_k^* = z_{k-1}$ for and obtains the sum

$$\frac{L(C)}{n} \sum_{k=1}^{n} f(z_{k-1}) \frac{(z_k - z_{k-1})}{|z_k - z_{k-1}|}.$$

Taking the limit for $n \longrightarrow \infty$ yields

$$\frac{1}{L(C)} \int_C f(z)\,\mathrm{d}z = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(z_{k-1}) \frac{z_k - z_{k-1}}{|z_k - z_{k-1}|} = \operatorname*{av}_{z\in C}\left(f(z)T(z)\right).$$

If we use the language of the complex path integral as defined on a (piecewise) continuously differentiable path, we obtain the formula

$$\frac{1}{L(\gamma)} \int_\gamma f(z)\,\mathrm{d}z = \operatorname*{av}_{t\in[a,b]} \left(f(\gamma(t)) \frac{\gamma'(t)}{|\gamma'(t)|} \right) = \operatorname*{av}_{t\in[a,b]} \left(f(\gamma(t)) T(\gamma(t)) \right). \tag{8.25}$$

Here, we used that $T(\gamma(t)) = |\gamma'(t)|^{-1}\gamma'(t)$ for all $t \in [a, b]$.

We emphasise that the complex path integral is *not* realised as the average of the integrand f along the curve *but* instead of the function $f \cdot T$ on $C = \operatorname{tr}(\gamma)$. We also emphasise that Gluchoff (1991) did not define the expression $\operatorname{av}_{z\in C}(f(z)T(z))$. Hence, it seems likely that the use of an average for the realisation of complex path integrals causes commognitive conflicts for learners who have learned previously that mean value interpretations for Riemann integrals involve the mean value of the integrand over an interval.

A suitable definition for $\mathrm{av}_{z\in C}(f(z)T(z))$ is

$$\underset{z\in C}{\mathrm{av}}(f(z)T(z)) = \frac{1}{L(C)}\int_C f\cdot T\,\mathrm{d}s,$$

where $\int_\gamma \cdot\,\mathrm{d}s$ signifies the path integral from Equation 8.14 naturally extended to complex functions, that is, $\int_\gamma g\,\mathrm{d}s = \int_\gamma \mathrm{Re}(g)\,\mathrm{d}s + i\int_\gamma \mathrm{Im}(g)\,\mathrm{d}s$ (Ahlfors, 1979, p. 104). This follows immediately from the following computation

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\\ &= \int_a^b f(\gamma(t))\frac{\gamma'(t)}{|\gamma'(t)|}|\gamma'(t)|\,\mathrm{d}t = \int_\gamma f\cdot T\,\mathrm{d}s.\end{aligned}$$

Since multiplication of complex numbers has the geometric interpretation of rotation-dilation (see also Section 8.2.1), Equation 8.24 allow us to interpret the complex path integral as a measure for the amount the tangent vector field associated to γ is rotated and dilated by f (modulo the constant $L(\gamma)$). This also corresponds to the ship metaphor of one of the experts interviewed by Oehrtman et al. (2019) (Section 5.4.3).

We will now review some examples for Equation 8.24 and then analyse the connections between Gluchoff's (1991) mean value interpretation and other mean value interpretations for other integrals.

For the ease of notation, let us write $\mathrm{av}(f,C)$ and $\mathrm{av}(f,\gamma)$ for the right-hand sides in Equation 8.24 and Equation 8.25.

Example 8.5. Let us look at a constant function, that is, $f(z)\equiv c$ for a $c\in\mathbb{C}$. Naturally, we may assume that the average of f on any curve is c.

The (complex) Riemann integral of f over the interval $[a,b]$ is $\int_a^b f(t)\,\mathrm{d}t = c(b-a)$, hence $\frac{1}{b-a}\int_\gamma f(t)\,\mathrm{d}t = c$. That is, the value we expect to be the average of f, namely c, is equal to the Riemann integral of f divided by the difference of b and a, which also happens to be the length of the interval.

For the case of complex path integrals and Gluchoff's average, we have $\int_C f(z)\,\mathrm{d}z = c(B-A)$, where A is the initial and B is the terminal point of γ, and

$$\mathrm{av}(f,C) = \frac{1}{L(C)}\int_\gamma f(z)\,\mathrm{d}z = \frac{c(B-A)}{L(C)}.$$

Hence, $\mathrm{av}(f,C)$ is in general different from c, which we identified as the natural average of f on C. ◇

Without having defined $\mathrm{av}(f,C)$ precisely, Gluchoff (1991) assumes that it can be inferred from the context and illustrates Equation 8.24 with several examples. The examples in his paper basically fall into four categories:

(1) The function $z\longmapsto f(z)T(z)$ reduces to a constant: Then, $\mathrm{av}\,(f,C)$ is this constant.

(2) The function $z \longmapsto f(z)T(z)$ reduces to (a constant multiple of) z: In this case, Gluchoff simply assumes directly that the average of z on the (directed) line segment between the complex numbers A and B is $\frac{A+B}{2}$.

(3) For power functions $z \longmapsto z^n$, $n \neq -1$, along the unit circle traversed counterclockwise Gluchoff (1991, p. 643) assumes that $\mathrm{av}_{|z|=1}(iz^{n+1}) = 0$ because z^{n+1} "spreads around the unit circle around itself uniformly $n+1$ times", thus $\int_{|z|=1} z^n \,\mathrm{d}z = 0$.

(4) Some averages are considered to be intuitively clear whereas other are not. For example, at one point in Gluchoff's (1991) paper the average of $t \longmapsto 1/(t\sqrt{2})$ for t ranging over $[1, 2]$ has to be evaluated. Gluchoff (1991, p. 643) describes that "it is not clear what the average" $\mathrm{av}_{1\leq t\leq 2}(1/t\sqrt{2})$ should be. On the other hand, he describes that "it is intuitively clear that [$\mathrm{av}_{z\in C}\, T(z)$; EH] is 0" when C is "any simple closed path" (Gluchoff, 1991, p. 643). It seems not clear to us why the first average is not clear to Gluchoff while the other is (note in particular that the second assertion is about any curve not only a special case).

Even if some of the examples by Gluchoff (1991) seem plausible from basic experiences with averages, we consider it problematic that neither a definition for $\mathrm{av}(f, C)$ nor a clear method for its computation is given. We believe it is unlikely that students in their second or third year in an undergraduate programme in mathematics find a definition of $\mathrm{av}(f, C)$ on their own. The readers of Gluchoff's (1991) paper are also left with the question of why certain averages should be clear and others not.

Comparison of Gluchoff's mean value interpretation to mean value interpretations of other integrals

Conflicts between the mean value interpretation for complex path integrals and mean value interpretations for other integrals

Recall that the basic idea of average for an integrable real function $g\colon [a, b] \longrightarrow \mathbb{R}$ ($a < b$, Section 6.3), states that the mean value $m(g) := \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x$ satisfies

$$(b-a)m(g) = \int_a^b m(g)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Moreover, the mean value theorem from real analysis guarantees that there exists a $c \in [a, b]$ such that

$$g(c) = m(g) = \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x$$

if g is continuous (Forster, 2016, p. 213).

A direct analogue to this property is not available for complex path integrals as the example $f(z) = \mathrm{e}^{iz}$ and $\gamma(t) = t$ for $0 \leq t \leq 1$ shows: There is no $c \in \mathrm{tr}(\gamma)$ such that

$$f(c) = \int_\gamma f(z)\,\mathrm{d}z$$

(Brown & Churchill, 2009, p. 120). The same example show that no similar mean value theorem holds for Gluchoff's average.[91] Nevertheless, there is a variant of the mean value theorem for

91 Admittedly, the mean value theorem for Riemann integrals in this simple form does not transfer immediately to integrals in multivariate analysis, too, but see for example Janković and Merkle (2008) and Merkle (2015) for some generalisations. Likewise, the mean value theorem for derivatives (e.g., Forster, 2016, p. 181) also does not transfer directly to complex variables (see Qazi, 2006).

complex path integrals: Let L denote line segment from some complex number α to another β (parametrised from α to β), then there are $z_1, z_2 \in L$ such that

$$\int_L f(z)\,\mathrm{d}z = (\beta-\alpha)\,(\mathrm{Re}(f)(z_1) + i\,\mathrm{Im}(f)(z_2))$$

(Rodríguez et al., 2013, p. 109).

More generally, for the Lebesgue integral of a measurable function $h\colon (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B})$, where $(\Omega, \mathcal{A}, \mu)$ is a finite measure space such that $\mu(\Omega) \neq 0$ and $\mathcal{B}$ is a σ-algebra on the real numbers, the mean value of h over Ω is

$$M(h) := \frac{1}{\mu(\Omega)} \int h\,\mathrm{d}\mu.$$

Accordingly, mean value here is the unique number such

$$\mu(\Omega)M(h) = \int M(h)\,\mathrm{d}\mu = \int h\,\mathrm{d}\mu.$$

This mean value property generalises the one for Riemann integrals. In both cases, the reciprocals of the scaling factors in front of the integrals, $(b-a)$ and $\mu(\Omega)$, are the integrals of the function with constant value 1. However, this property does not generalise to Gluchoff's average $\mathrm{av}(f,\gamma)$. Rather, we have

$$\int_\gamma 1\,\mathrm{d}z = \gamma(b) - \gamma(a),$$

and

$$\int_\gamma \mathrm{av}(f,\gamma)\,\mathrm{d}z = (\gamma(b) - \gamma(a))\,\mathrm{av}(f,\gamma),$$

which is in general different from $L(\gamma)\,\mathrm{av}(f,\gamma)$. Yet, $L(\gamma)$ is the scaling factor of $\mathrm{av}(f,\gamma)$ according to Gluchoff's interpretation, not $\gamma(b)-\gamma(a)$.

Average of a complex-valued function along a path

Let us now compare $\mathrm{av}(f,\gamma)$ to the actual *average of the function f along the trace of γ*. We will denote this average by $\widetilde{\mathrm{av}}(f,\gamma)$. It is given as follows. Recall that the real path integral of first kind for a real-valued function g on the trace of γ (Equation 8.14) can be interpreted as the oriented area between the graph of g and the trace of γ (with the usual convention on positive and negative values; see Section B.2.1, Figure B.2).

Let γ be a simple or simple closed path such that the integrals in the following exist. Extending the definition of the path integral of first kind to complex-valued functions f on the trace of γ results in the definition

$$\int_\gamma f\,\mathrm{d}s := \int_\gamma \mathrm{Re}(f)\,\mathrm{d}s + i\int_\gamma \mathrm{Im}(f)\,\mathrm{d}s. \tag{8.26}$$

This is an integral of a complex function with respect to arc length, which is also signified in the literature as $\int_\gamma f|\mathrm{d}z|$ (e.g., Ahlfors, 1979, p. 104; González, 1992, p. 420).

Then, we can define the mean value of f along γ as

$$\widetilde{\mathrm{av}}(f,\gamma) := \frac{1}{L(\gamma)}\int_\gamma f\,\mathrm{d}s = m_u + i m_v. \tag{8.27}$$

Here,

$$m_u = \frac{1}{L(\gamma)} \int_\gamma u \, \mathrm{ds} \quad \text{and} \quad m_v = \frac{1}{L(\gamma)} \int_\gamma v \, \mathrm{ds}$$

are the mean values of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ along γ.

The definition of $\widetilde{\mathrm{av}}(f,\gamma)$ in Equation 8.27 is compatible the previously established mean value properties of Riemann and Lebesgue integrals. Namely, it satisfies

$$\widetilde{\mathrm{av}}(f,\gamma)L(\gamma) = \int_\gamma \widetilde{\mathrm{av}}(f,\gamma) \, \mathrm{ds} = \int_\gamma f \, \mathrm{ds}.$$

Gluchoff's average $\mathrm{av}(f,\gamma)$ may then also be expressed

$$\mathrm{av}(f,\gamma) = \frac{1}{L(\gamma)} \int_\gamma f \cdot T \, \mathrm{ds}. \tag{8.28}$$

For this average, we found contrariwise

$$\mathrm{av}(f,\gamma)\,(\gamma(b) - \gamma(a)) = \int_\gamma \mathrm{av}(f,\gamma) \, \mathrm{d}z.$$

Example 8.6. Let us again consider the example from Example 8.2 and Example 8.4, that is, f is given by $f(z) = x + 2y + iy^2$ ($x = \mathrm{Re}(z), y = \mathrm{Im}(z)$) and γ is the standard parametrisation of the unit circle. We already know that $\int_\gamma f(z)\,\mathrm{d}z = -2\pi + i\pi$. The average of f along γ is

$$\begin{aligned}
\widetilde{\mathrm{av}}(f,\gamma) &= \frac{1}{L(\gamma)} \int_\gamma f \, \mathrm{ds} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\cos(t) + 2\sin(t) + i\sin(t)^2\right) \cdot 1 \, \mathrm{d}t = \frac{i}{2} \\
&\neq -1 + \frac{i}{2} = \frac{1}{L(\gamma)} \int_\gamma f(z) \, \mathrm{d}z.
\end{aligned}$$

In particular, we see that $\widetilde{\mathrm{av}}(f,\gamma) \neq \frac{1}{L(\gamma)} \int_\gamma f(z)\,\mathrm{d}z$ as expected. ◇

Finishing this section, we stress that $\widetilde{\mathrm{av}}(f,\gamma)$ is the accurate interpretation of the average of the function f along a path γ in our view since it fits to the average interpretations for the other integrals we recalled. Clearly, Gluchoff (1991) does not describe that his average formula involves the mean value of f along γ, but of $f \cdot T$. But we think that the average $\widetilde{\mathrm{av}}(f,\gamma)$ fits more closely to students' precedents with mean values for other integrals. This average however does not relate in a simple way to the complex path integral.

Mean values, Cauchy's integral theorem, and Cauchy's integral formula

Another mean value interpretation of complex-valued functions can be obtained from Cauchy's integral formula (Theorem A.22). However, it requires that the function is holomorphic. Let $f\colon \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and $B_r(z_0)$ an open ball such that $\overline{B_r(z_0)} \subseteq \Omega$. Then, Cauchy's integral formula is

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

for $z_0 \in B_r(z_0)$. That is, the value of f at any point in the circle can be realised as the complex path integral of $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ along the positively oriented boundary of $B_r(z_0)$ divided by $2\pi i$. Now, we choose z to be the centre of the ball, $z = z_0$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{it}\right) \mathrm{d}t.$$

According to this formula, $f(z)$ is the mean value of f along the boundary of the circle $B_r(z)$ because the integrand here runs through the function values of f on the boundary of the ball divided by 2π (not $2\pi r = L(\partial B_r(z))$ though if $r \neq 1$). This mean value is also not obtained as a complex path integral of f alone but of $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$. In fact, f itself cannot be the right choice for the integrand because Cauchy's integral theorem (Theorem A.17) implies that $\int_{\partial B_r(z)} f(\zeta)\, \mathrm{d}\zeta$ vanishes.

Example 8.7. Let us continue with Example 8.5.[92] We will compute the complex path integral and Cauchy's integral formula for the constant function $f \equiv c$ along the boundary of $B_r(0)$ as well as the two versions of average $\mathrm{av}(f, \partial B_r(0))$ and $\widetilde{\mathrm{av}}(f, \partial B_r(0))$ we considered previously. Note that $T(z) = iz$ for $z \in \partial B_r(0)$.

- By Cauchy's integral theorem or a direct computation, we see that

$$\int_{\partial B_r(z)} f(z)\, \mathrm{d}z = 0.$$

- This agrees with Gluchoff's average value

$$\mathrm{av}(f, \partial B_r(0)) = \frac{1}{L(\partial B_r(0))} \int_{\partial B_r(0)} f \cdot T\, \mathrm{d}s = \frac{1}{2\pi r} \int_0^{2\pi} cre^{it}\, \mathrm{d}t = 0.$$

- The second average value we defined computes to

$$\widetilde{\mathrm{av}}(f, \partial B_r(0)) = \frac{1}{L(\partial B_r(0))} \int_{\partial B_r(0)} f\, \mathrm{d}s = \frac{1}{2\pi r} \int_0^{2\pi} cr\, \mathrm{d}t = c,$$

- which is in line with Cauchy's integral formula

$$\frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(z)}{z - 0}\, \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} c\, \mathrm{d}t = f(0) = c. \qquad \diamond$$

8.3 COMPLEX PATH INTEGRALS AND PRIMITIVE FUNCTIONS

In this section, we analyse under whether circumstances the anti-derivative aspect (Section 6.3.2) may be extended to complex analysis. Therefore, we will compare the conditions for the existence of anti-derivatives for real-valued functions of one real variable, for vector fields of two real variables, and for complex functions. We have already argued that the anti-derivative aspect is not fully appropriate for Riemann integrals at university level and that it

92 This example is also discussed in (Hanke, 2022b).

may rather serve as a partial aspect. In particular, in order to develop an aspect from the formula in (6.1) we need to guarantee the existence of a primitive function or for a given function. We explained that this is indeed possible for holomorphic functions on simply-connected domains (Section 7.5.2). However, the existence of primitive functions for complex functions may be approached with the similar question from real analysis in one and two variables and thus it is worthwhile to study these connections with regards to potential vertical coherence in the teaching of integrals in undergraduate mathematics curricula. Additionally, we emphasise many equivalent conditions about the independence of complex path integrals with respect to the chosen path of integration, the existence of anti-derivatives, and the integrand being holomorphic, which Rodríguez et al. (2013) summarised as the Fundamental theorem of complex function theory (Theorem A.26).

8.3.1 *Existence of holomorphic primitives*

The fundamental theorem of calculus Theorem B.3 has a direct analogue in the complex setting. One part of the fundamental theorem of integral calculus (Theorem B.3) asserts that if $g\colon [a,b] \longrightarrow \mathbb{R}$ is a continuous function and $c \in [a,b]$ is any point, then the function $G\colon [a,b] \longrightarrow \mathbb{R}$ defined by

$$G(x) := \int_c^x g(t)\,\mathrm{d}t \tag{8.29}$$

is a primitive function for g. Moreover, for any $\alpha, \beta \in [a,b]$, we have

$$\int_\alpha^\beta g(t)\,\mathrm{d}t = G(\beta) - G(\alpha).$$

Complex version of the fundamental theorem of calculus

The second part transfers directly to complex path integrals: If $f\colon \Omega \longrightarrow \mathbb{C}$ is continuous, $F\colon \Omega \longrightarrow \mathbb{C}$ is an anti-derivative for f, and $\gamma\colon [a,b] \longrightarrow \Omega$ is a (piecewise) continuous differentiable path, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{8.30}$$

This follows from the fact that $t \longmapsto F(\gamma(t))$ is a primitive of $t \longmapsto f(\gamma(t))\gamma'(t)$ on $[a,b]$ and an application of the real version of the fundamental theorem to the real and imaginary part of $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$. In particular, the complex path integral only depends on the initial and terminal point of γ in this case, but not the rest of the path. Some authors use this proposition to justify the signifier

$$\int_{z_0}^z f(z)\,\mathrm{d}z,$$

where the integral is the complex path integral along any (piecewise) continuously differentiable path from z_0 to z in Ω (e.g., Silverman, 1974, p. 64). Consequently, if the integrand possesses a primitive, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ whenever γ is closed. Silverman (1974, p. 64) summarises Equation 8.30 as follows (considering the special case of a holomorphic integrand though):

> This immediately leads to the formula
>
> $$\int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta = \Phi(z) - \Phi(z_0)$$
>
> expressing the 'definite integral' on the left as a difference between the values of the indefinite integral at the end points of the path of integration. Thus, confining ourselves to the case of functions analytic [holomorphic; EH.] in a simply-connected domain, we see that complex integration (just like real integration) can be regarded both as a process of summation [...] and as the operation inverse to differentiation.

However, in complex analysis a major difference to analysis in one real variable occurs. It is *not* the case that the continuity of the integrand f guarantees the existence of an antiderivative. Similarly, we know that continuity is too weak to guarantee the existence of potentials for vector field. Here, we need to require additionally that the vector field is continuously differentiable, satisfies the integrability condition and its domain is simply-connected (Theorem B.13).

Example 8.8. Let us consider two simple examples. The function $z \longmapsto \overline{z}$ is continuous on the simply-connected domain $\mathbb{C}$ and

$$\int_{|z|=r} \overline{z}\,\mathrm{d}z = 2\pi r^2 i.$$

If the integrand had a primitive, then the integral should have been 0. The function $z \longmapsto \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but this domain is not simply-connected (and cannot be extended further). In this case, the integral along the same path does not evaluate to 0 either:

$$\int_{|z|=r} \frac{1}{z}\,\mathrm{d}z = 2\pi i. \qquad \diamond$$

Construction of primitive functions

Existence of primitive function for holomorphic functions

The existence of a primitive of f is guaranteed though, if Ω is *simply-connected*, or can be enlarged to a simply-connected domain, on which f is *holomorphic*. In this case, a primitive can be constructed as follows, mimicking the construction of a primitive or potential in real analysis (Equation 8.29): Fix any point $\omega \in \Omega$ and set

$$F(z) := \int_{\gamma_{\omega,z}} f(\zeta)\,\mathrm{d}\zeta \tag{8.31}$$

for any (piecewise) continuously differentiable path $\gamma_{\omega,z}$ from ω to z in Ω. It can be shown that the integral does not depend on the choice of the path unless it starts at ω and ends at z, thus F is a well-defined function on Ω. Moreover, F is holomorphic and $F' = f$.

The crucial point here is that the well-definedness of F means that the integral in Equation 8.31 does not depend on the actual choice of the path from ω to z (except for that its initial point is ω and its terminal point is z). This independence property is equivalent to the vanishing of the complex path integral for closed (piecewise) continuously differentiable paths in the domain of f, that is, to Cauchy's integral theorem (Theorem A.17). In fact, many more conditions on f or path integrals involving f are equivalent in this context. These connections are summarised in the Fundamental theorem of complex function theory (Theorem A.26)

(Rodríguez et al., 2013, pp. 3–4). In a way, this means that holomorphicity and the property that complex path integrals only depend on the integrand and the initial and terminal point of a path are very strong and essentially equivalent properties of a continuous function and even equivalent to the existence of a holomorphic primitive function.

Example 8.9. There is no primitive for the function f from Example 8.2, Example 8.4, and Example 8.6, even though its domain $\mathbb{C}$ is simply-connected and f is continuous. This follows from the fact that $\int_\gamma f(z)\,\mathrm{d}z \neq 0$, where γ parametrises the unit circle counterclockwise, but also from the observation that f is not holomorphic because it does not satisfy the Cauchy-Riemann differential equations (Equation A.6). ◇

We emphasise again that we needed to impose additional constraints on f beyond its continuity in order to ensure that is has a primitive. Since the holomorphicity of f and the simple connectedness of Ω are restricting the general setting for which we would like to define complex path integrals quite a lot (i.e., continuous integrands and (piecewise) continuously differentiable paths), Equation 8.30 cannot be used as a full aspect of complex path integrals. Rather, this equation is only endorsable as a partial aspect of the complex path integral under the additional constraints we have just mentioned.

Moreover, if we defined complex path integrals in terms of the difference between the values of a primitive function for the integrand, we would not be able compute integrals like

$$\int_{\partial B_r(c)} \frac{1}{z-c}\,\mathrm{d}z = 2\pi i, \tag{8.32}$$

which are at the heart of complex analysis. In fact, the property that this integral does not vanish is essential for complex analysis and needed to prove many theorems like Cauchy's integral formula (Theorem A.22) and Residue theorem (Theorem A.29) etc.:

> *If the integral [in Equation 8.32; EH.] vanished, too, there would be no complex analysis at all.* (Remmert & Schumacher, 2002, p. 156, own transl., emph. orig.)

Similarly, Klazar (2019b, p. 2) appreciates the result $\int_{\partial B_1(0)} \frac{1}{z}\,\mathrm{d}z \neq 0$ as

> an under-appreciated pillar of complex analysis: if [$\int_{\partial B_1(0)} \frac{1}{z}\,\mathrm{d}z$; EH.] were 0, no Cauchy formulae [...] would exist and the complex analysis would collapse.

Defining complex path integrals along arbitrary paths with the help of primitive functions—even without integration at all

Anti-derivative partial aspect of complex path integrals

Nevertheless, there is a theoretical question, for which it is very helpful to realise complex path integrals as the difference of values of primitives. The question is whether the definition of complex path integrals can be extended to *arbitrary* paths instead of (piecewise) continuously paths or rectifiable paths. The answer is affirmative in case we restrict the class of integrands to *holomorphic* functions (Section A.10). The main ingredient for this extension is the property that holomorphic functions on simply-connected domains possess primitives (Theorem Existence of primitives for holomorphic functions (Theorem A.20)). Since balls are simply-connected, there is an open ball around each point in the domain of a holomorphic function, on which it has a primitive. Then, we may cover the trace of an arbitrary path with partly overlapping balls and add up local versions of Equation 8.30 for each local primitive.

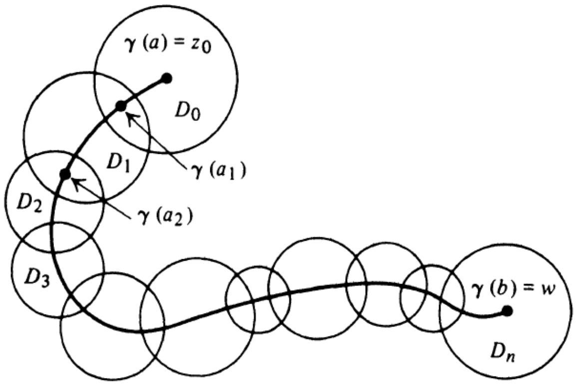


Figure 8.14: Covering of a path with partly overlapping balls (Figure 1 from Lang (1999, p. 323)).

More precisely, assume that $\gamma\colon [a,b] \longrightarrow \Omega$ is any path in the domain of a holomorphic function $f\colon \Omega \longrightarrow \mathbb{C}$. We can find a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$ and open balls $D_1, \ldots, D_n \subseteq \Omega$ such that $\gamma([t_{k-1}, t_k]) \subseteq D_k$ and f has a primitive function F_k on each D_k locally for each $k = 1, \ldots, n$ (see Figure 8.14). Then, one can prove that the definition

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{k=1}^{n} \left(F_k(t_k) - F_k(t_{k-1})\right) \tag{8.33}$$

is indeed well-defined (i.e., the right-hand side of this equation does not depend on the choice of the subdivision of $[a,b]$, the balls, and the local primitives). Additionally, it is sufficient that the path is continuous. (Lang, 1999, ch. III. §4; see Section A.10 for more details and another alternative to Equation 8.33)

Moreover, since the existence of local primitive functions for holomorphic functions can be proven without the use of integration (Section 7.5.2), Equation 8.33 may in fact serve as a partial aspect for complex path integrals of holomorphic functions along arbitrary paths, which we may call the *anti-derivative partial aspect*.

8.3.2 *Comparison to analysis of two real variables*

It is instructive to compare the existence of primitives for holomorphic functions to the existence of potentials for vector fields of two real variables. After all, any complex function may be naturally identified with a vector field (or the corresponding Pólya vector field), and we have seen how this change of perspective leads to several realisations of complex path integrals in terms of real path integrals of second kind (Section 8.1.4, Section 8.2.2) (see Ahlfors, 1979, ch. 4.1.3; Apostol, 1981; Lang, 1999, app. §6; Forster, 2017a; Königsberger, 2004b, for references to this subsection).

Assume that $\mathbf{F} = (M, N)^T\colon \Omega \longrightarrow \mathbb{R}^2$ is a continuously differentiable vector field on a domain $\Omega \subseteq \mathbb{R}^2$. A *potential for* $\mathbf{F}$ is a continuously differentiable function $\varphi\colon \Omega \longrightarrow \mathbb{R}$ whose gradient is $\nabla\varphi = \mathbf{F}$. In this case, $\mathbf{F}$ is also called a gradient field. We say that $\mathbf{F}$ satisfies the *integrability condition* if the partial derivatives satisfy $\partial_2 M = \partial_1 N$ on Ω. Note that this corresponds to one of the Cauchy-Riemann differential equations (Equation A.6) for the complex function $M - iN$.[93] We may also phrase this in terms of differential forms: $M\,\mathrm{d}x + N\,\mathrm{d}y$ is

93 The minus sign in $M - iN$ should not be surprising here: $\mathbf{F} = (M, N)^T$ is the Pólya vector field associated to $M - iN$ (see Section 8.1.4, Section 8.2.2).

called *exact* if $\mathbf{F}$ is a gradient field, and $M\,\mathrm{d}x + N\,\mathrm{d}y$ is called *closed* if $\mathbf{F}$ satisfies the integrability condition.

Furthermore, if $\mathbf{F}$ has the potential φ, then the real path integral of second kind of $\mathbf{F}$ along γ can be realised as

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \varphi(\gamma(b)) - \varphi(\gamma(a)),$$

similar to Equation 8.30.

The Poincaré lemma (Theorem B.13) guarantees the existence of a potential for $\mathbf{F}$ if $\mathbf{F}$ satisfies the integrability condition and Ω is simply-connected. Hence, the existence of a potential for a continuously differentiable vector field is also tied to the simple connectedness of its domain and to a differential equation, just like we have seen for the existence of a primitive function for a holomorphic function. However here, only one instead of two differential equations (Equation A.6) has to be satisfied though.

Let us now assume that Ω is simply-connected. Then, again mimicking Equation 8.29, we can construct a potential φ for $\mathbf{F}$ as follows: We fix any $(x_0, y_0)^T \in \Omega$ and set

$$\varphi(x, y) := \int_{(x_0,y_0)^T}^{(x,y)^T} M\,\mathrm{d}x + N\,\mathrm{d}y \tag{8.34}$$

for $(x, y)^T \in \Omega$, where the integral is taken along any (piecewise) continuously differentiable path in Ω with initial point $(x_0, y_0)^T$ and terminal point $(x, y)^T$. Similar to the complex case, the integral here is independent of the actual choice of the path from $(x_0, y_0)^T$ to $(x, y)^T$ in Ω and it can be shown that $\nabla\varphi = \mathbf{F}$.

We will now transfer these propositions about vector fields and potentials to the complex setting. In other words, if these results about vector fields and potentials have been established previously, one can use them to answer the existence problem of primitive functions for holomorphic functions, too. Hence, we will present another way to substantiate the existence of primitive functions for holomorphic functions. For that purpose, let $f = u + iv$ be a holomorphic function. Since we required that the vector fields were continuously differentiable, we also require here that f' is continuous.[94] We have already seen in Equation 8.17 that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (u, -v)^T\,\mathrm{d}\mathbf{T} + i\int_\gamma (v, u)^T\,\mathrm{d}\mathbf{T}$$

for any (piecewise) continuously differentiable path γ in Ω. By the Cauchy-Riemann differential equations Equation A.6, $(u, -v)^T$ and $(v, u)^T$ each satisfy the integrability condition. Thus,

$$\varphi(x, y) := \int_{(x_0,y_0)^T}^{(x,y)^T} u\,\mathrm{d}x - v\,\mathrm{d}y, \quad \text{and} \quad \psi(x, y) := \int_{(x_0,y_0)^T}^{(x,y)^T} v\,\mathrm{d}x + u\,\mathrm{d}y$$

94 In fact, the continuity of the derivative is a consequence of the holomorphicity of f. However, for the transfer of propositions about the existence of a potential, we need to require this technical constraint additionally at this point. This said, the assumption that f' is continuous is stronger than the holomorphicity of f alone. We go on with this condition here nevertheless because we consider it beneficial to discuss the search for potentials and primitive functions from the point of view of vector analysis, too. Establishing these points of view may allow students to deepen the connections they see between real and complex analysis.

give rise to potentials φ and ψ of $(u, -v)^T$ and $(v, u)^T$ on Ω. Consequently, identifying complex numbers with vectors in $\mathbb{R}^2$, we see that

$$\begin{aligned} F(z) &:= \int_{z_0}^{z} f(z)\,\mathrm{d}z := \int_{(x_0,y_0)^T}^{(x,y)^T} u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_{(x_0,y_0)^T}^{(x,y)^T} v\,\mathrm{d}x + u\,\mathrm{d}y \\ &= [\varphi(x,y) + i\psi(x,y)] \end{aligned}$$

gives rise to a primitive F of f on Ω.

Example 8.10. Let us look another time at the function from Example 8.2, Example 8.4, Example 8.6, and Example 8.9. The entries of the vector fields $\mathbf{w} = (u, -v)^T$ and $(v, u)^T$ are given by $u(x, y) = x + 2y$ and $v(x, y) = y^2$. We have $\partial_1 u(x, y) = 1$, $\partial_2 u(x, y) = 2$, $\partial_1 v(x, y) = 0$, and $\partial_2 v(x, y) = 2y$. Hence, neither $(u, -v)^T$ nor $(v, u)^T$ is a gradient field because the integrability conditions for these two vector fields, $\partial_2 u = -\partial_1 v$ and $\partial_2 v = \partial_1 u$, are not fulfilled. These are either never fulfilled or only on a non-open set. Furthermore, $u + iv$ is not holomorphic. $\diamond$

From the existence of potentials for vector fields in real analysis, the existence of a primitive of a complex-valued holomorphic function could be proven. This was possible because the Cauchy-Riemann differential equations for $f = u + iv$ corresponded to the integrability conditions for the vector fields $(u, -v)^T$ and $(v, u)^T$.

Last but not least, we may ask whether it is also possible to start with the proposition on the existence of primitives for holomorphic functions in order to prove the existence of potentials for vector fields satisfying the integrability condition. This is not possible if we try to reverse the procedure above because in the complex case, the real and imaginary part of $f = u + iv$ had to satisfy two differential equations instead of only one, which is the case in the integrability condition for vector fields. In this sense, the proposition on the existence of potentials of continuously differentiable vector fields is stronger than the proposition about the existence of primitives for holomorphic functions (at least, if we require the continuity of f). This was also observed by Pringsheim (1903) in his paper on Theorem Goursat's lemma (Theorem A.19). Pringsheim developed of proof of Goursat's lemma though, which can be transferred into a proof of an analogue of Goursat's lemma for vector fields.

Example 8.11. In this example, we illustrate the case that a vector field $(u, -v)^T$ may satisfy the integrability condition but that $u + iv$ is not holomorphic (the minus sign in the vector field is only chosen for convenience, because then $(u, -v)^T$ is the Pólya vector field for $u + iv$).

Define u and v on $\mathbb{R}^2$ by $u(x, y) = 2xy$ and $v(x, y) = -x^2$. Then, $(u, -v)^T$ is a gradient field because it satisfies the integrability condition $\partial_2 u(x, y) = 2x = \partial_1(-v)(x, y)$ on $\mathbb{R}^2$. A potential is given by $\varphi(x, y) = yx^2$. However, $u + iv$ is not holomorphic: Only one of the Cauchy-Riemann differential equations (Equation A.6) is fulfilled, namely $\partial_2(x, y) = -\partial_1 v(x, y)$, but the other is not, namely $\partial_1 u(x, y) = 2y \neq 0 = \partial_2 v(x, y)$ (except for $y = 0$). $\diamond$

Overall, we think that it is important to point out in class the commonalities and differences in the existence problems of anti-derivatives in analysis of one and two variables and in complex analysis. It is particularly relevant to emphasise that the constraints on the function, for which we wish to find an anti-derivative, become stronger, and to point to the core reasons why this is the case. The three cases are as follows:

- Every *continuous* function $g\colon [a, b] \longrightarrow \mathbb{R}$ has a primitive function.

- Every *continuously differentiable* vector field $\mathbf{F}\colon \Omega \longrightarrow \mathbb{R}^2$ on a *simply-connected domain* $\Omega \subseteq \mathbb{R}^2$ has a potential.
- Every *holomorphic* function $f\colon \Omega \longrightarrow \mathbb{C}$ on a *simply-connected domain* $\Omega \subseteq \mathbb{C}$ has a holomorphic primitive function.

The key observation is that the candidates for the anti-derivatives in Equation 8.29, Equation 8.31, and Equation 8.34 are well-defined. In the last two cases, this well-definedness is equivalent to the independence of the path integrals from the chosen path (except for its dependence on the initial and terminal point), which is in turn equivalent to the vanishing of the path integrals along closed paths (see also Theorem A.26 and Theorem B.12).

8.4 A COVARIATIONAL POINT OF VIEW ON COMPLEX PATH INTEGRATION

In this last section of the epistemological analysis, we take a bird's eye view on the use of complex path integrals in an introductory course or textbook on complex analysis. Here, we see three milestones, which are dealt with in basically all first courses and textbooks on complex analysis: (1) The statement and proof of a version of Goursat's lemma (Theorem A.19) and Cauchy's integral theorem (Theorem A.17) each, (2) the proof holomorphic functions are analytic, and (3) the statement and proof a variant of the Residue theorem (Theorem A.29) and its applications. Of course, this is a rather short list and contains a lot of implicit contents to be covered. Nevertheless, our observations are in line with Bottazzini and Gray (2013, ch. 10.1), who describe that there is "remarkable degree of consensus [..] about what constitutes elementary complex function theory" (Bottazzini & Gray, 2013, p. 691). Additionally, we consider a comprehensive account on the equivalent properties, which characterise the holomorphicity of complex functions, which include properties related to complex path integrals as well (e.g., such a list is presented in Theorem A.26).

Our goal in this section is to analyse the properties of the complex path integral, which are used to achieve these milestones. Thus, we will detour a little from the substantiation of the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ as a complex number for a given path and function at first. Instead, we are interested here in the *covariation*[95] of mappings of the form $I\colon (\gamma, f) \longmapsto I(\gamma, f)$, whose arguments are the integrands and the paths. In this vein, Greenleaf (1972, p. 265, italics orig.) describes that

> [i]t is appropriate to think of the line integral [complex path integral; EH.] as an **operation** [emph. EH.] that puts together a contour γ and a function $f(z)$ to give a complex number $\int_\gamma f(z)dz$ (subject to the requirement that f be defined on the trajectory of γ, of course). This operation depends *linearly* on the integrand [...] and *additively* on the contour [...].

In this section, we thus ask which properties (i.e., object-level rules) of this "operation" are used in complex analysis and which completely characterise the complex path integral. We consider the study of the covariation of mappings of the form as above as important since it

95 *Covariation* is one of the three basic ideas (Section 2.2.2) for the notion of function (e.g., Greefrath et al., 2016b; vom Hofe, 1995; vom Hofe & Blum, 2016; Vollrath, 1989; Carlson et al., 2002, see also). This basic idea emphasises that a change in the arguments of a function changes its function values in a specific manner and asks for a description of this dependency (Greefrath et al., 2016b, p. 48; vom Hofe & Blum, 2016, p. 248). The other basic ideas of functions are *mapping*, which emphasises that a function assigns one value for a given input, and *object*, which emphasises that a function is a mathematical object "as a whole" (vom Hofe, 2003; Vollrath, 1989; Greefrath et al., 2016b, pp. 47–50). Whereas we do not follow basic ideas as a theoretical element for our epistemological analysis, this idea nevertheless captures quite well what we are about to analyse in this section.

amounts to study the use of the properties of complex integrals. This is a beneficial point of view for students but also for lecturers creating a "red thread" in their lectures.

Since it is possible to characterise Riemann integrals for continuous functions abstractly (Section 8.4.1), we ask whether this is also possible for complex path integrals. The guiding question is which conditions need to be imposed on such a mapping in order to fully characterise $\int_\gamma f(z)\,\mathrm{d}z$ (of course, except for simply saying that $I(f,\gamma) := \int_\gamma f(z)\,\mathrm{d}z$). In particular, we review one approach in this direction by Klazar (2019a, 2019b, 2020), which we will further discuss and generalise (Section 8.4.2). We also derive a list of properties of complex path integrals, which characterise its use in many textbooks or lecture notes (Section 8.4.3) and exemplify this use with a proof of Cauchy's integral formula (Theorem A.22). Then, in Section 8.4.4, we will answer the quest for an axiomatic characterisation of complex path integrals in the affirmative, however restricting the class of integrands to holomorphic function (Theorem 8.13).

8.4.1 *Intermezzo: Axiomatic characterisations of the Riemann integral*

Axiomatic characterisations of the Riemann integral

Let us start by briefly recalling two possibilities to define the Riemann integral for certain classes of functions on an interval of real numbers $[a, b]$ axiomatically for motivational purposes. We use the word "axiomatic" relatively loosely here in order to imply that we are interested in a list of properties for a mapping, which guarantees that this mapping agrees with the integral in a way to be specified in context. The first possibility works for continuous integrands and the other for regulated functions as an extension of a positive linear functional on the space of step functions on an interval of real numbers (Section 6.3.2).[96]

For this purpose, let $f\colon D \longrightarrow \mathbb{R}$ be a real function defined on an interval D of real numbers.

Axiomatic definition of the Riemann integral for continuous functions

Assume that f is continuous. Assume further that the function

$$\begin{aligned} \mathrm{S}\colon \underline{D} := \left\{(a,b) \in D^2 : a \le b\right\} &\longrightarrow \mathbb{R}, \\ (a,b) &\longmapsto \mathop{\mathrm{S}}_{a}^{b} f \end{aligned}$$

satisfies the following to two properties:

(1) $\mathrm{S}_a^b f + \mathrm{S}_b^c f = \mathrm{S}_a^c f$ for all $(a,b), (b,c) \in \underline{D}$ and

(2) for all $(a,b) \in \underline{D}$ and constants $m, M \in \mathbb{R}$ the inequality $m \le f(x) \le M$ for all $x \in [a,b]$ implies $m(b-a) \le \mathrm{S}_a^b f \le M(b-a)$.

Then, there is one and only one such function S and it satisfies $\mathrm{S}_a^b f = \int_a^b f(x)\,\mathrm{d}x$ (Gillman, 1993; Herfort & Reinhardt, 1980; Hernández Rodríguez & López Fernández, 2013; Tietze et al., 2000, p. 200). Consequently, the Riemann integral for *continuous functions* can be defined via the two properties above.[97]

96 For didactical impulses from this axiomatic point of view on Riemann integrals see Gillman (1993), Herfort and Reinhardt (1980), Pickert (1976), and Tietze et al. (2000). See also Burgin (2012), Daniell (1918), Leinert (1995), Shenitzer and Steprāns (1994), and Taylor (1985) for more about axiomatic characterisations of integrals.

97 For the usability of this approach see for example Wasserman et al. (2022, ch. 13). They prove the existence of primitive functions for continuous real-valued functions of a real variable on intervals of real numbers by essentially resorting to these two properties only.

Extension of a positive linear functional

A variant of the Riemann integral using approximations with step functions was briefly touched in Section 6.3. We define a positive linear form L on the real vector space $\mathcal{T}(D)$ of real-valued step functions on D. For a step function $t \in \mathcal{T}(D)$ there exist a partition $a = x_0 \leq \ldots \leq x_m = b$ and real numbers $c_k \in \mathbb{R}$ such that $t(x) \equiv c_k$ for $x \in (x_{k-1}, x_k)$ $(k = 1, \ldots, m)$, and we set

$$L(t) := \sum_{k=1}^{m} c_k (x_k - x_{k-1})$$

which is well-defined (Amann & Escher, 2006, p. 18). Vividly spoken, L measures the area that is enclosed by the pieces of the graph of a step function t, the horizontal axis, and the vertical axes at $x = a$ and $x = b$, where the areas above the x-axis are weighted positively and the areas below are weighted negatively. L is linear, which means that $L(\alpha s + t) = \alpha L(s) + L(t)$ for step functions $s, t \in \mathcal{T}(D)$ and $\alpha \in \mathbb{R}$. L is also positive, which means that $L(t) \geq 0$ for $t \in \mathcal{T}(D)$ such that $t \geq 0$.

Since $\mathcal{T}(D)$ is dense in the Banach space of regulated functions $\mathcal{R}(D)$ (see footnote 64 and footnote 65), L extends to a uniquely to a linear functional on $\mathcal{R}(D)$, namely $f \longmapsto \int_a^b f(x)\,\mathrm{d}x$ (Amann & Escher, 2006, ch. VI).

The quintessence here is that it is possible to define a rather "simple form of integral", namely $L(t)$ for step functions $t \in \mathcal{T}(D)$, which then extends uniquely to a larger set of functions, and hence defines an integral for these functions.

Let us now turn again to the case of complex path integrals.

8.4.2 *Mappings modelling complex path integrals*

Klazar (2019a, 2019b, 2020) used a covariational point of view in his lectures on complex integration and his approach serves as a model for ours in Section 8.4.4. His goal was to include three lectures on complex analysis into his course on real analysis in order to prove the proposition that holomorphic functions are analytic. To derive this theorem he defines complex path integrals along line segments and rectangular paths only and proves versions Cauchy's integral theorem and integral formula for rectangular paths.[98]

Klazar (2019b) defines the set

$$H := \{f \colon \mathbb{C} \setminus A \longrightarrow \mathbb{C} : f \text{ holomorphic}, \ A \subseteq \mathbb{C} \text{ compact}\}$$

and the mapping

$$\oint\!\!\!\!\!\square \colon H \longrightarrow \mathbb{C}, \quad f \longmapsto \oint\!\!\!\!\!\square\, f := \int_{\partial R} f(z)\,\mathrm{d}z,$$

where $f \colon \mathbb{C} \setminus A \longrightarrow \mathbb{C}$ is holomorphic for some compact set $A \subseteq \mathbb{C}$, R is a rectangle such that $A \subseteq \operatorname{int}(R)$, and ∂R denotes a closed path that winds around the rectangle once counterclockwise. In fact, $\oint\!\!\!\!\!\square$ is well-defined, that is, it does not depend on the choice of the rectangle with the required condition. Klazar (2019b) proves this with a variant of Cauchy's integral theorem

98 More precisely, Klazar uses the approach to define complex path integrals via complex Riemann sums (Section 8.1.1). In the third lecture, Klazar (2020) remarks that it was not really necessary to introduce the mapping $\oint\!\!\!\!\!\square$. While this may be true, we consider it didactically very valuable to list the properties of $\oint\!\!\!\!\!\square$ needed later and to emphasise their use.

(Theorem A.17) for rectangular paths. Then, Klazar (2019b, 2020) lists and proves the following properties of $\oint$ and the complex path integral:

1. $\oint$ is $\mathbb{C}$-linear, that is, $\oint(\alpha f + \beta g) = \alpha \oint f + \beta \oint g$ for $\alpha, \beta \in \mathbb{C}$ and $f, g \in H$;
2. $\oint f = 0$ for every holomorphic function of the form $f\colon \mathbb{C} \setminus A \longrightarrow \mathbb{C}$, $A = \{a\}$, which is bounded in a neighbourhood of $a \in \mathbb{C}$;
3. $\oint f = \rho$ for $f\colon \mathbb{C} \setminus A$, $z \longmapsto \frac{1}{z-a}$, $A = \{a\}$, $a \in \mathbb{C}$, and a constant $\rho \in \mathbb{C} \setminus \{0\}$ independent of a; and
4. $\lim_{n\to\infty} \oint f_n = \oint f$ for all $f, f_1, f_2, \ldots\colon \mathbb{C} \setminus A \longrightarrow \mathbb{C}$, $A \subseteq \mathbb{C}$ compact, $R \subseteq \mathbb{C}$ a rectangle such that $A \subseteq \operatorname{int}(R)$, and $f_n \longrightarrow f$ uniformly on ∂R;
5. and Darboux's inequality (Equation A.16) for line segments and rectangular paths.

With these properties, Klazar (2020) proves Cauchy's integral formula (Theorem A.22) for rectangular paths, Liouville's theorem (Lang, 1999, p. 130), the analyticity of an entire functions, and Residue theorem (Theorem A.29) for rectangular paths (he proved a variant of Goursat's lemma (Theorem A.19) and Cauchy's integral formula before introducing $\oint$). It is instructive to see that proofs of these important theorems of complex analysis can be done with a rather formal approach using only a few properties of $\oint$ and the complex path integral for rectangular paths (but of course, these properties need to be derived in one way or another, too).

Modelling the complex path integral with residues and winding numbers

Note that the mapping $\oint$ models a special case of the Theorem Residue theorem (Theorem A.29): If $A \subseteq \mathbb{C}$ is finite and $f\colon \mathbb{C} \setminus A \longrightarrow \mathbb{C}$ is holomorphic, then

$$\oint f = 2\pi i \sum_{\omega \in A} \operatorname*{Res}_{\omega}(f),$$

where the integral on the left hand side is taken along the path that winds once around a rectangle R such that $A \subseteq \operatorname{int}(R)$ counterclockwise.

More generally, we may consider the mapping

$$\int_{\mathrm{res}}^{\Omega}\colon (\gamma, f, A) \longmapsto 2\pi i \sum_{\omega \in A} \operatorname*{Res}_{\omega}(f) \operatorname*{Ind}_{\gamma}(\omega), \tag{8.35}$$

where Ω is a simply-connected domain, γ a closed (piecewise) continuously differentiable path in Ω, and $A \subseteq \Omega$ is a finite set, and $f\colon \Omega \setminus A \longrightarrow \mathbb{C}$ is holomorphic. This values of this mapping correspond to the evaluation of the integral $\int_\gamma f(z)\,\mathrm{d}z$ with the help of the Residue theorem (Theorem A.29). The mapping $\int_{\mathrm{res}}^{\Omega}$ is thus a more general version of $\oint$ because it is also applicable to other domains than complements of compact sets and functions defined on these and other paths than rectangular paths.

Here, $\operatorname{Res}_\omega(f)$ is the residue of f at ω and $\operatorname{Ind}_\gamma(\omega)$ is the winding number of γ around ω (see Section A.9). Both of these can and are occasionally defined in terms of complex path integrals, namely

$$\operatorname*{Res}_{\omega}(f) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(\omega)} f(z)\,\mathrm{d}z$$

and

$$\operatorname*{Ind}_{\gamma}(\omega) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-\omega} \,\mathrm{d}z$$

(e.g., Pöschel, 2015, pp. 326, 331). In this case, the mapping $\int_{\text{res}}^{\Omega}$ depends explicitly on complex path integrals.

Residue-type partial aspect of complex path integrals

However, this dependence can be overcome because residues and winding numbers can also be defined without integration. Doing so, the right hand side of Equation 8.35 can be used to determine $\int_{\gamma} f(z)\,\mathrm{d}z$ completely under the constraints imposed by the residue theorem. The residue is

$$\operatorname*{Res}_{\omega}(f) = b_{-1},$$

where

$$f(z) = \sum_{k\in\mathbb{Z}} b_k (z-\omega)^k$$

is the Laurent series expansion of f on a punctured ball $B_{\varepsilon}(\omega) \setminus \{\omega\}$ in $\Omega \setminus A$ (e.g., Freitag & Busam, 2006, p. 164). Since we have learned in Section 7.5.2 that Laurent series expansions can be proven without integration, this definition of a residue can also be made without integration. Additionally, the winding number can be defined as follows: If the domain of γ is $[a, b]$, we can find two continuous functions $r\colon [a, b] \longrightarrow [0, \infty)$ and $\theta\colon [a, b] \longrightarrow \mathbb{R}$ such that $\gamma(t) = \omega + r(t)\mathrm{e}^{i\theta(t)}$ for all $t \in [a, b]$ (Stewart & Tall, 2018, ch. 7.4). We may then set

$$\operatorname*{Ind}_{\gamma}(\omega) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

This definition is also geometrically more intuitive than the previous formula for the winding number: We can interpret r as a function that measures the distance of γ to ω and θ as a continuous choice of argument for the function $[a, b] \longrightarrow \mathbb{R}$, $t \longmapsto \gamma(t) - \omega$. With this definition, it is plausible that $\operatorname{Ind}_{\gamma}(\omega)$ indicates how often γ winds around ω (this number is positive if there are more net rounds in counterclockwise direction than in clockwise direction).[99]

In this sense, we may interpret the mapping $\int_{\text{res}}^{\Omega}$ as a model for complex path integrals for closed (piecewise) continuously differentiable paths in Ω and holomorphic integrands on $\Omega \setminus A$ for a finite set $A \subseteq \mathbb{C}$, which is indeed *definable* without any integral at all. Henceforth, this definition is a partial aspect of complex path integrals, which we baptise the *residue-type partial aspect*. However, this would almost completely blur the origin of the value $2\pi i \sum_{w\in A} \operatorname{Res}_{\omega}(f) \operatorname{Ind}_{\gamma}(\omega)$.

8.4.3 *Condensation of properties of complex path integrals used in complex analysis*

Let us now summarise properties of the complex path integral with respect to their dependency on the integrand and the path of integration from a bird's eye view. We need to fix some terminology.

We define

$$\mathcal{P}(\Omega) = \{\gamma\colon [a, b] \longrightarrow \Omega : \gamma \text{ cont., (piecew.) cont. diff. path}\}$$

99 It goes without saying that both formulas for the residue and both formulas for the winding number coincide respectively.

to be the set of (piecewise) continuously differentiable paths in the domain Ω and for $\gamma \in \mathcal{P}(\Omega)$ we define

$$\mathcal{C}_\gamma = \{f\colon \operatorname{tr}(\gamma) \to \mathbb{C} : f \text{ continuous}\},$$

to be the set of continuous functions on the trace of γ. Finally, we define

$$\underline{\mathcal{P}(\Omega) \times \mathcal{C}} := \bigcup_{\gamma \in \mathcal{P}(\Omega)} \{\gamma\} \times \mathcal{C}_\gamma,$$

the set of all pairs of (piecewise) continuously differentiable paths and continuous functions on their trace, and set

$$\begin{aligned} \int^\Omega \colon \underline{\mathcal{P}(\Omega) \times \mathcal{C}} &\longrightarrow \mathbb{C}, \\ (\gamma, f) &\longmapsto \int^\Omega (\gamma, f) := \int_\gamma f := \int_\gamma f(z)\,\mathrm{d}z. \end{aligned}$$

Frequently used properties of the complex path integral as a mapping

To the best of our knowledge, we find that the following properties of $\int^\Omega$ characterise the use of the complex path integral in introductory courses and textbooks on complex analysis with the aim to prove the three milestones from the beginning of this section.[100]

Frequently used properties of complex path integrals

Properties 8.12. The mapping $\int^\Omega$ has the following properties:

1. *(Recovery of the real Riemann integral)*. The restriction of $\int^\Omega$ to real intervals $[a, b] \subseteq \Omega$ and continuous real-valued functions $g\colon [a, b] \longrightarrow \mathbb{R}$ yields the usual Riemann integral

$$\int^\Omega \left(\mathrm{id}_{[a,b]}, g\right) = \int_a^b g(x)\,\mathrm{d}x.$$

2. *($\mathbb{C}$-linearity in the integrand)*. For every $\gamma \in \mathcal{P}(\Omega)$ the mapping

$$\int^\Omega (\gamma, \cdot)\colon \mathcal{C}_\gamma \longrightarrow \mathbb{C}, \quad f \longmapsto \int_\gamma f$$

is $\mathbb{C}$-linear, that is, $\int^\Omega(\gamma, \alpha f + g) = \alpha \int^\Omega(\gamma, f) + \int^\Omega(\gamma, g)$ for $f, g \in \mathcal{C}_\gamma$ and $\alpha \in \mathbb{C}$.

100 Similar lists appear in the literature, too. For example, two sections in González's (1992) textbook are entitled "elementary" and "further" properties of complex integrals (chs. 7.7.–7.8). Isaev (2017, ch. 6) lists our properties 2., 3., 4., 9., and a computation similar to 6. from Properties 8.12 directly after his definition of the complex path integral (see also Freitag & Busam, 2006, ch. II. §1; Remmert & Schumacher, 2002, ch. 6.3; Lang, 1999, ch. III. §2; Remmert, 1998, ch. 6.§2, and many others).

3. *(Effects of reparametrisation).* For every $\gamma \in \mathcal{P}(\Omega)$ and the path with reversed parametrisation $-\gamma$ (see Equation A.12)

$$\int^{\Omega}(\gamma,\cdot) = \int_{\gamma}\cdot = -\int_{-\gamma}\cdot = -\int^{\Omega}(-\gamma,\cdot).$$

More generally, if γ is defined on $[a,b]$ and η is an orientation pre- or reserving reparametrisation of γ (i.e., $\eta = \gamma \circ \tau$ for a piecewise continuously differentiable and strictly monotonous function $\tau\colon [a,b] \longrightarrow [a,b]$), then

$$\int^{\Omega}(\gamma,\cdot) = \int_{\gamma}\cdot = \pm\int_{\eta}\cdot = \pm\int^{\Omega}(\eta,\cdot),$$

where $+$ is taken if τ is strictly monotonically increasing and $-$ is taken if τ is strictly monotonically decreasing.

4. *(Additivity in paths).* For every $\gamma, \eta \in \mathcal{P}(\Omega)$ such that the initial point of η is the terminal point of γ, and every continuous function $f\colon \operatorname{tr}(\gamma) \cup \operatorname{tr}(\eta) \longrightarrow \mathbb{C}$, it holds that

$$\int^{\Omega}(\gamma \oplus \eta, f) = \int_{\gamma\oplus\eta} f = \int_{\gamma} f + \int_{\eta} f$$

for the juxtaposition $\gamma \oplus \gamma$ of γ and η (see Equation A.13).

5. *(Calculation 1: Recovery of difference between terminal and initial points).* For every path $\gamma \in \mathcal{P}(\Omega)$ with initial point A and terminal point B it holds that

$$\int^{\Omega}(\gamma, z \mapsto 1) = \int_{\gamma} 1 = B - A,$$

and

$$\int^{\Omega}(\gamma, z \mapsto z) = \int_{\gamma} z = \frac{1}{2}\left(B^2 - A^2\right).$$

More generally,

$$\int^{\Omega}(\gamma, z \mapsto z^n)\,\mathrm{d}z = \frac{1}{n+1}\left(B^{n+1} - A^{n+1}\right)$$

for a (piecewise) continuously differentiable path $\gamma \in \mathcal{P}(\Omega)$ with initial point A and terminal point B, and $n \neq -1$.

6. *(Calculation 2: Laurent monomials).* For every $z_0 \in \Omega$ and $r > 0$ such that the ball $B_r(z_0)$ and its boundary is contained in Ω it holds that

$$\int^{\Omega}(\partial B_r(z_0), z \mapsto (z - z_0)^n) = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1, \end{cases}$$

where $\partial B_r(z_0)$ denotes any path in $\mathcal{P}(\Omega)$ that traverses the boundary of $B_r(z_0)$ once counterclockwise.

7. *(Homotopy invariance).*[a] For every two paths $\gamma, \eta \in \mathcal{P}(\Omega)$ that are homotopic relative to initial and terminal points or free homotopic, and a holomorphic function f on a neighbourhood of the image of the homotopy between γ and η in Ω, the integrals do not change:

$$\int^{\Omega}(\gamma, f) = \int_{\gamma} f = \int_{\eta} f = \int^{\Omega}(\eta, f).$$

8. *(Commutativity of $\int^{\Omega}$ with uniform limits of sequences of continuous functions).* For every $\gamma \in \mathcal{P}(\Omega)$ the mapping $\int^{\Omega}(\gamma, \cdot)$ commutes with uniform limits, that is, for every uniformly convergent sequence $(f_n)_{n\in\mathbb{N}} \in (\mathcal{C}_\gamma)^{\mathbb{N}}$ it holds that

$$\int^{\Omega}\left(\gamma, \lim_{n\to\infty} f_n\right) = \int_{\gamma} \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_{\gamma} f_n = \lim_{n\to\infty} \int^{\Omega}(\gamma, f_n).$$

9. *(Darboux's inequality).* For a continuous function f on the trace of $\gamma \in \mathcal{P}(\Omega)$, we have

$$\left|\int^{\Omega}(\gamma, f)\right| = \left|\int_{\gamma} f\right| \leq L(\gamma) \max_{\zeta\in\mathrm{tr}(\gamma)} |f(\zeta)|. \tag{8.36}$$

[a] This property depends on the topology of the domain, too (the same paths may be homotopic relative to initial and terminal points or free homotopic in some domains but not in all). That is, if we wish to emphasise this dependence further, we could modify the definition of $\int^{\Omega}$ slightly:

$$\int : \underline{\mathcal{D}\times\mathcal{P}\times\mathcal{C}} := \bigcup_{\Omega\in\mathcal{D}} \{\Omega\} \times \underline{\mathcal{P}(\Omega)\times\mathcal{C}} \longrightarrow \mathbb{C}.$$

Here, $\mathcal{D}$ denotes the set of domains in $\mathbb{C}$. Then, we have the additional requirements $\int(\Omega, \gamma, f) = \int^{\Omega}(\gamma, f)$, and $\int(\Omega, \gamma, f) = \int(\Omega', \gamma, f)$ for $\Omega, \Omega' \in \mathcal{D}$ whenever both $\int(\Omega, \gamma, f)$ and $\int(\Omega', \gamma, f)$ are defined.

The properties in this list are not intended to be minimal. For example, 8. follows from 9. (Remmert & Schumacher, 2002, p. 162). Rather, we believe that these properties describe the use of complex path integrals thoroughly.

As an example for how (some of) these properties can be used and were collected, let us look at the main steps in a proof of Cauchy's integral formula (Theorem A.22) (e.g., Freitag & Busam, 2006, pp. 86–88). Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function, C a compact circle in Ω, and ∂C any (piecewise) continuously differentiable path that winds around the boundary of C once counterclockwise, and $z_0 \in C$. By item 7 and item 9, we have

$$0 = \int_{\partial C} \frac{f(z) - f(z_0)}{z - z_0}$$

since $z \longmapsto \frac{f(z)-f(z_0)}{z-z_0}$ can be extended holomorphically to Ω and ∂C is homotopic to a constant path in Ω. Applying item 7 again yields

$$0 = \int_{\partial B_r(z_0)} \frac{f(z)-f(z_0)}{z-z_0},$$

where $\partial B_r(z_0)$ is a (piecewise) continuously differentiable path traversing once counterclockwise around the circle $B_r(z_0)$ for any $r > 0$ such that $\overline{B_r(z_0)} \subseteq C$. By item 2, the last equation is equivalent to

$$0 = \int_{\partial B} \frac{f(z)}{z-z_0} - f(z_0) \int_{\partial B} \frac{1}{z-z_0}$$

and by item 6 to

$$0 = \int_{\partial B} \frac{f(z)}{z-z_0} - 2\pi i f(z_0).$$

Hence, Cauchy's integral formula is verified.

As another example, Goursat's lemma (Theorem A.19) can be proven similarly with Properties 8.12. This theorem asserts that $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$ if Δ is a compact triangle contained in the domain of the holomorphic function f. In the proof of this theorem, we may use the property that complex path integrals of constant functions and of the identify function vanish along closed (piecewise) continuously differentiable paths (item 5), the $\mathbb{C}$-linearity of the complex path integral with respect to the integrand (item 2), and Darboux's inequality (item 9). In an iterative process replacing the original triangle Δ with a sequence of nested triangles in the interior of Δ, whose diameters converge to 0, the modulus of the complex path integral in question is estimated to be smaller than any given positive number, hence, it vanishes (e.g., Freitag & Busam, 2006, pp. 74–76). In this estimation, the linear approximation of holomorphic functions (Proposition A.7) is used, too. This proof is the most important point in complex analysis, where the definition of holomorphic functions (or an equivalent characterisation) is actually used. Afterwards, Goursat's lemma may be used to prove Cauchy's integral theorem (Theorem A.17) and it is therefore also used implicitly for Cauchy's integral formula (Theorem A.22) and other theorems up to the Residue theorem (Theorem A.29). We mimic the proof of Goursat's lemma in our proof of the axiomatic characterisation of complex path integrals for holomorphic functions in Theorem 8.13.

8.4.4 *Axiomatic characterisation of complex path integrals*

Axiomatic partial aspect

We will now prove the axiomatic characterisation of complex path integrals for holomorphic functions. This means, we will obtain yet another partial aspect here, the *axiomatic partial aspect*.

In the following, we assume that all boundaries of triangles are oriented counterclockwise.

Axiomatic characterisation of complex path integrals

Theorem 8.13. Let Ω denote a domain in $\mathbb{C}$, $\mathcal{H}(\Omega)$ the $\mathbb{C}$-vector space of holomorphic functions on Ω, and $\mathcal{P}(\Omega)$ the set of (piecewise) continuously differentiable paths in Ω.

Then, there is one and only one mapping

$$\wr\colon \mathcal{P}(\Omega) \times \mathcal{H}(\Omega) \longrightarrow \mathbb{C}$$

such that:

1. For every $\gamma \in \mathcal{P}(\Omega)$ the function $\wr(\gamma, \cdot)$ is $\mathbb{C}$-linear (i.e., $\wr(\gamma, \alpha f_1 + f_2) = \alpha\wr(\gamma, f_1) + \wr(\gamma, f_2)$ for all $f_1, f_2 \in \mathcal{H}(\Omega)$ and $\alpha \in \mathbb{C}$);
2. for every $f \in \mathcal{H}(\Omega)$ the function $\wr(\cdot, f)$ is additive (i.e., $\wr(\gamma_1 \oplus \gamma_2, f) = \wr(\gamma_1, f) + \wr(\gamma_2, f)$ for all $\gamma_1, \gamma_2 \in \mathcal{P}(\Omega)$);
3. for every $\gamma \in \mathcal{P}(\Omega)$ and $f \in \mathcal{H}(\Omega)$ it holds that $\wr(\gamma, f) = -\wr(-\gamma, f)$;
4. for path along a line segment $\gamma \in \mathcal{P}(\Omega)$, $\gamma\colon [a, b] \longrightarrow \Omega$, it holds that $\wr(\gamma, z \longmapsto 1) = \gamma(b) - \gamma(a)$;
5. for every compact triangle $\Delta \subseteq \Omega$ it holds that $\wr(\partial\Delta, \mathrm{id}) = 0$;
6. for every $\gamma \in \mathcal{P}(\Omega)$ and every $f \in \mathcal{H}(\Omega)$ it holds that $|\wr(\gamma, f)| \leq L(\gamma) \max_{z\in \mathrm{tr}(\gamma)} |f(z)|$; and
7. for every closed path $\gamma \in \mathcal{P}(\Omega)$ and every $f \in \mathcal{H}(\Omega)$ there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{P}(\Omega)^{\mathbb{N}}$ of closed polygonal paths such that $\wr(\gamma_n, f) \longrightarrow \wr(\gamma, f)$ for $n \longrightarrow \infty$.

This mapping satisfies

$$\wr(\gamma, f) = \int_\gamma f(z)\,\mathrm{d}z$$

for all $\gamma \in \mathcal{P}(\Omega)$ and $f \in \mathcal{H}(\Omega)$. ■

Remark 8.14. The properties $\wr$ in Theorem 8.13 are motivated by frequently used properties of complex path integrals (e.g., those in Properties 8.12). For property 7., we notice that for every rectifiable simple closed path in Ω and every continuous functions $f\colon\ \mathrm{tr}(\gamma) \cup \mathrm{int}(\gamma) \longrightarrow \mathbb{C}$ there is a sequence of closed polygonal paths $(\gamma_n)_{n\in\mathbb{N}}$ in the interior of γ such that $\int_{\gamma_n} f(z)\,\mathrm{d}z \longrightarrow \int_\gamma f(z)\,\mathrm{d}z$ for $n \longrightarrow \infty$ (Wendland & Steinbach, 2005, p. 518). Similarly, for every rectifiable path in Ω, there is a sequence of polygonal paths $(\Gamma_n)_{n\in\mathbb{N}} \in \mathcal{P}(\Omega)^{\mathbb{N}}$, each of which starts at the initial point of γ and ends at the terminal point of γ, such that $\int_{\Gamma_n} f(z)\,\mathrm{d}z \longrightarrow \int_\gamma f(z)\,\mathrm{d}z$ for $n \longrightarrow \infty$ (Conway, 1973/1978, p. 65).

Consequently, the usual complex path integral $(\gamma, f) \longmapsto \int_\gamma f(z)\,\mathrm{d}z$ satisfies the axioms of $\wr$ in Theorem 8.13 and only the uniqueness of $\wr$ has to be proven here. ◇

Proof of the uniqueness in Theorem 8.13

In our proof, we mimic the argumentation in Freitag and Busam (2006, pp. 72–80, 237–241). We begin by mimicking Goursat's lemma (Theorem A.19). First, we derive a special case for affine linear functions.

Lemma 8.15. For all compact triangles $\Delta \subseteq \Omega$ and all $\alpha, \beta \in \mathbb{C}$ we have

$$\wr(\partial\Delta, z \longmapsto \alpha z + \beta) = 0. \qquad \blacksquare$$

Proof. Let $\Delta \subseteq \Omega$ denote a compact triangle and let $\alpha, \beta \in \mathbb{C}$. We enumerate the vertices of Δ in counterclockwise direction with A, B, C and we denote by $\sigma_{A,B}, \sigma_{B,C}$, and $\sigma_{C,A}$ the paths along the directed line segments from A to B, B to C, and C to A. Then, $\partial\Delta = \sigma_{A,B} \oplus \sigma_{B,C} \oplus \sigma_{C,A}$ and properties 2., 4., and 5. imply

$$\begin{aligned}
\wr(\partial\Delta, z \longmapsto \alpha z + \beta) &= \alpha \wr(\partial\Delta, \mathrm{id}) + \beta \wr(\partial\Delta, z \longmapsto 1) \\
&= \beta \wr(\partial\Delta, z \longmapsto 1) \\
&= \beta \left(\wr(\sigma_{A,B}, z \longmapsto 1) + \wr(\sigma_{B,C}, z \longmapsto 1) + \wr(\sigma_{C,A}, z \longmapsto 1)\right) \\
&= \beta \left((B - A) + (C - B) + (A - C)\right) \\
&= 0.
\end{aligned} \qquad \square$$

Next, we extend the previous lemma to holomorphic functions.

Lemma 8.16 (Goursat's lemma for $\wr$)**.** Let $\Delta \subseteq \Omega$ be a compact triangle and $\wr$ as in Theorem 8.13. Then, for every $f \in \mathcal{H}(\Omega)$,

$$\wr(\partial\Delta, f) = 0. \qquad \blacksquare$$

Proof. Let $f \in \mathcal{H}(\Omega)$ and $\varepsilon > 0$. We construct a sequence of triangles $\Delta_0, \Delta_1, \ldots$ via the following procedure. Let $\Delta_0 = \Delta$. Given that Δ_n is already constructed for an $n \in \mathbb{N}_0$, we set Δ_{n+1} to be one of the four subtriangles Δ_n^j of Δ_n ($j = 1, 2, 3, 4$) obtained by halving the edges of Δ_n such that

$$|\wr(\partial\Delta_{n+1}, f)| = \max_{j=1,2,3,4} \left|\wr(\partial\Delta_n^j, f)\right|.$$

As a result, we obtain a strictly decreasing sequence of triangles

$$\Delta_0 \supseteq \Delta_1 \supseteq \ldots$$

with respect to inclusion. The intersection of all triangles in this decreasing sequence is a singleton, in other words, there is a $z_0 \in \Omega$ such that

$$\bigcap_{n \in \mathbb{N}_0} \Delta_n = \{z_0\}.$$

Then, for $n \in \mathbb{N}_0$, the perimeter of Δ_n is $L(\Delta_n) = 2^{-n} L(\partial\Delta)$ and it holds that

$$\begin{aligned}
|\wr(\partial\Delta_n, f)| &= \left|\wr(\partial\Delta_n^1, f) + \wr(\partial\Delta_n^2, f) + \wr(\partial\Delta_n^3, f) + \wr(\partial\Delta_n^4, f)\right| \\
&\leq 4 \left|\wr(\partial\Delta_{n+1}, f)\right|
\end{aligned}$$

by properties 2. and 3. of $\wr$. Consequently,

$$|\wr(\partial\Delta, f)| \leq 4^n \, |\wr(\partial\Delta_n, f)| \,. \tag{8.37}$$

for $n \in \mathbb{N}_0$.

Since f is complex differentiable at z_0, there is a continuous function $\varphi\colon \Omega \longrightarrow \mathbb{C}$ such that $f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z)$ as in Proposition A.7. Accordingly, there is a $\delta > 0$ such that $B_\delta(z_0) \subseteq \Omega$ and

$$|\varphi(z)| \leq \varepsilon |z - z_0| \tag{8.38}$$

for $z \in B_\delta(z_0)$. Now, we choose an $N \in \mathbb{N}$ such that $\Delta_N \subseteq B_\delta(z_0)$; then:

$$\begin{aligned}
|\wr(\partial\Delta, f)| &\leq 4^N \, |\wr(\partial\Delta_N, f)| && \text{(by Equation 8.37)}\\
&= 4^N \left|\wr(\partial\Delta_N, z \longmapsto f(z_0) + f'(z_0)(z - z_0)) + \wr(\partial\Delta_N, \varphi)\right| && \text{(by 1.)}\\
&= 4^N \, |\wr(\partial\Delta_N, \varphi)| && \text{(by Lemma 8.15)}\\
&\leq 4^N \frac{L(\partial\Delta)}{2^N} \max_{z \in \partial\Delta_N} |\varphi(z)| && \text{(by 6.)}\\
&\leq 4^N \frac{L(\partial\Delta)}{2^N} \varepsilon \frac{L(\partial\Delta)}{2^N} && \text{(by Equation 8.38)}\\
&= L(\partial\Delta)^2 \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the lemma is proven. □

Corollary 8.17. Let A, B, C be the corners of a compact triangle in Ω, $\gamma_{A,B}$ be a simple path along the line segment from A to B, likewise $\gamma_{B,C}$ and $\gamma_{A,C}$, and $f \in \mathcal{H}(\Omega)$. Then, we have

$$\wr(\gamma_{A,C}, f) = \wr(\gamma_{A,B}, f) + \wr(\gamma_{B,C}, f) \,,$$

where $\wr$ is a mapping that satisfies the properties in Theorem 8.13. ■

Proof. By Lemma 8.16, it holds that $0 = \wr(\gamma_{A,B} \oplus \gamma_{B,C} \oplus (-\gamma_{A,C}), f)$. Then, properties 2. and 3. of $\wr$ imply $0 = \wr(\gamma_{A,B}, f) + \wr(\gamma_{B,C}, f) - \wr(\gamma_{A,C}, f)$. □

Now, we proceed to prove Theorem 8.13 for star domains first and then extend it to general domains.

Proof of the uniqueness of $\wr$ in Theorem 8.13 for star domains. We assume that Ω is a star domain and let $f \in \mathcal{H}(\Omega)$. The outline of the proof is as follows:

- Step 1: We begin by constructing a primitive function F for f on Ω.
- Step 2: Then, we evaluate $\wr(\gamma, f) = F(B) - F(A)$, where $\gamma \in \mathcal{P}(\Omega)$ is the path along the line segment from A to B in Ω.
- Step 3: Finally, we extend our previous calculation to general paths $\gamma \in \mathcal{P}(\Omega)$.

STEP 1 We fix a star centre ω of Ω. For every $z \in \Omega$ we set η_z to be the path along the line segment from ω to z in Ω. Furthermore, we set

$$\begin{aligned} F \colon \Omega &\longrightarrow \mathbb{C}, \\ z &\longmapsto \wr(\eta_z, f). \end{aligned}$$

and choose $z_0 \in \Omega$. We have to show

$$\lim_{\substack{z \in \Omega \\ z \to z_0}} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

Let $r > 0$ such that $B_r(z_0) \subseteq \Omega$. For $z \in B_r(z_0)$, let $\sigma_{z_0,z}$ be the path along the line segment from z_0 to z in $B_r(z_0) \subseteq \Omega$. It follows from properties 2. and 3. of $\wr$, and Lemma 8.16 that

$$0 = \wr(\eta_{z_0} \oplus \sigma_{z_0,z} \oplus (-\eta_z), f) = \wr(\eta_{z_0}, f) + \wr(\sigma_{z_0,z}, f) - \wr(\eta_z, f);$$

thus

$$F(z) - F(z_0) = \wr(\eta_z, f) - \wr(\eta_{z_0}, f) = \wr(\sigma_{z_0,z}, f)$$

for $z \in B_r(z_0)$. Then, properties 1. and 4. of $\wr$ imply

$$\wr(\sigma_{z_0,z}, z \longmapsto f(z_0)) = f(z_0)(z - z_0),$$

for $z \in B_r(z_0)$. Putting everything together, we obtain for $z \in B_r(z_0) \setminus \{z_0\}$ that

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \wr(\sigma_{z_0,z}, f) - \frac{1}{z - z_0} \wr(\sigma_{z_0,z}, z \longmapsto f(z_0)) \right| \\ &= \left| \frac{1}{z - z_0} \wr(\sigma_{z_0,z}, z \longmapsto f(z) - f(z_0)) \right| \\ &\leq \frac{1}{|z - z_0|} |z - z_0| \max_{\xi \in \operatorname{tr}(\sigma_{z_0,z})} \{|f(\xi) - f(z_0)|\} \\ &\overset{z \to z_0}{\longrightarrow} 0. \end{aligned}$$

The second last line follows from property 6. and the last line follows from the continuity of f. Hence, we have shown $F' = f$.

STEP 2 Now, let $\gamma \in \mathcal{P}(\Omega)$ be the path along the line segment from $A \in \Omega$ to $B \in \Omega$. Then,

$$\wr(\gamma, f) = \wr(-\eta_A, f) + \wr(\eta_B, f) = \wr(\eta_B, f) - \wr(\eta_A) = F(B) - F(A) \tag{8.39}$$

by Corollary 8.17, and thus $\wr(\cdot, f)$ is uniquely determined for line segments in Ω.

More generally, if $\gamma \in \mathcal{P}(\Omega)$ is a polygonal path from A to B in Ω, there is a $k \in \mathbb{N}$ and there are points $A = z_0, z_1, \ldots, z_k = B$ such that $\gamma = \sigma_{z_0,z_1} \oplus \sigma_{z_1,z_2} \oplus \cdots \oplus \sigma_{z_{k-1},z_k}$. Using property 2. and Equation 8.39 we get

$$\wr(\gamma, f) = \sum_{\nu=1}^{k} \wr(\sigma_{z_{\nu-1},z_\nu}, f) = \sum_{\nu=1}^{k} (F(z_\nu) - F(z_{\nu-1})) = F(B) - F(A).$$

In particular, $\wr(\gamma, f) = 0$ if $\gamma \in \mathcal{P}(\Omega)$ is a closed polygonal path in Ω.

STEP 3 Let $\gamma \in \mathcal{P}(\Omega)$ with initial point A and terminal point B. Then, the path $\eta_A \oplus \gamma \oplus (-\eta_B)$ is closed and an element of $\mathcal{P}(\Omega)$. Therefore, according to property 7. and the step 2, there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{P}(\Omega)^{\mathbb{N}}$ of closed polygonal paths such that

$$\wr(\eta_A \oplus \gamma \oplus (-\eta_B), f) = \lim_{n\to\infty} \wr(\gamma_n, f) = 0.$$

Using the properties 2. and 3. of $\wr$, we get

$$\wr(\gamma) = \wr(\eta_B) - \wr(\eta_A) = F(B) - F(A). \tag{8.40}$$

This finishes the proof of Theorem 8.13 for star domains. □

Finally, we extend the previous proof of the theorem to general domains (see also Section A.10).

Proof of Theorem 8.13 for arbitrary domains. Let Ω be any domain in $\mathbb{C}$, $\gamma \in \mathcal{P}(\Omega)$, and $f \in \mathcal{H}(\Omega)$. We recall from Freitag and Busam (2006, p. 237) that there is an $\varepsilon > 0$ and a partition $a = t_0 < t_1 < \ldots < t_n = b$, $n \in \mathbb{N}$, such that

$$\operatorname{tr}\left(\gamma_{|[t_{\nu-1},t_\nu]}\right) \subseteq B_\varepsilon\left(\gamma(t_{\nu-1})\right) \cap B_\varepsilon\left(\gamma(t_\nu)\right) \subseteq \Omega$$

for all $\nu = 1, 2, \ldots, n$. This means that we can cover the trace of γ with balls of the same radius ε centred at $\gamma(t_\nu)$ for $\nu = 1, 2, \ldots, n$, each of which is completely contained in Ω, and the restrictions of γ to the intervals given by the partition are paths in two succeeding balls.

According to the proof of Theorem 8.13 for star domains, there is primitive F_ν for f on $B_\varepsilon\left(\gamma(t_\nu)\right)$ such that $\wr\left(\gamma_{|[t_{\nu-1},t_\nu]}, f\right) = F_\nu(\gamma(t_\nu)) - F_\nu(\gamma(t_{\nu-1}))$ for $\nu = 1, 2, \ldots, n$. Since adding a constant to the primitive functions does alter the number $F_\nu(\gamma(t_\nu)) - F_\nu(\gamma(t_{\nu-1}))$, we may assume that $F_\nu(\gamma(t_\nu)) = F_{\nu+1}(\gamma(t_\nu))$ for $\nu = 1, 2, \ldots, n-1$.

Then, using property 2. of $\wr$ and Equation 8.40, we get

$$\begin{aligned}
\wr(\gamma, f) &= \sum_{\nu=1}^{n} \wr\left(\gamma_{|[t_{\nu-1},t_\nu]}, f\right) \\
&= \sum_{\nu=1}^{n} F_\nu(\gamma(t_\nu)) - F_\nu(\gamma(t_{\nu-1})) \\
&= F_\nu(\gamma(b)) - F_1(\gamma(a)).
\end{aligned}$$

Hence, the value $\wr(\gamma, f)$ is completely determined by f and the initial and terminal point of γ alone. □

9

ASPECTS AND PARTIAL ASPECTS OF COMPLEX PATH INTEGRALS

In this chapter, we will summarise the aspects and partial aspects of complex path integrals we have identified in our epistemological analysis in the last two chapters. These aspects and partial aspects are the most prevalent ways to introduce complex path integrals.

We recall that Greefrath et al. (2016a, p. 101) defined an aspect of a mathematical concept as a "subdomain of the concept that can be used to characterize it on the basis of mathematical content". We reconceptualised this definition with regards to the commognitive framework as an object-level narrative, which can function as a definition or as a narrative, which is equivalent to a definition, and which originates in a surrounding discourse about the mathematical object in question (see Section 6.3.1). In particular, one can differentiate the aspects of complex path integrals in terms of the keywords or signifiers used in the definiens (i.e., the defining utterances), which realise the definiendum (i.e., the complex path integral). Furthermore, we differentiated aspects from *partial aspects* (Roos, 2020). These are narratives to characterise the mathematical object in question, but which is only endorsable under additional constraints. One may say that an aspect allows for a more general definition than a partial aspect. For example, it is in principle possible to define a Riemann integral in terms of the difference of the values of a primitive function at the upper and lower bound of integration—but only in case the existence of the primitive is guaranteed (e.g., as an additional constraint on the integrand).

Mathematical context for the aspects of complex path integrals

In order to be able to differentiate aspects from partial aspects, a certain mathematical context has to be fixed. Here, the mathematical object to be defined is the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$. We specify the context for the aspects to be endorsable in terms of the properties of the function f and the path γ: In this case, we require that the aspects of complex path integrals should be endorsable for a *(piecewise) continuously differentiable path* γ in a domain $\Omega \subseteq \mathbb{C}$

and a *continuous complex function* f on the trace of γ. We argue that these constraints on the integrand and the path are the constraints we encountered most often in our review of complex analysis literature. Other constraints are possible, but these constraints are the well-balanced blend of generality, technical difficulty, and relevance for the further use of complex path integrals in theory development in complex analysis.

Therefore, let $\gamma\colon [a, b] \longrightarrow \Omega$ be a (piecewise) continuously differentiable path in a domain $\Omega \subseteq \mathbb{C}$ with real part γ_1 and imaginary part γ_2, and let $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$ be a continuous function with real part u and imaginary part v.

Section 9.1 deals with the *four aspects* of complex path integrals, which characterise complex path integrals in general. In Section 9.2, we present the *four partial aspects*, which characterise the complex path integral or integration of complex functions under additional constraints on the integrands and the paths.

9.1 ASPECTS OF COMPLEX PATH INTEGRALS

Aspects of complex path integrals

Four aspects for complex path integrals could be found:

1. The *product sum aspect* (Section 9.1.1),
2. the *substitution aspect* (Section 9.1.2),
3. the *vector analysis aspect* (Section 9.1.3), and
4. the *mean value aspect* (Section 9.1.4).

We will now summarise each of them.

9.1.1 *The product sum aspect*

Product sum aspect of complex path integrals

The *product sum aspect* characterises the complex path integral as the limit of complex Riemann sums (Section 8.1.1):

Product sum aspect

The complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ of a complex function f defined on the trace of a (piecewise) continuously differentiable or rectifiable path γ is the limit of the sums

$$\int_\gamma f(z)\,\mathrm{d}z := \lim_{\substack{P^{(n)},\xi^{(n)} \\ n\longrightarrow\infty}} \sum_{k=1}^{\nu_n} f\left(\gamma\left(\xi_k^{(n)}\right)\right)(\Delta\gamma)_k^{(n)}, \tag{9.1}$$

where the limit is taken for any sequence of partitions $P^{(n)}\colon a = t_0^{(n)} < t_1^{(n)} < \ldots < t_{\nu_n}^{(n)} = b$ of $[a, b]$ of length $\nu_n \in \mathbb{N}$, tags $\xi_k^{(n)} \in [t_{k-1}, t_k]$, and $(\Delta\gamma)_k^{(n)} = \gamma(t_k) - \gamma(t_{k-1})$ for $k = 1, 2, \ldots, \nu_n$, $n \in \mathbb{N}$, such that the norm $\ell(P^{(n)})$ converges to 0 as $n \longrightarrow \infty$.

This aspect is endorsable in a complex analysis discourse, in which this new integral is related discourses about other integrals such as Riemann or real path integrals, which are analogously defined using limits of product sums. This definition of the complex path integral refines Cauchy's (1825) definition (Section 7.4) in such a way that it applies to a more general class of paths and where the tag vectors can be chosen arbitrarily.[101]

The convergence of the sequences of the complex Riemann sums in Equation 9.1 is guaranteed and independent of the choices of sequences of partitions and tag vectors. We may substantiate this convergence with a direct proof, mimicking the corresponding proofs from real analysis. Having endorsed the convergence of Riemann sums for real integrals, we may also decompose the sequences of complex Riemann sums into their real and imaginary parts, whose convergence is then guaranteed since they converge to the real path integrals $\int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y$ and $\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y$. The convergence is however also guaranteed in case γ is rectifiable (e.g., González, 1992, ch. 7.6) and thus this aspect is endorsable for continuous complex functions on traces of rectifiable paths, too.

The definiens of the product sum aspect includes keywords and signifiers such as "limit", "complex Riemann sum", "partition", and "norm of a partition". This aspect does not require any integral in its definiens. Nevertheless, these keywords and the whole aspect is clearly rooted in the discourse on real integrals; hence, integrals are at least implicitly present here, too. Visual mediators for this aspect may be plots of the path, partitions of its trace, and plots of the integrand (e.g., as a vector field with vectors drawn at the tags) as in Figure 8.1. The addends in the complex Riemann sums in Equation 9.1 may be realised as copies of $(\Delta\gamma)_k^{(n)}$, which were rotated by the argument and dilated by the modulus of $f(\xi_k^{(n)})$. Visual mediators for these quantities such as Figure 8.3, Figure 8.4, and Figure 8.5b show these sums as the concatenations of the corresponding vectors.[102]

9.1.2 *The substitution aspect*

We call the aspect, which defines complex path integrals in terms of a Riemann integral of a complex-valued function, the *substitution aspect* (see Section 8.1.2):

Substitution aspect of complex path integrals

Substitution aspect

The complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ of a complex function f defined on the trace of a (piecewise) continuously differentiable path γ is

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t. \tag{9.2}$$

The substitution aspect is substantiated by the formal substitution $z = \gamma(t)$, hence $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, where z runs through the trace of γ if t varies on the domain of γ. This aspect is endorsable once the Riemann integral of a complex-valued function has been introduced to a discourse on complex analysis. That is, the object-level rule $\int_a^b g(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}\,(g(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}\,(g(t))\,\mathrm{d}t$ for a continuous complex-valued function g on the real interval $[a, b]$ has

101 Essentially, Cauchy's (1825) only allowed paths with monotonous real and imaginary parts and tags corresponding to the points at the boundary of a given partition.

102 Clearly, figures showing the trace of γ are useful visual mediators for all aspects and partial aspects. Hence, we do not list them each time.

to be established previously. Furthermore, it is assumed here that the right-hand side Equation 9.2 is not affected by the points t, where γ is not differentiable.

The definiens of the substitution aspect includes keywords and signifiers related to the Riemann integral of a complex-valued functions. That is, in addition to the newly to be introduced integral in its definiendum, this aspect includes an integral in its definiens. The substitution aspect does not include keywords and signifiers related to limits or sums explicitly (although these may be used realise the right-hand side in Equation 9.2 further). Additionally, bearing in mind the product structure in the definientia in Equation 8.14 and Equation 8.15, one can also endorse Equation 9.2 by arguing that it mimics the definitions of real path integrals in the sense that the multiplication of real numbers or the Euclidean scalar product is replaced with complex multiplication.

Visual mediators in addition to the formulas presented above may be graphs of the real and imaginary part of $t \longrightarrow f(\gamma(t))\gamma'(t)$ and the shaded areas under these graphs according to the basic idea of area as in Figure 8.2. However, these graphs are in fact hardly used in complex analysis literature. The substitution aspect works whenever the real and imaginary part of $t \longrightarrow f(\gamma(t))\gamma'(t)$ are Riemann-integrable, as is the case for our choice of f and γ.

9.1.3 *The vector analysis aspect*

Vector analysis aspect of complex path integrals

The *vector analysis aspect* of complex path integrals characterises them by defining their real and imaginary parts in terms of real path integrals of second kind (Section 8.1.4):

Vector analysis aspect

The complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ of a complex function f defined on the trace of a (piecewise) continuously differentiable path γ is

$$\int_\gamma f(z)\,\mathrm{d}z := \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y. \tag{9.3}$$

Using signifiers for the Pólya vector field $\mathbf{w}_f = (u, -v)^T$ associated to f and the matrix $\mathbf{J}$, which describes a counterclockwise rotation by $\pi/2$, the right-hand side in Equation 9.3 may also be realised as

$$\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N}$$

or

$$\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{J}\mathbf{w}_f\,\mathrm{d}\mathbf{T}.$$

Mimicking the definition of real path integrals of second kind (Equation 8.15) for a complex instead of a real vector field, one may equally realise the definition of the complex path integral based on the vector analysis aspect as

$$\int_\gamma f(z)\,\mathrm{d}z := \int_\gamma \begin{pmatrix} f \\ if \end{pmatrix} \mathrm{d}\mathbf{T} := \int_a^b \left\langle \begin{pmatrix} f \\ if \end{pmatrix}, \begin{pmatrix} \gamma'(t) \\ \gamma_2'(t) \end{pmatrix} \right\rangle \mathrm{d}t.$$

The vector analysis aspect is endorsable in a discourse on complex analysis that extends a discourse on vector analysis to complex vector fields. Especially the use of the Pólya vector field in relation to defining complex path integrals is endorsable with respect to the physical meaning of real path integrals as flow, flux, or work we explored in Section 8.2.2.

This aspect is also endorsable in a discourse on complex differential forms (Section 8.1.4), which is isomorphic to the discourse about vector fields: Identifying differential forms $P\,\mathrm{d}x + Q\,\mathrm{d}y$ with vector fields $(P, Q)^T$ (P and Q are either real- or complex-valued functions of two real or one complex variable), the narratives about differential forms may be translated into narratives about vector fields and vice versa. Since real path integrals of second kind can be defined for continuous vector fields and rectifiable paths, the vector analysis aspect is suitable for rectifiable paths, too.

The main keywords and signifiers in the definientia according to the vector analysis aspect are related to real path integrals of second kind and vector fields. Hence, as in the substitution aspect, other integrals are used in the definientia here explicitly.

Visual mediators for the vector analysis aspect comprise plots of vector fields related to f (i.e., $\mathbf{f} = (u, v)^T$ and $(v, u)^T$, but in particular $\mathbf{w}_f$). These plots help interpret real path integrals as flow, flux, or work, and therefore, they may be used to realise the real and imaginary part of $\int_\gamma f(z)\,\mathrm{d}z$) (see Figure 8.9, Figure 8.10, Figure 8.12, Figure 8.11, and Figure 8.13).

9.1.4 *The mean value aspect*

Mean value aspect of complex path integrals

We have developed the *mean value aspect* of the complex path integral for regular, simple or simple closed paths γ in Ω (Section 8.2.3). Let $T(z)$ denote the complex number, which represents the unit tangent vector to $\mathrm{tr}(\gamma)$ at $z \in \mathrm{tr}(\gamma)$ with respect to the direction induced by the path. This means that $T(\gamma(t)) = |\gamma'(t)|^{-1}\gamma'(t)$ for every $t \in [a, b]$:

Mean value aspect

The complex path integral of a continuous function f on the trace of a regular, simple or simple closed (piecewise) continuously differentiable path γ is the mean value of the function $f \cdot T$ along the oriented trace of γ, where T is the unit tangential field induced by γ, scaled by $L(\gamma)$; in other words,

$$\int_\gamma f(z)\,\mathrm{d}z = L(\gamma) \underset{z\in\mathrm{tr}(\gamma)}{\mathrm{av}} (f(z)T(z)) \,. \tag{9.4}$$

This definition is based on a previous substantiation of the average on the right-hand side of this equation, namely (Equation 8.28)

$$\underset{t\in[a,b]}{\mathrm{av}} (f(\gamma(t))T(\gamma(t))) := \frac{1}{L(\gamma)} \int_\gamma f \cdot T\,\mathrm{d}s.$$

In other words, the defining equation according to the mean value aspect may also be written as

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f \cdot T\,\mathrm{d}s.$$

This aspect originates in discourses on mean values and the geometry of paths and goes back to a paper by Gluchoff (1991). Technically, the definition of average we use here works for regular paths, which are not simple or not simple closed, too (footnote 90). Since paths with vanishing derivatives are rare in complex analysis or can often be avoided by subdividing the path into pieces, the restriction to regular paths is of minor importance. In order to apply the mean value aspect for piecewise regular paths, one may adopt these formulas to each piece and add up. Therefore, we still consider the mean value aspect an aspect of complex path integrals (even though we require that γ is regular).

In Section 8.2.3, we substantiated that this mean value interpretation conflicts with other mean value interpretations related from real analysis in one or two variables or measure theory. The main reason for these conflicts is that Equation 9.4 contains an average for $f \cdot T$ but not f (see also Hanke, 2022b).

The main keywords in the definiens are "mean value" and "unit tangential field". No keywords related to integrals appear explicitly in the definiens. However, an integral is implicitly involved because the average in Equation 9.4 is realised with the help of $\int_\gamma f \cdot T \, \mathrm{d}s$. Visual mediators for the mean value aspect may especially include plots of $f \cdot T$ and of the unit tangential vector field T attached to $\mathrm{tr}(\gamma)$.[103]

9.2 PARTIAL ASPECTS OF COMPLEX PATH INTEGRALS

We will now focus on stronger constraints on the integrands, paths, and occasionally the domains of the integrands. Therefore, the following characterisations of complex path integrals are *partial aspects*.

Partial aspects of complex path integrals

Four partial aspects of complex path integrals arose during the epistemological analysis:

1. The *anti-derivative partial aspect* (Section 9.2.1),
2. the *Green-type partial aspect* (Section 9.2.2),
3. the *residue-type partial aspect* (Section 9.2.3), and
4. the *axiomatic partial aspect* (Section 9.2.4).

In all these partial aspects, the integrand f is required to be defined on a domain Ω. In the first, third, and partly in the fourth of these partial aspects, the class of integrands is restricted to holomorphic functions on Ω (or even restricted further to holomorphic functions on simply-connected domains). In the second partial aspect, the real part u and the imaginary part v of f are required to be continuously differentiable. Hence, the class of admissible integrands is thus restricted here for all partial aspects compared to the aspects we considered before.[104]

103 To be precise, since the current discourse about the mean value aspect is restricted to the paper by Gluchoff (1991) only, no visual mediators beyond formulas have been used so far.

104 However, some authors explicitly argue that holomorphic functions play the most important role in complex analysis (e.g., "*'Complex analysis' is the theory of analytic [holomorphic; EH.] functions*" Freitag and Busam, 2006, p. XIV, emph. orig.). Others argue that non-holomorphic or non-analytic functions are a mere curiosity in complex analysis. For example, Jeffrey (1992, p. 306) describes that

> […] the theory of complex analysis, and of complex integration in particular, applies only to such [holomorphic / analytic; EH.] functions [… and the example of an integral of a nonanalytic function; EH.] is of no significance for what follows.

Thus, some authors would consider the restriction of defining complex path integrals only for holomorphic functions to be insignificant. We will see later that also Uwe, one of the experts, we interviewed for our empirical part, shares this opinion (Chapter 13).

Accordingly, we assume for the rest of the section that $f\colon \Omega \longrightarrow \mathbb{C}$ is a complex function on a domain $\Omega \subseteq \mathbb{C}$.

9.2.1 *The anti-derivative partial aspect of complex path integrals*

Anti-derivative partial aspect of complex path integrals

We discussed in Section 8.3 that there are no anti-derivatives for continuous, but not holomorphic, complex functions f. However, if there is an anti-derivative F for f, then the complex path integral of f along γ can be expressed in terms of differences of values of F.

We recall that if f is holomorphic, then it has local anti-derivatives in neighbourhoods (e.g., disks) of every point in Ω; if Ω is also simply-connected, then there is a global anti-derivative for f on Ω (Section 8.3.1; Theorem A.20). Hence, we now assume that f is holomorphic. Then, the *anti-derivative aspect* of the complex path integral can be stated as follows:

Anti-derivative partial aspect

Let γ denote a path in Ω. If F is a global anti-derivative of f on Ω, then the complex path integral is fully characterised as

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))\,. \tag{9.5}$$

More generally, if $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of $[a,b]$, $n \in \mathbb{N}$, $D_1, \ldots, D_n \subseteq \Omega$ are open disks that satisfy $\gamma([t_{k-1}, t_k]) \subseteq D_k$, and F_k is a local anti-derivative for f on each D_k for each $k = 1, \ldots, n$, then the complex path integral of f along γ can be realised as

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{k=1}^{n} \left(F_k(t_k) - F_k(t_{k-1})\right). \tag{9.6}$$

The anti-derivative partial aspect is endorsable in a discourse on holomorphic functions. It mimics the formula to evaluate an integral from the fundamental theorem of calculus (Equation B.2, Equation A.17). Drawing on works in topological analysis in the 20th century (Section 7.5.2), we argued in Section 8.3 that these local or global anti-derivatives can be constructed without any prior theory of complex integration. Hence, the anti-derivative aspect is actually endorsable without any circular reasoning involving integration. Moreover, we remark once again that it suffices that γ is merely continuous (see Section 8.3.1, Section A.10; Lang, 1999, ch. III. §4).[105]

We would also like to emphasise that Equation 9.5 immediately implies that the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ vanishes if there is a global primitive function for f and γ is closed. This existence is guaranteed if Ω is simply-connected. However, the existence of local prim-

105 Nevertheless, we recall that the assignment

$$F\colon \Omega \longrightarrow \mathbb{C}, \quad z \longmapsto F(z) := \int_{\gamma_{\omega,z}} f(\zeta)\,\mathrm{d}\zeta$$

for a fixed $\omega \in \Omega$ and any (piecewise) continuously differentiable path from ω to z yields an anti-derivative for F on simply-connected domains Ω. That is, Equation 9.5 and Equation 9.6 are endorsable as theorems in a discourse, in which anti-derivatives are constructed in terms of integrals, too.

itives for f does not necessarily imply that the right-hand side of Equation 9.6 vanishes. For instance, this would contradict the equation $\int_{\partial B_1(0)} \frac{1}{z}\, \mathrm{d}z = 2\pi i$.[106]

The main keywords used in this partial aspect are "global anti-derivative", "local anti-derivative" and "partition". This aspect is mainly realised symbolically. A figure showing the cover of the trace of γ with the disks $D_1, \ldots, D_n$ (Figure 8.14) may be used to accompany the narrative of the anti-derivative partial aspect.

Remark 9.1. If F is a primitive function for f and $\gamma\colon [a, b] \longrightarrow \Omega$ is (piecewise) continuously differentiable, then the fundamental theorem of calculus (Theorem B.3) and the chain rule imply $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(F \circ \gamma)(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$. Hence, the substitution aspect and the anti-derivative partial aspect imply each other if a primitive function for the integrand is known, using the chain rule and a proposition from real analysis. As we have explained above, the existence of primitive functions for holomorphic integrands on simply-connected domains is guaranteed. Hence, the anti-derivative partial aspect implies the formula from the substitution aspect. $\diamond$

9.2.2 *The Green-type partial aspect of complex path integrals*

Green-type partial aspect of complex path integrals

In the preceding partial aspect, it was required that the integrands were holomorphic. Following Trahan (1965), we require a different differentiability condition here, namely that the real and imaginary part of f are continuously differentiable. Additionally, we assume that γ is a simple rectifiable path in Ω, whose interior is also contained in Ω.

The *Green-type partial aspect* is as follows (Section 8.1.5):

Green-type partial aspect

If u and v are continuously differentiable functions on Ω and γ is a rectifiable path, then the complex path integral for $f = u + iv$ along γ may be defined as

$$\int_\gamma f(z)\,\mathrm{d}z = 2i \iint_{\mathrm{int}(\gamma)} \bar{\partial} f\,\mathrm{d}\mathcal{A}. \tag{9.7}$$

Here, $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ is one of the Wirtinger differential operators and the integral on the right-hand side is a double integral for complex-valued functions of two real variables.

Using Trahan's (1965) own terminology, Equation 9.7 may be equally realised as

$$\int_\gamma f(z)\,\mathrm{d}z = \iint_{\mathrm{int}(\gamma)} \delta f\,\mathrm{d}A. \tag{9.8}$$

At this point, δ denotes the differential operator acting on f by $\delta f = \partial_1 u + i\partial_1 v - (\partial_2 v - i\partial_2 u)$ and $\mathrm{d}A = i\,\mathrm{d}x\,\mathrm{d}y$ (i.e., we have the relations $\delta f = 2\bar{\partial} f$ and $\mathrm{d}A = i\,\mathrm{d}\mathcal{A}$).

106 In this context, the reader is invited to recall the notion of *analytic continuation* (e.g., Lang, 1999, ch. XI), but we will forego a discussion here.

This partial aspect is endorsable in a discourse based on real analysis in two variables. The endorsability stems from Green's theorem (Theorem B.15). In particular, the notion of a double integral of a complex function h on a region D of the plane has to be endorsed previously (i.e., $\int_D h \, \mathrm{d}\mathcal{A} := \int_D \mathrm{Re}(h) \, \mathrm{d}\mathcal{A} + i \int_D \mathrm{Im}(h) \, \mathrm{d}\mathcal{A}$ in case the two integrals on the right side exist) in order to be able to use it in the right hand sides of Equation 9.7 and Equation 9.8 (Section 8.1.5).

Keywords and signifiers in the definiens of the Green-type partial aspect centre around continuous differentiability of real-valued functions of two real variable, differential operators, as well as double integrals. Visual mediators for this partial aspect may include plots of the real and imaginary part of $\bar{\partial} f$ on the interior of γ: The graphs of these functions enclose a certain volume with the plane, which, when weighted by sign, realise the double integrals $\iint_{\mathrm{int}(\gamma)} \mathrm{Re}(\bar{\partial} f) \, \mathrm{d}\mathcal{A}$ and $\iint_{\mathrm{int}(\gamma)} \mathrm{Im}(\bar{\partial} f) \, \mathrm{d}\mathcal{A}$ (and similarly for δf).[107]

9.2.3 *The residue-type partial aspect*

Residue-type partial aspects of complex path integrals

In this aspect, Ω is again required to be simply-connected, f to be holomorphic with the exception of a finite subset $A \subseteq \Omega$, and γ to be closed. The following is the *residue-type partial aspect* (Section 8.4.2):

Residue-type partial aspect

The complex path integral of a holomorphic function $f : \Omega \setminus A \longrightarrow \mathbb{C}$ along a closed (piecewise) continuously differentiable path γ in $\Omega \setminus A$ is the finite sum

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi i \sum_{\omega \in A} \operatorname*{Res}_{\omega}(f) \operatorname*{Ind}_{\gamma}(\omega). \tag{9.9}$$

This partial aspect is endorsable according to the formula from the Residue theorem (Theorem A.29). It applies to holomorphic functions with singularities (which are represented by the set A). Similar to the anti-derivative partial aspect, the residue-type partial aspect characterises the complex path integral in terms of a finite sum. Since residues and winding numbers can be defined without complex path integrals, as we have argued in Section 8.4.2, the potential definition in Equation 9.9 is not circularly depending on integration.

The residue theorem usually appears within or at the end of a first course on complex analysis. Hence, this partial aspect seems rather grotesque for a definition of an integral and would rather hide the origin of Equation 9.9. A visual mediator that shows the trace of the path and how much it winds around each point in A is useful to illustrate the relationship between the integrand in terms of its singularities and the path.

9.2.4 *The axiomatic partial aspect of complex path integrals*

Axiomatic partial aspect of complex path integrals

The last partial aspect of complex path integrals is motivated by the *covariation* of the mappings

107 To be precise here again, the current discourse on the Green-type partial aspect seems to be restricted to Trahan's (1965) paper only, in which we do not find any visual mediators except for formulas. Even though the discussion of Green's theorem in relation to complex path integrals is not at all restricted to this paper, we could hardly find the visual mediators just proposed in the literature.

$$\oint\Box\colon H \longrightarrow \mathbb{C},$$

$$\int^{\Omega} : \underline{\mathcal{P}(\Omega) \times \mathcal{C}} \longrightarrow \mathbb{C},$$

or

$$\wr\colon \mathcal{P}(\Omega) \times \mathcal{H}(\Omega) \longrightarrow \mathbb{C}$$

from Section 8.4. Inspired by the axiomatic characterisations of Riemann integrals (Section 8.4.1; e.g., Gillman, 1993; Herfort and Reinhardt, 1980; Pickert, 1976; Tietze et al., 2000), Heffter's (1960) and Klazar's (2019a, 2019b, 2020) goal to only use a few properties of complex path integrals to prove analyticity of holomorphic functions or to minimise the requirements to even define complex path integrals (Section 7.5, Section 8.4.2), and the basic idea of *covariation* for functions (e.g., Carlson et al., 2002; Greefrath et al., 2016b; vom Hofe, 2003; vom Hofe & Blum, 2016; Vollrath, 1989), these three mappings illustrate how the values of complex path integrals vary when the integrands or paths vary. We argued that the covariations of mappings represent the use of complex path integrals in complex analysis, that is, they describe the fundamental object-level rules about complex path integrals (in particular Section 8.4.3).

Moreover, we found a set of properties, which characterise complex path integrals for holomorphic functions completely (Theorem 8.13). This is the *axiomatic partial aspect*:

Axiomatic partial aspect

Let $\Omega \subseteq \mathbb{C}$ be a domain, $f\colon \Omega \longrightarrow \mathbb{C}$ a holomorphic function, and γ a path in Ω. Then, the complex path integral can be defined as

$$\int_{\gamma} f(z)\,\mathrm{d}z = \wr(\gamma, f), \tag{9.10}$$

Here, $\wr$ is the unique mapping $\mathcal{P}(\Omega) \times \mathcal{H}(\Omega) \longrightarrow \mathbb{C}$ with the following properties:

1. *For every $\gamma \in \mathcal{P}(\Omega)$ the function $\wr(\gamma, \cdot)$ is $\mathbb{C}$-linear (i.e., $\wr(\gamma, \alpha f_1 + f_2) = \alpha\wr(\gamma, f_1) + \wr(\gamma, f_2)$ for all $f_1, f_2 \in \mathcal{H}(\Omega)$ and $\alpha \in \mathbb{C}$);*
2. *for every $f \in \mathcal{H}(\Omega)$ the function $\wr(\cdot, f)$ is additive (i.e., $\wr(\gamma_1 \oplus \gamma_2, f) = \wr(\gamma_1, f) + \wr(\gamma_2, f)$ for all $\gamma_1, \gamma_2 \in \mathcal{P}(\Omega)$);*
3. *for every $\gamma \in \mathcal{P}(\Omega)$ and $f \in \mathcal{H}(\Omega)$ it holds that $\wr(\gamma, f) = -\wr(-\gamma, f)$;*
4. *for path along a line segment $\gamma \in \mathcal{P}(\Omega)$, $\gamma\colon [a, b] \longrightarrow \Omega$, it holds that $\wr(\gamma, z \longmapsto 1) = \gamma(b) - \gamma(a)$;*
5. *for every compact triangle $\Delta \subseteq \Omega$ it holds that $\wr(\partial\Delta, \mathrm{id}) = 0$;*
6. *for every $\gamma \in \mathcal{P}(\Omega)$ and every $f \in \mathcal{H}(\Omega)$ it holds that $|\wr(\gamma, f)| \leq L(\gamma) \max_{z \in \mathrm{tr}(\gamma)} |f(z)|$; and*

7. *for every closed path $\gamma \in \mathcal{P}(\Omega)$ and every $f \in \mathcal{H}(\Omega)$ there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \mathcal{P}(\Omega)^{\mathbb{N}}$ of closed polygonal paths such that $\wr(\gamma_n, f) \longrightarrow \wr(\gamma, f)$ for $n \longrightarrow \infty$.*

This is a partial aspect and not a full aspect because it characterises complex path integrals only for holomorphic functions.

As described above, the axiomatic partial aspect is endorsable in a discourse on holomorphic functions and its intersection with the discourse on axiomatic foundations of integrals (e.g., Gillman, 1993; Heffter, 1960; Herfort & Reinhardt, 1980; Pickert, 1976; Shenitzer & Steprāns, 1994; Taylor, 1985; Tietze et al., 2000). To our knowledge, Theorem 8.13 is unprecedented in the literature.

The main keywords in the definiens of the axiomatic partial aspect are "mapping $\mathcal{P}(\Omega) \times \mathcal{H}(\Omega) \longrightarrow \mathbb{C}$" as well as the keywords listed in the properties of $\wr$ such as "$\mathbb{C}$-linear" or "additive". No keywords related to integrals appear in this definiens, similar to the anti-derivative and residue-type partial aspect. No particular visual mediators except for the symbolic significations seem suitable here.

This partial aspect is mainly of theoretical interest. It does not immediately allow any concrete calculations because no rule or procedure for the evaluation of $\wr(\gamma, f)$ is part of this partial aspect. However, the proof of Theorem 8.13 contains the construction of a primitive F for f and then evaluates $\wr$ in terms of differences of the form $F(\gamma(b)) - F(\gamma(a))$. Hence, the anti-derivative aspect is the essential approach to evaluate $\wr$—at least theoretically. Simply stating the properties of the mappings $\fint$, $\int^{\Omega}$, or $\wr$ without any motivation or what they have to do with "integrals" seems to us to be rather unhelpful for learners. However, reviewing the covariation of complex path integrals towards the end of a course in complex analysis may enable the discursants to focus on very important properties of complex path integrals.

9.3 SUMMARY

We have summarised all the aspects and partial aspects we discovered in our epistemological analysis in commognitive terms. Hence, we have accomplished our second goal for this thesis, namely the comprehensive analysis of approaches to complex path integrals as present in the literature and partly beyond. This list of (partial) aspects shows how complex path integrals are connected to various mathematical discourses. In the future, these (partial) aspects may be used for the design of teaching and learning materials on complex path integrals or for further research in complex analysis education.

Part III

EXPERTS' INTUITIVE MATHEMATICAL DISCOURSES ABOUT INTEGRATION IN COMPLEX ANALYSIS

OVERVIEW OF PART III

This part covers our empirical study on experts' intuitive mathematical discourses about complex path integrals. Figure III shows the outline of this part:

- We describe the study design as a multi-case study as well as the methodological background on expert interviews and data analysis in Chapter 10.
- The planning of the interviews, the interview guideline, the selection of the sample, and the conduction of the analysis is presented in Chapter 11.
- The results are contained chapters 12 to 15. First, we introduce the main results in Chapter 12. In particular, we summarise the *discursive frames* governing our experts' construction of *discursive images* about complex path integrals and their intuitive substantiations of integral theorems in complex analysis here. Then, we present the case studies for each expert in Chapter 13, Chapter 14, and Chapter 15. In these chapters, we analyse in detail how the discursive frames were enacted by each participant as well as which discursive images about complex path integrals and intuitive substantiations of Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and the Existence of primitives for holomorphic functions (Theorem A.20) they constructed.

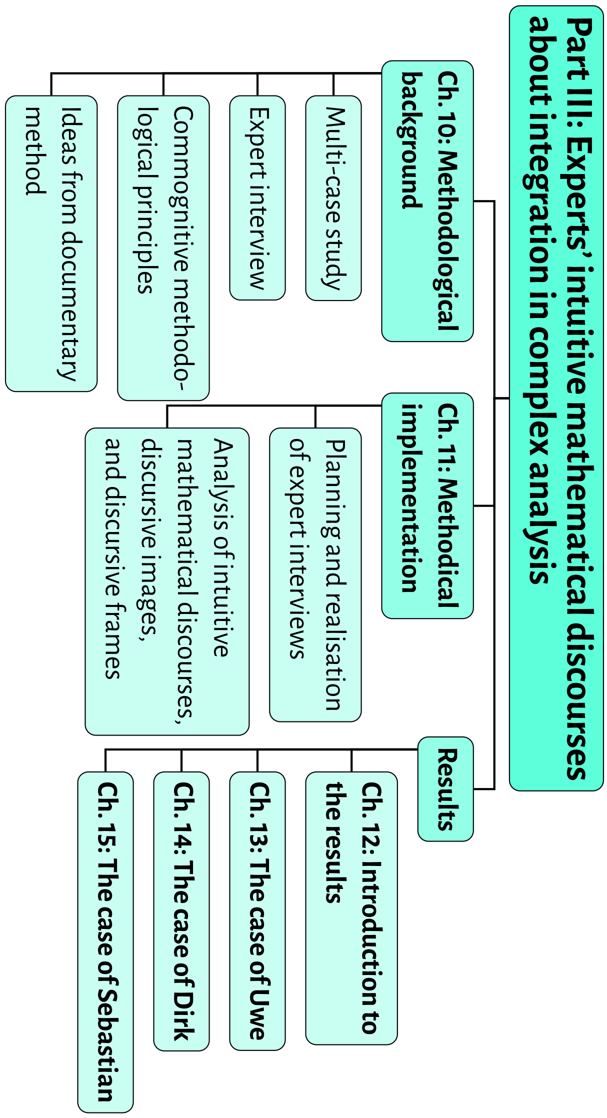


Figure III: Overview of Part iii.

10 METHODICAL AND METHODOLOGICAL BACKGROUND

This chapter deals with the methodological background on our empirical study. This study can be described as

- an *instrumental multi-case study* (Section 10.2), in which
- data is collected in *semi-structured guided expert interviews on equal footing* between expert mathematicians and a mathematics educator (Section 10.3).

We analyse the expert interview in terms of methodological principles of commognitive research (Section 10.4) and furthermore inspired by two steps of interpretation from documentary method (Section 10.5). The aim of the present chapter is to explain the overall methodological perspective. Afterwards, we cover the implementation of the study in detail in Chapter 11.

Since mathematics was conceptualised as a form of discourse, which is led by professional mathematicians at universities and research institutions in the field of research but also in teaching, and learning mathematics is the individualisation public mathematical discourses (e.g., Chan & Sfard, 2020; Sfard, 2008), it is important to find out the ways in which expert mathematicians talk about of mathematical objects and propositions (Sfard, 2014). More generally, all kinds of social domains are perpetuated by experts' activities (e.g., Ericsson et al., 2018), and experts have been the focus of several studies in mathematics education (e.g., Burton, 2004; Heintz, 2000; Kiesow, 2016; Mejía-Ramos et al., 2019; Mejía-Ramos & Inglis, 2017; Mejía-Ramos & Weber, 2020; Nardi, 2008; Oehrtman et al., 2019; Rupnow, 2021; Soto-Johnson et al., 2016; Stillman et al., 2020; Weber et al., 2020; Weber et al., 2014, and many more). This study contributes to this body of research. Understanding how experts work mathematically is especially important because students' possible engagements in activities related to those by experts can only be studied and valued after it has been more thoroughly understood in which ways mathematicians "attach[]" understandings to their "formalism" (Weber et al., 2014,

p. 38; see also Mejía-Ramos & Weber, 2020; Weber et al., 2020). Thus, engaging experts in communication about their idiosyncratic ways of understanding of mathematical objects and propositions—in other words: intuitive mathematical discourses (Chapter 4)—is a vital point of departure in university mathematics education.

Based on our theoretical development in Part i, we can rephrase our initial complex of research questions B (Section 1.2) as follows:

B': i) What characterises experts' intuitive mathematical discourses about complex path integrals?

ii) What are their discursive images about complex path integrals?

iii) How do experts substantiate central integral theorems from complex analysis intuitively?

In order to answer these questions, we reconstruct experts' discursive images of the complex path integral, their intuitive substantiations of integral theorems in complex analysis, and underlying discursive frames guiding their explorations of discursive images or other intuitive interpretations of complex path integrals and integral theorems (see Chapter 4).

We choose a qualitative, reconstructive approach (Bohnsack, 2014b; Kleemann et al., 2013; Przyborski & Wohlrab-Sahr, 2014; Rosenthal, 2014). As "intersubjective transparency", "indication of the research process", "empirical grounding", and considerations of "relevance" and "limitation", "coherence", and "reflected subjectivity" (Steinke, 2017, p. 328–331, own transl., emph. omitted) are central for the quality of qualitative research, the methodological ideas for the selection and conduction of data collection and data analysis methods will be comprehensively discussed in this and the next chapter (see also Section 11.5; cf. Flick, 2019; Helfferich, 2011; Misoch, 2015).

Since our research question addresses mathematical communication, data collection and analyses have to be chosen in such a manner that they respect this specific research focus: The data collection should support communication (spoken, written, and visual mediators), and the data analysis should allow to identify patterns in individuals' intuitive mathematical discourses. Our researcher's task is then to produce research-based stories about discursive actions of the mathematical actors in our study (Sfard, 2018). Borrowing a phrase by Barwell (2013a, p. 211), our "interest is not so much in what participants mean, as in how 'what they mean' is discursively constructed and dealt with through interaction".

10.1 INTRODUCTION AND OVERVIEW OF METHODOLOGY

10.1.1 *Overview of methodology*

The study is designed as a *multiple case study* with mathematicians who do research and teaching in complex analysis or related fields (Section 10.2). Data is collected in semi-structured expert interviews (Section 10.3, Bogner et al., 2009; Gläser and Laudel, 2010; Meuser and Nagel, 1991; Nohl, 2017; Pfadenhauer, 2009). The interviews are then transcribed and analysed with respective to the principles of commognitive methodology (Section 10.4; Sfard, 2008, 2013) and inspired by steps of interpretation from documentary method (Section 10.5; e.g., Bohnsack, 2014b; Bohnsack et al., 2010; Kleemann et al., 2013; Nohl, 2017; Przyborski and Wohlrab-Sahr, 2014). On the one hand, we focus on the thematic content contained in experts' narratives, and on the other hand, we are interested in the ways the experts actually fabricated their talk about complex path integration. That is, one the one hand, we focus on their choice of keywords, visual mediators, and narratives, everything that is factually produced, and on the other hand, we look for explicitly stated or implicitly guiding metarules with the help of which we can describe the experts' intuitive mathematical discourses from the interviews. In this step we also analyse their particular use of keywords, potential commognitive conflicts, and reconstruct discursive frames. This distinction between the actually produced discursive entities (keywords and visual mediators) as well as their fabrication in context (e.g., in terms of discursive frames) is suggested from the point of view of commognition (Lavie et al., 2019; Sfard, 2008) and is also seen as relevant in the formulating and reflecting interpretation from documentary method.

10.1.2 *Two remarks on reconstruction and methodological terminology*

Two remarks are in order: one about the understanding of *reconstruction* in this thesis, the other about the terminological interplay between terms from qualitative, sociologically oriented methodology, and commognition.

A remark on reconstruction

Reconstruction

Discursive actions are based on individuals' interpretations of task situations according to their own previous actions in precedents or other discursants' actions in those situations (Lavie et al., 2019). This may include obeying to explicitly or implicitly known rules. These ruled-based activities based on precedents may be described as individuals' constructions of first kind (in the sense of Schütz) structuring individuals' actions, which however may not be explicitly known or communicable (Przyborski & Wohlrab-Sahr, 2014, pp. 12–13). Reconstructions in this sense are thus constructions of second kind (Przyborski & Wohlrab-Sahr, 2014, pp. 12–13; cf. Bohnsack, 2014b, pp. 22–32).

In commognitive terms, we understand reconstruction as commognitive researchers' discursive activity to construct and substantiate narratives about individuals' actions in a discourse (cf. Sfard, 2013, 2018). The product of such a reconstructive procedure are interpretations of the empirically gathered data; in other words, *third person's* or *researchers' narratives* about the discourses and discursants, which were observed and analysed in the process of research (cf. Kontorovich, 2021b; Sfard, 2008, 2020a; Sfard & Prusak, 2005). In order to arrive at these reconstructions, we distinguish what is said during the interviews from how it is said (cf. Section 10.5.2). This is compliant with the commognitive framework and also corresponds to our distinction between discursive images (i.e., what the interlocutors say about the complex path integrals in relation to what they perceive as their mental images or intuitive interpretations) and discursive frames (i.e., the sets of metarules governing interlocutors' intuitive mathematical discourses, in particular for the construction of discursive images and intuitive substantiations of mathematical propositions).

A remark on methodological terminology

In this chapter on methodology, we draw on ideas from qualitative research and we will mostly follow the terminology from there. Especially with respect to expert interviews, the word "knowledge" will appear. These are sensitising concepts on their own (Blumer, 1954). Instead of being fully defined concepts, they give "the user a general sense of reference and guidance in approaching empirical instances" (Blumer, 1954, p. 7). Commognitively speaking, we may say that having knowledge of a mathematical topic means being able to perform discursive actions with regard to this topic. Cooper and Lavie (2021, p. 2) describe "[k]nowledge in a discipline such as mathematics [...] as participation in a (mathematical) discourse, recognized by its special characteristics", namely its keywords, routines, visual mediators, and endorsed narratives. In our view, it is slightly preferable not to equate knowledge with participation but rather with explicated or visible outcome of such participation. That is, a piece of knowledge may be seen as a narrative or a routine in the surrounding discourse (e.g., complex analysis) about the objects or participants of the respective discourse.

10.2 STUDY DESIGN

Let us now describe the general design of our empirical study (Przyborski & Wohlrab-Sahr, 2019). As already explained in Chapter 5, Oehrtman et al. (2019) conducted the only study on experts' idiosyncratic interpretations of complex integration. The authors showed that most of their experts struggled to interpret the complex path integral intuitively. Since little empirical research is available and none is studying this phenomenon through the commognitive lens, a *case study* is a proper choice of research design for our study, too (Denscombe, 2010; Flyvbjerg, 2006; Grandy, 2010; Hering & Jungmann, 2019; Przyborski & Wohlrab-Sahr, 2019; Stake, 1995).

Instrumental multi-case study

Case study is as a method to collect data along *cases*, but it is not implying a procedure of data analysis (Gerring, 2014). A case study is "*as an intensive study of a single unit for the purpose of understanding a larger class of (similar) units*" (Gerring, 2014, p. 342, emph. orig.). More precisely, the case study design in this part of the thesis is an *instrumental case study* (Grandy, 2010; Stake, 1995). Here, the single case, in terms of a person or an organisation, is of second interest for the researcher. Rather, this type of case study is characterised by its interest in a previously determined phenomenon. While instrumental case studies "do not permit generalization in a statistical sense" it aims to "identify patterns and themes" (Grandy, 2010, p. 474). Instrumental case studies persuade by "richness rather than generalizability" (Grandy, 2010, p. 475):

> In developing new theory or testing out existing theory, it [the instrumental case study; EH.] allows researchers to use the case as a comparative point across other cases in which the phenomenon might be present. (Grandy, 2010, p. 475)

Since we address phenomena of mathematical discourses, it is useful to study the discourses of multiple individuals. Hence, a case study design with multiple cases (Stake, 2006) allows to integrate several cases into one study with the aim to get insight into a variety of manifestations of intuitive mathematical discourses and initiate empirically grounded comparisons. Therefore, the overall design of the empirical part of this thesis can be seen as an *instrumental multi-case study* (Grandy, 2010; Stake, 1995, 2006). Our multi-case study will be "*[d]iscovery led*" (descriptive, exploratory, and partly comparative) (Denscombe, 2010, p. 55, emph. orig.) since we will analyse the intuitive mathematical discourses about complex path integrals by different individuals and compare them with each other. It is also "[t]heory led" (Denscombe, 2010, p. 55, emph. orig.) as it illustrates the application of the commognitive framework to the study of experts' intuitive interpretations of complex path integrals. With this regard, our case study does not aim to generalise but to gain fundamental theoretical insights into a practically unexplored area of research in mathematics education (cf. Eisenhardt, 1989; Eisenhardt & Graebner, 2007; Flyvbjerg, 2006, 2011; Grandy, 2010; Rittberg & Van Kerkhove, 2019; Stake, 1995, 2006; Ylikoski, 2019).

10.3 EXPERT INTERVIEWS

10.3.1 *Basic assumptions*

Expert interviews

Qualitative interviews are an important method for data collection when researchers are interested in specialised forms of knowledge, narrations, or other types of information (Hopf, 2017; Przyborski & Wohlrab-Sahr, 2014). An *expert interview* is a specific type of semi-structured guideline interview (Bogner et al., 2009; Gläser & Laudel, 2010; Meuser & Nagel, 1991, 2009a; Misoch, 2015; Nohl, 2017; Przyborski & Wohlrab-Sahr, 2014, ch. 3.4.5), which are characterised by the interviewees, namely, the *experts*. Expert interviews are often conducted in order to obtain information that is only accessible via direct questioning of certain persons. According to Gläser and Laudel (2010),

> *'[e]xpert' describes the specific role of the interview partner as source of special knowledge of the social circumstances being researched. Expert interviews are a method to exploit this knowledge* (p. 12; own transl., emph. orig.)

In other words, expert interviews are a method of data collection that is suitable for gathering information about special types of knowledge or actions, which are tied to the role of persons in special social settings.

Meuser and Nagel (2009b) differentiate that experts are leading the knowledge production in their field, for one thing because of their special profession specific knowledge and acting, but then again also by virtue of their social position, for example, as professor or lecturer at university. Having obtained such a position is an indicator for us as a researcher that have had specific insight into the subject matter. Moreover, these professionals need to act variably and also in non-standardised ways when they face teaching and research problems (cf. Pfadenhauer, 2009, pp. 87–90). The experts "stand for a problem perspective that is typical for the institutional context in which she or he gained her or his knowledge and in which she or he acts" (Meuser & Nagel, 2009a, p. 469, own transl.). In our case, the experts are two lecturers and one professor in mathematics from two German universities (see Section 11.1.1), who taught complex analysis before and whose research area is connected to complex analysis, too. In alignment with the goals of an instrumental case study (Section 10.2), this also means that we are not interested in the single person in his role as an expert, but rather the discursive activities they engage in when facing the problem to explain complex path integrals in a way they deem intuitive for themselves and potentially for others.[108]

In order to elicit this specific view on a topic and their communicative actions, it is important to let the expert narrate; it is not suitable if the interview is guided too much or reduced towards a standardised inquiry (Meuser & Nagel, 1991). Rather, an "open-ended guideline interview is the appropriate instrument for data collection" (Meuser & Nagel, 2009a, p. 472, own

108 In the context of complex analysis, it is clearly not the case that only experts may be described as particularly knowledgeable; good students may also be proficient in complex analysis discourse, too. However, an expert is someone who "possesses knowledge, which she or he may not necessarily possess alone, but which is not available for everybody in the interesting field of action" (Meuser & Nagel, 2009a, p. 467, own transl.), but whose experience as a researcher and lecturer and her or his participation as experienced agents in complex analysis discourses is what sets them apart.

transl.). Therefore, it is useful to prepare questions and other stimuli, which serve as an initial guide, but also leave room such that the interviewees can include their own perspective to the interviews (Gläser & Laudel, 2010; Meuser & Nagel, 1991, 2009a; Misoch, 2015; Pfadenhauer, 2009).

10.3.2 *Conducting expert interviews "on equal footing"*

Interviews at the eye-level / at equal footing

Since we are not interested in factual knowledge about complex path integrals, our study demands that the interviews are conducted in a rather "relaxed", "'everyday[ed]'" way as "quasi-normal conversation" (Pfadenhauer, 2009, p. 84), in which the interviewer and interviewee communicate "at the eye-level", that is, "on equal footing" (Pfadenhauer, 2009, p. 86):

> Insofar as the epistemological interest of the expert interview is directed at issues that are regarded as relevant or are being debated among experts, the accompanying basic concern of the expert interview is to create an interview setting that approaches the conversation situation among experts as closely as possible (Pfadenhauer, 2009, p. 86).

In this context, Kontorovich (2021a, p. 218) describes that the communication between a mathematician and a mathematics educator "requires a professional vulnerability, openness, and trust from the interlocutors". The style of conversation should be rather free such that the interviewees feel comfortable to express their thoughts, even if these are not mathematically rigorous. At no point should the interviewees be afraid of expressing their idiosyncratic perspectives (Kontorovich, 2021a; Pfadenhauer, 2009). In order to achieve such a situation, the interviewee should acknowledge the research interest of the interviewer, which is best acquired if the interviewer demonstrates genuine interest and competency in the interviewee's field as well.

Hence, since the interviewer is also an insider of complex analysis discourse, he can ask deeper questions or give new prompts if the conversation gets stuck (e.g., a provocative question or a statement from another interviewee to test the current interviewee's reaction to another experts' opinion). Eventually this means that the interviewer is not completely neutral in asking and prompting. However, it is exactly this kind of conversations making our interviews valuable because the interlocutors can freely interact with the mathematically fluent interviewer and thus they can also direct the conversation with respect to what is relevant to them personally.

A first remark on analysing expert interviews

The analysis of expert interview is often described to be guided with a subsumptive focus. Gläser and Laudel (2010) suggest using *qualitative content analysis*, which they describe as "an extraction, that is, as a procedure that extracts information from the text and processes it separately from the text", and which is essentially based on coding methods (Gläser & Laudel, 2010, p. 271, own transl.). Similarly, Meuser and Nagel (1991, p. 455, own transl.) describe that the analyses of expert interviews focus on "empirical knowledge but not the theoretical explanation of generalisation of the empirical facts". Therefore, these directions of analyses of expert interviews are mostly oriented towards thematic units, extraction of information, and coding rather than sequential analysis.

These analytical principles do not primarily aim at the logic of the case. They aim at a reduction of complexity by thematic grouping, paraphrasing, or coding. However, a primarily subsumptive focus does not fit to the rationale of our empirical investigation. Rather, in order to analyse experts' intuitive mathematical discourses and henceforth the discursive mental images and discursive frames as instantiated in the expert interviews, the enacted discourses in the expert interviews need to be the unit of analyses. The very discursive nature of the dictum *thinking as communicating* (Sfard, 2008) requires that we focus on the discursive features (keywords, narratives, visual mediators, routines) from the commognitive framework, and the notions of discursive images and discursive frame we developed in Chapter 3. For this to be done, several guiding principles of commognitive data analysis have to be followed, which we will explain in the next section.

10.4 METHODOLOGICAL COMPONENTS OF COMMOGNITIVE DATA ANALYSIS

Methodological components of commognitive data analysis

In sum, there is not one method of data analysis that fits the commognitive framework. Notwithstanding, empirical commognitive research is based on several principles, which place discourse at the centre of analysis and move away from assertions about what discursants *possess* or *are* towards detailed accounts of what they *do* (Sfard, 2013, 2020a). Sfard (2013, pp. 158–161, emph. orig.) characterises methodological aspects of commognitive research as follows:

- "*principle of disobjectification*": Commognitive researchers should rather report about what interlocutors *do* instead of what they *have* or *are*.
- principle of "*operationality*": Terminology used in educational research should be explained as clearly as possible. They should allow the researcher to identify "situations in which these terms may be used as descriptors of participants' actions" (Sfard, 2013, p. 160).
- "*verbal fidelity*": Interlocutors' utterances should be reported as accurate as possible.
- "*principle of multimodality*": Not only verbal utterances but also non-verbal actions should be made visible in research reports.
- "*discourse completeness*": The object of study is discourse as a whole rather than isolated bits or the mathematical concept.
- "*principle of interdiscursivity*": It is likely that multiple discourses overlap and that multiple discourses influence interlocutors' discursive actions.
- "*principle of alternating perspectives*": The researcher should alternate between the perspectives of an insider and an outsider of a discourse. For example, the commognitive researcher should bear in mind that her or his own use of keywords may differ from that of the discursants she or he observes. Therefore, the commognitive researcher should be attentive towards possible incommensurabilities between different interlocutors' mathematical discourses.
- "*principle of ontological fidelity*" and prevention of "*ontological collapse*": By way of participating in mathematical discourses, an interlocutor tells a story about mathematical objects or participants of that discourse. The commognitive researcher in turn should aim to tell stories about these stories. The latter should not be confused with the former, and hence, the authors of the stories should not be neglected.

- "*principle of generalizability*": The commognitive researcher should aim to look for patterns that go beyond single cases only (Sfard, 2013, p. 161).
- "*principle of empirical accountability*": Every assertion the researcher makes about data should be substantiated with explicit references to pieces of data.
- "*theoretical accountability*": Attempts towards theoretical generalisation should be evidenced as much as possible. For example, the commognitive researcher should "seek analytic explanation or support her local story with similar stories from additional sources (this, if you wish, is the quantitative component of discursive research)" (Sfard, 2013, p. 161).

Commognitive analyses often focus on one or more of the four features of discourses (keyword usage, endorsed narratives, visual mediators, and routines). The research process usually consists of the transcription of data, a sequential analysis of interlocutors' discursive as well as their accompanying practical actions, and comparisons between different fragments of data (within case and across case).

Sfard (2008, p. 139, emph. orig.) suggests one method to approach data, which she calls "*interpretative elaboration*": Performing an interpretative elaboration means to go through the text "utterance by utterance" and "elaborate[] on the text produced by the interlocutors" (Sfard, 2008, p. 139). In this context, Morgan and Sfard (2016, p. 104, emph. orig.) suggest to "(1) specify *the aspects of discourse* on which the analysis should focus; (2) formulate *questions* about each of these aspects; and (3) operationalise these questions by defining their central notions with the help of *textual indicators*".[109] For example, one guiding question for the analysis of visual mediators is "To what extent does the discourse make use of specialised mathematical modes?", which can be found via textual indicators such as the "presence of table, diagrams, algebraic notation, etc." (Morgan & Sfard, 2016, p. 106). Concerning (endorsed) narratives, one may ask "What is the degree of alienation of the discourse?", which can be answered via "mathematical objects as agents in processes", whose agency might be obscured by "non-finite verb forms" or "passive voice" (Morgan & Sfard, 2016, p. 107). Concerning routines, one may ask whether an interlocutors' goal is to produce a new narrative, which hints at an exploration, or whether she or he is interested in performing a certain process, possibly to please another interlocutor, which hints at a ritual (Lavie et al., 2019; Morgan & Sfard, 2016; Sfard, 2008).

Changing from an insider to an outsider while analysing data (*principle of alternating perspectives*) is important for finding commognitive conflicts: An insider is likely to react to the utterances of her or his interlocutors based on her or his own precedents. An outsider is more neutral with regards to her or his precedents, for example, she or he may not know which discursive reaction is suitable for one of the discursive actions she or he witnesses. Taking both perspectives, the researcher is able to spot commognitive conflicts (by taking an outsider perspective) and its potential origins (by taking an insider perspective to reconstruct why and how interlocutors may have produced their discursive actions).

109 To be precise, Morgan and Sfard (2016) deal with written discourses in the form of examination scripts. Nevertheless, the questions and indicators they list (except for those directly aimed at the examination) are suitable for the analysis of other discourses more generally, too.

10.5 DOCUMENTARY METHOD

We add to our analytical stance methodological considerations from *documentary method*. We note though that we do not use documentary method as our principal method for data analysis explicitly, but we will borrow some of its methodological ideas to guide our data analysis later (see Section 11.3 and Section 11.4). In particular, we relate the documentary method's focus on the *objective meaning* of a text, i.e., the *what*, and the *documentary meaning*, i.e., the *how* of the interlocutors' action to produce the text (Bohnsack, 2014b) to the commognitive framework and our conceptualisation of intuitive mathematical discourses from Chapter 4.

10.5.1 *Basic assumptions*

Documentary method

Originally developed for the analysis of group discussions (Bohnsack, 2014b), *documentary method* is now used in many different contexts (cf. Kleemann et al., 2013, ch. 5; Przyborski & Wohlrab-Sahr, 2014, ch. 5.4; Przyborski & Slunecko, 2020) and also for the analysis of interviews (Nohl, 2017). The general idea of *documentary method* is to gain

> access to that knowledge, which is not given to us lexically, conceptually, but is embedded into our immediate practice of acting as implicit knowledge, which is shared with others (Przyborski & Wohlrab-Sahr, 2014, p. 13, own transl.).

Therefore, one of the main assumptions of documentary method is that people live in *conjunctive spaces of experience*, in which they gain and share *atheoretical* knowledge from everyday practice that "serves as an orientation for our practical action and [...] practice" (Bohnsack, 2014a, p. 4). More precisely, this knowledge is "intuitive" and "pre-reflexive", that is, embedded into practice and therefore not necessarily conscious (Bohnsack, 2014a, p. 5). Atheoretical knowledge is described as

> a general term, including the incorporated knowledge, which we acquire in a valid way through the medium of material pictures, as also the implicit or metaphoric knowledge, which we acquire through the medium of mental images as we can find them in narrations and descriptions – that is to say, in texts (Bohnsack, 2014a, p. 5).

Atheoretical knowledge is thus depending on individuals' experiences. Their experiences in collective situations enable them to build up parts of their knowledge, which may not be explicit and verbalisable but are guided by their presence in the collective (Przyborski & Slunecko, 2020, p. 542). Hence, individuals sharing such atheoretical knowledge build up conjunctive experiences (Bohnsack, 2014a, 2014b; Nohl, 2017). Accordingly, documentary method acknowledges two strands of knowledge, explicit and verbally expressible knowledge, and knowledge in and for activities, which may not be explicitly verbalisable by the participants of an activity.[110]

These ideas are in line with the commognitive perspective on individuals' discourses as being subsequent, individualised versions of collective discourses, in which the individuals participate. When faced with a new task situation, the individuals choose and adapt their discursive

110 An example for athereotical knowledge is that of tying a knot. To know how to tie a knot is to perform that action, which is nevertheless difficult to verbalise exactly. The documentary meaning is the process of producing the knot; as such, it is tied to practical action. We may know how to tie it but nevertheless experience difficulty in describing this procedure. This example illustrates the task of researchers to reconstruct how people do or tell what they do or tell. (Nohl, 2017, p. 5)

actions based on their precedent-search-spaces, which constitute of precedents the interlocutors identify as similar enough to guide their current actions (Lavie et al., 2019).

Documentary method is not interested in truth or normative correctness. Rather, truth and normative correctness are "bracketed" from the analysis (Bohnsack, 2010, p. 108). The focus of interest is directed towards *what* is said or done and *how* it is said or done. This means that in a first step of interpretation data is analysed with respect to what is objectively expressed and then the fabrication of the talk is analysed. The two steps of documentary method, the *formulating interpretation* and the *reflecting interpretations*, address these two layers, the when and the how, of analysis (see Section 10.5.2).

In this regard, documentary method distinguishes two kinds of meaning, which may be traced in texts such as transcripts of interviews, and which we see connected to the commognitive point of view of human communication as organised in discourses: *immanent* and *documentary meaning*. Immanent meaning is divided into *intentional expressive meaning* and *objective meaning*. The first of these signifies what an interlocutor actually meant when saying something; this is empirically unascertainable, and therefore does not qualify for research, and is bracketed from analyses. The objective meaning though is what is said and which can be identified thematically. For instance, related to mathematical discourses, the objective meaning can be directly read off from textual data in form of the keywords interlocutors use as well as the narratives and visual mediators they produce. Narratives represent communicative knowledge, that is, forms of knowledge an interlocutor is actually able to produce.

Contrary, documentary meaning is expressed in the actions of the interlocutors. The documentary meaning refers to the act of the production of an action: "documentary meaning gauges the action or text according to the process by which it came about" (Nohl, 2010, p. 201). Reconstructing the documentary meaning comprises the identifications of regularities: "this involves identifying continuities across a series of action sequences or narrative sequences about such actions" (Nohl, 2010, p. 208). Even if interlocutors are not able to explicate what they believe to know or to do, they are able to act according to their interpretations of task situations (cf. Lavie et al., 2019) and guiding frames (cf. Bohnsack, 2014b). Researchers may then inspect these actions and look for potential regularities to account for the observed actions.

Relation to the commognitive framework

Let us now rephrase these layers of meaning in terms of the commognitive framework. This may partly change the original intentions of documentary method, but this rephrasing of its elements is essential in order to be coherent with the theoretical point of view in this thesis.

As explained above, documentary analyses ask for what and how something is said (Bohnsack, 2014b; Nohl, 2017). It is thus concerned with how interlocutors give birth to their utterances, whether and how public discourses manifest themselves in these utterances, and whether and which idiosyncrasies constitute an interlocutor's individualised discourse. The objective meaning of utterances is thus what the discursants actually produce: It appears as the chosen keywords and visual mediators, as well as the constructed narratives, which may be further endorsed. The documentary meaning corresponds to the ways, in which individuals fabricate their discursive actions, in other words, to the enacted metarules and routines. With respect to the analysis of intuitive mathematical discourses, this means that we have to inspect how the individuals shape this discourse, for instance, from which discourses they borrows keywords, (endorsed) narratives, and visual mediators, and which metarules they enact. Of particular interest to us is how our experts initiate and conduct their explorations about potential mental images or other intuitive interpretations of complex path integrals and integral theorems in their intuitive mathematical discourse, and on the discursive images about complex path integrals (Section 4.2) they produce. That is, we are interested in what they construct and which discursive frame govern the how of these constructions (see Section 4.4).

Eventually this also means that we have to take into account that the interviewees may not even know what they know or how they proceed in constructing narratives about mathematical objects even if they are perfectly capable of doing it. Przyborski and Slunecko (2020, p. 541) even argue that the process of making explicit what interlocutors have been doing may actually detain them during their actions (Przyborski & Slunecko, 2020, p. 541). In other words, research participants will often not make explicit what they know or do, especially not in terms of researchers' theoretical terms (Bohnsack, 2014b; Kleemann et al., 2013; Nohl, 2017; Przyborski & Wohlrab-Sahr, 2014).

The commognitive researcher analyses the publicly observable stories offered by the research participants and their discursive actions. In line with commognitive principles of data analysis (Section 10.4), it is the researcher's task

> · to base her or his statements as far as possible on the interlocutors' activities,
>
> · to show that her or his interpretations are fully compatible with the details of the interactive events,
>
> · to make explicit that and how the utterances of the participants in the conversation can be interpreted in terms of her or his statements (Deppermann, 2008, p. 51, own transl., bullet points in orig.).

Our commognitive analysis will eventually be inspired by two steps of analysis, namely the formulating and the reflecting interpretation of our interview transcripts, to be described in the next subsection.

10.5.2 *Formulating and reflecting interpretation*

Formulating and reflecting interpretation

Two steps of interpretation are distinguished in documentary method (Bohnsack, 2014b; Nohl, 2017):

- the formulating interpretation and
- the reflecting interpretation.

We adapt these two steps for our data analysis. In order to emphasise that we borrow ideas from the following two steps of interpretation and combine them with the commognitive analysis, we will refer to our steps of analysis later as *initial interpretation* and *fine interpretation* (Section 11.3).

Formulating interpretation

The *formulating interpretation* is the first step of data analysis in documentary method. Its main purpose is the identification of themes during an interview (Nohl, 2017, p. 30). In the first step of formulating interpretation, data is segmented into smaller passages (i.e., episodes). Themes can be entries of the interview guideline or topics discussed during a sequence of utterances.

Moreover, it should be taken into account whether a topic from a previously analysed episode or interview appears in another (Nohl, 2010, p. 204).[111]

Then, in the following *formulating fine interpretation* (Nohl, 2017, p. 31, own. transl.), the episodes may be analysed further thematically by identifying potential smaller chunks of an episode and thematic subunits. In our reading of documentary method, the main topics identified in an episode (or sub-episode) can be keywords related to the research objectives, elements from the theory (e.g., commognition) used by the researcher, or a short paraphrase attached to the chunk of data. Thus, the goal of the formulating interpretation is to identify themes, to sensitise the interpreter for the observed interlocutors' word usages, and to alienate the researcher from the text. After all, "the thematic content is not self-evident but in need of an interpretation" (Nohl, 2017, p. 31, own transl.). In Bohnsack's (2014b, p. 35., own transl.) words, one stays in the "reference system of the text"; that is, "[o]ne summarises what competent members of a linguistic community understand rather indisputably about the text" (Przyborski & Slunecko, 2020, p. 546), and one respects who is the author of the respective text.

In terms of the different meanings that are distinguished in documentary method, the use of words and the uttered narratives constitute the objective meaning in an episode. In commognitive terms, one identifies its main keywords, narratives, and the mathematical discourses the keywords and narratives may belong to. This way, the commognitive researcher begins to construct her or his own story about the discursive actions and stories present in the data. Nevertheless, this is only an initial step: The formulating interpretation mainly functions as a tool to select episodes for further and finer analysis in alignment with the research questions.

Reflecting interpretation

What is said can be separated from how it is said. Accordingly, the *reflecting interpretation* is intended to reconstruct the ways the interlocutors talk about what is being talked about: "the reflective interpretation aims at the reconstruction and explication of the *frame*, within which the topic is dealt with, the way *how* [...] the topic is treated" (Bohnsack, 2014b, p. 137, own transl., emph. orig.). These ways of fabrication of talk are understood as a *frame of orientation* (Bohnsack, 2010, 2014b; Nohl, 2010, 2017). We have defined a commognitive version of such frames in Section 4.4, namely *discursive frames* as sets of metadiscursive rules governing experts' intuitive mathematical discourses.[112]

The reflecting interpretation is the step of analysis, where data is examined more closely. We may for instance examine the four features of discourses (Section 3.4). In our case, we base our further interpretations of the data on the concepts offered by the commognitive framework: We analyse our experts' use of keywords related to complex path integration and what they perceive and describe as their mental images or otherwise intuitive interpretations of complex path integrals. At this point, we highlight the discursive images about complex path integrals our experts construct and observe whether they use visual mediators to accompany their discursive images. Furthermore, once we have carved out a set of metarules, which may have caused our experts to construct their discursive images or other intuitive interpretations, we hypothesise that we have found a discursive frame of their intuitive mathematical discourses

111 Clearly, the exact grouping of text passages into an episode is not unique. However, this grouping as such is not of too much importance because the text will be of course analysed and interpreted much further. In principle, identifying episodes is a pragmatic enterprise.

112 Hence, our inclusion of discursive frames to the notion of intuitive mathematical discourse is also justified retrospectively from the basic tenets of documentary method.

about the complex path integral. Finally, we look for other episodes of the same interview or in other interviews in order to identify other discursive actions which may also be described in terms of this frame. In Kontorovich's (2021b, p. 5) words, we strive for "analytical statement[s] with which the observed performance is consistent".

Reflecting interpretations are carried out *sequentially* and *comparatively* (Nohl, 2017, pp. 35–41).

Sequentiality

In a sequential analysis of utterances, only previous utterances from the same text can be taken into account. Hence, it is analysed how a discursant reacts to the directly preceding utterances but possibly also utterances further ago: "Every apparent individual action has been sequentially connected in the sense of a well-formed, rule-like linkage to a preceding action and in turn opens up a leeway for well-formed, rule-like connections" (Oevermann, 2000, cited by Sammet & Erhard, 2018, p. 32, own transl.).

A commognitive version of reflective interpretation also needs to take into account discursants' precedents (Lavie et al., 2019). However, in interviews, these precedents are hardly visible due to the short time the discursants are recorded (in principle, longitudinal observations would be needed to identify precedents and situations, in which an interlocutor considers these precedents for further discursive actions). Practically, for the analysis of interviews, precedent-search-spaces are only recognisable in case the discursants explicitly refer to precedents (and even in this case, only the retrospective description of the precedent is made available). Otherwise, they have to be hypothetically derived from the interlocutors' discursive actions (cf. Kontorovich, 2021b). Therefore, in the sequential interpretation of an interview excerpt, an utterance is analysed in relation to previous utterances by the same or the other interlocutor. For example, the reappearance of a certain topic signals that this topic may be important for an interlocutor and may hint towards a discursive frame.

Comparativity

Comparativity signals that the researcher looks for comparisons both within the same case as well as across cases. It is precisely through this comparative perspective that similarities and differences in dealing with a topic become visible. This way, theoretical conclusions are empirically grounded and validated with respect to multiple occurrences in data. In particular, the reconstruction of one discursive frame becomes more comprehensible compared to other discursive frames as reconstructed in different portions of data:

> The increase in the validity of a case analysis is thus not only linked to the increasing empirical foundation of the respective case itself but also to the increasing *empirical foundation* of the horizons of comparison [...]. (Bohnsack, 2014b, p. 139, own transl., emph. in orig.)

Comparisons within and between cases are based on the search for *homologies*, that is, thematically comparable beginnings of episodes and their further elaboration during the episodes. In principle, homologies can also be obtained in thought experiments (Nohl, 2017, pp. 35–41). For example, in our case the interviewer referred to the basic idea of area for Riemann integrals (Section 2.2.3) in all interviews, thus initiating episodes with a comparable beginning, but each interviewee reacted to this prompt quite differently as we will see in our results. Comparing homologous episodes, we can determine the "implicit regularity" in individuals' discourses (e.g., in which situations they follow a certain metarule) (Nohl, 2017, p. 37, own transl.).[113]

In sum, during a reflecting interpretation, the researcher examines the fabrication of talk and the specific ways in which interlocutors address a topic. Accordingly, the "orientation patterns [...; in our context: discursive frames; EH.] developing processually in the course of the discourse" (Bohnsack, 2014b, p. 39, own transl.) are worked out (see Section 4.4).

113 For example, Uwe and Sebastian (Chapter 13, Chapter 15) immediately continued to value this interpretation as not helpful for complex path integrals, whereas Dirk (Chapter 14) tried to transfer it to complex path integrals as well.

11

METHODICAL IMPLEMENTATION

Whereas the last chapter dealt with methodological thoughts about the study design, data collection, and data analysis, this chapter is devoted to the implementation of the study. In Section 11.1, the planning of the interviews including the choice of the sample and the interview guideline is described. Then, Section 11.2 covers the actual data collection in terms of the conduction of interviews, videography, and transcription. In Section 11.3, we describe more closely how we analysed the transcripts. The chapter closes with remarks on the presentation of the results (Section 11.4) and the quality of the research process (Section 11.5)

11.1 PLANNING OF INTERVIEWS

The interviews were planned and conducted in German. In the following presentation of the guideline and analyses, all excerpts from the transcripts and interview guideline are of course presented in English. The full interview guideline can be found in the appendix (in German and in English; Appendix D).

11.1.1 *Sample*

A snowball sampling technique (Akremi, 2019, pp. 320–321; Misoch, 2015, pp. 193–194) was applied to recruit three mathematicians from two different German universities. They have different research areas related to complex analysis and were required to have teaching experience in complex analysis. Two of the experts were chosen through personal acquaintance, another was recommended by colleagues.

Table 11.1 shows a list of the participants, Uwe, Dirk, and Sebastian (all names are pseudonyms), their main research areas, as well as the approximate duration of their autonomous teaching experience. All of them have taught various courses in the field of analysis,

Table 11.1: Overview of the interviewees.

	Research area	Teaching experience
Uwe	Applied analysis & functional analysis	≈ 17 years
Dirk	Analysis & geometry	≈ 18 years
Sebastian	Applied analysis & dynamical systems	≈ 14 years

including complex analysis. Uwe and Dirk are lecturers in mathematics, Sebastian is a professor. As such the taken sample contains a selection of persons regularly teaching at German universities.

All participants were contacted by mail or personally. The experts were motivated to participate in the expert interviews with the fact that there is very little empirical research on the teaching and learning of complex analysis and that this research is therefore important for mathematics education at university level. When the experts were contacted, it was explained that the research is about "mental imagery" (German: Vorstellungen) and "intuitive approaches" to complex analysis in general but no additional information was given. The participants signed a consent form, in which they have been assured that all data is treated according to scientific and ethical standards, is used for research in mathematics education only, and in which they agreed to participate in this study.

Even though the sample is quite small, it may count as representative sample of university teachers for complex analysis. However, since we planned our study as an exploratory case study, a small number of participants is still adequate, to put the theoretical elements we developed in Chapter 4 to practice. We cannot answer whether the sample is maximally contrasting or whether we have found collected a saturated set of data—in whatever sense—because criteria of comparisons about experts' intuitive understanding of complex path integrals were not available. Since research area, affiliation, and the concrete teaching experiences vary among the sample, it is likely that the selection of cases is not biased in an à priori unknown direction. After beginning to analyse the data, we also noted that our sample produced a variety of discursive images about complex path integrals, which were not overlapping between the participants explicitly, and a variety of discursive frames in experts' intuitive mathematical discourses about complex path integrals, which partly overlapped (see Chapter 12 for an initial overview of the results). Therefore, our cases were sufficient for individual and cross-case analyses since we could concentrate on the logic of each case but also on shared discursive frames between the cases. In sum, we consider our sample to satisfy the needs for this study (cf. Akremi, 2019; Grandy, 2010; Helfferich, 2011, pp. 172–175). We address the questions on limitations and further options for research with a further varied sample in Section 11.5 and Section 16.4.

11.1.2 *Interview guideline*

In this section, we outline the interview guideline. As discussed in Section 10.3, the guideline serves to control the topics during the interviews and to have impulses ready in case the conversation gets stuck. If an interviewee raised a topic, which was originally intended for a later part of the interview, the questions prepared for this topic were flexibly adapted to this situation.

⋙ *Excerpts from the interview guideline will be presented in italics.*

General outline of the interview guideline

The interviews included several topics from complex analysis, including holomorphic functions and complex path integration. The guideline contained six parts and can be seen in full in Appendix D. However, for this thesis, only the part about integration in complex analysis will be considered as well as other parts, in which the participants addressed complex path integrals or integral theorems themselves.[114]

The six parts of the interviews were:

0. Introduction,
I. central contents of a first course in complex analysis,
II. mental images for holomorphicity and beliefs about the role of mental images in mathematics,
III. tasks on holomorphic functions,
IV. mental images about complex path integrals and intuitive explanations of integral theorems in complex analysis,
V. closing.

In part 0., the introduction of the interview, the interviewer recalled the general research interest. In particular, the word "mental image(s)" (German: Vorstellung(en)) was described with several keywords such as pictures, associations, symbols, actions, operations, formulas, schemes, or models. We quote the introduction from the interview guideline:

⋙ *[The interview] is about how experts in complex analysis, that is, researchers and lecturers, understand basic concepts and which "mental images" they have.*

When I say "mental images", then I do not only mean pictures you may "have in mind" or could draw. "Mental images" should rather be understood much more broadly here. In addition to pictures, it can also be about associations, symbols, actions, operations, formulas, schemes, models or the like that are of central significance for you with respect to a concept.

[...]

Please answer the questions as thoroughly as possible. For instance, you may use paper and pencil for your explanations. It is very important for me to get insight into your "world of thoughts", therefore I would like to ask you to express your thoughts, spontaneous ideas, etc. aloud.

With these introductory remarks, we aimed to sensitise the interviewees that many different issues can be understood with the word "mental image(s)" and that we care about their individual use of it. In particular, the use of this (and related) word(s) should not have been restricted in any way during the interview. Furthermore, the interviewer explained that there was not much research in mathematics education about university mathematics beyond first

114 The interview guidelines in the preliminary reports in Soto-Johnson et al. (2012), Soto-Johnson et al. (2011) of the study by Oehrtman et al. (2019) and the guideline by Hancock (2018) were used as an inspiration for the construction of the interview guideline of this empirical investigation.

year level. This was also intended to raise the experts' interest in this research project (see Appendix D for the rest of the introduction). This introduction was not read to the interviewees exactly, but it was uttered rather freely to initiate a comfortable way of talking to each other during the interview.

The interviewees were also encouraged to use paper and pencil during the interview whenever they felt it was appropriate. On some occasions, the interviewer started to draw or write something, too; hence, he additionally encouraged the production of visual mediators.

Then in part I., the experts were asked to describe what they consider the most important topics of a first course on complex analysis. This prompt engaged the experts in a first narration. Based on the experts' answers, prompts or questions from the interview guideline could be connected to the start of the interview. Also during the first interview part, general questions about the importance of mental images in complex analysis and mathematics in general were addressed. Holomorphicity was addressed in parts II. and III., which we skip at this point.

In general, the interviewer included several prompts that directly hinted at "mental images" or "intuition" (and of course other German variants of these words) during the interview. This was partly due to the colloquial nature of the conversation, but it was also done to encourage the experts to use these words in relation to complex path integrals themselves.

Interview prompts for the part on complex path integration

Now, we examine the questions in part IV., the part on complex path integration, more closely. For the sake of reference, these questions will be enumerated from **IQ1** to **IQ4** here ("**I**ntegral **Q**uestion").

IQ1*: Individual interpretations of* $\int_\gamma f(z)\,\mathrm{d}z$

The first question in interview part IV. addressed experts' interpretations of complex path integrals, in particular of $\int_\gamma f(z)\,\mathrm{d}z$ as a complex number.

⋙ *In calculus / real analysis, we can interpret the definite integral of a real-valued function as the measure for the (signed) area under the graph.*
Which geometric meaning does the complex number

$$\int_\gamma f(z)\,\mathrm{d}z$$

for a (piecewise continuously differentiable) path $\gamma\colon [a,b] \longrightarrow \Omega$ *and a continuous function* $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$ *have for you?*

The question is introduced with a reference to the basic image of area for Riemann integrals (Section 2.2.3).[115] Nevertheless, it was not explicitly asked to relate this basic idea to complex path integrals. The experts could freely respond to this task. For instance, the corresponding question sheet for this interview prompt (Figure 11.1) did not contain a reference to the basic image of area for Riemann integrals.

115 The comparison to real integrals was also used as a stimulus by Hancock (2018, p. 456), who asked his students "What does the integral of a complex valued function represent? How is this different than the integral of a real valued function? How is it the same?" or by Soto-Johnson et al. (2012, p. 447), who asked their experts "Sometimes in calculus we can interpret the definite integral of a real-valued function to represent the area under the curve. What geometric representation or explanation might be useful to understand the complex number obtained as an answer to a definite integral of a complex valued-functions?"

Which geometric meaning does the complex number

$$\int_{\gamma} f(z)\,\mathrm{d}z$$

for a (piecewise continuously differentiable) path $\gamma\colon [a,b] \longrightarrow \Omega$ and a continuous function $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$ have for you?

Figure 11.1: Question sheet on the potential meaning of $\int_{\gamma} f(z)\,\mathrm{d}z$ to elicit experts' mental images about the complex path integral.

The written question contains constraints on γ and f. In the interviews though, the question was uttered more colloquially, hinting rather roughly at potential constraints on γ and f. In particular, the interviewees were encouraged to use the constraints on the paths, functions, or domains they deemed appropriate and whenever they felt the need to.

To be precise, we also remark that the written prompt in Figure 11.1 contained the word "geometric meaning" of the complex path integral $\int_{\gamma} f(z)\,\mathrm{d}z$. However, we emphasise again that we do not assume that there really is a "meaning" behind complex path integrals (this would also not comply with the commognitive framework). Rather, this word was used to instantiate a rather free way of talking about complex path integrals and to evoke experts' individual interpretations of it. In this context, we also remark again that the interviewees were encouraged to interpret and use words from the word field around "mental images", "meaning" etc. as they felt appropriate. In particular, the interviewees were encouraged to digress from "geometric meaning" to what they deemed appropriate instead.[116]

After the initial responses, follow-up questions focused on experts' ways of working with complex path integrals or deepened the narratives the experts had already produced.

After having discussed the mathematical object "complex path integral" itself, the interviews covered the experts' intuitive substantiations of Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20).

***IQ2**: On Cauchy's integral theorem*

Cauchy's integral theorem (Theorem A.17) states that the complex path integral of a holomorphic function on a simply-connected domain along a closed path always vanishes.

The next question aimed at how the experts intuitively substantiate this theorem, potentially using one of their individual interpretations addressed earlier. Since Cauchy's integral theorem is one of the cornerstones in complex analysis and basically all further in complex analysis related to integrals are based upon it (cf. e.g., Freitag & Busam, 2006; Lang, 1999), it is important to find out how experts may explain this theorem intuitively.

116 Contrary to the advice of some authors not to ask about theoretical concepts directly (e.g., "Do not ask about theoretical categories […], rather ask about concrete aspects from the lifeworld of your interlocutor [German: Lebenswelt Ihres Gegenübers; EH.]" (Hermanns, 2017, p. 368, own transl.)), this and other questions during the interviews about potential mental images and intuitive explanations for concepts in complex analysis were explicitly included because the conceptualisation of intuitive mathematical discourses requires that situations of communication are set, in which the experts can engage in communicating about complex path integrals in relation to what they consider as their mental images, intuitive explanations etc.

One version of Cauchy's integral formula is

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

for $f: \Omega \longrightarrow \mathbb{C}$ holomorphic, $z_0 \in \Omega$, $r > 0$ such that $B(z_0, r) \Subset \Omega$ and $z \in B(z_0, r)$.
How do you imagine the assertion of this formula?
How would you argue vividly or with one of your mental images that this formula holds?

Figure 11.2: Question sheet on Cauchy's integral formula.

⋙ *One of the central theorems, which is frequently needed for further theorems, is Cauchy's integral theorem.*
Which version of Cauchy's integral theorem do you prefer and why?[117]
How could one explain vividly that this theorem is correct?

At this point, one deviation from the interview guideline occurred. In the actual conversations, it was not directly discussed which version of Cauchy's integral theorem the experts preferred and view. This was no harm for the analysis later, because we could reconstruct from the transcripts, which constraints on γ, f, or Ω the experts did or did not take into account by examining their use of the respective keywords related to paths, integrands, and domains (see Section 13.6, Section 14.6, Section 15.4).

IQ3: *On Cauchy's integral formula*

Cauchy's integral formula (Theorem A.22) expresses a value of a holomorphic function in terms of a certain complex path integral along the boundary of a circle. A similar formula does not exist in the context of Riemann integrals (i.e., there is no direct relationship between the Riemann integral of a real-valued function on an interval and its values at the boundary points of the interval). Similar to Cauchy's integral theorem, Cauchy's integral formula is another cornerstone in complex analysis (cf. e.g., Freitag & Busam, 2006; Lang, 1999).

⋙ *One version of Cauchy's integral formula is*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

for $f: \Omega \longrightarrow \mathbb{C}$ holomorphic, $z_0 \in \Omega$, $r > 0$ such that $B(z_0, r) \Subset \Omega$ and $z \in B(z_0, r)$.[118]
How do you imagine the assertion of this formula?
How would you argue vividly or with one of your mental images that this formula holds?

Cauchy's integral formula is also remarkable because it is in a sense universal for all holomorphic functions f on neighbourhoods of balls in the complex plane and points z in the interiors

117 Potential suggestion from the interviewer: *I would suggest the following version: If $f: \Omega \longrightarrow \mathbb{C}$ is a holomorphic function on a simply-connected domain $\Omega \subseteq \mathbb{C}$, then it holds that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every piecewise continuously differentiable path γ in Ω.*
118 Here, the utterance $B(z_0, r) \Subset \Omega$ is used to signify that the compact closure of the ball is contained in Ω.

of these balls. More precisely, the integrand in Cauchy's formula is a function of f and z alone, namely

$$(f,z) \longmapsto \left(\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}\right).$$

Also note that there is an interpretation of Cauchy's integral formula via mean values for $z = z_0$ (e.g., Remmert & Schumacher, 2002, p. 183): The function value $f(z_0)$ is equal to the mean of the values of f along the boundary of the circle $B_r(z_0)$. Namely, in this case, Cauchy's formula reduces to

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{it}\right) \mathrm{d}t. \tag{11.1}$$

This and other potential relationships between complex path integrals and mean values were discussed in Section 8.2.3 and Section A.6.

Recall furthermore that Equation 11.1 looks very similar to the formula in the mean value property of harmonic functions: If $\varphi\colon U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}^2$ open, is harmonic, $a \in U$, and $r > 0$ such that $\overline{B_r(a)} \subseteq U$, then

$$\varphi(a) = \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(a + r(\cos(t), \sin(t))^T\right) \mathrm{d}t.$$

For $v \in B_r(a)$, there is an analogue to Cauchy's integral formula for harmonic functions as well, namely *Poisson's integral formula*. (Forster, 2017b, §6).

Henceforth, **IQ3** also includes the potential to discuss relationships between integrals and mean values in complex or vector analysis.

***IQ4**: Existence of holomorphic primitives*

There is a primitive function for every continuous real-valued functions on an interval of real numbers by the fundamental theorem of calculus (Theorem B.3). But in complex analysis, there is not a holomorphic primitive function for every continuous function on a connected set of complex numbers (not even on simply-connected domains; see Section 8.3, Section A.6, and in particular the Theorem Fundamental theorem of complex function theory (Theorem A.26)). Rather, a holomorphic primitive function for a complex function on a simply-connected domain exists if and only if the function is holomorphic itself (see Existence of primitives for holomorphic functions (Theorem A.20)). Hence, there is a major difference in the requirements that guarantee the existence of primitive functions in real and complex analysis (continuity on intervals in the real case vs. holomorphicity on simply-connected domains in the complex case). Therefore, the next question aimed to elicit how experts argue intuitively that this is the case.[119] Moreover, the interview prompt hinted at a potential analogy between real and complex analysis the experts could or could not take up.

⋙ *There are primitive functions for continuous functions $f\colon I \longrightarrow \mathbb{R}$ ($I \subseteq \mathbb{R}$ open interval) as ones learns from the fundamental theorem of calculus.*
What could an "analogous" situation for complex functions look like?
Why isn't there a primitive function for every continuous function $f\colon \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ domain)?
Where does this "analogue" situation fail?

119 A similar prompt was also used by Hancock (2018, p. 456), who asked his students "When does a complex valued function have an antiderivative? Why would this be useful to know?".

Table 11.2: Duration of the interviews.

	Interview time
Uwe	92 min.
Dirk	123 min.
Sebastian	88 min.

11.2 REALISATION OF INTERVIEWS

11.2.1 *Conduction of interviews*

The interviews lasted for approximately 90 to 120 minutes (Table 11.2). The running cameras did not seem to disturb the experts and they seemed to feel comfortable during the conversation. The interview part on integration was originally scheduled last, but it was pulled forward when the experts began to discuss complex path integrals or integral theorems. Of course, all prompts, **IQ1** to **IQ4**, were included in all interviews nevertheless.

The possible problem that an expert may not want to explain his mental images because she or he may be afraid of uttering formally incorrect narratives or of embarrassing himself, can never be ruled out in general. Nevertheless, we consider that the interviews proceeded on a basis of trust and all the experts were very open and approachable. For instance, the experts occasionally mentioned that they were not sure whether their interpretations were endorsable in any sense or whether they could be judged as intuitive (by themselves or others). Therefore, we hypothesise that the experts made explicit their occasional hesitation or their struggles to provide intuitive interpretations of the complex path integral or of the integral theorems.

The interviewer reacted to the utterances of the experts in various ways. Following Helfferich (2011, pp. 102–114), his reactions included questioning an issue not having understood completely, the demand to explain a certain keyword in more detail, the confrontation with possible contradictions, paraphrasing, the offer of explanations or alternatives, or questions to evaluate a statement (see also Denscombe, 2010, ch. 14; Gläser & Laudel, 2010, ch. 4; Misoch, 2015, ch. 9).

11.2.2 *Videography and transcription*

The interviews were videotaped with two cameras, which captured both interlocutors, their gestures, as well as what was written down during the interviews. The interviewer provided paper and pens in different colours for he expert and himself. The interviewees were encouraged to use these materials whenever they would like to. All of these notes were collected.

The interviews were fully transcribed with the help of a student assistant. Overall, the style of transcript may be described as an extended scientific transcript (Fuß & Karbach, 2019). An even more detailed system for transcription (e.g., for linguistic analyses) was not necessary (cf. Bogner et al., 2014, pp. 41–43). Utterances were transcribed as verbatim as possible. Incomplete sentences, grammatical errors, or idiosyncrasies of the German language were not corrected during the translation process. Punctuation was standardised as far as possible though. Sounds (e.g., agreeing "mmh" or questioning "uhm?") were literally transcribed, and gestures were indicated in brackets (e.g., "[circular gesture with right index finger in the air]"). Truncated words were indicated with / (e.g., "differen/"). Incomprehensible words are indicated as such in brackets; when the audio allowed a guess, this was indicated, too (e.g., "[incompr., likely: ...]"). Overlapping speech acts were indicated within the previous speech act in case the interrupting speech act was very short and the first interlocutor continued speaking. Pauses were indicated in round brackets: (.), (..), and (...) denote pauses for one, two, and three seconds, and (xs) denotes a pause for x seconds ($x \geq 4$).

Mathematical keywords were written as pronounced. In particular, spoken letters in mathematical context were written in the same font as the rest of the transcript (e.g., "function f"; "path gamma"). In case such an utterance was accompanied with written notation, this notation was also indicated within brackets or shown in figures accompanying the transcripts (e.g., in this case letters realising a mathematical object were denoted in mathematical font such as "f" or "γ").

If someone spoke and wrote or sketched something at the same time, this was indicated with brackets in the transcripts, too. Figures with screenshots of one of the video cameras or excerpts from the collected notes were included in the transcripts to make the process of writing or sketching visible. For example, when a sketch was drawn during the course of multiple utterances or some parts of the figure were added later, multiple screenshots were included in the transcript in order to make the genesis of the visual mediator visible. Care was taken that the interviewees are not identifiable on the screenshots.

11.3 ANALYSIS OF INTUITIVE MATHEMATICAL DISCOURSES

We will now turn to the analysis of experts' intuitive mathematical discourses from the interviews as presented in our three case studies in Chapter 13, Chapter 14, and Chapter 15. Figure 11.3 shows a schematic overview, the elements of which we now describe closely. Even though this figure suggests a linear procedure, these steps were nevertheless carried out circularly as will be explained in the following. For this purpose, we recall that a discursive image about complex path integrals was defined as potentially visually mediated narrative in intuitive mathematical discourses. In a discursive image, an interlocutor describes what she or he considers a mental image or an intuitive interpretation complex path integrals (see Section 4.2). A discursive frame for intuitive mathematical discourse was defined as a set of metarules guiding discursants' explorations for the construction of a discursive image or the production of an intuitive substantiation of a mathematical propositions (see Section 4.4).

11.3.1 *Initial interpretation*

Our initial interpretation (i.e., our adaption of the formulating interpretation) consisted mainly of the identification of episodes in the transcripts for further detailed analyses. This identification was based on the interview prompts (in particular **IQ1** to **IQ4**) and interlocutors' usages of keywords related to integration (e.g., "path integral", "Cauchy's integral theorem", or integrals for real functions), and the explorations the experts started in response to the task situations set by the interviewer.

The identified parts of the interviews were mostly located in interview part IV. However, there were other relevant episodes not from this part, too. In particular, the parts on complex path integration from each interview were divided into episodes by identifying topics and subtopics. The central topics were *interpretations of complex path integrals*, *intuitive substantiation*

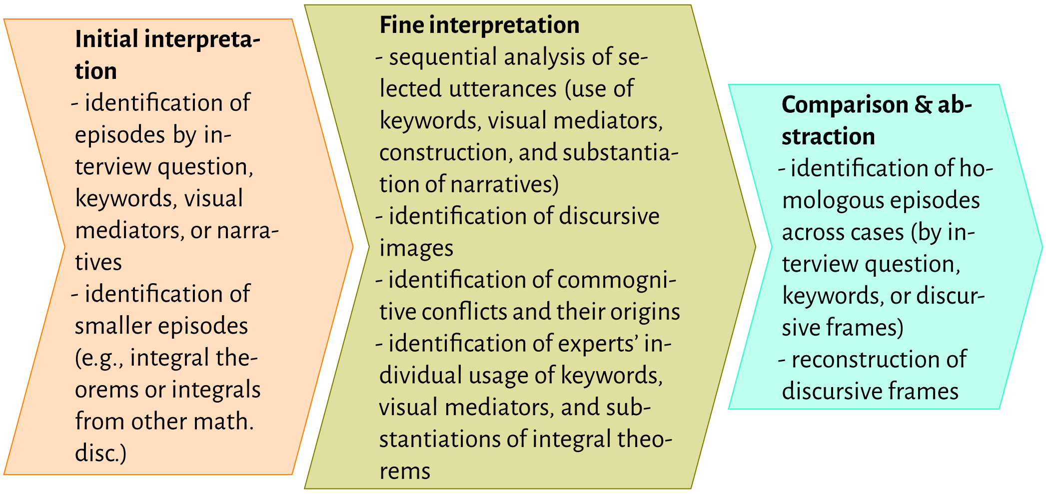


Figure 11.3: Flowchart of the analysis of the interviews.

of Cauchy's integral theorem, *intuitive substantiation of Cauchy's integral formula*, and *intuitive substantiation for the existence of holomorphic primitive functions* in line with our interview questions **IQ1** to **IQ4**. Corresponding episodes were subdivided into smaller episodes based on changes of the general topic into subtopics as was indicated by interlocutors' use of keywords, visual mediators, and narratives, or when the interviewer introduced another question. We identified these smaller episodes for instance when an expert changed his initial response to a task situation and proceeded with an alternative or additional interpretation, or when the interviewer provided another prompt or asked about something the expert said. References to mathematical discourses other than complex analysis (e.g., vector analysis) also marked a change from one smaller episode to another.

11.3.2 *Fine interpretation*

Having selected episodes for further analysis, we analysed how the experts organised their intuitive mathematical discourses about complex path integrals. That is, we analysed *how* the interlocutors said *what* they said or mediated visually. In this fine interpretation (i.e., our adaption of the reflecting interpretation), we analysed which *discursive images about complex path integrals* appeared in each interview. We also identified potential sets of metarules with the help of which we can describe the experts' discursive actions during the interview. Below, we describe the procedure for identifying discursive images and discursive frames with more details.

In this step of our analysis, we examined experts' intuitive mathematical discourse about complex path integrals (usage of keywords, narratives, visual mediators, metarules / routines). We will now describe the analysis of these features, in particular discursive images, metarules and discursive frames, and commognitive conflicts.

Discursive images

According to our conceptualisation in Section 4.2, a discursive image about complex path integrals is a narrative in intuitive mathematical discourses about complex path integrals the discursant judges as a mental image or otherwise intuitive explanation. It may be assisted with visual mediators. Hence, we had to look for our interviewees' use of the keyword "complex path integral", "mental image", "intuitive", or other linguistic variants, and visual mediators for the realisation of complex path integrals in the same utterance or in utterances close to each other. We also looked for narratives about the complex path integral, which the experts produced directly responding to the task situation in context of **IQ1**. Furthermore, we also identified discursive images during the conversation about Cauchy's integral theorem with Uwe and Sebastian: Both experts engaged in certain substantiations here from which we could derive one discursive image each. In other words, these discursive images are joint constructions of the expert and the interviewer-researcher.

When these utterances were distributed over several utterances, they had to be combined into one narrative. For example, our experts occasionally constructed narratives about complex path integrals and said shortly afterwards or before that they considered these as a mental image or an intuitive explanation of complex path integrals.

Metarules

Recall that metarules are rules about mathematists; more precisely, they "*define patterns in the activity of the discursants trying to produce or substantiate object-level narratives*" (Sfard, 2008, p. 201, emph. orig.). In particular, the metarules in intuitive mathematical discourses we focus on contain descriptions about how the experts proceeded during the construction of their discursive images and intuitive substantiations of mathematical propositions.

Routines are defined as "set[s] of metarules that describe a repetitive discursive action" (Sfard, 2008, p. 208). Whether the metarules identified in the expert interviews really are repetitive and stable over a longer period of time cannot be investigated here. The amount of repetition to be observed is of course limited to the interview itself. Therefore, we consider metarules here without a strict requirement of repetition over time. Clearly, certain discursive actions appeared more than once during the interview, which was an indicator for repetition, but it not possible to know whether these repeated actions really are repetitive beyond the interview situation. Nevertheless, routines such as realising mathematical objects graphically, substantiations of propositions, and other general mathematical routines could be identified as such.

Metarules are often tacit and thus results of commognitive researchers' interpretative acts (Sfard, 2008). Nevertheless, discursants may however describe the governing mechanisms of their own discursive actions. Furthermore, we suspect that the open and trusting conversation we initiated in the interviews led the experts to describe their own actions rather openly and clearly. The experts produced narratives on meta-level quite often during the interviews and thus they gave hints at potential metarules. Hence, we could distinguish whether the experts made metarules explicit themselves or not (e.g., the experts frequently used "I" or "one" for a generic mathematist, that is, they explicitly constructed narratives on meta-level; this may be an indicator for their awareness of their own discursive actions and potentially for governing metarules). We also had to make our own hypotheses about the rules having guided interlocutors' discursive actions. In this case, the metarules are researchers' accounts for our experts' intuitive mathematical discourses (see also the methodological remarks on finding metarules in Section 4.4 and Section 10.4).

Based on the analysis of the metarules which we reconstructed to account for the description of experts' exploration of discursive images and intuitive substantiations of integral theorems, we finally formulated the discursive frames as follows.

Discursive frames

As explained before, discursive frames are sets of metarules governing the construction of discursive images and intuitive substantiations of mathematical propositions (see Section 4.4). We developed this theoretical notion to account for our initial observation that neither our experts nor those interviewed by Oehrtman et al. (2019) shared a set of (discursive) mental images about complex path integrals. Hence, discursive frames are a commognitive feature of intuitive mathematical discourses, which we have used to account for the complexity of intuitive mathematical discourses about complex path integrals and particularly for experts' explorations of discursive images and intuitive substantiations.

Discursive frames were reconstructed by first identifying a theme in experts' utterances when discussing their intuitive understanding of complex path integrals (e.g., references to vector analysis discourse, a general view on integration as a cross-curricular topic (Kontorovich, 2018b), application of complex path integrals, etc.). Metarules arising from these themes were hypothesised and compared with what the interlocutors said and did. In particular, we formulated potential discursive frames and analysed whether our experts' actions could be described as obeying to the rules of this frame. As such, discursive frames can be described as "analytical statement[s] with which the observed performance is consistent" (Kontorovich, 2021b, p. 5). They are idealised sets of rules to explain experts' discursive activities for their construction and substantiation of their discursive images and other explorations in their intuitive mathematical discourses.

In the presentation of the analyses, each reconstructed discursive frame will be entitled with a keyword or keyphrase and a description of its central metarules, that is, an explanation of what it means to be guided by the respective discursive frame. All reconstructions of discursive frames will be evidenced with lots of examples from the transcripts.

We will now give some examples. One of the reconstructed discursive frames in experts' intuitive mathematical discourse is the "area"-frame. A general metarule for this frame consists of recalling the interpretation of Riemann integrals as an area (Section 2.2.3) and then trying to adapt it to complex path integrals. This is a metarule of intuitive mathematical discourses about complex path integrals because it relates complex path integrals to what a mathematist considers as a useful interpretation of another kind of integral. Assuming that the area interpretation may be transferable to complex path integrals, interlocutors following this frame may start exploring how complex path integrals and areas relate to each other. In our study, Dirk engaged in such an exploration: He tried to produce a still to him unbeknownst narrative about complex path integrals and areas (Section 14.3). On the contrary, Uwe and Sebastian rejected the appropriateness of the area interpretation for complex path integrals. Sebastian went on to argue that it is more appropriate to relate integrals to mean values and the act of measuring (Section 15.2). Hence, these two interlocutors did not follow this frame. As it will turn out, none of the experts produced an object-level narrative or a visual mediator that relates the keywords "complex path integral" and "area" to each other—excerpt, of course, for rejecting the appropriateness of the basic idea of area for complex path integrals.

Another example is the "vector analysis"-frame. It consists of various metarules to relate complex path integrals to integrals from vector analysis. For instance, Uwe used this frame to identify complex path integrals and real path integrals as instance of the more general concept of path integral, and created the discursive image that complex path integrals are path integrals of third kind.

In sum, we constructed eight discursive frames for experts' intuitive mathematical discourses about complex path integrals. These will be introduced in the first chapter on results (Chapter 12) and then exemplified in the case studies for each expert.

Commognitive conflicts

Commognitive conflicts, that is, situations in which interlocutors "use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to differing rules" (Sfard, 2008, p. 161), were also analysed. They may occur either within the narratives of one interlocutor ("intra-commognitive conflict"; Kontorovich, 2021b) or between different interlocutors (Sfard, 2008, pp. 255–258). Commognitive conflicts are identifiable when narratives are constructed with the help of incommensurable metarules. According to Sfard (2021, p. 15), "[t]he most obvious sign of incommensurability, and thus of commognitive conflict, would be the presence of several irreconcilable uses of the same words."

For example, Uwe formulated narratives about the vanishing of complex path integrals for holomorphic functions along closed paths according to different usages of the word "holomorphic" (see Section 13.4): On the one hand, he used "holomorphic" as an adjective to signify a holomorphic function in the endorsed way in complex analysis (i.e., a complex differential function defined on an arbitrary open set) and constructed a discursive image about the corresponding complex path integral as a "weighted sum of residues". On the other hand, he used it at another point to signify a complex differentiable function defined on a simply-connected domain (or without isolated singularities), which he did not say explicitly but could be reconstructed from the context. Then, he concluded that the corresponding complex path integral along a closed path "typically evaluates to zero". Both usages of the word "holomorphic" cannot be endorsed at the same time (in fact, only the first one is endorsed in complex analysis): According to the first narrative, the complex path integral is a certain sum (hence, no claim is made whether this sum is zero), while according to the second narrative, the complex path integrals is said to be (typically) zero.

11.4 PRESENTATION OF ANALYSES AND RESULTS

The presentation of the analyses and results is organised into four chapters, one introductory chapter (Chapter 12) and one for each expert (Chapter 13, Chapter 14, Chapter 15).

In Chapter 12, we summarise our main results. In particular, we provide an overview of the discursive frames and the discursive images about complex path integrals we identified in our three interviews.

The case studies in Chapter 13, Chapter 14, and Chapter 15 may be regarded as a comprehensive fine interpretation of episodes from each interview with a focus on how the experts' employed the discursive frames, which discursive images about complex path integrals they produced, and other features of their intuitive discourses (e.g., idiosyncratic word usages, certain special types of substantiations, or commognitive conflicts in relation to their discursive images). These chapters begin with an introduction to each expert, where we also describe their individual views on complex analysis or their pedagogical metarules on the use of graphical realisations of mathematical objects. Then, we organise the subsequent sections according to discursive frames, which however often overlap and are tied to the inner logic of each expert's intuitive mathematical discourse, and therefore prohibit isolated reporting. Experts' intuitive substantiation of Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), or the Existence of primitives for holomorphic functions (Theorem A.20) are presented in three further sections in each chapter. Each chapter ends with a summary.

Discursive images about complex path integrals

In each case study, the discursive images about complex path integrals will appear in the margin.

In order to ease their recognition, he discursive images about complex path integrals emphasised in *marginal notes*. In most cases, these are verbatim citations from the transcripts or linguistically smoothed variants. In a few cases though, these discursive images were distributed over several utterances or reconstructed during the analysis and were then presented as concise narratives. The discursive images are also contained in Figure 16.2.

Discursive frames

The names of the discursive frames are usually printed in **bold** for easier recognition when reading the following chapters.

Numbering of excerpts of transcripts

All of the excerpts are then numbered line by line within the chapters on each expert. The number in front of the first speaker indicates the number of the turn. For example, in

1 312–Uwe: There are path integrals of first, second, and third kind, I like to say.
2 [...]

the number 312 indicates that this excerpt starts at turn 312 of the transcript of the interview with Uwe.

Comparisons

Comparisons between the cases are presented in the following way. We start with the case of Uwe (Chapter 13). Afterwards in Chapter 14, the analysis of the interview with Dirk is presented and homologous episodes are compared to the interview with Uwe. Then, the analysis of the interview with Sebastian is presented in Chapter 15, which contains comparisons with the previous two interviews.

11.5 REFLECTIONS ON THE QUALITY OF THE RESEARCH PROCESS

Let us now reflect on the quality of our empirical research (Flick, 2019; Helfferich, 2011; Misoch, 2015; Steinke, 2017; Strübing et al., 2018). We follow Steinke's (2017) criteria. She avoids the use of the keywords "objectivity", "validity", and "reliability". Instead, she replaces them with seven criteria to make sure that the choice and procedure of data collection and analysis, the theoretical embedding, the presentation of results, and the researcher' role become as comprehensible as possible. Steinke (2017, pp. 328–331, own transl.) suggests discussing the following aspects: "intersubjective transparency", "indication of the research process", "empirical grounding", "limitation", "coherence", "relevance", and "reflected subjectivity". We have addressed the relevance of our research in the introduction and at multiple other points and will reflect on the contribution of our results to mathematics education in the discussion (Chapter 16). Therefore, we will now only discuss the other criteria.

Intersubjective transparency

Intersubjective transparency means that the process of how the research was conducted becomes transparent to the recipients of the research report. For this to be done, we described in detail why we chose commognition (e.g., Lavie et al., 2019; Sfard, 2008, 2013, 2020a) to ground our focus on experts' individual interpretations of complex path integrals on a discursive framework. As such, we have made our conceptualisation of mathematical thinking and the discursive nature of complex analysis discourses explicit. Additionally, this and the preceding chapter document how data was collected in semi-structured expert interviews between mathematicians and a mathematics educator on equal footing. In particular, we described how the interview guide was constructed and it is reproduced in its entirety in Appendix D. Moreover, we highlighted methodological principles of commognitive data analysis (Section 10.4).

Indication of the research process

Indication of the research process is the criterion which urges the researchers to argue that their choice of data collection, transcription, and methods for analysis are appropriate for the research goals. We have described these points during this and the last chapter. In particular, we argued that expert interviews are a valuable source of information for this study because they allowed us to organise a setting which guarantees that the mathematicians and the mathematics educator engage in the type of communication sought for. In this setting, the experts were not restricted by any means and hence they could contribute to our research not only by discussing integration in complex path but also by indicating what they value as mental images or intuitive explanations in this mathematical context. No previous conception of what mental images might be for the experts was presupposed; occasionally though, the interviewer included prompts, which were based on the area interpretation for Riemann integrals or asked for visualisations.

The analysis of our central theoretical concepts, that is, discursive images about complex path integrals and discursive frames in intuitive mathematical discourse, was also described precisely (Section 11.3).

Since quantitative representativity do not belong to our goals, it is appropriate to have chosen a small sample of research participants. This allowed us to analyse these cases deeply in order to contribute to the still scarce literature on complex analysis education by empirically grounding the notion of intuitive mathematical discourse about complex path integrals (see also Section 16.4).

Empirical grounding

In order to ensure the empirical grounding of our study, we include many excerpts from the interview transcripts in Chapter 13, Chapter 14, and Chapter 15. All theoretical insights we developed are thus directly linked to samples from the data. Additionally, data segments from the same and the other interviews were used to check whether one discursive frame reconstructed in one interview appeared in another and how they were put to practice each time. As such, data analyses were carried out low-inferential, that is, close to the utterances and actions of the discursants. In particular, one of the reasons for why we included the notion of discursive frame to our theoretical repertoire for the analyses of the intuitive mathematical discourses was that our initial search for discursive mental images revealed, in accordance with Oehrtman et al. (2019), that there was no consistent set of discursive images about complex path integrals across our research participants. Hence, we needed another way to look at our data in order to see more general pattern in our experts' intuitive mathematical discourses about complex path integrals.

In sum, our results are empirically and theoretically grounded and they are not limited to the particularities of the single cases but linked to theoretical elements from the commognitive framework, too (cf. Przyborski & Wohlrab-Sahr, 2014, p. 25). Hence, our empirical study includes both, local commognitive findings about each expert's discourse (e.g., idiosyncratic usages of certain keywords or commognitive conflicts), but also those at the level of discursive frames and other elements to characterise discourses (e.g., certain differences between some substantiations in the intuitive mathematical discourse to substantiations; see also Section 16.4).

We did not opt for communicative validation with the except themselves in the form of "member checks" because the experts were not familiar with the commognitive framework or mathematics education research in general, but in the end, this could have validated the appropriateness of our reconstructions of discursive frames (Flick, 2019; Meyer, 2018; Misoch, 2015, ch. 10.3.1). Instead, we performed "peer debriefing" with mathematics educators, one of which has special expertise in the commognitive framework, another with expertise in complex analysis, and other members of the local team of mathematics educators at the University of Bremen (Flick, 2019; Meyer, 2018; Misoch, 2015, ch. 10.3.1). For example, we conjectured that Uwe used the keyword "holomorphic" in different ways at different points in the interview and discussed with the expert in complex analysis and mathematics education for agreement.

Limitation

The criterion of limitation addresses how generalisable and representative the results are. Since our sample is quite small, we cannot generalise our results quantitatively. According to Przyborski and Wohlrab-Sahr (2014, p. 363, own transl.), "fact-findings and the analyses of single cases have always been relevant to scientific work" (see also Section 10.2). In line with this assessment of the importance of findings for a rather small sample, our study is valuable for basic research in complex analysis education in particular and university mathematics education at large. We present a theoretical framework and initial "fact-findings" (Przyborski & Wohlrab-Sahr, 2014, p. 363, own transl.) for experts' intuitive mathematical discourses about complex analysis. Together with Oehrtman et al.'s (2019) study about experts' "construction of mathematical meaning" for the complex path integrals as the authors write in the title of their research paper, we have helped to deepen empirical research in a field, which is still almost unexplored, with a novel theoretical approach (see also Section 16.4).

Our results are limited to mathematicians, whose research area is related to complex analysis and who teach this subject at university. It is conceivable that the intuitive mathematical discourses of other persons using and teaching complex path integrals (e.g., physicists) and those of students could be quite different. It is also conceivable that our experts would have answered differently if they had been able to prepare for the interview in advance. However, we decided against this option in the assumption that the experts would be able to recall the most important parts of what they count as their individual interpretations of complex path integrals without preparation.

Our case studies in Chapter 13, Chapter 14, and Chapter 15 will show that we could reconstruct various discursive frames and identified various discursive images from our small sample. Therefore, our results show the width and subtleties of intuitive mathematical discourses about complex path integrals experts may engage in.

Coherence

Our empirical results are coherent with the commognitive framework, that is, this framework was appropriate to account for our research questions and data. Furthermore, our empirical study in Part iii and our epistemological study in Part ii form a coherent portray of two facets of experts' mathematical discourses in complex analysis unified by the theoretical lens from commognition.

Reflected subjectivity

Reflected subjectivity deals with the role of the researcher as the constitutive element during the research process. It is therefore particularly important that the empirical research is conducted as open and reflected as possible. The researcher has to question at all times whether she or he approaches the data with her or his own knowledge of or attitudes towards the research questions and data. Researchers should strive for a *methodically controlled understanding of the other* (Przyborski & Wohlrab-Sahr, 2014, pp. 16–17, own transl.). For instance, this means that several hypotheses should be developed and tested before a conclusion about the data is drawn. In particular, the author of this thesis was the interviewer, and hence, he already brought his initial interpretations of experts' utterances to the conversations.

The author has demonstrated his familiarity with discourses about complex path integrals during the epistemological analyses in Part ii, which was needed for the conduction and analysis of the expert interviews. In order to account for the complexity and to remain open for the experts' individual interpretations, the interviews had to be analysed not using a normative point of view. Even though the expert interviews were analysed independently from the epistemological analysis, a partly comparative perspective was at times useful though: By comparing the experts' narratives to narratives from scholarly complex analysis discourse, we could identify, for example, idiosyncratic word uses or special forms of substantiations in intuitive mathematical discourses about complex path integrals (see the next chapters).

We emphasise that this additional comparative perspective on the interviews was not intended to highlight deficient aspects in our experts' intuitive mathematical discourses, but rather to underline their peculiarities. After all, our mathematical experts are highly proficient participants of complex analysis discourse. Moreover, the interviews were explicitly designed and conducted in such a way that the experts were encouraged to deviate from the scholarly discourse if they felt that this corresponded to their intuitive understanding of complex path integrals.

12 INTRODUCTION TO THE RESULTS

We introduce our results with an overview. Overall, we provide four central results:

- *nine discursive images* about complex path integrals, none of which was endorsed in more than one interview (see Figure 12.1);
- *eight discursive frames* governing experts' construction of discursive images about complex path integrals and intuitive substantiations of integral theorems in complex analysis (see Figure 12.1 and Table 12.1);
- experts' intuitive substantiations of central theorems from complex analysis (Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20)), as well as
- additional commognitive features of intuitive mathematical discourses (e.g., idiosyncratic word uses or substantiations) for experts' constructions of discursive images or their substantiations of integral theorems in their intuitive mathematical discourses about complex path integrals.

At this point, we briefly recall that discursive images about complex path integrals are narratives in intuitive mathematical discourses, possibly assisted with visual mediators, in which the discursants describe what they perceive to be their mental images or otherwise intuitive explanations about complex path integrals (see Section 4.2). A discursive frame is a set of metarules in intuitive mathematical discourses governing the construction of discursive images and other intuitive interpretations or substantiations of integral theorems (see Section 4.4). We emphasise once again that discursive frames are commognitive researchers' categories and we borrow Kontorovich's (2021b, p. 5) words to describe them as "an analytical statement with which the observed performance is consistent".

We intend to teaser the discursive frames in the rest of this chapter and then deepen our analysis of them in the following chapters. In line with our planning of the case study, the following three chapters cover each of our experts' intuitive mathematical discourses about complex path integrals: We cover the interview with Uwe in Chapter 13, the interview with Dirk in Chapter 14, and the interview with Sebastian in Chapter 15. In order to account for the complexity of these individual enactments of these discourses, these chapters are organised in such a way that each individual's narratives and metarules can be understood in context.[120] Accordingly, these three chapters contain detailed commognitive analyses. Readers who are mainly interested in the discursive images and discursive frames can skip these details in the local analyses on first reading.

120 In principle, it would have been possible not to divide the results into chapters according to experts. Nevertheless, we consider that our presentation fits better to the complexity of the respective discourses. In this way, we can also present each experts' explorations of discursive images of complex path integrals and work out whether and how the experts draw on their discursive images when substantiating the integral theorems.

Chapter 13, Chapter 14, and Chapter 15 are organised in a similar way:

- We start with an introduction to each expert. In this introduction, we present a summary of how they describe their view on complex analysis, on the role of mental images or intuition in complex analysis, and pedagogical metarules on realising mathematical objects visually.[121] These observations serve as a reference for later parts of the analyses.
- Then, we will illustrate the *discursive frames* as realised in each of the experts' intuitive mathematical discourses about complex path integrals. For this to be done, we separate each of the case studies in Chapter 13, Chapter 14, and Chapter 15 into several sections, which centre about one or more of these discursive frames, and analyse how the frames were enacted by each expert and led to the construction of the discursive images about complex path integrals.
- We include three sections in each of the case studies to cover our experts' intuitive substantiations of Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20). Analysing these substantiations, we highlight experts' applications of the discursive frames again.
- Each of the chapters ends with a summary of the respective intuitive mathematical discourse about complex path integrals.

We discovered that our experts did not produce a consensual set of discursive images about complex path integrals during the interviews. That is, none of the discursive images from one interviewee were explicitly constructed by the other interviewees. We would like to caution, however, that first, we did not systematically ask the experts to evaluate the discursive images from other interviews (except for a few cases, which resulted from the idiosyncrasies of each conversation), and that second, we do not imply an absence of consensual discursive images about complex path integrals among the experts in general. Rather, we did not observe them in the interview setting in this study. This means, that at least during the interviews, for which the experts could not prepare, all experts followed quite different routes for the task situations we had prepared, namely, to explicate what they consider their mental images or other intuitive explanations of complex path integrals and integral theorems.

Then again, our observation that no consensual set of discursive images was produced during the interviews also means that we could not use discursive images as a unit of analysis to reconstruct potential commonalities between experts' intuitive mathematical discourses about complex path integrals. Therefore, it proved to be crucial to include discursive frames governing experts' explorations of discursive images and intuitive substantiations of integral theorems to the centre of our attention. Doing so, we found eight discursive frames, some of which appeared in more than one interview, but also individual discursive frames, which appeared in one interview only.

Table 12.1 shows a list of the discursive frames and who followed them during the interviews. The discursive frames used by more than one expert have been put at the top of the table, and the others are listed from Uwe over Dirk to Sebastian. However, we do not want to give the impression that the discursive frames are organised hierarchical. The discursive frames were

121 Here, we use "pedagogical metarule" (cf. Gallego-Sánchez et al., 2022; Heyd-Metzuyanim & Shabtay, 2019; Viirman, 2015) to signify an experts' narrative about her or his own actions as lecturers such as a narrative about their use of visual mediators (see Section 4.3).

Table 12.1: Overview of discursive frames in experts' intuitive mathematical discourses about complex path integrals.

Discursive frames	Uwe	Dirk	Sebastian
(F1) "restriction of generality"	x	x	x
(F2) "theorematic"	x	x	x
(F3) "vector analysis"	x	x	
(F4) "tool"	x		
(F5) "no meaning"	x		
(F6) "area"		x	
(F7) "mean value"			x
(F8) "holomorphicity ex machina"			x

used for different purposes during the interviews, namely either for experts' intuitive interpretations of complex path integrals as such, eventually resulting in discursive images about them, or for intuitive substantiations of at least one of the integral theorems (see the description of these frames below). We will explore how these frames came into being in each interview and for which purposes they were used by each expert in the case studies for each of them.

Figure 12.1 provides an overview of the discursive frames and the discursive images about complex path integrals produced in terms of the respective discursive frames by each expert. In this figure, the discursive frames are shown as large grey blocks and the discursive images, which resulted from the application of the respective discursive frames, are shown as coloured boxes (orange for Uwe, blue for Dirk, and green for Sebastian). The discursive frames, which appeared in more than one interview, are marked with an asterisk.

Some remarks about the figure are in order. At multiple points during the interviews, several discursive frames overlapped. We show one particular overlap in Figure 12.1 explicitly, namely between the "restriction of generality"-frame and two other frames (the "theorematic"- and the "holomorphicity ex machina"-frame). The reason for this explicit highlighting of the overlap is that the "restriction of generality"-frame did not lead to a discursive image alone but only in conjunction with the other two discursive frames (see below). We also point out that we grouped tightly connected discursive images here within the same boxes. Therefore, some of our nine the discursive images are in fact small sets of tightly connected discursive images.

We note that even though Dirk may be described to have followed the "area"- and "vector analysis"-frame while exploring his intuitive understanding of complex path integrals he did not end up with a discursive image about complex path integrals based on these two discursive frames. However, we included two blue boxes with dashed edges within these in Figure 12.1 in order to account for Dirk's constant effort to derive a graphical realisation of complex path integrals. Although he did not produce such a graphical realisation, he reflected on the difficulties of a potential transfer of the visual mediators he produced for other integrals to complex path integrals (see Chapter 14).

Discursive frames appearing in more than one interview were usually enacted differently. This could already be anticipated from what we described before: If experts had enacted the discursive frames the same way, the experts would also have produced similar discursive images.

We would like to baptise and summarise the eight discursive frames as follows:

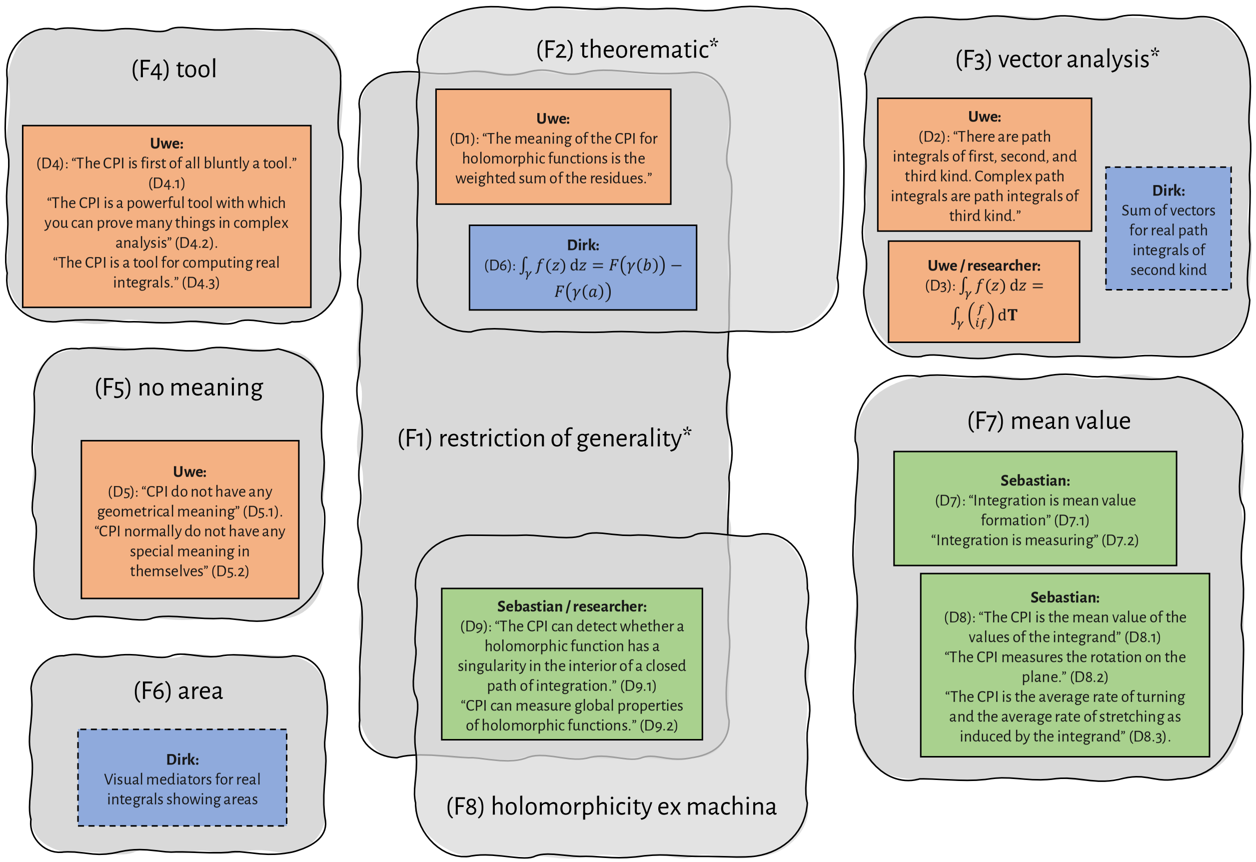


Figure 12.1: Discursive frames and discursive images from the three expert interviews.[a]

a "CPI" is an abbreviation for "complex path integral(s)". Some of the discursive images are linguistically smoothed variants of the original utterances from the interviews. (D3), (D8.3), and (D9) are joint constructions with the interviewer-researcher.

(F1) The **"restriction of generality"-frame** consists of a rather general set of metarules in mathematics, namely to explore more general set of conditions by restriction to special cases. Proponents of this frame engage in construction routines to derive narratives about complex path integrals for a more restricted class of functions or paths than continuous functions or piecewise continuously differentiable paths. This discursive frame was not used alone for the construction of a discursive image about complex path integrals, but in combination with the "theorematic"- or "holomorphicity ex machina"-frame. This discursive frame was also used when the experts discussed special cases of the integral theorems.

(F2) The **"theorematic"-frame** contains metarules on the extraction of a potential meaning of the complex path integral based on theorems in complex analysis or to provide an intuitive explanation for a mathematical proposition. It is considered a discursive frame for intuitive mathematical discourses about complex path integrals because the theorems used are neither constructed nor directly substantiated themselves; instead, they are tied to an interlocutors' assessment of a "meaning" of complex path integrals or are considered as a suitable reaction to the task situation to produce a mental image or to provide an intuitive explanation. Following such a metarule, Uwe produced the discursive image *(D1) "The meaning of the complex path integral for holomorphic functions is the weighted sum of the residues"*. Dirk produced the discursive image *(D6)* "$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$", where F is a primitive function for f. We call these two discursive images, which are based on the "theorematic"-frame, "theorematic images".

(F3) The **"vector analysis"-frame** consists of the set of metarules according to which mathematicians explore their potential mental images or intuitive explanations of complex path integrals by recalling a narrative from vector analysis. Oftentimes, this may include finding an analogy between a narrative from complex analysis and a narrative from vector analysis and then constructing a discursive image. For example, Uwe samed integrals from vector analysis and complex analysis and constructed the discursive image *(D2) "There are path integrals of different kinds. The complex path integral is a path integral of third kind"*. When substantiating Cauchy's integral theorem, Uwe performed a certain exploration, the result of which effectively yielded the narrative *(D3)* "$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (f, if)^T\,\mathrm{d}\mathbf{T}$", in other words, another discursive image about the complex path integral reconstructed from Uwe's utterances. Here, the complex path integral is realised as a real path integral of second kind for a complex vector field.

(F4) The **"tool"-frame** consists of metarules according to which the complex path integral is valued and potentially used as a tool in complex analysis. Accordingly, working with the complex path integral means using it as a tool for something else, which is attached more importance. Uwe used this frame and produced the discursive images *(D4.1) "The complex path integral is a powerful tool with which you can prove many things in complex analysis"*, *(D4.2) "The complex path integral is first of all bluntly a tool"*, and *(D4.3) "The complex path integral is a tool for computing real integrals"*.

(F5) The central metarule of the **"no meaning"-frame** is to reject a meaning for complex path integrals. Proponents of this discursive frame may reject the meaning in different ways: They may reject a potential geometrical meaning (e.g., an area as for Riemann integrals), a physical meaning (e.g., work as for real path integrals of second kind), or any intrinsic meaning of complex path integrals at all. Following this frame, mathematists may further substantiate why they value the complex path integral to not have any of these or other meanings. They may switch to another discursive frame and construct discursive images about complex path integrals nevertheless. For example, Uwe even stopped answering after having rejected any meaning of the complex path integrals but switched discursive frames after multiple questioning by the interviewer. The discursive images Uwe produced here were *(D5.1) "Complex path integrals do not have a geometric meaning"* and *(D5.2) "Complex path integrals normally do not have any special meaning in themselves"*.

(F6) The **"area"-frame** consists of explorations for finding a relationship between the complex path integral as a certain area, or possibly another geometrical quantity. While the other experts rejected a meaning of the complex path integral as an area, Dirk was the only one who tried to transfer the basic idea of area to complex path integrals. He used the routine of drawing graphs of functions to produce multiple visual mediators showing certain areas (Figure 14.3, Figure 14.4b) or a plot of a vector field (Figure 14.5), which rather realise Riemann integrals and real path integrals of first or second kind. Dirk reflected on their usability for the case of complex path integrals. However, he did not produce a discursive image of complex path integrals related to a geometrical feature such as an area. For this reason, we do not count the narratives he produced following the "area"-frame as discursive images about complex path integrals (rather, they could be counted as discursive images about other integrals), and Dirk also did not value them to be appropriate for complex path integrals either. However, he continuously looked for potential visual mediators besides formulae to realise complex path integrals and to substantiate integral theorems intuitively during the interview.

(F7) The **"mean value"-frame** contains metarules according to which interlocutors identify integrals with mean values or a process of measuring. Users of this frame unify integrals from different mathematical discourses with the interpretation of them as mean values. In particular, they apply this interpretation to complex path integrals. This discursive frame also contains metarules about how discursants may determine the respective mean value for the case of complex path integrals. Sebastian used this discursive frame to construct a rather general discursive images about integration, namely *(D7.1) "Integration is mean value formation"* and *(D7.2) "Integration is measuring"*. More specifically, he also produced discursive images for the complex path integral: *(D8.1) "The complex path integral is the mean value of the values of the integrand"*, *(D8.2) "The complex path integral measures the rotation on the plane"* and, jointly with the interviewer-researcher, *(D8.3) "The complex path integral is the average rate of turning and the average rate of stretching as induced by the integrand"*.

(F8) The **"holomorphicity ex machina"-frame** consists of metarules according to which holomorphicity is valued as a very strong and rigid property of complex functions, so strong that propositions about complex path integrals, in which holomorphic functions play a role, can be virtually expected. Accordingly, this discursive frame contains substantiations of integral theorems in complex analysis based on this perceived rigidity of holomorphic functions. This frame was enacted by Sebastian and led to discursive images about complex path integrals, which were constructed jointly by him and the interviewer-researcher: *(D9.1) "The complex path integral can detect whether a holomorphic* *function has a singularity in the interior of a closed path of integration*" and *(D9.2)* "*Complex path integrals can measure global properties of holomorphic functions*".

13 THE CASE OF UWE

This chapter presents the case study on Uwe’s intuitive mathematical discourse about complex path integrals. The interview part on complex path integrals was the last part in the actual conversation with a duration of approximately 24 minutes for 110 turns ($\approx$ turns 303–414).

We begin the chapter with a description of Uwe’s pedagogical metarules concerning the use of visualisation in complex analysis in Section 13.1. Then, we analyse in detail how Uwe carried out his intuitive mathematical discourse about complex path integrals and we highlight where we have reconstructed the discursive frames. These frames overlap at multiple points, which is reflected in the organisation of the following sections. We already note here that Uwe’s initial

Figure 13.1: Discursive frames from the interview with Uwe.

reaction to the task situation to describe a potential (geometrical) interpretation of the complex path integral was defensive: He rejected any meaning for the number $\int_\gamma f(z)\,dz$. However, in the course of the interview Uwe created several discursive images about complex path integrals, which we may describe in terms of various discursive frames.

Accordingly, we begin the analysis of Uwe's explorations of discursive images about complex path integrals with the **"no meaning"-frame** in Section 13.2 and then proceed with those discursive frames, which were used during his further explorations. The most dominant discursive frames he unfolded were the **"restriction of generality"-**, the **"vector analysis"-**, and the **"theorematic"-frame**. These frames and their entanglement are the subject of Section 13.3 and Section 13.4. We will discuss the **"tool"-frame** which also overlapped with other discursive frames from the previous sections in Section 13.5.

Afterwards, we analyse Uwe's intuitive substantiations of Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20) in Section 13.6, Section 13.7, and Section 13.8. In Section 13.9, we summarise Uwe's intuitive mathematical discourse about complex path integrals.

Figure 13.1 shows an overview of Uwe's intuitive mathematical discourse about complex path integrals. The bubbles close to the centre represent a discursive frame and the bubbles emanating from these bubbles summarise the discursive images Uwe constructed or indicate his usage of the frames otherwise.[122]

13.1 INTRODUCTION TO UWE

Let us start with general findings on the elements related to complex path integrals Uwe considers to be relevant in a course on complex analysis and how Uwe envisions the role of imagination in complex analysis. Recall that the interview started with a description of a first course on complex analysis and also included the role of imagination in complex analysis (Section 11.1.2). This includes Uwe's general view towards visualising functions in complex analysis and mathematical objects related to complex path integrals.

Related to integration, Uwe mentions Cauchy's integral theorem, Cauchy's integral formulas, and the residue theorem as core elements in a lecture on complex analysis. Whereas he does not mention the complex path integral itself directly at this point, it is clearly needed for discussing integral theorems (turns 9 to 13). Yet, the fact that he considers theorems on complex path integral as important but not the concept of complex path integral itself hints towards a view on complex path integrals Uwe presents later in the interview: Roughly speaking, Uwe explains that complex path integrals are of no intrinsic "meaning" in themselves but instead get "meaning" from the residue theorem and as "tools" in complex analysis (see especially Section 13.2, Section 13.4, Section 13.5).

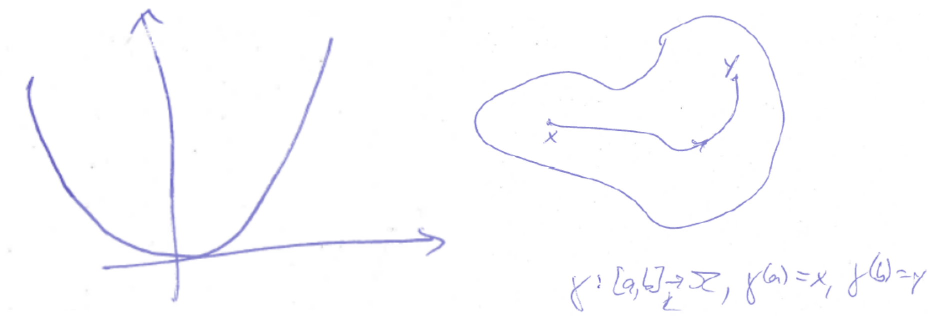
13.1.1 *On the role of imagination in complex analysis*

When the interviewer asks whether Uwe considers it important for complex analysis to have mental images (German: Vorstellungen) and whether Uwe uses them (turns 105ff.), Uwe reports that mental images are important in complex analysis and that he frequently uses them. He argues that graphs of functions cannot be visualised for reasons of dimension. However, this does not imply that it is impossible to draw pictures related to functions; it is still possible to draw pictures of the domains of functions:

106–Uwe: Well. (.) I illustrate lots of concepts with pictures, so imagination
(German: Vorstellung) is always important, no matter in which part of
mathematics and [Int.: Mmh.] now complex analysis is one area in which
one can draw a picture particularly often [Int.: Mmh.], but not from the
functions because it goes from R-two to R-two and then [incompr.]
four-dimensional [Int.: Mmh.] and soon it's completely over with the
imagination, which one has in the real case, so these imaginations, which
one has from the real case, that one can really draw a graph of a function
and says: Yes, I imagine the function f of x equals x-square like this [draws
a parabola shape]. That is of course a mental image (German: Vorstellung),
which one can never have in the complex case of that quality, because the
four/ [Int.: Mmh.] the four-dimensional imagination is not there,[123] but at
least one can simply draw the domains of the function.

122 Preliminary results on Uwe's discursive images about the complex path integral and exemplary aspects of his interpretation of Cauchy's integral formula can be found in Hanke (2019, 2020a).

123 "[...] because the four-dimensional imagination is not there" is the literal translation of the idiosyncratic German "[...] weil die vierdimensionale Vorstellung nicht da ist". Here, and throughout many of the translations the original idiosyncrasies will be respected to the extent possible. In this case, Uwe's utterance is a rather idiosyncratic version for "One cannot imagine four-dimensional objects".



(a) Uwe's visual mediator for the graph of $f(x) = x^2$

(b) Uwe's visual mediator for a path from x to y.

Figure 13.2: Uwe's visual mediators for the graph of real function and for a path between two points.

Uwe explains in this turn that he illustrates concepts from complex analysis (domains of functions and paths) with pictures. In particular, he describes adaptions to the routine of drawing graphs of functions from real to complex analysis. Visual mediators for functions in the "real cases" like Figure 13.2a, showing the graph of the function with the term $f(x) = x^2$, are not possible in complex analysis. This visual mediator is used as an exemplar for a "mental image, which one can never have" in complex analysis (lines 10f.). Uwe substantiates this by saying that "four-dimensional imagination is not there" (line 12) and also that "it's [soon] completely over with the imagination" (lines 6f.).

Nevertheless, Uwe offers an alternative for this particular visual mediator. He switches the context to "path integrals" and describes that one can still draw a picture for the domain of functions (see the blob-shape in Figure 13.2b). While this does not realise the function itself, it realises its domain:

106–Uwe: [...] And path integrals play a role and then I simply draw [draws the outer contour in Figure 13.2b]. So, okay, this is what my set looks like and, alright, it's connected. This means I can draw a path from here to there, which runs from x to y [draws the "path" inside the previously draw contour; Figure 13.2b]
[...]
So for me it is like this, (.) whether I write down this [likely the figure just drawn] or write down gamma is a path from an interval a b, uh, into omega and, uh, gamma of a equals x, gamma of b equals y [writes down the inscriptions in Figure 13.2b], then I have this picture in mind, and when I see this picture, then I have this formula in mind.

He introduces the keyword "path integral" and draws the curved line connecting two points labelled x and y realising a path between the two points in a given set, which can also be identified as "connected". Accordingly, Uwe explicates another version of the routine of drawing a function; here, the function is a path and the picture shows the trace of the path. For Uwe, the drawing in Figure 13.2b and the inscription $\gamma\colon [a, b] \longrightarrow \Omega, \gamma(a) = x, \gamma(b) = y$ are mutually realising a "path from an interval a b, uh, into omega and, uh, gamma of a equals x, gamma of b equals y" (lines 21ff.): Directly referring to his mindset, Uwe describes that one of these realisations evokes the other and vice versa.

Note that Uwe uses the keywords "mental images" / "imagination" explicitly[124], as well as pictures and formulas "in mind" (lines 23f.). Hence, this is a prime example for intuitive mathemat-

124 More precisely, Uwe used the German word "Vorstellung" both times, which can be translated to "mental image" or "imagination", depending on context.

ical discourse, that is, discourse about mathematical objects in relation to what is perceived as one's imagination / mental images.

Moreover, Uwe describes a pedagogical metarule for discussing mathematical concepts in class (it was left out in the transcript excerpt above at line 19):

> 106–Uwe: [...] it is always important to illustrate those things with pictures. [Int.: Mmh.] Simply, if one/ If one is offered things in different ways, then more connections simply build up in mind I think. [...] That on the one hand you have the, uhm, the formal, (.) correct mathematical notion and on the other hand you simply have a picture to go with it. [...]

The metarule described here consists of the realisation of mathematical objects in various ways (such as with different modalities, e.g., inscriptions and pictures) with the aim to complement a "formal, correct mathematical concept" with a "picture". Henceforth, Uwe values different modalities as complementing formality in mathematics, all of which are important for learning, namely to "build up [connections] in mind" (line 27).

However, Uwe describes scenarios, in which pictures are important or possible. More precisely, the interviewer asked whether there are differences for the use of imagination (German: Vorstellungen) in real and complex analysis and whether complex analysis is particularly suitable for producing mental images (German: Vorstellungen). Uwe asserts that complex analysis is a mathematical domain, in which one can draw many pictures, also quantitatively comparable to real analysis (turns 133–141):

> 134–Uwe: Yes, yes, in complex analysis one can draw incredibly many pictures, so [Int.: Mmh.] actually as much as in real analysis. [Int.: Mmh.] They are just pictures of different kind (German: einfach anders geartete Bilder), with which one works. [Int.: Mmh.] But compared to other more abstract, uh, things, it is simpler. Functional analysis, there one can draw very much less, if everything is suddenly infinite dimensional, [Int.: Mmh.] then it is much harder to develop intuition with pictures (German: Anschauung aufzubauen über Bilder). In complex analysis, one can work much more directly with pictorial intuition (German: bildlicher Anschauung).

In contrast to functional analysis, in which "everything is suddenly infinite dimensional", Uwe argues that one can work with "pictorial intuition" in complex analysis. According to Uwe, one can draw pictures in both real and complex analysis pictures , but they are of "different kind" each, as was also illustrated before making use of the visual mediators in Figure 13.2a and Figure 13.2b.

13.1.2 *On the limited usefulness of the routine of plotting holomorphic functions*

Uwe limits the application of the routine of drawing plots of functions however quite considerably based on its usefulness. In particular, Uwe devalues the use of certain possibilities to plot functions (such as plots of graphs of the real part, imaginary part, or modulus of a complex functions, or the coloured analytical landscape (Appendix C)) as not helpful because, according to him, these plots do not provide additional information to what he already knows about holomorphic functions (turn 110). For example, the interviewer explains that graphs of real functions can be helpful for identifying its differentiability, but Uwe rejects this approach, because the functions he likes to consider in complex analysis are already assumed to be holomorphic, so there is no need to find out whether they are differentiable:

> 114–Uwe: I don't know which properties some people wish to recognise, I have no idea. So I/

Int.: Well, in the real world, one kind of wishes, okay, the function looks nice, curvy and smooth here, so to speak, so it is likely that it is differentiable.

Uwe: Yes, but holomorphic functions are already holomorphic, so, as nice as possible and then one can only ask about certain points, where/ I don't know, where are the roots, where are the poles, whatever. [Int.: Mmh.] And that/ roots locally look all the same. If a holomorphic function has a root in some point, then you already know that, yes, if this is a simple root, then it is boring at this point, then it is conformal there. [Int.: Yes.] If it is a multiple root, then it essentially looks like z to the k-th power if it is a root of order k. [Int.: Mmh.] In small neighbourhoods one knows what the stuff looks like anyway [Int.: Mmh.] and therefore, (.) I don't know what I should draw a picture for.

In substantiating the uselessness of the aforementioned plots of complex functions, Uwe makes use of a rather deep object-level rule about holomorphic functions, which in turn substantiate why Uwe calls holomorphic functions to be "as nice as possible" (line 43): The object-level rule is that a holomorphic functions is locally conformal around a point in its domain (if the function has a simple root at that point) or behaves like a function of the form $z \longmapsto z^k$ for some $k \in \mathbb{N}$; therefore, the behaviour of a holomorphic function are already determined a lot.[125] Being aware of this object-level rule about holomorphic functions, Uwe does not recall any properties of holomorphic functions potentially to be observed or studied with the help of plots of the real part, imaginary part, or modulus (or coloured analytical landscapes) of the function in question—not because he cannot do so but because he does not see an additional value of these plots.

In sum, two important observations can be made here: The first is about the transfer of the routine of drawing plots of functions from real to complex analysis. Uwe considers pictures to be a useful means in complex analysis in general. However, he values the routine of drawing pictures related to holomorphic functions (for example, those endorsed in the literature; e.g., Bornemann, 2016, pp. 10–17; Freitag and Busam, 2006, pp. 53–57; Wegert, 2012; Appendix C) as not helpful.[126] On the other hand, the visual mediators for domains of complex functions or paths (Figure 13.2a, Figure 13.2b) are instances of what Uwe considers to be pictures of "different kind" (line 32) compared to those from real analysis. Hence, Uwe does not blindly attempt to transfer the routine of drawing plots of functions to complex analysis because this routine is valued as not helpful, at least not in the case of holomorphic functions.

The second observation, which will become very important later in the context of integration, is that Uwe restricts his attention to holomorphic functions. Uwe repeats his restriction of interest to holomorphic functions in the next utterances and even claims that "non-smooth functions", which the interviewer brings up in the context of real analysis, do not even exist in complex analysis.[127] However, Uwe uses non-holomorphic functions "at the very beginning [of a complex analysis course; EH.]" in order to distinguish holomorphic functions from them

125 Note that Uwe is only referring to the local behaviour of holomorphic functions in neighbourhoods of *roots*. Therefore, the behaviour of holomorphic functions is not yet characterised in neighbourhoods of points, which are no roots of the function. However, in this case, one can translate the function and apply the same argument to this translated function. That is, if z_0 is in the domain of a holomorphic function f but no root, then $\widetilde{f}\colon z \longmapsto f(z) - f(z_0)$ has a root at z_0 and the object-level rule described by Uwe can be applied to $\widetilde{f}$ (see also the next subject-matter didactic embedding).

126 We are not saying that these plots are indeed helpful in one way or another only because they appear in textbooks. We are just trying to recapitulate Uwe's substantiation why he does not consider these visualisations helpful.

127 Clearly, non-smooth functions (in the sense of not infinitely differentiable functions) exist in complex analysis (e.g., $z \longmapsto \overline{z}$ is nowhere complex differentiable), but Uwe makes a claim here about the class of functions he assumes to be most important in theory-building in complex analysis.

(lines 59f.). Moreover, Uwe considers plots of functions as unimportant for local inspection as discussed above but rather for global reasons, the discussion of which will be left aside here.

118–Uwe: Locally holomorphic functions are boring so to speak, because one knows exactly how they look like. Either it is an amplitwist or z to the k, one or the other.
[...]
122–Uwe: [...] complex analysis is about holomorphic functions and not about anything else. [Int.: Mmh.] I mean why should I talk about other functions? As I said, I do this at the very beginning in order to differentiate holomorphic functions from the others, [Int.: Mmh.], and then, we know, okay, then, we know what holomorphic functions are, and from now on, everything is holomorphic.
Int.: Well, there are non-smooth functions in real analysis, too, which one can look at nevertheless.
Uwe: [partly overlapping with the last turn: Yes, possible,] but not in complex analysis.
Int.: Okay, this/
Uwe: In every theorem containing a function, the function is typically holomorphic or meromorphic or whatever. [Int.: Mmh.] So holomorphic, where it does not have singularities. Therefore, I do not need to think about other functions. [Int.: Mmh.] Like I said, producing pictures of functions is something important globally. [Int.: Mmh.] [...]

Hence, we can hypothesise that when talking about functions, Uwe may occasionally use the word "function" to signify holomorphic (or meromorphic) functions since "everything is holomorphic" (line 62) after holomorphic functions have been introduced in class. This may even be an idiosyncratic metarule for Uwe, namely to equate functions with holomorphic (or meromorphic) functions in discussing complex analysis on an intuitive, non-rigorous level as here during the interview.[128] In fact, we will see that Uwe rejects any geometrical meaning of complex path integrals and he will not offer any picture directly related to complex path integrals beyond the visual mediator in Figure 13.2b to realise a path. Instead, he creates various other discursive images about complex path integrals.

13.2 THE **"NO MEANING"-FRAME**

Directly before the following episode, the interviewer describes that he would now like to talk about "integrals". Uwe asked "Path integrals?", hence he already specifies the keyword "integral" to find out which integrals the interviewer is referring to. The interviewer agrees: "Exactly, path integrals in C, uhm, in the complex setting." and the following episode took place. In this episode, Uwe rejects a meaning of the complex path integral very firmly:

305–Int: Uhm, (..) in analysis, in real analysis, in the one-dimensional at least, one often has the mental image (German: Vorstellung) that the integral somehow, uhm, represents the signed area (German: vorzeichenbehafteter Flächeninhalt) [Uwe: Oh, god.] of a function. So somehow what lies above the x-axis [Uwe: Yes, yes.] is plus and then below is negative. [Uwe: Yes, yes, yes.] Uhm, now, what meaning does – where is my little sheet? – this complex number have [shows the question sheet; see Figure 11.1], which you get through such an integration process in the complex setting? So let's say the path is piecewise/

128 Noteworthy, later in the interview part on complex integration, Uwe also considers the set of holomorphic functions to be the set of functions interesting enough to be studied with complex path integration (see Section 13.4). As explained in Section 9.2, similar points of view on the role of holomorphic function are found in the literature, too.

Uwe: None.
Int: continuously differentiable or something [encircles the task]
Uwe: None. (.) None.
Int: None? (.) But there happens to come a complex number (German: Aber da kommt ja jetzt eine komplexe Zahl raus)/
Uwe: None. No geometric meaning at all.
Int: Uh-hum. (..)
Uwe: Zero.
Int: But there happens to come out a number. What does [it] mean/
Uwe: Yes, yes, alright. Nothing.

The interviewer introduces the interpretation of Riemann integrals as an oriented area. While the interviewer describes this as a common interpretation, Uwe rejects any geometric meaning of complex path integrals outright: Before the interviewer could even finish his question, Uwe immediately interrupts with repeated "yes" to indicate that he is aware of the area interpretation of integrals as described by the interviewer. Uwe interrupts again with "Oh, god." (line 76), which indicates some concern about the area interpretation of integrals in the context of complex path integration. The interviewer goes on to ask about a possible "meaning" of "this complex number", which is signified as $\int_\gamma f(z)\,\mathrm{d}z$ on the exercise sheet put on the table (Figure 11.1) and as the result of "an integration process in the complex setting" (line 80). The interviewer also specifies the path to be "piecewise continuously differentiable or something", hence beginning to realise the inscription on the paper orally because the sheet is just about to be handed out and it is unlikely that Uwe could have yet read it.

Uwe repeatedly replies "None". After the interviewer has repeated Uwe's answer as a question and emphasised that there is a complex number at play, Uwe specifies his answer to "No geometric meaning". The interviewer addresses the complex number, which "come[s] out" of the process of integration, and Uwe sticks to his answer.[129]

Consequently, this episode can be summarised as a clear rejection of a potential geometric meaning of the complex path integral. An object-level rule consistent with the shown episode is that the path integral is not related to any kind of geometrical situation, hence no narrative about a geometrical interpretation of the complex path integral can be constructed. Moreover, Uwe makes his refusal without hesitation and remains stable even when being challenged by the interviewer's questioning insisting questioning. This episode may thus be framed with the **"no meaning"-frame**, whose principal metarule is to reject a potential meaning for complex path integrals and. Here, following this rule, Uwe does not even bother to look for a potential (geometrical) meaning of the number $\int_\gamma f(z)\,\mathrm{d}z$, and thus, he does not start a further exploration yet. After all, having been challenged by the interviewer, it would have been possible for Uwe to reconsider his rejection of any geometrical meaning of the complex path integral.

Complex path integrals do not have any geometrical meaning.

Henceforth, it is plausible to conclude that Uwe endorses the discursive image "Complex path integrals do not have any geometrical meaning.", which was reconstructed from his utterances "None.", and "No geometric meaning at all." the interviewer's repeated questioning.

13.3 THE "VECTOR ANALYSIS"-FRAME

In this section, we describe the **"vector analysis"-frame** and exemplify its use in different episodes. This discursive frame in intuitive mathematical discourses about the complex path

129 At this point, the constraints on the path or the integrand seem to have no relevance for Uwe. At most points, the interviewer mentions the technical conditions or has written them down on the questions sheets. Later in the interview, however, Uwe will occasionally specify constraints on the paths and the integrands, even if only tacitly.

integral is characterised by the usage of elements from vector analysis discourse: Interpretations for complex path integrals are constructed by making use of keywords (e.g., “(real) path integral”) or endorsed narratives from vector analysis (e.g., the “integrability conditions” for exact vector fields, see Definition B.10, or the proposition that the real path integral of second kind of an exact vector field along a closed path vanishes, see Theorem B.12) paths mentioned are at least tacitly assumed to be piecewise continuously differentiable (see also Section 13.4.3).

13.3.1 *“Path integrals of different kinds”*

In this subsection, we see how Uwe applies the metarules of saming and baptising, according to which he identifies real path integrals and complex path integrals as two instances of the same concept, namely *path integrals*.

This episode connects directly to the preceding one. After Uwe has multiply rejected a potential geometric meaning of complex path integrals, the interviewer’s third reaction to this rejection opens up the perspective from a potential geometric meaning of complex path integrals to other interpretations. The interviewer asks “Nothing at all?” instead of “No geometric meaning at all?”, and now, Uwe seems to comply with the implicit request to explain other possible meanings of the path integral from his point of view and he changes frames.

311–Int.: Nothing at all?
Uwe: There are path integrals of first, second, and third kind, I like to say. [Int.: Aha.] Of first kind is a scalar, uhm, path integral, which, whoa, no idea, is especially important for calculating the arc length, [Int.: Mmh.] where the number one is simply integrated along the path, (.) and then there is the path integral of second kind, which is incredibly important in physics for some kind of work along some kind of paths, where one has a scalar product, and then there is the complex path integral, and for this, one does not have any imagination at all at first. [Int.: Mmh.] There is complex multiplication, so to speak [points to $f(z)\,\mathrm{d}z$ on the question sheet in Figure 11.1]/ This is, so to speak, if you like, f of z is complex multiplied by dz [Int.: Mmh.] and there one best doesn’t imagine anything at all [giggles].

Uwe mentions the existence of path integrals of different kinds, namely of first, second, and third kind. Recall the discursive action of *saming*, which consists of “assigning one signifier (giving one name) to a number of things previously not considered as being ‘the same[]’” (Sfard, 2008, p. 302). As a result, Uwe sames path integrals of different kinds. Clearly, the signifiers “path integral of first kind”, “path integral of second kind”, and “complex path integral” stem from real and complex analysis discourse (possibly also in varying terminology) and are not new as such, but here, these three are grouped together under the signifier “path integral”. Hence, while Uwe clearly differentiates three kinds of path integrals, he nevertheless considers them to be similar, and accordingly, he performs a saming routine at this point. Note furthermore that Uwe “likes to say” that there are different path integrals, which indicates that he has samed these kind of integrals before.

There are path integrals of first, second, and third kind. Complex path integrals are path integrals of third kind.

Moreover, the excerpt just shown indicates a baptising routine apart from the assignment of the signifier “path integral” to the three kinds of path integrals. One the one hand, Uwe baptises the real path integral of first kind as “scalar path integral” and describes its use for the computation of the “arc length” (of paths) (lines 94f.). The path integral of second kind is not named further but is said to be important for “work along any paths” (lines 97ff.). In particular, even though saying “no idea” once, Uwe performs a recalling routine by stating endorsed narrative about a geometrical or physical application of path integrals of first and second kind. The

$$\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$

Figure 13.3: Formula for the complex path integral involving a complex times.

complex path integral is mentioned at the third place in this list; thus, we infer that it is a "path integral of third kind". Hence, we conclude that Uwe endorses the discursive image "Complex path integrals are path integrals of third kind".

Whereas Uwe describes the first two kinds of path integrals in terms of applications, namely computing a geometrical (arc length) and physical quantity (work), for "this", the complex path integral, "one does not have any imagination at all at first" (lines 100 and 103). This is in accordance with the rejection of a geometrical meaning of complex path integrals as in the episode in Section 13.2. However, Uwe additionally begins to substantiate why he claims a lack of imagination for complex path integrals: He explains that the multiplication of the integrand with dz, "f of z is complex multiplied by dz", is where "one best doesn't imagine anything at all" (lines 102f.). Accordingly, Uwe considers the complex multiplication (verbally uttered and also symbolically signified by pointing to $f(z)$ dz in $\int_\gamma f(z)\,\mathrm{d}z$; see line 102f.) to be the reason for the lack of imagination of the complex path integral.

Also note that Uwe does not say that *he* cannot or does not imagine anything for the complex path integral, but rather "one", in other words, a generic mathematist (lines 100f. and 103f.). Thus, Uwe positions his rejection of imagination within a (not specified) community of mathematists and advises that it is even "best" to not imagine anything at all under the complex path integral (line 103).

In the interview, Uwe goes on the describe the "meaning" (word used by Uwe) of the complex path integral in terms of his interpretation of the residue theorem. The analyses of this attribution of meaning will be postponed to Section 13.4 and we continue here with other episodes related to the "vector analysis"-frame.

13.3.2 *Multiplying f and dz*

The interviewer asks what Uwe meant with "complex multiplication", which Uwe realised previously with the utterance "f of z is complex multiplied by dz" (see line 102). We note though that this product of complex numbers in relation to the integrands were also discussion at another point (see line 181):

327–Int.: Mmh. (.) Yes, what did you mean with complex multiplication between f and dz?

Uwe: Well, I/ [writes the formula in Figure 13.3 while speaking] integral a to b, f of gamma of t times gamma prime of t, dt. And there is always the question: Which times [points to the dot in the formula just written] do I have? [Int.: Mmh.] And, uhm, there one can, uhm/ In the path integral of second kind one had [points to the same dot] a scalar product of the two expressions [points alternately to the inscription $f(\gamma(t))$ and $\gamma'(t)$ in Figure 13.3] because f is a vector function and gamma prime a vector. [Int.: Mmh.] There one has a scalar product [Int.: Mmh.] and here [shows again to the dot] one has the complex times. [Int.: Mmh, I see.] And that is of course, first of all, something different from the scalar product.

In order to illustrate the "complex multiplication" between f and $\mathrm{d}z$, Uwe produces the visual mediator in Figure 13.3, the inscriptions of which he also pronounces verbally. Then, he realises the dot (the "times") in Figure 13.3 in two different ways, depending on two path integrals under consideration (108ff.). For the "path integral of second kind" Uwe realises the dot as the "scalar product of the two expressions" $f(\gamma(t))$ and $\gamma'(t)$ (this requires that the "expressions" are vectors, which Uwe also confirms by calling $\gamma'(t)$ a vector and f a vector function), and for the case "here" (i.e., the complex path integral) Uwe realises the dot as the "complex times" (in which case f corresponds to a complex-valued function and $\gamma'(t)$ to a complex number).

This bears a potential commognitive conflict: The dot in Figure 13.3 is realised as two different products and accordingly the inscriptions $f(\gamma(t))$ and $\gamma'(t)$ are realised as vectors or as complex numbers. This corresponds to an overloaded usage of the same symbols in different contexts, or in other words, to two different metarules for the use of the same signifiers. Hence, interlocutors who are only familiar with the use of the dot $\cdot$ for either the first or the second type of product would experience a commognitive conflict here. In particular, this may likely be the case when the double realisation is performed to differentiate two kinds of path integrals from each other as in the shown episode. Yet, it is precisely this twofold realisation of the same signifier that enables Uwe to recognise the same structure between the two different path integrals. Hence, it is reasonable to regard the two integrals in question as analogue (cf. Bartha, 2019). Accordingly, the production of this analogy is in line with the **"vector analysis"-frame**. More explicitly, Uwe explicates the metarules for the use of the signifiers for the multiplication and the objects being multiplied in order to recognise a similar structure for the realisation of two kinds of path integrals (see also Section 8.1.4).

Unfortunately, no more precise substantiation is given for why the complex multiplication, or that of f with $\mathrm{d}z$, causes Uwe to advise to not imagine anything here.

We would like to finish this section by embedding Uwe's response to the question whether complex path integrals defined analogous to real path integrals to the textbook literature. We have described in Section 8.1.1 that Needham (1997) starts the chapter on integration in his textbook by telling his readers that he is going to "blindly attempt to generalize real integrals (via Riemann sums)" and to find out afterwards "what we have created" (Needham, 1997, p. 383). Hence, as in the case of Uwe, Needham (1997) does not give a geometrical motivation (even though he recalls the area interpretation for Riemann integrals), but still, his approach is motivated by a formal analogy to real analysis. Doing so, Needham (1997) connects the newly to be introduced content (i.e., the complex path integral) with a definition readers are assumed to be familiar with (i.e., the definition for Riemann integrals). However, we may not forget at this point that Needham (1997) does in fact interpret the complex Riemann sums geometrically as concatenations of vectors in the plane later in his book.

13.3.3 *Rewriting the integrand: A change in visual mediators*

In the excerpt shown last in this section, Uwe is asked to explain what he has meant with "complex multiplication between f and dz" (line 104). Here, Uwe begins to rewrite the complex multiplication between $f(\gamma(t))$ and $\gamma'(t)$ in complex path integral in terms as a real path integral of second kind involving a certain scalar product and a complex vector field.[130]

The following lines continue the previous transcript excerpt from lines 113ff. onward:

130 The signifier *complex vector field* is used here to denote a function from a subset of $\mathbb{C}$ to $\mathbb{C}^2$.

328–Uwe: (...) and here [shows again to the dot in Figure 13.3] one has the complex times. [Int.: Mh, I see.] And that is of course, first of all, something different from the scalar product. With the right interpretation, you can somehow rewrite it into a scalar product and then you can interpret it again a little bit as a path integral of second kind. [I: Mmh.] One can do that (.) and then, as I said, one can use the Cauchy-Riemann differential equations in a somehow suitable way.

Thus, Uwe claims that it is possible to rewrite the complex times (between $f(\gamma(t))$ and $\gamma'(t)$) in terms of a scalar product and accordingly to obtain a realisation of the complex path integral as a path integral of second kind—again in line with the **"vector analysis"-frame**. His discursive actions moreover hint at a deed (i.e., "*an action resulting in a physical in objects*" Sfard, 2008, p. 236): Uwe wants to "rewrite" "it" (i.e., the "complex times", which amounts to a change of the visual mediator $f(\gamma(t)) \cdot \gamma'(t)$; line 119). Uwe argues that once this change of a product of complex numbers into a scalar product is achieved, one arrives at an interpretation of the complex path integral in terms of a real path integral of second kind. Then, Uwe says, one can go on and use the "Cauchy-Riemann differential equations in a somehow suitable way" (line 121f.). Hence, the change in visual mediators ultimately establishes a novel identity such that "one" can use the Cauchy-Riemann differential equations (even though it is not clear yet why Uwe attempts to do so). Hence, the deed is intertwined with an exploration routine leading to a narrative about the complex path integrals in terms of a real path integral of second kind. We will see in Section 13.6 that Uwe uses the narrative, which ultimately relates complex path integrals to real path integrals of second kind, to substantiate Cauchy's integral theorem.

The interviewer claims that one can "split up the integral" (i.e., $\int_\gamma f(z)\,\mathrm{d}z$) into real and imaginary part. This indicates a change in visual mediators, too, and would ultimately lead to the narrative $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y$ (if $f = u + iv$), in other words, an identity of the complex path integral in terms of real path integrals of second kind (Equation 8.17). However, Uwe does not appreciate this particular change of visual mediators:

329–Int.: Yes, one can disassemble f and gamma [shows to Figure 13.3] into its real and imaginary parts and then split up the integral into the imaginary and the real/ real part.

Uwe: Yes, that, uhm, is not even the smartest thing, I believe. I am/ (..) Oh, well, right now, I am/ [Int.: Mmh.] So, somehow one probably has to interpret the complex multiplication as matrix, or something like that. I don't know at the moment. [Int.: Mmh.] But like I said, this/ I have done this before and this is also quite nice, but it is also only helpful when people already know their way around with conservative vector fields and stuff like that. [Int.: Mmh.] If people already know this well, then it's of no use whatsoever.

In this episode, the interviewer takes up the melange of a deed and exploration routine Uwe has started earlier: The interviewer describes the action to "disassemble" f and γ and to "split up" the integral into real and imaginary parts, hence producing a change in visual mediator ultimately producing a narrative about complex path integrals. However, even though Uwe agrees that such a split is possible, he devalues it as "not [..] the smartest thing". Rather Uwe suggests to "interpret the complex multiplication as a matrix, or something like that" (line 126ff.) (see Section 13.6 for the continuation of this episode).

13.3.4 *A pedagogical metarule on the use of the* ***“vector analysis”-frame***

In this section, we discuss Uwe’s own reflection on the usage of the **“vector analysis”-frame** in class. The following turns took place after Uwe has produced the discursive images about path integrals of third kind and the residue image (Section 13.3, Section 13.4). Hence, connections between real and complex analysis have already been addressed before.

The interviewer asks “And what makes you think that it [the complex path integral; EH.] has to be defined the way it is defined?” and Uwe responds like this:

318–Uwe: Uhm, honestly I have never thought about it before. Uh. (...) At some point I understood why/ So at some point I think I have understood some theorems, uh, a bit better, but these are things that are actually already quite difficult. So this is nothing I would ever somehow, uh, even start to discuss at (.) a normal lecture level at all. I mean, fine, so one idea (German: Vorstellung) [points to the integral on the question sheet in Figure 11.1], which one could develop, is that this indeed has something to do with path integrals of second kind, because the Cauchy-Riemann differential equations hold. So you can do something, oh but I don’t know whether I can do it now. So you can say: If you already know, uh, that for exact vector fields path integrals along closed paths [circular gesture with the pen] are zero, then you can derive Cauchy’s integral theorem from it. That’s also a quite nice way to go.
[... Utterances removed for anonymity ...]
One can do it this way, and this is actually quite nice, yes. [Int.: Mmh.] But it is only nice in case you have understood this with the, uh, with the, uh, exact vector fields (.) and (4s) everything about path integrals of second kind.

Although Cauchy’s integral theorem could be derived from the Cauchy-Riemann differential equations (Equation A.6) and proposition from vector analysis that path integrals of second kind of exact vector fields along closed paths vanish (Theorem B.12; lines 137ff.)—an approach Uwe finds attractive (line 147), he chooses *not* to do so in class. That is, even though the **“vector analysis”-frame** is constitutive for Uwe’s own intuitive understanding of the mathematical object complex path integral and integral theorems, he does not attempt to discuss it with students. The reason for this is that one has to have understood the discourse about exact vector fields and real path integrals of second kind before. Uwe perceives the difficulty of this approach to Cauchy’s integral theorem as so high that he would “never (...) even start to discuss [it] at normal lecture level at all” (lines 136ff.). In fact, he even argues that he may have understood this once better than he does now. This underlines the difficulty to combine the discourses on vector analysis and complex analysis in class. We conclude that Uwe is aware that he cannot simply integrate the discourse on vector analysis to his teaching of complex analysis without running the risk of overwhelming the students.

13.4 THE **“THEOREMATIC”-FRAME** AND THE **“RESTRICTION OF GENERALITY”-FRAME**

In this section, two other discursive frames appear, namely, the **“theorematic”-frame** and the **“restriction of generality”-frame**. The first of these discursive frames governs the use of a theorem to construct a discursive image and the second consists of metarules to restrict the level of generality imposed on the class of mathematical objects under consideration. Accordingly, users of these discursive frames engage in construction routines based on a particular restricted set of prerequisites they impose on the mathematical objects in question. We find that restricting the generality is a quite common scholarly mathematical metarule as well because oftentimes, when a narrative cannot be substantiated in general, an object-level rule may be constructed and endorsed for a less general case. In particular, such a restriction of generality fits to the application of theorems, because theorems usually come with their own set of constraints on mathematical objects (e.g., the Residue theorem (Theorem A.29) is a narrative about complex path integrals of *holomorphic* functions but not of a more general set of functions).

In the episodes presented in this section, Uwe's discursive actions may be described in terms of these two frames. He restricts the generality of the set of integrands or the paths to be integrated along. In combination with the "theorematic"-frame, he then constructs the discursive image that the "meaning" of a complex path integral is a "weighted sum of residues".

13.4.1 *Weighted sum of residues*

The next episode is directly following the first episode in Section 13.3, where Uwe has explained the existence of path integrals of three kinds. After Uwe has said that one should not imagine anything at all in line about the complex path integral 103 (line 133 in the next excerpt), the interviewer starts to pose a new question, but Uwe already interrupts. We may hypothesise that Uwe begins to respond to the interviewer's repeated request for an interpretation of complex path integrals:

312–Uwe: [...] and there one best doesn't imagine at all [giggles].
Int.: Aha. Okay. Is there/
Uwe: So, I mean, I am of course, of course, only interested in this for holomorphic functions [points to the question sheet], because this is, this is simply a tool [knocks on the question of the question sheet] in complex analysis [Int: Mmh.], path integrals. This is nothing more than a tool actually. And, uhm, therefore this is only interesting for holomorphic functions and, well, there one knows the residue theorem and it tells one exactly which mental image (German: Vorstellung) one should have of it, namely: If the path is only passing around isolated singularities of f, (.) I simply have to look at f in the singularities and calculate the residues there, then I also know what this, what this integral means, what comes out of the integral. In the end, this is what the path integral means. The sum of the residues, the weighted one.
Int.: But this is not how it is defined [Uwe: Nope.], the path integral.
Uwe: Nope, but as said before, this is first of all only a/ bluntly only a tool, this integral.

The complex path integral is first of all bluntly a tool.

First, we notice that Uwe rates the complex path integral as a tool and produces the discursive images: It is "simply a tool in complex analysis", "nothing more than a tool actually", and "bluntly only a tool" (lines 137, 138, and 148). Hence, already at this point, he can infer that Uwe endorses the role of complex path integrals as a tool in complex analysis (see also Section 13.5 for more on the **"tool"-frame**). However, this excerpt contains a very explicit attribution of "meaning" to the complex path integral, namely that of a weighted sum of residues.

More precisely, Uwe clarifies that he is only interested "in this for holomorphic functions", where "this" is most likely signifying a complex path integral. In particular, Uwe restricts the generality here: He does not focus on complex path integrals for arbitrary continuous functions as suggested by the interviewer (as is printed on the exercise sheet; Figure 11.1), but restricts the context to holomorphic functions only. This restriction of generality was also present at the beginning of the interview when Uwe discussed the usefulness of plots of holomorphic functions

(Section 13.1.2) and hints again at the **"restriction of generality"-frame**. This restriction to holomorphic functions is substantiated with the narrative that the complex path integral is a tool (line 137), and which is also evident from the next lines, when Uwe explains that it (the complex path integral) is only "therefore" interesting for holomorphic functions. Moreover, the restriction to holomorphic function is substantiated with the statement that "one knows the residue theorem", one of whose requirements is that the integrand of a complex path integral is holomorphic (Residue theorem (Theorem A.29)).

Then, Uwe performs a recall routine to rephrase the residue theorem. This theorem is entering the scene as a mathematical protagonist, which "one knows". It is personified for it "tells one exactly which mental image you should have of it" (lines 140f.). As opposed to alienation (i.e., the process of "representing phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 44)), which is a part of the process of objectification of mathematical objects (Section 3.3), here Uwe describes the residue theorem rather as a *humanised* participant of the mathematical discourse, who "tells us" which "mental images one should have of it". A few lines later, this "mental image" is explicitly realised as an outcome of the integral (line 144): "In the end, this is what the path integral means. The sum of the residues, the weighted one" (line 145). These residues are in turn realised as the results of a computation at the "isolated singularities of f". In particular, Uwe describes the computation of the residues in terms of a metarule, namely that he has to "look at f in the singularities and calculate the residues there" (lines 142f.).

Theorematic discursive image: The meaning of the complex path integral for holomorphic functions is the weighted sum of the residues.

Combining Uwe's utterances, he has eventually produced the discursive image "The meaning of the complex path integral for holomorphic functions is the weighted sum of the residues". Since a theorem from complex analysis is used here to describe the "meaning" of the complex path integral, we call this discursive image a *theorematic discursive image*. By this we mean a narrative in an intuitive mathematical discourse which attributes meaning (or potentially another interpretation) to the mathematical object at hand in terms of the statement of a theorem. Here, one could call it the *residue theorem image*. It is a result of the discursive actions we describe in terms of the **"theorematic"-** and **"restriction of generality"-frame**.

In sum, here the valuation of the complex path integral as a tool, the restriction of interest to the class of holomorphic functions, and the description of the "meaning" or the "mental image" of the complex path integral are very tightly interwoven and led to the utterance of the residue theorem image.

Moreover, it is remarkable that the Uwe identifies the "meaning" of the complex path integral here with a theorem, which appears later in courses on complex analysis, not close to the definition. Thus, the meaning he identifies here is not something learners could possibly grasp when complex path integrals are introduced.

We also note that Uwe's personal involvement fluctuates quite a lot in this (but also other) episodes. He switches from statements, in which he describes his own "interest" in complex path integrals for holomorphic functions only, to the general advise for generic mathematists not to "imagine [anything] at all" (line 133). Similarly, his utterances vary between static descriptions (e.g., the recalled theorem) and procedural descriptions (e.g., that the path is passing around singularities).

13.4.2 *Implicit constraints lead to commognitive conflicts—Different usages of the keyword "holomorphic function"*

The episode shown at the beginning of this section contains commognitive conflicts: First, it is not sufficient to require the integrand to be holomorphic (on its domain), but the path needs to be closed, too. Even though the interviewer tried to include the constraints on the path into the discussion (on the question sheet and while posing the very first question in this part of the interview), constraints on the path are not explicitly considered by Uwe here (see also Section 13.4.3 for more on the implicitly constraints on paths). Technically, the narrative "this is what the path integral means, the sum of the residues, the weighted one" is as a narrative about complex path integrals in general. However, as just explained, Uwe implicitly assumes holomorphic functions and closed paths here, but this is not reflected directly in the discursive image finally produced. An interlocutor not knowing these tacit assumptions would hence not be able to endorse this narrative. As such, Uwe's theorematic image is not endorsable in general because it relies on a tacit constraint on the class of functions and paths, which are only known to us from the context of the episode. Hence, even though the theorematic image is concise, it needs to be placed into the surrounding narratives, that is, it needs to be understood in its discursive context.

Let us compare Uwe's theorematic image with another sequence of turns, which indicates a hidden intra-personal commognitive conflict (Kontorovich, 2019):

332–Uwe: And yes, like I said, in the end this has no, no/ I mean with a holomorphic function [shows to $\int_\gamma f(z)\,\mathrm{d}z$ in Figure 11.1], the integral typically evaluates to zero (German: bei einer holomorphen Funktion kommt typischerweise null raus bei dem Integral), from this point of view: What should be interesting about it?
Int.: Yes, fine, if the path is closed. If it is not closed, then/
Uwe: Yes, but essentially this is only interesting for closed paths. [...]

Whereas Uwe has previously substantiated the meaning of the complex path integral for holomorphic function to be a weighted sum of residues, here, he realises it as "typically [...] zero" (lines 150ff.; see Cauchy's integral theorem (Theorem A.17)). The interviewer interjects that the path needs to be closed, thus adds a constraint to the prerequisites for Uwe's previous narrative (line 155). Uwe agrees but again restricts the generality of his exploration to closed paths instead of more general paths (see Section 13.4.3).

Hence, at this point, a commognitive conflict has arisen between the narrative "with a holomorphic function, the integral typically evaluates to zero" just shown and that of the weighted sum of residues. The commognitive conflict is a consequence of different tacit realisations of the keyword *holomorphic function*, which include potential tacit constraints on the existence of (isolated) singularities of the integrand (with non-vanishing residues). For the following argument, we assume that the path of integration is closed:

- If the keyword *holomorphic function* is used in the endorsed scholarly way, that is, there may be singularities of the functions the path of integration winds around, then the evaluation of complex path integrals as a weighted sum of residues is endorsable (Section 13.4).

- If the keyword *holomorphic function* is used to signify a *holomorphic function without (isolated) singularities* or *on a simply-connected domain* the narrative that the complex path integral of these functions evaluates to zero (line 150ff.) is endorsable.[131]

- The narratives conflict each other and both are not endorsable at the same time. Hence, here we have an instance of an intra-commognitive conflict (Kontorovich, 2021b). It occurs between two different usages of the keyword "holomorphic function" at two different points in the same interlocutor's intuitive mathematical discourse about the complex path integral. However, remember that the interview is about Uwe's idiosyncratic interpretations of complex path integrals and therefore, it could have been expected that narratives occasionally conflict with each other.

We can also observe a commognitive conflict at meta-level, namely in Uwe's talk about his interest as a mathematist. In line 135ff., Uwe argued that he is only interested in complex path integrals for holomorphic functions, but in lines 150ff., he asks what should be interesting about the case of holomorphic functions. Hence, it is plausible once again that Uwe uses the word "holomorphic function" not always according to the same rule in his intuitive mathematical discourse about the complex path integral.

13.4.3 *A focus on closed paths*

Uwe has explained previously that he is only interested in complex path integrals of holomorphic function and made use of the residue theorem to create a narrative about the meaning of the complex path integral as a weighted sum of residues (Section 13.4). At this point, Uwe followed a **"restriction of generality"-frame**. Similarly, in order to apply the residue theorem or Cauchy's integral theorem for a complex path integral of a holomorphic function along a (piecewise continuously differentiable) path, this path has to be closed—a constraint only partially present in the excerpts before.[132]

In this subsection, we explore how this second restriction of generality related to paths is substantiated in Uwe's intuitive mathematical discourse. In particular, Uwe substantiates his tacit use of the keyword "path" to signify a "closed path".

The next transcript excerpt continues the excerpt that ended in line 156:

333–Int.: Yes, fine, if the path is closed. If it is not closed, then/
Uwe: Yes, but essentially this is only interesting for closed paths. In complex analysis, in the end, only closed paths or cycles play a role. [Int.: Mmh.] The others appear in the definition, yes, of course. Because one calculates integrals along other paths, uh, as an intermediate step sometimes, too, but [Int.: Mmh.] there is no theorem, in which some arbitrary paths, uh, some things would play a role, where there isn't a closed path behind the scenes (German: im Hintergrund) somehow. [Int.: Mmh.] Simply doesn't exist in complex analysis.
Int.: Yes, so (5s)/
Uwe: I mean, I can use such a path integral [points to the question sheet in Figure 11.1], to define a primitive function. There, I have to integrate from

131 It is also endorsable if its residues, possibly weighted with the winding numbers given by the path, add up to zero, but this seems a rather artificial condition and this case was also never addressed as a potential condition for a complex path integral to compute to zero.

132 In line 142, Uwe describes the condition "[i]f the path is only passing around isolated singularities of f" though, which at least indicates that the path may be assumed to be closed because of the word "around". However, it is not explicitly mentioned. The interviewer mentions it explicitly in line 155.

point to another, but I do this/ when I want to check whether it is really a primitive function, I trace it back to a closed path.
Int.: Yes, okay. Mmh.
Uwe So, usually a closed path hides behind the scenes somewhere.
Int.: Mmh. So kind of (..) I, I want to use Cauchy's integral theorem finally. So, I want that it computes to zero for closed paths.
Uwe: Yes, or not if there are singularities in the way.

Here, the interviewer does not endorse Uwe's utterance that the complex path integral of a holomorphic function "typically evaluates to zero". Rather, he adds the condition "if the path is closed" (line 157). While this is in accordance with scholarly complex analysis discourse (see Theorem Cauchy's integral theorem (Theorem A.17)) and Uwe agrees, he also substantiates his previous narrative by arguing that "this [a complex path integral; EH.] is only interesting for closed paths" (line 158). This substantiation is based on Uwe's observation that in each theorem, in which a path occurs, a closed path is of importance.

In particular, Uwe and the interviewer follow different metarules for the use of the keyword "path" or for which paths are to be considered important. This becomes apparent when the interviewer adds the constraint "closed" in line 157. It may thus be hypothesised that Uwe may have tacitly assumed that the paths in his previous narratives were closed. After all, in Uwe's words, "usually a closed path hides behind the scenes" (lines 164, 172). This use of the keyword "path" is also compatible with the residue theorem image (Section 13.4), where the path also needs to be closed in order to be endorsable in complex analysis discourse. However, we do not imply here that Uwe *always* uses the keyword "path" as a substitute for "closed path". Rather, the word "path" is used in some places as a proxy word where "closed path" is necessary in formal complex analysis discourse.

Nevertheless, Uwe identifies a situation, where non-closed paths appear in complex analysis, namely for the definition of complex path integrals, or as an "intermediate step" in certain calculus or to define and prove the existence of certain primitive functions (lines 160f., 168ff.; see e.g., Existence of primitives for holomorphic functions (Theorem A.20)).

13.5 THE **"TOOL"-FRAME**

In this section, we revisit the **"tool"-frame** in Uwe's intuitive mathematical discourse, which has also been present in Section 13.4. Herse, it shows up in conjunction with the **"vector analysis"-frame**.

The next turns centre around the question whether complex path integrals are defined the way they are by means of analogy to real path integrals and the multiplication of f by dz is also taken up again by the interviewer in order to elicit a substantiation for Uwe's claim why "best doesn't imagine at all" for this multiplication. Instead of valuing the complex path integral as important for geometrical or physical computations, Uwe values it as a "tool".

321–Int.: When I [take] the path integral of a real function, which is defined on R-two, let's say, and maps to R, and we integrate along a path, then/ (..) Well, then, (.) then there I have, if the path is continuously differentiable, the, uhm, gamma prime as a factor in the integral so to speak, when I plug in the parametrisation.
Uwe: [with emphasis: Yes, alright (German: schon)], but there is a complex times and that makes everything a little weird.
Int.: Mmh. (.) So, it is not (.) defined that way for reasons of analogy? (..) In your opinion? So that it is simply adopted visually in a similar way, just

like the difference quotient is adopted, too? And then you find out: that happened to be good. Or would that be too/

Uwe: So how people came to do it that way back then, I honestly have no idea. [I: Mmh.] Whether they just said it out of analogy: Oh, we'll see what happens, or/ So I honestly/ I honestly really don't know. [I: Mmh.] It just turns out [points quickly to the question sheet in Figure 11.1] that this is a powerful tool in complex analysis with which you can prove many things, but/

Int.: Mmh. (.) So, so to speak, for us today it is a strong tool because we know: This object exists, so to speak, and it has proven itself (German: das hat sich bewährt)?

Uwe: Yes, yes. Well, I have to say honestly that I have never dealt with how that came about and such. Would be an interesting question.

The interviewer asks whether the definition of the complex path integral is chosen "for reasons of analogy" and "visually in a similar way" (compared to real path integrals) as he argues is the case for the difference quotient (lines 183ff.). He also judges that the definition (of complex path integrals, as can be inferred from the context) "happened to be good". Even though Uwe does not know whether the complex path integral was defined for reasons of analogy by the "people [...] back then", he emphasises that "this" (the complex path integral) is a "powerful tool in complex analysis with which you can prove many things" (line 191). This is a very explicit discursive image about the complex path integral and it relates the mathematical object to its using mathematists. This narrative is co-constructed by Uwe and the interviewer: The interviewer offers a substantiation for the definition of the complex path integrals first, namely the apparent analogy to a "path integral of a real function" and that it turned out later that this procedure led to a "good" definition, and Uwe adds the interpretation of a complex path integral as a "tool".

The complex path integral is a powerful tool with which you can prove many things in complex analysis.

A compatible metarule to describe Uwe's discursive behaviour here is that the complex path integral is a mathematical object to *use* in order to do something, here proving, but not to question why this tool is defined the way it is or whether it may have a geometrical meaning. This is the first time that hints at the **"tool"-frame**, that is, the frame according to which mathematists use complex path integrals to perform certain discursive actions in complex analysis. Here, Uwe puts this metarule to practice by stating the discursive image that complex path integrals are tools in complex analysis used for proving (no proof is actually produced, but this has also not been the task put forward by the interviewer).

*A melange of the **"no meaning"-frame** and the **"tool"-frame***

We have already seen that Uwe regards complex path integrals as tools (e.g., for proving; line 191) and uses this valuation to restrict the class of integrands to holomorphic functions (Section 13.4). Another use of this **"tool"-frame** appeared again during the interview:

342–Uwe: I would never have the motivation to calculate an integral anyway. So a complex path integral is the most boring thing that you can give, uh, there can be. [Int.: Mmh.] There are real integrals that you can calculate and then somehow cleverly transform them into complex integrals and then you can calculate these because there are good theorems for this. Then you can derive the real integral from it again. [points several times to the question sheet in Figure 11.1] But/

Int.: But why is that better? Why is the real/ Why is it more interesting to calculate real integrals?

Uwe: Well, because they, uh, have the meaning you started with, [I: Ah.] that this is an area under a graph. [Int.: Mmh.] But I (.) don't know any task

209 that says: Calculate this complex path integral. That gives now/ That has
210 just no meaning. That is/ (.) [Int.: Mmh.] Complex path integrals normally
211 do not have any special meaning in themselves.
212 Int.: Uhm, okay.
213 Uwe: At least I don't know of any application there.

Complex path integrals normally do not have any special meaning in themselves.

According to Uwe, it is "better" or "more interesting" (adjectives suggested by the interviewer though) to calculate real integrals because they have the meaning the interviewer suggested at the beginning, namely the "area under a graph" (line 208). On the contrary, Uwe does not only describe complex path integrals to not have "any special meaning in themselves" (line 210f.) but even to be "the most boring thing [...] there can be" (line 199f.), which clearly hints at the **"no meaning"-frame** again. Nevertheless, Uwe values complex path integrals as a tool again: Whereas Uwe has described previously that these tools can be used for proving (Section 13.3), here, the tool has a practical application, namely to compute real integrals by "cleverly transform[ing] them into complex integrals" and using "good theorems" (lines 200). The purpose of complex path integrals as tools for proving, which we have seen in Section 13.3, is now complemented with their usability in the routine of computing real integrals. Hence, it is plausible that Uwe endorses the discursive image "The complex path integral is a tool for computing real integrals".

The complex path integral is a tool for computing real integrals.

INTERMEDIATE BREAK

Until here, we have reconstructed Uwe's usage of discursive frames for the construction of discursive images for complex path integrals. In the following three sections, we examine how Uwe has intuitively substantiated the three integral theorems (Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20)) during the interview.

13.6 SUBSTANTIATING CAUCHY'S INTEGRAL THEOREM WITH THE **"VECTOR ANALYSIS"-** AND **"THEOREMATIC"-FRAME**

This section covers Uwe's intuitive explanation of Cauchy's integral theorem. Here, Uwe continues to explore the approach to derive Cauchy's integral theorem from Section 13.3.4 and the idea to rewrite the integrand of a complex path integral in one for real path integrals of second kind for a complex vector field from Section 13.3.3.

198 347–Int.: Uhm. Yes, alright. Uhm, at/ In Cauchy's integral theorem, why is it
199 somehow intuitively [or: vividly (German: anschaulich)] (..) to explain that
200 for closed paths (.) the integral over a holomorphic function, uhm, is zero?
201 Uwe: Yes, like I said, if one makes a connection to conservative vector fields, then
202 (.) then you get/ then you can motivate that a little bit. [Int.: Yes.] (.) But
203 it is only then of any use if one (.) believes to have already understood it.
204 Int.: (.) But when I have a conservative vector field, well, then is the/ then is/
205 So conservative is defined somehow, there is a primitive function or
206 something like that, what [incompr., probably: what a gradient]/
207 Uwe: Yes, and that, that, that one can simply, uhm, characterise it via the
208 partial derivatives/ [...]

Uwe explains that one can motivate Cauchy's integral theorem with "a connection to conservative vector fields", but this "is only then of any use if one believes to have already understood it". Hence, the vector analysis discourse is guiding Uwe again. He claims that an understanding of a fact related to conservative vector fields can possibly be used to motivate Cauchy's integral theorem. The interviewer hints towards a possible definition of the word "conservative" with the existence of a primitive function, but this remains unclear at this moment, and in fact, the definition of a conversative vector field is not brought up later again. Uwe nevertheless agrees and hints towards a "characteris[ation]" of "it" (likely a gradient field) via "partial derivatives" (lines 207f.). It can be hypothesised that the keywords "conservative vector field" and "partial derivatives" hint at the so-called *integrability condition* for vector fields (Definition B.10). This will be confirmed soon.

Rewriting the integrand of a complex path integral in terms of a scalar product

In the next turn (continuing the turn in line 207f.), Uwe rewrites the integrand of a complex path integral in terms of a scalar product. Recall that $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ is one possible realisation of the complex path integral of f along γ and $\int_a^b \langle \mathbf{F}(\gamma'(t)), \gamma'(t)\rangle\,\mathrm{d}t$ is one realisation for the real path integral of second kind for a vector field $\mathbf{F}$ along γ.

350–Uwe: Yes, and that, that, that one can simply, uhm, characterise it via the partial derivatives/ [Int.: Mmh.] Ok, will I manage to do it? I definitely can't do it all. What does one have to do? (9s) A complex multi/ So when I want to multiply with the vector f-one, f-two [writes down the vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$], then this corresponds to the multiplication with the, uhm, matrix f-one, minus f-two, f-two, f-one, or so [writes the first two lines in Figure 13.4]. [Int.: Mmh.] And then gamma-one prime, gamma-two prime, you can do something like that [writes the first line in Figure 13.4]. (...) But do I really want this? You can interpret this as a scalar product, namely the scalar product of [writes the ⟨-symbol in the second line of the same figure] (.) oh, (...) uhm, so I now have [crosses out the ⟨-symbol]/ This is f-one, f-two times gamma-one prime plus, uhm, minus f-two, f-one times gamma-two prime and I now can interpret this somehow as a scalar product [writes the rest of the second line in Figure 13.4]. (4s) But, uhm, well ok. I can interpret this as a scalar product and, uhm, yes, now this is/ puh. (..) Yes, yes, I can interpret this [likely, the vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$] a scalar/ so f-one, f-two is f back again and here [likely the vector $\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$] is i f, right? [writes the vector $\begin{pmatrix} f \\ if \end{pmatrix}$ in Figure 13.4] [Int.: Mmh.] This vector here is i f, thus there is a scalar product of the vector i f, f i f with gamma-prime [writes the rest in Figure 13.4]. [Int.: Mmh.]

Here, Uwe transforms the product of complex numbers $f \cdot \gamma'$ into of a scalar product and finally obtains the following equations (Figure 13.4):[133]

$$\begin{pmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{pmatrix} \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \gamma_1' + \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \gamma_2' = \left\langle \begin{pmatrix} f \\ if \end{pmatrix}, \gamma' \right\rangle . \tag{13.1}$$

First, he realises the integrand f as a vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where f_1 and f_2 are the real and imaginary part of f, but then he changes this realisation immediately into the shown matrix in (lines 211). Then, he argues that multiplication with the vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ corresponds to a multiplication with

133 Note that the arguments $\gamma(t)$ and t of f and γ' are omitted in Uwe's notation.

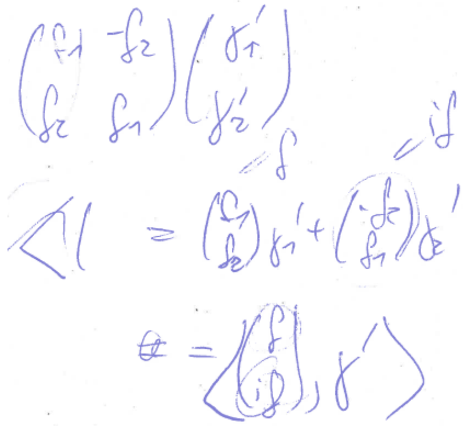


Figure 13.4: $f \cdot \gamma'$ as a matrix-vector-product.[a]

[a] The portion of the figure to the left of the equal sign in the second line is crossed out. The $= f$ and $= if$ above the second line are added later (lines 234ff.).

the matrix $\begin{pmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{pmatrix}$. In sum, Uwe realises the product $f \cdot \gamma'$ as the shown matrix-vector-product and then further as a scalar product.

Here, Uwe's word use indicates multiple times that he "interprets" complex numbers as vectors or matrices (lines 217, 221, 223) or that multiplication with a vector "corresponds" to a matrix multiplication (line 213). In fact, this word use does not yet indicate that Uwe could be *equating* complex numbers (or functions) f and γ' with vectors or matrices (or vector- and matrix-valued functions).

Let us examine this chain of equations in Figure 13.4 further. The first equation is endorsable with the usual rules of matrix-vector-multiplication. The second equality, however,

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \gamma_1' + \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \gamma_2' = \left\langle \begin{pmatrix} f \\ if \end{pmatrix}, \gamma' \right\rangle, \tag{13.2}$$

causes commognitive conflicts related to the use of signifiers for complex numbers and vectors: At first glance, the right-hand side signifies a scalar product of a vector in $\mathbb{C}^2$, $\begin{pmatrix} f \\ if \end{pmatrix}$, and a complex number, γ' (at least, γ' has not been realised differently before). However, since Uwe has realised the function f as $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ before, it is likely that he realises γ' here as the vector $\begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix}$, too. Then, the right-hand side in Equation 13.2 is endorsable as a scalar product of

$$a+ib=\begin{pmatrix} a \\ b \end{pmatrix}$$

Figure 13.5: Complex number = vector.

vectors, which is an element of $\mathbb{C}$ (also note that $\mathbb{R}^2 \subseteq \mathbb{C}^2$). Thus, Equation 13.2 eventually signifies an equation between a vector in $\mathbb{R}^2$ and a complex number $\mathbb{C}$:

$$\begin{aligned}\mathbb{R}^2 \ni \begin{pmatrix} f_1 & -f_2 \\ f_2 & f_1 \end{pmatrix} \begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} &= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \gamma_1' + \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} \gamma_2' \\ &= \left\langle \underbrace{\begin{pmatrix} f \\ if \end{pmatrix}}_{\in \mathbb{C}^2}, \underbrace{\gamma'}_{\in \mathbb{C} \cong \mathbb{R}^2 \subseteq \mathbb{C}^2} \right\rangle \in \mathbb{C}.\end{aligned}$$

We argued before that Uwe's previous word use indicates that he *identifies* or *interprets* complex numbers and real vectors, not that he actually *equates* them. Rather, it seems that Uwe endorses the metarule to tacitly *equate* complex numbers with real vectors whenever suitable.

We conclude that the narrative in Equation 13.2 conflicts the metarule to differentiate between complex numbers and vectors explicitly, which is likely enacted by the interviewer. In fact, the interviewer even asks about Uwe's intermingling usage of vectors in $\mathbb{R}^2$, in $\mathbb{C}^2$, and complex numbers. Now, Uwe finally explains the metarule to identify complex numbers with vectors, which we have anticipated earlier, explicitly:

355–Int. Mmh. I was just asking myself about the scalar product [points to the scalar product in Figure 13.4], what that is supposed to mean. So, I do as if f and i f are quasi real (.) entries?
Uwe: [overlapping with the previous turn: No, complex] entries again.
Int.: So, I am/
Uwe: [overlapping with the previous turn: Because that/] f-one/ This vector f-one, f-two, this is again the vector f and here is the vector i f [finally adds $= f$ and $= if$ above the second line in Figure 13.4]
[...]
363–Int.: Okay, this is a little bit mixed here, C and R, but well.
Uwe: Yes, but a real vector is also in/ lies also in C-two, therefore.
Int.: All right. Okay, uh, mmh.
Uwe: Yes, well, there are of course some re-interpretations on the way, which are actually no real re-interpretations. So I really write a plus i b is equal to the vector a, b [writes Figure 13.5]. [Int.: Mmh.] So this really is the same. [Int.: Uhm, okay.] Because i is the vector zero, one.

Here, Uwe realises the functions f and if explicitly as $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$, which has been implicit before, and additionally, he calls f and if "vector[s]", too. He also explains more generally that he endorses the object-level rules $a + bi = \begin{pmatrix} a \\ b \end{pmatrix}$ and $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (Figure 13.5). However, since both, f and its components f_1 and f_2 (and similarly, γ', γ_1', and γ_2') appear within and outside vectors in Figure 13.4, the application of this metarule is inherently vague as to *when* exactly Uwe applies this metarule. For example, if he had consistently identified complex number as vectors in $\mathbb{R}^2$, then the vector $\begin{pmatrix} f \\ if \end{pmatrix}$ would ultimately be a vector, whose entries were other vectors.

Rewriting the complex path integral as a real path integral of second kind

$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \begin{pmatrix} f \\ if \end{pmatrix} \mathrm{d}\mathbf{T}$

Since the left-hand side of the equation

$$f \cdot \gamma' = \left\langle \begin{pmatrix} f \\ if \end{pmatrix}, \gamma' \right\rangle$$

from Figure 13.4 appears in the integrand $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$, Uwe has essentially established the following narrative

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \begin{pmatrix} f \\ if \end{pmatrix} \mathrm{d}\mathbf{T} \tag{13.3}$$

(see Equation 8.19), even though it is not stated explicitly. Accordingly, we conclude that Uwe endorses the discursive image in Equation 13.3, which we have however reconstructed as a third-person narrative (i.e., it is a researcher's narrative about Uwe's previous discursive actions). It is again a consequence of the **"vector analysis"-frame**.

Integrability conditions and Cauchy-Riemann differential equations

Finally, Uwe uses another analogy between vector and complex analysis: The analogy is about the proposition about the vanishing of real path integrals of second kind for conservative vector fields along closed paths (Theorem B.12) and Cauchy's integral theorem. Essentially, Uwe argues that Cauchy's integral is no longer "surprising" because of this analogue theorem from vector analysis. Therefore, we may describe Uwe's discursive actions here with an overlap of the **"theorematic"-** and **"vector analysis"-frame**.

More precisely, Uwe performs a recalling routine of the notion of the integrability condition for conversation vector fields (Definition B.10), the Cauchy-Riemann differential equations (Equation A.6), and the previously mentioned theorem from vector analysis. The following turn continues the utterance in line 228:

350–Uwe: (...) So, I can interpret it this way and now, uhm, there is the
integrability condition for exact, for conservative vector fields, namely that
if I derive the second function [points to if in the last line in Figure 13.4]
with respect to the first variable, it comes out the same as if I derive the
first function [points to f in the last line in Figure 13.4] with respect to the
second variable. (.) And then there comes out exactly d-two f is equal to i
d-one f [writes the formula in Figure 13.6] and these are exactly the
Cauchy-Riemann differential equations.

Uwe follows the metarule just described and produces the visual mediator in Figure 13.6, namely the "Cauchy-Riemann differential equations" (see Equation A.7). The interviewer wonders why Uwe has derived this formula and asks for its "quintessence" and whether it may possibly justify why the complex path integral is defined the way it is:

368–Uwe: No, this is a motivation for Cauchy's integral theorem. This, uhm, this
vector field that you have here is always the vector field f, i f. [Int.: Mmh.]

$$\boxed{\partial_2 f = i\partial_1 f}$$

Figure 13.6: Cauchy-Riemann differential equation.

That is/ There is, if you use this multiplication, there is this vector field in both components f , i f [points to the vector $\begin{pmatrix} f \\ if \end{pmatrix}$ in Figure 13.4] behind the scenes and this vector field satisfies the integrability condition [Int.: Mmh.] because of the Cauchy-Riemann/ Riemannian, uhm, because of Cauchy's differential equations. So when one is familiar with, with vector analysis, then one is no longer surprised [Int.: Mmh.] by Cauchy's integral theorem.

Hence, Uwe has finally substantiated Cauchy's integral theorem: He has rewritten $f \cdot \gamma'$ as $\left\langle \begin{pmatrix} f \\ if \end{pmatrix}, \gamma' \right\rangle$, eventually realising Equation 13.3, then, he has applied the integrability condition to $\begin{pmatrix} f \\ if \end{pmatrix}$, which turns out to be the Cauchy-Riemann differential equation in Figure 13.6, and since a theorem for exact vector fields guarantees the vanishing of real path integrals of second kind (Theorem B.12), Uwe concludes that "one is no longer surprised" by Cauchy's integral theorem.[134]

13.7 SUBSTANTIATING CAUCHY'S INTEGRAL FORMULA—A **MELANGE OF FRAMES**

Let us now turn to Uwe's substantiations of Cauchy's integral formula. The accompanying question sheet, which was shown during the interview, can be seen in Figure 11.2.

13.7.1 *Values "inside" are determined by values on the boundary and a retrospective substantiation*

Uwe argues that Cauchy's integral formula relates the function values of the respective function inside the circle to the values on the boundary of the circle.

387—Int.: [...] Okay, (.) this is roughly a version, let's say, of Cauchy's integral formula for an integral over such a circle, which is centred in z-zero, radius r and so on and so forth [see Figure 11.2]. (..) Um, (..) what is my question here? How do you imagine the statement of the formula?

Uwe: For example: One can calculate the function in the interior in terms of the function values on the boundary, [Int.: Mmh.] or construct them.

Int.: Yes. (..) And that is/ (..) So that is the statement of this formula for you so to speak.

Uwe: Well, one of the essential statements. [Int.: Mmh.] And it is also the central statement [points to the formula in Figure 11.2] with which one, with the help of which one, uhm, proves analyticity of holomorphic function. [Int.: Mmh.] Because the right/ Because it is very easy to see that the right-hand side is analytic.

134 To be precise, we recall once again that f is holomorphic if and only if it satisfies the Cauchy-Riemann differential equation $\partial_2 f = i\partial_1 f$ if and only if $\begin{pmatrix} f \\ if \end{pmatrix}$ satisfies the integrability condition. We also note that the two discursants did not explicitly discuss that the domain of f has to be simply-connected in Cauchy's integral theorem.

Int.: Which is?
Uwe: Well, because one can develop the integrand [shows to the same formula]
into a power series, into a geometric series, and then it is just integrated
(German: und dann wird nur noch weg integriert) and the series is pulled
out of the integral and done.

Being asked how he imagines the formula, Uwe describes that one can compute the function "in the interior in terms of the function values at the boundary" or that one can "construct them". While Uwe values this instruction for computation or construction as one of the "essential statements" of Cauchy's integral formula, he argues further that this formula can be used to prove the "analyticity of holomorphic function". Uwe describes this proof as a metarule about what the discursants have to do in this proof, namely, "one can develop the integrand into a power series", which is realised as a "geometric series", and then, the series is "pulled out of the integral" (lines 275ff.).

The two descriptions Uwe offers for Cauchy's integral formula characterise this formula as a tool: 1) for the computation of the function "inside" the circle and 2) for a proof of the analyticity of holomorphic functions. This is compliant with Uwe's previous valuation of the complex path integral as a tool, however, here it is for a formula rather than a mathematical object. Hence, he is at least partially guided by the **"tool"-frame** here, too, even though he does not use the word "tool" himself.[135]

Next, the interviewer asks whether there is a vivid explanation or whether one can justify the formula with a mental image:

393–Int.: Mmh. (.) Uhm, and the formula itself, uh, is this somehow vividly
(German: anschaulich) plausible [Uwe laughs shortly]? Can one justify it
with a mental image (German: Vorstellung)?
Uwe: Pff. Well, I mean, I'll put it this way: Completely inadmissible, one can
deduce it from the residue theorem, [Int.: Mmh.] that if you have
understood one, then it fits good to the other, but/ (.) I mean, it has a lot
to do with the, uh, with the, uhm, mean value property of uh, (..) uh,
harmonic functions. I mean, real and imaginary part of holomorphic
functions are harmonic functions and there corresponding formulas hold
[makes a circular movement over the full text in Figure 11.2]. [Int.: Mmh.]
But again these are just relationships to other things, so that is intuitively
(German: anschaulich) not at all clear that something like this holds.

Uwe rejects an intuitive explanation for "something like this" (i.e., Cauchy's integral formula). Rather, Uwe explains that one can deduce "it" from the residue theorem and either of the theorems (Cauchy's integral formula and the residue theorem) fit to each other. The deduction from the residue theorem is however judged as "[c]ompletely inadmissible". Likely, Uwe considers this derivation "inadmissible" because the residue theorem is proven in complex analysis courses or textbooks after Cauchy's integral formula has been proven. In particular, Cauchy's integral formula is a special case of the residue theorem.[136] We would like to call this substantiation of Cauchy's integral formula in terms of the residue theorem a *retrospective substantiation*. Using this term, we indicate that the substantiation is based on a mathematical proposition, which is itself derived as a corollary of what the substantiation should substantiate. Even

Retrospective substantiation

135 A few utterances later, the interviewer asks whether Uwe considers Cauchy's integral formula to have a "tool character" (German: Hilfsmittelcharakter), to which Uwe agrees.

136 For this to be done, we observe that the function $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is holomorphic on $\Omega \setminus \{z\}$ and has a simple pole at z with residue $f(z)$. Hence, the residue theorem yields $\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 2\pi i f(z)$, which is equivalent to Cauchy's integral formula.

though Uwe regards this derivation as "completely inadmissible", it fulfils a substantiating purpose here in Uwe's intuitive mathematical discourse about the complex path integral.

13.7.2 *A mean value interpretation of Cauchy's integral formula*

Additionally, Uwe argues that Cauchy's integral formula has "a lot to do with the [...] mean value property of [...] harmonic functions". This latter claim is substantiated with the fact that real and imaginary part of holomorphic functions are harmonic, for which "corresponding formulas hold". Positioning harmonic functions in vector analysis discourse, Uwe contextualisation of Cauchy's integral formula to the mean value property of harmonic functions is another at least partly guided by the **"vector analysis"-frame** here, too.

The interviewer wants to know which mean value Uwe has refers to more precisely:

395–Int.: Uh, what does mean value mean here? Mean value of [emphasised: what]?
Uwe: So the simplest thing would be that one takes z for the, uh, centre of the circle [points to the z in $f(z)$ in Figure 11.2] (.) [Int.: Mmh.] and then computes the value at the centre of the circle with the values from the circle/ over the boundary of the circle and then one figures that this, uh, is just an, an integral mean (German: Integralmittel) over the values over the boundary of the circle.
Int.: (..) Mmh. And integral mean means (...) the integral along the path divided by the length of the path?
Uwe: Yes, something like that, exactly.
Int.: Something like that, mmh. (..) But now, if z is not the centre, then (..) [Uwe: Then not, no, no exactly.] then it is not the case.
Uwe: Then it is more complicated, exactly. Then, this would correspond to the, uh, Poisson integral formula for harm/ for harmonic functions [Int.: Mmh.] where one can construct a harmonic function for given boundary values, but this/ yes. [Int.: Mmh.] Like I said, there is nothing vivid (German: anschaulich ist da nichts), or there is nothing clear. First of all (..) this no theorem you would immediately come up with.

Uwe begins with what he considers the "simplest thing", namely the case when z in Figure 11.2 is the "centre of the circle". In this case, Cauchy's integral formula realises the function value at this bound as an "integral mean" of the "values over the boundary of the circle" (possibly divided by the length of the path).

For this interpretation of Cauchy's integral formula, Uwe has again **restricted the level of generality**, hence was led by the corresponding frame. Accordingly, the interviewer asks for the case Uwe has not considered yet, namely, when z is not the centre of the circle. Uwe describes that this is "more complicated" and would correspond to "Poisson's integral formula", where "one can construct a harmonic function for given boundary values". Hence, in this case, Uwe recalls another theorem from vector analysis. But Uwe objects again that this does not yield a "vivid" interpretation of Cauchy's integral theorem.

We note that we have not described Uwe's discursive action here in terms of the **"mean value"-frame**, which we will reconstruct during our analysis of the interview with Sebastian (Section 15.2). This is because we understand the "mean value"-frame as a frame for constructing a discursive image about complex path integrals more generally, not for the explanation of a single formula.

13.7.3 *A qualitative version of the identity theorem and a second retrospective substantiation*

The interviewer still wants to know in more detail why the function values at the boundary of the circle determine the values inside the circle:

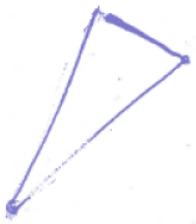
405–Int.: Mmh. (..) Mmh. But now, why should the values at the boundary determine the values inside, so to speak?
Uwe: Well, this is the identity theorem, one knows that very few values already determine the function. Here, one has a concrete formula, that, if you have the values at the boundary of the circle, that, then you can const/ really constructively determine the function. (.) That if one not only knows that it is determined but that one also knows in which way it is determined.
Int.: Oh, I know, it would be on the boundary and now I can calculate how it would be somewhere else, so to speak [Uwe: Yes, exactly.]. Mmh.
Uwe: It is a concrete computation rule.
Int.: Uhm, yes. Interesting. (..)
Uwe: That, that it [the function; EH.] is determined by the values on the boundary, that tells us the identity theorem. One has this [theorem; EH.] only later, but/
Int.: Does one have a circular reasoning here so to speak? So does one/
Uwe: Whoa, there/ Yes, of course, the power series development shows up at the identity theorem, and it [the power series development; EH] follows from Cauchy's integral formula, sure. [Int.: Mmh.] Of course you can try to prove other things in a different way first, one can do many things in many orders in complex analysis. [Int.: Mmh.] But/
Int.: But then this formula perhaps is an indicator that something as in the identity theorem, uh, could be true, so to speak. [Uwe: Puh.] [Incompr., probably: Like a] determination property.
Uwe: One sees it at this point for the first times, yes, yes, exactly.

Retrospective substantiation

Uwe answer the question why the values at the boundary determine the values inside by recalling the identity theorem in an idiosyncratic version: "very few values already determine the function" (see Identity theorem (Theorem A.23)). Since the boundary of a circle consists of accumulation points, the identity theorem is indeed applicable. Thus, we have another instance of a *retrospective substantiation*: The identity theorem, usually proven later in a course or textbook on complex analysis, is used to anticipate Cauchy's formula. Moreover, Uwe describes Cauchy's integral formula as "a concrete formula" for how the values of a holomorphic function a determined by others, namely the values at the boundary of the circle. As such, the assertion of Cauchy's integral formula is substantiated with another theorem. Again, we may ground this specific recall and description of Cauchy's integral formula in the **"theorematic"-frame**. We conclude that Uwe values Cauchy's integral formula as a *quantitative version of the identity theorem*.

The interviewer recognises that Uwe's substantiation is retrospective and that there is a thread of a circular reasoning here. Uwe responds that the power series development, derived from Cauchy's integral formula, is used in the identity theorem (more precisely, of holomorphic functions), and acknowledges that there are several ways to organise the course of reasoning in complex analysis ("one can do many things in many orders in complex analysis"). We interpret that Uwe argues that one could in principle avoid the alleged circular reasoning the interviewer has mentioned. Nevertheless, both interlocutors endorse that Cauchy's integral theorem is the "first time" one sees the "determination property" of the identity theorem.

Lastly, we notice that Uwe does not consider any of the substantiations he has provided to be intuitive: According to him, "that is intuitively not at all clear that something like this holds" (lines 289f.).



(a) Triangle in the complex plane.

(b) Degenerate triangle on the real line.

Figure 13.7: A triangle and a degenerate triangle.

13.8 SUBSTANTIATING THE EXISTENCE OF HOLOMORPHIC PRIMITIVES WITH THE **"THEOREMATIC"-FRAME**—A STORY ABOUT TRIANGLES

Uwe substantiates that continuous complex-valued functions of a continuous variable do not necessarily have primitives, even though real-valued functions of one real variable always possess primitives on intervals (see Theorem B.3). His central argument is that a particular condition has to be satisfied: The complex path integrals of the function along triangular paths have to vanish. Whereas the analogue condition in real analysis is always satisfied for continuous functions because all triangles on the real line are degenerate, in complex analysis the condition is not trivially satisfied.[137]

The interviewer introduces the respective part of the interview by recalling how a primitive function is constructed for continuous real-valued functions of a real variable. Then, the interviewer asks why there is no "analogue" statement in complex analysis.

375–Int.: [...] Uhm, yes, uh, maybe something about primitive functions now. (.) Fine, in the real case one somehow knows: Continuous functions always possess primitive functions on intervals because I can get these by the integral [Uwe: Yes.] with fixed lower bound [Uwe: Yes.]. Why can't I get something analogue in the complex case? So why doesn't every continuous function possess primitives there?

Uwe: Uhm, yes, that's the joke of it. Prim/ uh. Uh, when one does this with the primitive functions, then one wants to integrate from a fixed point to somewhere else [draws the right side of the triangle in Figure 13.7a] and integrate from a fixed point to somewhere else [draws the left side of the triangle in Figure 13.7a] [Int: Mmh.], and then one wants to find out whether it is a primitive function and then one has to form a difference quotient here [draws the short side of the triangle in Figure 13.7a]. And one has to be able to express the integral along this line segment [points with to the short side of the triangle in Figure 13.7a] in terms of the integral along this plus integral along this line segment [points to the other two long sides of the triangle in Figure 13.7a]. [Int.: Mmh.] And this only works if the integral along this triangular path [runs with the pen along the sides of the triangle in Figure 13.7a] is zero. This is exactly the Goursat. [...]

Uwe agrees to the interviewer and describes a similar procedure for the construction of primitive functions for the complex case (see Theorem A.20). He describes the metarule one has to integrate from a fixed point to "somewhere" or "somewhere else". Uwe illustrates this with the

137 In fact, it can be shown that the following conditions for a continuous complex-valued function f on domain Ω are equivalent: (i) for every $\omega \in \Omega$ there is an open neighbourhood $U(\omega) \subseteq \Omega$ of ω such that the restriction $f_{|U(\omega)}$ has primitive function, (ii) $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$ for every compact triangle $\Delta \subseteq \mathbb{C}$, and (iii) f is holomorphic. If Ω is also simply-connected, then f has a primitive function on all of Ω. (Freitag and Busam, 2006, p. 108; see also Goursat's lemma (Theorem A.19) and the Fundamental theorem of complex function theory (Theorem A.26))

visual mediator in Figure 13.7a, which shows two lines emanating from one point, the "fixed point" (bottom left) to two other points. Hence, he recalls the constructive part of Existence of primitives for holomorphic functions (Theorem A.20). He argues that one has to check whether the function constructed this way is actually a primitive for the integrand. In order to check this, one has to look at the difference quotient of this new function, which essentially amounts to check whether the complex path integral along a triangle (Figure 13.7b) vanishes. This last assertion is referred to as Goursat's lemma (Theorem A.19). Therefore, we conclude that Uwe's substantiation here is based on the **"theorematic"-frame**: Goursat's lemma is transformed to a story about the vanishing of complex path integrals along triangular paths, a condition which guarantees the existence of a primitive function.

Now, having recapitulated the construction of primitive functions and Goursat's lemma, Uwe explains the difference to the case of real analysis in more detail. Consequently, he creates a story line linking the existence of primitive functions in real and complex analysis:

376–Uwe: [...] But when I do this in the one-dimensional case [draws a line] and
integrate from this fixed point [marks the point at the left end of the drawn
line in Figure 13.7b] one time to this [marks the point at the right end of
the line in Figure 13.7b] and one time to this point [marks the third point
close to the point drawn second], then I only have a degenerate triangle
and when I integrate from here to there [points from the left to the right
point in Figure 13.7b] and subtract the integral from here to there [points
from the left to the middle point in Figure 13.7b], then, of course, it
computes to the integral from here to there [points to the small segment
between the middle and right point in Figure 13.7b]. [Int.: Mmh.] But here
[points to Figure 13.7a of course not.]
[...]
382–Uwe: The geometry of C is more complicated than that of R. That is the
reason.
Int.: Mmh. (..) Yes.
Uwe: I always explain this in my lecture, but I do not know how much reaches
the students, but/

Uwe describes that in the "one-dimensional case" (i.e., the case of real analysis), all triangles are "degenerate" (see also the quote by Tao (2013) in Section A.6). To substantiate this claim, Uwe draws the visual mediator in Figure 13.7b, where dimension-wise, all three points (the point to the left, at which the integration is described to start, and the two points where the integrals end) are on the same line. In this case, Uwe argues that the difference between the integrals from the left to the right and the middle point computes to the integral along the smaller line segment, a well-known property of Riemann integrals.

Even though Uwe does not explain explicitly why holomorphicity instead of continuity has to be required for the existence of a primitive function, this can be inferred from the context: Holomorphicity belongs to the prerequisites of Goursat's theorem. Unfortunately, Uwe is not aware to what extend his story about the vanishing of triangles reaches his students when he explains it in his lectures.

Last but not least, in his story about the vanishing of integrals along triangular paths, Uwe consistently uses the keyword "integral" instead of more refined keywords such as "complex path integrals" etc. Hence, possibly implicitly, Uwe links the two discourses (real and complex analysis) together also by *not* distinguishing the respective names for the integrals in these discourses.

Table 13.1: Overview of discursive frames in Uwe's intuitive mathematical discourse about complex path integrals.

Discursive frame	Construction of a discursive image	Substantiation of an integral theorem
(F1) "restriction of generality"	Section 13.4	Section 13.7 (Cauchy's integral formula)
(F2) "theorematic"	Section 13.4	Section 13.6 (Cauchy's integral theorem), Section 13.7 (Cauchy's integral formula)
(F3) "vector analysis"	Section 13.3	Section 13.6 (Cauchy's integral theorem), Section 13.7 (Cauchy's integral formula)
(F4) "tool"	Section 13.4, Section 13.5	Section 13.7 (Cauchy's integral formula)
(F5) "no meaning"	Section 13.2, Section 13.5	
(F6) "area"		
(F7) "mean value"		
(F8) "holomorphicity ex machina"		

13.9 SUMMARY OF UWE'S INTUITIVE MATHEMATICAL DISCOURSE ABOUT COMPLEX PATH INTEGRALS

Uwe's intuitive mathematical discourse about complex path integrals can be characterised by the usage of multiple frames. Most characteristic for his intuitive mathematical discourse was his initial rejection of any potential meaning of complex path integrals followed by the usage of a variety of discursive frames, with the help of which he produced several discursive images about complex path integrals.

Here, we reiterate each of these discursive frames and discursive images from his interview, followed by brief summary of Uwe's intuitive substantiations of the three integral theorems. For this purpose, Table 13.1 shows the discursive frames from Uwe's intuitive mathematical discourse about complex path integrals. It shows whether he followed a discursive frame for the task situation to describe his mental images or otherwise intuitive explanations of complex path integral and/or for his intuitive substantiation of at least one of the integral theorems.

*The **"restriction of generality"-frame***

This discursive frame contains the metarule to restrict the generality of the mathematical objects under consideration followed by exploration routines on the mathematical objects from this restricted context. We do not claim that this frame is particularly specific for intuitive mathematical discourses; it may equally well be used in scholarly mathematical discourses (e.g., it may be used for the construction of theorems which could not be endorsed otherwise or to simplify a certain proof, etc.), but it occurred in Uwe's and the other two intuitive mathematical discourses. This discursive frame usually appears in combination with another frame, which enables the discursants who restricted the level of generality to explore the mathematical object or proposition further. For example, combining it with the "theorematic"-frame, Uwe attributed a meaning to complex path integrals based on the residue theorem (see below).

Uwe argued that complex path integrals are only "interesting" for holomorphic functions and that there usually is a closed path "behind the scenes". Hence, having restricted the generality accordingly, he only needed to construct narratives about $\int_\gamma f(z)\,\mathrm{d}z$ for holomorphic f and closed γ.

The ***"theorematic"-frame***

The theorematic frame consists of the metarules to recall theorems and to use them for further explorations in intuitive mathematical discourses. That is, theorems are used to derive discursive images or substantiate other propositions. For example, Uwe recalled the residue theorem to construct a narrative about the "meaning" of complex path integrals (see below). As another example, Uwe recalled a theorem about the vanishing of real path integrals for conservative vector fields along closed paths, the integrability condition, and the Cauchy-Riemann differential equations to construct the narrative that Cauchy's integral is "not surprising".

Using the "theorematic"-frame, Uwe constructed the following discursive image about complex path integrals:

- *"The meaning of the complex path integral for holomorphic functions is the weighed sum of residues."* This narrative is a discursive image because Uwe addresses the keyword "meaning" in relation to complex path integrals here explicitly. It is bond to the restriction to holomorphic functions (and to closed paths, which are not explicitly contained here).

We call a discursive image based on a theorem a *theorematic image*. Accordingly, the previous discursive image might be referred to as the *residue theorem image*.

The ***"vector analysis"-frame***

This discursive frame is characterised by a discursant's effort to connect the complex path integral or a proposition from complex analysis to vector analysis. This can be done, for example, by finding an analogy (cf. Bartha, 2019) between mathematical objects or narratives from the two discourses (complex analysis and vector analysis). Accordingly, this frame includes the recalling of objects or narratives from the vector analysis. For example, Uwe recognised an analogy between the definitions of complex and real path integrals, samed these objects, and produced the following discursive images:

- *"There are path integrals of first, second, and third kind. The complex path integral is a path integral of third kind."* Here, complex and real path integrals (of first and second kind) are samed as under the object "path integral". Therefore, this discursive image unites and distinguishes complex path integrals from the others at the same time. Uwe explained in this context that all path integrals share the same structure in their integrands: They involve a product of the integrand composed with the path (e.g., $f(\gamma(t))$) and the derivative of the path (e.g., $\gamma'(t)$), but the choice of multiplication depends on the path integral. Moreover, physical interpretations for real path integrals of second kind in terms of work or flux were not transferred to complex path integrals.

- "$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (f, if)^T\,\mathrm{d}\mathbf{T}$." This narrative was reconstructed from Uwe's substantiation of Cauchy's integral theorem. He aimed to equate $f \cdot \gamma'$ with $\langle (f, if)^T, \gamma' \rangle$, both of which appear as realisations of the integrands in a complex path integral and a real path integral of second kind (the latter for a $\mathbb{C}^2$-valued vector field though). Here, Uwe multiply acted according to his metarule to equate vectors with complex numbers and rewrote a product of complex numbers in terms as a matrix-vector-product.

*The **"tool"-frame***

Using the "tool"-frame means using or valuing complex path integrals as tools with the help of which other mathematical actions can be carried out (e.g., computing real integrals, proving theorems in complex analysis, etc.). Uwe also recognised that Cauchy's integral formula can be used as a tool to compute certain values of a holomorphic function or to prove the analyticity of holomorphic functions.

- Uwe's discursive images "*The complex path integral is first of all bluntly a tool*", "*the complex path integral is a powerful tool with which you can prove many things in complex analysis*", and "*the complex path integral is a tool for computing real integrals*" explicitly realise complex path integrals as tools for different purposes.

*The **"no meaning"-frame***

To follow this discursive frame means to deprive complex path integrals of an own meaning and potentially to advise others to not imagine anything about them. Although Uwe initially adhered to these metarules, he unfolded a rich intuitive mathematical discourse afterwards from which we could reconstruct the most discursive images about complex path integrals among our three experts.

- "*Complex path integrals do not have a geometric meaning*" and "*Complex path integrals normally do not have any special meaning in themselves.*" Whereas Uwe acknowledged the area image for Riemann integrals and that real path integrals of first and second kind describe arc lengths or work, complex path integrals are explicitly described as mathematical objects without any geometrical or intrinsic meaning.

Intuitive substantiations of the three integral theorems

Uwe combined the **"theorematic"-** and **"vector analysis"-frame** to substantiate Cauchy's integral theorem by recalling an analogue theorem about the vanishing of real path integrals of second kind for vector fields satisfying the integrability condition. Essentially, he realised the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ as $\int_\gamma (f, if)^T\,\mathrm{d}\mathbf{T}$ and concluded that $(f, if)^T$ formally satisfies the integrability condition $\partial_2 f = i\partial_1 f$ as a consequence of the Cauchy-Riemann differential equations. According to Uwe, "one is no longer surprised" by Cauchy's integral theorem.

The applications of rules from four discursive frames could be observed when Uwe explained Cauchy's integral formula: He established a connection to the mean value property of harmonic functions for the special case $z = z_0$ (**"restriction of generality"-frame**, (**"vector analysis"-frame**), differentiated between the interpretation and identified Cauchy's integral formula as a quantitative version of the identity theorem using *retrospective substantiation* (**"theorematic"-frame**). He also values the formula as a tool to compute function values and to prove the analyticity of holomorphic functions (**"tool"-frame**).

Last but not least, Uwe's story about the vanishing of integrals along triangles is based on a theorem in complex analysis relating the existence of primitive functions to the vanishing of complex path integrals along boundaries of triangles (**"theorematic"-frame**).

Closing remarks

Uwe's usage of the words "(mental) image" (German: Vorstellung) or "meaning" indicates his general awareness for potential interpretations of complex path integrals. Although he initially regarded complex path integral only as tools and attached no meaning to them, his intuitive mathematical discourse is rich and variable in the sense that he can unfold multiple discursive frames.

As a result of his metarules to restrict the paths to closed ones or of integrands to holomorphic functions (sometimes with the option for singularities, sometimes without), the corresponding idiosyncratic usage of the keywords "holomorphic function" or "path", and his metarule to occasionally equate complex numbers with vectors some of his narratives caused commognitive conflicts between him and the interviewer, between his intuitive mathematical discourse during the interview and scholarly complex analysis discourse, or even intra-discursive (cf. Kontorovich, 2021b). These conflicts were partly resolved when the interviewer pointed to the technical constraints, which were not explicitly present in Uwe's narratives, or when Uwe explained the metarule on equating complex numbers with vectors explicitly.

14

THE CASE OF DIRK

This chapter covers the interview with Dirk. In the actual conversation, the interview part on complex path integrals and integral theorems was divided into two parts, one rather at the beginning, the other rather at the end. In total, these parts lasted approximately 43 minutes and spanned roughly 60 turns ($\approx$ turns 16–25, 136–188).

As in the analyses of the interview with Uwe, we first introduce Dirk and turn to general findings related to the role of mental images or visualisation in complex path integrals (Section 14.1). Then, we continue with Dirk’s production of a discursive image using the **“theorematic”-frame** in Section 14.2 and his attempt to transfer the area interpretation for real integrals to complex path integrals applying the **“area”-frame** in Section 14.3. Afterwards, we discuss Dirk’s application of the **“vector analysis”-frame**, according to which he describes integration as an infinitesimal summation of vectors. However, similar to his application of the area-frame, he did not construct an explicit narrative about the complex path integral. In Section 14.5, we finally briefly cover Dirk’s rejection of the **“tool”-frame**.

Dirk’s substantiations of the three integral theorems are mainly characterised by the **“restriction of generality”-frame**, to which other frames and the general routine to realise mathematical objects graphically are added. In Section 14.6, we discuss Dirk’s substantiation of

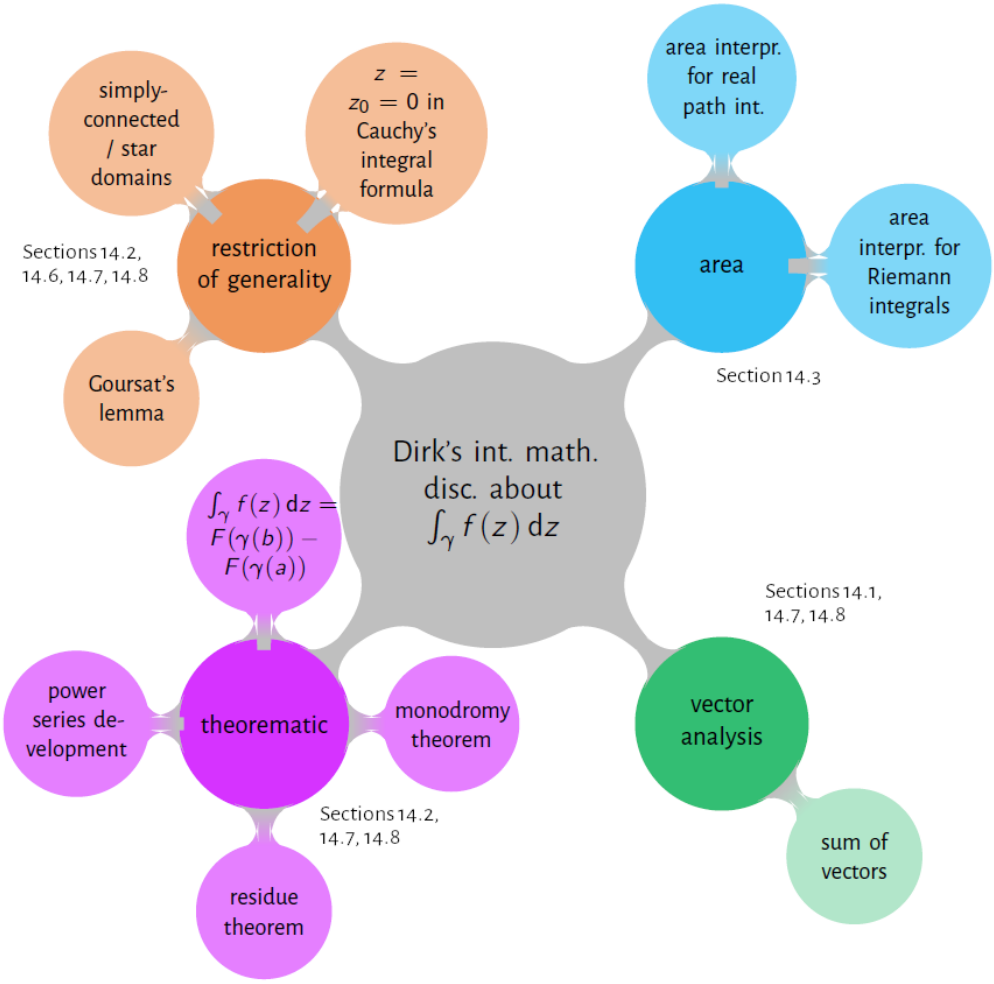


Figure 14.1: Discursive frames from the interview with Dirk.

Goursat's lemma as a special case of Cauchy's integral theorem. Then, Section 14.7 and Section 14.8 cover Dirk's intuitive substantiations of Cauchy's formula and the existence of primitive functions. During the analyses, we will also compare commonalities and differences between Uwe's and Dirk's enactment of the discursive frames. In Section 14.9, we summarise Dirk's intuitive mathematical discourse about complex path integrals.

As in Chapter 13, we illustrate Dirk's usage of the discursive frames with **??**. Even though he used various discursive frames during his explorations about complex path integrals, we have identified only one explicit discursive image about complex path integrals in this interview, which appears in Section 14.2, namely the narrative from the fundamental theorem of calculus for complex functions (Proposition A.16) according to which $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(b))$ for a function f with primitive F.[138]

14.1 INTRODUCTION TO DIRK

In this introductory section, we describe briefly how Dirk compares complex and real analysis as well as how he describes his use of visual mediators in mathematics in general and complex

138 Preliminary results on Dirk's search for suitable interpretations of the complex path integral and of Cauchy's integral formula can be found in Hanke (2019, 2020a).

analysis in particular. Dirk aims to provide "pictures" (German: Bilder), a keyword he uttered multiple times, whenever possible, and the quest for pictures is present during the whole interview. Dirk describes himself as a visual thinker who tries to convey vivid, intuitive reasoning in his lectures on complex analysis (Section 14.1.2). He sees complex analysis as a "new world" compared to real analysis, even though many technical terms are defined similarly as in real analysis (Section 14.1.1). Throughout the interview, he tries to derive a still to him unknown visual mediator multiple times, with the help of which he can realise complex path integrals as a certain area or another geometric object. Even though the only discursive image about complex path integrals Dirk utters explicitly is a formula (Section 14.2), his intuitive mathematical discourse is strongly influenced by visual mediators and the search for them. The visual mediators he actually produces, however, mostly realise domains in $\mathbb{C}$, traces of paths, or are related to integrals from real or vector analysis, and Dirk is careful to not mistakenly transfer them to complex path integrals. Rather, he explains carefully why the visual mediators he produces for real integrals do not transfer to the complex case.

In addition, Dirk's discourse is characterised by lots of explorations based on the use of different theorems from complex analysis. That is, he is frequently employing a version of the **"theorematic"-** and the **"restriction of generality"-frame**. Using these frames, he reduces the generality for his subsequent explorations. At different points, in particular during his substantiations of Goursat's lemma (Section 14.6), we observe a specific usage of keywords, which we will call *pars pro toto usage*. That is, he uses certain keywords not primarily for the realisation of mathematical objects appearing in certain proofs but rather as pointers to full narratives about these keywords in the respective proofs.

14.1.1 *A "new world"*

Discussing the contents of a course on complex analysis, Dirk mentions several topics related to complex path integrals such as isolated singularities, residues, the identity theorem, and the proposition that holomorphic functions analytic (e.g., line 112f.). Additionally, he baptises complex analysis as a "new world" compared to calculus / real analysis, which he substantiates by recalling several theorems from complex analysis, which "shock[ed]" Dirk: "back then, it was a first shock [for me] that uh, simply assuming that a function is, uh, complex differentiable, implies that it is analytic" and the "next shock" for him was "the identity theorem" (turn 4). He concludes that "this, uh, is quite a different world than real analysis, this/ the concept of holomorphicity, is very strong, right", which one should emphasise time and again in teaching:[139]

4–Dirk: [...] This is different here, it's a new world. It is not real analysis.

Accordingly, Dirk separates real / vector analysis and complex analysis from each other. Nevertheless, he is fully aware of the connections between the two discourses and potential links for teaching. For instance, Dirk claims that there are two ways to prove Cauchy's integral theorem. The first is via Goursat's lemma, which he considers to be intrinsically complex analytic, not making use of real path integrals, and the second is based on real path integrals. In this regard, acknowledges explicitly that "many things" (e.g., proof methods) are analogue (see Section 8.1.4):

6–Dirk: [...] It is not the usual way in a lecture on complex analysis, but, uhm, if
there is time left, uh, it is interesting, to also talk a little bit about, uhm

139 See Section 15.3, where Sebastian discusses how this perceived "rigidity" of holomorphic functions is enacted as a whole discursive frame influencing his intuitive substantiations of integral theorems.

gradient fields, or conservative, uh, fields in the first place and, uh, path integral, scalar or vectorial in the real case, right, because many things are analogue, proof methods are analogue, too. This, uh, would be interesting to build this bridge [...]

Recall from Section 13.3.4 that Uwe prefers not to include real path integrals to his teaching class for reasons of difficulty and required previous knowledge. In contrast, Dirk does not reject the usage of real path integrals in complex path, rather he values this usage as a potential for teaching ("interesting to build this bridge", line 6f.), which may be included to a course on complex analysis in case enough resources are left (e.g., time).

14.1.2 *"Thinking in pictures"*

We introduced earlier that Dirk considers himself a visual thinker. Consequently, Dirk aims to develop "pictures" of the mathematical objects he investigates and also to include visual mediators to teaching. In addition, he finds that complex analysis fits to his own way of thinking and that complex analysis is one branch of mathematics, which is particularly suitable for the inclusion of pictures.

The following episode took place after Dirk described how he imagines holomorphic functions. He described them as particularly "rigid" and that they locally behave like amplitwists (i.e., small circles are mapped to small circles instead of ellipses; see Section 5.1.3). Then, Dirk explains more generally that he considers himself a visual thinker:

6–Dirk: [...] I think in pictures. [Int: Mmh.] So, I am actually a geometer and I think in pictures.

Accordingly, it is plausible that Dirk generally attempts to explain mathematical notions in terms of pictures. When asked whether he regards complex analysis differently than other branches of mathematics with respect to the potential inclusion of pictures and mental images, Dirk again underlines his preference for geometrical aspects and describes a general metarule, namely to develop a picture for the mathematical notions first before formalising them:

57–Int.: Mmh. (.) Would you say that complex analysis, uh, is different from other branches of mathematical for you personally? With respect to how you treat pictures and mental images? Or is this (.)/
[...]
58–Dirk: Uhm, well, so I/ (...) Uhm. It depends on what, what you mean with "different". Uhm, it is similar in all things [mathematical topics; EH.] that/ that I deal with in teaching and research [Int.: Mmh.] and that, that's also because of, of the afore/ aforementioned structure of my thinking, so to speak. I, I don't do abstract algebra for good reasons [laughs]. [Int.: Mmh.] Not in research. I regard algebra as a very, very useful and meaningful instrument to do geometry. [...] and therefore complex analysis fits very well into my personal thought pattern (German: Denkschema) [Int.: Yes.], and, uh, therefore, I love to teach complex analysis. And that, uh, uhm, (.) and, and, uh, I'd say it is different from abstract algebra, uh, in that, that one really/ that it's just the right thing to develop these pictures (German: dass es sich anbietet, solche Bilder zu entwickeln). Uhm (...) fine, my/ my research area is ***, which has a lot to do with complex analysis, with Riemann surfaces at so on, right, this, this is tightly interwoven, right, but it is a similar pattern of thought that you develop the picture of what you investigate first of all, right, [Int.: Mmh.] and then you try to, uh, formalise these things [...] I would perhaps note that as a, uh, uh, difference between complex analysis and other, perhaps somewhat drier [mimics quotation

marks with his fingers] areas of mathematics, right, that one can, uh, develop these images and work with them. [...] But one should not stick with the pictures.

Dirk claims to not only work with pictures in research and teaching generally, but moreover he argues that one can “develop” and “work” with pictures in complex analysis particularly well (line 33) as opposed to “drier” areas of mathematics. Consequently, he loves to teach it. Nevertheless, Dirk cautions that “one should not stay with the pictures” (line 33). He describes furthermore that he experienced students to fail to be able to formalise mathematical arguments if they only stick with pictures:

60–Dirk: [...] lecturers should, should clearly realise this danger, right, that, uh, if they work too intuitively and convey too many pictures, then, some, uh, students get the, the impression: Yes, this is somehow like a comic book, and pf, I do not need to care about details now. [Int.: Mmh.] This is dangerous.

We conclude that Dirk endorses the metarule to visualise mathematical notions in both, teaching and research, but additionally to not stop at visualising concepts but complement to visual mediators with formalisations, too. Otherwise, Dirk sees the danger that students may not learn to formalise mathematics.[140]

We may hypothesise at this moment that Dirk would have developed “pictures” for the complex path integral, which he could or has already used in teaching. However, this is not the case as we show in the following sections. Nevertheless, we will discuss in Section 14.3 how Dirk enacted the **“area”-frame** in order to realise complex path integrals as a certain area.

14.2 A HESITANT APPLICATION OF THE **“THEOREMATIC”-FRAME**

In this section, we will see an application of the **“theorematic”-frame**. Dirk recalls the complex analytic version of the fundamental theorem of calculus (Equation A.17) to create a narrative about the complex path integral in response to the interviewer’s quest for a potential meaning of the complex path integral. Since the function in this narrative requires a primitive, it is necessarily also derived in combination with **“restriction of generality”-frame**.

Similar to the interview with Uwe, the interviewer introduced the question about a potential meaning of the complex path integral with a description of the area interpretation for Riemann integrals and hands out the corresponding question sheet (Figure 11.1):

136–Int.: [...] Which geometrical meaning or which vivid (German: anschauliche) meaning does this integral have [encircles the inscription $\int_\gamma f(z)\,\mathrm{d}z$], this complex number that comes out, for a, let’s say, (.), yes, as you wish [points to γ in Figure 11.1], continuously differentiable path, or piecewise, or whatever you would like to take, and a function, that is defined on its image, defined on the image of the path. Now, what can one imagine [Dirk: imagine here] here? Is there a nice meaning?

Dirk: (13s.) Uhm, so in strict complex analytical context, what is the meaning? Alright, uhm, (5s.) if one talks, uhm, about path integrals in vector fields, then it has a physical meaning, but what, what would it be here? (23s.) Uhm. (7s.)

140 We cannot tell exactly what Dirk considers “formal”. We can only hypothesise that what Dirk considers as formalised mathematics may correspond to scholarly mathematical discourses with precise definitions and deductive reasoning (cf. Sfard, 2014; Viirman, 2021) as opposed to the use of visual mediators only.

We observe two important aspects here. On the one hand, Dirk is hesitant and takes a long time to answer (see also the long pauses in Section 14.5). He interrupts approximately 50 seconds of silence only by repeating the interviewer's question or recalling a potential physical interpretation for another kind of integrals. In particular, Dirk does not question whether there is a potential meaning of complex path integrals. On the other hand, Dirk separates the complex analysis and vector analysis discourses. We conclude this from his utterances that "path integrals in vector fields" (i.e., real path integrals of second kind) have a "physical meaning" but he wonders what a meaning might be here in "strict complex analytical context". Similar to Uwe, who also acknowledged a physical meaning of real path integrals of second kind (Section 13.2), Dirk does not transfer an interpretation from vector analysis to complex path integrals. In such, we infer that Dirk does not follow the "vector analysis"-frame at this point (but see Section 14.4).

The interview seems to get stuck and Dirk is very uncertain and hesitant in responding. Accordingly, the interviewer introduces two prompts to the interview. The first prompt deals with the potential discursive image that complex path integrals are "tools", which we already discussed in Section 14.5. The second prompt deals with Cauchy's integral theorem (see line 123):

142–Int.: Let me ask you a follow-up question I might have asked [laughingly: mmh]. Namely: Uh, we already addressed Cauchy's integral theorem. [Dirk: Yes.] Depending on the version you prefer the assertion eventually is that the integral vanishes for closed paths, depending/ under mild technical conditions. Can one/ I mean, why is the integral for holomorphic functions zero [draws a circular gesture with his hand in the air] along a closed path? [Dirk: Uhm.] Under the technical conditions one might still have? [Dirk: short laughter] So something like star-shaped domain or something like that.

Dirk: Yes, uhm. So, I believe, I know what this boils down to. I mean if one knows that, uh, f has a primitive function [writes f and F in Figure 14.2], [Int.: Mmh.] if one starts off from this knowledge, right, then it can, uhm, be relatively quickly shown/ [writes $\int_\gamma f(z)\,\mathrm{d}z$ in Figure 14.2]. Uh, sorry, right, uhm, let's say [encircles the γ in $\int_\gamma f(z)\,\mathrm{d}z$ with the pen in the air], yes exactly, and gamma is, uh, is not necessarily closed, right? [writes the lower line in Figure 14.2] [Int.: Mmh] [incompr.] is the domain, the, uh, primitive function [writes "Stammfkt" in Figure 14.2], (..) right. That this is (...) [writes $F(\gamma(b)) - F(\gamma(a))$ in Figure 14.2]. Uhm, yes, one can use this perhaps to he/ help with the imagination (German: Vorstellung), right, but (.) [...]

The interviewer asks for a substantiation of Cauchy's integral theorem, which he recalls idiosyncratically in terms of the utterances "for closed paths the integral vanishes [...] under mild technical conditions" and the "integral for holomorphic functions [is] zero along a closed path". He also produced a visual mediator in form of a gesture tracing a closed curve in the air. Additionally, Dirk is asked to choose himself which technical conditions he wants to add, and he suggests that domain may be star-shaped himself. Dirk now produces the narrative

Theorematic discursive image: $\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{14.1}$$

Hence, he enacts a recalling routine and states the formula from the complex version of the fundamental theorem of calculus (Equation A.17) in Figure 14.2. This narrative realises the complex path integral as the sum of the differences of a primitive function F of the integrand f.

Since, Dirk adds the constraint that the γ "is not necessarily closed", it is likely that Dirk does not follow the interviewer's prompt to discuss Cauchy's integral theorem at this point but actually still replies to the previous question on a potential meaning of the complex path inte-

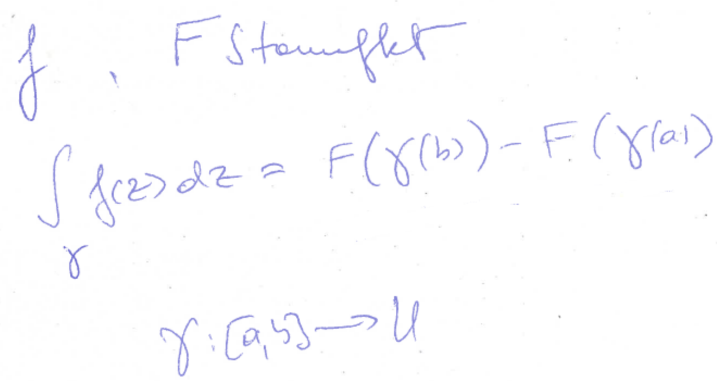


Figure 14.2: Realisation of a complex path integral as a difference of the values of a primitive function for the integrand.[a]

[a] “Stammfkt.” is an abbreviation for the German word “Stammfunktion”, which translates to “primitive function”.

gral. After all, one prerequisite in all versions of Cauchy’s integral theorem is that the path is closed.[141] Our interpretation is also supported by Dirk’s utterance believes “I know what this boils down to” (line 60), which we interpret as a moment of rather sudden insight. Additionally, Dirk claims that Equation 14.1 “could help with the imagination” (lines 68ff.). Even though it is not stated precisely what the “imagination” is about, we classify the narrative in Equation 14.1 as a discursive mental image about the complex path integral (i.e., a theorematic image since it is based on a theorem). Since Equation 14.1 is only valid for functions f with a primitive function F, we also attribute that Dirk has necessarily restricted the level of generality and his narrative does not apply for functions without a primitive function. However, since the interviewer mentioned the constraint from Cauchy’s integral theorem on the integrand to be holomorphic, it is also possible that Dirk based his narrative on this premise.

In the next section, we discuss the episode, which directly continues the previous excerpt at line 70. Based on this episode, we can finally confirm that Dirk is still trying to construct a narrative for the geometric meaning of complex path integrals, not a substantiation of Cauchy’s integral theorem.

14.3 THE **“AREA”-FRAME**: TRANSFERRING THE AREA INTERPRETATION

In this section, we analyse how Dirk enacts the **“area”-frame**. He employs the routine of graphing and tries to transfer the area image of Riemann integrals (Section 2.2.3) to complex path integrals.

143–Dirk: [...] but (.) in principle one would like to, uhm, resort to such a picture, right [draws Figure 14.3]. [incompr., likely: And now] it’s, uhm, not an interval in R, but a path. Let’s say, this here is C, right [draws Figure 14.4a], and, uhm, this is necessarily helpful, such a picture, since the values are complex. Uhm, well, that one says: Okay, here I have a function on a domain [encircle a part of the sketched plane and adds the letter U; Figure 14.4b] here to R on the/ This is the trace of the path, right [writes γ in Figure 14.4b] [Int.: Mmh.] and that, that the function has certain values

141 Although Dirk could a set of prerequisites for Cauchy’s integral theorem himself, it is not plausible that he, as an expert, would endorse a version of Cauchy’s integral theorem for non-closed paths. Hence, this supports our interpretation that Dirk does not discuss Cauchy’s integral theorem at this point.

Figure 14.3: Dirk's "such a picture" showing the graph of a real-valued function on an interval.

> and that one somehow computes this area [points with the pen to the shaded area in Figure 14.4b]. Uhm, this is maybe a crutch, but, uh, well, one has to consider [writes the $\mathbb{C}$ to the axis pointing upwards; Figure 14.4b] that the values here [taps on the last written $\mathbb{C}$] are complex, right. [Int.: Mmh.] So one cannot draw it like that. [...] Uh, well (..) what is a good picture? (..) I believe, it is, uh, difficult, to draw a picture because, uh, (..) well, that uh/ Here we have real-valued functions of one real variable [points to Figure 14.3], so the graph is in two-space, one can draw that, but the equivalent would be four-dimensional. One can't draw that.

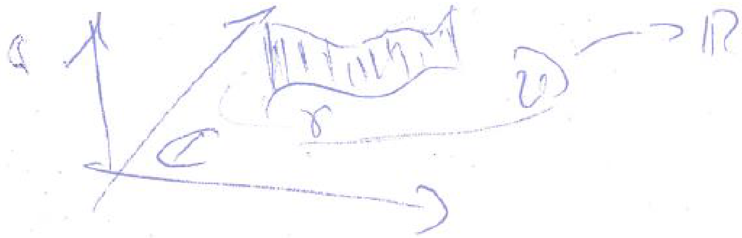
Dirk claims that "one" (i.e., a generic mathematist) "would, uhm, like to resort such a picture" like that in Figure 14.3, in which he realised a real-valued function of a real variable, whose graph is in "two-space" (lines 85ff.). Hence, Dirk performs the routine of sketching a graph of a real-valued function of a real variable and shading the area below the graph and the x-axis, which is frequently seen in visual mediators for the area interpretation of Riemann integrals. It is evident that this interpretation and figures showing the area below a graph belong to Dirk's precedent search space for the task to describe the geometric meaning of $\int_\gamma f(z)\,dz$, after all, similar visual mediators are often drawn in real analysis discourse to realise the Riemann integral as a certain area. Here, Dirk engages in an exploration routine investigating whether this interpretation for Riemann integrals can be transferred to complex analysis.

Dirk modifies Figure 14.3 to Figure 14.4a by replacing the "interval in R" with "a path" in $\mathbb{C}$ (line 72). Figure 14.4a also shows a shaded area. This visual mediator realises the graph of a function with positive real values defined on the trace of a path in the complex plane. Dirk acknowledges though that Figure 14.4a is "not necessarily helpful [...] since the values are complex" (lines 74f.) and that "one cannot draw it like that" (line 83) because the values of the function (i.e., the integrand in a complex path integral) are complex. He realises the co-domain of the function to be integrated with the letter $\mathbb{C}$ at the axis pointing upwards in Figure 14.4b. Nevertheless, this axis is 1-dimensional and the function actually realised in this figure has real values, which Dirk indicates with the arrow and $\mathbb{R}$ at the right side of the figure.[142] He argues that an appropriate figure in the context he started to investigate would show a four-dimensional graph.

Dirk's attempt to transfer Figure 14.3 to a visual mediator realising the graph of a complex-valued function or an area corresponding to a complex path integral is not successful, but he reflects why Figure 14.4b is not fully appropriate and may only be a "crutch" (line 80) for the visual realisation of a complex path integral. In commognitive terms, Dirk resolves a commognitive conflict: The narrative about complex path integrals to be constructed and the visual mediators stem from different discourses, namely complex and namely real / vector analysis. The

142 In our view, the visual mediator in Figure 14.4b realises a real path integral of first kind and we conjecture that Dirk knows this even though he does not say it explicitly (see Figure B.2).

(a) Dirk’s visual mediator showing a curved shaded area over the trace of a path in the complex plane.



(b) Dirk’s visual mediator showing a curved shaded area over the trace of a path in the complex plane (continued).

Figure 14.4: Dirk’s attempt to transfer “such a picture” to the complex setting (right).

visual mediators realise graphs of functions in 2- or 3-dimensional space; but the graphs of the integrands of complex path integrals are sets in 4-dimensional space. Hence, Dirk detects an incommensurability between the narrative to be constructed and the visual mediators produced.

Nevertheless, Dirk is certain that a visual mediator can be found, which serve to realise complex path integrals:

143–Dirk: [...] Nevertheless (.) is has to be possible (..) uh [taps on the integral $\int_\gamma f(z)\,\mathrm{d}z$ on the question sheet] to develop a picture for that. (8s.)

In summary, Dirk engaged in an exploration with the aim to transfer the area interpretation for Riemann integrals (and likely for real path integrals of first kind even though he did not utter it explicitly) and to obtain a narrative about the complex path integral. That is, he tried to produce an analogy between realisations of two integrals (cf. Bartha, 2019): While he knew the first realisation (the area interpretation), the second was still unbeknownst to him (the realisation of the complex path integral).

Hence, we conclude that Dirk followed the **“area”-frame**: It contains the metarule of finding an area and constructing a narrative about the complex path integral in relation to this area. It also consists of metarules for realising mathematical objects (e.g., graphs of functions) with the help of visual mediators besides mathematical formulae. More generally, this frame is a special case of the general metarule of building an analogy between two mathematical discourses by recognising two narratives as being similar with respect to a specific feature (cf. Bartha, 2013). Here, the analogy is between the discourse on Riemann integrals and/or real path integrals of first kind and the discourse on complex path integrals.[143] Nevertheless, Dirk

143 Recall that Uwe was also intensively building an analogy between different kinds of integrals, but in his case, this analogy dealt with the product structure in different path integrals. Here, in the case of Dirk, it is about a geometric interpretation.

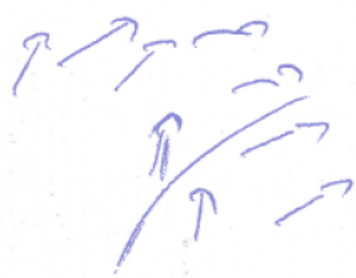


Figure 14.5: Dirk's visual mediator for a path in a vector field.

did not produce a discursive images about complex path integrals with the help of the "area"-frame. The reason for our conclusion here is he did not explicitly realise complex path integrals in any of his utterances and that he did not consider any of the visual mediators he drew to be adequate.

14.4 THE **"VECTOR ANALYSIS"-FRAME** AND INFINITESIMAL SUMMATION

Not having succeeded in producing a visual mediator apart from a formulae to realise the complex path integral with the help of the area-frame, Dirk produced another visual mediator, in this case, one of a "vector field" and a "path".

143–Dirk: [...] if one looks at scalar or vectorial (German: vektorielle) integrals in a vector field [draws the arrows in Figure 14.5], right, so (..) [looks up, thinking], uh (7s.), right, now first of all vectorial. The equivalent here is a vectorial path integral [Int.: Mmh.], right? (...) Then/ Integration is some kind of a, uh, infinitesimal summation [marks the last two words with his fingers in quotation marks], right, and then one sums up [draws the curve without arrowhead in Figure 14.5] how these vectors here along this path [points to the curve drawn last] stick out from the/ from the [Int.: Mmh.] tangential vector, right, that is then, so to speak, uhm (4s.), this picture there and I believe, uhm/ Well, if/ There is of course an equivalent if one talks about winding numbers here [points to $\int_\gamma f(z)\,\mathrm{d}z$ in Figure 11.1], right, and then, one does something similar, right, when one integrates, (..) right, along a path [traces some circles with his fingers in the air], right, and one counts [similar gesture], uh, uh, how often one has turned. [Int.: Mmh.] [...]

Here, Dirk enters the discourse about real path integrals, in particular real path integrals of second kind, which he baptises "vectorial integrals", again. He switches to this discourse because he considers "vectorial integrals" to be the equivalent to complex path integrals. Henceforth, it is apt to describe Dirk's discursive actions in terms of the **"vector analysis"-frame**. Dirk's discursive actions here can be described in terms of the metarule to create an analogy between complex path integrals to real path integrals and to recall an interpretation of the latter. In this context, he describes "[i]ntegration [as] some kind of a, uh, infinitesimal summation" of "how these vectors here along this path stick out from the tangential vector" (lines 94ff.). Hence, he constructs a narrative about "integration" as a process, namely infinitesimal summation and recalls the interpretation of real path integrals of second kind as work or flow in an idiosyncratic manner, namely in terms of a certain sum, in which it is summed up "how these vectors here along the path stick out [..] from the tangential vector field" (lines 97; see Section B.2.2). Similar to many other utterances he produced before, he does not speak of his own interpretation here but phrases them as general statements independent of himself. However, the narratives produced here are not about complex path integrals. Hence, we cannot identify a discursive image about complex path integrals here again.

Whereas summation could also be used to describe complex path integrals (see the definition via Riemann sums in Section 8.1.1), Dirk does not transfer this idea to the complex path integral directly. Rather, he changes context again, now using the keyword "winding numbers" (line 101; see Definition A.28). In this context, he claims to identify an "equivalent" again; however, we are not certain what exactly he considers equivalent here. Nevertheless, integration is again mentioned as a process, namely an action of the generic mathematist "one" to integrate along a path and to count "how often one has turned" (lines 102ff.).[144]

14.5 REJECTING THE **"TOOL"-FRAME**

In this section, we present a strong contrast to Uwe's intuitive mathematical discourse, namely Dirk's firm rejection of the **"tool"-frame**. While Dirk eventually rejects this frame to be part of his intuitive mathematical discourse, he nevertheless shares the idea that certain the notion of complex path integral and in particular Cauchy's integral theorem are a "central technical notion" and a "building block" in complex analysis:

4–Dirk: [...] we have Cauchy's integral theorem, this is the basic building block for
almost everything that comes after, right, [...] Okay, uh, speaking of the
path integral, uh, of course one should introduce the path integral in the
context of Cauchy's integral theorem [Int: Yes.] and also explain, right, that
it, uh, is the central notion, uh, probably, uh, the central technical notion
of complex analysis, and, uhm, okay, this/ And when one has Cauchy's
integral theorem, right, then, the, uh, power series development theorem is
not far away anymore [...]

Dirk values the (complex) path integral as "the central concept, uh, probably, uh the central technical concept of complex analysis" (line 110f.). Even though this may at first look similar to Uwe's valuation of the complex path integral as a "tool" (Section 13.5), Dirk does not endorse it as an intuitive explanation or a mental image of the complex path integral: When discussing a possible geometric interpretation, the interviewer challenges Dirk whether the complex path integral is just a tool, but Dirk considers this point of view unsatisfactory.

The next excerpt continues the last excerpt we showed in Section 14.2.

138–Int.: I have already heard people say: Okay [coughs], now, this is simply a
technical tool, this integral, with the help of which you can prove things
which matter in complex analysis in the end.
Dirk: Fine, one may take up this position, but, uhm, (4s.) don't know how, uh,
satisfying this is. Uhm. (11s.) Mmh. (18s.)

144 We remark that Dirk restricts the generality of paths here. Winding numbers only make sense for closed path, but Dirk only uses the keyword "path" instead of "closed path". This hints at potential commognitive conflict with scholarly complex analysis discourse based on an imprecise usage of the keyword "path" and it also conflicts with the visual mediator in Figure 14.5, which does not show a closed path. Rather, it seems that the quick change of contexts here caused the non-endorsable usage of keywords or construction of narratives here. This commognitive conflict is similar to those we analysed in Uwe's intuitive mathematical discourse about the complex path integral. Uwe also produced narratives that could only have been endorsed if some keywords had been specified additionally (e.g., see Section 13.4 or Section 13.4.3).
In principle, Dirk restricts the class of the paths here in the context of winding numbers from arbitrary (piecewise continuously differentiable; this additional requirement remained tacit at most parts of the interview) paths to closed paths. However, we refrain from describing this part of the interview with Dirk in terms of the **"restriction of generality"-frame** because Dirk did not produce a narrative about complex path integrals here.

Int.: So, if you have not thought about this before, that is also a valid answer so to speak.
Dirk: [laughs] Uhm, I actually believe to have thought about it before, I have just forgotten how I explained it back then. Uhm. (21s.)
[...][145]
143–Dirk: this is, uhm, probably the same idea we talked about before, right, that this is a technical tool somehow, which, which makes several things, uhm, accessible, but it shouldn't be this way, [one; EH.] should (...) be able to deliver a good picture of what this means. [Int.: Mmh.] Well, and this, uhm/ I believe this is not always done in textbooks, right, (.) uhm, (4s.)/
Int.: Yes, so it is indeed rare to find something about this.
Dirk: (4s.) The question is also whether it is helpful or rather confusing. [Int.: Mmh.] (5s.) What this is (..) about. (13s.) Uhm, uh, no, I would, would have [Int.: Mmh.] to think about it longer how to teach/convey this.

Dirk acknowledges that one may regard the complex path integral as a tool to "make things accessible", but also that it "shouldn't be this way" only. For him, it is preferable to "deliver a good picture" (lines 125ff.). Hence, even though Dirk calls the narrative that the complex path integral is a tool, we cannot to count it as a discursive image of his.

However, we also have to note that Dirk claims that "it is not always done in textbooks" and he conjectures that "it", likely again a potential picture, may in fact be confusing or unhelpful, where we interpret "it" as the provision of a "good picture of what this [the complex path integral; EH.] means".

INTERMEDIATE BREAK

Having reconstructed Dirk's usage of discursive frames to explore potential discursive images for complex path integrals, we will now analyse how he substantiated the three integral theorems (Goursat's lemma (Theorem A.19), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20)) intuitively. The overarching frame Dirk enacts in his intuitive substantiations of the integral theorems is the **"restriction of generality"-frame**. Following this frame, Dirk concentrates on a substantiation of Goursat's lemma rather than the more general Cauchy's integral theorem. Similarly, Dirk focuses on the case of a power series when discussing Cauchy's integral formula. In line with his general pedagogical metarule to provide pictures whenever possible, his explorations of the three integral theorems are also characterised by multiple graphical realisations of mathematical objects.

14.6 SUBSTANTIATING A SPECIAL CASE OF CAUCHY'S INTEGRAL THEOREM WITH THE **"RESTRICTION OF GENERALITY"-FRAME**: GOURSAT'S LEMMA

In this section, we analyse how Dirk intuitively substantiated Goursat's lemma. We recall that Goursat's lemma states that $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$ if Δ is a compact triangle contained in the domain of a holomorphic function f. After Dirk recalled central theorems from complex analysis, the interviewer asked for a potential intuitive substantiation. In his responses, Dirk introduced Cauchy's integral theorem and Goursat's lemma. In particular, he starts to discuss the proof of Goursat's lemma instead of the more general Cauchy' integral theorem.

145 Here, the interviewer begins to ask about Cauchy's integral theorem as another prompt. We discuss this in Section 14.2.

14.6.1 Pars pro toto *usage of keywords—recalling central parts of the proof of Goursat's lemma*

The interviewer asks for vivid explanations of these theorems. Dirk argues that the "heart of the proof" of the power series development of holomorphic functions is Cauchy's theorem, and that Goursat's lemma is central for this theorem.

16–Int.: [...] Can one explain these big properties, like for example the identity theorem or power series development vividly (German: anschaulich) without having to comprehend a proof?

Dirk: Uh. (4s.) Mmh. (..) That is a good question. Without giving the proof. What is a vivid idea? (..) Power series development, yes, unfortunately, it is a long way to get there, right. One needs/ First, on needs to, uh, understand Cauchy's integral theorem and after that one notices, uh, that this, uh, is the heart of the proof. [Int.: Mmh.] (.) Uhm, (4s.) yes, (.) it's a good question, I, uhm, I never tried to simply explain the power series development theorem intuitively, right, there is this and that reason why one should expect it. [Int.: Mmh.] It, it is not easy because the way to it is long, to, to the power series development theorem. [Int.: Yes.] Uh, (..) this is a good starting point thought, I might think about it, longer, how one really/ [...]

Dirk recalls a chain of arguments for the proof of the power series development (of holomorphic functions). He repeats the interviewer's quest for an explanation "without giving the proof" rephrases the question as to "why one should expect it" (i.e., the power series development theorem). He adds that this is "not easy" because a "long way" is to go, namely via the "heart of the proof", namely Cauchy's integral theorem (line 140), and that he never tried to give an intuitive explanation for the power series development before.[146]

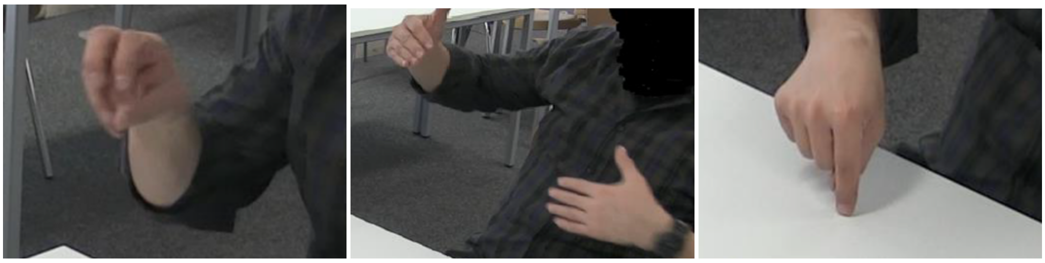
However, Dirk then refers to yet another theorem, namely Goursat's lemma, and substantiates it more detailed, but, as requested, he does not give a full proof:

17–Dirk: [...] Before we start with the proof, right, here, even though/ One can, one can already, uh/ One has Goursat's lemma, right, this idea with the triangles getting smaller and smaller, right, and, uh, no matter whether one takes triangles or rectangles and so forth [forms a triangle with the hands], and then one chooses one [gripping gesture with a hand; Figure 14.6a], right, and, uh/ Okay, one can, one can explain Goursat's lemma in a few steps intuitively, this would be closed, right, and then we get, uh, with this picture with the, uh, a bit of topology comes into play [huge gestures with firmly held hands; Figure 14.6b], right, with nested intervals and so forth, but in the one point, there the complex differentiability [Int.: Mmh] finally plays a role, right, [taps a finger on the table; Figure 14.6c], and uh, and so one can explain this, exactly, then/ [...]

In this excerpt, Dirk substantiates Goursat's lemma in terms of an idiosyncratic description of central elements from the proof of Goursat's lemma.[147] In this substantiation, he barely uses keywords related to integrals (except for the names of theorems). Nevertheless, he claims to be presenting "a few steps" of an "intuitive" explanation of the lemma (lines 152f.). A few utterances later, he also emphasises to have described "the intuition [...] one has to develop" for

146 When discussing the central elements of a course on complex analysis previously, Dirk argued that the power series development theorem is "not far away" if Cauchy's integral theorem has been established (lines 111f.). Thus, it seems that the longest part of the way to prove the power series development theorem Dirk describes at this point refers to the substantiation of Cauchy's integral theorem.

147 We recall that Goursat's lemma can be proven by estimating the modulus of $\int_{\partial\Delta} f(z)\,\mathrm{d}z$ to be smaller than any given positive number by inductively subdividing the triangle Δ with a sequence of nested subtriangles, whose diameters converge to 0 (see the end of Section 8.4.3 and our proof of Lemma 8.16).



(a) Dirk reaches for a point in the air. **(b)** Dirk illustrates nested intervals. **(c)** Dirk points to a spot on the table.

(d) Dirk folds his hands together.

Figure 14.6: Dirk's gestures accompanying Goursat's lemma.

Goursat's lemma (lines 171, see below).[148] Moreover, he does not refer to himself here but again to the generic mathematist "one".

The substantiation starts with a general reference to "the idea with the triangles [or rectangles, EH.] getting smaller and smaller" (line 148). The choice of one of these triangles is accompanied with a bodily movement, that is, his gesture in Figure 14.6a, which looks like he is picking something from the air. Likely, this gesture underlines his utterance that "one chooses one". Subsequently, he refers to other components of the proof, namely, "topology" and "nested intervals", again accompanied by bodily movements: Figure 14.6b shows that Dirk firmly holds his hands upright followed by a reduction of space between the hands, which fits well to the keyword "nested intervals", which we interpret here as a sequence of triangles getting smaller and smaller.[149]

Then, the utterance "but in one point, there the complex differentiability finally plays a role" is then accompanied with the next gesture, namely spotting a point on the table with one finger (Figure 14.6c). Dirk concludes that "so one can explain this". Hence, he believes that the keywords just mentioned, "topology", "nested [triangles]", and "complex differentiability" constitute the essential elements of the proof of Goursat's lemma. However, Dirk did neither explain these keywords nor how they connect to the proof. It is likely that he expected his interlocutor to not need a further explanation and the interviewer also did not ask.

A few turns later, the interviewer mentions "nested intervals" himself in one of his idiosyncratic descriptions of the proof of Goursat's lemma. Dirks takes up this keyword again and describes it more detailed:

21–Dirk: Uh, exactly, (.) and one realises that/ One can actually/ It is the same as on the smaller one, one the smaller one, on the smaller one [shrinking movement with his hands, see Figure 14.6d] [Int.: Right.], right, and then one realises: Okay, this gets arbitrarily small, so it has to be zero.

148 The keyword "path integrals" appears once during the substantiation of Goursat's lemma in the utterance "Yes, okay, this is the connection from the, uh, surprising property, right, that the path integrals of closed paths are zero, simply from the complex differentiability in one point" (lines 163; see below).

149 An endorsed definition of "nested intervals" is that they form a sequence of intervals such that one interval in the sequence contains the next interval and their diameters converge to zero (e.g., Forster, 2016, pp. 50–51). However, based on our knowledge of the proof of Goursat's lemma, we conclude that Dirk misspoke here and rather meant "nested triangles" instead of "nested intervals".

Here, Dirk is thus realising more specifically that something gets smaller and smaller, and we can infer from the context that this something is a triangle. This turn shows again a gesture accompanying the utterance "on the smaller one, on the smaller one", in which Dirk moves his hands together until they are folded on the table (Figure 14.6d). The narrative "[i]t is the same as on the smaller one" requires some interpretation though. We hypothesise that "it" is actually the complex path integral, and Dirk claims that this does not change when the triangle, whose boundary is the path of integration, is replaced by a smaller one in the sequence of nested triangles. While this is true—after all, all these integrals are 0 according to Goursat's lemma—this is in fact not yet known during its proof. Finally, the following utterance "this gets arbitrarily small, so it has to be zero" seems to correspond to the object-level rule needed at the end of the proof of Goursat's lemma, namely, if the modulus of the integral in question is smaller than an arbitrary constant, it has to vanish.

Pars pro toto

We conclude from the episodes described so far in this section that Dirk is not constructing narratives about complex path integrals or about the theorems mentioned, but instead, his utterances are often incomplete and serve as pointers to parts of the proof. That is to say that Dirk performs a special recalling routine, in which he does not explain the steps of the proof precisely but rather presents parts of the proof in terms of the central keywords. The central keywords function as a *pars pro toto* each. Moreover, the order, in which Dirk introduces them, corresponds to their order in the proof of Goursat's lemma. As such, each keyword is used to replace a whole narrative, which is part of the proof.[150] Both interlocutors have at least implicitly agreed that the other is familiar with complex analysis, and during the course of the conversation, and in particular Dirk seems to interpret the interview situation in such a way that he experts his interlocutor to be able to replaces the pars pro toto keywords with the partial narratives from the proof of Goursat's lemma.[151]

14.6.2 *Interplay of local and global differentiability*

Next, Uwe substantiates why this nesting of intervals/triangles to a point is important. In particular, he emphasises once again what he considers the "intuition" for Goursat's lemma, namely that holomorphic functions locally behave like amplitwists (see also Section 5.1.3).

> 17–Dirk: [...] Yes, okay, this is the connection (German: das ist der Bogen, den man schlägt) from the, uh, surprising property, right, that the path integrals of closed paths [quick circular gesture with one hand] are zero, simply from the complex differentiability in one point [repeats previous tapping gesture]. [Int.: Mmh.] Right, and then one sees: I do in fact need the complex differentiability, and this has to be exactly an amplitwist [touches the table with fingertips and turns his wrist] [Int.: Mmh.], the linear, uh, approximation, this cannot be any diffeomorphism, for example. So, there/ this would be the intuition, so to speak, which one has to develop. One has to understand a bit of topology, understand the nested intervals and then in that point: Yes, okay, there I have/ the function has to be complex differentiable there, so it is an amplitwist, right [turns his wrist in the air]. This, uh, this would be the intuition so to speak [...]

150 We cannot know for sure which proof of Goursat's lemma Dirk recalls exactly, but many proofs make use of a nested sequence of triangles or rectangles (see e.g., Fischer & Lieb, 2003; Freitag & Busam, 2006; Jänich, 2004; Lang, 1999; Remmert & Schumacher, 2002; see also Gray, 2000)

151 We also remark that this particular way of communicating is not a unique feature of Dirk's discourse compared to the other two experts. In Dirk's discourse it appeared most clearly though. For example, Uwe also hinted at some steps of the proof of the existence of primitive functions for holomorphic functions, too, but overall his descriptions were more detailed.

One has to understand topology a little, there with the nested intervals and then understand in one point [turning gesture with his rand]: Yes, okay, there I have/ the function has to be complex differentiable there, so it is an amplitwist [similar gesture], right, [Int.: Mmh.] this, uh, this would be the intuition [...]

Here, Dirk produces a narrative about the "surprising property" that "the path integrals of closed paths are zero, simply from the complex differentiability in one point". This alleged object-level rule is substantiated with the metarule that "I [i.e., Dirk] do in fact need the complex differentiability, and this has to be an amplitwist". However, this narrative is not fully endorsable because complex differentiability in one point alone does not suffice for the complex path integrals along triangular paths to vanish.[152] Dirk is aware of this conflict fact and he clarifies how differentiability is used in the proof of Goursat's lemma more detailed in his next turns. Likely, this explanation was initiated by the interviewer's question on the condition of holomorphicity instead of real differentiability (lines 189ff.). In particular, Dirk emphasises that the nesting triangles could in principle intersect to an arbitrary point in the interior of the original triangle, which substantiates why the condition of complex differentiability of the integrand at one point only is not sufficient:

19–Dirk: [...] so complex differentiable is usually normally defined locally at one point, uh, like with the real differentiability, right, but holomorphicity is usually, so to speak, right, defined as global property, but on open sets, [Int.: Mmh.]. [incompr.] Because this is the only way to unleash its full power, [Int. Mmh.] right, uh, if you only have this, uh, punctually in some points [quickly points to different locations on the table], that does not mean anything.
[...]
Int.: Exactly. The question I might ask myself now would be: Why is that the case for holomorphic functions (.) but not for real differentiable functions?
21–Dirk: [...] this has to be, uh, complex differentiable everywhere, right, that this depends, uh, on the argument, because this can zoom in anywhere, anyplace [gestures wildly through the air] [Int.: Mmh.]. One does not have any control to which point one zooms into [reaches to different spots in the air]. Right, one has to assume that this holds everywhere, that, uh, that the linear approximation is an amplitwist, right, and the/ But this, this is the connection to, to, to the concept, to the local concept of complex differentiability [pointing gestures], right.

In these and the previous turns, we observe traces of the **"restriction of generality"-frame** again. Above, Dirk's utterance "surprising property [..] that the path integrals of closed paths are zero, simply from the complex differentiability at one point" (lines 163ff.) contains a very general class of paths, namely the closed paths, while his further substantiations of Goursat's lemma (e.g., lines 148ff.) deal with the smaller class of triangular paths. Hence, beginning with the general class of paths, which appears in Cauchy's integral thereon, Dirk restricts them to triangular paths, and then switches back to the more general case again.

14.7 SUBSTANTIATING CAUCHY'S INTEGRAL FORMULA WITH A **MELANGE OF FRAMES**

Cauchy's integral formula was covered last in the interview with Dirk.

152 For example, $f : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto |z|^2$, is complex differentiable only at 0, but $\int_{\partial\Delta} |z|^2 \, dz = \frac{4}{3}$ for the triangle with vertices $-1-i, 1-i$, and i, whose boundary is traversed counterclockwise.

14.7.1 *Cauchy's integral formula is a mean value property*

The interviewer began the part of the interview on a potential intuitive explanation of Cauchy's integral formula with the utterance "that a function value is expressible as integral, where I integrate along a boundary of a circle" (lines 201), and then asks how Dirk imagines the assertion of this formula or how he would present or imagine it. Therefore, the interviewer already mentioned a possible interpretation.

161–Int.: Okay. So my final question would be: Uhm, Cauchy's integral formula, (...) uhm/ (.) I also prepared a sheet for that [shows the question sheet in Figure 11.2] This looks familiar to you, in all likelihood, so that a function value is expressible as integral, [Dirk: Mmh.] where I integrate along a boundary of a circle and that is of this form, so (4s.) yes, the question is: How do you imagine the assertion of this formula? (...) Or how can one clarify (..) that this formula holds/ or first of all the proposition itself, how would you present or imagine it?

Dirk: Well, the first thing to notice is that one can, uhm, uh (.) the, uh, express the function value, uh, in, uh, in z with the function values on, uh, this circle, right, the boundary of the ball, right [Int.: Mmh.], right, and, uh, one can/ It depends on how you design the lecture, then you can connect/, uh, (5s.) it to, uh, to, uh, he/ harmonic functions, so the real concept (German: der reelle Begriff) and then one can probably say that uh/ Yes, exactly, you can also, also say this earlier somehow, that, uh (.), uh, real and imaginary part of a holomorphic function/ well/ not earlier because earlier one does not know what, uh/ that, uh that the, uh, Laplace is defined at all. One already has to know that this thing is twice differentiable [Int.: Mmh.]. Yes, yes, uhm, so okay. One can, one can argue, right, that, uh, that it is such a mean value property, just like with harmonic, uh, functions, namely, uh, one integrates over the boundary of the ball and gets the function value. [Int.: Mmh.] [...]

Here, Dirk realises Cauchy's integral formula as a formula, namely "that one can [...] express the function value [...] in z with the function values on, uh this circle" (lines 207). Hence, this utterance contains a metarule about the generic mathematist "one" to express a function value with the help of a formula. Moreover, Dirk realises this formula as "such a mean value property, just like with harmonic, uh, functions, namely, uh, one integrates over the boundary of the ball and gets the function value" (lines 218). Immediately, after describing this mean value property, Dirk realises harmonic functions as "the real concept", likely as a counterpart to holomorphic functions, for which a similar mean value property holds (lines 218ff.). In particular, Dirk does not stop by realising Cauchy's integral formula as a mean value property, but instead he connects this property to a similar property for harmonic functions. That is, he incorporates the discourse on vector analysis, which deals with harmonic functions, too, into his discourse at this point. That is, he enacts the **"vector analysis"-frame** again. In particular, he also connects his substantiation of Cauchy's formula in terms of an analogue proposition from vector analysis to the design of a lecture as well.[153]

153 More precisely, Dirk implicitly recalls the generally endorsed definition of harmonic functions as twice continuously differentiable functions $u\colon \Omega \longrightarrow \mathbb{R}$ satisfying *Laplace's equation*, that is, $\frac{\partial}{\partial x^2}u + \frac{\partial}{\partial y^2}u = 0$ (e.g., Lang, 1999, p. 241; Freitag & Busam, 2006, p. 48). However, holomorphic functions are first of all defined as once complex differentiable on their domain (e.g., Lang, 1999, pp. 27, 30; Freitag & Busam, 2006, pp. 35, 45) (see Theorem A.25). This explains Dirk's objection that one has to know whether "this thing" is twice differentiable, where "thing" is most likely either a holomorphic or harmonic function (line 216). Notwithstanding, holomorphic functions are in fact infinitely differentiable, but this is usually not yet proven in a course on complex analysis when it deals with Cauchy's integral formula. Rather the opposite, the infinite differentiability of holomorphic functions is proven with the help of Cauchy's integral formula (e.g. Lang, 1999, pp. 128–129; Freitag & Busam, 2006, pp. 106–107, and many others).

Dirk notices however, that the z in Cauchy's integral formula (Figure 11.2) should also appear as the centre of the circle over whose boundary the integral in Cauchy's integral formula is taken in order to establish the connection between Cauchy's formula and the mean value property of harmonic functions:

164–Dirk: [...] Well, whereby the z does not appear here [points to the right z in the displayed formula in Figure 11.2], right, it would be interesting [continuous to point to this z] if one takes the z to be in the centre/ and then one connects it to harmonic functions. [Int.: Mmh.] Okay.
Int.: Well, as I wrote it, the z does not have to be the centre of the ball/
Dirk: Exactly, exactly. But there, to present this connection, it should be this way. [Int.: Mmh.] Uhm, (.) right, this is the/ the point is the general assertion [points to the displayed formula in Figure 11.2]. Uhm/

Hence, at this point, Dirk is not restricting the generality himself to explain Cauchy's integral formula but rather he is led to require that $z = z_0$ (lines 223, 223f.) is needed for the mean value property of harmonic functions (cf. Forster, 2017b, p. 196). Here, for the case of holomorphic functions, Dirk recognises that the assertion is more general (lines 227f.).[154]

The special case $z = z_0$ is brought up by the interviewer again though:

173–Int.: So if I remember correctly and one equates z and z-zero and parametrises the circle explicitly, (.) this term [points to the integrand in Cauchy's integral formula] vanishes because of this e to the two pi i thing and somehow a mean value of [Dirk: Yes, exactly, exactly.], between the values on the circle stays, so to speak. All values on the boundary of the circle are being integrated (German: aufintegriert) [Dirk: Yes, yes, yes, yes.] modulo/

Here, the interviewer recalls the evaluation of the integral in Cauchy's integral formula, namely that "[a]ll values on the boundary are being integrated", to which Dirk agrees. This verbal utterances can by symbolically realised as

$$\frac{1}{2\pi i}\int_{\partial B(z,r)}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} f\left(z + re^{it}\right)\mathrm{d}t,$$

where $\partial B(z,r)$ is parametrised via $[0, 2\pi] \longrightarrow \mathbb{C}, t \longmapsto z + re^{it}$ (e.g., Remmert & Schumacher, 2002, p. 183). Hence, the integrand here is composed of all values of f on $\partial B(z,r)$ divided by 2π.

In sum, Dirk is giving meaning to Cauchy's integral formula by baptising it a "mean value property" and describing an analogy to harmonic functions. which itself are realised as real and imaginary part of holomorphic functions, and for which also a mean value property holds. As such, Dirk is following metarules from the **"vector analysis"-frame** to explore and substantiate an analogy between complex and vector analysis related to mean values and integral formulas—very similar to Uwe's substantiation of Cauchy's integral formula.

14.7.2 *Using power series development and a potential failure of intuition*

In this subsection, we discuss how Dirk enacted the **"restriction of generality"-frame** and **"theorematic"-frame** as part of his substantiation of Cauchy's integral formula. The restriction

154 We recall that Uwe also distinguished the two cases $z = z_0$ and $z \neq z_0$ in Cauchy's integral formula. For the first case, he mentioned the mean value property for harmonic functions like Uwe, and for the second case, he baptised the analogue formula for harmonic functions as the "Poisson integral formula" (Section 13.7).

is again based on the case $z = z_0$ and also on the power series development of a holomorphic function.

The interviewer asks for potential "vivid meaning" of or a potential "mnemonic" for Cauchy's integral formula:

165–Int.: What is/ So the vivid meaning (German: anschauliche Bedeutung) of this formula for you is which, so to speak? [Dirk: Uh.] So how can one interpret this formula somehow contentwise (German: inhaltlich irgendwie anschaulich)? [waits for Dirk, 6s.] Well, you talked about mean values. Now, the question is mean values of what?

Dirk: (13s., exhales whistling) Well [incompr.], uhm, uh. Later, this will be applied actually, uh, for, uh, the centre, [Int.: Mmh.] for z-zero, right, and, uhm (7s.), uhm. What is the interpretation of this assertion? (4s.)

167–Int.: Well, a student could ask himself: How do I recall this formula, for example. And then it would probably be good if he had some mnemonic (German: inhaltliche Eselsbrücke) so to speak. (4s.) [Dirk: Uh.] Or/ (..) Well.

Dirk: (17s.) Well, so, uhm (...), mmh. (9s.) Alright, I take z-zero equals zero and z [writes while speaking; see Figure 14.7] [Int.: Mmh.], right, and then we take zeta divided by zeta, (.), right, we integrate on the boundary of this ball, right, (4s.), right, and now one can imagine, our goal is the power series development, right, and then, then we can say: Okay, this already is a power series, right, and then I divide by, by the variable, (.), right, [...] [Dirk produces the chain of equations in Figure 14.7]

Dirk thinks for a long time. Then, he first recalls a potential application of Cauchy's integral formula for the case $z = z_0$, but does not explain what the application is, and repeats the question for an "interpretation of this assertion" (lines 241ff.). Then, after the second question by the interviewer and a long pause again, Dirk finally restricts the generality for the following construction of a narrative even further than $z = z_0$: He uses the constraint $z = z_0 = 0$ to construct the narrative in Figure 14.7, namely

$$\frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{f(\zeta)}{\zeta}\,\mathrm{d}\zeta = a_0.\text{[155]} \tag{14.2}$$

In particular, Dirk does not seem to look for the potential mnemonic but rather continues to apply the power series development, which he described previously as a "goal" (line 251). At first, it seems that he connects his exploration at this point to the derivation of the power series development theorem (Theorem A.25). However, then he already uses a power series, namely $f(\zeta) = \sum a_n \zeta^n$, instead of deriving it from Cauchy's formula and develops Equation 14.2. Therefore, we conclude that Dirk enacts the **"theorematic"-frame** here as well because he uses a theorem, namely the power series expansion of holomorphic functions, to proceed with his intuitive substantiation of Cauchy's integral formula. Since he identifies the power series expansions as a "goal" but already uses it for the construction of a narrative, we also conclude that he performs a *retrospective substantiation*, a phenomenon we observed in Uwe's intuitive mathematical discourse about Cauchy's integral formula, too (Section 13.7): A theorem usually derived later in a course on complex analysis (the power series development theorem) is used for a substantiation of a special case of an earlier theorem (Cauchy's integral formula). In the interview with Uwe, the theorem used was the residue theorem and here it is the power series development theorem.

Retrospective substantiation

155 To be precise, Dirk first computed the first term in the second line of Figure 14.7 to be $\frac{a_1}{2\pi i}\int_{\partial B(0,r)} \frac{\mathrm{d}\zeta}{\zeta}$, but corrected the a_1 then to a_0 after the interviewer objected.

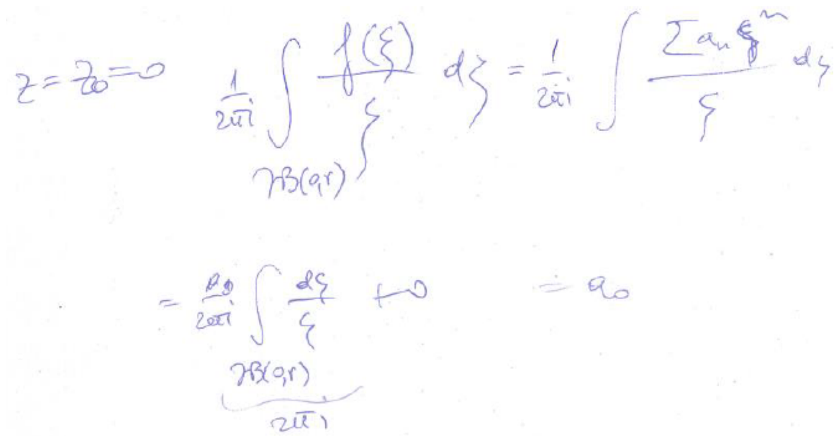


Figure 14.7: Dirk examines Cauchy's integral formula for $z = z_0 = 0$ using a power series.[a]

a The inscription directly next to the integral in the second line should read "$+0$".

14.7.3 *"How can I imagine this statement?"—Drawing new visual mediators*

Even though Dirk produced the narrative in Equation 14.2, he is not satisfied and still asks for the "idea of that" (i.e., of Cauchy's integral formula) and "how can I imagine this statement?" (line 262) and acknowledges that his preceding argumentation was in fact "anticipatory" 261):[156]

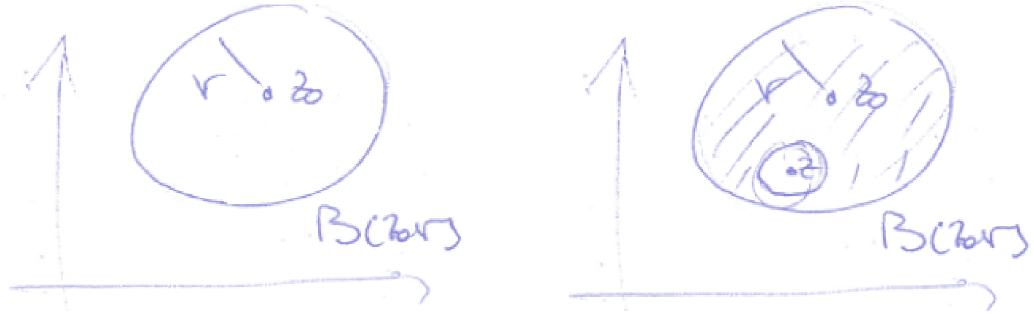
170–Dirk: [...] What is the idea of that? (German: Was ist davon die Vorstellung?)
[likely referring to the term $\frac{a_1}{2\pi i}\int_{\partial B(0,r)} \frac{d\zeta}{\zeta}$] (..) Well, we can calculate that
explicitly, (4s.) that is two pi i [writes $2\pi i$ under the integral sign in the
second line of Figure 14.7] [Int.: Mmh.] [laughs], right, and then it simply
computes to a-zero [writes a_0 in Figure 14.7]
[...]
Okay, uh, but this is of course anticipatory (German: vorgegriffen), this,
uh/ Now, what is the uh/ How can I imagine this statement?

Dirk does not achieve an answer he is satisfied with, and then, the interviewer challenges him whether Cauchy's formula is a "borderline case, where intuition seems to fail":

178–Dirk: Yes. (6s.) But why does the formula hold intuitively? [Int.: Mmh.] This
is the next question, right?
Int.: Yes, I mean, (.) are we in a borderline case here, where intuition seems to
fail or where one is no longer able to interpret this, where one has to realise:
Okay, now I have to rely on calculation (German: Kalkül[157])?
Dirk: Uh, this is of course, uh, uh, (.) the easiest way out so to speak. But is this
really the case, or can one expect that this/ that this is the case here? (7s.)
Well. Knowing that, uh, Cauchy's integral formula holds, that, uh formula,
that Cauchy's integral theorem holds. [Int.: Yes.] [...] Well, it is the question
whether one can argue intuitively here (5s.) or not, rather [Int.: Mmh.]
convey an intuitive image [...] (7s.). No, I think (..) it only helps to (..) put z
equal to z-zero, right, and uh, work with this image [points to Figure 14.8a].
But (13s.) well, [...] one in fact takes a z [marks a point in the interior of

156 The German word "vorgegriffen" refers to the fact that the reasoning is based on something not yet known. In this case, Dirk is likely referring to the fact that he used the power series development of a holomorphic function in his argument, whereas this is a proposition still to be shown, or in his words, a "goal".

157 The German word "Kalkül" could be translated with calculatio, calculation, or calculus. It does not refer to the mathematical discipline of calculus but to an umbrella term for formal calculations or a system of formal rules.



(a) A circle with radius r and centre z_0.

(b) A circle inside a circle.

Figure 14.8: Dirk's draws two circles to substantiate Cauchy's integral formula.

> the circle in Figure 14.8a and draws a circle around it; see Figure 14.8b], uh, and then one applies uh, the, uh, Cauchy's integral theorem, right, to a domain between two curves, right [shades the interior of the larger circle in Figure 14.8b]? But why exactly this? [laughs] [Int.: Mmh.] Uh, how/ what is something handy that one can say here (22s.) Uhm.

Dirk considers the interviewer's proposal as "the easiest way out" and asks himself whether "one can expect that this" (likely Cauchy's integral formula) "is the case". He uses Cauchy's integral theorem as a basis for his further explorations and argues that "it only helps to (..) put z equal to z-zero" (lines 273f.). Even though he re-asserts the condition $z = z_0$, he draws the visual mediator shown in Figure 14.8, where z and z_0 are drawn as distinct points: In this visual mediator, Dirk realises the ball $B(z_0, r)$ and a smaller ball centred at z contained in the other ball. He recalls a part of the proof of Cauchy's integral theorem, in which Cauchy's integral theorem is applied to the region between the two "curves" (i.e., $\partial B(z_0, r)$ and the boundary of the other ball).[158]

In sum, Dirk's discourse about Cauchy's integral formula proved to be very rich: He restricted the level of generality, applied two theorems, recalled the important use of Cauchy's integral formula to derive the power series development of holomorphic functions, and recalled an important step in the proof of this formula. Nevertheless, we conclude that he did not construct a narrative he accepts as an intuitive explanation of this formula because he repeated this quest over and again.

We also emphasise that Dirk does never question the possibility for an intuitive explanation of Cauchy's integral formula. This contrasts Uwe's valuation that Cauchy's integral formula is not intuitively clear at all (Section 13.7). Even more, when the interviewer offered the potential interpretation that Cauchy's integral formula might be an instance where intuition might fail, Dirk does not agree with him and continuous to ask for a "handy" explanation (line 280). He also explains that

> 184–Dirk: [...] I usually try to convey the, these plausibility, uh, considerations, right, uhm, but if this is not obvious, right, so, then I probably did not do it in the past. [...]

Hence, Dirk explains a pedagogical metarule in his intuitive discourse about the complex path integral, namely that an intuitive, handy explanation of Cauchy's integral formula, in his words, "plausibility considerations", should not only be possible but also be taught, and that he

158 Indeed, in the proof of Cauchy's integral formula, it is shown that $\frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial B(z,s)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$, where $s > 0$ is chosen such that $\overline{B(z,s)} \subseteq B(z_0, r)$ (e.g., Remmert & Schumacher, 2002, pp. 180–181).

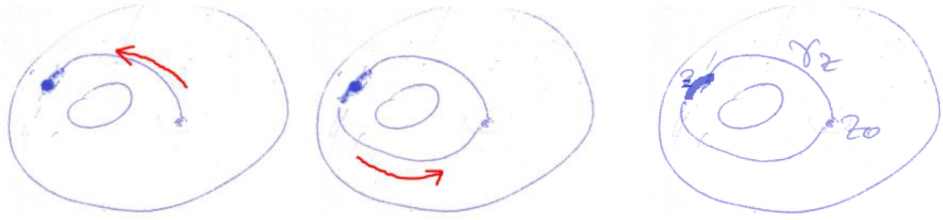
tries to teach those, even though he does not find an intuitive explanation during the interview. As such, he reinforces what he said about the complex path integral earlier, namely that one "should (...) be able to deliver a good picture of what this means" (lines 126f.).

14.8 SUBSTANTIATING THE EXISTENCE OF HOLOMORPHIC PRIMITIVES WITH THE **"THEOREMATIC"-** AND **"RESTRICTION OF GENERALITY"-FRAME**—A STORY ABOUT SIMPLY-CONNECTED DOMAINS

The interview with Dirk dealt with the substantiation of the existence of primitive functions for holomorphic functions after he produced the theorematic image "$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$" (Section 14.2) and his attempt to produce a visual mediator for the complex path integral (Section 14.3). Dirk has thus already applied the "theorematic"-frame in form of the fundamental theorem of calculus here (see Equation A.17). The interviewer introduces his question about the existence of holomorphic primitive functions with the second part of the version from calculus (Theorem B.3), namely that there are primitive functions for continuous real functions on intervals. He asks whether there is an analogue situation in complex analysis and already gives away part of a possible answer, namely that there need not even exist primitive functions for holomorphic functions on domains in $\mathbb{C}$. Dirk's answer is also based on the **"theorematic"-frame** initiated by recalling the notion of simple connectedness a potential definition of a primitive, and the monodromy theorem / homotopy invariance of complex path integrals (see Theorem A.30, Equation A.20):

146–Int.: [...] So maybe one question on primitive functions: In the real case it is somehow like is: A real differentiable function, or even continuous function, on an interval (.) has a primitive function [Dirk nods] as you learn from the fundamental theorem. [Dirk: Mmh.] Is there an analogue situation in the complex setting? (...) Or asked differently: For an open set in C, or even a domain, it does not have to be the case that the function has a primitive function.
Dirk: So only continuous/
Int.: [incompr.] even if it is holomorphic, yes.
Dirk: Yes, fine, okay. Right.
Int.: Where does this fail or isn't the situation analogue at all?
151/152–Dirk: Mmh. (12s.) Well, this, uh, is, uh, the, uh/ When one goes from one dimension to two, then the question of simple connectedness immediately arises, right, and that's where it goes wrong. I mean if the domain is simply-connected, then [Int.: Mmh.] we have everything, mmh, right, but in R [draws a line with an interval] there are only the intervals and they are already simply-connected.

First of all, Dirk repeats the keyword "continuous", suggesting that he is taking up the constraint on the function from the interviewer, which the interviewer in turn refined to "holomorphic", and to which Dirk agrees. Dirk rephrases the analogy in terms of a shift from "one dimension to two", which is "where it goes wrong" (lines 295ff.). Furthermore, he introduces the keyword "simple connectedness" as a pars pro toto, similarly to how he introduced certain keywords to his substantiation of Goursat's lemma (Section 14.6), but he will explain why this notion is relevant later. This property is identified as the cause for the potential non-existence of primitive functions, namely "if the domain is simply-connected, then we have everything"; (lines 297f.). He also recalls that intervals are "already simply-connected". From this we conclude that Dirk, like Uwe, recognises a certain property of functions as always fulfilled in the



(a) Dirk draws two paths around a "hole" (red arrows indicate direction of drawing).

(b) Dirk's draws γ_z.

Figure 14.9: Dirk's visual mediator for the construction of a primitive function.

$$F_{:}(z) := \int_{\gamma_z} f(z) dz$$

Figure 14.10: Dirk's construction of a primitive function.

context of real analysis but not complex analysis. In Uwe's case, this property was the vanishing of integrals along triangular paths (Section 13.8), and here it is the simple connectedness.

153–Int.: And why is the simple connectedness so important?

Dirk: Mmh. [Int. incompr., probably "in C"] (4s) Well, so, uhm, (...) one can (.) define the primitive (.) with the help of path integrals (..) and/ but it has to be, uh, exactly, uh, unique, right/ so if we have a hole here [draws a circular shaped blob and a smaller blob inside as a "hole"], right, and then define once along this path and once along that path and then it computes to something else [draws two semi-circular shaped lines around the previously drawn "hole", one above and one below; Figure 14.9a], right, then we do not have a unique primitive function [Int.: Mmh.]. Or it can happen, but, uh, if this is simply-connected, then the monodromy theorem [points to ...] tells us that it always has to compute to the same.

Int.: Monodromy theorem for you means invariance with respect to homotopy or/

Dirk: Exactly. [Int.: Mmh.] (..) Uhm, yes, fine, [incompr., likely "right"] one does not have to go/ Fine, well, one does not have to go that far. But, uhm, that, that, uh (...) Although, one can also/ uhm. Wait a moment. Does one have to go that far? (.) Uhm, (..) what is it? So, (.) uh [begins to write the equation in Figure 14.10] So, of course F of z, right/ that, uh, I assume/ this is z-zero and this is gamma-z to z-, right [add the inscriptions to the right figure in Figure 14.9a, see Figure 14.9b]. (5s) okay. [Int.: Mmh.] Right? [...]

After the interviewer asks why the simple connectedness of domains is important, Dirk recalls the metarule how to define primitive functions "with the help of path integrals" (Theorem A.20) and eventually produces the definition of a primitive function in Figure 14.10: Dirk draws the visual mediator in Figure 14.9a and describes that the primitive obtained this way has to be "unique" (lines 302ff.).[159] Then, he describes two potential paths of integration, whose corresponding integrals are said to not compute to the same value "if we have a hole here" and in which case "we do not have a unique primitive function" (lines 304ff.).[160] Dirk substantiates

159 Recall that if f is holomorphic on a simply-connected domain Ω and $z_0 \in \Omega$ is a fixed, then $z \longmapsto \int_{\gamma_z} f(\zeta)\, d\zeta$ defines a primitive function for f on Ω (Theorem A.20).

160 To be precise, the two integrals described by Dirk may be equal even if the domain is simply-connected. Dirk seems to notice this as well and because it weakens his previous utterance to "it can happen".

Figure 14.11: Dirk's illustrates a star domain.

the equality of these different integrals with the monodromy theorem and simple connected domains, which guarantee that "it [likely the primitive function; EH.] always has to compute to the same" (line 310f.). Hence, he incorporates yet another theorem to substantiate the uniqueness of a potential primitive function.

Lastly, we observe that Dirk follows the **"restriction of generality"-frame** once again. Having produced the defining utterance for a primitive function in Figure 14.10, Dirk considers a special case of domains, namely star domains.

156–Dirk: [...] And, uhm, (.) okay, with star domains, there I say/ there I restrict myself [underlines γ_z in Figure 14.10] to straight paths [draws Figure 14.11], right, and when I want to argue more generally, right, then, then I have to, uh/ if that is supposed to be just any path [points to γ_z in the same figure], the I have to, uh, monodromy, the I have to [encircles z_0 in the same figure], uh, uh, continue analytically and by that it computes to the same for F and therefore it is the same/ so this definition is unique [points to Figure 14.10].

For the case of star domains, Dirk argues that the paths in the primitive in Figure 14.10 can be taken to be straight line segments. He enacts again the routine of drawing to realise the mathematical objects from his utterances (here, a star domain as a circular shape and multiple lines starting at a common point; Figure 14.11). However, immediately after presenting this special case, Dirk resorts back to the general case, where "this [γ_z; EH.] is a just any path" and "monodromy" or "analytic continuation" the paths drawn in Figure 14.8 guarantee that F is well-defined (see Theorem A.30), which is the case, as we know, for simply-connected domains.

Moreover, in order to differentiate the two approaches for defining a primitive function via path integrals, Dirk presents a counterexample: He argues that he cannot construct a "unique primitive" for the function $\mathbb{C} \setminus \{0\}, z \longmapsto 1/z$, because "the zero is in the way".

156–Dirk: [...] if I want to build a, a primitive for this [$1/z$] and start at one, uhm, then, then, uh, the zero is in the way, then I cannot, uh, define a unique primitive function on the negative real numbers, on the negative real half/ uh, axis [Int.: Mmh.]. Certainly not with the/ with the idea with the star-shaped [points to Figure 14.11], with the straight paths, but also not with the, uh, uh, [Int.: Mmh.] generalised idea. [Int.: Mmh.] Uhm, and (...), well, the domain of definition is, uh, not simply-connected [draws a circular shaped with a dot inside and writes $\mathbb{C} \setminus \{0\}$].

Here, Dirk explicates the metarule that one can neither apply "the idea with the star-shaped, with the straight paths, but also not with the, uh, uh, generalised idea". More precisely, for star domains, he substantiates that the primitive cannot be defined uniquely on the negative real axis, and in the more general case, it cannot be defined because the domain is not necessarily simply-connected. Hence, Dirk uses a special case to differentiate between the construction of primitives via complex path integrals along straight paths or "just any path[s]" (line 325). Consequently, even though this example contains a single function only, Dirk uses it to exemplify the differences and the failure of the construction of a primitive function, which he described before and which resulted in the equation in Figure 14.10.

In sum, Dirk creates a link between the prompt brought up by the interviewer, namely the portion of the fundamental theorem of calculus about the existence of primitive functions for continuous real-valued functions on intervals, and complex analysis ("from one dimension to two"; line 295), by creating a sequence of narratives with accompanying visual mediators about simply- or not simply-connected domains, the monodromy theorem, and in particular the uniqueness of a potential primitive function.

14.9 SUMMARY OF DIRK'S INTUITIVE MATHEMATICAL DISCOURSE ABOUT COMPLEX PATH INTEGRALS

The **"theorematic"-**, **"vector analysis"-**, and **"restriction of generality"-frame**, which appeared in Uwe's intuitive mathematical discourse about complex path integrals appeared in the interview with Dirk, too. Additionally, we found the **"area"-frame** in the case study of the interview with Dirk.

Table 14.1 shows an overview of the discursive frames from Dirk's intuitive mathematical discourse the complex path integrals. As in the case of Uwe, the table also shows whether Dirk followed a discursive frame for the construction of a discursive image about complex path integrals and/or for the substantiation of one or more of the integral theorems. We note that even though Dirk followed the **"area"-** and the **"vector analysis"-frame** for the construction of a discursive image, he rather described interpretations for the Riemann integrals or real path integral of first or second kind in terms of a sum of vectors, but did not produce an analogue statement for complex path integrals. Therefore, we do not attribute a discursive image about the complex path integral to him based on these two frames. The theorematic image we recall below was realised in terms of the other two frames he used for the exploration of a discursive image about complex path integrals.

We have taken care to describe the ways in which Uwe and Dirk used the shared discursive frames throughout our analyses in this and the previous chapter. For summarising purposes, we focus in this section of the case study of Dirk on the new frame and the only narrative we classified as Dirk's discursive image about complex path integrals.

The "area"-frame

In line with his self-description as a visual thinker, Dirk's main goal throughout the interview was to produce a visual realisation of the complex path integral. In this context, he often used the keyword "picture" and he engaged in multiple ways to construct a still to him unknown visual mediator, with the help of which he could realise complex path integrals as certain areas or other geometrical objects. That is, we describe the "area"-frame as a set of metarules according to which mathematists produce visual mediators and construct narratives to related complex path integrals to areas or another geometrical quantities. A central metarules is to try to transfer the basic idea (see Section 2.2.3) of area to complex path integrals.

Dirk did in fact try to transfer the area interpretation for Riemann integrals but also the area interpretation for real path integrals of first kind, which he realised in Figure 14.3 and Figure 14.4b, to the complex path integral. He reflected that these figures are not appropriate mainly because the integrands in complex path integrals are complex-valued and therefore the graphs of complex functions lie in four-dimensional space, which were not realised in his sketches.

Table 14.1: Overview of discursive frames in Dirk's intuitive mathematical discourse about complex path integrals.

Discursive frame	Construction of a discursive image	Substantiation of an integral theorem
(F1) "restriction of generality"	Section 14.2	Section 14.6 (Cauchy's integral theorem), Section 14.7 (Cauchy's integral formula), Section 14.8 (existence of holom. primitives)
(F2) "theorematic"	Section 14.2	Section 14.7 (Cauchy's integral formula), Section 14.8 (existence of holom. primitives)
(F3) "vector analysis"	Section 14.4	Section 14.7 (Cauchy's integral formula), Section 14.8 (existence of holom. primitives)
(F4) "tool"		
(F5) "no meaning"		
(F6) "area"	Section 14.3	
(F7) "mean value"		
(F8) "holomorphicity ex machina"		

The theorematic-image "$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$"

The only explicit discursive image we found in Dirk's interview was based on the complex version of the fundamental theorem of calculus: "$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$". This narrative expresses the complex path integral of f along γ as the difference between the value of a primitive function F for f at the terminal and initial point of γ. We classified this narrative as a discursive image in Dirk's intuitive mathematical discourse because it was produced as a response to the task situation to describe his mental images about complex path integrals. Since f is required to have a primitive here, this narrative is a result of the application of the **"restriction of generality"-** and the **"theorematic"-frame**. This discursive image was practically not used further in the interview though.

Intuitive substantiations of the three integral theorems

Dirk's intuitive substantiation of Cauchy's integral theorem mainly consisted of a substantiation of Goursat's lemma. That is, essentially Dirk followed the **"restriction of generality"-frame** here and explored a special case of the theorem. In our analysis of this substantiation, we noticed a specific way to recall the proof of Goursat's lemma, which we called *pars pro toto* usage of keywords: Assuming that his interlocutor is familiar with the proof of the lemma, Dirk used keywords such as "nested intervals [triangles; EH]" or "topology" as a replacement for full narratives in the order of their appearance in this proof.

Similar to Uwe, Dirk described Cauchy's integral formula as a mean value formula for holomorphic functions, which is also endorsed in the literature (e.g., Remmert & Schumacher, 2002, pp. 182–183), and related it to the mean value property of harmonic functions. Hence, Dirk realised the **"vector analysis"-frame** in the context of Cauchy's integral formula, too. **Restricting the generality** to the special case $z = z_0 = 0$ and using the power series development of a holomorphic function (which Dirk actually considered a corollary to Cauchy's integral formula), Dirk used the **"theorematic"-frame** to engage in a *retrospective substantiation* (i.e., a substantiation, in which a proposition is justified with another proposition usually proven later).

Dirk's intuitive substantiation of the theorem on the existence of primitive functions for holomorphic functions and their non-existence for only continuous functions was mainly based on the **"theorematic"-** and **"restriction of generality"-frame**. He recalled the construction of potential primitive functions in terms of integrating along paths with a fixed initial point and argued about their uniqueness with the help of the monodromy theorem or analytic continuation.

Closing remark

Even though Dirk explained his general pedagogical metarule to realise mathematical objects graphically, both for himself and for students, he did not memorise or construct a graphical realisation for complex path integrals during the interview. He did not produce a narrative he valued explicitly as a mental image or an intuitive interpretation of complex path integrals. Instead, he produced several visual mediators for real integrals, domains of functions, or paths, and reflected on their usability for the realisation of complex path integrals.

15

THE CASE OF SEBASTIAN

In this chapter, we present the case study on Sebastian's intuitive mathematical discourse about complex path integrals. The respective part took place in the middle of the interview and covered roughly 25 minutes scattered over approximately 70 turns ($\approx$ turns 126–197).

In this chapter, we find two additional discursive frames, which did not appear in the other two interviews: the **"mean value"-** and the **"holomorphicity ex machina"-frame**. While the intuitive mathematical discourses about complex path integrals from the interviews with Uwe and Dirk could be explained in terms of a usage of various frames, Sebastian's intuitive mathematical discourse is primarily characterised by these two new discursive frames.

In Section 15.1, we introduce Sebastian's valuation of the most important elements of a course on complex analysis and his general point of view on complex analysis. Then, we introduce the "mean-value"-frame in Section 15.2 and the "holomorphicity ex machina"-frame in Section 15.3. Sebastian's intuitive explanations of the integral theorems are contained in Section 15.4, Section 15.5, and Section 15.6. In Section 15.7, we summarise the case of Sebastian's intuitive mathematical discourse about complex path integrals.

As in the previous two chapters, we illustrate the frames Sebastian used during the interview in Figure 15.1.

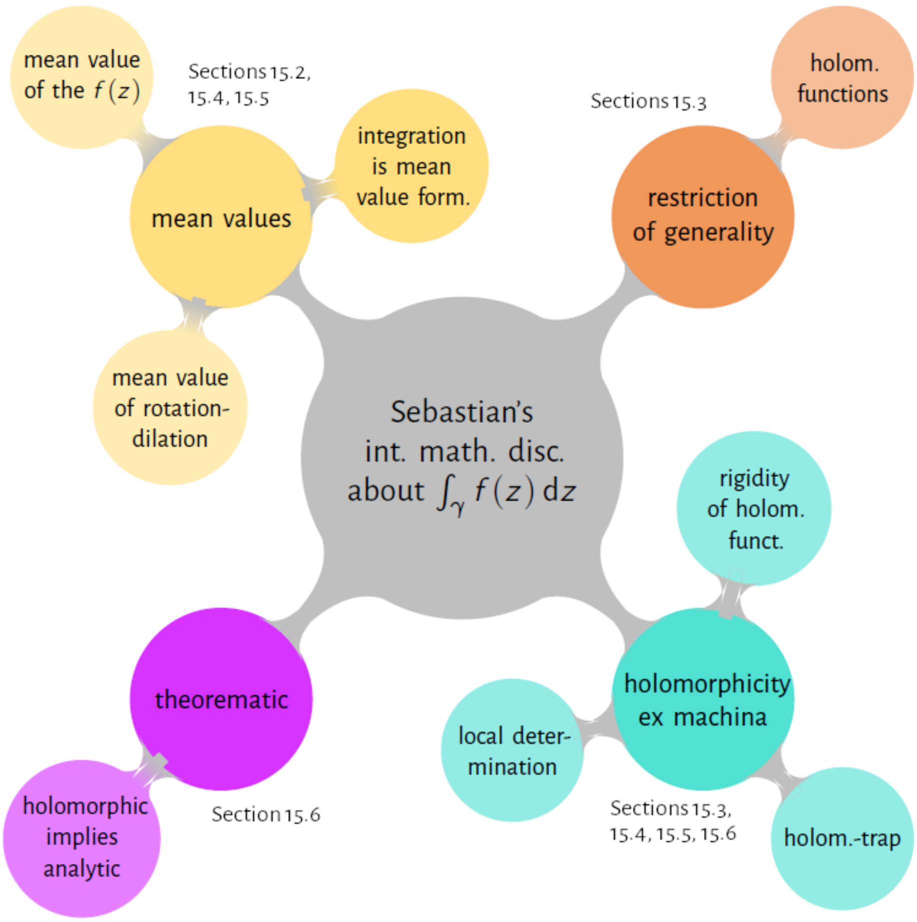


Figure 15.1: Discursive frames from the interview with Sebastian.

15.1 INTRODUCTION TO SEBASTIAN

Sebastian identifies several topics to be central for a course on complex analysis including differentiation and integration, Cauchy's integral theorem, Laurent series, and analytic functions. In particular, he considers the *Riemann mapping theorem*[161] as a satisfactory highlight at the end of such a course. However, the calculus of residues, which we interpret as the residue theorem and its applications, is considered satisfactory, too:

2–Seb.: [...] The goal should be the Riemann mapping theorem, [Int.: Mmh.] but, uhm, I have not gotten this far in a lecture I once gave, but it was more a question of tapping into path integrals, calculus of residues, [Int.: Mmh.] so to speak, and by the time you get there you've already come quite far. (.) Yes, so, differentiation, integration, path integral, residue, [Int.: Mmh.] and the of course, connected to that, uhm, (.) what are they called? (..) Laurent series [Int.: Mmh.] or uh, analytic functions more generally, this is, so to speak, accompan/ uh, goes hand in hand with Cauchy's integral theorem. [...]

4–Seb.: Of course, that would actually be/ The Riemann mapping theorem would really be satisfactory. But, uh, as I said, I/ In an introductory course you

161 The *Riemann mapping theorem* states that any simply-connected open set $\emptyset \subsetneq U \subsetneq \mathbb{C}$ is analytically isomorphic to the unit disc $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$; in other words, there exists a biholomorphic function $U \longrightarrow \mathbb{E}$ (e.g., Lang, 1999, ch. X).

can also come along by without it and then the satisfactory, uh, goal would be the calculus of residues. [...]

While discussing the central elements of complex analysis, he mentions the complex path integral at several points explicitly. Moreover, he even explains them as a "picture" he associated with holomorphic functions, and adds more topics to be covered in a course on complex analysis:

32–Seb.: [...] So this is again related very closely to the path integral somehow. So maybe this is the next picture that I associate with holomorphic functions, always is actually the complex path integral. [Int.: Mmh.] Uhm, mean value property and the next steps are then derived from this. Yes, and the important theorems, uhm, Cauchy-Riemann differential equation, Cauchy's integral theorem, uh, maximum principle, (.) yes, and then residue theorem, if you want, that's all/ This is some sort of a cascade, right? [...]

The results in complex analysis seem like "some sort of a cascade" to him (line 20). We conclude that Sebastian sees theorems about holomorphic functions and complex path integrals as closely related. This close connection almost constantly reappears during the interview, and we describe the corresponding metarules with the help of which Sebastian organises his intuitive mathematical discourse during the interview in terms of the **"holomorphicity ex machina"-frame** (Section 15.3).

In addition, Sebastian values complex analysis as an important part of mathematics training at university, both, for reasons of aesthetics and practical relevance (31ff.):

120–Int.: [...] Do you generally consider it important that one builds up mental images (German: Vorstellungen) for complex analysis, as a student or as lecturer, whatever, or is this (.) so for complex analysis particularly important, or [Seb.: Yes, so/] is this different with respect to other mathematical domains?

Seb.: So/ (.) Well, I think what probably everyone always says and that/ uh, I agree with it that complex analysis simply has a seductive elegance, yes, where one always thinks: That's not even possible (German: Das gibt's doch gar nicht) that everything connects to each other so nicely. Therefore the previous comment with the [Int.: Mmh.], uh, uniform convergence[162] and, uhm, therefore I find complex analysis extremely important as an aesthetic mathematical education. And this goes hand in hand with mental images (German: Vorstellungen). Uhm, so on one side it is of practical relevance of course, so, uh, without complex analysis one cannot understand many other things. [Int.: Mmh.] And therefore I consider this is a very fundamental part of mathematical education.

Sebastian reacts to the interviewer's prompt on the role of mental images in complex analysis by recalling the idiom that complex analysis a mathematical discourse of "seductive elegance" (line 27), in which mathematical objects have particularly nice properties and the results are almost unbelievable.

In addition, he argues that one can understand other mathematical discourses, in particular partial differential equations, better if one has seen analogue results in complex analysis before. Again, he mentions integration explicitly and claims that theorems from real analysis (e.g., Stokes' or Gauß' theorems), partial differential equations etc. would be easier to understand in case one has come into contact with integration theory in complex analysis before:

162 The "comment" Sebastian refers to the narrative "Complex analysis is when everything converges uniformly" (turn 32) from one of his colleagues, which Sebastian calls an "mental image" (German: Vorstellung) of complex analysis, too.

124–Int.: Yes, I mean, I would probably ask: What is it that one cannot understand if one does not know complex analysis?

Seb.: Yes, okay, so, uh, I would say partial differential equations. [Int.: Mmh.] Therefore/ So this link, uh, between differential equations and properties of integrals, Cauchy's integral theorem, Cauchy-Riemann differential equations and harmonic and harmonicity, let's say, [Int.: Mmh.] of the, the, uh, holomorphic functions. One has a hard time when one, uh, suddenly comes to real partial differential equation analysis and sees things like maximum principle or something like that. And then, the analogy to complex analysis has always been helpful for me. [Int.: Mmh.] And the next thing is integration theory in general of course. Without, uh/ So in general what a path integral should be in the real case (German: Wegintegral im Reellen) and is, I find this much more natural in complex analysis, [Int.: Mmh.] in function theory.[163] Uhm, and then it goes on, like Stokes' theorem, Gauß' theorem, (.) you are better off if you have seen something like this before. [Int.: Yes.] From my point of view. Fine, and then it goes further because these ideas, uh, complex differentiability or Cauchy's integral theorem, these are used in operator theory or in solution theory of partial differential equations. [Int.: Mmh.] And that was incredibly important to me to have known something like this before, [Int.: Mmh.] otherwise it would have been very difficult. There is the so-called Dunford integral and [incompr., likely: the] functional calculus and all of this is borrowed from this, uh, path integration of complex analysis.

We derive three observations from the remarks in this section: First, Sebastian considers complex analysis as a particularly elegant mathematical discourse, in which, second, propositions about holomorphic functions and complex path integrals are closely related; and third, Sebastian claims that prior familiarity with integration theory in complex analysis promotes the understanding of other mathematical discourses in which integrals occur.

15.2 INTEGRATION IS TAKING THE MEAN VALUE—THE **"MEAN VALUE"-FRAME**

In this section, we will see how Sebastian follows the **"mean value"-frame** in order to explain his interpretations of complex path integrals. He creates very explicit discursive images about complex path integrals as certain mean values, which are inferred from the general discursive image according to which Sebastian identifies integration with *mean value formation*.

The "mean value"-frame is characterised by metarules according to which discursants identify integration as a process of taking a mean value of certain quantities associated to the integrand of an integral or as the process of averaging these quantities. More precisely, mathematists enacting the "mean value"-frame construct narratives with the help of which they describe their personal interpretations of integrals as mean values or results of measuring processes.

The "mean value"-frame arose from Sebastian's initial rejection of the interpretation of an integral as an area. Therefore, we put the "mean value"-frame into context and discuss this initial rejection first of the area interpretation first.

As in the other interviews, the interviewer explains the area interpretation for Riemann integrals and before even asking for a potential interpretation of complex path integrals, Sebastian interrupts to say that he does not favour this interpretation for integration in general:

126–Int.: [...] Uhm, so for path integration, uh, in real analysis, so R-one-valued functions, let's say, we often interpret the integral as an oriented area under the graph of the function. [Seb.: Yes. [shakes head]] So plus and minus are

163 The German word for "complex analysis" usually is "Funktionentheorie". In contrast to "real analysis" you can also translate it to "complex analysis". Sebastian used both words here.

weighted separately [points twice into the air, the second time below the first]/

Seb.: Yes, I find this bad [quietly, presumably: but well]. [laughs]

Int.: Yes, but this interpretation is standard, [Seb.: I know, I know, I know.] [laughs] it has somehow, how shall I say it [both laugh], is so canonical that one does not/ So, I mean, so/ so canonical that one does not even need to mention it, I'd say.

Seb.: Yes, yes, exactly. But I find that one/ I would always tell my pupils:[164] Actually, one should think about mean values, in particular when one has Lebesgue integration in mind. [Int.: Mmh.] [emphatic: And measures.] [Int.: Yes.] It is about measuring. And, uhm, this geometric intuition (German: geometrische Anschauung) can destroy this higher dimensional situation (German: das Höherdimensionale). [Int.: Mmh.] So if one imagines that one measures some area, uh, that an integral is the same as a determination of area, uh, then this could be really impeding with path integration (German: bei der Wegintegration schon sehr im Weg stehen). [Int.: Mmh.] And therefore I find it much better if one imagines: integration is mean value formation. [Int.: Mmh.] And by coincidence it turns out to be the area in the one-dimensional setting.

Here, similar to Uwe's immediate rejection of geometrical meaning of the number $\int_\gamma f(z)\,dz$, Sebastian values the area image to be "bad" (line 6). We have seen a similar rejection of the area interpretation in Uwe's intuitive mathematical discourse. However, since Uwe mainly rejected the area meaning for complex path integrals and not for integration in general, Sebastian's rejection is of a different kind: He rejects the area interpretation for the complex path integral only but for integrals more generally. After all, he does not specify the keyword "integration" further except for "Lebesgue integration". Instead of thinking about area, Sebastian explains a very explicit metarule in form of an advice for learners: "one should think about mean values" and "measuring" in context of integrals (lines 11ff.).

In particular, we note that Sebastian's use of keywords for the mathematists here does not suggest that he is referring to his idiosyncratic interpretation of integration. Instead he uses keywords for a larger class of mathematists: One the one hand, he refers directly to the "pupils" and one the other hand he refers to the generic mathematist "one" (e.g., "one imagines that one measures some area" (line 16) or "one imagines: integration is mean value formation" (line 20). Sebastian substantiates his advice by valuing "this geometric intuition" (likely, the interpretation of an integral as an area) as inappropriate or even "bad" (line 6): It would "destroy this higher dimensional situation" (likely, he refers to $\mathbb{C}$ as a two-dimensional real vector space). Then, finally, he relates his general utterances to "path integration" (lines 16ff.): In this case, he considers the narrative that an integral is "the determination of area" to be non-endorsable because it "could be really impeding with path integration". Furthermore, the area interpretation for the "one-dimensional setting", which we interpret here as Riemann integrals, is said to be a coincidence.

Integration is mean value formation
Integration is measuring.

We conclude that Sebastian has endorsed several discursive images about integration (i.e., the process of computing integrals) here, namely "Integration is mean value formation" and "Integration is measuring", which we have constructed from Sebastian's utterances above. We emphasise that these narratives contain the keyword "integration" rather generally instead of keywords for particular instances of integrals (e.g., Riemann integrals, complex path integrals, etc.).

164 Sebastian used the word "Schüler" here, which rather refers to learners in school than students at university. We suspect though that he used this word synonymously for "students" at university.

Figure 15.2: Sebastian's traces a path on the table.

Up to now, it has not been clarified what the mean value is taken of in any of the integrals mentioned before. With this regard, the interviewer asks directly for it:

130–Int.: And which/ Mean value of what?
Seb.: Yes, of what's, uh, in the integrand, so to speak.
Int.: Mmh. (.) So, (...) yes, I will give you the sheet for this. So the/ Asked very concretely: How is it in the complex setting, how would one, uh, imagine this integral? Uh, where is my sheet? So when I have a path in the complex plane [puts the question sheet on the table; see Figure 11.1] [Seb.: Mmh.], continuously differentiable or piecewise continuously differentiable for my sake, whatever, and a continuous function on the image of this path, what does this complex number mean, [points to $\int_\gamma f(z)\,\mathrm{d}z$ on the question sheet] which this happens to compute to, this complex number?
Seb.: Yes, uh, for me this is simply the mean value of the complex numbers, which I grab along this path [traces a circular curve with his finger on the table, see Figure 15.2]. [Int.: Mmh.] Right, and therefore can/ Therefore this is again a complex number because it does not have a geometric/ area meaning (German: geometrischen/ Inhaltsbedeutung), but mean value formation over the objects, which one quasi sees along the path [encircles the inscription $\int_\gamma f(z)\,\mathrm{d}z$]. [Int.: Mmh.] And, uhm, in my view this has nothing to do with area, [Int.: Mmh.] because one/ You can see this with Stokes' theorem, too. This is actually the rotation [traces again a circular curve on the table] that one measures on the plane.

The complex path integral is the mean value of the values of the integrand.

Sebastian states that the mean value is based on the integrand. Thus, Sebastian has effectively constructed a discursive image about the complex path integral: "The complex path integral is the mean value of the integrand."

Following that, Sebastian repeats the previously stated discursive image about the complex path integral as the mean value of the integrand in terms of a metarule about his own discursive actions related to the complex path integral: "the mean value of the complex numbers, which I grab along this path" (lines 33f.) or in terms of a generic mathematists actions, namely "mean value formation over the objects, which one quasi sees along the path". Then, he argues that "this [the mean value; EH.] is again a complex number" because it is a mean value of complex numbers (line 36f.). The result being a complex number is moreover substantiated a second time with the utterance that "it does not have a geometrical meaning". Hence, at this point, Sebastian does not directly substantiate the absence of a geometrical meaning with the fact that the complex path integral is a complex number, but rather the other way around.

The complex path integral measures the rotation on the plane

Another discursive image is present at this point, too, namely "[t]his [the complex path integral; EH.] is actually the rotation that one measures on the plane.". This discursive mental image is also substantiated with a theorem, namely "Stokes' theorem".[165]

Now, having established his discursive mental image that complex path integrals are mean values of function values, he interprets these values geometrically.

> 133–Seb.: [...] here we are again at what we discussed previously, that these, uh, complex numbers always have this character of an amplitwist (German: Drehstreck-Charakter). Yes, so in this respect I find this, uh, cannot be interpreted geometrically as an area, but rather as this rotation and path/ rotational stretching along the path, for these numbers f of z that one, uh, sees there.

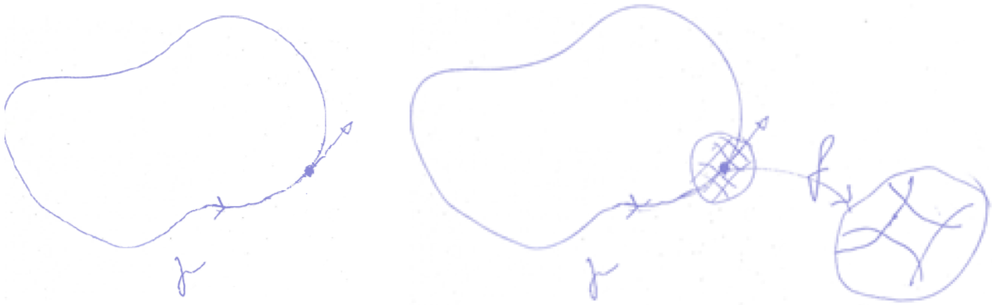
At this point, Sebastian refers to a part of the interview, in which the interlocutors discussed that complex numbers can be interpreted as amplitwists (i.e., multiplying with a complex number w amounts to stretch with the factor $|w|$ followed by a rotation by $\mathrm{Arg}(w)$). Sebastian uses this interpretation of a complex function locally as an amplitwists to substantiate once again "this [the complex path integral; EH.] cannot be interpreted geometrically as an area".

Next, the interviewer begins to paraphrase Sebastian's previous narratives, but is immediately interrupted by Sebastian:

> 134–Int.: And this means that the amplitwist along the path (.)/
>
> Seb.: Yes, so I, I measure the rotation so to speak, right, so I have/ when I have this path here, uh, gamma, [draws Figure 15.3a; the small arrows at the right is drawn a bit later] uh, piecewise continuous, okay, let's say [Int.: Somehow, yes, right.], exactly, uhm, then at each point I have [draws the arrow at the right of Figure 15.3a], uh/ On the one hand, the parametrisation of the path is in it, uh, [Int.: Yes.], but it will be quasi cancelled or neutralised right by the definition of the path integral. And on the other hand, I just have these values of the function f of z and, uh, f of z does now what we have seen previously [points to the stack of sheet accumulated during the interview], yes, this now maps some portion of what one has here, uh, with this grid, uh, grid images (German: mit diesen Gitter-, äh, vorstellungen) [draws the ball-shape at the thickened point in Figure 15.3b], on my behalf. [Int.: Mmh.] And, uhm, geometrically speaking this involves such a stretching and a twist, uh, in some sense probably, yes, so (.) at the point, where the points of the grids are distorted, [Int.: Yes.] or so [completes the sketch in Figure 15.3b]. And I average this effect [points to the circular shape in Figure 15.3b] along this path so to speak. This is my mental image at least. [Int.: Mmh.] (.) So this here is the f, so, yes [writes the f in Figure 15.3b].

First, Sebastian repeats the action of measuring the rotation. Then, he engages in the routine of graphically realising mathematical objects: He realises a path γ as a closed curve and

165 One version of Stokes' theorem can be phrased as follows: Let $U \subseteq \mathbb{R}^3$ be an open set, $F\colon U \longrightarrow \mathbb{R}^3$ a continuously differentiable vector field, and $A \subseteq U$ a two-dimensional oriented compact submanifold with smooth boundary ∂A, then, $\int_A \mathrm{rot}(F)\,\mathrm{d}S = \int_{\partial A} F\,\mathrm{d}r$ (Jänich, 2005, p. 177). Here, the left-hand side is an integral of F on the submanifold A and the right-hand side is a path integral along the boundary of A. A two-dimensional variant of this theorem is Green's theorem (Theorem B.15), according to which the real path integral of a continuously differentiable vector field, which is defined on a domain containing the trace and interior of a closed (piecewise) continuously differentiable Jordan path, can be computed in terms of a double integral. Furthermore, a variant for complex functions exists, too (see Equation A.31). Unfortunately, we do not know which version of Stokes' theorem Sebastian refers to. Since Sebastian explains a few utterances later that complex functions induce a rotation-dilation locally at each point on the trace of a path of integration, it is also possible that he uses the keyword "rotation" for the rotation component of a rotation-dilation instead of a differential operator from a variant of Stokes' theorem. We are also not certain whether Sebastian is considering a closed path here. Therefore, we are hesitant to describe Sebastian's narrative here as a result of the "theorematic"- or "restriction of generality"-frame.



(a) Sebastian's sketch of a path γ.

(b) Sebastian's sketch of an amplitwist at a point of the path.

Figure 15.3: Sebastian's visual mediator for this *mean value image*.[a]

[a] To be precise, the arrowhead to the curved line below the inscription f and the circle around the deformed grid to the right are actually added in turn 138 (see lines 72 and 77).

the "parametrisation of the path" as the arrow attached to the curve at the thickened point (Figure 15.3a). Such an arrow is often used to realise a vector of the tangential field γ' (if γ is differentiable, which seems to be taken for granted here, even though, to be precise, Sebastian says that γ is piecewise continuous (line 52)). The function f is realised as a mapping that distorts a small grid to around the thickened point on the curve (line 65; Figure 15.3b). This distortion is realised graphically with the deformed grid to the right of the signifier f in Figure 15.3b. Having visually realised the function f and the path γ, Sebastian is now able to reiterate his previous discursive image about complex path integrals in terms of a metarule about his own mathematical actions: He "averages this effect [of distortion; EH.] along this path". Moreover, he calls this narrative his "mental image". Hence, we conclude that Sebastian endorses the discursive image "The complex path integral is the average rate of turning and the average rate of stretching as induced by the integrand" (see lines 87ff.).

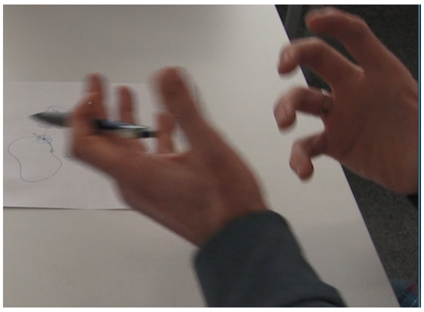
The complex path integral is the average rate of turning and the average rate of stretching as induced by the integrand.

We would like to add a remark about line 54ff. in order to underline the consistency with which Sebastian describes the complex path integral as an average of function values of f. He claims that the "parametrisation" of the path is "cancelled" or "neutralised", and that he "just ha[s] these values of the function f of z". We hypothesised before that "parametrisation" may signify the tangential field γ'. However, the formula $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ implies that γ' actually is a constitutive part of complex path integrals (Section 8.1.3). Since Sebastian describes the complex path integral in terms of a mean value of the function values *alone* though, the cancellation of the parametrisation fits to his discursive image though.

The interviewer would like to know whether the number $\int_\gamma f(z)\,\mathrm{d}z$ is realised in Figure 15.3b as well:

137–Int.: Yes. (5s.) Naively asked: Where is this complex number [points to the inscription $\int_\gamma f(z)\,\mathrm{d}z$ in Figure 11.1], so can one see it in the sketch or is it just/

138–Seb.: Well [draws the arrowhead to the curved line below the inscription f in Figure 15.3b], [...] Okay, for me the imagination is just so that I think of a small neighbourhood at each point [points to the thickened point in Figure 15.5] and from that I see [points to the right part of the figure] how the effect of f is in some abstract [plane; EH.] lying somewhere else [encircles the deformed grid in Figure 15.3b, see Figure 15.5], [Int.: Mmh.] so the path is in one plane so to speak and in my imagination that what f does is in another plane, somehow. [Int.: Mmh.] But one can surely combine this, I have not thought about that before, uh. [Int.: Mmh.] (9s.) Simply/ this four dimensionality is a hurdle, at least for me, to imagine this



(a) Sebastian’s gesture for an amplitwist.

(b) Interviewer’s gesture for turning.

(c) Interviewer’s gesture for stretching.

Figure 15.4: Interviewer’s and Sebastian’s gestures for an amplitwist.

better. In particular when it comes to those path integration issues. (9s.)
So the number f of z, yes, the number f of z has/ uh, really is a linear
amplitwist for me [turns his hands; Figure 15.4a]. And this effect is
averaged along this path and this is what the integral means to me.
[...]
141–Int.: Mmh. So when I try to para/ paraphrase what this mean value means, is
it for you like that at each point of the path the mapping [shows to the ”f”
in Figure 15.3b] somehow turns and stretches (German: drehstreckt) [taps
at several point of the curve in Figure 15.5] [Seb.: Yes.] and when I do this
along the whole path [circular movement with a finger above the table] and
compute the integral [points to Figure 11.1], the number that it computes
to is the average rate of turning [gesture for turning; Figure 15.4b] and the
average rate of stretching [Figure 15.4c] at the same time?
Seb.: [nods] Somehow in this direction, exactly. So, this is at least my/ the image
I have in mind (German: Vorstellung, die ich im Hinterkopf habe). This
does not necessarily, uhm/ I don’t claim that this is completely consistent,
but this is my mental image (German: Vorstellung) so to speak.

Here, Sebastian explains again the effect of f as a mapping between two planes. Both interlocutors use similar gestures to visually and bodily realise the turning and stretching component of amplitwists (Figure 15.4). In particular, the interviewer’s paraphrase of Sebastian’s discursive image confirms our interpretations so far. In particular, Sebastian asserts several times that “this is what the integral means to [him]” (lines 84f.), “this is [...] the image I have in mind” and “this is my mental image” (lines 95f.). Noteworthy, at this point, Sebastian relates his narratives explicitly to himself as opposed to generic mathematicians or learners, and he is aware that his interpretation may not be fully consistent.

Last but not least, he claims that “four dimensionality” is a hurdle for him to image “those path integration issues” (lines 81f.). Likely, Sebastian refers to the property of f to map a two-dimensional domain to another plane.

To sum up, following the **“mean value”-frame**, Sebastian offered three discursive images about integration or the complex path integral. Not only did Sebastian explicitly describe the discursive images we found to be his “mental images” of complex path integrals, but he is also reflective and cautions his interlocutors that his interpretation may not be “completely consistent” (lines 95ff.):

- The first discursive image characterises integration in general as mean value formation and measuring. This narrative was also presented as an advice for students.
- The second discursive image specifies this general discursive image about integration to the complex path integral: Sebastian describes $\int_\gamma f(z)\,\mathrm{d}z$ as the mean value of the values of f along the path γ.
- Having realised the “effect” (line 66) of the integrand f in Figure 15.5, the third discursive image characterises the complex path integral as the average rate of rotation and dilation induced by f. It was uttered by the interviewer and basically also approved by Sebastian.

Lastly, we remark that Sebastian’s mean value interpretation conflicts with the endorsed one by Gluchoff (1991), which is contained in the mean value aspect of complex path integrals (Section 9.1.4): According to this aspect, the complex path integral $\int_\gamma f(z)\,\mathrm{d}z$ is the mean value of the function $f \cdot T$, where T is the unit tangential field induced by γ, along the oriented trace of γ, that is, $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f \cdot T\,\mathrm{d}s$. In this narrative, the complex path integral is not the mean value of f but of $f \cdot T$.

Mean value interpretations in the other two interviews

We recall that Uwe and Dirk used the keyword “mean value” when substantiating Cauchy’s integral formula, too. According to them, this formula states that a function value of a holomorphic function inside a ball can be calculated as the mean value of the function values along the boundary of the ball (Section 13.7, Section 14.7). However, we decided not to argue that Uwe and Dirk enacted the “mean value”-frame: Neither of them explicitly expressed the idea of integration as averaging or measuring in general and neither of them interpreted complex path integrals as certain mean values. Uwe and Dirk mentioned mean values only in the context of Cauchy’s integral formula. Sebastian, on the other hand, interpreted integrals as mean values in general and used this interpretation to construct discursive images about complex path integrals in particular. Additionally, he made use of these discursive images explicitly for the substantiation of Cauchy’s integral theorem and Cauchy’s integral formula (see Section 15.4, Section 15.5).

15.3 THE “HOLOMORPHICITY EX MACHINA”-FRAME

In this section, we introduce the **“holomorphicity ex machina”-frame**. This frame is especially important for the analysis of Sebastian’s substantiations of the three integral theorems in Section 15.4, Section 15.5, and Section 15.6. Sebastian observes a “mysterical connection” between holomorphic functions and complex path integrals (e.g., line 132 below), which is the basis for the usage of the “holomorphicity ex machina”-frame in his subsequent intuitive mathematical discourse about complex path integrals. This discursive frame consists of the valuation of *holomorphicity* as a strong and rigid property of complex functions that virtually enforces the integral theorems to be endorsable, in which holomorphic functions play a role. The frame also consists of substantiations based on this rigidity property. Our name for this discursive frame stems from our perception that "holomorphicity" appears as a "deus ex machina" in the intuitive substantiations based on this frame.[166]

We note further that this discursive frame does not necessarily intersect with the **"restriction of generality"-frame**. The reason is that if the context of the narratives is already based on holomorphic functions, for instance when the two interlocutors in the interview discuss an integral theorem about holomorphic functions, the interlocutors do not necessarily restrict this context further. However, in Section 15.3.2 both frames do intersect when Sebastian and the interviewer-researcher construct a discursive image about complex path integrals following both frames. The reason here is that the interlocutors create a discursive image about complex path integrals as such, which is based on the restriction of the class of integrands and the class of paths to holomorphic functions and closed paths.

In order to explain the application of the "holomorphicity ex machina"-frame in more detail, we need to explain first how Sebastian interprets holomorphic functions.

15.3.1 *Intermezzo: Rigidity of holomorphic functions*

Sebastian describes holomorphic functions as particularly rigid functions. For example, rather at the beginning of the interview, describing the main contents of a course in complex analysis, he argues that in the definition of "complex differentiation [...] there is more in the limit than for real differentiation" (turn 2). He argues more precisely:

14–Seb.: [...] At first, you don't notice it, right? But of course, everything is
already in there. That complex differentiation is a much, much more rigid
notion that two-dimensional real differentiation [Int.: Mmh.] and, uhm, if
you write down the difference quotient, you don't see that you actually, uh,
do much more than you first of all naively think. [Int.: Mmh.] Well, and
this leads of course, uh, to Cauchy-Riemann differential equations and then
to harmonic analysis [...] So I also see the close connection to Gauß'
theorem, Stokes' theorem, mean value properties and those things, which
are, uh, important in real analysis in higher dimensions.

Sebastian differentiates the notion of complex differentiability from "two-dimensional differentiation", which we interpret as the real total differentiability of $\mathbb{R}^2$-valued functions of two real variables.[167] Already at this point, Sebastian identifies integral theorems such as Gauß' theorem or Stokes' theorem, mean value properties and others as consequences of the complex differentiability (i.e., holomorphicity), which are then in turn important for higher-dimensional real analysis.

In addition to this initial identification of rigidity (line 100), when asked for potential pictures of symbols connected to the keyword "holomorphic", Sebastian identifies another very important property of holomorphic functions for him (besides the series development),

166 The *Duden*, a dictionary of the Standard High German language, explains "deus ex machina" as follows (https://www.duden.de/rechtschreibung/Deus_ex_Machina, retrieved 01/25/2022; own transl.): "unexpected help appearing at the right moment in an emergency; surprising, unexpected solution to a difficulty".

167 Recall that a complex function f is complex differentiable at a point z if the corresponding vector field $\mathbf{f}$ is totally real differentiable and the Cauchy-Riemann differential equations are satisfied (Section A.3). In this sense, complex differentiability is a stronger property than real total differentiability.

namely, that local properties of holomorphic functions imply global properties of these functions, or in his words, “local properties say it all” (113ff.):

29–Int.: Mmh. Are there other pictures or symbols or so, which you directly connect to the word “holomorphic”? Or/

Seb.: Well, what immediately comes to my mind for holomorphic, uh, is Taylor series or rather series development. [Int.: Mmh.] Yes and this is not a geometric picture, but this mean, similar to here, that for a complex analytic function, uh, the local properties say it all, so to speak. So when I know/ If I know everything about a function in one point, [Int.: Mmh.] locally with the derivatives, then I know the function in total, [Int.: Mmh.] on its domain of holomorphicity at least and as such, uh, this is always a quite localised thing for me, that/ there are no, uhm, global, uh, properties of an analytic function because it/ Everything can be captured locally. [Int.: Mmh.] Except for those things like branch points or branch cuts for Riemann surfaces, so. [Int.: Mmh.] That is again rather a global property that cannot be reflected here, right, but if you restrict it to the domain of holomorphicity and you know everything from one point, then everywhere.

The central narrative here is Sebastian’s description about his local and global “knowledge” of holomorphic functions: He argues that he knows a function globally if he knows it locally at one point. This local knowledge is furthermore realised in terms of the keywords “derivatives” (line 115). Hence, he endorses the object-level rule on holomorphic functions that local properties imply global properties. Moreover, Sebastian expresses this interplay of local and global properties the other way around, too, namely, that there are (almost) no global properties of holomorphic functions, which are not yet captured in local information. However, Sebastian also mentions a few examples to the rule, namely branch points or branch cuts.[168]

15.3.2 *Complex path integrals make singularities visible*

In this section, we analyse how Sebastian and the interviewer created a discursive image about complex path integrals based on the **“holomorphicity ex machina”-** and the **“restriction of generality”-frame**. While the “holomorphicity ex machina”-frame clearly requires the functions to be holomorphic, it seems that The next utterance connects to line 86.

138/139–Seb.: [...] And therefore, also on closed paths [points to Figure 15.3b] it is not surprising that one somehow gets zero, for example, if, if it is holomorphic. But this is again related/ this is again related to, of course, it may be that I have a singularity[169] here or something like that [adds the shaded blob shape to Figure 15.3b, see Figure 15.5], [Int.: Mmh.] or that there is a pole, or whatever. And then it is of course hard to imagine that from the points, which are out here [points to the thickened point at the right of the “path”, see Figure 15.5], you can see this here [points to the newly drawn blob in Figure 15.5]. [Int.: Mmh.] Well, and/ But this comes again from the mysterical connection (idiosyncratic German: mysterische Zusammenhang) that a complex-analytic function is locally determined completely. This means it possible to know from a local inspection [points

168 At this point, we are reminded of Dirk’s pointing usage of keywords to replace entire narratives. Here, Sebastian’s personal metarule “[i]f I know everything about a function in one point, locally with the derivatives, then I know the function in total, on its domain of holomorphicity at least” (lines 114ff.) seems to point to the Identity theorem (Theorem A.23): This theorem asserts that a holomorphic function is completely determined by the values of its derivatives at one point only.

169 The original German word at this point was “Definitionslücke”. This word signifies a point or a set of points, on which a function is not defined. In the context of holomorphic functions, we consider the words “Definitionslücke” and “Singularität” to be exchangeable though.

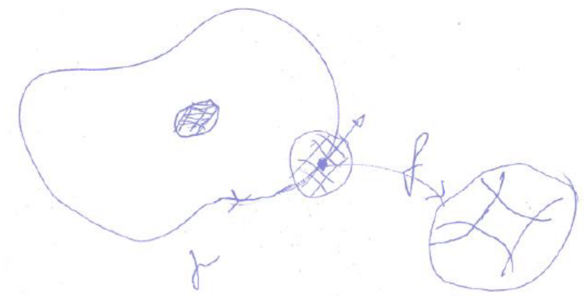


Figure 15.5: Sebastian's adds a "singularity" inside a path γ.[a]

[a] The dark spot within the curve on the left is drawn in turn 139, which we discuss in Section 15.3.2.

> again to the thickened point] to know what is happening here inside [points again to the newly drawn blob].

The narrative in lines 123ff., "on closed paths it is not surprising that one somehow gets zero, for example, if, if it is holomorphic", is an idiosyncratic version of Cauchy's integral theorem (Theorem A.17). However, it is not fully endorsable in this form since one constraint is missing. It has to be required further that there are no singularities in the interior of the path.

Sebastian seems to have noticed this condition immediately, too, because he then mentions that "of course, it may be that I have a singularity here or something like that" (lines 125ff.). He realises such a potential singularity with a crossed-out blob shape in the interior of the previously realisation of γ (Figure 15.5). According to Sebastian, the "local inspection" of the function at a point on the trace of γ it is possible to "know what is happening here inside" (i.e., the existence of a singularity). He substantiates this with the "mysterical connection that a complex-analytic [holomorphic; EH.] function is locally determined completely" (132ff.). This narrative is in complete accordance with the object-level rules on holomorphic functions Sebastian established previously, namely that local properties of a holomorphic function enable mathematists to know the function globally (Section 15.3.1). Nevertheless, Sebastian finds it "hard to imagine that from the points, which are out here, you can see this [singularity] here" (lines 128ff.).

The complex path integral can detect whether a holomorphic function has a singularity in the interior of a closed path of integration.

Effectively, Sebastian states here the metarule according to which singularities of a holomorphic function in the interior of a path can be detected with the help of the complex path integral. This assertion is substantiated with the holomorphicity of the function; a logical argument is not present. We summarise the previous observations in form of a new discursive image of complex path integrals: "The complex path integral can detect whether a holomorphic function has a singularity in the interior of a closed path of integration".[170] However, since Sebastian does not explicitly utter this narrative explicitly, we regard is as a third-person narrative, jointly constructed by Sebastian and the interviewer-researcher. Hence, this discursive image is a result of the **"holomorphicity ex machina"-frame** and the **"restriction of generality"-frame**.

INTERMEDIATE BREAK

Here, we insert again a short break. We have described two new discursive frames, which appeared in Sebastian's intuitive mathematical discourse about the complex path integral only—

170 We consider this narrative to be endorsable in scholarly complex analysis a well. If f is a holomorphic function on a domain containing the trace a closed (piecewise) continuously differentiable path γ, then $\int_\gamma f(z)\,\mathrm{d}z \neq 0$ implies that f has at least one singularity in the interior of γ.

the **"mean value"-** and the **"holomorphicity ex machina"-frame**. Both discursive frames were used in explorations that eventually led to discursive images about the complex path integral. In the following three sections, we examine how Sebastian enacted these frames to derive substantiations for the three integral theorems (Cauchy's integral theorem (Theorem A.17), Cauchy's integral formula (Theorem A.22), and Existence of primitives for holomorphic functions (Theorem A.20)) during the interview.

15.4 SUBSTANTIATING CAUCHY'S INTEGRAL THEOREM WITH THE **"MEAN VALUE"-** AND **"HOLOMORPHICITY EX MACHINA"-FRAMES**

In this section, we discuss Sebastian's intuitive substantiation of Cauchy's integral theorem.

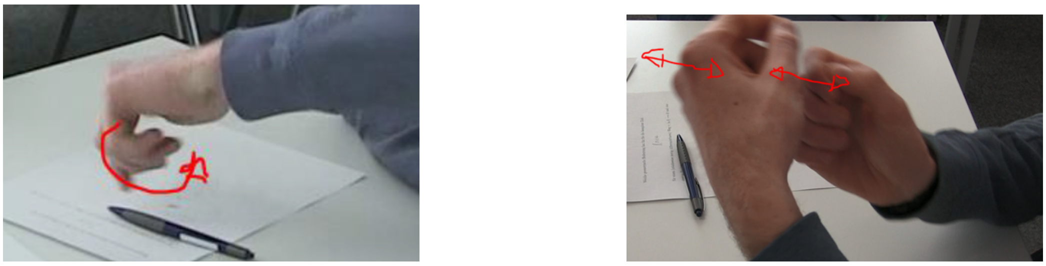
15.4.1 *"Averaging out" and driving a car on a mountain trail*

Sebastian's mean value interpretation of complex path integral has already been discussed at this point during the interview. Now, the interviewer challenges Sebastian to explain Cauchy's integral theorem on the basis of this mean value interpretation. Accordingly, the interviewer introduces the question on the intuitive substantiation of Cauchy's integral theorem with the **"mean value"-frame** himself. He describes the vanishing of the complex path integral as if "on average nothing happened at all":

143–Int.: [...] Fine and for closed paths there is Cauchy's integral theorem. [Seb.:
Mmh.] So what does it say? (...) Maybe in this version [incompr.] on star
domains, closed paths [Seb.: Mmh.] the integral gets zero. [Seb.: Mmh.]
Let's imagine this was the case here, why is it zero now? [Seb.: Mmh.] So
this would mean, on average (German: mittelwertmäßig) nothing happened
at all [Seb.: Mmh.] or how, how's that?
Seb.: [nods] Yes, exactly [both grin]. [Int.: Mmh.] (...) Yes, the rotation effect
[turns his hand; Figure 15.6a], so to speak, I can most easily imagine it to
be averaged out (German: ausgemittelt) and therefore it disappears, and
that this/ with this stretch effect [pulls hands together and apart;
Figure 15.6b] that it is averaged away to zero (German: ausgemittelt wird
zu Null). (..) Yes, okay, that's kind of like driving a car on a closed, uh,
mountain road, yes [Int.: Mmh.], then everything one has lost, so to speak,
uh, one has to get back again sometime. [Int.: Mmh.] Well, that's my naive
imagination (German: naive Vorstellung) about it, anyway.
Int.: Yes. (..) and what do I lose so to speak? What does this mean?
Seb.: Yes, so that I/ For once, I have this turning, which I forget about for the
moment, yes, [Int.: Mmh] and now it's only about this other effect and
what I so to speak/ what I mean with that is that one, uh, simply has to
come back to the same height [Int.: Mmh.], so to speak. Uhm, for me this is
the intuitive explanation for this, so to speak, that this effect is also
averaged to zero.

Sebastian agrees to the interviewer's adaption of the mean value interpretation. Just like before when he explained the complex path integral as the mean value of the rotation and dilation of the integrand (see Section 15.2), Sebastian argues here that both, "the rotation effect" (line 143) and "this stretch effect" (line 146) are "averaged away". Both effects are illustrated with gestures (Figure 15.6).

Then, Sebastian provides a metaphor from real life: He compares "it" (likely, the process of averaging) to driving a car on a "mountain road". Whereas he argues first in lines 143ff. that he "can most easily imagine" the "rotation effect" to vanish, in his subsequent utterances in



(a) Sebastian rotates his wrist to illustrate the "rotation effect".

(b) Sebastian moves his hands together and apart to illustrate the "stretching effect".

Figure 15.6: Sebastian's gestures for averaging the "rotation effect" and "stretching effect".

lines 153ff., he neglects "this turning" though for the moment and focuses on the "height" of the car instead. In this case, he argues that any loss or change in height has to be made up when returning to the starting point (lines 149f., 155f.).

Two important observations can be made here in addition to Sebastian's metaphor: First, he stays in the discourse about mean values and averaging and does not use any keyword related to integrals explicitly. We may hypothesise that at least here the discourse about mean values and the discourse about complex path integrals are isomorphic for Sebastian (cf. Sfard, 2008, p. 122). Second, Sebastian values his descriptions from turns 144 and 146 explicitly to be his "naive imagination" and "intuitive explanation" (lines 150, 157).

We also note that neither the interviewer nor Sebastian mentioned the condition of the integrand to be holomorphic here. However, this condition has already been stated before in connection to Cauchy's integral theorem (e.g., lines 123ff.).

15.4.2 *Expecting Cauchy's integral theorem*

The interviewer takes it up again and explicitly asks why holomorphicity and not continuity is the appropriate condition for the functions:

147–Int.: Yes. (..) Yes, then it is/ So why does this only apply to holomorphic functions? Why isn't it sufficient to be continuous for example?
Seb.: Oh, yes, yes, that's right, isn't it? That's a good point. (.) Uhm, (..) this is/ So why/ why the holomorphicity plays into it, so to speak, that you somehow still have a global, uh/ For me at least, the interpretation is that you can only expect this averaging to work like this, uh, if, uh, nothing bad can happen somehow, so to speak. [Int.: Mmh.] And the holomorphic properties[171] then force, uh, that everything is already determined locally, so [I: Mmh.] if I know what happens in one place, then I know what happens everywhere. Um, that's my naive interpretation of it for now. So only continuity/ (..) Good point actually, yes. [...]

Sebastian agrees that the question why the functions have to be holomorphic instead of continuous only is valid (both interlocutors do not focus on additional constraints on topology of the domain or potential singularities here though). He substantiates the constraint on the integrand to be holomorphic idiosyncratically with an expectation: He argues that he can "expect" the "averaging" only to work out properly "if, uh, nothing bad can happen" (lines 164f.). This in turn is substantiated with the property of holomorphicity itself. That is, Sebastian's substanti-

171 The German original here was "die holomorphen Eigenschaften". This is probably an idiosyncratic way of saying "holomorphicity".

ation of Cauchy's integral theorem here is based on the property of holomorphicity itself: He values it as a property enforcing that "nothing bad can happen" and the averaging for the complex path integral acts out in such way that it vanishes.[172]

We can infer a corresponding metarule for Sebastian's intuitive mathematical discourse about the complex path integral here: If holomorphicity belongs to the constraints on the functions in a narrative (here, Cauchy's integral theorem), then this narrative can be considered as endorsable because holomorphicity is valued as such a strong property that it essentially forces the narrative in question to be plausible. We consider this metarule the characteristic underlying metarule of the **"holomorphicity ex machina"-frame**.

15.4.3 *A metarule on potentially misleading imagination*

In the rest of the discussion on Cauchy's integral theorem, no further intuitive substantiations were produced. Rather, Sebastian explains a general metarule for his intuitive mathematical discourse: He warns that the explanations he deemed intuitive before may be too naive to account for the mathematical details actually required to substantiate the theorem.

The following lines of transcript immediately continue the previous excerpt in line 169:

148–Seb. [...] Of course, one had to construct a simple counterexample now. (8s.)
Yes, this probably has to do something with the fact that this idea I have is
simply too naive, yes, it is not enough for the explanation of this averaging.
And, uh, then/ finally everything is contained in Goursat's lemma or
Morera or wherever, [Int.: Mmh.] uhm, in the subtleties, which one puts
aside in the imagination (German: Vorstellung), yes.
[... both interlocutors recall Goursat's lemma and its proof ...]
160–Seb.: [...] So this also shows that these heuristic/ these intuitions, which one
may probably have for general paths, uh, (.) can be misleading (German:
verfälschend) because for the real proof you have to get into the fine
structure [Int.: Mmh.] and this then also explains that one does not get far
with continuity only, right
[...]
165–Int.: So the subtlety involved in the proof is the actual reason why one needs
the holomorphicity, so to speak?
Seb.: Yes, this uniformi/ this universality, this universal claim (German:
Universalitätsanspruch), right, so if you want to have a theorem that says
for all paths, then this is a universal claim, where you, mustn't, uh, wonder
if a strong prerequisite is necessary. [I: Mmh.] (..) This is, so to speak, an
interpretation of why one cannot avoid getting involved into the subtleties,
because, uh, this universality one wants to prove [I: Mmh.] also requires,
uh, that one really looks for everything possible (German: alles abgrast,
was möglich ist).

At first, Sebastian sets himself the task to provide a "simple counterexample" (we suspect for Cauchy's integral theorem if holomorphicity was replaced with continuity) (line 170) and thinks about for a moment in silence. He does not take up this idea further though. Instead, Sebastian makes explicit another metarule of his intuitive mathematical discourse: He hypothesises that his "idea" might be too naive and that the subtleties from the corresponding proofs

172 We are aware that holomorphicity of the integrands alone is not sufficient to endorse Cauchy's integral theorem (besides the conditions on the path) since the function may still have singularities in the interior of the path. Previously though, the interviewer mentioned the constraint "star domains" (line 138) for the domain of the functions and Cauchy's integral theorem is endorsed for holomorphic functions on star domains. Hence, it is either possible that Sebastian took this additional condition for granted or it was not considered as particular relevant at this point—after all, he was encouraged to depart from scholarly discourse whenever he saw fit.

(he mentions Goursat's lemma and Morera (likely, he refers to Morera's theorem here)) may not be present "in the imagination" (lines 174f.). Instead, a careful inspection of a proof may reveal the details (i.e., the "fine structure", line 179), which are missed in the imagination, but which are necessary to account for the universality of Cauchy's theorem.

We conclude that Sebastian is very reflective about what he considers his imagination: On the one hand, he can describe it very clearly—we have heard him utter explicitly what he counts to belong to it—and on the other hand, he cautions himself that it does not account for the technical details in proofs, which in turn show why weaker conditions may indeed not be sufficient.

15.5 SUBSTANTIATING CAUCHY'S INTEGRAL FORMULA WITH THE **"MEAN VALUE"-** AND **"HOLOMORPHICITY EX MACHINA"-FRAMES**

Sebastian and the interviewer discussed Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \tag{15.1}$$

(see Figure 11.2) at the end of the interview. The interviewer introduced the question about Cauchy's integral formula as follows:

181–Int.: [...] Okay, maybe lastly now, uhm, to Cauchy's integral formula, maybe once, (.) uhm, (..) in the end it says something like this: [Seb.: Mmh.] So I have a function, holomorphic, [Seb.: Mmh.] open domain, or open set at least, a circle, [Seb.: Mmh.] which is now centred around z-zero here, radius r, I integrate along its boundary [Seb.: Mmh.] and I get the function value at one point with respect to this integrand here [shows the question sheet, Figure 11.2]. [Seb.: Mmh.] Now, what's my question? How do you imagine the statement of this formula? Can one somehow argue vividly (German: anschaulich) that this formula here makes sense now?

In particular, the interviewer realised Cauchy's integral himself with an utterance on meta-level, namely "I integrate along [the circle's] boundary" and "I get the function value at one point with respect to this integrand". This way, he already draws attention to the integrand in Equation 15.1.

Sebastian offers two explanations of Cauchy's integral formula. Doing so, he intertwines the **"mean value"-frame** and the **"holomorphicity ex machina"-frame**. One the one hand, he describes Cauchy's integral formula as "some kind of a mean value theorem" (line 203). In contrast to Uwe and Dirk (see Section 13.7, Section 14.7), Sebastian argues that the mean value here is derived from the function $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$, *not* f itself. As such, his explanation of Cauchy's integral formula is consistent with his general mean value interpretation of complex path integrals (see Section 15.2) but differs from Uwe's and Dirk's interpretation, even though these two experts also described Cauchy's integral formula with the help of mean values. One the other hand, he also describes it as a formula, which might have been expected due to the global determination of holomorphic functions in terms of the local information about it on the set $\partial B(z_0, r)$ (see Section 15.3.1).

15.5.1 *Averaging in Cauchy's integral formula*

According to Sebastian's discursive images from Section 15.2, the complex path integral is the mean value of the values of the integrand along the path of integration. Hence, we may hypothesise that if he follows the **"mean value"-frame** again, he will interpret $\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta$ as a mean value of the function $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ along the boundary of the circle $B(z_0, r)$. This is indeed his initial reaction:

> 183/184/185–Seb.: Mmh. (6s.) Well, intuitively (German: anschaulich)/ So my first reaction was to say: For me, uh, this is some kind of a mean value theorem, yes, so the mean value property [Int.: Mmh.] of a holomorphic function.
> [...]
> And that is how I finally imagine this assertion, that I encircle this point so to speak and now I do not average over the function values themselves, this would be zero, but I measure with this singular factor. [Int.: Mmh.] Uh, not singular, uh, which is singular at this point. [Int.: Mmh.] If you chose the averaging along this point correctly, then one can, uh, also measure this value. [Int.: Yes.] It always amazes me that there is a universal averaging function so to speak or at least modification of the function. [...]

Sebastian called Cauchy's integral formula a "some kind of a mean value theorem" and "the mean value property of a holomorphic function" (lines 203f.). These utterances appeared similarly in the interviews with Uwe and Dirk, too. However, Sebastian then realises this mean value property differently: He describes one meta-level that he "encircle[s] this point" (likely, z) and he "do[es] not average over the function values themselves, this would be zero, but [he] measure[s] with this singular factor" (lines 207ff.). That is, the special function integrated in Cauchy's integral formula (i.e., $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$) is said to involve a "singular factor" or at least a factor, which is "singular at this point" (lines 210). We conclude from the context that this factor is the term $\frac{1}{\zeta - z}$, which has a pole at z as a function of ζ.[173]

As we have argued before, Sebastian is led by the **"mean value"-frame** and his narratives are consistent with the discursive images from Section 15.2. But additionally, he substantiates why, in his view, the mean value in Cauchy's integral formula cannot involve the function f itself: According to him, the result of the complex path integral "would be zero". This narrative in turn corresponds to Sebastian's mean value interpretation of Cauchy's integral theorem (Section 15.4).

We remark however that Sebastian's mean value interpretation here is in conflict with the endorsed mean value interpretation of Cauchy's formula from the literature. For this, we consider the case $z = z_0$ (after all, Sebastian did not rule out this case). We recall that Cauchy's integral formula can then be rewritten as $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{it}\right) \mathrm{d}t$; in other words, the integral here actually involves the values of the function f on the boundary of the ball itself (Remmert & Schumacher, 2002, p. 83). Uwe's and Dirk's mean value interpretations were rather consistent with this narrative from the literature.

15.5.2 *Expecting Cauchy's integral formula*

The **"mean value"-frame** and the **"holomorphicity ex machina"-frame** were entangled in Sebastian's intuitive explanation of Cauchy's integral theorem, and so are they here. Indeed, the following utterances took place in turn 183 at line 206, which we left out in the previous section to discuss the application of the "mean value"-frame.

173 Since Sebastian does not mention the fraction $\frac{1}{2\pi i}$ explicitly, we cannot know for sure whether this fraction is also part of the "singular factor" (e.g., $\frac{1}{2\pi i(\zeta - z)}$ would be a plausible candidate for the "singular factor", too).

As in the case of Cauchy's integral theorem, Sebastian argues that the interplay between local and global properties of holomorphic functions cause him to expect that "something like this" (i.e., Cauchy's integral formula) (line 223) is plausible, too:

> 183–Seb.: [...] So this means, if I move here on the boundary, uhm, then I can, through this boundary so, as we have seen previously, yes, no/ there are no local or global properties, it's all always the same, I can gather enough information, uh, on this path here about this point, so to speak. [Int.: Mmh.] And, uhm, I am doing this while I am, uh, including a pole there. [Int.: Mmh.] Well, uhm, (.) this is probably no good, uh, not a particularly fine intuitive explanation, but this is the imagination (German: Vorstellung) for me at least, that this, uh, with the help of these properties, that, uh, local properties are global properties, (.) I, at least, uh, can expect that something like this is possible. [Int.: Mmh.] (...)

Sebastian describes that he can "gather enough information, uh, on this path here about this point" (likely, the point z), while "including a pole there" (lines 216ff.). We interpret Sebastian's narrative here as follows: The information contained in the complex path integral of the function $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ along $\partial B(z_0, r)$ (divided by $2\pi i$) yields a piece of information about the singularity at z, namely its function value $f(z)$. We based this interpretation on the discursive image we reconstructed in Section 15.3.2, namely that complex path integrals of holomorphic functions along closed (piecewise continuously differentiable) paths can detect a singularity in the interior of the path.

Again referring to the interplay between local and global properties (of holomorphic functions), Sebastian argues that "something like this" (i.e., Cauchy's integral formula or at least a similar formula) could be expected (line 223). Thus, again, he seems to follow the metarule according to which the strong property of holomorphicity leads him to virtually expect Cauchy's integral formula in the context of his intuitive mathematical discourse.

In order to check, whether he is able to follow Sebastian's line of argumentation, the interviewer paraphrases Sebastian's previous narratives by asking whether the holomorphicity property of complex functions is causing Cauchy's integral formula to be "not unbelievable":

> 188–Int.: Mmh. So is it for you like that not/ simply because holomorphicity has these particular properties that it is not unbelievable that such a formula holds, so to speak?
>
> Seb.: [partly overlapping with the previous turn: Exactly, that is where I start,] so to speak. Uh, I would also say, uh, to students first of all, uh: Well, could you imagine that something like this could be true? [Int.: Mmh.] And, uhm, answer: Yes, because, uh, holomorphic functions have the property that one can also measure global properties with the help of path integrals.

Sebastian agrees to the interviewer's paraphrasing question, which reinforces our previous analyses.

Complex path integrals can measure global properties of holomorphic functions

Then, Sebastian explains another pedagogical metarule about his teaching of complex analysis. He describes that he asks his students whether they could imagine "something like [Cauchy's integral formula; EH.] this to happen" and that the potential answer is affirmative. In particular, Sebastian repeats a variant of discursive image about the complex path integral as a detector for singularities (Section 15.3.2, which we can phrase as a discursive image about complex path integral, which is based on the **"holomorphicity ex machina"-frame**: "Complex path integrals can measure global properties of holomorphic functions".

Last but not least, we notice that Sebastian is certain to have described how he "imagine[s]" Cauchy's integral formula (lines 207, 220), in line with his previous intuitive mathematical discourse about complex path integrals. Even though, he cautions that his explanation may not be "a particularly fine intuitive explanation" (lines 219), he values it appropriate enough to discuss it with students.

15.5.3 *Asking the physicists*

However, there is one aspect for which Sebastian does not find to have an intuitive explanation, namely, why the integrand in Cauchy's integral is the way it is.

189–Seb.: [...] Fine, so this depends a little on how one approaches this further, uhm, I think one can probably get a geometrical intuition (German: geometrische Anschauung) [incompr., probably: today] over this imagination (German: Vorstellung) of/ about Stokes' theorem so to speak from two-dimensional analysis, uh, real analysis. (.) The engineers know that, I do not know this by heart, uh, but that one imagines here quasi: Okay, now here I have the circulation and when I measure the circulation with this factor correctly, so f of zeta divided/ then, uh, I can deduce that/ I needed to think about it again, I don't know it right away. But I think that one/ as a lecturer I still had the change to produce more intuition (German: mehr Intuitionsbildung zu produzieren), [Int.: Mmh.] without doing the proof formally, that is what is all about.
[...]
192–Int.: [...] The physicists always have nice mental images (German: gute Vorstellungen), right, this goes in direction of potential theory, [Int.: Mmh.] that one measures, uh, the mass, [Int.: Mmh.], which one walks around, and I think one can make use of this physical intuition. But I don't have that ready. [...]
Int.: Mmh. Okay. (..) To sum it up, it is not miraculous that there is something like it [points to the formula Figure 11.2], [Seb.: Mmh.] but the concrete shape of how I get it is just the way it is.
Seb.: This is just/ Here I know/ Yes, I have not yet managed to provide myself with a direct illustration (German: Veranschaulichung) in my career so far, so to speak. As I said, my/ my addressee would be the physicists. I think they would have a [I: Mmh.] way to explain it so that I can see it immediately/ that one can see it immediately, that this [points to the formula in Figure 11.2] is the only right way.

Sebastian is again reflecting about the potential limitation of what he considers his intuition. He claims it might be possible to explain the factor $\frac{1}{\zeta - z}$ in Cauchy's integral formula with Stokes' theorem and the "circulation" (another word for the differential operator rotation) or that physicists or engineers (potentially working in potential theory) might be able to provide an explanation for the particular shape of the integrand in Cauchy's integral formula.

15.6 SUBSTANTIATING THE EXISTENCE OF HOLOMORPHIC PRIMITIVE FUNCTIONS WITH THE **"HOLOMORPHICITY EX MACHINA"-FRAME**—A STORY ABOUT THE "HOLOMORPHICITY TRAP"

In this last section, we analyse how Sebastian argues about the existence of primitive functions for holomorphic functions. He gives two reasons why continuous, but not holomorphic, functions cannot have primitives: The first is based on the "holomorphicity-trap" metaphor (Section 15.6.1) and the other is based on the construction "pseudo primitives" (Section 15.6.2). Similar to the substantiations of the previous two integral theorems, his argument is based on the **"holomorphicity ex machina"-frame** and the **"theorematic"-frame**.

15.6.1 *The holomorphicity-trap*

Sebastian introduces a metaphor, the "holomorphicity-trap", in order to substantiate that there are primitive functions for holomorphic functions (omitting technical details) but not for only continuous functions. This metaphor is an idiosyncratic way for saying that derivatives of holomorphic functions are automatically holomorphic, too; hence, there cannot be primitive functions for non-holomorphic complex functions.

The interviewer introduces the interview part on the existence of primitive functions with the fundamental theorem of calculus (Theorem B.3), according to which there are primitive functions for continuous functions on intervals in $\mathbb{R}$, and asks for a potential analogue to complex analysis. In particular, in this interview, he did not yet mention himself that continuity is not enough in complex analysis.

167–Int.: [...] Okay, continuous functions defined on an interval of R [Seb.: Mmh.] and mapping to R always have primitive functions. [Seb.: Mmh.] One can get it via, well, yes, as the fundamental theorem says, via the integral with a fixed lower bound, uh, of the integral. (.) What could a possible analogue situation for complex functions look like?
Seb.: Well, this does/ So my first answer is: This simply doesn't work anymore [Int.: Mmh.], uh, on the contrary one is caught in the holomorphicity-trap so to speak, uh, if you have a primitive function, then the function, uh, must have already been holomorphic. And, uhm, (.) that, uhm, yes, I don't really have a better explanation for that than, uhm, that somehow turns out afterwards that a holomorphic function is analytic.
Int.: Mmh. (.) So this means infinitely many/ this is the property of infinite differentiability.
Seb.: Right, then/ If you know this, then it is immediately clear, that this cannot be true. [...] That this in retrospect naive imagination (German: Vorstellung) from real analysis [Int.: Mmh.] cannot be transferred. [Int.: Mmh.] And for me this is always the explanation: Well, it is simply the case that this complex differentiation is an extremely rigid property. Which one can see again from the analogy to two-dimensional differentiability, because there are less parameters, possible so to speak [Int. Mmh.], this structure properties of the Jacobian is very rigid.

Sebastian argues that the situation described by the interviewer does not "work anymore" because "one [a general mathematist; EH.] is caught in the holomorphicity-trap" (line 265). In particular, this metaphor results from a recall of Theorem A.25, according to which holomorphic functions are analytic (line 269). Hence, Sebastian's reasoning is based on the **"holomorphicity ex machina"-** and the **"theorematic"-frame** here: He uses a theorem to construct a narrative in response to the task situation initiated by the interviewer to look for a potential analogue to the theorem the interviewer mentioned. In order to justify this metaphor, Sebastian resorts again to the narrative that "complex differentiation is an extremely rigid property" (line 275f.). We have observed equivalent narratives at multiple point before, in particular when Sebastian substantiated the other two integral theorems. The importance of the "holomorphicity ex machina"-frame in Sebastian's intuitive mathematical discourse is moreover supported with his utterance that "for me this is always the explanation" (line 275).

Similar to some of the substantiation by Uwe and Dirk (see Section 13.7, Section 14.7.2), Sebastian's substantiation is again one we call *retrospective*, and which he also recognises as such ("that somehow turns out afterwards that a holomorphic function is analytic", lines 268f.): Usually the analyticity of holomorphic functions is proven later than propositions about the existence of primitive functions for holomorphic functions (under suitable technical conditions, e.g., simple connectedness of the domain) in complex analysis textbooks or lectures, but Sebastian uses it here to explain the existence of primitives functions for holomorphic functions.

15.6.2 *Pseudo primitive functions*

Even though Sebastian describes that he does not "really have a better explanation for that" (lines 267f.)—likely his initial reaction to the task situation as given by the interviewer—he does in fact provide another explanation after the interview recalls the constructive of primitive as in Equation B.1 in terms of actions a mathematist may carry out. More precisely, Sebastian argues that the functions which are defined in an analogous manner for complex functions, which are not holomorphic, would result in "pseudo primitive functions".

171–Int.: Okay, but now I could say: Okay, I do it the same way in the complex setting. I have my function, I fix a point in my domain, let's say for the sake of simplicity, [Seb.: Mmh.] (.) and integrate along a path to somewhere else. [Seb.: Mmh.] Now, why isn't this a primitive function (.) in general? Why doesn't that work?

Seb.: Alright, okay, this is Cauchy-Riemann again, yes, because with differentiation, uh, one does not only see this path but all directions, so to speak. [Int.: Mmh.] Yes, of course one has created a pseudo primitive function with respect to this path integral, so to speak, [Int.: Mmh.] that is clear, but it is not complex differentiable, because complex differentiable sees the other directions,[174] too, so to speak. [Int.: Mmh.] At least this is my imagination, [incompr.] whether this is correct.

Int.: Okay, yes, this is interesting, yes. So, (..) since I can have different paths [Seb.: Yes.] my function changes differently at each point of these paths, probably/

Seb.: Yes and Cauchy's integral theorem says that if/ assume it was differentiable, then it wouldn't matter which path I had taken, [Int.: Mmh.] but this also means that this integral sees everything, so to speak. [Int.: Mmh.] And, uhm, you cannot expect from a naive integral that this can be done. [Int.: Mmh.] Yes and at this point the weakness of continuity becomes apparent, so to speak.

Int.: So the holomorphicity of the function enables the integral to see (.) what happens everywhere. [Seb.: Yes.] Just because it went there along one path.

Seb.: Exactly, that's how I would say it. And if one starts with a continuous function and integrates a path (German: einen Weg integriert), uh, and then, uh, wonders that it isn't holomorphic, [Int.: Mmh.] then you may say: Well, this is because the holomorphicity sees everything.

Even though Sebastian knows that the construction of a potential primitive functions suggested by the interviewer does not give rise to a complex differentiable function if the original function is not holomorphic but continuous only, he takes up this suggestion nevertheless and baptises this potential primitive function as a "pseudo primitive function" (line 287). He explores this new function and describes it to be at least differentiable with respect to the direction of the path used to construct the pseudo primitive function.[175]

Using specifically mathematical signifiers, we may rephrase Sebastian's narrative as follows: If f is a continuous complex function, the pseudo primitive function is given by $F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta$, where γ_z is a certain path from a fixed point to z in the domain of f (the path is not

174 In the original German, Sebastian used the adjective "komplex differenzierbar" as a noun: "[...] weil komplex differenzierbar eben sozusagen auch die anderen Richtungen sieht". Therefore, "complex differentiable" is used here as if it was the noun "complex differentiability".

175 The keyword "Cauchy-Riemann" also signals that Sebastian refers to directional derivatives in this episode.

further specified though). Then, F is differentiable at z with respect to the direction induced by γ_z, but not necessarily with respect to other directions.[176]

In order to substantiate, why the construction of F works for holomorphic functions, Sebastian follows again the **"holomorphicity ex machina"-frame**: The complex differentiability (of the original function) "sees" the other directions, too (line 289f.). We hypothesise that "other directions" refers to the directions not induced by the path of integration.

One very apparent feature of this episode is the way Sebastian talks about holomorphicity. He talks about it as if it were a mathematist who can "see" and look into different directions and check whether the pseudo-primitive function is differentiable with respect to these directions. This way of talking about mathematical objects as if they were human agents was also present in Uwe's intuitive mathematical discourse about complex path integrals. In Uwe's case, however, it was residue theorem telling him the meaning of complex path integrals (Section 13.4), whereas here in Sebastian's case, it is in particular the property of holomorphicity, which he described as if it was human and which could "see". Eventually, this way of talking about properties of functions was also taken up by the interviewer when paraphrasing Sebastian's narratives (e.g., in line 301f., the interviewer hypothesised that the holomorphicity of the original functions enables the (complex path) integral to "see" as well).

15.7 SUMMARY OF SEBASTIAN'S INTUITIVE MATHEMATICAL DISCOURSE ABOUT COMPLEX PATH INTEGRALS

In this chapter, two new frames for guiding intuitive mathematical discourse about complex path integrals were reconstructed: the **"mean value"-frame** and the **"holomorphicity ex machina"-frame**.

As in the two previous cases, we show the distribution of discursive frames from the interview with Sebastian in Table 15.1 and highlight whether they were used for the construction of a discursive image about complex path integrals and/or for the substantiation of one or more of the integral theorems.

The "mean value"-frame

The **"Mean value"-frame** consists of metarules for the construction and endorsement of narratives about complex path integrals, mean values, measuring, and the mathematists working with them. It contains the interpretation that integration in general is the process of forming a mean value or of measuring. Accordingly, users of this discursive frame may adapt this general interpretation to the specific case of complex path integrals as well.

The discursive images Sebastian produced enacting this discursive frame can be summarised as follows:

- "*Integration is mean value formation*" and "*Integration is measuring*". Sebastian prefers to regard integration as mean value formation and a process of measuring. He explains that he advises his students to follow this interpretation, too, particularly, when working with Lebesgue integrals. Moreover, Sebastian describes the area interpretation for Riemann integrals as a coincidence.

176 Here, we call F differentiable at z with respect to the direction induced by γ_z if the limit $\lim_{h\to 0, z+h\in \mathrm{tr}(\gamma)} \frac{F(z+h)-F(z)}{h}$ exists. We need to assume further that each γ_{z+h} is the juxtaposition of γ_z followed by a segment between z and $z+h$ on $\mathrm{tr}(\gamma)$ (González, 1992, ch. 6.9).

Table 15.1: Overview of discursive frames in Sebastian's intuitive mathematical discourse about complex path integrals.

Discursive frame	Construction of a discursive image	Substantiation of an integral theorem
(F1) "restriction of generality"	Section 15.3.2	
(F2) "theorematic"		Section 15.6 (existence of holom. primitives)
(F3) "vector analysis"		
(F4) "tool"		
(F5) "no meaning"		
(F6) "area"		
(F7) "mean value"	Section 15.2	Section 15.4 (Cauchy's integral theorem), Section 15.5 (Cauchy's integral formula)
(F8) "holomorphicity ex machina"	Section 15.3	Section 15.4 (Cauchy's integral theorem), Section 15.5 (Cauchy's integral formula), Section 15.6 (existence of holom. primitives)

- "*The complex path integral is the mean value of the values of the integrand*", "*The complex path integral measures the rotation on the plane*", and "*The complex path integral is the average rate of turning and the average rate of stretching as induced by the integrand*". This set of discursive images explicitly relates complex path integrals to mean values. The mean value is realised in terms of three closely related quantities, namely the function values of the integrand on the trace of the path of integration, the rotation as induced by the function values considered as multiplication with a complex number, and as paraphrased by the interviewer, the turning and stretching part of this rotation separately.

The "holomorphicity ex machina"-frame

The **"holomorphicity ex machina"-frame** contains of the valuation of the holomorphicity property as a very rigid property, according to which local properties of complex functions transfer to global properties. Users of this frame present this property of complex functions as a deus ex machina, which virtually enforce an integral theorem to be true or to be at least plausible and potentially to be expected. It is moreover particularly characteristic for this frame that no or almost no further reasons are given for the substantiations the user of this frame is involved in.

Sebastian and the interviewer-researcher jointly produced the following discursive images related to this discursive frame:

- "*The complex path integral can detect whether a holomorphic function has a singularity in the interior of a closed path of integration*" and "*Complex path integrals can measure global properties of holomorphic functions*". These narratives related complex path integrals to properties of holomorphic functions. Both share the underlying idea that information about the holomorphic functions (e.g., whether it has a singularity or in terms the values along a path) can be related to complex path integrals of these functions. Since the sets of functions and paths are restricted to holomorphic functions and closed (piecewise continuously differentiable) paths, these discursive are a product from the "holomorphicity ex machina"- and the "restriction of generality"-frames.

Intuitive substantiations of the three integral theorems

Sebastian used the **"mean value"-** and **"holomorphicity ex machina"-frame** to substantiate the three integral theorems in a consistent way with respect to the discursive images previously produced. In accordance with the **"mean value"-frame**, he substantiated Cauchy's integral theorem in terms of the narrative that the integrand did not change on average on the path of integration and with the help of the metaphor of a car driving on a mountain trail. The integral in Cauchy's integral formula was realised as the mean value of the transformed integrand $\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$ rather than f itself. Therefore, Sebastian's mean value interpretation differed from Uwe's and Dirk's (and also the interpretation endorsed in the literature) that the integral in Cauchy's integral formula realised the mean value of f along the circle of integration.

Having endorsed the narrative that local information about holomorphic functions determine a holomorphic function as a whole, Sebastian also applied the **"holomorphicity ex machina"-frame** to substantiate Cauchy's integral theorem, Cauchy's integral formula as well as the existence of primitive functions for holomorphic functions (given suitable technical constraints) and the non-existence of primitive functions for merely continuous functions. For the latter, he also derived the metaphor of a "holomorphicity trap", which essentially replaces the narrative that holomorphic functions are analytic and therefore, continuous functions, which are not holomorphic, cannot have holomorphic primitives. In all of the propositions, Sebastian identified the rigidity of holomorphic functions and the interplay of their local and global properties as the reason to outright expect these or similar theorems.

Closing remark

Altogether, Sebastian reflected clearly, which narrative he considered to belong to his mental images or intuitive explanations of complex path integrals and integral theorem, some of which he even advises students to use, too. He also reflected that his intuitive explanations of the integral theorems may not be fine-grained enough to account for the technical details needed to endorse them in scholarly discourse (e.g., this concerned the question whether continuity instead of holomorphicity of the integrand was sufficient in Cauchy's integral theorem).

Part IV

DISCUSSION

16

DISCUSSION AND DIRECTIONS FOR FURTHER RESEARCH

In this last chapter, we discuss the main contributions of this thesis, its limitations, and opportunities for further research.

16.1 OVERVIEW

This thesis contributes to the education of complex analysis, a topic that has not yet received much attention in university mathematics education so far, namely to broaden

- our understanding of how the topic of complex path integrals is structured from an epistemological perspective, and
- our understanding of how experts interpret complex path integrals individually and may realise them intuitively for themselves and potentially for others.

As such, this thesis is a melange of pure basic research and use-inspired basic research (Stokes, 1997). The relevance of our thesis stems from the need to teach complex path integrals in a better way to students (Gluchoff, 1991, cf. e.g.,), and more generally, to find didactical approaches for how to connect the topic to previous discourses from real analysis. The thesis also contributes to basic research for mathematics education at the tertiary level by combining German subject-matter didactics (e.g., Greefrath et al., 2016a, 2016b; Hefendehl-Hebeker, 2016; Hußmann et al., 2016) with the commognitive framework (e.g., Lavie et al., 2019; Sfard,

2008) for both, the reconceptualisation of normative aspects of a mathematical notion and individuals' interpretations thereof.

Epistemological and empirical research in mathematics education has endorsed various interpretations of real integrals of functions of one real variable (e.g., Greefrath et al., 2016a, 2016b; Jones, 2013, 2015, 2018; Thompson, 1994; Thompson & Silverman, 2008; Thompson & Harel, 2021, and lots of others). For instance, based on German subject-matter didactics, mathematics educators have endorsed various approaches to define and teach real integrals as well as investigated learner's and expert's individual ≪mental images≫ [177] of these integrals (e.g., Greefrath et al., 2016a, 2016b, 2021a). Opposed to these findings, epistemological and empirical research on educational issues on complex analysis is still rare. Reports in the literature have underlined for roughly 40 years now (e.g., Braden, 1987; Gluchoff, 1991; Hancock, 2018; Howell et al., 2017a; Oehrtman et al., 2019; Soto & Oehrtman, 2022) that complex analysis, and in particular complex path integrals, are a demanding and even "mystif[ying]" (Gluchoff, 1991, p. 641) experience for students and experts. Even though some meaningful interpretations of complex path integrals were found (e.g., the flow and flux of vector fields, Braden, 1987; Needham, 1997; Polya and Latta, 1974, or mean values, Gluchoff, 1991), our review of the literature on complex analysis has underlined that these interpretations have not received widespread attention. In this line, Oehrtman et al.'s (2019) case study on experts' construction of meaning for complex path integrals shows that even experts consider it a hard task to interpret them geometrically or physically. Their interviewed experts rather resorted to formal analogies to real integrals and only one of their experts provided an idiosyncratic metaphor about the navigation of a ship (Oehrtman et al., 2019). Empirical research in complex analysis thus did not arrive at a consensual set of interpretations for complex path integrals within experts.

Furthermore, there has not yet been a comprehensive epistemological analysis of complex path integrals which could elicit potential approaches to introduce the complex path integral in relation to varying mathematical prerequisites (e.g., on the class of functions to be integrated, the class of paths to be considered). However, as various instances of the cross-curricular topic "integrals" (cf. Kontorovich, 2018b) appear in undergraduate mathematics curricula, we present various potential definitions or interpretations of complex path integral, which may be grounded in definitions or interpretations for their real analogues.

In sum, we addressed two complementary research questions (see Table 16.1). To address these questions, the thesis is divided into three parts. Part i covers the necessary theoretical conceptualisations for the thesis, in particular for our empirical investigation in Part iii on research question B. Part ii deals with research question A and is also partly based on the commognitive framework presented in Part i as well as subject-matter didactics. Henceforth, this thesis contains a systematic and comprehensive analysis of potential approaches to and interpretations of complex path integrals as endorsed by experts in complex analysis literature (and partly beyond since we found a new axiomatic approach). It also contains three case studies of experts' intuitive interpretations of complex path integrals and how they make sense of these objects for themselves and potentially for others. The focus of the commognitive framework on discourses allows us to tackle our questions from an overarching theoretical perspective, in which public endorsed discourses and individuals' discourses are considered as two facets of the same.

177 Recall that we use the brackets ≪. . . ≫ in order to refer to a keyword, which appears in colloquial discourses and which has been conceptualised in several different manners in mathematics education. Hence, ≪. . . ≫ does not signify any of these conceptualisations in particular.

Table 16.1: Research questions.

Research questions	Answered in
A: What are the subject-specific approaches to complex path integrals and how are they embedded into different mathematical areas?	Part ii: Epistemological analysis of complex path integrals and previous research in complex analysis education
B: i) What are experts' individual ways to interpret complex path integrals? ii) What do their mental images of these mathematical objects look like? iii) How to experts explain central integral theorems from complex analysis intuitively?	Part i: Theoretical framework Part iii: Experts' intuitive mathematical discourses about integration in complex analysis

For research question A, our integration of German subject-matter didactics with a commognitive perspective allowed us to close the above mentioned research gap and to perform a solid epistemological analysis of the topic of complex path integrals. Based on our theoretical considerations we could specify the first part of research question A to

A': Which mathematical *aspects* characterise complex path integrals and the use of complex path integrals in complex analysis?

We identified four *aspects* and four *partial aspects* characterising the endorsability of the notion of the path integral within different mathematical discourses. Here, aspects of complex path integrals are object-level narratives, which may function as a definition of the complex path integral of a continuous function along a (piecewise) continuously differentiable path or turn out to be equivalent to another definition. Aspects are distinguishable by the use of signifiers and keywords in their definientia, which accordingly stem from different discourses (see Section 6.3). We considered an aspect of complex path integrals as a partial aspect if additional constraints on the paths or integrands are needed for their endorsement. From a discursive perspective, this epistemological analysis equipped us with a thorough understanding of the endorsed discourse on complex analysis in the literature at large, which could then also be used as a comparative perspective for parts of our following commognitive analysis of the expert interviews.

For the complex of research questions B, we used the commognitive perspective to develop the notion of *intuitive mathematical discourses* for our study of experts' intuitive approaches to complex path integrals. In our instrumental case study (Grandy, 2010; Stake, 1995, 2006) with three expert interviews "at the eye-level" (Gläser & Laudel, 2010; Pfadenhauer, 2009), we provided a "*non-deficit, non-prescriptive, context-specific, example-centred* and *mathematically focused*" account of experts' intuitive mathematical discourses about complex path integrals, as Nardi (2016, p. 361, emph. orig.) would call it. Here, our central contribution is that we took a discursive perspective, which allowed us to conceptualise (1) *discursive images* as the commognitive counterparts to ≪mental images≫ and (2) *discursive frames*, which are sets of metarules governing experts' constructions of discursive images and interpretations of complex path integrals. While our results on experts' discursive images agree with Oehrtman et al. (2019) in the sense that our experts did not produce a consensual set of discursive mental images of complex path integrals, the notion of discursive frames turned out to be instrumental to identify commonalities in how experts structure their intuitive discourses about complex path integrals. Based on these theoretical constructs, we can specify the complex of research questions in B to

B': i) What characterises experts' intuitive mathematical discourses about complex path integrals?

ii) What are their discursive images about complex path integrals?

iii) How do experts substantiate central integral theorems from complex analysis intuitively?

In sum, we discovered nine discursive frames and ten discursive images about complex path integrals. In particular, these discursive frames governing experts' intuitive mathematical discourses about the complex path integral illustrate both, idiosyncratic and inter-individual metarules our experts followed when describing their intuitive understanding of these mathematical objects. We discuss our empirical findings including these discursive frames and discursive mental images Section 16.4. Thus, our commognitive perspective extends our knowledge about experts' intuitive interpretations of complex path integrals significantly with respect to how they construct discursive images in conversations with a mathematics educator on equal footing (Pfadenhauer, 2009).

During the course of the thesis, we analysed two kinds of discourses on complex path integrals as led by experts, namely the endorsed scholarly discourse and exemplars of their intuitive mathematical discourses. The results thus obtained are important because experts are leading figures in teaching mathematics at university and catalysts for students' learning of complex analysis. In the long run, we hypothesise that the results from this thesis can be used to clarify how experts' may design their teaching of complex path integrals to support the learning of integrals in complex analysis, as well as to incorporate the learners' perspectives (cf. Hahn & Prediger, 2008; Kattmann & Gropengießer, 1996; Kattmann et al., 1997; Prediger, 2005; Reinfried et al., 2009).

We understand our epistemological analysis as a specific approach to the historical-genetic development of the discourse on complex path integrals at large. By reconceptualising the notions of (partial) aspect (Greefrath et al., 2016a, 2016b; Roos, 2020; Section 16.3) and of ≪mental image≫, in discursive terms (Lavie et al., 2019; Sfard, 2008; Section 16.2), we made concepts from subject-matter didactics understandable from the commognitive framework (cf. Bikner-Ahsbahs & Prediger, 2010, 2014; Prediger et al., 2008). Nevertheless, in Part ii the subject-matter didactical focus dominates, while in Part iii the commognitive focus dominates.

We will now subdivide our discussion of our main results into three parts in accordance with the structural organisation of this thesis at large (Part i, Part ii, and Part iii):

1. *Contribution to theory* including a reconceptualisation of individuals' mental images of a mathematical notion from a discursive perspective in terms of intuitive mathematical discourses, which includes the broadening of the unit of analysis from a cognitive unit to discourse (Section 16.2);

2. *Contribution to subject-matter didactics of complex analysis* including an epistemological analysis of complex path integrals from a normative, subject-matter didactical point of view to understand complex analysis discourse as endorsed at large (Section 16.3);

3. *Contribution to empirical research in complex analysis* including a reconstruction of discursive frames and discursive frames in experts' intuitive mathematical discourses about complex path integrals (Section 16.4).

16.2 CONTRIBUTION TO THEORY

In reviewing the literature on complex analysis education and ≪mental images≫ in the broadest sense, we observed the empirical gap that little was known about potential ≪mental images≫ or intuitive interpretations of complex path integrals as opposed to their counterparts in real analysis. However, we need to know such images or interpretations in order to develop teaching-learning materials for improving teaching of complex analysis in the long run.

Several theoretical approaches dealing with ≪mental images≫ of mathematical objects and related concepts have been proposed in mathematics education literature before. Our review in Chapter 2 has shown that many of them were either dominated by a prescriptive view or the notion of ≪mental images≫ itself was only weakly or little conceptualised and made researchable in terms of their empirical ascertainability. Additionally, none of them accounted for the complexity of mathematical objects at tertiary level and allowed us to investigate mathematical experts' individual interpretations of these objects in a theory-driven and exploratory way. Additionally, a first review of the data from our interviews supported Oehrtman et al.'s (2019) finding that experts do not share a consistent set of mental images for complex path integrals or even struggle to express such at all. Therefore, we found that a discursive approach would account for the complexity of our data for it shifted our view towards a broader unit of analysis, namely discourse instead of the single mental image or interpretation (Sfard, 2008). In particular, since many studies have shown that mathematists at all levels rely heavily on their individual interpretations and idiosyncratic approaches to mathematical concepts (for the case of experts see e.g., Burton, 2004; Davis et al., 2012; Fischbein, 1987; Heintz, 2000; Hersh, 2014; Kiesow, 2016; Nardi, 2008; Sfard, 1994; Weber et al., 2014, and others), which are not necessarily based on mathematical rigour as required in scholarly discourses (cf. Sfard, 2014; Viirman, 2021), it is plausible that these specific ways of mathematical thinking are indeed special types of mathematical discourses. Thus, in order to investigate research question B we first needed to conceptualise these discourses and a commognitive counterpart for ≪mental image≫ and we arrived at B'. Doing so, we closed a gap in existing theoretical frameworks on mental images and the commognitive framework, and it enables us to perform non-prescriptive, exploratory, theory-driven account for individuals' interpretations of mathematical objects.

This theoretical work allowed us to conceptualise intuitive mathematical discourses (including their constituents discursive images and discursive frames) about a mathematical notion (Chapter 4), which is illustrated in Figure 16.1:

- *Intuitive mathematical discourses* about a mathematical notion are mathematical discourses, in which metarules of scholarly mathematical discourses are optional. Instead, mathematists may use other metarules, they believe are better suited to make the mathematical object accessible. Individuals' engaging in intuitive mathematical discourses aim to make the mathematical object accessible for themselves or for others.

- Narratives in intuitive mathematical discourses, in which the discursant relates the mathematical object in question to what she or he considers to be her or his mental

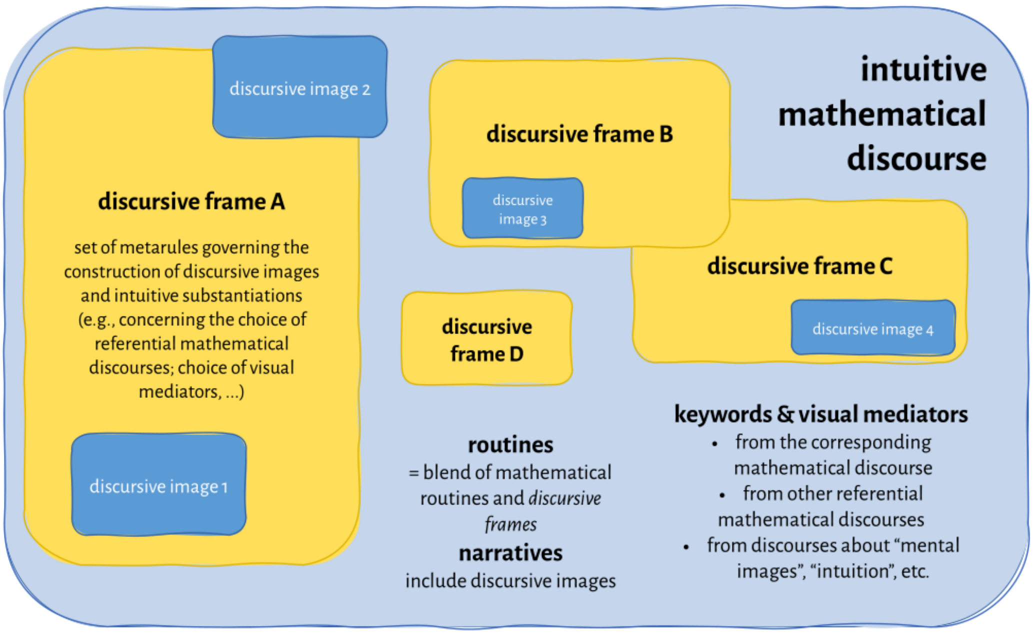


Figure 16.1: Venn diagram illustrating an intuitive mathematical discourse (repetition of Figure 4.1).

image or otherwise intuitive interpretations, are called *discursive (mental) images*. Discursants may clearly use visual mediators to reinforce their discursive images and hence use other than linguistic means to express themselves. Henceforth, discursive images are not based on a previously fixed version of what a mental image or intuitive interpretation may be for an individual, they do not have to be pictorial, but they are products of the intuitive mathematical discourses as enacted by the respective discursants. In particular, we may detect discursive images as narratives quite directly in empirically gathered data and do not have to look for them in discursants' heads.[178]

- Inspired by Heyd-Metzuyanim et al.'s (2018) commognitive reformulation of the idea of *frame* (cf. Goffman, 1974/1986; Hammer et al., 2005), we defined a *discursive frame* in an intuitive mathematical discourse as a set of metarules guiding mathematists' explorations (constructions and substantiations) of discursive images or mathematical propositions. Discursive frames and other mathematical routines constitute the routines in intuitive mathematical discourses.

  With this regard, enacting a certain discursive frame may lead to the construction and/or substantiation of a discursive image, but does not necessarily have to (see below for the summary of discursive frames and discursive images). The notion of discursive frame particular takes into account our need for a theoretical construct in order to analyse commonalities in our experts' intuitive discourses. The notion of discursive image alone did

178 Even though we use the adjective "mental" here, this is intended as no more than a reminder that these special narratives reflect the individuals' judgements of what is, for themselves, a mental image about the mathematical objects in question. That is, we do not assume that ≪mental images≫ exist but that mathematists use this and related keywords in order to describe mathematical object in a special way, which is potentially set apart from scholarly mathematical discourses based on rigorous definitions and strict metarules for explorations (Sfard, 2014; Viirman, 2021). Hence, discursive images in our sense may result from discursants' precedents in other discourses, in which mathematical objects were related to the keyword "mental image" (or similar keywords), too (e.g., precedents from real analysis, where the basic idea of area is endorsed as one of several instances of mental images for Riemann integrals).

> not suffice because our experts' discursive images turned out to be mostly distinct. Thus, focusing on discursive frames enabled us to analyse the fabrication of our experts' intuitive mathematical discourses about complex path integrals in more detail. This notion enabled us to arrive at a clearer picture of commonalities and differences in our interlocutors' discourses, which would have remained hidden if we had not broadened the theoretical perspective on mental images and intuition in relation to the specific mathematical notion under investigation (see Section 16.4).
>
> As discursive rules "are not anything that would be followed by discourse participants in a conscious, intentional way", we understand discursive frames rather as "retroactively written into interlocutors' past activities", which may "reappear, possibly in a slightly modified version, in these interlocutors' future actions" (Sfard, 2008, p. 203). In other words, discursive frames are "analytical statement[s] with which the observed performance is consistent" (Kontorovich, 2021b, p. 5).

Keywords and metalevel rules in these intuitive discourses about a mathematical notion may be borrowed from mathematical discourses about that notion. However, as described above, mathematical routines may be enhanced or replaced by discursive frames, depending on the respective implementation of the intuitive mathematical discourse (see Figure 16.1). Intuitive mathematical discourses may be enacted in various ways (e.g., as parts of mathematical lecturers, as conversations between two experts, etc., or in terms of interviews "at the eye-level" (Pfadenhauer, 2009) as in the empirical part of this thesis).

This new discursive conceptualisation has several advantages compared to acquisitionist conceptualisations. It focuses on the empirically ascertainable and proposes a new mathematics educational discourse on ≪mental images≫ and ≪intuition≫ that is free of invisible acquisitionist or cognitive categories (cf. Sfard, 2006, 2009a, 2009b).

Our conceptualisation of discursive image is oriented towards discursants' individual thinking (= communication) processes and their own judgements of what they count to belong to their mental images etc. reflected in their use of the respective keywords (Section 2.3; cf. Wittgenstein, 1953/2009). In particular, it is not primarily subsumptive in the sense that individuals' narratives are subsumed under previously formulated ≪mental images≫ of one kind or another. Since the unit of analysis has changed to discourse, our new point of view on ≪mental images≫ is also intrinsically interactive. Moreover, conceptualising this special type of thinking as an own type of discourse finally allows to apply the repertoire theoretical notions provided by the commognitive framework. That is, we get more insight into how experts govern their intuitive interpretations or individual ways to make sense of a mathematical notion since we extend our focus. For instance, we refrain from focusing on images in a pictorial sense and turn towards experts' (construction and endorsement of) narratives more generally. This was indeed very beneficial for our study since the visual mediators our experts used hardly ever realised the number $\int_\gamma f(z)\,\mathrm{d}z$ directly apart from mathematical formulae (cf. Section 16.4). Thus, the discursive perspective allowed us to capture the abstractness of the mathematical object "complex path integral".

Focusing on intuitive mathematical discourses instead of single ≪mental images≫, we also take into account the "*principle of interdiscursivity*" (Sfard, 2013, p. 159, emph. orig.): According to this principle, "instances of 'pure' discourses are rare, that real-life conversations make use of multiple discourses, and that our participation in one discourse is affected, sometimes only from afar, by [a; EH.] plethora of other discourses" (Sfard, 2013, pp. 159–160). Here, such an interdiscursivity in intuitive mathematical discourses is likely because there a rather well-known interpretations for other integrals and previous research has shown that discursants refer to these integrals when asked for interpretations of complex path integrals (Hancock, 2018; Oehrtman et al., 2019). From a scholarly perspective, we observed this principle in our epistemological analysis in Part ii, too. This is also reflected in some of the (partial) aspects, which are endorsable in different related mathematical discourses (see the next section).

16.3 CONTRIBUTION TO SUBJECT-MATTER DIDACTICS OF COMPLEX ANALYSIS

Since we deal with experts' intuitive mathematical discourses on complex path integrals on the one hand, it is also expedient to have analysed the approaches to complex path integrals in scholarly discourse about complex path integrals, too—after all, this is another type of experts' discourse. This is our second main contribution in form of an epistemological analysis of complex path integrals (Part ii). It is in line with Winsløw et al.'s (2021, p. 74) demand for "content-specific mathematics education knowledge (didactics) [... for] more advanced subjects (e.g., linear algebra) and their teaching at university level". This study has made an important contribution for this endeavour. Our comprehensive analysis has been unprecedented in complex analysis education and is a useful starting point but also point of reference for our and future empirical studies (cf. Hochmuth, 2021b): We need to focus on endorsed narratives about complex path integrals and metarules for how mathematists can operate with them as the background for the study of experts‘ intuitive discourses. After all, it is reasonable to assume that experts are familiar with a lot of facets of endorsed scholarly discourses and since our interviews were planned as conversations "at the eye-level", we had to ensure that we had reconstructed the approaches to complex path integrals in the scholarly discourses beforehand. We also claim that in the future we will be able to develop didactically informed incentives for teaching complex analysis based on our analysis.

Since definitions are central in scholarly mathematical discourses (Sfard, 2008; Viirman, 2021) one of our main foci in the epistemological analysis were definitions and their substantiations. The other were geometrical and physical interpretations from the literature. To this end, we have analysed approximately 50 textbooks, lecture notes, historical sources, and journal articles to reconstruct the main ways of introducing complex path integrals (see Section 6.2). In our analysis, we examined differences in the definitions and other object-level narratives as well as their substantiations with respect to different constraints on the integrands and paths, grouped them, and thus abstracted the aspects and partial aspects (see also Figure 6.1). We also highlighted curricular cross-connections to other mathematical discourses throughout the epistemological analysis (e.g., real analysis or measure theory). Additionally, we traced the historical developments of the discourse about complex path integrals from the first use of integrals of complex functions over Cauchy's first definition of complex path integrals to its use in modern complex analysis texts and their role in theory development of complex analysis related to analytic functions. Overall, our epistemological analysis systematically explores the approaches to ad interpretations of complex path integrals, which are consolidated today. A central mathematical result from our epistemological analysis led to a new axiomatic characterisation of complex path integrals for which we were inspired by the axiomatic approaches to integrals known from real analysis or measure theory (Burgin, 2012; Daniell, 1918; Gillman, 1993; Herfort & Reinhardt, 1980; Pickert, 1976; Shenitzer & Steprāns, 1994; Taylor, 1985). Hence, this thesis contributes to research mathematics in complex analysis as well.

In order to summarise our central findings of the epistemological analysis, we reconceptualised the notion of aspects and partial aspects in commognitive terms. These notions were first introduced by Greefrath et al. (2016a, 2016b) and Roos (2020). Based on the most frequent case observed in the literature and used in complex analysis, we have specified that a general definition for complex path integrals should realise the signifier $\int_\gamma f(z)\,\mathrm{d}z$ is defined for a continuous integrand f and an at least piecewise continuously differentiable path γ. In this context, we say that an *aspect for complex path integrals* is a narrative, which can function as a definition or turns out to be equivalent to a definition of the complex path integral in the general case. Aspects are set apart by the keywords and signifiers used in their definientia (Section 6.3.1) and stem from different mathematical discourses. A *partial aspect for complex path integrals* was defined as an aspect which is however only endorsable for additional constraints on the integrands or the paths. In this respect, our epistemological analysis consisted of a critical examination of current as well as historical documents on complex path integrals, which in sum describes the phylogenetic development of discourses on complex path integrals from which the respective aspects emerge.

16.3.1 *Discussion of the aspects and partial aspects for complex path integrals*

Let $f = u + iv$ denote a continuous complex function on the trace of a path $\gamma\colon [a, b] \longrightarrow \mathbb{C}$.

Aspects

We reconstructed four aspects for complex path integrals (Chapter 9).

PRODUCT SUM ASPECT The product sum aspect can be understood as a transfer of the product sum aspect as described by Greefrath et al. (2016a, 2016b) for Riemann integrals. It characterises the complex path integral as a limit of complex Riemann sums of the form $\sum_{k=1}^{n} f(\xi_k)(\gamma(t_k) - \gamma(t_{k-1}))$ (Section 9.1.1). Since other integrals are also defined via limits of product sums, this aspect is endorsable in every discourse, in which such a definition is endorsed for these other integrals. As such, this aspects fits particularly nicely in most mathematics curricula. The geometric interpretation of complex product sums in terms of concatenations of vectors by Needham (1997) seems to be little known or used in the literature though.

SUBSTITUTION ASPECT The substitution aspect is motivated by a formal application of the substitution rule from real analysis (Section 9.1.2). Since complex analysis is usually taught after calculus / real analysis, this aspect is endorsable right away because it only requires the notion of Riemann integral, because this aspect essentially boils down to the definition $\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$. The textbook analysis has revealed that this aspect is widely used as the definition of complex path integrals in many of textbooks.

VECTOR ANALYSIS ASPECT The vector analysis aspect is based on the separation of the complex path integral into real and imaginary part, each of which are a real path integrals of second kind, one with respect to the tangential field and the other with respect to the normal vector field associated to γ (Section 9.1.3): $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{T} + i\int_\gamma \mathbf{w}_f\,\mathrm{d}\mathbf{N}$. The integrands in these two real path integrals are not f viewed as a vector field $(u, v)^T$ but instead has to be replaced by the so-called Pólya vector field $\mathbf{w}_f = (u, -v)^T$ associated to f ((Braden, 1987; Polya & Latta, 1974); Section 8.1.4, Section 8.2.2). This aspect is endorsable in vector analysis discourses, which provide the definition of the real path integrals appearing at the right-hand side of the defining equation for $\int_\gamma f(z)\,\mathrm{d}z$. It is also endorsable from the formal calculation $f\,\mathrm{d}z = (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) = u\,\mathrm{d}x - v\,\mathrm{d}y + i(v\,\mathrm{d}x + u\,\mathrm{d}y)$.

MEAN VALUE ASPECT The mean value aspect (Section 9.1.4) is related to what Greefrath et al. (2016a, 2016b) call the basic idea of "average value" (Section 2.2.3). However, here, a full aspect could be developed based on the idea of averaging. Following Gluchoff (1991), the mean value aspect of complex path integrals does not use the mean value of the integrand f itself but instead of $f \cdot T$. Here, T denotes the unit tangent vector attached to the path of integration viewed as a complex number (Section 8.2.3).

Partial aspects

We reconstructed four partial aspects for complex path integrals, each of which is endorsable under additional constraints on the class of integrand or paths (Section 9.2).

ANTI-DERIVATIVE PARTIAL ASPECT The anti-derivative partial aspect characterises the complex path integral of a function f along γ as the difference $F(B) - F(A)$ whenever F is a holomorphic primitive function for f such that its domain contains the trace of γ, A is the start, and B the end point of γ (Section 9.2.1). The existence of a holomorphic primitive function for f is however only guaranteed if f is holomorphic on (or holomorphically extendable to) a simply-connected domain in $\mathbb{C}$ hence, this partial aspect is only suitable for holomorphic function (Section 8.3.1). Since holomorphic functions can be shown to have primitives on simply-connected domains (or locally on more general domains) without the use of integration, this aspect is in fact endorsable in a complex analysis discourse dealing with holomorphic functions.

GREEN-TYPE PARTIAL ASPECT The Green-type partial aspect is based on ideas by Trahan (1965) (Section 9.2.2). It characterises complex path integrals for functions f, whose corresponding vector field $\mathbf{f}$ is continuously partially differentiable, and simple closed (piecewise) continuously differentiable (or more generally simple, closed, rectifiable) paths in terms of a double integral of δf over the interior of the path, where δ is a certain differential operator (Section 8.1.5).

RESIDUE-TYPE PARTIAL ASPECT The residue-type partial aspect characterises complex path integrals for closed (piecewise) continuously differentiable paths γ and holomorphic (instead of real totally differentiable integrands f as in the Green-type partial aspect), whose domain includes the topological closure of the interior of γ (Section 9.2.3): It is the finite sum $2\pi i \sum_{\omega \in A} \mathrm{Res}_\omega(f)\,\mathrm{Ind}_\gamma(\omega)$, where A corresponds to the set of (isolated) singularities of f (Residue theorem (Theorem A.29)). This aspect is usually a theorem in complex analysis discourses. However, it can be made at least plausible that such a formula should hold, and since the residues and winding numbers can be defined integration-free, the formula here may in fact be used as a definition, if one wishes to follow a rather unusual way.

AXIOMATIC PARTIAL ASPECT The axiomatic partial aspect comprises our new axiomatic characterisation of complex path integrals in terms of properties of mappings of the form $I : (\gamma, f) \longmapsto I(\gamma, f) \in \mathbb{C}$ (Section 9.2.4). Here, there are two inputs, namely paths and integrands. We were inspired by the list of frequently used object-level rules about complex path integrals we condensed to Properties 8.12, which characterise the use of complex path integrals for theory development in complex analysis (cf. Klazar, 2019a, 2019b, 2020), and the axiomatic approaches to Riemann or Lebesgue integrals (e.g., Daniell, 1918; Gillman, 1993; Herfort & Reinhardt, 1980; Pickert, 1976; Shenitzer & Steprāns, 1994; Taylor, 1985). We asked ourselves whether a subset of these object-rules about complex path integrals is sufficient to guarantee that there is one and only one mapping of the form I. Indeed, we proved an axiomatic characterisation of complex path integrals in terms of a mapping of the form $\wr\colon \mathcal{P}(\Omega) \times \mathcal{H}(\Omega) \longrightarrow \mathbb{C}$ (Theorem 8.13). Here, $\mathcal{P}(\Omega)$ denotes the set of paths in Ω and $\mathcal{H}(\Omega)$ the set of holomorphic functions on Ω.

We have also found several interpretations of complex path integrals in geometrical or physical contexts, which are partly based on the aspects: One describes the real and imaginary part as the flow (or work) and flux of the Pólya vector field associated to the integrand (Braden, 1987; Polya & Latta, 1974) and hence stems from the vector field aspect. The second corresponds directly to the mean value aspect and describes the complex path integral as a mean value (Gluchoff, 1991). A third interpretation is based on the geometric realisation of the product sum aspect, where each complex Riemann sum is realised as a concatenation of vectors in the place (Needham, 1997, ch. 8). Apart from the first, the other interpretations have hardly been received in the mathematics and mathematics education literature.[179]

We note though that none of our three cases and also the experts from Oehrtman et al.'s (2019) study could recite any of these interpretations correctly. There were a few exceptions though: Some experts in our and Oehrtman et al.'s (2019) study were aware that there is a physical application of complex path integrals; Sebastian (from our study) created an own idiosyncratic story about how complex path integrals represented certain mean values, and Rafael (from Oehrtman et al. (2019)) constructed an idiosyncratic version of the product sum aspect.

The mean value aspect provides opportunities for transferring mean value interpretations for Riemann or measure theoretic integrals to complex path integrals. The analogy is very subtle and also Cauchy's integral formula gives rise to another, but frequently mentioned mean value interpretation for the complex path integral. That is, lecturers may use the mean value aspect not as a definition but to evoke commognitive conflicts in order to elicit and highlight the specificities of the mean value interpretations in complex analysis (see Hanke, 2022b)

We hypothesise that the geometrical, physical, and mean value interpretations from the literature, in particular by Braden (1987), Gluchoff (1991), and Needham (1997) do not belong to experts' common knowledge in complex analysis, or maybe even that authors of complex analysis texts do not see a value in their inclusion to their texts. It remains to be investigated in further research how lecturers in complex analysis value these interpretations.

16.3.2 *Further perspectives for research about (partial) aspects of complex path integrals*

While we have compared the discursive images from our expert interviews with the (partial) aspects whenever points of connections were observed, it remains a task for further research to find out which precedents influences mathematists' (potentially unconscious) choices for framing their intuitive mathematical discourses about complex path integrals. While mathematics education for calculus and analysis has a long tradition and many researcher have contributed to it, this is not yet the case for complex analysis, and we cannot yet answer questions about longitudinal developments of intuitive mathematical discourses in complex analysis.

179 For instance, we could only find very few references to Gluchoff's (1991) paper, in which case this paper was not discussed content-wise (e.g., Burckel, 2021).

For instances, we may hypothesise that experts' special research areas influence how they interpret complex path integrals.[180]

It may be suspected that emphasising several of the (partial) aspects in a course on complex analysis explicitly is beneficial for students in the sense that they perceive more vertical coherence between the courses in their study programmes and that they can import techniques from one discourse on integrals to that of complex path integrals. This way, cross-curricular linkages between different courses of undergraduate mathematics curricula can be established, which may facilitate students' transitions from calculus / real analysis courses to complex analysis (cf. Hochmuth, 2021b; Hochmuth et al., 2021). This can be one step in creating coherence between different courses and prevent that students learn or memorise them in isolation (cf. Kondratieva & Winsløw, 2018; Kontorovich, 2018b; Winsløw et al., 2021).

Endorsement of (partial) aspects in university classrooms

A *first block* of questions for further research deals with the teaching and learning of complex path integrals depending on the choice of aspect for their introduction in a university classroom. A natural first step may be to evaluate the acceptance or use of the (partial) aspects within lecturers of complex analysis at large. We may hypothesise that different curricula may reinforce lecturers to choose one of the aspects over another and we may ask for lecturers' sensitivity towards their choices of definition in the given curricular context. Classroom observations or interviews with lecturers may then reveal how (partial) aspects are factually enacted in courses on complex analysis. Interviews with students or the collection of solutions to tasks may then give insights how learners re-enact their lecture's discourse on complex path integrals.

Another valuable step is to develop and conduct tasks or teaching experiments, in which single aspects are emphasised or in which connections to real analysis are made explicit. In this context, Soto-Johnson and Hancock (2018) have already presented a teaching proposal for the introduction of the amplitwist concept, and Hochmuth (2021a) and Kondratieva and Winsløw (2018) have done similar pioneering work on the vertical coherence between real analysis and nonlinear approximation theory, Fourier analysis, and vector analysis. For instance, we may construct tasks for students to compute the same complex path integral(s) with the help of different (partial) aspects, to engage them in exploring the prerequisites for the partial aspects, and to make them check whether they are satisfied or not.

Notwithstanding, we need to critically question the role of different aspects in teaching. Students are certainly confronted with increased demands if they have to be able to distinguish several aspects or identify them as equivalent. We must therefore ask whether students benefit at all if different (partial) aspects are addressed in teaching. It may well turn out that this well-intentioned approach may benefit some students but overwhelm others. In the latter case, it may be more promising to address the complex path integral consistently with regard to a single aspect.[181]

180 We evidence for this hypothesis: Sebastian framed parts of his intuitive discourse about complex path integrals with the mean value frame, which he used to subsume complex path integrals under the idea of averaging.

181 The teaching of complex analysis in service courses deserves separate attention, which we reserve here for the specialists in this area of mathematics education.

Analogies learners see and develop between real and complex analysis

A *second block* of questions centres around the analogies between real and complex analysis. More specifically, we may investigate whether and how students recognise the plethora of connections between real and complex analysis we exhibited in our epistemological analysis (e.g., the product sum aspect of integrals or analogical statements and proofs of important theorems such as the existence of primitives or potentials) or how they may be assisted in doing so. In this respect, the study by Soto and Oehrtman (2022) is promising: The authors observed that students can anticipate the product sum aspect of complex integrals from their precedents in real analysis.

Globally, we may inspect how large analysis cycles, which are quite common for German mathematics degrees and which are taught for several consecutive semesters, establish vertical coherence between integrals and how they introduce complex path integrals given they have introduced other integrals before. Locally, we may inspect whether and how lecturers construct tasks about complex path integrals for students with the aim to connect them to other integrals and even to which extent they consider this a useful and manageable task for their teaching practice.

Integration of the commognitive framework and subject-matter didactics

A *third block* of potential future research deals with the affordances of the use of the commognitive framework in subject-matter didactics. For well-elaborated theoretical constructs (e.g., the aspects for Riemann integrals; Greefrath et al., 2016a, 2016b), it is questionable which benefits the commognitive perspective entails. For example, in quantitative studies, in which students' answers are subsumed under previously established categories (e.g., these aspects), subject-matter didactical research has worked well so far (e.g., Greefrath et al., 2021a). Nevertheless, for qualitative studies in which we intend to find out which aspects are individualised by which students to which extend, a commognitive perspective such as the one we proposed for (partial) aspects is useful. Mathematists' narratives can then be examined with regard to their discursive origins (e.g., on the basis of textual indicators; cf. Morgan and Sfard, 2016) and compared with the narratives found in the (partial) aspects. For example, we may study mathematists' use of signifiers that appear in the (partial) aspects and related keywords or signifiers. One may also investigate in depth which task situations trigger a certain mathematist to use or construct certain narratives and we may identify the metarules mathematists use for the construction or use of the (partial) aspects.

Coherence in mathematics education teaching programmes

Last but not least, we believe that the case of complex analysis is suitable to discuss a *fourth issue*, coherence in mathematics study programmes, in general. For instance, the rich connections to other mathematical discourses we have shown in our epistemological analysis may reinforce lecturers of complex analysis to teach complex analysis in favour of "Klein's plan B" which aims at vertically connected curricula (Kondratieva & Winsløw, 2018). However, complex analysis may also be taught rather intrinsically, omitting these various connections, and lecturers may favour "Klein's plan A" instead, which is in alignment with the modularisation of study programmes and students' varying mathematical backgrounds who attend courses on complex analysis. More precisely, "Plan A" refers to

> a more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part for itself, with a minimum of resources and with all possible avoidance of borrowing from neighbouring fields (Klein, 1908/1932, cited by Kondratieva and Winsløw, 2018, p. 121).

As opposed to Plan A, "Plan B" refers to "the organic combination of the partial fields", which supports "an understanding of several fields under a uniform point of view" and "the comprehension of the sum total of mathematical science as a great connected whole" (Klein, 1908/1932, cited in Kondratieva and Winsløw, 2018, p. 122).

16.4 CONTRIBUTION TO EMPIRICAL RESEARCH IN COMPLEX ANALYSIS

Our instrumental case study (Grandy, 2010; Stake, 1995) strengthens our knowledge about mathematical experts' individual interpretations of complex path integrals. Our sample contained the three mathematical experts Uwe, Dirk, and Sebastian from two German mid-size universities with teaching experience in complex analysis. Methodically, we accessed experts' intuitive mathematical discourses in semi-guided expert interviews "at the eye-level" (Pfadenhauer, 2009). At the beginning of the interviews, it was specified that the interviewer was interested in what the experts would consider their potential mental images or otherwise intuitive explanations about central notions in complex analysis. A comfortable interview setting allowed both interlocutors in each interview to speak freely and to discuss their individual interpretations of complex path integrals. The interviewer started the part of the interview on complex path integrals with the basic idea of area (see Section 11.1.2), but also encouraged the experts explicitly to digress from this context.

In line with Oehrtman et al.'s (2019) findings, our initial review of the data from the interviews revealed no consistent set of discursive images about complex path integrals across the experts. Rather, we found nine quite different discursive images, none of which was explicitly endorsed in more than one interview.[182] The notion of discursive frame enabled us to find commonalities in experts' intuitive mathematical discourses about the complex path integral, which would have remained hidden if we had not broadened the theoretical perspective to metarules governing discourses (see Section 16.2). We have found eight discursive frames. Henceforth, the discursive conceptualisations in this thesis led to a new and rewarding perspective on individualisations of mathematical discourses.

16.4.1 *Discussion of discursive frames and discursive images in experts' intuitive mathematical discourses about the complex path integral*

Three of the discursive frames we identified occurred in more than one interview. We emphasise that these discursive frames were identified in intuitive mathematical discourses about complex path integrals but not for other mathematical objects. Hence, they are first of all specific for complex path integrals. However, we hypothesise below whether certain of these discursive frames might also be found in intuitive mathematical discourses about other mathematical objects as well.

182 Sometimes, we even observed the opposite. For instance, Uwe constructed the discursive image (D1) that the complex path integral is only a tool in complex analysis (see below). The interviewer asked Dirk whether he would approve this narrative. Dirk rejected it because in his opinion, this idea would be more of an excuse for having no mental image rather than a mental image at all.

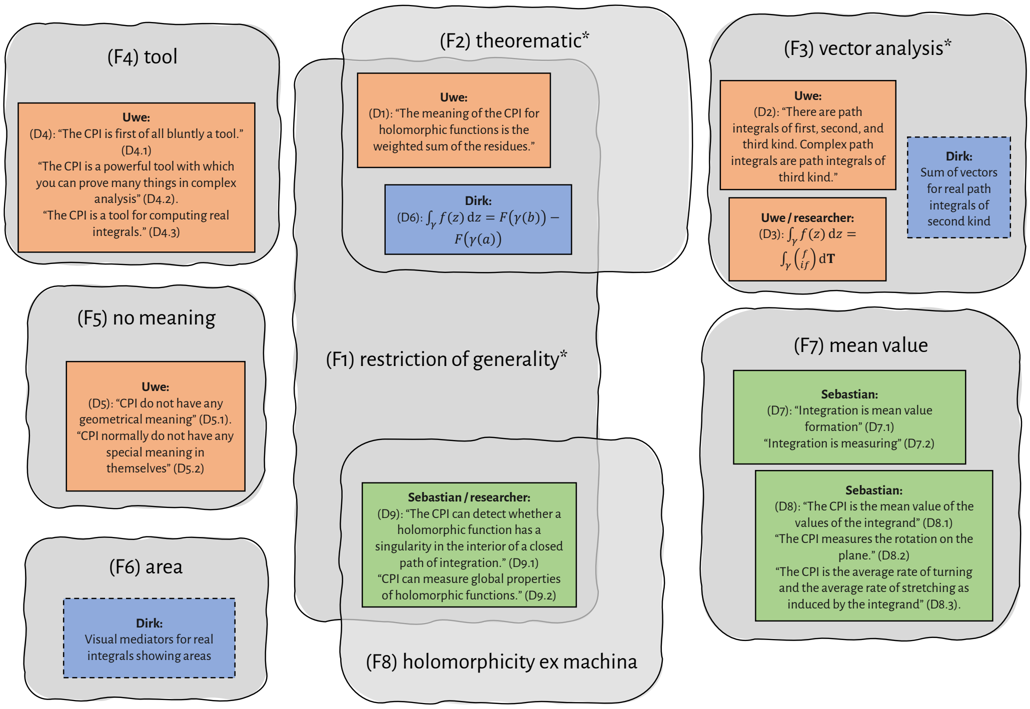


Figure 16.2: Discursive frames and discursive images from the three expert interviews (repetition of Figure 12.1).

Figure 16.2 shows the discursive frames from the three expert interviews as well as the discursive frames, which resulted from experts' applications of these frames (see Chapter 13, Chapter 14, and Chapter 15). The frames with an asterisk appeared in two interviews, while the frames without an asterisk appeared in only one interview. Since our sample contained only three experts, we may hypothesise that the frames, which more than one participant followed, are potentially shared across experts, but we cannot know whether the others are idiosyncratic only because of the small sample size.

Discussion of discursive images about complex path integrals

The discursive images are narratives about complex path integrals, which were produced as results of exploration routines in experts' intuitive mathematical discourses. That is, they are narratives about the complex path integrals, which the experts have endorsed as one of their mental images or as an explanation they deemed intuitive (according to their own usage of the keywords mental image, intuitive, or a similar keyword). The discursive images can be grouped with respect to the usage in the discursive context, in which they were produced (see Figure 16.2 for numbering of discursive images):

i. The potential "meaning" of a complex path integrals is derived from theorems (D1, D6).

ii. The complex path integral is valued with respect to applications (D4, D9).

iii. The complex path integral is connected to other mathematical concepts (real path integrals in (D2, D3); overarching interpretation of integrals in terms of mean values or measures (D7, D8)).

iv. The complex path integral is denied a meaning in its own right (D5).

In this respect, our experts have produced discursive images to describe a potential meaning of complex path integrals or an absence thereof (i. and iv.). Other discursive images are more functional and place complex path integrals into the context of their use in complex analysis or beyond (ii.). The remaining discursive images (iii.) may be described as links between the imaginary boundaries of mathematical discourses in the sense that they link complex path integrals to other integrals or to interpretations of other integrals. Especially, these discursive images suggest that its proponents identify complex path integrals as instances of a integrals as a "cross-curricular concept" (cf. Kontorovich, 2018b).

Discussion of discursive frames in experts' intuitive mathematical discourses about complex path integrals

We now turn to the discussion of discursive frames. Recall that we have summarised each of them before in the introduction to the results (Chapter 12) and in the case studies for each expert (Section 13.9, Section 14.9, Section 15.7).

We note already here that the discursive frames overlapped at multiple points during the interview. Accordingly, metarules were often superimposed to each other for the construction of individual interpretations of complex path integrals and for the intuitive substantiations of integral theorems in complex analysis (see also Section 16.4.2).

On the multiple used discursive frames: the "restriction of generality"-, "theorematic"-, and "vector analysis"-frames

We begin to discuss the three discursive frames, which appeared in more than one interview. These frames are thus universal for our sample of experts. We hypothesise though that these discursive frames are also universal beyond our sample (i.e., other mathematists may apply them in their intuitive mathematical discourses about complex path integrals, too):

(F1) "RESTRICTION OF GENERALITY"-FRAME This frame is characterised by the general mathematical metarule to restrict the attention to special cases before engaging in further explorations. Having restricted the level of generality, one is left with special cases and can try to explore mathematical narratives for these cases.

This general metarule is also frequently applied in scholarly mathematical discourses. For instance, mathematical objects in theorems are also usually restricted in their level of generality so that an object-level rule about them can be constructed and substantiated. Another application of this metarule is the construction of corollaries or the visual realisations of mathematical objects, which may occasionally not be obtainable in the general case. It is thus plausible that experienced interlocutors of scholarly mathematical discourses such as our experts in complex analysis transfer this general metarule to the construction of narratives in their intuitive mathematical discourses.

The "restriction of generality"-frame was only used together with the "theorematic"- and "holomorphicity ex machina"-frame for the construction of discursive images. For instance, Uwe restricted the class of paths to *holomorphic* functions instead of *continuous* functions at multiple points during the interview (sometimes explicitly, sometimes implicitly). Furthermore, he restricted the class of (piecewise continuously differentiable) paths to *closed* paths at various points during the interview, too. The discursive image (D1), in which Uwe identifies a meaning for complex path integrals in terms of a "weighted sum of residues", is a result of such a restriction of generality for the paths and integrands of complex path integrals. In a similar way, Sebastian and the interviewer-researcher constructed the discursive images in (D9), according to which complex path integrals can detect singularities of holomorphic functions in the interior of a path or that it can measure global properties of holomorphic functions. Moreover, the discursive image (D6), which realises a complex path integral as the difference of values of a primitive function of the integrand is also only endorsable if the integrand has a primitive function.

The "restriction of generality"-frame was also used in experts' intuitive substantiations of integral theorems. For example, Dirk's exploration of the vanishing of complex path integrals for holomorphic functions along closed paths was mostly restricted to a discussion of Goursat's lemma (Theorem A.19), or it was to realise Cauchy's integral formula (Theorem A.22) as a mean value property of complex path integrals for the special case $z = z_0$.

(F2) "THEOREMATIC"-FRAME Discursants, whose discursive behaviour during parts of the interview can be described with the "theorematic"-frame make use of theorems in complex analysis to construct discursive images or in their intuitive substantiations of other integral theorems.

The appearance of the "theorematic"-frame is noteworthy because with the help of this frame the experts govern their construction of narratives in intuitive mathematical discourses, too, not only in scholarly discourses. The theorematic images (D1) and (D6), based on this frame, are closely linked to the residue-type and anti-derivative partial aspect, which essentially amount to the same narratives. The difference between the construction of these narratives as partial aspects in scholarly discourses or in intuitive mathematical discourses is that in the former case they are endorsable as potential definitions and in the latter case they are constructed as explanations for the potential meaningfulness of the notion of complex path integral itself.

Similar to the "restriction of generality"-frame, the "theorematic"-frame may be used in scholarly mathematical discourses as well. Therefore, it can be assumed that experienced interlocutors may also include this frame in their intuitive mathematical discourses.

(F3) "VECTOR ANALYSIS"-FRAME The "vector analysis"-frame contains metarules with the help of which mathematical objects or propositions from complex analysis are connected to vector analysis and used to substantiate a narrative about complex path integrals in intuitive mathematical discourses. This discursive frame occurred for instance when discursants made use of integral theorems from vector analysis and applied them in their substantiations of theorems in complex analysis by means of analogical reasoning (cf. Bartha, 2019). The discursive image (D3) "$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma (f, if)^T\,\mathrm{d}\mathbf{T}$" was reconstructed from Uwe's discursive action to rewrite the integrands of a realisation of complex path integrals as the integrands of a real path integral of second kind for complex vector fields. Accordingly, he used a variant of (D3) to substantiate the narrative that Cauchy's integral theorem is no longer surprising once one knows that real path integrals of second kind of exact vector fields along closed paths vanish.

The "vector analysis"-frame also occurred in conjunction with the metarules of saming and baptising when Uwe identified different path integrals from vector and complex analysis as instances of an overarching mathematical concept named "path integrals". He noticed an analogy in the formation of their integrands based on the apparent same structure involving products of real numbers, complex numbers, or the scalar product of vectors (Section 13.3). In this sense, Uwe samed path integrals as a "cross-curricular concept" (Kontorovich, 2018b).

As another example, Uwe and Dirk related Cauchy's integral formula to the mean value property of harmonic functions (or Poisson's integral formula) from vector analysis. Sebastian, on the other hand used his own mean value interpretation for complex path integrals here and did not refer to propositions about harmonic functions (see below).

In view of the vector analysis aspect of complex path integrals (Section 9.1.3), propositions about complex path integrals can be transferred to propositions about real path integrals of second kind, which we have explored in Chapter 8, too. Therefore, it is also likely that mathematists other than our research participants, who are familiar with complex and vector analysis discourses, incorporate the "vector analysis"-frame into their intuitive mathematical discourses, too. Moreover, integrals in vector analysis can be used in physical or geometrical interpretations, hence, mathematists may use these to explore potentially similar narratives about complex path integrals as well. For instance, this was done by some of the experts in Oehrtman et al.'s (2019) study.

We may even describe our own discursive action, namely to identify partial aspects of complex path integrals (Section 9.2) depending on constraints on the classes of paths, functions, or domains, with a scholarly variant of the "restriction of generality"- and "theorematic"-frames: These additionally imposed constraints were needed for the endorsability of the partial aspects and may count as theorems (i.e., the fundamental theorem of calculus, the residue theorem, or Green's theorem for complex functions) in case one of the aspects of complex path integrals (Section 9.1) is used to define them.

On the other discursive frames: the "tool"-, "no meaning"-, "area"-, "mean value"-, and "holomorphicity ex machina"-frames

The other discursive frames appeared in one interview only. This indicates that these frames may be more idiosyncratic than the multiple used frames. Whereas it is conceivable that the discursive actions of other mathematicians in intuitive mathematical discourses about complex path integrals may also be described with these frames, we suspect however that the corresponding metarules are not endorsed to the same extent as the three previously discussed frames might be. In other words, we consider it likely that other mathematicians may not describe their own intuitive understanding of complex path integration with the corresponding metarules from these discursive frames.

(F4) "TOOL"-FRAME The "tool"-frame consists of metarules that arise from the valuation of complex path integrals as tools in complex analysis. They may be applied in mathematical routines such as proving or computations. As such, the "tool"-frame is more specific than a general belief of mathematics as a tool because here a particular object (i.e., the complex path integral) is seen as a particular tool for particular tasks. For instance, Uwe explicitly described complex path integrals as tools for proving or the computation of real integrals (D4). Contrariwise, Dirk considered this too easy a way out and hence did not follow this frame. Thus, at least idealised, proponents of the "tool"-frame value the complex path integral as an aid but not necessarily as intrinsically important itself.

The use of complex path integrals for proofs and computations (e.g., to prove the analyticity of holomorphic functions via Cauchy's integral formula or the computation of Riemann integrals as an applications of the residue theorem) is endorsed in the literature (cf. Chapter 7, Chapter 8, Appendix A). Therefore, the valuation of complex path integrals as a tool does not surprise us in general. It appears to be rather idiosyncratic though in response to the task situation to express what one considers a mental image or an intuitive explanation.

The rejection of intrinsic importance of the complex path integral in intuitive mathematical discourses is even more evident in the next frame.

(F5) "NO MEANING" FRAME Like the "tool"-frame, the "no meaning"-frame appeared only in Uwe's intuitive mathematical discourse about complex path integrals. It contains the metarule according to which any intrinsic meaning for complex path integrals is denied and Uwe produced the discursive images in (D5). Neither of the other two experts in this study rejected a potential meaning or intuitive explanation as firmly as Uwe. However, we have to emphasise at this point that Uwe deviated from his initial rejection of intrinsic meaning of the complex path integral during the interview. In fact, compared to the other two experts, his intuitive mathematical discourse contained the most discursive images about complex path integrals.

We consider the "no meaning"-frame an idiosyncratic discursive frame for intuitive mathematical discourses because the quest for interpretations of mathematical objects seems to be rather frequent in mathematics (cf. Chapter 2). This frame contradicts this view. Nevertheless, Oehrtman et al. (2019) also report that one of their interviewed experts "would not think of integration geometrically" and questioned whether there was "'deep insight'" (Oehrtman et al., 2019, p. 415).

(F6) "AREA"-FRAME The "area"-frame consists of explorations for finding a relationship between the complex path integral as a certain area, or possibly another geometrical quantity. While the other experts rejected a meaning of the complex path integral as an area, Dirk tried to transfer the basic idea of area to complex path integrals. He used the routine of drawing graphs of functions to produce visual mediators showing certain areas (Figure 14.3, Figure 14.4b) or a plot of a vector field (Figure 14.5). These visual mediators rather realise integrals for real functions though (Riemann integrals or real path integrals of first or second kind). Dirk reflected on the usability of these visual mediators for the case of complex path integrals in detail. However, in the end, he did not produce a discursive image about complex path integrals related to a geometrical object such as an area explicitly. For this reason, we did not opt to count his utterances at this point as discursive images about complex path integrals. Rather, his utterances could have counted as discursive images about real integrals. In this context, Dirk did also not endorse any of his utterances as intuitive interpretations or mental images for complex path integrals, except for that they may be a "crutch" (Section 14.3).

While we could find a geometrical / physical interpretation for the real and imaginary parts of $\int_\gamma f(z)\,\mathrm{d}z$ in terms of real path integrals of second kind of the Pólya vector field associated f (Section 8.2.2) and it is clearly possible to plot the real and imaginary parts of $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ as areas enclosed by the graphs of $t \longmapsto \operatorname{Re}(f(\gamma(t))\gamma'(t))$ and $t \longmapsto \operatorname{Im}(f(\gamma(t))\gamma'(t))$ and the horizontal axis (cf. Example 8.2) as in the basic idea of area for Riemann integrals (Section 2.2.3), we have not found support in the literature for an area interpretation of complex path integrals. Nevertheless, since geometrical / physical interpretations of real integrals are widely known, it is conceivable that other mathematists attempt to adapt them to complex path integrals in their intuitive mathematical discourses (Oehrtman et al., 2019).

(F7) "MEAN VALUE"-FRAME The "mean value"-frame was used by Sebastian only. It consists of metarules with the help of which mathematists may relate complex path integrals to mean values or processes of measuring. Following this discursive frame, Sebastian identified integrals as "cross-curricular concepts" (Kontorovich, 2018b), which are connected in terms of the narrative that are realisable as mean values. On the one hand, Sebastian produced the discursive images in (D7), which identify integration as mean value formation and measuring in general. On the other hand, he also specified (D7) to the case of complex path integrals in (D8). According to these discursive images, complex path integrals are mean values of the values of the integrand or the average rates of rotation and dilation induced by the integrand (the latter interpretation was produced together with the interviewer).

Remarkably, Sebastian's use of the "mean value"-frame is very consistent throughout the interview. Nevertheless, the discursive images in (D8) differ from the endorsed mean value aspect of complex path integrals and Sebastian's intuitive substantiation of Cauchy's integral formula differed from the endorsed mean value interpretation of Cauchy's integral formula in complex analysis literature (Gluchoff, 1991, see Section 9.1.4,Section 15.2, Section 15.5.1). Notwithstanding, since there is the endorsed interpretation of complex path integrals in terms of the mean value aspect and there are mean value interpretations for other integrals, is probable that other mathematists may attempt to interpret complex path integrals with the "mean value"-frame, too.

(F8) "HOLOMORPHICITY EX MACHINA"-FRAME The "holomorphicity ex machina"-frame appeared in Sebastian's intuitive mathematical discourse only. Its main characteristic is the valuation of holomorphic functions as particularly rigid and that local information about holomorphic functions imply global properties (Section 15.3). Sebastian described that this rigidity led him to anticipate that there are object-level rules on holomorphic functions such as Cauchy's integral theorem or Cauchy's integral formula. Additionally, Sebastian and the interviewer-researcher constructed the discursive images in (D9) we presented above.

This discursive frame seems rather idiosyncratic. For instance, even though Uwe and Dirk also interpreted holomorphic functions as particularly rigid, they did not use this valuation for the interpretation of integral theorems or complex path integrals in general. On the contrary, Uwe argued that Cauchy's integral is a formula, which one would rather not expect (Section 13.7).

16.4.2 *General observations on experts' intuitive mathematical discourses about complex path integrals*

Let us now discuss characteristics of our experts' intuitive mathematical discourses about the complex path integral in more general terms. In sum, four features are noteworthy.

Interdiscursivity of intuitive mathematical discourses

Our experts frequently digressed from complex analysis discourses. That is, they combined complex analysis discourse with other mathematical discourses such as real analysis and vector analysis. This observation corroborates Oehrtman et al.'s (2019) findings. However, while some of the experts in their study claimed that they were thinking about complex integration as if it were real integration, no such exclamation was made in our sample. Nevertheless, Dirk's and Sebastian's explorations based on the "area"- and "mean value"-frame support that they were trying to generalise an interpretation for real or Lebesgue integrals to complex path integrals as well. Additionally, none of our experts falsely transferred physical interpretations for real path integrals of second kind to complex path integrals.

We emphasise again that discursive frames often overlapped during the interviews. Hence, different metarules were at play at the same time when the experts engaged in their explorations about interpretations of complex path integrals and intuitive substantiations of integral theorems. This resonates with our hypothesis that intuitive mathematical discourses about complex path integrals are indeed realised by the *principle of interdiscursivity* (Sfard, 2013; Section 16.2). Since we observed this principle also from the scholarly perspective in Part ii, in particular in the aspects and partial aspects of complex path integral, we may thus conclude that scholarly and intuitive mathematical discourses follow the same principle of interdiscursivity. Since experts are familiar with the scholarly discourse, it is likely that they transported this interdiscursivity to their intuitive mathematical discourses. It would be interesting to find out in further research how intuitive mathematical discourses may affect scholarly mathematical discourses in more detail (see also Section 16.4.4).

Deviations from scholarly metarules

While experts are usually expected to exemplify endorsed scholarly mathematical discourses in research and teaching (Sfard, 2014; Viirman, 2021) the analyses in this thesis showed that their intuitive mathematical discourses about complex path integrals and integral theorems in complex analysis are not characterised by the same level rigour. This could have been expected because the setting of the interviews encouraged the experts to digress from scholarly discourse if this would fit their individual interpretations of complex path integrals. On the contrary, intuitive mathematical discourses lived from imprecise use of keywords and several other features that distinguish them from scholarly mathematical discourses. For example, we exhibited commognitive conflicts in Uwe's usage of the word "holomorphic" or "holomorphic function": Sometimes his narratives about holomorphic functions were only endorsable for holomorphic functions with no singularities. Similarly, he used the word "path" several times in narratives, whose endorsability required "closed path", and we observed that this was no simple slip of the tongue because he substantiated his narrative (in the intuitive, not the scholarly discourse!) with his perceived fact that there is always a "closed path behind the scenes" (Section 13.4.3). Next to this observation, Dirk used certain keywords as a "pars pro toto". That is, he used the words "topology", "nested intervals", or "complex differentiability" as overarching keywords to denote certain steps ("pars") in proofs of Goursat's lemma ("toto"), but did not explain these parts because he assumed that his interlocutor, the interviewer, was familiar with them (Section 14.6 and Section 14.7). Finally, when explaining why continuous functions, which are not holomorphic, cannot have a primitive function, Sebastian constructed a new mathematical object, which he baptised "pseudo-primitive". This keyword signifies a candidate for a primitive function, while scholarly complex analysis has endorsed that no primitive exists, and Sebastian added "pseudo" to account for this fact (Section 8.3.1).

Similarly, substantiation routines in the intuitive mathematical discourses differed from scholarly substantiations. For example, we observed substantiations by means of analogy and retrospective substantiations. In the first of these substantiation routines, experts noticed an analogy between mathematical objects and narratives in complex analysis discourse and another mathematical discourse and substantiated a proposition in complex analysis with the existence of an analogue proposition in the other discourse (Section 13.6). The second of these substantiation routine is characterised by a reversion of logical order: A theorem from complex analysis, which is usually proven at a later part of complex analysis was used to substantiate a proposition, which occurs earlier in complex analysis (Section 13.7).

Awareness for non-endorsability in scholarly discourses

The experts in this study were well aware that their intuitive explanations are not always endorsable in scholarly complex analysis discourse. That is, they constructed discursive images about complex path integrals, which are not endorsable in scholarly mathematical discourse, and acknowledged that this is the case. Henceforth, they considered certain narratives or other explanations to belong to their intuitive understanding of complex path integrals even though they were aware that they are not normatively correct. For instance, Sebastian acknowledged his uncertainty whether some of his intuitive interpretations about mean values or holomorphic functions were endorsable in general. Even though he did not endorse a discursive image about complex path integrals in relations to areas, Dirk produced the visual mediator in Figure 14.4b, which would have been endorsable for an area interpretation of real path integrals of first kind. He described necessary changes in the figure that would have been necessary for the endorsement of a discursive mental image for complex path integrals (namely one of the coordinate axes should have represented complex instead of real numbers but which could not have appropriately been drawn on paper).

Pedagogical awareness

All three experts emphasised pedagogical decision they face in teaching at least at one point during their interviews. They acknowledged that visual mediators are important resources for teaching and included some of them in their intuitive discourses. They produced several visual mediators, in which paths or functions were realised, but the complex path integral as a complex was hardly ever realised explicitly. Rather, the visual mediators, which did realise complex path integrals, were either mathematical symbols or realisations of mathematical objects associated to complex path integrals (e.g., paths or function values in terms of vectors).

Our sample also allows to hypothesise that lecturers in complex analysis consider the possibility of vertically coherent mathematics training related to integrals, even though they reacted differently to it. On the one hand, Uwe's narrative that complex path integrals are path integrals of third kind validates that he is aware of the connections between different kinds of path integrals. This narrative offers a potential motivation for the definition of complex path integral via the substitution or product sum aspect (Section 16.3). He also argued that the connections to vector analysis are a valuable source for a substantiation of Cauchy's integral theorem, but that one has to have a solid background in vector analysis. For that reason he does not consider emphasising connections to vector analysis for the substantiation of Cauchy's integral theorem though. Uwe is aware of the necessary precedents and suspects that students are not sufficiently familiar with vector analysis when they attend his lectures. It looks like he designs his lecture on complex analysis as a self-contained course in such a way that students are not overloaded with an optional linkage to vector analysis (cf. "Klein's Plan A"; see the end of Section 16.3.2). On the other hand, Sebastian emphasised a unifying point of view of integration as mean value formation and measuring, and used this interpretation in his intuitive explanation of the number $\int_\gamma f(z)\,\mathrm{d}z$ as well as Cauchy's integral theorem. This unifying point of view of integration, which he captured in the discursive images (D7) and (D8), rather approves "Klein's plan B". However, we do not know how both experts really teach complex path integrals or integral theorems, and we have to acknowledge that our expert interviews cannot be extrapolated to how the mathematicians act in practice (cf. Mejía-Ramos & Weber, 2020).

To finish the discussion of our results and before moving on to the limitations of our study and further research, a note of caution is due. The preceding discussion may have led to the impression that the experts in this study but also the experts interviewed by Oehrtman et al. (2019) are not very familiar with endorsable interpretations of complex path integrals or partly overgeneralise interpretations for other integrals. While this is correct to some extent, it is not what we wish to conclude from our study. Rather, we conclude that interpreting complex path integrals is a difficult task even for experts, but also that experts use combine various mathematical discourses with complex analysis in their intuitive discourses and can thus productively initiate exploration routines about what they consider their individual interpretations of complex path integrals. Hence, when looking at both kinds of experts' discourses—the scholarly discourse arising in a long historical process (Part ii) and the intuitive mathematical discourses (Part iii)—we see that complex analysis is usually not seen as separate from other discourses.

16.4.3 *Limitations of the empirical study*

The small sample size is clearly a limitation for our empirical study. Nevertheless, quantitative generalisation did not belong to our goals. Rather, drawing on our theoretical contribution in Part i, we intended to contribute to current empirical research with a fine-grained multi-case study of experts' intuitive mathematical discourses about complex path integrals, which complemented the endorsed point of view in Part ii. That is to say, we have applied a new commognitive approach to clarify the phenomenon of intuitive, non-rigorous interpretations of the mathematical notion at hand (cf. Eisenhardt, 1989; Flyvbjerg, 2006; Grandy, 2010; Ridder, 2017). For this initial step of theory building, a fine-grained analyses required a small sample. Theoretically, a more refined sampling strategy would have allowed for a larger variety in the sample (Misoch, 2015, ch. 7). Nevertheless, our sample was already contrasting enough for the variety of discourses we could present.

In line with instrumental case study methodology, our empirical study "facilitates understanding of a particular phenomenon", namely intuitive mathematical discourses about complex path integrals, which emphasises "richness rather than generalizability" (Grandy, 2010, p. 475). We agree that detailed analyses tracing the particularities of each case, eventually endanger that the resulting theoretical account "is very rich in detail, but lacks the simplicity of overall perspective" (Eisenhardt, 1989, p. 547). Therefore, we did not stop with isolated descriptions of each experts' discourse but identified patterns with respect to the elements of the commognitive theory (in particular the discursive frames, the discursive narratives, and the substantiations of these narratives). Thus, the level of generality in our study is "based on theoretical ideas about social mechanisms", which are potential ways experts interpret complex path integrals for themselves or for others, "not on formal empirical generalization to similar cases" (Ylikoski, 2019, p. 13).

Corroborating the findings by Oehrtman et al. (2019) (and similar observations in Hancock (2018) and Soto and Oehrtman's (2022) results in the case of students), we consider two findings as representative for the sample of experts having participated in empirical research in complex analysis education so far: First, experts likely do not share a consistent set of mental images of complex path integrals, which was also supported by our extensive literature review in Part ii, and second, experts organise their intuitive discourses around various discursive frames.

A further weakness of our study is that not all metarules in the discursive frames become fully apparent. A larger data set, in particular, more observations over time or other communicational activities such as solving mathematical tasks would have enabled us to find out more about these particular types of routines: For example, we have not gained any insight into experts' precedents directly but might have obtained if we had observed or asked the experts at other times, too (cf. Lavie et al., 2019). A further study, for example using a variant of "member checks" (e.g., Flick, 2019; Meyer, 2018; Misoch, 2015, ch. 10.3.1), may increase the validity of our results with respect to whether experts in complex analysis agree with our findings. However, since the experts are neither familiar with the commognitive framework nor with empirical research in complex analysis education, member checks were not conducted (Steinke, 2017). Instead, we validated our reconstructions of the discursive frames with researchers in mathematics education (one with expertise in complex analysis and one with expertise in the commognitive framework, as well as others in the field of mathematics education in general) and in discussions at mathematics education conferences ("peer debriefing"; e.g., Flick, 2019; Meyer, 2018; Misoch, 2015, ch. 10.3.1).

Even though our interviews "at the eye-level" (Pfadenhauer, 2009) allowed for a professional exchange between the mathematicians and a mathematics educator, our methodical approach to data collection was a laboratory setting nevertheless. Hence, we did not observe experts in their everyday work or in interaction with others. With respect to this issue, it would be interesting to find out which additional interpretations the experts would have given if they had had additional resources at their disposal or had been able to prepare for the interview. Other methods of collecting data such as interactions between experts, between experts and students, or group discussions would have been useful, too, and would have likely elicited other facets of intuitive mathematical discourses, which we were not able to observe in this study (see below).

16.4.4 *Further directions for research*

We would like to finish this chapter with directions for further research.

Validation and intensifying our understanding of experts' intuitive mathematical discourses about complex path integrals

The *first block* of questions is intended to deepen our understanding of experts' intuitive discourses in complex analysis. Validations of our results with a different and larger sample of experts is due. Interviewing more experts may show whether the same discursive frames or discursive images appear in other experts' intuitive mathematical discourses, too, or show new ones. In the long run, we would obtain more representative results for larger groups of complex analysis lecturers. In particular, we may also examine quantitatively whether and how much the discursive images, which were produced in our expert interviews, are endorsed by other mathematical experts. For this to be done, we may produce a questionnaire or conduct "acceptance interviews / surveys" (Jung, 1992; Wiesner, 1995; Wiesner & Wodzinski, 1996; cf. Hanke & Schäfer, 2017) in order to ask experts to rate their acceptance of the discursive images about complex path integrals and to substantiate what they do not endorse about it.

Intuitive mathematical discourses may be also enacted in communicational situations other than interviews. Our methodical approach only reveals what the experts also count as their mental images and intuitive explanations and consider worth communicating (Hanke, 2020a). For instance, interactions between discursants without the presence of an interviewer would have been a more realistic type of communicative activity, where one discursant's individual interpretation of complex path integrals might have been directly evaluated by other interlocutors (e.g., by other experts or students, which belong to the natural professional environment of mathematical experts). We hypothesise that intuitive understanding might also show itself when mathematists solve tasks in a cleverly manner, find proofs (cf. Kidron & Dreyfus, 2014), in conversations with colleagues, during lectures or in office hours when students ask for intuitive explanations, or probably also in lecturers "between the lines", and mathematical communication for non-specialised audiences (cf. Barwell, 2013a). All of these contexts provide fruitful opportunities for research. Nevertheless, during the interviews the interviewer ensured that we were actually observing intuitive mathematical discourses, which would have to be guaranteed in other methods of data collection otherwise, too.[183]

Intuitive mathematical discourses about other mathematical notions

A *second block* of further research asks which properties are characteristic for intuitive mathematical discourses more generally. We have observed

- *idiosyncratic usage of keywords* (e.g., the word "holomorphic" was sometimes used to realise a function with or without singularities, or the word "topology" was used not to refer to the mathematical discourse of topology but rather to as a pars pro toto for a part of the proof of Goursat's lemma);
- *keyword construction on-the-fly* (i.e., Sebastian baptised a function a "pseudo-primitive function" when discussing the non-existence of primitive functions for non-holomorphic functions),
- *retrospective substantiations* (e.g., when the residue theorem was used to substantiate Cauchy's integral formula as opposed to the sequence how the theorems are usually proven in complex analysis), and
- *substantiations based on analogies* without scholarly rigour (e.g., when a Cauchy's integral theorem was substantiated as true by recalling an analogue theorem from vector analysis).

We conjecture that these discursive mechanisms were coping mechanisms for the apparently difficult task situation to produce discursive mental images or visual mediators for complex path integrals. Therefore, we cannot be sure yet whether these mechanisms are specific for complex integration or are used by experts for other mathematical topics as well. Presumably, experts may use mathematical keywords in intuitive mathematical discourses about other topics also in an idiosyncratic manners, which are not endorsable in scholarly discourse. Moreover, we consider it very important to find out about the use of visual mediators to realise advanced mathematical objects in these discourses apart from mathematical formulae.

183 Even though complex path integrals are advanced mathematical objects from the point of view of mathematics curricula (they usually appear after the first year of mathematical studies), they are quite elementary compared to what many experts work with on a daily basis. Hence, it is more difficult to ensure that we would have observed experts communicate about complex path integrals in their daily work unless we ask them about these objects explicitly.

Students' intuitive mathematical discourses and the impact of experts' intuitive mathematical discourses on students' learning of complex analysis

In this empirical study, we focused on the discursive frames governing *experts'* construction of discursive images about complex path integrals. Further work is required to find out how students can benefit from discursive images or discursive frames experts may offer to them in teaching. With this regard, a *third block* of questions for future research centres around the interplay between experts' intuitive mathematical discourses and students' learning. Clearly, at first we need to find out how students' individual interpretations of complex path integrals may look like. There are several ways to examine students' individualisations of the discourse on complex path integrals, in which they participate in their lectures. For instance, we may conduct brief interviews directly after the complex path integral has been introduced in a lecture and ask the students to repeat what they perceived as the core ideas from the lecture, how they connect the newly introduced integral to other integrals etc. After students have worked with complex path integrals for a longer period of time, we may conduct interview similar to the ones in our study and ask for their potential mental images or intuitive explanations directly. Additionally, acceptance interviews / surveys (Jung, 1992; Wiesner, 1995; Wiesner & Wodzinski, 1996; cf. Hanke & Schäfer, 2017) (see above) about the discursive images we exhibited in this study may be conducted.

Further studies are recommended to find out about possible implications of experts' intuitive mathematical discourse for teaching. For instance, Sebastian used the mean value frame (F9) and the corresponding discursive images (D7, D8) and Uwe used the vector analysis frame (F3) and the discursive image (D5) to substantiate parts of Cauchy's integral theorem and in particular that they were not surprised by this theorem anymore. With this regard, these discursive frames and images were beneficial for the experts' idiosyncratic substantiations of Cauchy's integral theorem. However, Uwe valued his interpretation to be only helpful for students with a solid background in vector analysis, which clearly limits the potential the discursive frames and images had for him. Sebastian's mean value interpretation was mostly at odds with endorsed discourse on complex analysis, and thus it may cause students to repeat the same non-endorsable chain of reasoning. We wonder whether these interpretation can nevertheless be productively used in teaching. Generally speaking, we may suspect that discursive images are narratives assisting students' transition from novices to proficient participants in complex analysis discourses. After all, if narratives from intuitive mathematical discourses may support students' individualisation of scholarly complex analysis discourse, then it would be beneficial if these narratives told in teaching as well. However, much more research remains to be done to operationalise and explore these issues further.

Alignment of experts' scholarly and intuitive mathematical discourses on complex analysis

To develop a fuller picture of experts' mathematical discourses on complex path integrals, there is another facet, which we have not explored in this thesis. This facet are the local discourses lecturers establish in their courses. It comprises the *fourth* and last block for further research we describe here. A lot of research has already focused on advanced mathematics lectures. For example, diagrams or other informal sources are used by mathematicians to present or motivate proofs or definitions (e.g., Burton, 2004; Chorlay, 2022; Davis et al., 2012; Heintz, 2000; Kiesow, 2016; Sfard, 1994, and many others). Literature on teaching mathematics in advanced lectures at university advocates that students need to know both formal and informal content and mathematical reasoning (cf. Fukawa-Connelly et al., 2017; Melhuish et al., 2022).[184] Concerning the communication in mathematics lectures, Artemeva and Fox (2011) found that mathematics lecturers typically engage in "chalk talk" during their lectures, "a practice in which mathematicians wrote mathematical work on the board while simultaneously providing a running commentary on what their inscriptions meant and occasionally stopping to offer metacommentary on how their work fit into the broader mathematical framework" (Woods & Weber, 2020, p. 3). However, we have noted in our literature review reports of complex analysis lecturers, which indicate that students are "mystified on first exposure" to complex path integrals (Gluchoff, 1991, p. 641; cf. Braden, 1987), and therefore, at least some mathematicians aim to include more intuitive explanations to their teaching in complex analysis. Nevertheless, in the context of "limits" though, Chorlay (2022, p. 1077) cautions us that "most informal formulations are structurally incongruent with the formal definition, thus deepening the gap between intuition and conceptual understanding".

In this context, Lai and Weber (2014) and Woods and Weber (2020) also found that lecturers align their pedagogical decisions in proof presentation to their audiences.[185] These findings from the literature underpin our observations on our experts' descriptions of their own pedagogical practice. However, to date, we do not know what the discourses experts establish in their complex analysis lectures look like. We suspect however that intuitive mathematical discourse is a subdiscourse of what has previously been called "chalk talk" or "informal content".

In sum, we believe it is very important and promising to analyse complex analysis lectures commognitively (cf. Karavi et al., 2022; Pinto, 2019; Viirman, 2021) with regards to whether and how they introduce complex path integrals in their lecturers and how they mediate between the scholarly complex analysis discourse and the additional talk they deliver. For example, Karavi and Mali (2022) identified the subroutine "setting the proof" of the routine of proving, with the help of which a lecturer in complex analysis initiated substantiations of integral theorems in complex analysis, namely with by including visual mediators or recalling analogue theorems from real analysis. In the context of calculus and analysis, Chorlay (2022) and Viirman (2021) observed that lecturers accompany the routine of defining with examples or additional talk to support "conceptual understanding" or the "raison d'être" of a definition (Chorlay, 2022, p. 1085–1086, following Chevallard, 2005).[186] However, Fukawa-Connelly et al. (2017) and Lew et al. (2016) have also shown that students were rarely able to recall the metarules lecturers presented orally because the students tended to not include oral comments in their notes. Therefore, we ask whether and how students in complex analysis actually benefit when lecturers include intuitive mathematical discourse to their lectures.

184 "Informal" is defined rather broadly here. Basically everything besides definitions, proofs, and theorems are regarded as informal content in mathematics lecturers. For example, this may include lecturers' identification of analogies between previous and still to be learned content or colloquial statements to provide meaning of the content of the mathematics lecture in one way or another (Fukawa-Connelly et al., 2017).

185 At this point, Lai and Weber (2014) talk about "pedagogical proofs", in other words, "proofs that transform mathematical knowledge into ways of 'representing ideas so that the unknowing can come to know, those without understanding can comprehend and discern, and the unskilled can become adept' (Shulman, 1987, p. 7)" (Lai & Weber, 2014, p. 93).

186 Raison d'être refers to answers to the questions "What is its meaning and import in mathematics? What are its connections to other concepts? What are its meaning and relevance for the learning of mathematics?" lecturers may potentially address (Chorlay, 2022, p. 1079).

BIBLIOGRAPHY

Abraham, A. (Ed.). (2020). *The Cambridge handbook of imagination*. Cambridge University Press. https://doi.org/10.1017/9781108580298

Ahlfors, V. (1979). *Complex analysis. An introduction to the theory of analytic functions of one complex variable* (3rd ed.). McGraw-Hill.

Ahmad, M. (1955). Cauchy's theorem and its converse. *Acta Mathematica*, *93*(1), 15–25. https://doi.org/10.1007/BF02392518

Akcoglu, M. A., Bartha, P. F. A., & Ha, D. M. (2009). *Analysis in vector spaces. A course in advanced calculus*. Wiley. https://doi.org/10.1002/9781118164587

Akremi, L. (2019). Stichprobenziehung in der qualitativen Sozialforschung. In N. Baur & J. Blasius (Eds.), *Handbuch Methoden der empirischen Sozialforschung* (pp. 313–331). Springer VS. https://doi.org/10.1007/978-3-658-21308-4_21

Allmendinger, H. (2013). *Felix Kleins „Elementarmathematik vom höheren Standpunkte aus". Eine Analyse aus historischer und mathematikdidaktischer Sicht* (Doctoral dissertation). Universität Siegen. Siegen, Germany, universi Universitätsverlag Siegen. Retrieved 12/02/2021, from https://dspace.ub.uni-siegen.de/handle/ubsi/917

Allmendinger, H. (2016). Die Didaktik in Felix Kleins „Elementarmathematik vom höheren Standpunkte aus". *Journal für Mathematik-Didaktik*, *37*(1), 209–237. https://doi.org/10.1007/s13138-016-0089-1

Allmendinger, H. (2019). Examples of Klein's practice *Elementary Mathematics from a Higher Standpoint: Volume I*. In H.-G. Weigand, W. McCallum, M. Menghini Marta and Neubrand, & G. Schubring (Eds.), *The legacy of Felix Klein* (pp. 203–213). Springer. https://doi.org/10.1007/978-3-319-99386-7_14

Amann, H., & Escher, J. (2006). *Analysis II* (2., korrigiert Aufl.). Birkhäuser. https://doi.org/10.1007/3-7643-7402-0

Antonini, S. (2019). Intuitive acceptance of proof by contradiction. *ZDM*, *51*(5), 793–806. https://doi.org/10.1007/s11858-019-01066-4

Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education*, *6*(1), 1–29. https://doi.org/10.1007/s40751-019-00054-3

Apostol, T. M. (1981). *Mathematical analysis. A modern approach to advanced calculus* (2nd ed.). Addison-Wesley.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*, 215–241. https://doi.org/10.1023/A:1024312321077

Arens, T., Hettlich, F., Karpfinger, C., Kockelkorn, U., Lichtenegger, K., & Stachel, H. (2018). *Mathematik* (4th ed.). Springer Spektrum. https://doi.org/10.1007/978-3-662-56741-8

Arsove, M. G. (1955). On the definition of analytic function. *The American Mathematical Monthly*, *62*(1), 22–25. https://doi.org/10.2307/2308011

Artemeva, N., & Fox, J. (2011). The writing's on the board: The global and the local in teaching undergraduate mathematics through chalk talk. *Written Communication*, *28*(4), 345–379. https://doi.org/10.1177/0741088311419630

Arzarello, F., Bazzini, L., & Chiappini, G. (1995). The construction of algebraic knowledge: Towards a socio-cultural theory and practice. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education (PME 19, July 17–22, 2017)* (pp. 119–134). PME.

Axler, S. (2020). *Measure, integral & real analysis*. SpringerOpen. https://doi.org/10.1007/978-3-030-33143-6

Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: How dynamic interactive mediators can fulfill this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics*, *106*, 291–312. https://doi.org/10.1007/s10649-020-10009-w

Bak, J., & Popvassilev, G. S. (2017). The evolution of Cauchy's closed curve theorem and Newman's simple proof. *American Mathematical Monthly*, *124*(3), 217–231. https://doi.org/10.4169/amer.math.monthly.124.3.217

Barakat, M. (2014). *Funktionentheorie. Vorlesungsskript Wintersemester 2013/14* (lecture notes). TU Kaiserslautern. Retrieved 02/02/2021, from https://algebra.mathematik.uni-siegen.de/barakat/Lehre/WS13/Funktionentheorie/Skript/Funktionentheorie.pdf

Barnett, J. H., Can, C., & Clark, K. M. (2021). "He was poking holes ..." A case study on figuring out metadiscursive rules through primary sources. *The Journal of Mathematical Behavior*, *61*, 100838. https://doi.org/10.1016/j.jmathb.2020.100838

Barney, I. (1914). An extension of Green's theorem. *American Journal of Mathematics*, *36*(2), 137–150. https://doi.org/10.2307/2370236

Bartha, P. (2013). Analogical arguments in mathematics. In A. Aberdein & I. Dove (Eds.), *The argument of mathematics. Logic, epistemology, and the unity of science* (pp. 199–237). Springer. https://doi.org/10.1007/978-94-007-6534-4_12

Bartha, P. (2019). Analogy and analogical reasoning. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Spring 2019). Metaphysics Research Lab, Stanford University. Retrieved 02/17/2022, from https://plato.stanford.edu/archives/spr2019/entries/reasoning-analogy/

Barwell, R. (2009). Researchers' descriptions and the construction of mathematical thinking. *Educational Studies in Mathematics*, *72*(2), 255–269. https://doi.org/10.1007/s10649-009-9202-4

Barwell, R. (2013a). The academic and the everyday in mathematicians' talk: The case of the hyper-bagel. *Language and Education*, *27*(3), 207–222. https://doi.org/10.1080/09500782.2012.696656

Barwell, R. (2013b). Discursive psychology as an alternative perspective on mathematics teacher knowledge. *ZDM*, *45*, 595–606. https://doi.org/10.1007/s11858-013-0508-4

Bärwolff, G. (2017). *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (3rd ed.). SpringerSpektrum. https://doi.org/10.1007/978-3-662-55022-9

Bauersfeld, H. (1983). Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In H. Bauersfeld, H. Bussmann, G.

Krummheuer, J. H. Lorenz, & J. Voigt (Eds.), *Lernen und Lehren von Mathematik. Analysen zum Unterrichtshandeln II* (pp. 1–56). Aulis Verlag Deubner.

Beardon, A. F. (1979). *Complex analysis. The argument principle in analysis and topology*. Wiley.

Beck, M., Marchesi, G., Pixton, D., & Sabalka, L. (2018). *A first course in complex analysis* (Version 1.54). Retrieved 07/10/2022, from https://matthbeck.github.io/papers/complex.pdf

Beckenbach, E. F. (1943). The stronger form of Cauchy's integral theorem. *Bulletin of the American Mathematical Society*, *49*(8), 615–618. https://doi.org/10.1090/S0002-9904-1943-07992-X

Beesack, P. R. (1972). The Laurent expansion without Cauchy's integral theorem. *Canad. Math. Bull.*, *15*(4), 473–480. https://doi.org/10.4153/CMB-1972-085-x

Behrends, E. (2007). *Analysis Band 2. Ein Lernbuch* (2., akt. Aufl.). Vieweg. https://doi.org/10.1007/978-3-8348-9177-8

Bender, P. (1990). Zwei "Zugänge" zum Integral-Begriff? *mathematica didactica*, *13*, 102–127.

Bender, P. (1991). Ausbildung von Grundvorstellungen und Grundverständnissen – ein tragendes didaktisches Konzept für den Mathematikunterricht – erläutert an Beispielen aus den Sekundarstufen. In H. Postel, A. Kirsch, & W. Blum (Eds.), *Mathematik lehren und lernen. Festschrift für Heinz Griesel* (pp. 48–60). Schroedel.

Bender, P. (1997). Grundvorstellungen und Grundverständnisse für den Stochastikunterricht. *Stochastik in der Schule*, *17*, 8–33.

Bender, P. (1998). Basic imagery and understandings for mathematical concepts. In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view. Proceedings of the Topic Group 6 at the Eight International Congress on Mathematical Education (ICME 8, July 14–21, 1996)* (pp. 105–127). Åbo Akademi University.

Benítez, J., Giménez, M. H., Hueso, J. L., Martínez, E., & Riera, J. (2013). Design and use of a learning object for finding complex polynomial roots. *International Journal of Mathematical Education in Science and Technology*, *44*(3), 365–376. https://doi.org/10.1080/0020739X.2012.729678

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal for Research in Mathematics Education*, *36*(3), 176–247.

Ben-Zeev, T., & Star, J. R. (2001). Intuitive mathematics: Theoretical and educational implications (B. Torff & R. J. Sternberg, Eds.), 29–56.

Ben-Zvi, D., & Sfard, A. (2007). Ariadne's thread, daedalus' wings, and the learner's autonomy. *Éducation en didactique*, *1*(3), 117–134. https://doi.org/10.4000/educationdidactique.241

Berenstein, C. A., & Gay, R. (1991). *Complex variables. An introduction*. Springer. https://doi.org/10.1007/978-1-4612-3024-3

Bergsten, C. (2020). Mathematical approaches. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 498–505). Springer. https://doi.org/10.1007/978-3-030-15789-0_95

Bezuidenhout, J., & Olivier, A. (2000). Students' conceptions of the integral. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME 24, July 23–27, 2000)* (pp. 73–80). PME.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 483–506). Springer. https://doi.org/10.1007/978-3-642-00742-2_46

Bikner-Ahsbahs, A., & Prediger, S. (Eds.). (2014). *Networking of theories as a research practice in mathematics education. Authored by the Networking Theories Group*. Springer. https://doi.org/10.1007/978-3-319-05389-9

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68*(19), 19–36. https://doi.org/10.1007/s10649-007-9112-2

Biza, I. (2021). The discursive footprint of learning across mathematical domains: The case of tangent line. *Journal of Mathematical Behaviour*, *62*, 100870. https://doi.org/10.1016/j.jmathb.2021.100870

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016). *Research on teaching and learning mathematics at the tertiary level. State-of-the-art and looking ahead*. SpringerOpen. https://doi.org/10.1007/978-3-319-41814-8

Blumer, H. (1954). What is wrong with social theory? *American Sociological Review*, *19*(1), 3–10.

Bôcher, M. (1896). On Cauchy's theorem concerning complex integrals. *Bulletin of the American Mathematical Society*, *2*(5), 146–149. Retrieved 07/04/2022, from https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-2/issue-5/On-Cauchys-theorem-concerning-complex-integrals/bams/1183414568.full

Bogner, A., Littig, B., & Menz, W. (Eds.). (2009). *Interviewing experts*. Palgrave Macmillan. https://doi.org/10.1057/9780230244276

Bogner, A., Littig, B., & Menz, W. (2014). *Interviews mit Experten*. Springer. https://doi.org/10.1007/978-3-531-19416-5

Bohnsack, R. (2010). Documentary method and group discussions. In R. Bohnsack, N. Pfaff, & W. Weller (Eds.), *Qualitative analysis and documentary method in international educational research* (pp. 99–124). Barbara Budrich. https://doi.org/10.3224/86649236

Bohnsack, R. (2014a). Documentary method. In U. Flick (Ed.), *The SAGE handbook of qualitative data analysis* (pp. 217–233). SAGE. https://doi.org/10.4135/9781446282243.n15. (page numbers in the text refer to the online edition)

Bohnsack, R. (2014b). Rekonstruktive Sozialforschung: Einführung in qualitative Methoden (9., überarb. und erw. Auflage). https://doi.org/10.36198/9783838585543

Bohnsack, R., Pfaff, N., & Weller, W. (Eds.). (2010). *Qualitative analysis and documentary method in international educational research*. Barbara Budrich. https://doi.org/10.3224/86649236

Bolt, M. (2017). van der Pauw's theorem on sheet resistance. *PRIMUS*, *27*(8-9), 792–800. https://doi.org/10.1080/10511970.2016.1204576

Bornemann, F. (2016). *Funktionentheorie*. Birkhäuser. https://doi.org/10.1007/978-3-0348-0974-0

Bottazzini, U. (1986). *The higher calculus: A history of real and complex analysis from Euler to Weierstrass* (W. van Egmond, Trans.). Springer.

Bottazzini, U. (2003). Complex function theory, 1780–1900. In H. N. Jahnke (Ed.), *A history of analysis* (pp. 213–259). American Mathematical Society.

Bottazzini, U., & Gray, J. (2013). *Hidden harmony—geometric fantasies. The rise of complex function theory*. Springer. https://doi.org/10.1007/978-1-4614-5725-1

Braden, B. (1985). Picturing functions of a complex variable. *The College Mathematics Journal*, *16*(1), 63–72. https://doi.org/10.1080/07468342.1985.11972856

Braden, B. (1987). Pólya's picture of complex contour integrals. *Mathematics Magazine*, *60*(5), 321–327. https://doi.org/10.1080/0025570X.1987.11977332

Branchetti, L., Calza, G., Martani, S., & Saracco, A. (2020). Continuity of real functions in high school: A teaching sequence based on limits and topology. In T. Hausberger, M. Bosch, & F. Chellougui (Eds.), *Proceedings of the Third Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2020, September 12–19, 2020)* (pp. 73–82). University of Carthage; INDRUM.

Brill, A., & Noether, M. (1894). Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit. In G. Reimer, W. Dyck, & E. Lampe (Eds.), *Jahresbericht der Deutschen Mathematiker-Vereinigung. Dritter Band 1892–93* (pp. 107–566). Georg Reimer.

Brilleslyper, M. A., Dorff, M. A., McDougall, J. M., Rolf, J., Schaubroeck, L. E., Stankewitz, R. L., & Stephenson, K. (2012). *Explorations in complex analysis*. MAA Press. https://doi.org/10.1090/clrm/040

Brilleslyper, M. A., & Schaubroeck, B. (2017). Explorations of the Gauss-Lucas theorem. *PRIMUS*, *27*(8-9), 766–777. https://doi.org/10.1080/10511970.2016.1234525

Briot, C., & Bouquet, J.-C. (1859). *Théorie des fonctions elliptiques*. Gauthier-Villars.

Brown, J. W., & Churchill, R. V. (2009). *Complex variables and applications* (8th ed.). McGraw-Hill International Edition.

Buchholtz, N., & Behrens, D. (2014). „Anschaulichkeit" aus der Sicht von Lehramtsstudierenden. Ein didaktisches Prinzip für lehramtsspezifische Lehrveranstaltungen in der Studieneingangsphase. *mathematica didactica*, *37*, 137–162. https://doi.org/10.18716/ojs/md/2014.1124

Büchter, A., & Henn, H.-W. (2010). *Elementare Analysis. Von der Anschauung zur Theorie*. Spektrum Akademischer Verlag. https://doi.org/10.1007/978-3-8274-2680-2

Büchter, A., & Henn, H.-W. (2015). Schulmathematik und Realität – Verstehen durch Anwenden. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 19–49). Springer. https://doi.org/10.1007/978-3-642-35119-8_2

Burckel, R. B. (1979). *A introduction to classical complex analysis. Vol. 1*. Academic Press.

Burckel, R. B. (2021). *Classical analysis in the complex plane*. Birkhäuser. https://doi.org/10.1007/978-1-0716-1965-0

Burg, K., Haf, H., Wille, F., & Meister, A. (2012). *Vektoranalysis. Höhere Mathematik für Ingenieure, Naturwissenschaftler und Mathematiker* (2., überarb. Auflage). Springer Vieweg. https://doi.org/10.1007/978-3-8348-8346-9

Burgin, M. (2012). Definite integral: Basic approaches. *Integration: Mathematical Theory and Applications*, *3*(4), 383–396. Retrieved 04/06/2022, from https://www.proquest.com/scholarly-journals/definite-integral-basic-approaches/docview/1625504852/se-2?accountid=14136

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education. *For the Learning of Mathematics*, 375–393.

Burton, L. (2004). *Mathematicians as enquirers*. Kluwer. https://doi.org/10.1007/978-1-4020-7908-5

Caglayan, G. (2016). Mathematics teachers' visualization of complex number multiplication and the roots of unity in a dynamic geometry environment. *Computers in the Schools*, *33*(3), 187–209. https://doi.org/10.1080/07380569.2016.1218217

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Casorati, F. (1868). *Teorica delle funzioni di variabili complesse. Volume primo*. Tipografia dei Fratelli Fusi. Retrieved 02/02/2021, from http://resolver.sub.uni-goettingen.de/purl?PPN308951581

Cauchy, A.-L. (1823). *Résumé des leçons données à l'Ecole Royale Polytechnique sur le calcul infinitesimal*. De Bure frères, libraires du Roi et de la Bibliothèque du Roi.

Cauchy, A.-L. (1825). *Sur les intégrales définies, prises entre des limits imaginaires*. De Bure frères, libraires du Roi et de la Bibliothèque du Roi.

Cauchy, A.-L. (1882). Mémoire sur les intégrales définies. In *Oevres complètes d'Augustin Cauchy* (pp. 329–506). Gauthier-Villars. (Original work published 1814)

Chan, M. C. E., & Sfard, A. (2020). On learning that could have happened: The same tale in two cities. *The Journal of Mathematical Behavior*, *60*, 100815. https://doi.org/10.1016/j.jmathb.2020.100815

Chen, W. (2021, December 2). *But what are Polya vector fields? (and how can they be used to visualize complex integration?)* [Video]. Retrieved 12/02/2021, from https://www.youtube.com/watch?v=itEqPTJpxUo

Chin, K. E., & Jiew, F. F. (2018). A framework of making sense of mathematics. In F.-J. Hsieh (Ed.), *Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education: EARCOME 8* (pp. 309–319). National Taiwan Normal University. Retrieved 01/27/2022, from https://eprints.qut.edu.au/203257/

Chorlay, R. (2022). Accounting for the variability of lecturing practices in situations of concept introduction. *International Journal of Mathematical Education in Science and Technology*, *53*(5), 1071–1091. https://doi.org/10.1080/0020739X.2021.2014584

Clark, K. M. (2019). History and pedagogy of mathematics in mathematics education: History of the field, the potential of current examples, and directions for the future. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)* (pp. 29–55). Freudenthal Group & Freudenthal Institute, Utrecht University; ERME.

Clemens, K. (1982a). Visual imagery and school mathematics. *For the Learning of Mathematics*, *2*(2), 2–9.

Clemens, K. (1982b). Visual imagery and school mathematics. *For the Learning of Mathematics*, *2*(3), 33–38.

Clüver, T., & Salle, A. (2020). Grundvorstellungen – sichere Br´ucken oder ungewissen Pfade in die Hochschulanalysis? In H.-S. Siller, W. Weigel, & J. F. Wörler (Eds.), *Beiträge zum Mathematikunterricht 2020* (pp. 1329–1323). WTM. https://doi.org/10.17877/DE290R-21268

Cobb, P. (1994). Where is the mind? constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 13–20. https://doi.org/10.3102/0013189X023007013

Cobb, P. (2006). Discursive perspectives on mathematical learning: Commentary on Sfard's and Lerman's chapters. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 189–201). Sense.

Cohen, P. J. (1959). On Green's theorem. *Proceedings of the American Mathematical Society*, *10*(1), 109–112. https://doi.org/10.1090/S0002-9939-1959-0104249-3

Colyvan, M. (2012). *An introduction to the philosophy of mathematics*. Cambridge University Press. https://doi.org/10.1017/CBO9781139033107

Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. In S. Lerman (Ed.), *Handbook of research on the psychology of mathematics education. Past, present and future* (pp. 305–345). Sense. https://doi.org/10.1163/9789087901127_012

Connell, E. H. (1961). On properties of analytic functions. *Duke Math. J.*, *38*, 73–81. https://doi.org/10.1215/S0012-7094-61-02807-1

Connell, E. H. (1965). A classical theorem in complex variable. *The American Mathematical Monthly*, *72*(7), 729–732. https://doi.org/10.1080/00029890.1965.11970599

Conway, J. B. (1978). *Functions of one complex variable I* (2nd ed.). Springer. https://doi.org/10.1007/978-1-4612-6313-5 (Original work published 1973)

Cooper, J. (2016). *Mathematicians and primary school teachers learning from each other* (Doctoral dissertation). Weizmann Institute of Science. Rohovot, Israel. https://doi.org/10.34933/wis.000040

Cooper, J., & Kontorovich, I. (Eds.). (2021). Advances in commognitive research. *The Journal of Mathematical Behavior*, *60-62*.

Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses – A commognitive look at instructional design in the zone of proximal development. *Journal of Mathematical Behaviour*, *61*, 100822. https://doi.org/10.1016/j.jmathb.2020.100822

Cufí, J., & Verdera, J. (2015). A general form of Green's formula and the Cauchy integral theorem. *Proceedings of the Mathematical Society*, *143*(5), 2091–2102. https://doi.org/10.1090/S0002-9939-2014-12418-X

Custy, J. (2011, March 7). *The geometry of integrating a power around the origin* (Wolfram Demonstrations Project, Ed.). Retrieved 04/10/2021, from https://demonstrations.wolfram.com/TheGeometryOfIntegratingAPowerAroundTheOrigin/

Danckwerts, R., & Vogel, D. (2005). *Elementare Analysis*. Books on Demand.

Danckwerts, R., & Vogel, D. (2006). *Analysis verständlich unterrichten*. Spektrum Akademischer Verlag.

Danenhower, P. (2000). *Teaching and learning complex analysis at two British Columbia universities* (Doctoral dissertation). Simon Fraser University, Burnaby, Canada. Retrieved

02/02/2021, from http://www.nlc-bnc.ca/obj/s4/f2/dsk1/tape3/PQDD_0008/NQ61636.pdf

Danenhower, P. (2006). Introductory complex analysis at two British Columbia universities: The first week–complex numbers. In F. Hitt, G. Harel, & A. Selden (Eds.), *Research in collegiate mathematics education. VI* (pp. 139–170). AMS.

D'Angelo, J. P. (2017). Complex variables throughout the curriculum. *PRIMUS*, *27*(8-9), 778–791. https://doi.org/10.1080/10511970.2016.1235642

Daniell, P. J. (1918). A general form of integral. *Annals of Mathematics*, *19*(4), 279–294. https://doi.org/10.2307/1967495

Davis, P., Hersh, R., & Marchisotto, E. A. C. (2012). *The mathematical experience. Study edition. Updated with epilogues by the authors*. Birkhäuser. https://doi.org/10.1007/978-0-8176-8295-8

Denscombe, M. (2010). *The good research guide for small-scale social research projects* (4th ed.). Open University Press.

Deppermann, A. (2008). *Gespräche analysieren. Eine Einführung* (4th ed.). VS Verlag für Sozialwissenschaften. https://doi.org/10.1007/978-3-531-91973-7

Dieudonné, J. (1966). *Foundations of modern analysis* (6th ed.). Academic Press.

Dittman, M., Soto-Johnson, H., Dickinson, S., & Harr, T. (2017). Game building with complex-valued functions. *PRIMUS*, *27*(8), 869–879. https://doi.org/10.1080/10511970.2016.1234527

Dixon, J. D. (1971). A brief proof of Cauchy's integral theorem. *Proceedings of the American Mathematical Society*, *29*(3), 625–626. https://doi.org/10.1090/S0002-9939-1971-0277699-8

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education*, *13*(5), 360. https://doi.org/10.2307/749011

Driver, D. A., & Tarran, D. S. G. (1989). Five approaches to the teaching of complex numbers. *Teaching Mathematics and its Applications*, *8*(3), 122–127. https://doi.org/10.1093/teamat/8.3.122

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 21, October 23–26, 1999)* (pp. 3–26). PME.

Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization. *ZDM*, *46*(1), 159–170. https://doi.org/10.1007/s11858-013-0565-8

Duval, R. (2017). *Understanding the mathematical way of thinking – The registers of semiotic representations*. Springer. https://doi.org/10.1007/978-3-319-56910-9

Eggleston, H., & Ursell, H. (1952). On the lightness and strong interiority of analytic functions. *Journal of the London Mathematical Society*, *27*(3), 260–271. https://doi.org/10.1112/jlms/s1-27.3.260

Eisenhardt, K. M. (1989). Building theories from case study research. *The Academy of Management Review*, *14*(4), 532–550. https://doi.org/10.5465/amr.1989.4308385

Eisenhardt, K. M., & Graebner, M. E. (2007). Theory building from cases: Opportunities and challenges. *Academy of Management Journal*, *50*(1), 25–32. https://doi.org/10.5465/amj.2007.24160888

Elstrodt, J. (2018). *Maß- und Integrationstheorie* (8., erw. und aktualisierte Aufl.). Springer Spektrum. https://doi.org/10.1007/978-3-662-57939-8

Erath, K., Prediger, S., Quasthoff, U., & Heller, V. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: The case of explaining to learn and learning to explain. *Educational Studies in Mathematics*, *99*(2), 161–179. https://doi.org/10.1007/s10649-018-9830-7

Ericsson, K. A., Hoffman, R. R., Kozbelt, A., & Williams, A. M. (Eds.). (2018). *The Cambridge handbook of expertise and expert performance* (2nd ed.). Cambridge University Press. https://doi.org/10.1017/9781316480748

Ernest, P. (1991). *The philosophy of mathematics education*. Routledge Falmer.

Ernest, P. (1994). Constructivism: Which form provides the most adequate theory of mathematics learning? *Journal für Mathematikdidaktik*, *15*(3-4), 327–342. https://doi.org/10.1007/BF03338812

Ernest, P. (2010). Reflections on theories of learning. In B. Sriraman & L. English (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 39–47). Springer. https://doi.org/10.1007/978-3-642-00742-2_4

Ettlinger, H. J. (1922). Cauchy's paper of 1814 on definite integrals. *Annals of Mathematics, Second Series*, *23*(3), 255–270. https://doi.org/10.2307/1967922

Farris, F. A. (1998). Visual complex analysis. By Tristan Needham. *The American Mathematical Monthly*, *105*(6), 570–576. https://doi.org/10.1080/00029890.1998.12004928

Farris, F. A. (2017). Domain coloring and the argument principle. *PRIMUS*, *27*(8-9), 827–844. https://doi.org/10.1080/10511970.2016.1234526

Fernández-León, A., Gavilán-Izquierdo, J. M., & Toscano, R. (2021). A case study of the practices of conjecturing and proving of research mathematicians. *International Journal of Mathematical Education in Science and Technology*, *52*(5), 767–781. https://doi.org/10.1080/0020739X.2020.1717658

Ferus, D. (2009). *Komplexe Aanalysis. Vorlesungsskript Sommersemester 2008* (tech. rep.). TU Berlin. Retrieved 02/02/2021, from https://page.math.tu-berlin.de/ferus/KA/komplexe_Analysis_2008.pdf

Feudel, F. (2020). *Die Ableitung in der Mathematik für Wirtschaftswissenschaftler. Analysen zum benötigten, gelehrten und von Studierenden erreichten Verständnis des Ableitungsbegriffs*. Springer Spektrum. https://doi.org/10.1007/978-3-658-26478-9

Feudel, F., & Biehler, R. (2021a). Students' understanding of the derivative concept in the context of mathematics for economis. *Journal für Mathematik-Didaktik*, *42*(1), 273–305. https://doi.org/10.1007/s13138-020-00174-z

Feudel, F., & Biehler, R. (2021b). Students' understanding of the economic interpretation of the derivative in the context of marginal cost. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-021-00144-x

Fischbein, E. (1987). *Intuition in science and mathematics. An educational approach*. Reidel.

Fischer, A. (2006). *Vorstellungen zur linearen Algebra: Konstruktionsprozesse und -ergebnisse von Studierenden* (Dissertation). Universität Dortmund.

Fischer, W., & Lieb, I. (2003). *Funktionentheorie: Komplexe Analysis in einer Veränderlichen* (8., neubearbeitete Auflage). Springer. https://doi.org/10.1007/978-3-322-96973-6

Fischer, W., & Lieb, I. (2010). *Einführung in die Komplexe Analysis: Elemente der Funktionentheorie*. Vieweg+Teubner. https://doi.org/10.1007/978-3-8348-9377-2

Flatto, L., & Shisha, O. (1973). A proof of cauchy's integral theorem. *Journal of Approximation Theory*, *7*(4), 386–390. https://doi.org/10.1016/0021-9045(73)90040-3

Flick, U. (2019). Gütekriterien qualitativer Sozialforschung. In N. Baur & J. Blasius (Eds.), *Handbuch Methoden der empirischen Sozialforschung* (2nd ed., pp. 473–488). Springer VS. https://doi.org/10.1007/978-3-658-21308-4_33

Flores H., L. (2001). The imagination's piano in Wittgenstein's Philosophische Untersuchungen. In R. Haller & K. Puhl (Eds.), *Wittgenstein and the future of philosophy. A reassessment after 50 years. Papers of the 24th International Wittgenstein Symposiums (August 12–18, 2001, Kirchberg am Wechsel)* (pp. 218–223). Österreichische Ludwig Wittgenstein Gesellschaft. Retrieved 09/05/2022, from https://www.alws.at/alws/wp-content/uploads/2018/06/papers-2001a.pdf

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry*, *12*(2), 219–245. https://doi.org/10.1177/1077800405284363

Flyvbjerg, B. (2011). Case study. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (4th ed., pp. 301–316).

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*(1), 97–124. https://doi.org/10.1007/s10649-012-9411-0

Forst, W., & Hoffmann, D. (2012). *Funktionentheorie erkunden mit Maple* (2., überarb. und aktualisierte Aufl.). Springer Spektrum. https://doi.org/10.1007/978-3-642-29412-9

Forster, O. (2016). *Analysis 1. Differential-und Integralrechnung einer Veränderlichen* (12., verbesserte Auflage). Springer. https://doi.org/10.1007/978-3-658-11545-6

Forster, O. (2017a). *Analysis 2. Differentialrechnung im* $\mathbb{R}^n$*, gewöhnliche Differentialgleichungen* (11., erw. Auflage). Springer. https://doi.org/10.1007/978-3-658-19411-6

Forster, O. (2017b). *Analysis 3. Maß- und Integrationstheorie, Integralsätze im* $\mathbb{R}^n$ *und Anwendungen* (8., verbesserte Auflage). Springer. https://doi.org/10.1007/978-3-658-16746-2

Freitag, E., & Busam, R. (2006). *Funktionentheorie 1* (4., korrigierte und erw. Aufl.). Springer. https://doi.org/10.1007/3-540-32058-X

Fritzsche, K. (2019). *Grundkurs Funktionentheorie. Eine Einführung in die komplexe Analysis und ihre Anwendungen* (2nd ed.). Springer Spektrum. https://doi.org/10.1007/978-3-662-60382-6

Fukawa-Connelly, T., Weber, K., & Mejía-Ramos, J. P. (2017). Informal content and student notetaking in advanced mathematics classes. *Journal for Research in Mathematics Education*, *48*(5), 567–579. https://doi.org/10.5951/jresematheduc.48.5.0567

Fuß, S., & Karbach, U. (2019). *Grundlagen der Transkription* (2nd ed.). Barbara Budrich. https://doi.org/10.36198/9783838550749

Galbis, A., & Maestre, M. (2012). *Vector analysis versus vector calculus*. Springer. https://doi.org/10.1007/978-1-4614-2200-6

Gallego-Sánchez, I., González, A., & Gavilán-Izquierdo, J. M. (2022). Analyzing pedagogical routines in the upper secondary school teacher's discourse using the commognitive approach. *International Journal of Instruction*, *15*(3), 291–306. https://doi.org/10.29333/iji.2022.15316a

Garcia, S. R. (2017). Advanced linear algebra: A call for the early introduction of complex numbers. *PRIMUS*, *27*(8-9), 856–868. https://doi.org/10.1080/10511970.2016.1235644

Garcia, S. R., & Ross, W. T. (2017). Approaching Cauchy's theorem. *PRIMUS*, *27*(8-9), 758–765. https://doi.org/10.1080/10511970.2016.1234525

Gathmann, A. (2017). *Einführung in die Funktionentheorie. Vorlesungsskript Wintersemester 2016/17* (lecture notes). TU Kaiserslautern. Retrieved 04/23/2021, from https://www.mathematik.uni-kl.de/gathmann/class/futheo-2016/futheo-2016.pdf

Gavilán-Izquierdo, J. M., & Gallego-Sánchez, I. (2021). How an upper secondary school teacher provides resources for the transition to university: A case study. *International Electronic Journal of Mathematics Education*, *16*(2), em0634. https://doi.org/10.29333/iejme/10892

Gerring, J. (2014). What is a case study and what is it good for? *American Political Review*, *98*(2), 341–354. https://doi.org/10.1017/S0003055404001182

Ghedamsi, I. (2017). A micro-model of didactical variables to explore the mathematical organization of complex numbers at upper secondary level. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1–5, 2017)* (pp. 2065–2972). DCU Institute of Education; ERME.

Giaquinto, M. (2020). The epistemology of visual thinking in mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Spring 2020). Metaphysics Research Lab, Stanford University. Retrieved 02/17/2022, from https://plato.stanford.edu/archives/spr2020/entries/epistemology-visual-thinking/

Gillman, L. (1993). An axiomatic approach to the integral. *The American Mathematical Monthly*, *100*(1), 16–25. https://doi.org/10.2307/2324809

Gläser, J., & Laudel, G. (2010). *Experteninterviews und qualitative Inhaltsanalyse als Instrumente rekonstruierender Untersuchungen* (4th ed.). VS.

Globevnik, J. (1990). Zero integrals on circles and characterizations of harmonic and analytic functions. *Transactions of the American Mathematical Society*, *317*(1), 313–330. https://doi.org/10.2307/2001464

Glock, H.-J. (Ed.). (1996). *A Wittgenstein dictionary*. Blackwell.

Gluchoff, A. (1991). A simple interpretation of the complex contour integral. *The American Mathematical Monthly*, *98*(7), 641–644. https://doi.org/10.1080/00029890.1991.11995771

Gluchoff, A. D. (1993). Complex power series – a vector field visualization. *Mathematics Magazine*, *66*(3), 189–191. https://doi.org/10.1080/0025570X.1993.11996116

Goffman, C. (1977). A bounded derivative which is not Riemann integrable. *The American Mathematical Monthly*, *84*(3), 205–206. https://doi.org/10.2307/2319494

Goffman, E. (1986). *Frame analysis. An essay on the organization of experience*. Northeastern University Press. (Original work published 1974)

González, M. O. (1992). *Classical complex analysis*. Marcel Dekker.

González-Velasco, E. A. (1980). The homotopic proof of cauchy's integral theorem. *International Journal of Mathematical Education in Science and Technology*, *11*(2), 189–191. https://doi.org/10.1080/0020739800110207

Gordon, R. A. (2016). A bounded derivative that is not Riemann integrable. *Mathematics Magazine*, *89*(5), 364–370. https://doi.org/10.4169/math.mag.89.5.364

Goursat, E. (1884). Démonstration du théorème de Cauchy. Extrait d'une lettre adressée à M. Hermite. *Acta mathematica*, *4*(11), 197–200.

Goursat, E. (1900). Sur la definition general des fonctions analytiques, d'après Cauchy. *Transactions of the American Mathematical Society*, *1*(1), 14–16. https://doi.org/10.2307/1986398

Grabiner, J. V. (2005). *The origin's of Cauchy's rigorous calculus*. Dover. (Original work published 1981)

Grandy, G. (2010). Instrumental case study. In A. J. Mills, G. Dureos, & E. Wiebe (Eds.), *Encyclopedia of case study research* (pp. 473–475). SAGE. https://doi.org/10.4135/9781412957397.n175

Grattan-Guinness, I. (2005a). A.-L. Cauchy, Cours d'analyse (1821) and Résumé of the calculus (1823). In I. Grattan-Guinness (Ed.), *Landmark writings in western mathematics 1640–1940* (pp. 341–353). Elsevier. https://doi.org/10.1016/B978-044450871-3/50106-6

Grattan-Guinness, I. (Ed.). (2005b). *Landmark writings in western mathematics 1640–1940*. Elsevier. https://doi.org/10.1016/B978-0-444-50871-3.X5080-3

Gray, J. (2000). Goursat, Pringsheim, Walsh, and the Cauchy integral theorem. *The Mathematical Intelligencer*, *22*, 60–66, & 77. https://doi.org/10.1007/BF03026773

Gray, J. (2015). *The real and the complex: A history of analysis in the 19th century*. Springer. https://doi.org/10.1007/978-3-319-23715-2

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2016a). Aspects and "Grundvorstellungen" of the concepts of derivative and integral. Subject matter-related didactical perspectives of concept formation. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 99–129. https://doi.org/10.1007/s13138-016-0100-x

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2016b). *Didaktik der Analysis. Aspekte und Grundvorstellungen zentraler Begriffe*. Springer. https://doi.org/10.1007/978-3-662-48877-5

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2021a). Basic mental models of integrals: Theoretical conception, development of a test instrument, and first results. *ZDM*, *53*(3), 649–661. https://doi.org/10.1007/s11858-020-01207-0

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2021b). Test zur Erfassung von Grundvorstellungen zu Ableitungen und Integralen (GV-AI), Empirische Erfassung von Grundvorstellungen zur ersten Ableitung einer Funktion an einer Stelle und zum bestimmten Integral. https://hal.archives-ouvertes.fr/hal-03103685

Greefrath, G., Oldenburg, R., Siller, H.-S., Ulm, V., & Weigand, H.-G. (2022). Mathematics students' characteristics of basic mental models of the derivative [online first]. *Journal für Mathematik-Didaktik*. https://doi.org/10.1007/s13138-022-00207-9

Greene, R. E., & Krantz, S. G. (2006). *Function theory of one complex variable*. American Mathematical Society.

Greenleaf, F. P. (1972). *Introduction to complex variables*. Saunders.

Griesel, H., vom Hofe, R., & Blum, W. (2019). Das Konzept der Grundvorstellungen im Rahmen der mathematischen und kognitionspsychologischen Begrifflichkeit in der Mathematikdidaktik. *Journal für Mathematik-Didaktik*, *40*(1), 123–133. https://doi.org/10.1007/s13138-019-00140-4

Griffiths, M. (2013). Intuiting the fundamental theorem of arithmetic. *Educational Studies in Mathematics*, *82*(1), 75–96. https://doi.org/10.1007/s10649-012-9410-1

Grønbaek, N., & Winsløw, C. (2015). Media and milieus for complex numbers: An experiment with Maple based text. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9, February 4–8, 2015)* (pp. 2131–2137). Charles University in Prague, Faculty of Education; ERME.

Güçler, B. (2016). Making implicit metalevel rules of the discourse on function explicit topics of reflection in the classroom to foster student learning. *Educational Studies in Mathematics*, *91*(3), 375–393.

Gutiérrez, A. (1996). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th International Conference of the PME (PME 20, July 8–12, 1996)* (pp. 3–19). University of Valencia.

Hackenberger, F. (n.d.). *QTikZ - Editor for the TikZ language*. Version 0.10.1. Retrieved 03/25/2020, from http://www.hackenberger.at/blog/ktikz-editor-for-the-tikz-language/

Hacker, P. M. S. (1990). *Wittgenstein. Meaning and mind: An analytical commentary on the* Philosophical Investigations. Basil Blackwell.

Hadamard, J. (1945). *The mathematician's mind. The psychology of invention in the mathematical field*. Princeton University Press.

Hahn, S., & Prediger, S. (2008). Bestand und änderung – Ein Beitrag zur Didaktischen Rekonstruktion in der Analysis. *Journal für Mathematik-Didaktik*, *29*(3-4), 163–198. https://doi.org/10.1007/BF03339061

Hales, T. C. (2007). Jordan's proof of the Jordan curve theorem. *Studies in Logic, Grammar and Rhetoric*, *10*(23), 45–60.

Hammer, D., Elby, A., Scherr, R., & Redish, E. F. (2005). Resources, framing, and transfer. In J. P. Mestre (Ed.), *Transfer of learning from a modern multidisciplinary perspective* (pp. 89–119). Information Age Publishing.

Hamza, S. F. (2012). *A study of concept images and concept definitions related to metric spaces* (Doctoral dissertation). National University of Ireland Maynooth. Retrieved 03/11/2021, from http://mural.maynoothuniversity.ie/4393/

Hamza, S., & O'Shea, A. (2016). Concept images of open sets in metric spaces. In E. Nardi, C. Winsløw, & T. Hausberger (Eds.), *Proceedings of the First Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2016, March 31 – April 2, 2016)* (pp. 113–122). University of Montpellier; INDRUM.

Hanche-Olsen, H. (2008). On Goursat's proof of Cauchy's integral theorem. *The American Mathematical Monthly*, *115*(7), 648–652. https://doi.org/10.1080/00029890.2008.11920575

Hancock, B. (2017). Undergraduates' reasoning about integration of complex functions within three worlds of mathematics. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1231–1237). SIGMAA.

Hancock, B. A. (2018). *Undergraduates' collective argumentation regarding integration of complex functions within three worlds of mathematics* (Doctoral dissertation). University of Northern Colorado, Greeley, CO. Retrieved 02/02/2021, from https://digscholarship.unco.edu/dissertations/492/

Hancock, B. (2019). From qualification to consensus: The role of multimodal uncertainty in collective argumentation regarding complex integration. *Journal of Mathematical Behaviour*, *55*, 100700. https://doi.org/10.1016/j.jmathb.2019.03.007

Hanke, E. (2016). *Ausprägung und Akzeptanz von Vorstellungen zur Stetigkeit bei Mathematikstudierenden* (Master's thesis). University of Bremen. Bremen, Germany.

Hanke, E. (2017). Empirie kommunikativer Abbilder am Beispiel von Vorstellungen von Mathematikstudierenden zur Stetigkeit. In U. Kortenkamp & A. Kuzle (Eds.), *Beiträge zum Mathematikunterricht 2017* (pp. 373–376). WTM. https://doi.org/10.17877/DE290R-18511

Hanke, E. (2018). "A function is continuous of and only if you can draw its graph without lifting the pen from the paper" – Concept usage in proofs by students in a topology course. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018, April 5–7, 2018)* (pp. 44–53). University of Agder; INDRUM.

Hanke, E. (2019). Anschauliche Deutungen des komplexen Wegintegrals und der Cauchyschen Integralformel von Expert*innen der Funktionentheorie. In A. Frank, S. Krauss, & K. Binder (Eds.), *Beiträge zum Mathematikunterricht 2019* (pp. 321–324). WTM. https://doi.org/10.17877/DE290R-20856

Hanke, E. (2020a). Intuitive mathematical discourse about the complex path integral. In T. Hausberger, M. Bosch, & F. Chellougui (Eds.), *Proceedings of the Third Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2020, September 12–19, 2020)* (pp. 103–112). University of Carthage; INDRUM.

Hanke, E. (2020b). Vorstellungen im intuitiven mathematischen Diskurs. In H.-S. Siller, W. Weigel, & J. F. Wörler (Eds.), *Beiträge zum Mathematikunterricht 2020* (pp. 385–388). WTM. https://doi.org/10.17877/DE290R-21347

Hanke, E. (2022a). Aspects of complex path integrals. *Proceedings of the Twelth Congress of the European Society for Research in Mathematics Education (CERME12, February 2–6, 2022)*. Retrieved 09/13/2022, from https://hal.archives-ouvertes.fr/hal-03754865. Poster available at https://doi.org/10.13140/RG.2.2.15916.74887

Hanke, E. (2022b). Vertical coherence in the teaching of integrals? An example from complex analysis [paper submitted for presentation at the Fourth Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2022, October 19–22, 2022)].

Hanke, E. (2022c). Vorstellungen und Aspekte vom komplexen Wegintegral [paper submitted for presentation at the GDM-Jahrestagung 2022 (GDM 2022, August 29 – September 02, 2022)]. *Beiträge zum Mathematikunterricht 2022*.

Hanke, E., Hehner, S., & Bikner-Ahsbahs, A. (2021). Reducing fragmentation in university pre-service teacher education – Conditions and strategies. *EDeR – Educational Design Research*, *5*(2). https://doi.org/10.15460/eder.5.2.1613

Hanke, E., & Schäfer, I. (2017). Students' view of continuity. An empirical analysis of mental images and their usage. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1–5, 2017)* (pp. 2081–2088). DCU Institute of Education; ERME.

Hanke, E., & Schäfer, I. (2018). Learning complex analysis in different branches – Project Spotlight-Y for future teachers. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018, April 5–7, 2018)* (pp. 54–63). University of Agder; INDRUM.

Hanna, G., & de Villiers, M. (Eds.). (2021). Springer. https://doi.org/10.1007/978-94-007-2129-6 (Original work published 2012)

Harel, G. (2013). DNR-based curricula: The case of complex numbers. *Journal of Humanistic Mathematics*, *3*(2), 2–61. https://doi.org/10.5642/jhummath.201302.03

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). National Council of Teachers of Mathematics.

Harmse, J. E. (2008). Extensions of the Cauchy-Goursat integral theorem. *Journal of Mathematical Analysis and Applications*, *339*(1), 429–437. https://doi.org/10.1016/j.jmaa.2007.06.063

Harré, R., & Gillett, G. (2010). *The discursive mind*. SAGE.

Hefendehl-Hebeker, L. (2016). Subject-matter didactics in German traditions. Early historical developments. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 11–31. https://doi.org/10.1007/s13138-016-0103-7

Hefendehl-Hebeker, L., vom Hofe, R., Büchter, A., Humenberger, H., Schulz, A., & Wartha, S. (2019). Subject-matter didactics. In H. N. Jahnke & L. Hefendehl-Hebeker (Eds.), *Traditions in German-speaking mathematics education research* (pp. 25–60). SpringerOpen. https://doi.org/10.1007/978-3-030-11069-7_2

Heffter, L. (1951). Zur Begründung der Funktionentheorie. *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse*, *6*. https://doi.org/10.1007/978-3-662-22636-0

Heffter, L. (1960). *Begründung der Funktionentheorie auf alten und neuen Wegen* (2. wesentlich verbesserte Auflage). Springer. https://doi.org/10.1007/978-3-642-49164-1

Heins, M. (1968). *Complex function theory*. Academic Press.

Heintz, B. (2000). *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*. Springer.

Helfferich, C. (2011). *Die Qualität qualitativer Daten. Manual für die Durchführung qualitativer Interviews* (4th ed.). VS Verlag für Sozialwissenschaften. https://doi.org/10.1007/978-3-531-92076-4

Herfort, P., & Reinhardt, G. (1980). *Mathematik. Studienbriefe zur Fachdidaktik für Lehrer der Sekundarstufe II: Zugänge zur Integralrechnung. Analysis MA 3*. Deutsches Institut für Fernstudien an der Universität Tübingen.

Hering, L., & Jungmann, R. (2019). Einzelfallanalyse. In N. Baur & J. Blasius (Eds.), *Handbuch Methoden der empirischen Sozialforschung* (2nd ed., pp. 619–632). Springer VS. https://doi.org/10.1007/978-3-658-21308-4_41

Hermanns, H. (2017). Interviewen als Tätigkeit. In U. Flick, E. von Kardorff, & I. Steinke (Eds.), *Qualitative Forschung. Ein Handbuch* (12th ed., pp. 360–369). rowohlt.

Hernández Rodríguez, O., & López Fernández, J. M. (2013). The mean value inequality and the fundamental theorem of calculus. Retrieved 04/06/2022, from https://www.researchgate.net/publication/309537032_THE_MEAN_VALUE_INEQUALITY_AND_THE_FUNDAMENTAL_THEOREM_OF_CALCULUS

Hersh, R. (1998). *What is mathematics, really?* Vintage.

Hersh, R. (2014). *Experiencing mathematics. What do we do, when we do mathematics?* AMS.

Heuser, H. (2003). *Lehrbuch der Analysis. Teil 1* (15., durchgesehene Aufl.). Teubner. https://doi.org/10.1007/978-3-322-96828-9

Heuser, H. (2008). *Lehrbuch der Analysis. Teil 2: Mehrdimensionale Integralrechnung und ihre Anwendungen* (14. Auflage). Vieweg + Teubner.

Heyd-Metzuyanim, E., & Graven, M. (Eds.). (2019). Rituals and explorations in mathematical teaching and learning. *Educational Studies in Mathematics*, *101*(2).

Heyd-Metzuyanim, E., Munter, C., & Greeno, J. (2018). Conflicting frames: A case of misalignment between professoinal development efforts and a teacher's practice in a high school mathematics classroom. *Educational Studies in Mathematics*, *97*(1), 21–37. https://doi.org/10.1007/s10649-017-9777-0

Heyd-Metzuyanim, E., & Shabtay, G. (2019). Narratives of 'good' instruction: Teachers' identities as drawing on exploration vs. acquisition pedagogical discourses. *ZDM*, *51*(3), 541–554. https://doi.org/10.1007/s11858-018-01019-3

Hirn, A., & Weiß, C. (2018). *Analysis – Grundlagen und Exkurse. Grundprinzipien der Differential- und Integralrechnung*. Springer Spektrum. https://doi.org/10.1007/978-3-662-55538-5

Hochkirchen, T. (2003). Theory of measure and integration from Riemann to Lebesgue. In H. N. Jahnke (Ed.), *A history of analysis* (pp. 261–290). American Mathematical Society.

Hochmuth, R. (2021a). Analysis tasks based on a theorem of mathematical practices in signal theory courses. *International Journal of Research in Undergraduate Mathematics Education*, *53*(5), 1113–1132. https://doi.org/10.1080/0020739X.2021.1978572

Hochmuth, R. (2021b). Fachliche Analysen als Grundlage hochschuldidaktischer Interventionen – Einführung. In R. Biehler, A. Eichler, R. Hochmuth, S. Rach, & N. Schaper (Eds.), *Lehrinnovationen in der Hochschulmathematik. praxisrelevant – didaktisch fundiert – forschungsbasiert* (pp. 9–17). Springer Spektrum. https://doi.org/10.1007/978-3-662-62854-6_2

Hochmuth, R., Broley, L., & Nardi, E. (2021). Transitions to, across and beyond university. In V. Durand-Guerrier, R. Hochmuth, E. Nardi, & C. Winsløw (Eds.), *Research and development in university mathematics education. Overview produced by the International Network for Didactic Research in University Mathematics* (pp. 193–215). Routledge. https://doi.org/10.4324/9780429346859-14

vom Hofe, R. (1992). Grundvorstellungen mathematischer Inhalte als didaktisches Modell. *Journal für Mathematik-Didaktik*, *13*(4), 345–364. https://doi.org/10.1007/BF03338785 vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*. Spektrum.

vom Hofe, R. (1998). On the generation of basic ideas and individual images: Normative, descriptive and constructive aspects. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 317–331). Kluwer.

vom Hofe, R. (2003). Grundbildung durch Grundvorstellungen. *Mathematik lehren*, *118*, 4–8.

vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a category of subject-matter didactics. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 225–254. https://doi.org/10.1007/s13138-016-0107-3

Holland, A. S. B. (1980). *Complex function theory*. Elsevier North Holland.

Hopf, C. (2017). Qualitative Interviews – ein überblick. In U. Flick, E. von Kardorff, & I. Steinke (Eds.), *Qualitative Forschung. Ein Handbuch* (12th ed., pp. 349–360). rowohlt.

Howell, R., Noell, A., & Zorn, P. (Eds.). (2017a). Revitalizing complex analysis. *PRIMUS*, *27*(8–9).

Howell, R., Noell, A., & Zorn, P. (Eds.). (2017b). Revitalizing complex analysis. *PRIMUS*, *27*(8–9), 755–757. https://doi.org/10.1080/10511970.2017.1312652

Howell, R. W., & Schrohe, E. (2017). Unpacking Rouché's theorem. *PRIMUS*, *27*(8-9), 814–826. https://doi.org/10.1080/10511970.2016.1235646

Hußmann, S., & Prediger, S. (2016). Specifying and structuring mathematical topics. A four-level approach for combining formal, semantic, concrete, and empirical levels exemplified for exponential growth. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 33–67. https://doi.org/10.1007/s13138-016-0102-8

Hußmann, S., Rezat, S., & Sträßer, R. (2016). Subject matter didactics in mathematics education. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 1–9. https://doi.org/10.1007/s13138-016-0105-5

Ioannou, M. (2012). *Conceptual and learning issues in mathematics undergraduates first encounter with group theory: A commognitive analysis* (Doctoral dissertation). University of East Anglia, Norwich, England. Retrieved 05/07/2021, from https://ueaeprints.uea.ac.uk/id/eprint/39453

Ioannou, M. (2018). Commognitive analysis of undergraduate mathematics students' first encounter with the subgroup test. *Mathematics Education Research Journal*, *30*(2), 117–142. https://doi.org/10.1007/s13394-017-0222-6

Ionescu, L.-M., Pripoae, C.-L., & Pripoae, G.-T. (2021). Classification of holomorphic functions as Pólya vector fields via differential geometry. *mathematics*, *9*(1890). https://doi.org/10.3390/math9161890

Isaev, A. (2017). *Twenty-one lectures on complex analysis. A first course*. Springer. https://doi.org/10.1007/978-3-319-68170-2

Jahnke, H. N. (Ed.). (2003). *A history of analysis*. American Mathematical Society.

Jänich, K. (2001). *Analysis für Physiker und Ingenieure. Funktionentheorie, Differentialgleichungen, Spezielle Funktionen* (4th ed.). Springer. https://doi.org/10.10071978-3-662-05703-2

Jänich, K. (2004). *Funktionentheorie. Eine Einführung* (6th ed.). Springer. https://doi.org/10.1007/3-540-35015-2

Jänich, K. (2005). *Vektoranalysis* (5th ed.). Springer. https://doi.org/10.1007/b138936

Janković, S., & Merkle, M. (2008). A mean value theorem for systems of integrals. *Journal of Mathematical Analysis and Applications*, *342*(1), 334–339. https://doi.org/https://doi.org/10.1016/j.jmaa.2007.12.012

Jaworski, B., Treffert-Thomas, S., Hewitt, D., Feeney, M., Shrish-Thapa, D., Conniffe, D., Dar, A., Vlaseros, N., & Anastasakis, M. (2018). Student partners in task design in a computer medium to promote foundation students' learning of mathematics. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of the Second Conference of the International Network for Didactic Research in University Mathematics (INDRUM 2018, April 5–7, 2018)* (pp. 316–325). University of Agder; INDRUM.

Jayakody, G. N. (2015). *University first year students' discourse on continuous functions: A commognitive interpretation* (Doctoral dissertation). Simon Fraser University, Burnaby, Canada. Retrieved 01/17/2020, from https://summit.sfu.ca/item/15768

Jeffrey, A. (1992). *Complex analysis and applications*. CRC.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behaviour*, *32*, 122–141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behaviour*, *38*, 9–28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R. (2018). Prototype images in mathematics education: The case of the graphical representation of the definite integral. *Educational Studies in Mathematics*, *97*(3), 215–234. https://doi.org/10.1007/s10649-017-9794-z

Jones, S. R. (2020). Scalar and vector line integrals: A conceptual analysis and an initial investigation of student understanding. *The Journal of Mathematical Behavior*, *59*, 100801. https://doi.org/10.1016/j.jmathb.2020.100801

Jung, W. (1992). Probing acceptance, a technique for investigating learning dificulties. In R. Duit, F. Goldberg, & H. Niedderer (Eds.), *Research in physics learning. Issues and empirical studies. Proceedings of an international workshop at the University of Bremen* (pp. 278–295). IPN.

Karakok, G., Soto-Johnson, H., & Dyben, S. A. (2013). In-service secondary teachers' conceptualization of complex numbers. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *16th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1–7). SIGMAA.

Karakok, G., Soto-Johnson, H., & Dyben, S. A. (2015). Secondary teachers' conception of various forms of complex numbers. *Journal of Mathematics Teacher Education*, *18*(4), 327–351. https://doi.org/10.1007/s10857-014-9288-1

Karatsuba, A. A. (1995). *Complex analysis in number theory*. CRC.

Karavi, T., & Mali, A. (2022). Investigation of the connections within proof in complex analysis lecturing. *Proceedings of the Twelth Congress of the European Society for Research in Mathematics Education (CERME12, February 2–6, 2022)*. Retrieved 09/13/2022, from https://hal.archives-ouvertes.fr/hal-03754691

Karavi, T., Mali, A., & Avraamidou, L. (2022). Commognition as an approach to studying proof teaching in university mathematics lectures. *Eurasia Journal of Mathematics, Science and Technology Education*, *18*(7), em2132. https://doi.org/10.29333/ejmste/12173

Kattmann, U., & Gropengießer, H. (1996). Modellierung der didaktischen Rekonstruktion. In R. Duit & C. von Rhöneck (Eds.), *Lernen in den naturwissenschaften* (pp. 180–204). IPN.

Kattmann, U., Duit, R., Gropenßer, H., & Komorek, M. (1997). Das Modell der Didaktischen Rekonstruktion - Ein Rahmen für naturwissenschaftliche Forschung und Entwicklung. *Zeitschrift für Didaktik der Naturwissenschaften*, *3*(3), 3–18. https://doi.org/10.2565/01:13710

Kemmer, N. (1977). *Vector analysis: A physicist's guide to the mathematics of fields in three dimensions*. University Press. https://doi.org/10.1017/CBO9780511569524

Kidron, I., & Dreyfus, T. (2014). Proof image. *Educational Studies in Mathematics*, *87*, 297–321. https://doi.org/10.1007/s10649-014-9566-y

Kidron, I. (2020). Calculus teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 87–93). Springer. https://doi.org/10.1007/978-3-030-15789-0_18

Kiesow, C. (2016). *Die Mathematik als Denkwert. Eine Studie zur kommunikativen und visuellen Performanz mathematischen Wissens*. Springer VS. https://doi.org/10.1007/978-3-658-11410-7

Kinney, W. M. (2013). Teaching complex analysis as a lab-type course with a focus on geometric interpretations using *Mathematica*. In E. Gossett (Ed.), *ACMS 19th Biennal Conference Proceedings (May 29–June 1, 2013)* (pp. 103–120). Association of Christians in the Mathematical Sciences. Retrieved 04/10/2021, from https://pillars.taylor.edu/acms-2013/6/

Kinney, W. M. (2017). Using modules in teaching complex analysis. *PRIMUS*, *27*(8), 880–888. https://doi.org/10.1080/10511970.2016.1249321

Kirsch, A. (2014). The fundamental theorem of calculus: Visually? *ZDM*, *46*(4), 691–695. https://doi.org/10.1007/s11858-014-0608-9

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics*, *80*(3), 327–249. https://doi.org/10.1007/s10649-011-9352-z

Klazar, M. (2018). *Complex derivatives are continuous — three self-contained proofs. Part 1* (arXiv preprint). https://doi.org/10.48550/ARXIV.1707.07017

Klazar, M. (2019a). *Mathematical analysis III. Lecture 11* (lecture notes winter term 2019/20). Charles University Prague. Retrieved 06/09/2020, from https://kam.mff.cuni.cz/~klazar/pr11_19_MAIII_eng.pdf

Klazar, M. (2019b). *Mathematical analysis III. Lecture 12* (lecture notes winter term 2019/20). Charles University Prague. Retrieved 06/09/2020, from https://kam.mff.cuni.cz/~klazar/pr12_19_MAIII_eng.pdf

Klazar, M. (2020). *Mathematical analysis III. Lecture 13* (lecture notes winter term 2019/20). Charles University Prague. Retrieved 06/09/2020, from https://kam.mff.cuni.cz/~klazar/pr13_19_MAIII_eng.pdf

Kleemann, F., Krähnke, U., & Matuschek, I. (2013). *Interpretative Sozialforschung. Eine Einführung in die Praxis des Interpretierens* (2., korr. u. akt. Aufl.). Springer VS. https://doi.org/10.1007/978-3-531-93448-8

Kleine, M., Jordan, A., & Harvey, E. (2005). With a focus on 'Grundvorstellungen'. Part 1: A theoretical integration into current concepts. *Zentralblatt für Didaktik der Mathematik*, *37*(3), 226–233. https://doi.org/10.1007/s11858-005-0013-5

Klinger, M. (2018). *Funktionales Denken beim übergang von der Funktionenlehre zur Analysis. Entwicklung eines Testinstruments und empirische Befunde aus der gymnasialen Oberstufe*. Springer Spektrum. https://doi.org/10.1007/978-3-658-20360-3

Klinger, M. (2019). Grundvorstellungen versus Concept Image? Gemeinsamkeiten und Unterschiede beider Theorien am Beispiel des Funktionsbegriffs. In A. Büchter, M. Glade, R. Herold-Blasius, M. Klinger, F. Schacht, & P. Scherer (Eds.), *Vielfältige Zugänge zum Mathematikunterricht: Konzepte und Beispiele aus Forschung und Praxis* (pp. 61–75). Springer Spektrum. https://doi.org/10.1007/978-3-658-24292-3_5

Ständige Konferenz der Kultusminister in der Bundesrepublik Deutschland (KMK). (2019). *Ländergemeinsame inhaltliche Anforderungen für die Fachwissenschaften und Fachdidaktiken in der Lehrerbildung (Beschluss der Kultusministerkonferenz vom 16.10.2008 i. d. F. vom 16.05.2019*. Berlin, Germany, KMK. Retrieved 04/22/2022, from https://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2008/2008_10_16-Fachprofile-Lehrerbildung.pdf

Kneser, H. (1966). *Funktionentheorie* (2., durchgesehene und erw. Auflage). Vandenhoeck & Ruprecht.

Knopp, K. (1996). *Theory of functions. Parts I and II. Two volumes bound as one* (F. Bagemihl, Trans.). Dover. (Original work published 1945–1947)

Kondratieva, M., & Winsløw, C. (2018). Klein's plan B in the early teaching of analysis: Two theoretical cases of exploring mathematical links. *International Journal for Research in Undergraduate Mathematics Education*, *4*(1), 119–138. https://doi.org/10.1007/s40753-017-0065-2

Königsberger, K. (2004a). *Analysis 1* (6., durchgesehene Auflage). Springer. https://doi.org/10.1007/978-3-642-18490-1

Königsberger, K. (2004b). *Analysis 2* (5., korrigierte Auflage). Springer. https://doi.org/10.1007/3-540-35077-2

Kontorovich, I. (2018a). Undergraduates' images of the root concept in $\mathbb{R}$ and in $\mathbb{C}$. *Journal of Mathematical Behaviour*, *49*, 184–193. https://doi.org/10.1016/j.jmathb.2017.12.002

Kontorovich, I. (2018b). Why Johnny struggles when familiar concepts are taken to a new mathematical domain: Towards a polysemous approach. *Educational Studies in Mathematics*, *97*(1), 5–20. https://doi.org/10.1007/s10649-017-9778-z

Kontorovich, I. (2019). What can be more challenging than square-rooting from squared things? In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia (MERGA, June 30 – July 4, 2019)* (pp. 412–419).

Kontorovich, I. (2021a). Minding mathematicians' discourses in investigations of their feedback on students' proofs: A case study. *Educational Studies in Mathematics*, *107*(2), 213–234. https://doi.org/10.1007/s10649-021-10035-2

Kontorovich, I. (2021b). Pre-university students square-root from squared things: A commognitive account of apparent conflicts within learners' mathematical discourses. *The Jour-*

nal of Mathematical Behavior, *64*, 100910. https://doi.org/10.1016/j.jmathb.2021.100910

Kortemeyer, J. (2020). *Komplexe Zahlen. Eine Einführung für Studienanfänger*innen*. Springer Spektrum. https://doi.org/10.1007/978-3-658-29883-8

Kortemeyer, J., & Frühbis-Krüger, A. (2021). Mathematik im Lehrexport – ein bewährtes Maßnahmenpaket zur Begleitung von Studierenden in der Studieneingangsphase. In R. Biehler, A. Eichler, R. Hochmuth, S. Rach, & N. Schaper (Eds.), *Lehrinnovationen in der Hochschulmathematik. praxisrelevant – didaktisch fundiert – forschungsbasiert* (pp. 19–46). Springer Spektrum. https://doi.org/10.1007/978-3-662-62854-6_3

Kouropatov, A., & Dreyfus, T. (2013). Constructing the integral concept on the basis of the idea of accumulation: Suggestion for a high school curriculum. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 641–651. https://doi.org/10.1080/0020739X.2013.798875

Kouropatov, A., & Dreyfus, T. (2015). Learning the integral concept by constructing knowledge about accumulation. *ZDM*, *46*(4), 533–548. https://doi.org/10.1007/s11858-014-0571-5

Kraines, D. P., Kraines, V. Y., & Smith, D. A. (1990). The Cauchy integral formula. *The College Mathematics Journal*, *21*(4), 327–329.

Krüger, H. W. (1995). Fragwürdige Bilder. Wittgenstein über den Inhalt der Vorstellung. In E. von Savigny & O. R. Scholz (Eds.), *Wittgenstein über die Seele* (3rd ed., pp. 72–83). suhrkamp.

Lai, Y., & Weber, K. (2014). Factors mathematicians profess to consider when presenting pedagogical proofs. *Educational Studies in Mathematics*, *85*(1), 93–108. https://doi.org/10.1007/s10649-013-9497-z

Lakatos, I. (2015). *Proofs and refutations. The logic of mathematical discovery* (J. Worrall & E. Zahar, Eds.). Cambridge University Press. (Original work published 1976)

Lang, S. (1995). *Introduction to modular forms* (2nd ed.). Springer. https://doi.org/10.1007/978-3-642-51447-0

Lang, S. (1999). *Complex analysis*. Springer. https://doi.org/10.1007/978-1-4757-3083-8

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526–550). National Council of Teachers of Mathematics.

Lass, H. (1953). A note on Cauchy's theorem. *The American Mathematical Monthly*, *60*(2), 110–112. https://doi.org/10.2307/2308265

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Lawrentjew, M. A., & Schabat, B. W. (1967). *Methoden der komplexen Funktionenlehre* (U. Pirl, R. Kühnau, & L. v. Wolfersdorf, Trans.). VEB Deutscher Verlag der Wissenschaften.

Lax, P. D. (2007). The Cauchy integral theorem. *The American Mathematical Monthly*, *114*(8), 725–727. https://doi.org/10.1080/00029890.2007.11920463

Lee, J. M. (2013). *Introduction to smooth manifolds* (2nd ed.). Springer. https://doi.org/10.1007/978-1-4419-9982-5

Leinert, M. (1995). *Integration und Maß*. Springer. https://doi.org/10.1007/978-3-663-14080-1

Leland, K. O. (1964). Topological analysis of analytic functions. *Fundamenta Mathematicae*, *56*(2), 157–182. Retrieved 02/21/2022, from http://eudml.org/doc/213805

Leland, K. O. (1965). A polynomial approach to topological analysis. *Compositio Mathematica*, *17*, 291–298. Retrieved 02/21/2022, from http://www.numdam.org/item/CM_1965-1966__17__291_0/

Leland, K. O. (1966). A characterization of analyticity. *Duke Mathematical Journal*, *33*(3), 551–565. https://doi.org/10.1215/S0012-7094-66-03365-5

Leland, K. O. (1971a). A polynomial approach to topological analysis, II. *Journal of Approximation Theory*, *4*(1), 6–12. https://doi.org/10.1016/0021-9045(71)90035-9

Leland, K. O. (1971b). A polynomial approach to topological analysis. III. *Journal of Approximation Theory*, *4*(4), 433–441. https://doi.org/10.1016/0021-9045(71)90008-6

Lemmaxiom. (2021, August 23). *Visualizing complex integrals* [Video]. Retrieved 09/16/2021, from https://www.youtube.com/watch?v=iIzbovM4R2U

Lensing, F. (2021). *Das Begreifen begreifen. Auf dem Weg zu einer funktionalistischen Mathematikdidaktik*. Springer VS. https://doi.org/10.1007/978-3-658-32807-8

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Ablex.

Lerman, S. (2010). Theories of mathematics education: Is plurality a problem? In B. Sriraman & L. English (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 99–109). Springer. https://doi.org/10.1007/978-3-642-00742-2_11

Lew, K., Fukawa-Connelly, T. P., Mejía-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education*, *47*(2), 162–198. https://doi.org/10.5951/jresematheduc.47.2.0162

Linnebo, Ø. (2017). *Philosophy of mathematics*. University Press.

Linnebo, Ø. (2018). Platonism in the philosophy of mathematics. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Spring 2018). Metaphysics Research Lab, Stanford University. Retrieved 02/02/2021, from https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics/

Loeb, P. A. (1991). A note on Dixon's proof of Cauchy's integral theorem. *The American Mathematical Monthly*, *98*(3), 242–244. https://doi.org/10.1080/00029890.1991.11995735

Loeb, P. A. (1993). A further simplification of Dixon's proof of Cauchy's integral theorem. *The American Mathematical Monthly*, *100*(7), 680–681. https://doi.org/10.2307/2323893

Lorenz, F. (1997). *Funktionentheorie*. Spektrum Akademischer Verlag.

Lovitt, W. V. (1915). A geometrical interpretation of Green's formula. *The American Mathematical Monthly*, *22*(5), 152–154. https://doi.org/10.2307/2973486

Lützen, J. (2003). The foundation of analysis in the 19th century. In H. N. Jahnke (Ed.), *A history of analysis* (pp. 155–195). American Mathematical Society.

Lvovski, S. (2020). *Principles of complex analysis* (N. Tsilevich, Trans.). Springer. https://doi.org/10.1007/978-3-030-59365-0

Macintyre, A. J., & Wilbur, W. J. (1967). A proof of the power series expansion without differentiation theory. *Proceedings of the American Mathematical Society*, *18*(3), 419–424. https://doi.org/10.2307/2035470

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 201–211. https://doi.org/10.1080/00207390802213591

Manderfeld, K. (2020). *Vorstellungen zur Mathematikdidaktik. Explorative Studien zu Beliefs, Einstellungen und Emotionen von Bachelor-Studierenden im Lehramt Mathematik*. Springer Spektrum. https://doi.org/10.1007/978-3-658-31086-8

Mariotti, M. A., & Pedemonte, B. (2019). Intuition and proof in the solution of conjecturing problems' [sic!] *ZDM*, *51*(5), 759–777. https://doi.org/10.1007/s11858-019-01059-3

Markusewitsch, A. I. (1955). *Skizzen zur Geschichte der analytischen Funktionen*. VEB Deutscher Verlag der Wissenschaften.

Martin, J. (2013). Differences between experts' and students' conceptual images of the mathematical structure of Taylor series convergence. *Educational Studies in Mathematics*, *82*(2), 267–283. https://doi.org/10.1007/s10649-012-9425-7

Martínez-Planell, R., & Trigueros, M. (2020). Students' understanding of Riemann sums for integrals of functions of two variables. *The Journal of Mathematical Behavior*, *59*, 100791. https://doi.org/10.1016/j.jmathb.2020.100791

Martínez-Planell, R., & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM*, *53*(3). https://doi.org/10.1007/s11858-021-01233-6

Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study. *Internationl Journal of Mathematical Education in Science and Technology*, *49*(7), 1069–1082. https://doi.org/10.1080/0020739X.2018.1426795

Martín-Molina, V., González-Regaña, A. J., Toscano, R., & Gavilán-Izquierdo, J. M. (2020). Differences between how undergraduate students define geometric solids and what their lecturers expect from them through the lens of the theory of commognition. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(12), em1917. https://doi.org/10.29333/ejmste/9159

math.stackexchange.com. (2021). Some threads on interpretations of complex path integrals and Cauchy's integral formula. Retrieved 18/03/2021, from https://math.stackexchange.com/questions/904493/geometric-interpretation-of-complex-path-integral, https://math.stackexchange.com/questions/111368/what-is-a-geometric-explanation-of-complex-integration-in-plain-english, https://math.stackexchange.com/questions/110334/line-integration-in-complex-analysis, https://math.stackexchange.com/questions/446724/what-is-contour-integration, https://math.stackexchange.com/questions/1377910/geometrical-interpretation-of-a-line-integral-issue, https://math.stackexchange.com/questions/2048396/relationship-between-cauchy-goursat-theorem-and-conservative-vector-fields, https://math.stackexchange.com/questions/1169483/what-is-the-value-of-a-complex-line-integral, https://math.stackexchange.com/questions/3322740/intuition-to-cauchys-integral-formula.

Mathemaniac. (2022, January 23). *The geometric intuition of complex integration* [Video]. Retrieved 01/23/2022, from https://www.youtube.com/watch?v=EyBDtUtyshk

Mathews, J. H., & Howell, R. W. (2012). *Complex analysis for mathematics and engineering* (6th ed.). Jones & Bartlett Learning.

Mattheis, M. (2019). Aspects of "Anschauung" in the work of Felix Klein. In H.-G. Weigand, W. McCallum, M. Menghini Marta and Neubrand, & G. Schubring (Eds.), *The legacy of Felix Klein* (pp. 93–106). Springer. https://doi.org/10.1007/978-3-319-99386-7_7

McGowen, M. A., & Tall, D. (2010). Metaphor or met-before? the effects of previouos [sic!] experience on pratice and theory of learning mathematics. *The Journal of Mathematical Behaviour*, *29*(3), 169–179. https://doi.org/10.1016/j.jmathb.2010.08.002

Mejía-Ramos, J. P., Alcock, L., Lew, K., Rago, P., Sagwin, C., & Inglis, M. (2019). Using corpus linguistics to investigate mathematical explanation. In E. Fischer & M. Curtis (Eds.), *Methodological advances in experimental philosophy* (pp. 239–264). Bloomsbury Academic. https://doi.org/10.5040/9781350069022.ch-009

Mejía-Ramos, J. P., & Inglis, M. (2017). "Explanatory" talk in mathematicis research papers. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (PME 41, July 17–22, 2017)* (pp. 233–240). PME.

Mejía-Ramos, J. P., & Weber, K. (2020). Using task-based interviews to generate hypotheses about mathematical practice: Mathematics education research on mathematicians' use of examples in proof-related activities. *ZDM*, *52*(6), 1099–1112. https://doi.org/10.1007/s11858-020-01170-w

Melhuish, K., Fukawa-Connelly, T., Dawkins, P. C., Woods, C., & Weber, K. (2022). Collegiate mathematics teaching in proof-based courses: What we now know and what we have yet to learn. *Journal of Mathematical Behaviour*, *67*, 100986. https://doi.org/10.1016/j.jmathb.2022.100986

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior*, *60*, 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Merkle, M. (2015). An axiomatic integral and a multivariate mean value theorem. *Journal of Inequalities and Applications*, *346*. https://doi.org/10.1186/s13660-015-0866-2

Meuser, M., & Nagel, U. (1991). ExpertInneninterviews - vielfach erprobt, wenig bedacht. Ein Beitrag zur qualitativen Methodendiskussion. In D. Garz & K. Kraimer (Eds.), *Qualitativ-empirische Sozialforschung. Konzepte, Methoden, Analysen* (pp. 441–471). Westd. Verl. https://doi.org/10.1007/978-3-322-97024-4_14

Meuser, M., & Nagel, U. (2009a). Das Experteninterview – konzeptionelle Grundlagen und methodische Anlage. In S. Pickel, G. Pickel, H.-J. Lauth, & D. Jahn (Eds.), *Methoden der vergleichenden Politik- und Sozialwissenschaft. Neue Entwicklungen und Anwendungen*. VS Verlag für Sozialwissenschaften. https://doi.org/10.1007/978-3-531-91826-6_23

Meuser, M., & Nagel, U. (2009b). The expert interview and changes in knowledge production. In A. Bogner, B. Littig, & W. Menz (Eds.), *Interviewing experts* (pp. 17–42). Palgrave Macmillan. https://doi.org/10.1057/9780230244276_1

Meyer, F. (2018). Yes, we can(?) Kommunikative Validierung in der qualitativen Forschung. In F. Meyer, J. Miggelbrink, & K. Beurskens (Eds.), *Ins Feld und zurück – Praktische Probleme qualitativer Forschung in der Sozialgeographie* (pp. 163–168). Springer Spektrum. https://doi.org/10.1007/978-3-662-55198-1_20

Microsoft Corporation. (2021). *Microsoft word: Microsoft office 2021*. Retrieved 10/07/2022, from https://www.microsoft.com/de-DE/microsoft-365/word

Miller, J. (2018). *Earliest known uses of some of the words of mathematics (H)*. Retrieved 06/02/2021, from https://jeff560.tripod.com/h.html

Minami, U. (1942). On the Cauchy's integral theorem. *Proceedings of the Imperial Academy*, *18*(8), 440–445. https://doi.org/10.3792/pia/1195573825

Misoch, S. (2015). *Qualitative Interviews*. de Gruyter Oldenbourg. https://doi.org/10.1515/9783110354614

Mittag-Leffler, G. (1875). Beweis für einen Cauchy'schen Satz. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, *3*, 65–73.

Mittag-Leffler, G. (1923). Der Satz von *Cauchy* über das Integral einer Funktion zwischen imaginären Grenzen. *Journal für Mathematik*, *152*(1/2), 1–5. https://doi.org/10.1515/crll.1923.152.1

Moore, E. H. (1900). A simple proof of the fundamental Cauchy-Goursat theorem. *Transactions of the American Mathematical Society*, *1*(4), 499–506. https://doi.org/10.2307/1986368

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*(3), 249–266. https://doi.org/10.1007/BF01273731

Morgan, C. (2020). Discourse analytic approaches in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 223–227). Springer. https://doi.org/10.1007/978-3-030-15789-0_50

Morgan, C., & Sfard, A. (2016). Investigating changes in high-stakes mathematics examinations: A discursive approach. *Research in Mathematics Education*, *18*(2), 92–119. https://doi.org/10.1080/14794802.2016.1176596

Müller, J. (2018). *Konzepte der Funktionentheorie. Reelle und komplexe Analysis einer Variablen*. Springer Spektrum. https://doi.org/10.1007/978-3-662-56260-4

Nachlieli, T., & Elbaum-Cohen, A. (2021). Teaching practices aimed at promoting meta-level learning: The case of complex numbers. *The Journal of Mathematical Behavior*, *62*, 100872. https://doi.org/10.1016/j.jmathb.2021.100872

Nahin, P. J. (2010). *An imaginary tale: The story of* $\sqrt{-1}$. Princeton University Press. https://doi.org/10.1515/9781400833894

Narasimhan, R., & Nievergelt, Y. (2001). *Complex analysis in one variable* (2nd ed.). Birkhäuser. https://doi.org/10.1007/978-1-4612-0175-5

Nardi, E. (2006). Mathematicians and conceptual frameworks in mathematics education . . . or: Why do mathematicians' eyes glint at the sight of concept image / concept definition? In A. Simpson (Ed.), *Retirement as process and concept. A Festschrift for Eddie Gray and David Tall* (pp. 181–189).

Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. Springer. https://doi.org/10.1007/978-0-387-37143-6

Nardi, E. (2016). Where form and substance meet: Using the narrative approach of re-storying to generate research findings and community rapprochement in (university) mathematics education. *Educational Studies in Mathematics*, *92*(3), 361–377. https://doi.org/10.1007/s10649-015-9643-x

Nardi, E., & Iannone, P. (2003). Mathematicians on concept image construction: Single *'landscape'* vs. *'your own tailor-made brain version'*. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA (PME43, PME-NA25, July 13–18, 2003)* (pp. 365–372). PME.

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of calculus. *Research in Mathematics Education*, *16*(2), 182–198. https://doi.org/10.1080/14794802.2014.918338

Needham, T. (1997). *Visual complex analysis*. Claredon Press.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, *21*(2), 287–323. https://doi.org/10.1080/10508406.2011.611445

Neuenschwander, E. (1978). The Casorati-Weierstrass theorem (Studies in the history of complex function theory I). *Historia Mathematica*, *5*, 139–166.

Neuenschwander, E. (1981). Studies in the history of complex function theory II: Interactions among the French school, Riemann, and Weierstrass. *Bulletin of the American Mathematical Society*, *5*(2), 87–105.

Neuenschwander, E. (1996). *Riemanns Einführung in die Funktionentheorie. Eine quellenkritische Edition seiner Vorlesungen mit einer Bibliographie zur Wirkungsgeschichte der Riemannschen Funktionentheorie*. Vandenboeck & Ruprecht.

Newton, P. K. (2017). Fluid mechanics and complex variable theory: Getting past the 19th century. *PRIMUS*, *27*(8-9), 814–826. https://doi.org/10.1080/10511970.2016.1235645

Newton, T., & Lofaro, T. (1996). On using flows to visualize functions of a complex variable. *Mathematics Magazine*, *69*(1), 28–34. https://doi.org/10.1080/0025570X.1996.11996376

Nöbeling, G. (1949). Eine allgemeine Fassung des Hauptsatzes der Funktionentheorie von Cauchy. *Mathematische Annalen*, *121*(1), 54–66. https://doi.org/10.1007/BF01329616

Nohl, A.-M. (2010). Narrative interview and documentary interpretation. In R. Bohnsack, N. Pfaff, & W. Weller (Eds.), *Qualitative analysis and documentary method in international educational research* (pp. 195–217). Barbara Budrich.

Nohl, A.-M. (2017). *Interview und Dokumentarische Methode. Anleitungen für die Forschungspraxis* (5., aktualisierte und erw. Aufl.). Springer VS. https://doi.org/10.1007/978-3-658-16080-7

Nordlander, M. C., & Nordlander, E. (2012). On the concept images of complex numbers. *International Journal of Mathematical Education in Science and Technology*, *43*(5), 627–641. https://doi.org/10.1080/0020739X.2011.633629

Oehrtman, M., Soto-Johnson, H., & Hancock, B. (2019). Experts' construction of mathematical meaning for derivatives and integrals of complex-valued functions. *International Jour-*

nal of Research in Undergraduadte Mathematics Education, *5*, 394–423. https://doi.org/10.1007/s40753-019-00092-7

Oh, H. M., Park, J. H., & Kwon, O. N. (2013). An analysis of pre-service teachers' reflective thinking for tasks on polar coordinates. *Journal of the Korean Society of Mathematical Education Series D: Research in Mathematical Education*, *17*(2), 119–131. https://doi.org/10.7468/jksmed.2013.17.2.119

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1–18. https://doi.org/10.1007/BF00704699

Padberg, F., & Wartha, S. (Eds.). (2017). *Didaktik der Bruchrechnung* (5th ed.). Springer. https://doi.org/10.1007/978-3-662-52969-0

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G.-P. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International Journal of Mathematical Education in Science and Technology*, *37*(6), 681–706. https://doi.org/10.1080/00207390600712281

Paoletti, T., Moore, K. C., Gammaro, J., & Musgrave, S. (2013). Students' emerging understandings of the polar coordinate systems. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *16th Annual Conference on Research in Undergraduate Mathematics Education* (p. 8). SIGMAA.

Papadaki, E. (2019). Mapping out different discourses of mathematical horizon (F. Curtis, Ed.). *Proceedings of the British Society for Research into Mathematics Learning*, *39*(1). Retrieved 02/23/2021, from https://bsrlm.org.uk/publications/proceedings-of-day-conference/ip39-1/

Papadaki, E., & Biza, I. (2020). Conceptualising the discourse at the mathematical horizon: Looking at one teacher's actions "beyond the mathematics of the moment". In R. Marks (Ed.), *Proceedings of the British Society for Research into Mathematics Learning*. Retrieved 02/22/2021, from https://bsrlm.org.uk/publications/proceedings-of-day-conference/ip40-1/

Pathak, H. K. (2019). *Complex analysis and applications*. Springer. https://doi.org/10.1007/978-981-13-9734-9

Pfadenhauer, M. (2009). At eye level: The expert interview — a talk between expert and quasi-expert. In A. Bogner, B. Littig, & W. Menz (Eds.), *Interviewing experts* (pp. 81–97). Palgrave Macmillan. https://doi.org/10.1057/9780230244276_4

Pfeffer, W. (2017). *Qualitative Entwicklung der Begriffsbildung im Fach Mathematik in der Studieneingangsphase* (Doctoral dissertation). Universität Passau. https://doi.org/10.13140/RG.2.2.10408.06403

Pickert, G. (1976). Analysis in der Kollegstufe. In W. Benz, P. L. Butzer, W. D. Geyer, & H. Schubert (Eds.), *Jahresbericht der Deutschen Mathematiker-Vereinigung* (pp. 173–192). Teubner.

Piña Aguirre, J. G. (2018). *Un estudio histórico-epistemológico sobre la construcción social de conocimiento en variable compleja: El caso del teorema integral de Cauchy*. Center for Research and Advanced Studies of the National Polytechnic Institute, Mexico City, Mexico. Retrieved 01/05/2022, from https://www.researchgate.net/publication/353983560_Un_estudio_historico-epistemologico_sobre_la_Construccion_Social_de_conocimiento_en_Variable_Compleja_El_caso_del_Teorema_Integral_de_Cauchy

Pino-Fan, L. R., Font, V., Gordillo, W., Larios, V., & Breda, A. (2018). Analysis of the meanings of the antiderivative used by students of the first engineering courses. *International Journal of Science and Mathematics Education*, *16*(6), 1091–1113. https://doi.org/10.1007/s10763-017-9826-2

Pino-Fan, L. R., Gordillo, W., Font, V., Larios, V., & Castro, W. F. (2017). The antiderivative understanding by students in the first university courses. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (PME 41, July 17–22, 2017)* (pp. 41–48). PME.

Pinto, A. (2019). Variability in the formal and informal content instructors convey in lectures. *Journal of Mathematical Behaviour*, *54*(100680). https://doi.org/10.1016/j.jmathb.2018.11.001

Plunkett, R. L. (1959). A topological proof of the continuity of the derivative of a function of a complex variable. *Bulletin of the American Mathematical Society*, *65*(1), 1–4. https://doi.org/10.1090/S0002-9904-1959-10251-2

Poelke, K., & Polthier, K. (2009). Lifted domain coloring. *Computer Graphics Forum*, *28*(3), 735–742. https://doi.org/10.1111/j.1467-8659.2009.01479.x

Poelke, K., & Polthier, K. (2012). Domain coloring of complex functions: An implementation-oriented introduction. *IEEE computer graphics and applications*, *32*(5), 90–97. https://doi.org/10.1109/MCG.2012.100

Pollard, S. (1923). On the conditions for Cauchy's theorem. *Proceedings of the London Mathematical Society*, *2*(1), 456–482. https://doi.org/10.1112/plms/s2-21.1.456

Polya, G., & Latta, G. (1974). *Complex variables*. John Wiley & Sons.

Ponce Campuzano, J. C. (2019a). *Complex analysis. A visual and interactive introduction*. Retrieved 06/20/2022, from www.complex-analysis.com

Ponce Campuzano, J. C. (2019b). The use of phase portraits to visualize and investigate isolated singular points of complex functions. *International Journal of Mathematical Education in Science and Technology*, *50*(7), 999–1010. https://doi.org/10.1080/0020739X.2019.1656829

Ponce Campuzano, J. C. (2021). Domain colouring for visualising and exploring the beauty of complex functions. *Gazette of the Australian Mathematical Society*, *48*(4), 162–170. Retrieved 10/21/2021, from https://austms.org.au/publications/gazette/gazette484/

Ponce Campuzano, J. C. (n.d.). *SciMS. Science and mathematics simulations. Complex analysis* (University of Queensland, Ed.; tech. rep.). Retrieved 06/20/2022, from https://teaching.smp.uq.edu.au/scims/index.html#course=Complex_analysis&lecture=introduction

Ponce Campuzano, J. C., Roberts, A. P., Matthews, K. E., Wegener, M. J., Kenny, E. P., & McIntyre, T. J. (2019). Dynamic visualization of line integrals of vector fields: A didactic proposal. *International Journal of Mathematical Education in Science and Technology*, *50*(6), 934–949. https://doi.org/10.1080/0020739X.2018.1510554

Ponnusamy, S., & Silverman, H. (2006). *Complex variables with applications*. Birkhäuser.

Porcelli, P., & Connell, E. H. (1961). A proof of the power series expansion without Cauchy's formula. *Bull. Amer. Math. Soc.*, *67*(2), 177–181. https://doi.org/10.1090/S0002-9904-1961-10559-4

Pöschel, J. (2015). *Noch mehr Analysis: Mehrdimensionale Integration – Fouriertheorie – Funktionentheorie*. Springer Spektrum. https://doi.org/10.1007/978-3-658-05854-8

Potter, J. (2012). Discursive psychology and discourse analysis. In J. P. Gee & M. Handford (Eds.), *The Routledge handbook of discourse analysis* (pp. 104–119). Taylor & Francis.

Prediger, S. (2005). „Auch will ich Lernprozesse beobachten, um besser Mathematik zu verstehen." Didaktische Rekonstruktion als mathematikdidaktischer Forschungsansatz. *mathematica didactica*, *28*(2), 23–47.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM*, *40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For The Learning of Mathematics*, *6*(3), 42–46.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*(6), 595–610. https://doi.org/10.1007/BF00540062

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205–235). Sense.

Presmeg, N. (2020). Visualization and learning in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 900–904). Springer. https://doi.org/10.1007/978-3-030-15789-0_161

Pringsheim, A. (1896). Ueber Vereinfachungen in der elementaren Theorie der analytischen Funktionen. *Mathematische Annalen*, *47*(1), 121–154. https://doi.org/10.1007/BF01445790

Pringsheim, A. (1901). Ueber den Goursat'schen beweis des Cauchy'schen integralsatzes. *Transactions of the American Mathematical Society*, *2*(4), 413–421. https://doi.org/10.2307/1986254

Pringsheim, A. (1903). Der Cauchy-Goursatsche integralsatz und seine übertragung auf reelle Kurvenintegrale. *Sitzungsberichte der mathematisch-physikalischen Klasse der K. B. Akademie der Wissenschaften zu München*, *XXXIII*, 673–682.

Pringsheim, A. (1920). Elementare Funktionentheorie und komplexe Integration. *Sitzungsberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München*, 145–182.

Pringsheim, A. (1925). *Vorlesungen über Funktionentheorie. Erste Abteilung. Grundlagen der Theorie der analytischen Funktionen einer komplexen Veränderlichen*. Teubner.

Przyborski, A., & Slunecko, T. (2020). Dokumentarische Methode. In K. Mey Günter and Mruck (Ed.), *Handbuch Qualitative Forschung in der Psychologie: Band 2: Designs und Verfahren* (2., erw. und überarb. Auflage, pp. 537–554). Springer. https://doi.org/10.1007/978-3-658-26887-9_45

Przyborski, A., & Wohlrab-Sahr, M. (2014). *Qualitative Sozialforschung. Ein Arbeitsbuch* (4. erw. Aufl.). Oldenbourg. https://doi.org/10.1524/9783486719550

Przyborski, A., & Wohlrab-Sahr, M. (2019). Forschungsdesigns für die qualitative Sozialforschung. In N. Baur & J. Blasius (Eds.), *Handbuch Methoden der empirischen Sozialforschung* (pp. 105–123). Springer VS. https://doi.org/10.1007/978-3-658-21308-4_7

Puiseux, V. (1850). Recherches sur les fonctions algébriques. *Journal de Mathématiques pures et appliqués*, *1*(15), 365–480.

Qazi, M. (2006). The mean value theorem and analytic functions of a complex variable. *Journal of Mathematical Analysis and Applications*, *324*(1), 30–38. https://doi.org/10.1016/j.jmaa.2005.11.049

Quine, W. V. (1948). On what there is. *The Review of Metaphysics*, *2*(5), 21–38. Retrieved 03/29/2021, from http://www.jstor.org/stable/20123117

Rasmussen, C., Marrongelle, K., & Borba, M. C. (2014). Research on calculus: What do we know and where do we need to go? *ZDM*, *46*(4), 507–515. https://doi.org/10.1007/s11858-014-0615-x

Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education (PME26, July 21–26, 2002)* (pp. 89–96). School of Education; Professional Development, University of East Anglia.

Read, A. H. (1961). Higher derivatives of analytic functions from the standpoint of topological analysis. *Journal of the London Mathematical Society*, *36*(1), 345–352. https://doi.org/10.1112/jlms/s1-36.1.345

Redheffer, R. (1969). The homotopy theorems of function theory. *The American Mathematical Monthly*, *76*(7), 778–787. https://doi.org/10.1080/00029890.1969.12000322

Reid, D. A., & Knipping, C. (2010). *Proof in mathematics education. Research, learning and teaching*. Sense. https://doi.org/10.1163/9789460912467

Reinfried, S., Mathis, C., & Kattmann, U. (2009). Das Modell der Didaktischen Rekonstruktion. Eine innovative Methode zur fachdidaktischen Erforschung und Entwicklung von Unterricht. *Beiträge zur Lehrerbildung*, *27*(3), 404–414. https://doi.org/10.2565/01:13710

Rembowski, V. (2015a). Begriffsbilder und -konventionen in Begriffsfeldern: Was ist ein Würfel? In M. Ludwig, A. Filler, & A. Lambert (Eds.), *Geometrie zwischen Grundbegriffen und Grundvorstellungen* (pp. 129–154). Springer Spektrum. https://doi.org/10.1007/978-3-658-06835-6_9

Rembowski, V. (2015b). *Eine semiotische und philosophisch-psychologische Perspektive auf Begriffsbildung im Geometrieunterricht. Begriffsfeld, Begriffsbild und Begriffskonvention und ihre Implikationen auf Grundvorstellungen* (Doctoral dissertation). https://doi.org/10.22028/D291-26661

Remmert, R. (1998). *Theory of complex functions* (R. B. Burckel, Trans.; 4th corr. printing). Springer.

Remmert, R., & Schumacher, G. (2002). *Funktionentheorie 1* (5., neu bearbeitete Aufl.). Springer. https://doi.org/10.1007/978-3-540-41855-9

Ridder, H.-G. (2017). The theory contribution of case study research designs. *Business Research*, *10*(2), 281–205. https://doi.org/10.1007/s40685-017-0045-z

Riemann, B. (1867). *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse.* (2. unveränderter Abdruck, Inauguraldissertation) [Transcribed by D. R.

Wilkins]. Georg-August-Universität Göttingen. Göttingen, Germany, Huth. Retrieved 05/21/2020, from https://www.emis.de/classics/Riemann/Grund.pdf

Rittberg, C. J., & Van Kerkhove, B. (2019). Studying mathematical practices: The dilemma of case studies. *ZDM*, *51*(5), 857–868. https://doi.org/10.1007/s11858-019-01038-8

Rodríguez, R. E., Kra, I., & Gilman, J. P. (2013). *Complex analysis. In the spirit of Lipman Bers* (2nd ed.). Springer. https://doi.org/10.1007/978-1-4419-7323-8

Roh, K. H., & Lee, Y. H. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 34–68. https://doi.org/10.1007/s40753-016-0039-9

Roos, A.-K. (2020). *Mathematisches Begriffsverständnis im übergang Schule–Universität. Verständnisschwierigkeiten von Mathematik an der Hochschule am Beispiel des Extrempunktbegriffs*. Springer Spektrum. https://doi.org/10.1007/978-3-658-29524-0

Rosenthal, G. (2014). *Interpretative Sozialforschung. Eine Einführung* (5., aktualisierte und ergänzte Aufl.). Beltz Juventa.

Ross, F., & Ross, W. T. (2014). The Jordan curve theorem is nontrivial. In M. Pitici (Ed.), *The best writing on mathematics 2013* (pp. 120–129). Princeton University Press. https://doi.org/doi:10.1515/9781400847990-014

Ross, K. A. (2013). *Elementary analysis: The theory of calculus* (2nd ed.). Springer. https://doi.org/10.1007/978-1-4614-6271-2

Rosseel, H., & Schneider, M. (2003). Ces nombres que l'on dit ≪ imaginaires ≫. *Petit x*, *63*, 53–71. Retrieved 03/29/2021, from https://irem.univ-grenoble-alpes.fr/revues/petit-x/consultation/numero-63-petit-x/4-ces-nombres-que-l-on-dit-imaginaire--515031.kjsp?RH=1550185063755

Roth, W.-M. (2013). *On meaning and mental representation. A pragmatic approach*. Sense. https://doi.org/10.1007/978-94-6209-251-8

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Sense. https://doi.org/10.1007/978-94-6091-565-2

Roy, R. (2013). Appendix: Historical notes by Ranjan Roy. In R. E. Rodríguez, I. Kra, & J. P. Gilman (Eds.), *Complex analysis. In the spirit of Lipman Bers* (2nd ed., pp. 6–14). Springer. https://doi.org/10.1007/978-1-4419-7323-8_1

Rudin, W. (1987). *Real and complex analysis* (3rd ed.). McGraw-Hill.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *The Journal of Mathematical Behavior*, *62*, 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Rütten, C. (2016). *Sichtweisen von Grundschulkindern auf negative Zahlen. Metaphernanalytisch orientierte Erkundungen im Rahmen didaktischer Rekonstruktion*. Springer Spektrum. https://doi.org/10.1007/978-3-658-14196-7

Ryle, G. (2009). *The concept of mind* (J. Tanney, Ed.; 60th anniversary ed.). Routledge. https://doi.org/10.4324/9780203875858 (Original work published 1949)

Ryve, A. (2011). Discourse research in mathematics education: A critical evaluation of 108 journal articles. *Journal for Research in Mathematics Education*, *42*(2), 167–198. https://doi.org/10.5951/jresematheduc.42.2.0167

Ryve, A., Nilsson, P., & Pettersson, K. (2013). Analyzing effective communication in mathematics group work: The role of visual mediators and technical terms. *Educational Studies in Mathematics*, *82*(3), 497–514. https://doi.org/10.1007/s10649-012-9442-6

Sachs, R. (2017). Complex variables throughout the curriculum. *PRIMUS*, *27*(8-9), 845–855. https://doi.org/10.1080/10511970.2016.1256008

Salle, A., & Clüver, T. (2021). Herleitung von Grundvorstellungen als normative Leitlinien – Beschreibung eines theoriebasierten Verfahrensrahmens. *Journal für Mathematik-Didaktik*, *42*(2), 553–580. https://doi.org/10.1007/s13138-021-00184-5

Sammet, K., & Erhard, F. (2018). Methodologische Grundannahmen und praktische Verfahren der Sequenzanalyse. Eine didaktische Einführung. In F. Erhard & K. Sammet (Eds.), *Sequenzanalyse praktisch* (pp. 15–72). Beltz Juventa.

Sauter, M. (2017). How twisted can a Jordan curve be? *Ulmer Seminare über Funktionalanalysis und Differentialgleichungen*, *20*, 133–139. Retrieved 03/14/2021, from https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.020/sauter/files/Sauter-TwistedJordan.2017.usem20.pdf

von Savigny, E. (1996). *Wittgensteins "Philosophische Untersuchungen". Ein Kommentar für Leser. Band II. Abschnitte 316 bis 693* (2., völlig überarb. und vermehrte Auflage). Vittorio Klostermann.

von Savigny, E. (2011). Sprachspiele und Lebensformen: Woher kommt die Bedeutung? In E. von Savigny (Ed.), *Ludwig Wittgenstein. Philosophische Untersuchungen* (pp. 7–32). Akademie.

Schäfer, I. (2011). Vorstellung von Mathematiklehramtsstudieren[!] zur Stetigkeit. In R. Haug & L. Holzäpfel (Eds.), *Beiträge zum Mathematikunterricht 2011* (pp. 723–726). WTM. https://doi.org/10.17877/DE290R-7933

Schäfer, I., & Hanke, E. (2022). Das Y-Modell im Bereich der fachlichen Lehrerbildung in Mathematik. In S. Halverscheid, I. Kersten, & B. Schmidt-Thieme (Eds.), *Bedarfsgerechte fachmathematische Lehramtsausbildung. Analyse, Zielsetzungen und Konzepte unter heterogenen Voraussetzungen* (pp. 369–383). SpringerSpektrum. https://doi.org/10.1007/978-3-658-34067-4_21

Schiralli, M., & Sinclair, N. (2003). A constructivist response to 'Where mathematics comes from'. *Educational Studies in Mathematics*, *52*(1), 79–91. https://doi.org/10.1023/A:1023673520853

Schmid, H. (2018). *Höhere Technomathematik. Von Funktionen mit mehreren Variablen über Differentialgleichungen bis zur Stochastik*. SpringerSpektrum. https://doi.org/10.1007/978-3-662-58010-3

Scholz, O. (1998). Vorstellungen von Vorstellungen. In E. von Savigny (Ed.), *Ludwig Wittgenstein. Philosophische Untersuchungen* (pp. 191–213). Akademie.

Schüler-Meyer, A. (2019). How do students revisit school mathematics in modular arithmetic? Conditions and affordances of the transition to tertiary mathematics with a focus on learning processes. *International Journal of Research in Undergraduadte Mathematics Education*, *5*, 163–182. https://doi.org/10.1007/s40753-019-00088-3

Schüler-Meyer, A. (2020). Mathematical routines in transition: Facilitating students' defining and proving of sequence convergence. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *39*, 237–247. https://doi.org/10.1093/teamat/hrz019

Schüler-Meyer, A. (2022). How transition students relearn school mathematics to construct multiply quantified statements. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-021-10127-z

Schwarz, B. B. (2020). Psychological approaches in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 696–701). Springer. https://doi.org/10.1007/978-3-030-15789-0_167

Sealey, V. (2014). A framework for characterizing student understanding of riemann sums and definite integrals. *Journal of Mathematical Behavior*, *33*, 230–245. https://doi.org/10.1016/j.jmathb.2013.12.002

Serhan, D. (2014). Students' understanding of the definite integral concept. *International Journal of Research in Education and Science*, *1*(1), 84–88.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715

Sfard, A. (1994). Reification as the birth of metaphor. *For the learning of mathematics*, *14*(1), 44–55.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behaviour*, *14*(1), 15–39. https://doi.org/10.1016/0732-3123(95)90022-5

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13. https://doi.org/10.3102/0013189X027002004

Sfard, A. (2006). Participationist discourse on mathematics learning. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 153–170). Sense.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences*, *16*(4), 565–613. https://doi.org/10.1080/10508400701525253

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourses, and mathematizing*. Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Sfard, A. (2009a). Metaphors in education. In H. Daniels, H. Lauder, & J. Porter (Eds.), *Educational theories, cultures and learning. A critical perspective* (pp. 39–49). Routledge.

Sfard, A. (2009b). Moving between discourses: From learning-as-acquisition to learning-as-participation. In M. Sabella, C. Henderson, & C. Singh (Eds.), *AIP Conference Proceedings* (pp. 55–58). American Institute of Physics. https://doi.org/10.1063/1.3266753

Sfard, A. (Ed.). (2012). Developing mathematical discourse–Some insights from communicational research. *International Journal of Educational Research*, *51-52*.

Sfard, A. (2013). Discursive research in mathematics education: Conceptual and methodological issues. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the*

International Group for the Psychology of Mathematics Education (PME 37, July 28 – August 02, 2013) (pp. 157–161). PME.

Sfard, A. (2014). University mathematics as a discourse – why, how, and what for? *Research in Mathematics Education*, *16*(2), 199–203. https://doi.org/10.1080/14794802.2014.918339

Sfard, A. (2015). Learning, commognition and mathematics. In D. Scott & E. Hargreaves (Eds.), *The SAGE handbook of learning*. SAGE. https://doi.org/10.4135/9781473915213.n12

Sfard, A. (2018). On the need for theory of mathematics learning and the promise of 'Commognition'. In P. Ernest (Ed.), *The philosophy of mathematics education today* (pp. 219–228). Springer. https://doi.org/10.1007/978-3-319-77760-3_13

Sfard, A. (2020a). Commognition. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 95–101). Springer. https://doi.org/10.1007/978-3-030-15789-0_100031

Sfard, A. (2020b). Discursive approaches to learning mathematics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 234–237). Springer. https://doi.org/10.1007/978-3-030-15789-0_52

Sfard, A. (2021). Taming fantastic beasts of mathematics: Struggling with incommensurability [online first]. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-021-00156-7

Sfard, A., & Kieran, C. (2001). Cognition as commucation: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, *8*(2), 42–76. https://doi.org/10.1207/S15327884MCA0801_04

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, *34*(4), 14–22. https://doi.org/10.3102/0013189X034004014

Shenitzer, A., & Steprāns, J. (1994). The evolution of integration. *The American Mathematical Monthly*, *101*(1), 66–72. https://doi.org/10.1080/00029890.1994.11996908

Shinno, Y., & Fujita, T. (2021). Characterizing how and when a way of proving developds in primary mathematics classroom: A commognitive approach. *Internationl Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2021.1941365

Shisha, O. (1989a). Erratum [Erratum to "Proof of power series and Laurent expansions of complex differentiable functions without use of Cauchy's integral formula or Cauchy's integral theorem"]. *Journal of Approximation Theory*, *58*(2), 246. https://doi.org/10.1016/0021-9045(89)90025-7

Shisha, O. (1989b). Proof of power series and Laurent expansions of complex differentiable functions without use of Cauchy's integral formula or Cauchy's integral theorem. *Journal of Approximation Theory*, *57*(2), 117–135. https://doi.org/10.1016/0021-9045(89)90051-8

Shorey, T. N. (2020). *Complex analysis with applications to number theory*. Springer. https://doi.org/10.1007/978-981-15-9097-9

Sill, H.-D. (2019). *Grundkurs Mathematikdidaktik*. Schöningh. https://doi.org/10.36198/9783838550084

Silverman, R. A. (1974). *Complex analysis with applications*. Prentice-Hall.

Simon, M. A. (2017). Explicating “mathematical concept” and “mathematical conception” as theoretical constructs for mathematics education research. *Educational Studies in Mathematics*, *94*(2), 117–137. https://doi.org/10.1007/s10649-016-9728-1

Smith, R. K. (2008, September 12). *Contour integration* (Wolfram Demonstrations Project, Ed.). Retrieved 04/10/2021, from https://demonstrations.wolfram.com/ContourIntegration/

Smithies, F. (2005). A.-L. Cauchy, two memoirs on complex-variable function theory (1825 and 1827). In I. Grattan-Guinness (Ed.), *Landmark writings in western mathematics 1640–1940* (pp. 377–390). Elsevier. https://doi.org/10.1016/B978-044450871-3/50109-1

Smithies, F. (2009). *Cauchy and the creation of complex function theory*. Cambridge University Press. https://doi.org/10.1017/CBO9780511551697 (Original work published 1997)

Snider, A. D. (1978). On the definition of analyticity. *International Journal of Mathematical Education in Science and Technology*, *9*(3), 373–374. https://doi.org/10.1080/0020739780090318

Soto-Johnson, H. (2014). Visualizing the arithmetic of complex numbers. *The International Journal for Technology in Mathematics Education*, *21*(3), 103–114.

Soto-Johnson, H., & Hancock, B. (2018). Research to practice: Developing the amplitwist concept. *PRIMUS*, *29*(5), 421–440. https://doi.org/10.1080/10511970.2018.1477889

Soto-Johnson, H., Hancock, B., & Oehrtman, M. (2016). The interplay between mathematicians’ conceptual and ideational mathematics about continuity of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 362–389. https://doi.org/10.1007/s40753-016-0035-0

Soto-Johnson, H., & Oehrtman, M. (2011). Construct analysis of complex variables: Hypotheses and historical perspectives. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 204–209). SIGMAA.

Soto, H., & Oehrtman, M. (2017). Mathematicians’ interplay of the three worlds of the derivative and integral of complex-valued functions. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1436–1441). SIGMAA.

Soto, H., & Oehrtman, M. (2022). Undergraduates’ exploration of contour integration: What is accumulated? *Journal of Mathematical Behaviour*, *66*, 100963. https://doi.org/10.1016/j.jmathb.2022.100963

Soto-Johnson, H., Oehrtman, M., Noblet, K., Roberson, L., & Rozner, S. (2012). Experts’ reification of complex variables: The role of metaphor. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 433–447). SIGMAA.

Soto-Johnson, H., Oehrtman, M., & Rozner, S. (2011). Dynamic visualization of complex variables: The case of Ricardo. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 488–503). SIGMAA.

Soto-Johnson, H., & Troup, J. (2014). Reasoning on the complex plane via inscriptions and gesture. *The Journal of Mathematical Behavior*, *36*, 109–125. https://doi.org/10.1016/j.jmathb.2014.09.004

Souto Rubio, B., & Gómez-Chacón, I. M. (2011). Challenges with visualization: The concept of integral with undergraduate students. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (CERME7, February 9–13, 2011)* (pp. 2073–2082). University of Rzeszów; ERME.

Spiegel, M. R., Lipschutz, S., Schiller, J. J., & Spellman, D. (2009). *Complex variables with an introduction to conformal mapping and its applications* (2nd ed.). McGraw-Hill.

Stäckel, P. (1900). Integration durch imaginäres Gebiet. *Bibliotheka mathematica*, *3*(1), 109–128.

Stake, R. E. (1995). *The art of case study research*. SAGE.

Stake, R. E. (2006). *Multiple case study analysis*. The Guilford Press.

Steinke, I. (2017). Gütekriterien qualitativer Forschung. In U. Flick, E. von Kardorff, & I. Steinke (Eds.), *Qualitative Forschung. Ein Handbuch* (12th ed., pp. 319–331). rowohlt.

Stewart, I., & Tall, D. (2018). *Complex analysis. the hitch hiker's guide to the plane* (2nd ed.). Cambridge University Press. https://doi.org/10.1017/9781108505468

Stillman, G., Brown, J., & Czocher, J. (2020). Yes, mathematicians do X so students should do X, but it's not the X you think! *ZDM*, *52*(6), 1211–1222. https://doi.org/10.1007/s11858-020-01183-5

Stillwell, J. (2020). *Mathematics and its history* (3rd ed.). Springer. https://doi.org/10.1007/978-1-4419-6053-5

Stokes, D. E. (1997). *Pasteur's quadrant. Basic science and technological innovation*. Brookings Institution Press.

Sträßer, R. (2020). Stoffdidaktik in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 806–809). Springer. https://doi.org/10.1007/978-3-030-15789-0_144

Strübing, J., Hirschauer, S., Ayaß, R., Krähnke, U., & Scheffer, T. (2018). Gütekriterien qualitativer Sozialforschung. Ein Diskussionsanstoß. *Zeitschrift für Soziologie*, *47*(2), 83–100. https://doi.org/10.1515/zfsoz-2018-1006

Swartz, C., & Thomson, B. S. (1988). More on the fundamental theorem of calculus. *The American Mathematical Monthly*, *95*(7), 644. https://doi.org/10.2307/2323311

Swidan, O. (2020). A learning trajectory for the fundamental theorem of calculus using digital tools. *International Journal of Mathematical Education in Science and Technology*, *51*(4), 542–562. https://doi.org/10.1080/0020739X.2019.1593531

Swidan, O., & Fried, M. (2021). Focuses of awareness in the process of learning the fundamental theorem of calculus with digital technologies. *The Journal of Mathematical Behavior*, *62*, 100847. https://doi.org/10.1016/j.jmathb.2021.100847

Swidan, O., & Naftaliev, E. (2019). The role of the design of interactive diagrams in teaching-learning the indefinite integral concept. *International Journal of Mathematical Education in Science and Technology*, *50*(3), 464–485. https://doi.org/10.1080/0020739X.2018.1522674

Swidan, O., & Yerushalmy, M. (2014). Learning the indefinite integral in a dynamic and interactive technology environment. *ZDM*, *46*(4), 517–531. https://doi.org/10.1007/s11858-014-0583-1

Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: Prologue. *Educational Studies in Mathematics*, *91*(3), 299–306. https://doi.org/10.1007/s10649-015-9638-7

Tall, D. (Ed.). (2002). *Advanced mathematical thinking*. Springer. https://doi.org/10.1007/0-306-47203-1 (Original work published 1991)

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5–24. https://doi.org/10.1007/BF03217474

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*. Cambridge University Press. https://doi.org/10.1017/CBO9781139565202

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

Tantau, T. (n.d.). *The tikz and PGF packages: Manual for version 3.1.8b*. Retrieved 12/27/2020, from https://github.com/pgf-tikz/pgf

Tao, T. (2013). *Compactness and contradiction* (Draft version from 2011). American Mathematical Society. Retrieved 03/19/2021, from https://terrytao.files.wordpress.com/2011/06/blog-book.pdf

Taylor, A. E. (1985). *General theory of functions and integration*. Dover.

te Molder, H. (2015). Discursive psychology. In K. Tracy, C. Ilie, & T. Sandel (Eds.), *The international encyclopedia of language and social interaction*. John Wiley & Sons. https://doi.org/10.1002/9781118611463.wbielsi158

Thoma, A. (2018). *Transition to university mathematical discourses: A commognitive analysis of first year examination tasks, lecturers' perspectives on assessment and students' examination scripts* (Doctoral dissertation). University of East Anglia, Norwich, England. Retrieved 05/07/2021, from https://ueaeprints.uea.ac.uk/id/eprint/70208

Thoma, A., & Nardi, E. (2018). Transition from school to university mathematics: Manifestations of unresolved commognitive conflict in first year students' examination scripts. *International Journal of Research in Undergraduate Mathematics Education*, *4*(1), 161–180. https://doi.org/10.1007/s40753-017-0064-3

Thomas, N. J. T. (2021). Mental imagery. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2021). Metaphysics Research Lab, Stanford University. Retrieved 02/02/2021, from https://plato.stanford.edu/archives/fall2021/entries/mental-imagery/ (Original work published 1997)

Thomassen, C. (1992). The Jordan-Schönflies theorem and the classification of surfaces. *The American Mathematical Monthly*, *99*(2), 11–130. https://doi.org/10.2307/2324180

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2-3), 229–274. https://doi.org/10.1007/BF01273664

Thompson, P. W. (2020). Constructivism in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 127–134). Springer. https://doi.org/10.1007/978-3-030-15789-0_31

Thompson, P., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 43–52). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.005

Thompson, P. W., & Harel, G. (2021). Ideas foundational to calculus learning and their links to students' difficulties. *ZDM*, *53*(3), 507–519. https://doi.org/10.1007/s11858-021-01270-1

Tieszen, R. L. (1989). *Mathematical intuition*. Kluwer.

Tietze, U. P., Klika, M., & Wolpers, H. (Eds.). (2000). *Mathematikunterricht in der Sekundarstufe II. Band 1: Fachdidaktische Grundfragen – Didaktik der Analysis* (2., durchgesehene Aufl.). Springer. https://doi.org/10.1007/978-3-322-90568-0

Tirosh, D., & Tsamir, P. (2020). Intuition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 428–433). Springer. https://doi.org/10.1007/978-3-030-15789-0_85

Trahan, D. H. (1965). A new approach to integration for functions of a complex variable. *Mathematics Magazine*, *38*(3), 132–140. https://doi.org/10.1080/0025570X.1965.11975612

Treffert-Thomas, S., Jaworski, B., Hewitt, D., Vlaseros, N., & Anastasakis, M. (2019). Students as partners in complex number task design. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)* (pp. 4859–4866). Freudenthal Group & Freudenthal Institute, Utrecht University; ERME.

Troup, J. (2015). *Students' development of geometric reasoning about the derivative of complex-valued functions* (Doctoral dissertation). University of Northern Colorado, Greeley, CO. Retrieved 02/02/2021, from https://digscholarship.unco.edu/dissertations/313/

Troup, J. (2017). Developing students' reasoning about the derivative of complex-valued functions with the aid of *Geometer's Sketchpad (GSP)*. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 972–981). SIGMAA.

Troup, J. (2019). Developing reasoning about the derivative of a complex-valued function with the aid of Geometer's Sketchpad. *International Journal of Research in Undergraduate Mathematics Education*, *5*, 3–26. https://doi.org/10.1007/s40753-018-0081-x

Troup, J., Soto-Johnson, H., Karakok, G., & Diaz, R. (2017). Developing students' geometric reasoning about the derivative of complex valued functions. *Digitital Experiences in Mathematics Education*, *3*, 173–205. https://doi.org/10.1007/s40751-017-0032-1

Tumanov, A. (2004). A Morera type theorem in the strip. *Mathematical Research Letters*, *11*, 23–29. https://doi.org/10.4310/MRL.2004.v11.n1.a3

Tverberg, H. (1980). A proof of the Jordan curve theorem. *Bulletin of the London Mathematical Society*, *12*(1), 34–38. https://doi.org/10.1112/blms/12.1.34

Universität Bremen. Fachbereich 03 - Mathematik/Informatik (UB). (2016). *Modul- und Veranstaltungskatalog. B.Sc. und M.c.[sic!] Mathematik. B.Sc. und M.Sc. Technomathematik.*

Stand: 19.08.2016. Bremen, Germany. Retrieved 08/22/2022, from https://www.uni-bremen.de/fileadmin/user_upload/fachbereiche/fb3/fb3/Dateien/Studienzentrum_Mathematik/Modulhandbuecher/2016/Modulhandbuch-MatheTechnomathe_2016_08_19_1_.pdf

Universität Bremen. Fachbereich 03 - Mathematik/Informatik (UB). (2021). *Modulhandbuch für das Studienfach Mathematik im Studiengang Master of Education. Lehramt an Gymnasien/Oberschulen. Stand: 08.06.2021*. Bremen, Germany. Retrieved 08/28/2022, from https://www.uni-bremen.de/fileadmin/user_upload/fachbereiche/fb3/fb3/Dateien/Studienzentrum_Mathematik/Modulhandbuecher/Modulkatalog_MA_Lehramt_GyOS_2013.pdf

Ullmann, P. (2015). Grundvorstellungen zur Schulgeometrie. „Situated cognition" in der Geometriedidaktik. In M. Ludwig, A. Filler, & A. Lambert (Eds.), *Geometrie zwischen Grundbegriffen und Grundvorstellungen* (pp. 13–28). Springer Spektrum. https://doi.org/10.1007/978-3-658-06835-6_2

Vaughan, R. C. (1997). *The Hardy-Littlewood method* (2nd ed.). Cambridge University Press. https://doi.org/10.1017/CBO9780511470929

Veblen, O. (1905). Theory on plane curves in non-metrical analysis situs. *Transactions of the American Mathematical Society*, *6*(1), 83–98. https://doi.org/10.2307/1986378

Viirman, O. (2014a). *The function concept and university mathematics teaching* (Doctoral dissertation). Karlstad University, Karlstad, Norway. Retrieved 12/19/2021, from http://kau.diva-portal.org/smash/record.jsf?pid=diva2%3A693890&dswid=-5091

Viirman, O. (2014b). The functions of function discourse–university mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, *45*(4), 512–527. https://doi.org/10.1080/0020739X.2013.855328

Viirman, O. (2015). Explanation, motivation and question posing routines in university mathematics teachers' pedagogical discourse: A commognitive analysis. *International Journal of Mathematical Education in Science and Technology*, *46*(8), 1165–1181. https://doi.org/10.1080/0020739X.2015.1034206

Viirman, O. (2021). Mathematics lecturing as modelling mathematical discourse. *International Journal of Research in Undergraduate Mathematics Education*, *7*(3), 466–489. https://doi.org/10.1007/s40753-021-00137-w

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*(3), 293–205. https://doi.org/10.1080/0020739830140305

Vinner, S. (2002). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Springer. https://doi.org/10.1007/0-306-47203-1_5 (Original work published 1991)

Vinner, S. (2020). Concept development in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 123–127). Springer. https://doi.org/10.1007/978-3-030-15789-0_29

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356–366. https://doi.org/10.5951/jresematheduc.20.4.0356

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths on the development of some simple geometrical concepts. *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education (PME 4, August 16–17, 1980)*, 177–184.

Vohns, A. (2016). Fundamental ideas as a guiding category in mathematics education—early understandings, developments in German-speaking countries and relations to subject matter didactics. *Journal für Mathematik-Didaktik*, *37*(Suppl. 1), 193–223. https://doi.org/10.1007/s13138-016-0086-4

Volchkov, V. V. (1991). Functions with zero integrals over cubes. *Ukranian Mathematical Journal*, *43*(6), 806–810. https://doi.org/10.1007/BF01058952

Volchkov, V. V. (1996). An extremum problem related to Morera's theorem. *Mathematical Notes*, *60*(6), 606–610. https://doi.org/10.1007/BF02305151

Volchkov, V. V. (2003). *Integral geometry and convolution equations*. Springer. https://doi.org/10.1007/978-94-010-0023-9

Volchkov, V. V., & Volchkov, V. V. (2018). Morera-type theorems in the hyperbolic disc (Russian). *Izvestiya: Mathematics*, *82*(1), 34–64. https://doi.org/10.1070/IM8484

Vollrath, H.-J. (1989). Funktionales Denken. *Journal für Mathematik-Didaktik*, *1*, 3–37.

von Glasersfeld, E. (2018). Einführung in den radikalen Konstruktivismus. In P. Watzlawick (Ed.), *Die erfundene Wirklichkeit. Wie wissen wir, was wir zu wissen glauben? Beiträge zum Konstruktivismus* (11th ed., pp. 16–38). Piper.

Vossenkuhl, W. (2003). *Ludwig Wittgenstein* (2. durchgesehene Aufl.). Beck.

Výborný, R. (1979). On the use of a differentiable homotopy in the proof of the Cauchy theorem. *The American Mathematical Monthly*, *86*(5), 380–382. https://doi.org/10.1080/00029890.1979.11994813

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Vygotsky, L. S. (1987). *Thinking and speech* (R. W. Rieber & A. C. Carton, Eds.). Plenum.

Wartha, S. (2007). *Längsschnittliche Analysen zur Entwickung des Bruchzahlbegriffs*. Franzbecker.

Wasserman, N. H., Fukawa-Connelly, T., Weber, K., Mejía Ramos, J. P., & Abbott, S. (2022). *Understanding analysis and its connections to secondary mathematics teaching*. Springer. https://doi.org/10.1007/978-3-030-89198-5

Weber, C. (2007). *Mathematische Vorstellungen bilden. Praxis und Theorie von Vorstellungsübungen im Mathematikunterricht der Sekundarstufe II*. h.e.p.

Weber, K., Dawkins, P., & Mejía-Ramos, J. P. (2020). The relationship between mathematical practice and mathematics pedagogy in mathematics education research. *ZDM*, *52*(6), 1063–1074. https://doi.org/10.1007/s11858-020-01173-7

Weber, K., & Inglis, M. (2020). Mathematics education research on mathematical practice. In B. Sriraman (Ed.), *Handbook of the history and philosophy of mathematical practice* (pp. 1–28). Springer. https://doi.org/10.1007/978-3-030-19071-2_88-1

Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*(1), 36–58. https://doi.org/10.1080/00461520.2013.865527

Wegert, E. (2012). *Visual complex functions: An introduction with phase portraits*. Birkhäuser. https://doi.org/10.1007/978-3-0348-0180-5

Wegert, E., & Semmler, G. (2011). Phase plots of complex functions: A journey in illustration. *Notices of the AMS*, *58*(6), 768–780.

Weigand, H.-G. (2015). Begriffsbildung. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch der Mathematikdidaktik* (pp. 255–278). Springer. https://doi.org/10.1007/978-3-642-35119-8_9

Weigand, H.-G. (2019). What is or what might be the legacy of Felix Klein. In H.-G. Weigand, W. McCallum, M. Menghini Marta and Neubrand, & G. Schubring (Eds.), *The legacy of Felix Klein* (pp. 23–31). Springer. https://doi.org/10.1007/978-3-319-99386-7_2

Weigand, H.-G., Greefrath, G., Oldenburg, R., Siller, H.-S., & Ulm, V. (2017). Aspects and basic mental models ("Grundvorstellungen") of basic concepts of calculus. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (PME 41, July 17–22, 2017)* (pp. 313–320). PME.

Weinberg, A., Wiesner, E., & Fukawa-Connelly, T. (2014). Students' sense-making frames in mathematics lectures. *The Journal of Mathematical Behavior*, *33*, 168–179. https://doi.org/10.1016/j.jmathb.2013.11.005

Wendland, W. L., & Steinbach, O. (2005). *Analysis: Integral- und Differentialrechnung, gewöhnliche Differentialgleichungen, komplexe Funktionentheorie*. Teubner. https://doi.org/10.1007/978-3-322-82962-7

Wessel, J. (2015). *Grundvorstellungen und Vorgehensweisen bei der Substraktion. Stoffdidaktische Analysen und empirische Befunden von Schülerinnen und Schülern des 1. Schuljahres*. Springer. https://doi.org/10.1007/978-3-658-11386-5

Whyburn, G. T. (1964). *Topological analysis* (revised ed.). University Press.

Widder, D. V. (1946). A simplified approach to Cauchy's integral theorem. *The American Mathematical Monthly*, *53*(7), 359–363. https://doi.org/10.2307/2305848

Wieleitner, H. (1927). Zur Frühgeschichte des Imaginären. In L. Bieberbach, O. Blumenthal, & G. Faber (Eds.), *Jahresbericht der Deutschen Mathematiker-Vereinigung. 36. Band 1975–76* (pp. 74–87). Teubner.

Wiesner, H. (1995). Physikunterricht – an Schülervorstellungen und Lernschwierigkeiten orientiert. *Unterrichtswissenschaft*, *23*(2), 127–145.

Wiesner, H., & Wodzinski, R. (1996). Akzeptanzbefragungen als Methode zur Untersuchung von Lernschwierigkeiten und Lernverläufen. In R. Duit & C. von Rhöneck (Eds.), *Lernen in den Naturwissenschaften* (pp. 250–274). IPN.

Wiggins, S., & Potter, J. (2017). Discursive psychology. In C. Willing & W. Stainton-Rogers (Eds.), *The SAGE handbook of qualitative research in psychology* (2nd ed., pp. 93–109). SAGE.

Wikipedia. (2021). Vorstellung — Wikipedia, die freie Enzyklopädie [Online; August 19, 2021]. Retrieved 02/28/2022, from https://de.wikipedia.org/w/index.php?title=Vorstellung&oldid=214752728

Wilkinson, S. (2011, March 7). *Pólya vector fields and complex integration along closed curves* (Wolfram Demonstrations Project, Ed.). Retrieved 04/10/2021, from https://demonstrations.wolfram.com/PolyaVectorFieldsAndComplexIntegrationAlongClosedCurves/

Willwacher, T., & Ohrt, J. (n.d.). *TikzEdt - a semigraphical Tikz editor*. Version 0.2.3.0. Retrieved 03/25/2020, from http://www.tikzedt.org

Wilzek, W. (2021). *Zum Potenzial von Anschauung in der mathematischen Hochschullehre. Eine Untersuchung am Beispiel interaktiver dynamischer Visualisierungen in der Analysis*. Springer Spektrum. https://doi.org/10.1007/978-3-658-35361-2

Winsløw, C. (2019). Questioning definitions at university: The case of real analysis. *Educaçao Matématica Pesquisa*, *21*(4), 142–156. https://doi.org/10.23925/1983-3156.2019v21i4p142-156

Winsløw, C., Biehler, R., Jaworski, B., Rønning, F., & Wawro, M. (2021). Education and professional development of university mathematics teachers. In V. Durand-Guerrier, R. Hochmuth, E. Nardi, & C. Winsløw (Eds.), *Research and development in university mathematics education. Overview produced by the International Network for Didactic Research in University Mathematics* (pp. 59–79). Routledge. https://doi.org/10.4324/9780429346859-6

Winsløw, C., & Rasmussen, C. (2020). University mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 881–890). Springer. https://doi.org/10.1007/978-3-030-15789-0_100020

Winsløw, C. (2020). Analysis teaching and learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 50–53). Springer. https://doi.org/10.1007/978-3-030-15789-0_100029

Wittgenstein, L. (1967). *Remarks on the foundations of mathematics* (G. H. von Wright, R. Rhees, & G. E. M. Anscombe, Eds.; G. E. M. Anscombe, Trans.). M.I.T. (Reprinted from *Remarks on the foundations of mathematics*, by G. H. von Wright, R. Rhees, & G. E. M. Anscombe, Eds., 1964, Oxford, UK, Basil Blackwell)

Wittgenstein, L. (2009). *Philosophical investigations. The German text, with an English translation by G. E. M. Anscombe, P. M. S. Hacker and J. Schulte* (P. M. S. Hacker & J. Schulte, Eds.; G. E. M. Anscombe, P. M. S. Hacker, & J. Schulte, Trans.; rev. 4th ed.). Wiley-Blackwell. (Original work published 1953)

Wittmann, E. (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics*, *12*(3), 389–397. https://doi.org/10.1007/BF00311068

Woods, C., & Weber, K. (2020). The relationship between mathematicians' pedagogical goals, orientations, and common teaching practices in advanced mathematics. *The Journal of Mathematical Behavior*, *59*, 100792. https://doi.org/10.1016/j.jmathb.2020.100792

Wußing, H. (2009). *6000 Jahre Mathematik: Eine kulturgeschichtliche Zeitreise – 2. Von Euler bis zur Gegenwart*. Springer. https://doi.org/10.1007/978-3-540-77314-6

Wyler, O. (1965). The Cauchy integral theorem. *The American Mathematical Monthly*, *172*(1), 50–53. https://doi.org/10.2307/2313002

Ylikoski, P. (2019). Mechanism-based theorizing and generalization from case studies. *Studies in History and Philosophy of Science*, *78*, 14–22. https://doi.org/10.1016/j.shpsa.2018.11.009

Yoshiaki, H. (2017, July 31). *Mapping contour integrals* (Wolfram Demonstrations Project, Ed.). Retrieved 12/10/2021, from https://demonstrations.wolfram.com/MappingContourIntegrals/

Zalcman, L. (1972). Analyticity and the Pompeiu problem. *Archive for Rational Mechanics and Analysis*, *47*, 237–254. https://doi.org/10.1007/BF00250628

Zalcman, L. (1982). Modern perspectives on classical function theory. *The Rocky Mountain Journal of Mathematics*, *12*(1), 75–92. Retrieved 03/03/2021, from http://www.jstor.org/stable/44237594

zetamath. (2022, June 1). *Complex integration and finding zeros of the zeta function* [Video]. Retrieved 06/15/2022, from https://www.youtube.com/watch?v=uKqC5uHjE4g

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APPENDIX

A SUMMARY OF COMPLEX ANALYSIS

As with the theory of differentiation for complex-valued functions of a complex variable, the integration theory of such functions begins by mimicking and extending results from the theory for real-valued functions of a real variable, but again the resulting theory is substantially different, more robust, and more elegant.—Rodríguez et al. (2013, p. 81)

This chapter is a summary of complex analysis. It contains the main definitions and theorems about complex path integrals and can be used as a reference for our epistemological analysis in Part ii and the empirical investigation in Part iii. General references for the contents of this chapter are textbooks on complex analysis (e.g., Burckel, 2021; Fischer & Lieb, 2003, 2010; Freitag & Busam, 2006; González, 1992; Lang, 1999; Müller, 2018; Remmert & Schumacher, 2002; Rodríguez et al., 2013) or textbooks on real analysis, which contain chapters on complex analysis (e.g., Forster, 2017b; Königsberger, 2004b; Pöschel, 2015; Rudin, 1987). While the contents are mostly classical, we will also present several details probably not known from the standard literature as well as philosophical and educational remarks. Any reader familiar with complex analysis may skip this chapter.

A.1 COMPLEX NUMBERS AND COMPLEX FUNCTIONS

Let us briefly recall how we may realise the field of complex numbers.

Complex numbers as tuples of real numbers

The set of tuples of real numbers endowed with componentwise addition forms an abelian group $(\mathbb{R}^2, +)$, on which the multiplication

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} := \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix}$$

provides a field structure $(\mathbb{R}^2, +, \cdot)$; the multiplicative inverse of a tuple $(a, b)^T \neq (0, 0)^T$ is given by $\big(a/(a^2 + b^2), -b/(a^2 + b^2)\big)^T$. Componentwise scalar multiplication with elements of $\mathbb{R}$ makes $(\mathbb{R}^2, +)$ into a two-dimensional $\mathbb{R}$ vector space with basis vectors $e_1 := (1, 0)^T$ and $e_2 := (0, 1)^T$. The one-dimensional $\mathbb{R}$ vector space $\mathbb{R}$ is embedded into the former by $x \longmapsto xe_1$. Abbreviating 1 for e_1, i for e_2, and x for xe_1 for any $x \in \mathbb{R}$, one obtains the two-dimensional $\mathbb{R}$ vector space $(\mathbb{C}, +)$, where $+$ denotes componentwise addition on the direct sum $\mathbb{C} := \mathbb{R} \oplus \mathbb{R}i$. Defining a multiplication $\cdot$ on $\mathbb{C}$ by $(a + bi) \cdot (c + di) = ad - bc + (ac + bd)i$ for $a, b, c, d \in \mathbb{R}$, we see that $(\mathbb{C}, +, \cdot)$ is a field. Accordingly, the mapping

$$\iota\colon \mathbb{C} \longrightarrow \mathbb{R}^2, \quad a + bi \longmapsto \begin{pmatrix} a \\ b \end{pmatrix} \tag{A.1}$$

is an isomorphism of fields. The field $(\mathbb{C}, +, \cdot)$—$\mathbb{C}$ for short—constructed as above is said to be the *field of complex numbers*; its elements are called *complex numbers*. In particular, one has $i^2 = -1$. For $z = a + bi \in \mathbb{C}$, we say $a = \mathrm{Re}(z)$ is the *real part* and $b = \mathrm{Im}(z)$ is the *imaginary part of* z.

Complex numbers as matrices

The set of *skew-symmetric* matrices

$$\mathcal{C} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

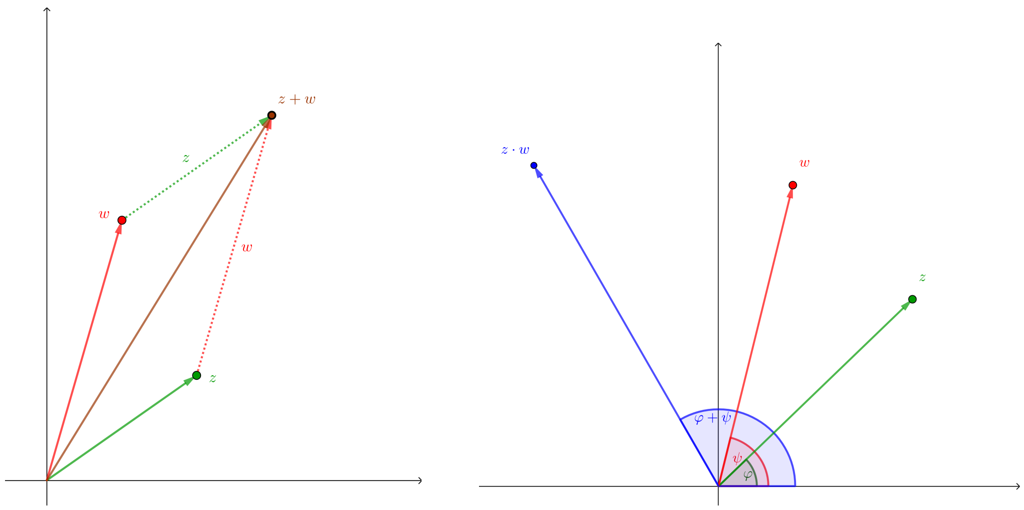
with addition and multiplication of matrices is a field. It is isomorphic as a field to $\mathbb{C}$ via the mapping

$$\kappa\colon \mathbb{C} \longrightarrow \mathcal{C}, \quad a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{A.2}$$

Henceforth, we may also identify complex numbers with skew-symmetric matrices. Furthermore, we recall that $\mathbb{R}$-linear maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, which are also $\mathbb{C}$-linear as a map $\mathbb{C} \longrightarrow \mathbb{C}$, are exactly those represented by elements of $\mathcal{C}$. Hence, applying ι and κ, we see that $\iota\left((a + bi)(c + di)\right) = \kappa(a + bi)\iota(c + di)$ for any $a, b, c, d \in \mathbb{R}$.

Geometry of addition and multiplication of complex numbers

Identifying complex numbers with the plane $\mathbb{R}^2$ shows that non-zero complex numbers can also be characterised by their distance to the origin and the angle they enclose with the positive abscissa. Recall that $\mathrm{e}^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$ for any $\varphi \in \mathbb{R}$.



(a) Addition of z and w represented as vectors.

(b) Multiplication of z and w represented as vectors.

Figure A.1: Addition and multiplication of complex numbers

Proposition & Definition A.1. For every complex number $z \neq 0$ there exist unique real numbers $r > 0$ and $\varphi \in [0, 2\pi)$ such that $z = re^{i\varphi}$.

The number $r =: |z|$ is called the *modulus of* z, and the number $\varphi =: \operatorname{Arg}(z)$ the *argument of* z. The tuple (r, φ) is called the *polar coordinates of* z. Additionally, we have $|0| = 0$; the argument of 0 is not defined. ■ ■

Addition and multiplication of complex numbers have a simple geometric interpretation.

Remark A.2 (Geometrical interpretation of addition and multiplication of complex numbers). For complex numbers $z = a + bi = re^{i\varphi}$ and $w = c + di = se^{i\psi}$ ($a, b, c, d, r, s, \varphi, \psi \in \mathbb{R}$), their sum and product are given by

Geometry of addition and multiplication of complex numbers

$$z + w = (a + c) + (b + d)i$$

and

$$zw = rse^{i(\varphi+\psi)}.$$

That is, the addition of z and w corresponds to the addition of vectors in $\mathbb{R}^2$ and the multiplication of z and w corresponds to a dilation of the vector $\iota(z)$ by $|z|$ followed by a rotation by φ in counterclockwise direction (see Figure A.1). ◇

A.2 SOME CONVENTIONS FOR COMPLEX NUMBERS AND COMPLEX FUNCTIONS

Let us fix some notations and conventions for this thesis (for reference see also Table D.1).

Convention A.3.

- When a complex number is written in the form $a + bi$, it will always be assumed that a and b are real numbers.
- The letter Ω will always denote a non-empty open subset of $\mathbb{C}$. Applying ι (but supressing it from the notation), Ω may also be a non-empty open subset of $\mathbb{R}^2$. The letter I will usually denote a non-empty open interval in $\mathbb{R}$.

- A *domain* is a non-empty open connected subset of $\mathbb{C}$ (or $\mathbb{R}^2$ respectively). Any *simply-connected set* is assumed to be connected (Lang, 1999, p. 118).

- We say that a *complex function* is a function $\Omega \longrightarrow \mathbb{C}$, and a *real function* is a function $I \longrightarrow \mathbb{R}$ or a function $\Omega \longrightarrow \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^2$.

  If $f = u + iv$ is a complex function, it will always be assumed that $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$.

- The letter x will signify the first variable in $\mathbb{R}^2$ and y the second. Additionally, we set $\partial_1 := \partial_x := \frac{\partial}{\partial x}$ and $\partial_2 := \partial_y := \frac{\partial}{\partial y}$. We will also apply these differential operators to complex functions. For this purpose, we set $\partial_1 f = \partial_1 u + i\partial_1 v$ and $\partial_2 f = \partial_2 u + i\partial_2 v$ for complex functions $f = u + iv$.

- We set

$$\mathbf{J} := \kappa(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

  This is the matrix of a counterclockwise rotation by $\pi/2$. ■

Vector field and Pólya vector field associated to a complex function

Definition A.4 (Vector fields associated to a complex functions)**.** Let $f = u + iv$ be a complex function.

- The vector field $\mathbf{f} = (u, v)^T$ is called the *vector field associated to f*.

- The vector field $\mathbf{w} := \mathbf{w}_f := (u, -v)^T$ is called the *Pólya vector field for f* (Braden, 1987; Polya & Latta, 1974). It is the vector field $\overline{\mathbf{f}}$ associated to the conjugate function $\overline{f} = u - iv$. ■

A.3 HOLOMORPHIC FUNCTIONS

From now on, let $\Omega \subseteq \mathbb{C}$ denote a domain and $f\colon \Omega \longrightarrow \mathbb{C}$ a complex function, and $z_0 \in \Omega$ an arbitrary point.

Limits of difference quotients

Since $\mathbb{C}$ is a field and therefore division by complex numbers is defined, it is possible to extend the definition of differentiability of real functions of one real variable to complex functions via limits of difference quotients.

Complex differentiability

Definition A.5 (Holomorphic functions)**.** The function f is called *complex differentiable at z_0* if the limit

$$\lim_{\substack{z \in \Omega \setminus \{z_0\} \\ z \to z_0}} \frac{f(z) - f(z_0)}{z - z_0}, \tag{A.4}$$

exists. In this case, the complex number $f'(z_0)$ via Equation A.4 is called the *complex derivative of f at z_0*.

Holomorphic function

If f is complex differentiable at every point in Ω, f is called *holomorphic (on Ω)* or a *holomorphic function (on Ω)*. In this case, we call the function $f'\colon \Omega \longrightarrow \mathbb{C}$, $z \longmapsto f'(z)$, the *derivative of f*.

(Holomorphic) primitive function

If a complex function F is holomorphic and $F' = f$, then F is called a *(holomorphic) primitive (function)* or an *anti-derivative* for f. ■

Some authors additionally require that f' is continuous instead of merely existent. The reason for this is that some proofs of integral theorems in complex analysis require this additional assumption (e.g., this is often the case when integral theorems from real analysis in two variables are used). However, derivatives of holomorphic functions are automatically continuous—moreover, they are automatically infinitely complex differentiable and analytic (see Theorem A.25).

Definition A.6 (Analytic functions)**.** The function $f\colon \Omega \longrightarrow \mathbb{C}$ is called *analytic (on Ω)* if for every $z_0 \in \Omega$ there is an open neighbourhood $U \subseteq \Omega$ of z_0 and a sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$ such that *Analytic function*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for every $z \in U$. ∎

We remark that U and $(a_n)_{n\in\mathbb{N}}$ are in fact allowed to vary if z_0 varies. Obviously, analytic functions are holomorphic. However, every holomorphic function is analytic, too (see for example Fundamental theorem of complex function theory (Theorem A.26)). This basically justifies defining holomorphic function either with Definition A.5 or Definition A.6. However, we believe that the way we presented here is more common in modern textbooks. Nevertheless, the terminology is sometimes ambiguous as well: Some authors call holomorphic functions "analytic functions", "regular functions", "monodronic functions", and others. According to Miller (2018), Briot and Bouquet (1859) were the first to use the adjective "holomorphic".

Linear approximation

As in real analysis, complex differentiable functions can be locally approximated by linear functions.

Proposition A.7 (Linear approximation)**.** The function $f\colon \Omega \longrightarrow \mathbb{C}$ is complex differentiable at z_0 with derivative $a \in \mathbb{C}$ if one of the following equivalent conditions is satisfied: *Linear approximation*

- There is a function $\psi\colon \Omega \longrightarrow \mathbb{C}$, which is continuous at z_0, satisfies

$$f(z) = f(z_0) + \psi(z)(z - z_0)$$

  for $z \in \Omega$, and $\psi(z_0) = a$;

- there is a function $\rho\colon \Omega \longrightarrow \mathbb{C}$, which is continuous at z_0, satisfies

$$f(z) = f(z_0) + a(z - z_0) + \rho(z)(z - z_0)$$

  for $z \in \Omega$, and $\rho(z_0) = 0$;

- there is a function $\varphi\colon \Omega \longrightarrow \mathbb{C}$ such that $f(z) = f(z_0) + a(z - z_0) + \varphi(z)$ for $z \in \Omega$ and

$$\lim_{\substack{z\in\Omega\setminus\{z_0\}\\ z\to z_0}} \frac{\varphi(z)}{z - z_0} = 0.$$

∎

∎

The last point of this proposition says that the derivative $f'(z_0)$ is the coefficient $\beta \in \mathbb{C}$ of the (affine) $\mathbb{C}$-linear function $\ell\colon \Omega \longrightarrow \mathbb{C}, \ell(z) = \alpha + \beta(z - z_0)$, which agrees with $f(z_0)$ at z_0 (i.e., $\alpha = f(z_0)$) and for which $\varphi\colon z \longmapsto f(z) - \ell(z)$ converges to 0 faster than linearly when $z \longrightarrow z_0$.

Cauchy-Riemann differential equations and skew-symmetric Jacobians

The identification of complex functions and vector fields suggests studying the question whether and how analytic properties of complex functions $f = u + iv$ can be described in terms of analytical properties of the corresponding vector field $\mathbf{f} = (u, v)^T$.

If f is complex differentiable at z_0, the limit of the difference quotient in Equation A.4 exists no matter how z approaches z_0; in other words, the limit exists and is equal to $f'(z_0)$ for any path along which z converges to z_0. Thus, taking the limit under the condition that the real part of z agrees with the real part of z_0 throughout, and taking the limit under the condition that the imaginary part of z agrees with the imaginary part of z_0, one obtains $f'(z_0)$ in both cases. Written out, this means that

$$\begin{aligned} f'(z_0) &= \lim_{\substack{h\in\mathbb{R}\setminus\{0\}\colon z_0+h\in\Omega \\ h\to 0}} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \partial_1 u(z_0) + i\partial_2 v(z_0) \\ &= \lim_{\substack{h\in\mathbb{R}\setminus\{0\}\colon z_0+h\in\Omega \\ h\to 0}} \frac{f(z_0+ih)-f(z_0)}{ih} \\ &= -i\partial_2 u(z_0) + \partial_2 v(z_0). \end{aligned} \tag{A.5}$$

In other words, we obtain $f'(z_0) = \partial_1 f(z_0) = -i\partial_2 f(z_0)$ and the two differential equations $\partial_1 u(z_0) = \partial_2 v(z_0)$ and $\partial_2 u(z_0) = -\partial_1 v(z_0)$.

In fact, these differential equations can be used to characterise whether f is differentiable at z_0 in terms of differentiability properties of $\mathbf{f}$. That is, f is complex differentiable at z_0 if and only if $\mathbf{f}$ is totally differentiable at $\iota(z_0)$ and

$$\partial_1 u(z_0) = \partial_2 v(z_0) \quad \text{and} \quad \partial_2 u(z_0) = -\partial_1 v(z_0).$$

This means that f is holomorphic if and only if $\mathbf{f}$ is totally differentiable and satisfies the so-called *Cauchy-Riemann differential equations*

Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{A.6}$$

on all points of Ω. The Cauchy-Riemann differential equations can also be rewritten as a single differential equation, namely

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}. \tag{A.7}$$

The Cauchy-Riemann differential equations for f are equivalent to the property of the Jacobian of the function $\mathbf{f}$ at $\iota(z_0)$, that is, the matrix

$$\begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix},$$

to be an element of $\mathcal{C}$. More precisely, if f is complex differentiable at $z_0 = x_0 + iy_0$, then we have $f'(z_0) = \partial_1 u(z_0 + i\partial_1 v(z_0)$ and

$$J_{\mathbf{f}}(x_0, y_0) = \begin{pmatrix} \partial_1 u(z_0) & \partial_2 u(z_0) \\ \partial_1 v(z_0) & \partial_2 v(z_0) \end{pmatrix} = \begin{pmatrix} \partial_1 u(z_0) & \partial_1 v(z_0) \\ -\partial_1 v(z_0) & \partial_1 u(z_0) \end{pmatrix}.$$

In particular, we have the following commutative diagram

$$\begin{tikzcd} \mathbf{f}(x,y) - \mathbf{f}(x_0,y_0) \arrow[r, equal] \arrow[d, mapsto] & J_{\mathbf{f}}(x_0,y_0)\,(x,y)^T + \Phi(x,y) \arrow[d, mapsto] \\ f(z) - f(z_0) \arrow[r, equal] & f'(z_0)(z - z_0) + \varphi(z) \end{tikzcd}$$

where the arrows pointing downward are given by ι^{-1}, φ is the function from Proposition A.7, and $\Phi = \iota \circ \varphi \circ \iota^{-1}$ on $\iota(\Omega)$.

Geometry of complex derivatives

Let $f\colon \Omega \longrightarrow \mathbb{C}$ be complex differentiable at $z_0 \in \Omega$. Then, $z \longmapsto f'(z_0)(z - z_0)$ linearly approximates $f(z) - f(z_0)$ by Proposition A.7. Since multiplication with a complex numbers geometrically corresponds to a dilation-rotation, the complex derivative can be interpreted as a rotation-dilation, too. In other words, given a small set Q of complex numbers in Ω containing z_0, then $f(Q)$ is approximately the set Q stretched by $|f'(z_0)|$, turned anti-clockwise by $\operatorname{Arg}(f'(z_0))$, and relocated by addition with $f(z_0)$. In this context, Needham (1997, ch. 4) describes holomorphic functions as *amplitwists*. *Amplitwist*

Furthermore, f is *conformal at* z_0 if $f'(z_0) \neq 0$. This means that f is angle- and orientation-preserving at z_0. In fact, f is conformal at z_0 if and only if f is complex differentiable at z_0 with non-vanishing derivative (Wegert, 2012, pp. 255–256). If f is holomorphic, $f'(z_0) = 0$, and f' has a root of order $k \in \mathbb{N}_{\geq 1}$ at z_0, then f multiplies angles at z_0 with factor $k+1$ (reduced modulo 2π) (Wegert, 2012, p. 258).[187] *Conformal map*

Wirtinger derivatives

The *Wirtinger operators*, named after Wilhelm Wirtinger (1865–1945), are the partial differential operators *Wirtinger operators and derivatives*

$$\partial := \frac{\partial}{\partial z} := \frac{1}{2}(\partial_x - \partial_y) \tag{A.8}$$

and

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\partial_x + i\partial_y). \tag{A.9}$$

If $\mathbf{f}$ is partially differentiable, then the *Wirtinger derivatives of f*

$$\partial f(z) = \frac{1}{2}(\partial_x f(z) - i\partial_y f(z))$$

187 Angle-preserving at z_0 means that if γ and η are two continuously differentiable paths in Ω through z_0 whose "tangential vectors" $\gamma'(z_0)$ and $\eta'(z_0)$ enclose an angle α, then the tangential vectors of $f \circ \gamma$ and $f \circ \eta$ enclose the same angle at $f(z_0)$ with respect to absolute value. Orientation-preserving means that these two angles are identical.

and

$$\bar{\partial} f(z) = \frac{1}{2}\left(\partial_x f(z) + i\partial_y f(z)\right)$$

for $z \in \Omega$.

A consequence of the Cauchy-Riemann differential equation in Equation A.7 is that the second of these Wirtinger derivatives vanishes.

Proposition A.8. The function f is complex differentiable at $z_0 \in \Omega$ if and only if $\bar{\partial} f(z_0) = 0$. In this case, the derivative of f at z_0 is $f'(z_0) = \partial f(z_0)$. ■ ■

The real part x and the imaginary part y of a complex variable $z = x + iy$ can be expressed in terms of z and $\bar{z}$ as well: $x = \frac{z+\bar{z}}{2i}$ and $y = \frac{z-\bar{z}}{2}$. Hence, one may use the two complex variables z and $\bar{z}$ instead of the two real variables x and y to represent at complex function, which is given in terms of x and y, too. Then, Proposition A.8 can be interpreted in the sense that a holomorphic function f is a function of z alone and not also of $\bar{z}$ (Snider, 1978). In fact, in case f is a function, which is represented on an open ball in terms of

$$f(z) = c + \sum_{n=1}^{\infty} a_n z^n + \sum_{m=1}^{\infty} b_m \bar{z}^m$$

for complex numbers $c, a_n, b_m \in \mathbb{C}$, $n, m \in \mathbb{N}$, it is holomorphic if and only if $b_m = 0$ for all $m \in \mathbb{N}$.

Let us assume that **f** is totally differentiable. Then, we define the *total differential of f* or the *complex differential form of f* as

$$\mathrm{d}f := \partial_x f \,\mathrm{d}x + \partial_y f \,\mathrm{d}y$$

(Fischer & Lieb, 2003, p. 26). In particular, we have $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\,\mathrm{d}y$ as well as $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$ and $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$. Thus, in terms of differential forms, one may also say that the Wirtinger operators are defined in such a way that they satisfy

$$\mathrm{d}f = \partial f \,\mathrm{d}z + \bar{\partial} f \,\mathrm{d}\bar{z}.$$

Solenoidal and rotation-free vector fields

f holomorphic $\Longleftrightarrow$ **f** *is solenoidal and rotation-free*

For $f = u + iv$, the holomorphicity of f is equivalent to the fact that the Pólya vector field $\mathbf{w}_f = (u, -v)^T$ associated to f is *solenoidal* and *rotation-free* (Lawrentjew & Schabat, 1967). That is because the equations

$$\mathrm{div}(\mathbf{w}_f) = 0 \quad \text{and} \quad \mathrm{rot}(\mathbf{w}_f) = 0,$$

which define solenoidal and rotation-free vector fields are equivalent to the Cauchy-Riemann differential equations Equation A.6 because $\mathrm{div}(\mathbf{w}_f) = \partial_1 u - \partial_2 v$ and $\mathrm{rot}(\mathbf{w}_f) = -\partial_1 v - \partial_2 u$. A physical interpretation of solenoidal and rotation-free vector fields can be found in Section B.2.4.

Harmonic functions

Harmonic function

A *harmonic function on* Ω is a twice continuously partially differentiable function $h =$

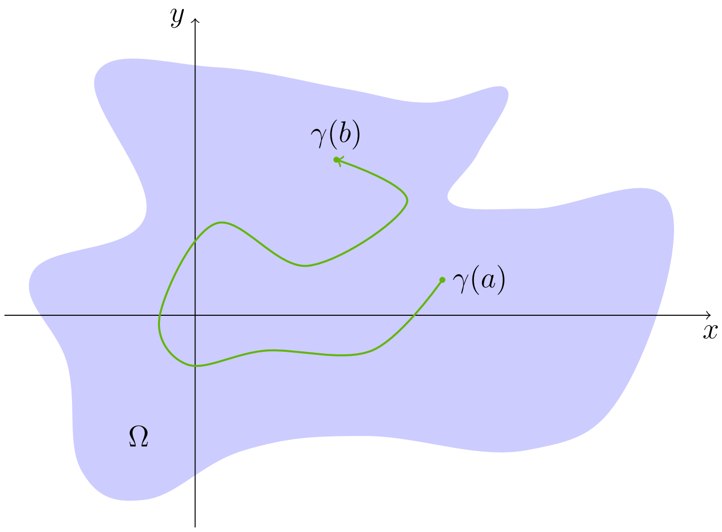


Figure A.2: The trace of a path γ in Ω.

$(h_1, h_2)^T : \Omega \longrightarrow \mathbb{R}^2$ that satisfying

$$\Delta(h) := \frac{\partial^2}{\partial x \partial x} h_1 + \frac{\partial^2}{\partial y \partial y} h_2 \equiv 0,$$

where $\Delta$is the *Laplace operator* (Forster, 2017a, p. 71). Real and imaginary part of a holomorphic function are harmonic. This follows from the fact that they are infinitely differentiable, satisfy the Cauchy-Riemann differential equations (Equation A.6), and Schwarz' theorem on the interchangeability of second partial derivatives (Forster, 2017a, p. 68). Vice versa, for every harmonic function u on a simply-connected domain (Definition A.13), there is another harmonic function v on this domain such that $u + iv$ is holomorphic. This follows from the existence of a holomorphic primitive for $\partial_1 u - i\partial_2 u$ on simply-connected domains (see Theorem A.20).

A.4 PATHS AND DOMAINS

In this section, we recall the notion of paths, some of their properties, and some notions from plane topology.

Definition A.9 (Paths)**.** A *path in* Ω is a *continuous* function $\gamma\colon [a, b] \longrightarrow \Omega$ for some interval of real numbers $[a, b]$. $\gamma(a)$ is called the *initial point* and $\gamma(b)$ the *terminal point of* γ. *Path*

The path γ is called *closed* if $\gamma(a) = \gamma(b)$, *simple* if γ is injective, and *simple closed* if γ is closed and $\gamma_{|[a,b)}$ is injective. The set $\mathrm{tr}(\gamma) := \gamma([a, b])$ is called the *trace of* γ. *Trace of a path*

A path is called *piecewise continuously differentiable* if there is a subdivision $a = t_0 < t_1 < \ldots < t_n = b$ ($n \in \mathbb{N}$) such that the restriction $\gamma_{|[t_{k-1}, t_k]}$ is continuously differentiable for all $1 \leq k \leq n$. *Piecewise continuously differentiable path*

A path is called *regular*, if it is continuously differentiable and its derivative does not vanish at any point. ■

An example of the trace of a path can be seen in Figure A.2.

Proposition & Definition A.10 (Rectifiable path and the length of a path)**.** The path γ is called *Rectifiable path and length of a path* *rectifiable* if

$$L(\gamma) := \sup\left\{\sum_{\nu=1}^{n-1} |\gamma(t_{\nu+1}) - \gamma(t_\nu)| \ : \ n \in \mathbb{N},\ a \le t_1 < t_2 < \ldots < t_n \le b\right\}$$

exists and is $< \infty$. The real number $L(\gamma)$ is said to be the *length of* γ.

If γ piecewise continuously differentiable with respect to the subdivision $a = t_0 < t_1 < \ldots < t_n = b$, $n \in \mathbb{N}$, then it is rectifiable and

$$L(\gamma) = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} |\gamma'(t)|\,\mathrm{d}t. \qquad ■$$

If $\gamma\colon [a, b] \longrightarrow \mathbb{C}$ is a piecewise continuously differentiable path, we may extend γ' to all of $[a, b]$ by setting $\gamma'(t) := 0$ for all $t \in [a, b]$ at which γ is not differentiable. This will not affect any of the integrals involving γ'.

Topology of domains

An open subset of $\mathbb{C}$ is a domain (i.e., connected) if and only if it is path-connected. However, oftentimes stronger properties than the connectedness are required in integral theorems. We define two of them: *star domains*, which are a subset of the set of *simply-connected domains* in $\mathbb{C}$.

Star domains

Definition A.11 (Star domains). Ω is a called a *star domain*, if there is a $z^* \in \Omega$ such that for all $z \in \Omega$ the line segment $\sigma_{z^*,z}$ from z^* to z (i.e., the path $\sigma_{z^*,z}\colon [0, 1] \longrightarrow \mathbb{C}$, $t \longmapsto z^* + t(z - z^*)$) lies in Ω. In this case, z^* is a called *star centre of* Ω. ■

Clearly, star domains are domains. Moreover, vividly speaking, a star domain can be shrunk down to any of its star centres. In order to make this idea rigorous, we need to define the notion of *homotopy*.

Homotopies

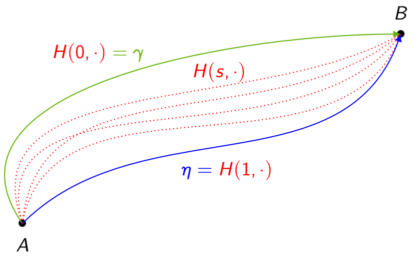
Definition A.12 (Homotopies). Let $\gamma, \eta\colon [a, b] \longrightarrow \Omega$ be two paths in Ω whose initial point A and terminal point B agree.

A *homotopy relative to initial and terminal points between* γ *and* η *in* Ω is a continuous function $H\colon [0, 1] \times [a, b] \longrightarrow \Omega$ such that $H(s, a) = A$, $H(s, b) = B$ for all $s \in [0, 1]$, and $H(0, t) = \gamma(t)$ and $H(1, t) = \eta(t)$ for all $t \in [a, b]$.

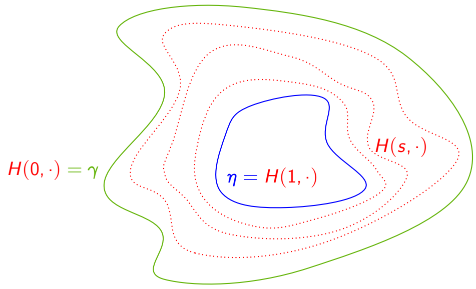
A *free homotopy between two paths* γ *and* η *in* Ω is a continuous function $H\colon [0, 1] \times [a, b] \longrightarrow \Omega$ such that $H(s, a) = H(s, b)$ for all $s \in [0, 1]$, and $H(0, t) = \gamma(t)$ and $H(1, t) = \eta(t)$ for all $t \in [a, b]$. ■

Figure A.3a and Figure A.3b illustrate the definitions. In particular, the red dotted lines indicate show some of the paths $H(s, \cdot)$ for some $s \in [0, 1]$. Intuitively, one can imagine that homotopic paths can be continuously deformed into each other without having to cross a “hole” in the domain.

If γ and η are two paths such that their initial and terminal points are all the same complex number, the notions of homotopy relative to initial and terminal points and free homotopy coincide. It can be shown that it suffices to assume that H is continuously differentiable if the paths under consideration are (piecewise) continuously differentiable because in this case both notions of homotopy (continuity only vs. continuous differentiability of H) give rise to the same pairs of homotopic paths in Ω (e.g., Gathmann, 2017, p. 29; Lee, 2013, p. 224).



(a) A homotopy relative to initial and terminal points between two paths γ and η, the paths $H(s,\cdot)$ are drawn in dotted red for six values of $s \in [0,1]$.



(b) A free homotopy between two paths γ and η, the paths $H(s,\cdot)$ are drawn in dotted red for five values of $s \in [0,1]$.

Figure A.3: Homotopies between paths.

Simply-connected domains

Definition A.13 (Simply-connected domain)**.** Ω is *simply-connected*, if every closed path in Ω is homotopic to a constant path in Ω. ■

Annuli

For example, all convex and all star domains in $\mathbb{C}$ are simply-connected. Neither the annuli $A_{r,s}(z_0) := \{z \in C : s < |z - z_0| < r\}$ for any $z_0 \in \mathbb{C}$ and $0 \leq s < r \leq \infty$ nor the sliced plane $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$ are simply-connected.

Jordan curve theorem

One may be tempted to believe that a simple closed path "divides" the plane into two components, one of which is the interior of the curve, and the other the exterior. This is the so-called *Jordan curve theorem* (some interesting background information is given by Hales, 2007; Sauter, 2017; Veblen, 1905).[188]

Jordan curve theorem

Theorem A.14 (Jordan curve theorem)**.** Suppose γ is a simple closed path $\mathbb{R}^2$. Then, the complement of the trace of γ, that is $\mathbb{R}^2 \setminus \operatorname{tr}(\gamma)$ has exactly two path-connected components, one of which is bounded. ■

Short more or less self-contained proofs are due to Thomassen (1992) and Tverberg (1980). Beardon (1979) gives a proof with methods of complex analysis.

Interior of a path

Definition A.15. In the setting of Theorem A.14, γ is called a *Jordan path*, the bounded component of $\mathbb{R}^2 \setminus \operatorname{tr}(\gamma)$ is called the *interior of* γ and will be denoted by $\operatorname{int}(\gamma)$. The unbounded component of $\mathbb{R}^2 \setminus \operatorname{tr}(\gamma)$ is called *exterior of* γ. ■

Theorem A.14 is insofar remarkable as simple closed paths can be rather complicated; for example, they do not even need to be rectifiable. An overview of such possibilities is given by Sauter (2017) and literature therein.

A.5 COMPLEX PATH INTEGRALS

Now, let us fix a piecewise continuously differentiable path $\gamma\colon [a,b] \longrightarrow \mathbb{C}$ and a continuous function $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$.[189]

188 Ross and Ross (2014) illustrate with beautiful artwork how complicated Jordan curves may be. This could be used to motivate that the Jordan curve theorem is actually not at all trivial.

189 Throughout the epistemological analysis in Part ii, we will also use different constraints on γ and f in the context of integration, when we discuss potential definitions of complex path integrals. Here, we restrict us to the rather conventional case that γ is (piecewise) continuously differentiable.

We define the integral of a complex-valued function of a real variable first: If $g\colon [a, b] \longrightarrow \mathbb{C}$ is a function such that $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$ are (Riemann) integrable, then we set

$$\int_a^b g(t)\,\mathrm{d}t := \int_a^b \mathrm{Re}(g)(t)\,\mathrm{d}t + i\int_a^b \mathrm{Im}(g)(t)\,\mathrm{d}t.$$

Complex path integral

We use this definition to define the *complex path integral of f along γ*. Since we deal with many ways to define the complex path integral in Part ii, we restrict ourselves here to one of the possibilities.[190] The complex path integral is denoted by

$$\int_\gamma f := \int_\gamma f\,\mathrm{d}z := \int_\gamma f(z)\,\mathrm{d}z.$$

If γ is continuously differentiable, it may be defined as

$$\int_a^b f\left(\gamma(t)\right)\gamma'(t)\,\mathrm{d}t, \tag{A.10}$$

and if $a = t_0 < t_1 < \ldots < t_n = b$ is a partition of γ such that $\gamma_{|[t_{k-1},t_k]}$ is continuously differentiable for all $1 \leq k \leq n$, then we may define it as

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Since the finite number of points, where γ' is not defined, does not change the integral at the right-hand side of this equation, we also write $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$ in this case. In particular, the complex path integral does not depend on the choice of the subdivision involved in the definition of a piecewise continuously differentiable path.

Invariance under reparametrisation

The complex path integral does not change under a reparametrisation of the path. A *reparametrisation* of a piecewise continuously differentiable path $\gamma\colon [a, b] \longrightarrow \Omega$ is a continuously differentiable function $\rho\colon [c, d] \longrightarrow [a, b]$ such that $\rho(c) = a$ and $\rho(d) = b$ (Freitag & Busam, 2006, pp. 65–66). Then, the chain rule immediately implies that

$$\int_\gamma f \quad \text{and} \quad \int_{\gamma\circ\rho} f$$

coincide.[191]

190 Some authors refer to complex path integrals as “complex line integrals”, “complex curve integrals”, or even omit the keyword “complex”. The latter practice easily leads to an overload of the same keywords from real analysis, which we do not prefer. Some authors define complex path integrals for *curves* C in Ω, that is, images of paths, where a parametrisation has to be given from the context. In this case, the notation $\int_C f(z)\,\mathrm{d}z$ may be used, too.

191 We note that for every piecewise continuously differentiable path γ there exists a reparametrisation $\psi\colon [a, b] \longrightarrow [a, b]$ such that $\gamma \circ \psi$ is continuously differentiable (Gathmann, 2017, p. 20; Burckel, 2021, p. 60). Therefore, since complex path integrals do not change under reparametrisation of the paths, it suffices to consider continuously differentiable paths only. However, this is not common practice in complex analysis texts, and thus, we will mostly assume paths to be piecewise continuously differentiable.

Functional properties of complex path integrals

Functional properties of complex path integrals

The complex path integral is a $\mathbb{C}$-*linear functional* on the $\mathbb{C}$ vector space of continuous functions defined on the trace of the same path. This means that

$$\int_\gamma (f + \alpha g) = \int_\gamma f + \alpha \int_\gamma g \tag{A.11}$$

for every piecewise continuously differentiable path γ in Ω, two continuous complex functions f, g on the trace of γ, and $\alpha \in \mathbb{C}$.

Reversed path and juxtaposition of paths

We recall two important ways to define new paths from old. Let $\gamma, \eta\colon [0,1] \longrightarrow \mathbb{C}$ be two paths (without loss of generality, they are defined on the interval $[0,1]$ here; otherwise we reparametrise it accordingly), such that $\gamma(1) = \eta(0)$. Then, the *reversed path* $-\gamma$ and the *juxtaposition of* γ *and* η are explicitly given by (see e.g., González, 1992, p. 420)

$$\begin{aligned} -\gamma\colon [0,1] &\longrightarrow \Omega, \\ t &\longmapsto \gamma(1-t) \end{aligned} \tag{A.12}$$

and

$$\begin{aligned} \gamma \oplus \eta\colon [0,1] &\longrightarrow \Omega, \\ t &\longmapsto \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2}, \\ \eta(2t-1), & \frac{1}{2} < t \le 1. \end{cases} \end{aligned} \tag{A.13}$$

Then, the following properties hold:

$$\int_\gamma f(z)\,\mathrm{d}z = -\int_{-\gamma} f(z)\,\mathrm{d}z. \tag{A.14}$$

and

$$\int_{\gamma\oplus\eta} f = \int_\gamma f + \int_\eta f \tag{A.15}$$

for all continuous complex functions f on $\mathrm{tr}(\gamma) \cup \mathrm{tr}(\eta)$. We say that the complex path integral is *additive* with respect to the paths of integration.

Moreover, the modulus of complex path integrals can be estimated in terms of a multiple of the length of the paths along we integrate:[192]

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \le L(\gamma) \max_{z\in\mathrm{tr}(\gamma)} |f(z)|. \tag{A.16}$$

One of the consequences of this inequality is that if $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence of continuous complex functions on $\mathrm{tr}(\gamma)$ with the limit function f, then

$$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f.$$

192 According to González (1992, p. 423), this inequality is named after Jean Gaston Darboux (1842–1917).

In particular, complex path integrals commute with series of functions. For a list of these and similar properties see for example Remmert and Schumacher (2002, ch. 6.3) and many other textbooks.

A.6 INTEGRAL THEOREMS IN COMPLEX ANALYSIS: CAUCHY'S INTEGRAL THEOREM, CAUCHY'S INTEGRAL FORMULA, GOURSAT'S LEMMA, AND THE EXISTENCE OF HOLOMORPHIC PRIMITIVE FUNCTIONS

The fundamental theorem of calculus readily transfers to integrals of complex functions.

Proposition A.16. If f has a primitive F an open neighbourhood of $\operatorname{tr}(\gamma)$, then we have

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))\,. \tag{A.17}$$

■

We call this result the *complex version of the fundamental theorem of calculus*. In particular, if the conditions in Proposition A.16 are satisfied and γ is closed, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

For example, the function $z \longmapsto z^n$ has a primitive for $n \in \mathbb{Z}\setminus\{-1\}$ on all of $\mathbb{C}$ and thus

$$\int_{\partial B} z^n\,\mathrm{d}z = 0,$$

where ∂B denotes the path along the boundary of the ball $B_r(c)$, which may for example be parametrised by $[0, 2\pi] \longrightarrow \mathbb{C}$, $t \longmapsto c + re^{it}$. On the other hand, for $n = -1$, a direct computation yields

$$\int_{\partial B} \frac{1}{z-c}\,\mathrm{d}z = 2\pi i.$$

This is a result, which is foundational to all the rest of complex analysis. For example, Remmert and Schumacher (2002, p. 156, own transl., emph. orig.) summarise the importance of the last equality neatly as follows:

> *If the integral $\int_{\partial B} \mathrm{d}\zeta/(\zeta - c)$ vanished, too, there would be no complex analysis at all.*

Vanishing complex path integrals

One remarkable fact is that complex path integrals of holomorphic functions along closed piecewise differentiable paths always vanish under relatively mild conditions. These results are often referred to as *Cauchy's integral theorem*, "the incomparable method of higher analysis" (Mittag-Leffler; cited in Bottazzini & Gray, 2013, p. 636).[193]

Cauchy's integral theorem

Theorem A.17 (Cauchy's integral theorem)**.** For every closed (piecewise) continuously differentiable path γ in a simply-connected domain Ω and a holomorphic function $f\colon \Omega \longrightarrow \mathbb{C}$, we have

$$\int_\gamma f(z)\,\mathrm{d}z = 0. \tag{A.18}$$

■

193 There are many other versions of this theorem in the literature. Often, it is stated and proven for a less general set of paths. The history of this theorem, different versions, and proofs are intensively discussed in the literature (e.g., Ahmad, 1955; Bak & Popvassilev, 2017; Beckenbach, 1943; Dixon, 1971; Garcia & Ross, 2017; Gray, 2000; Hanche-Olsen, 2008; Minami, 1942; Moore, 1900; Pollard, 1923; Pringsheim, 1901; Redheffer, 1969; Výborný, 1979, and many others). See Burckel (2021, p. 662), Cufí and Verdera (2015), Harmse (2008), and Nöbeling (1949) for a discussion of the most general versions.

Cauchy's integral theorem has no analogue in real analysis of one real variable, and this is essentially due to the existence of closed paths in the plane $\mathbb{C}$, which are not degenerate (that is, they do not only run from one point to the same on the real line). According to Tao (2013, p. 83), a "crucial reason" for why "complex analysis in one variable is significantly more powerful than real analysis in two variables", is that

> [...] in the complex domain, there exist non-trivial closed contours, whereas in the real domain, all closed contours are degenerate. Thus the *fundamental theorem of calculus* in real analysis gets augmented to the significantly more powerful *Cauchy theorem* in complex analysis, which is the basis for all the power of contour integration methods. By exploiting the additional dimension available in the complex setting, one can avoid being blocked by singularities or other obstructions, for instance by shifting a contour to go *around* a singularity rather than *through* it.

A simply-connected domain is a domain in which all paths are homotopic to a constant path. However, simply-connected domains can also be characterised in terms of complex path integrals (e.g., Freitag & Busam, 2006, p. 241; Remmert & Schumacher, 2002, p. 247):

Proposition A.18 (Characterisation of simply-connected domains)**.** A domain Ω is simply-connected if and only if all complex path integrals of holomorphic function on that domain vanish along every closed piecewise continuously differentiable path Ω. ■

A special version of this theorem is *Goursat's lemma*. Often, this lemma is proven first and then used to proof variants of Cauchy's integral theorem (e.g., for star domains Freitag & Busam, 2006, ch. II; Fischer & Lieb, 2010, ch. II; Remmert & Schumacher, 2002, ch. 7; or for convex domains Fischer & Lieb, 2003, ch. III).

Goursat's lemma

Theorem A.19 (Goursat's lemma)**.** Let $f\colon \Omega \longrightarrow \mathbb{C}$ be holomorphic and Δ a compact triangle in Ω. Let $\partial\Delta$ denote any path that winds around the boundary of Δ once counterclockwise; then

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0. \tag{A.19}$$

■

Furthermore, if γ and η are two homotopic paths in Ω (Definition A.12, see Figure A.3) in in Ω via the homotopy H and f is holomorphic on an open neighbourhood of the image of H, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\eta f(z)\,\mathrm{d}z \tag{A.20}$$

(Freitag & Busam, 2006, p. 240). Intuitively speaking, this means that the complex path integral of a holomorphic function does not change when the path is deformed continuously. Sometimes, this proposition is also referred to as the homotopic version of Cauchy's integral theorem.

Similar to the construction of primitives in real analysis, one can also construct holomorphic primitive for holomorphic functions by means of integration.

Existence of primitives for holomorphic functions

Theorem A.20 (Existence of primitives for holomorphic functions)**.** Let Ω be a simply-connected domain and $f\colon \Omega \longrightarrow \mathbb{C}$ a holomorphic function. Then, f has a primitive on Ω. If $\omega \in \Omega$ is fixed, a primitive of f is given by

$$\begin{aligned} F\colon \Omega &\longrightarrow \mathbb{C},\\ z &\longmapsto \int_{\gamma_{\omega,z}} f(\zeta)\,\mathrm{d}\zeta, \end{aligned}$$

where $\gamma_{\omega,z}$ is any (piecewise) continuously differentiable path in Ω with initial point ω and terminal point z. ■

In fact, the existence of a holomorphic primitive for a merely continuous function f on Ω is equivalent to the vanishing of all complex path integrals of f along any (piecewise) continuously differentiable paths in Ω. This is also equivalent to many other properties. We will list these properties below in the Fundamental theorem of complex function theory (Theorem A.26).

Let us look at an example for Theorem A.20.

Example A.21 (Principal branch of logarithm)**.** Let $\Omega := \mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$ be the *sliced plane* and let $f : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto \frac{1}{z}$. Then, f does not have a primitive on $\mathbb{C}$ because

$$\int_{\partial B_1(0)} \frac{1}{z} \, \mathrm{d}z = 2\pi i \neq 0,$$

as we have already seen. Otherwise, this would contradict Cauchy's integral theorem (Theorem A.17). Since Ω is simply-connected, there has to be a primitive for f restricted to Ω by Theorem A.20. For convenience, let us denote $f_{|\Omega}$ with f again. For $z \in \Omega$ let $\gamma_{1,z}$ denote the juxtaposition of a path along the line segment from 1 to $|z|$ following by the circle segment in Ω from $|z|$ to z (such that $\gamma_{1,z}$ is piecewise continuously differentiable). Then,

$$\begin{aligned} \operatorname{Log}\colon \Omega &\longrightarrow \mathbb{C}, \\ z &\longmapsto \int_{\gamma_{1,z}} \frac{1}{\zeta} \, \mathrm{d}\zeta \\ &= \int_1^{|z|} \frac{1}{x} \, \mathrm{d}x + i \int_0^{\operatorname{Arg}(z)} \mathrm{d}x \\ &= \ln(|z|) + i \operatorname{Arg}(z) \end{aligned}$$

is a primitive for f on Ω. Here, ln denotes the natural logarithm for positive numbers, and $\operatorname{Arg}(z)$ the argument of z. The function Log is called the *principal branch of the logarithm*. ◇

Relationship between function values of holomorphic functions

There is a fundamental relationship between the values of holomorphic functions and complex path integrals: *Cauchy's integral formula*.

Cauchy's integral formula

Theorem A.22 (Cauchy's integral formula)**.** Let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic and C a compact circle in Ω. Let ∂C denote a (piecewise) continuously differentiable path that winds around the boundary of C once in counterclockwise direction. Then, for any z_0 in the interior of C, *Cauchy's integral formula* holds:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial C} \frac{f(z)}{z - z_0} \, \mathrm{d}z. \tag{A.21}$$

■

We may interpret this theorem as follows: The values of the holomorphic function f in a compact circle C contained in its domain are determined by the values of the function at the boundary of the circle.

Mean value interpretation of Cauchy's integral formula

We obtain an interesting special case of Cauchy's integral formula for $z = z_0$ and $C =$

$B_r(z)$ such that $\overline{B_r(z)} \subseteq \Omega$. Then, the formula becomes

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z + re^{it}\right) \mathrm{d}t, \tag{A.22}$$

which Remmert and Schumacher (2002, p. 183, own transl., emph. omitted) call the "mean value equation" for holomorphic functions.

Cauchy's integral formula for derivatives

There is a generalisation Equation A.21 known as *Cauchy's integral formula for derivatives*: For every $n \in \mathbb{N}$ and $z_0 \in C$, the nth derivative of f at z_0 can be calculated as follows:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial C} \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z \tag{A.23}$$

More generally, if γ is a piecewise continuously differentiable path in Ω, Equation A.21 generalises to

$$\operatorname*{Ind}_{\gamma}(z_0) f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z \tag{A.24}$$

(see Definition A.28 for a reminder on the winding number $\mathrm{Ind}_\gamma(z_0)$).

There is also a converse for Theorem A.22: Any continuous function $g \colon \partial C \longrightarrow \mathbb{C}$ extends to one and only one function on C whose restriction to the interior of C is holomorphic; in other words, there is exactly one function $f \colon C \longrightarrow \mathbb{C}$ such that $f_{|C^\circ}$ is holomorphic and $f_{|\partial C} = g$ (Pöschel, 2015, p. 289). In fact, this function is given by $f(z) = g(z)$ for $z \in \partial C$ and

$$f(z) = \frac{1}{2\pi i} \int_{\partial C} \frac{g(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

for $z \in C^\circ$.

Cauchy's integral formula

> is remarkable in three ways. First, it expresses the function value at an arbitrary point of [C°; EH.] in terms of the function values at the boundary and therefore shows that two functions, which are holomorphic on [C; EH.] and agree on [∂C; EH.], also agree on [C°; EH.]. Second, the integral in [Equation A.21; EH.] depicts a discontinuous function, namely $f(z)$ in [C°; EH.] and 0 outside. Third, the variable z appears under the integral sign in a very simple and rational way, whereas the integral is an arbitrary holomorphic function (Kneser, 1966, p. 77, transl. EH.).

Similarly, Remmert and Schumacher (2002, p. 182, own transl.) observe that there is no analogue for Cauchy's integral formula in real analysis. They argue that Cauchy's integral formula is "a precursor of the identity theorem [Theorem A.23; EH.] and a first hint at the (sit venia verbo) 'analytical putty' between the range of holomorphic functions".

A rather practical application is that Equation A.21 can be used to calculate complex path integrals. If g is a holomorphic function on $\Omega \setminus \{z_0\}$ such that $f \colon z \longmapsto (z - z_0) g(z)$ can be extended holomorphically to Ω, then

$$\int_{\partial C} g(z) \,\mathrm{d}z = 2\pi i \lim_{z \to z_0} f(z).$$

For instance, Greenleaf (1972, p. 300) acknowledges this application as follows:

> The reader will find it instructive to try evaluating these integrals [examples based on Equation A.21; EH.] by direct calculations based on the defining formula [Equation A.10; EH.] for line integrals. This exercise should provide a keen appreciation of how much effort is saved by an appeal to Cauchy's Integral Formula (or Cauchy's Theorem) in evaluating line integrals.

We explained before that Cauchy's integral formula determines the value of a holomorphic function in the interior of a circle in terms of its values on the boundary (if the circle is completely contained in the domain of the function). This "uniqueness" property can be generalised:

Identity theorem

Theorem A.23 (Identity theorem)**.** Let Ω be a domain, and $f, g\colon \Omega \longrightarrow \mathbb{C}$ be holomorphic. Then, the following three conditions are equivalent:

1. $f = g$;
2. the set $\{z \in \Omega : f(z) = g(z)\}$ has an accumulation point in Ω;
3. there is a $\xi \in \Omega$ such that $f^{(n)}(\xi) = g^{(n)}(\xi)$ for all $n \in \mathbb{N}_0$. ■

Rigidity of holomorphic functions

Thus, the "overall run of an analytic function on a domain [Ω; EH.] is already completely determined if its values are known on a 'very small' subset of [Ω; EH.]" (Freitag & Busam, 2006, p. 120, own transl.)—there is "'*considerable solidarity*' between the function values [emph. in orig.]" (Freitag & Busam, 2006, p. 121, own transl.). Vividly speaking, we may say that holomorphic functions are very *rigid*. For example, a differentiable function $g\colon I \longrightarrow \mathbb{R}$ has at most one extension to a subset of the complex numbers.

Morera's theorem

We have seen in Goursat's lemma (Theorem A.19) and Cauchy's integral theorem (Theorem A.17) that the holomorphicity of a function implies that complex path integrals of this function along closed piecewise continuously differentiable paths vanish. There is a converse for these results, too:

Morera's theorem

Theorem A.24 (Morera's theorem)**.** Assume that $f\colon \Omega \longrightarrow \mathbb{C}$ is continuous. Then, f is holomorphic if and only if $\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0$ for all compact triangles $\Delta \subseteq \Omega$. ■

Accordingly, Morera's theorem guarantees that continuous functions are holomorphic if complex path integrals along boundaries of triangles in their domains vanish. Thus, Cauchy's integral and Morera's theorem hint strongly at a deep relationship between holomorphicity and complex path integration (see also Fundamental theorem of complex function theory (Theorem A.26)).[194]

194 Variants of Morera's theorem are subject of active research, namely about so-called "Morera type theorems" (e.g., Globevnik, 1990; Tumanov, 2004; Volchkov, 1996; Volchkov, 2003, pt. 5, ch. 4; Volchkov & Volchkov, 2018; Zalcman, 1972, 1982). For example, Volchkov (1991) showed that a complex function f on the open ball $B_1(0)$ is holomorphic if and only if $\int_{\partial Q} f(z)\,\mathrm{d}z = 0$ for every square $Q \subseteq B_1(0)$ of fixed side length $0 < d < 2/\sqrt{5}$ (if $d > 2/\sqrt{5}$, there exists a non-holomorphic function f on $B_1(0)$ such that $\int_{\partial Q} f(z)\,\mathrm{d}z = 0$ for every square of side length d).

A.7 POWER AND LAURENT SERIES EXPANSIONS

We have already hinted at the result that holomorphic functions are analytic. In particular, holomorphic functions are automatically infinite differentiable. This is a consequence of the following theorem about power and Laurent series expansions of holomorphic functions.

Power and Laurent series expansion

Theorem A.25 (Power and Laurent series expansion of holomorphic functions)**.** Let $f\colon \Omega \longrightarrow \mathbb{C}$ be a complex function.

1. If f is holomorphic, then, for each $z_0 \in \Omega$ there exists an $r = r(z_0) > 0$ and a sequence of complex numbers $a_k = a_k(z_0)$, $k \in \mathbb{N}_0$, such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for each $z \in B_r(z_0)$. The sum on the right-hand side is called a *power series expansion* or *Taylor series expansion of f in z_0*. The coefficients are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{\partial B_\tau(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, \mathrm{d}\zeta \tag{A.25}$$

for $k \in \mathbb{N}$ and any $0 < \tau < r$.

In particular, holomorphic functions are analytic.

2. If A is a discrete set in Ω and $f\colon \Omega \setminus A \longrightarrow \mathbb{C}$ is holomorphic,[195] then, for each $z_0 \in A$, $r = r(z_0) > 0$, and $s = s(z_0) > 0$ such that the annulus $A_{r,s}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < s\}$ is contained in Ω, there exists sequence of complex numbers $b_k = b_k(z_0)$, $k \in \mathbb{Z}$, such that

$$f(z) = \sum_{k=-\infty}^{\infty} b_k (z - z_0)^k$$

for each $z \in A_{r,s}(z_0)$. The sum on the right-hand side is called a *Laurent series expansion of f in z_0*. The coefficients are given by

$$b_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{\partial B_\tau(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, \mathrm{d}\zeta \tag{A.26}$$

for $k \in \mathbb{Z}$ and any $r < \tau < s$. ■

The usual proof of this theorem we encounter in textbooks or lecture notes on complex analysis begins with the expansion of the integrand in Cauchy's integral formula (Theorem A.22) into a geometric series and then the order of summation and integration is changed (e.g., Fre-

195 The elements of A are called (isolated) singularities of f. One distinguished three types: *removable singularities*, *poles*, and *essential singularities* (Freitag & Busam, 2006, pp. 131–145).

itag & Busam, 2006, pp. 106–107). For example, the power series development may be obtained like this:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B_\tau(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\partial B_\tau(z_0)} \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} (z - z_0)^k\,\mathrm{d}\zeta \\ &= \sum_{k=0}^{\infty} a_k (z - z_0)^k. \end{aligned}$$

However, in the middle of the 20th century, Taylor and Laurent series expansion theorems were proved *without* the use of complex path integration, too (see Section 7.5).

A.8 THE FUNDAMENTAL THEOREM OF COMPLEX FUNCTION THEORY

The following theorem underlines how closely holomorphicity and complex path integration in complex analysis are intertwined. Cauchy's integral theorem (Theorem A.17) and Morera's theorem (Theorem A.24) already hinted at this deep connection. In Section 7.5, we describe the mathematicians' efforts in the 19th and 20th century to reduce the amount of complex path integration for the study of holomorphic functions to a minimum. However, similar efforts were also undertaken in the other direction, namely to study developments of complex functions into series without the use of differentiation.

Fundamental theorem of complex function theory

The presentation of the following *Fundamental theorem of complex function theory* is due to Rodríguez et al. (2013, pp. 3–4).[196]

Fundamental theorem of complex function theory

Theorem A.26. For a domain $\Omega \subseteq \mathbb{C}$ and a continuous function $f\colon \Omega \longrightarrow \mathbb{C}$ the following conditions are equivalent:

1. f is *holomorphic*;
2. $\mathbf{f}$ is *totally differentiable and satisfies the Cauchy-Riemann equations* (Equation A.6);
3. for each simply-connected subdomain Ω' of Ω there is a *holomorphic primitive function for* $f_{|\Omega'}$, that is, there is a holomorphic function $F\colon \Omega' \longrightarrow \mathbb{C}$ such that $F'(z) = f(z)$ for $z \in \Omega'$ (such an F can be constructed as

$$F(z) := \int_{\gamma_{\omega,z}} f(\zeta)\,\mathrm{d}\zeta \quad (z \in \Omega')$$

for any fixed $\omega \in \Omega'$ and $\gamma_{\omega,z}$ any piecewise continuously differentiable path in Ω' which links ω to z);

196 Property 5. in Theorem A.26 is not included in the statement of the theorem by Rodríguez et al. (2013). We included it here because we study this property it in Section 8.3.

4. for every closed piecewise continuously differentiable path γ the *complex path integral* $\int_\gamma f(z)\,\mathrm{d}z$ *vanishes*;

5. complex path integrals *depend only on the initial and terminal point of the path* but not the concrete path; this means that for every two paths γ and η in Ω starting at the same point and ending at the same point the integrals $\int_\gamma f(z)\,\mathrm{d}z$ and $\int_\eta f(z)\,\mathrm{d}z$ are equal;

6. for every ball $B_r(z_0)$ ($z_0 \in \Omega, r > 0$) such that $\overline{B_r(z_0)} \subseteq \Omega$ and z in the interior of $B_r(z_0)$, *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

holds.

7. f is *infinitely differentiable*;

8. f is *analytic*; in other words, for every ball $B_r(z_0)$, $z_0 \in \Omega$, $r > 0$, such that $\overline{B_r(z_0)} \subseteq \Omega$, there is a sequence $(a_k)_{k\in\mathbb{N}}$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for every $z \in B_r(z_0)$, and the coefficients are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,\mathrm{d}\zeta$$

for every $0 < s < r$ and $k \in \mathbb{N}$;

9. let $(K_\lambda)_{\lambda\in\Lambda}$ be the family of connected components of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$, $s_\lambda \in K_\lambda$ ($\lambda \in \Lambda$) and $S = \{s_\lambda : \lambda \in \Lambda\}$; then, there is a sequence of rational functions whose uniform limit is f and whose singularities are contained in S.[a]

■

[a] A rational function is a quotient of two polynomial functions. The implication 1. to 9. is named after Carl Runge (1856–1927).

All of the conditions in this theorem are discussed either in this chapter or in the subject-matter didactic analysis Chapter 8, except for condition 9. The reason is that it will have no natural appearance in the reminder of the thesis and does not appear in the empirical study in Part iii.

A.9 RESIDUE THEOREM

Above, we chose the paths in Goursat's lemma (Theorem A.19), Cauchy's integral formula (Theorem A.22) and Equation A.23, and the power and Laurent series expansion in Theorem A.25 to be boundaries of circles mainly for reasons of simplicity. The results remain valid for other paths. We will not state these results explicitly for other paths but will proceed to state the *residue theorem*, which unifies Cauchy's integral theorem and integral formula.

Residue

Using the notation from Theorem A.25, it is immediate that all terms in a Laurent series, $f(z) = \sum_{k=-\infty}^{\infty} b_k(z-z_0)^k$, possess a primitive except for the term $z \longmapsto b_{-1}(z-z_0)^{-1}$ if $b_{-1} \neq 0$. This implies that

$$\int_{\partial B_\tau(z_0)} f(z)\,\mathrm{d}z = 2\pi i b_{-1}$$

for any $r < \tau < s$. Therefore, the coefficient b_{-1} is of particular importance and has a special name.

Residue **Definition A.27** (Residue)**.** Under the assumptions in the second part of Theorem A.25, we call $\operatorname{Res}_{z_0}(f) := b_{-1}$ the *residue of f at z_0*. ■

The residue has several other characterisations: $\operatorname{Res}_{z_0}(f)$ is the unique complex number ξ such that

$$z \longmapsto f(z) - \frac{\xi}{z-z_0}$$

has a primitive in an open neighbourhood of z_0 and

$$\operatorname*{Res}_{z_0}(f) = \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{\partial B_\varepsilon(z_0)} f(z)\,\mathrm{d}z$$

(Pöschel, 2015, pp. 331–332).

Winding number

Given a closed piecewise continuously differentiable path $\gamma\colon [a,b] \longrightarrow \Omega$ and $\omega \in \Omega$, there are two piecewise continuously differentiable functions $r, \theta\colon [a,b] \longrightarrow [0,\infty)$ such that

$$\gamma(t) = z + r(t)\mathrm{e}^{i\theta(t)} \tag{A.27}$$

for all $t \in [a,b]$ (Pöschel, 2015, p. 329). Hence, r is a function that measures the distance of γ to ω and θ measures the argument of $\gamma - \omega$.

Then, since γ is closed, $\theta(b) - \theta(a)$ is an integer multiple of 2π. Hence this difference measures the total change of argument of γ around ω. This measure counts multiplicity and is sensitive to clockwise and counterclockwise rotation.

Winding number **Definition A.28** (Winding number)**.** Assume that θ is the function from the previous discussion. Then

$$\operatorname*{Ind}_{\gamma}(\omega) := \frac{\theta(b)-\theta(a)}{2\pi}$$

is called the *winding number of γ around ω*. ■

A direct computation shows that

$$\operatorname{Im} \int_a^b \frac{\gamma'(t)}{\gamma(t) - \omega} \, \mathrm{d}t = \theta(b) - \theta(a)$$

and

$$\operatorname{Re} \int_a^b \frac{\gamma'(t)}{\gamma(t) - \omega} \, \mathrm{d}t = 0$$

(Pöschel, 2015, p. 329, ch. 9.7).[197] Hence, the winding number can also be expressed as a complex path integrals:

$$\operatorname*{Ind}_{\gamma}(\omega) = \frac{1}{2\pi i} \int_{\gamma} \frac{\mathrm{d}z}{z - \omega}.$$

The residue theorem

The *residue theorem* we have announced earlier is as follows (e.g., Freitag & Busam, 2006, p. 164; Pöschel, 2015, p. 333):

Theorem A.29 (Residue theorem)**.** Let Ω be a simply-connected domain in $\mathbb{C}$, A a finite set in Ω, $f\colon \Omega \setminus A \longrightarrow \mathbb{C}$ holomorphic, and γ a closed (piecewise) continuously differentiable path in $\Omega \setminus A$. Then, we have *Residue theorem*

$$\int_{\gamma} f(z) \, \mathrm{d}z = 2\pi i \sum_{\omega \in A} \operatorname*{Ind}_{\gamma}(\omega) \operatorname*{Res}_{\omega}(f). \tag{A.28}$$

■

Note that the sum in Equation A.28 consists of only finitely many addends. This is why the residue theorem is such a powerful tool for the computation of complex path integrals.

The residue theorem has remarkable consequences within complex analysis and many applications such as the computations of real integrals (e.g., Freitag & Busam, 2006, pp. 175–187). According to the philosopher of mathematics Colyvan (2012), this theorem is exemplary for the interconnections of different mathematical disciplines:

> The upshot is that the contour integral of a complex function depends only on what happens at a limited number of singularities inside the contour. So although a line integral looks as though it is local (i.e., depends on the behavior of the function along the contour in question), in fact it is determined by non-local features of the function (i.e., what happens at singularities remote from the contour in question). This is the mathematical equivalent of action at a distance.
>
> [...]
>
> This raises questions about intrinsic explanation in mathematics. You'd expect to be able to calculate real integrals via real methods and, moreover, any mathematical explanations of why the integrals take the values they do should be made in terms of real analysis. The Residue Theorem and its applications to real analysis raise doubts about both these expectations. Sometimes the only means we have of calculating real integrals is via the Residue Theorem and the associated excursion into the complex domain. Moreover, it can be argued that this excursion is more than just a useful calculational trick: in

197 One misprinted derivative sign in Pöschel (2015, p. 329) was corrected.

> at least some cases, the *explanation* of the answer is given by the Residue Theorem (or so it seems to me). This, in turn suggests that intrinsic explanations (in this case, real-analysis explanations of real-analysis facts) are not always possible. Alternatively, we might draw the conclusion that the explanations in question via the Residue Theorem are, despite appearances, intrinsic; it's just that real analysis and complex analysis are more closely connected than we might initially think. Such applications of the Residue Theorem to real analysis thus raise interesting questions about the boundaries between the various branches of mathematics and force us to ponder intra-mathematical explanations. (Colyvan, 2012, p. 160-161, emph. orig.)

A.10 COMPLEX PATH INTEGRALS ALONG ARBITRARY PATHS

Complex path integrals along arbitrary paths

The notion of complex path integral can be extended to arbitrary paths if the integrand is holomorphic. For this to be done, let $\gamma\colon [a,b] \longrightarrow \Omega$ be a *any* path in the domain Ω and let $f\colon \Omega \longrightarrow \mathbb{C}$ be holomorphic. We have seen that holomorphic functions may not have a primitive on all of Ω but at least in open balls in Ω because balls are simply-connected (see Theorem A.20). Since the trace $\mathrm{tr}(\gamma)$ is compact, one may choose a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,b]$ such that the traces of the restriction of γ to $[t_{k-1}, t_k]$, that is, $\gamma([t_{k-1}, t_k])$, is contained in an open ball $D_k \subseteq \Omega$ on which f possesses a primitive F_k locally for $k = 1, 2, \ldots, n$. Then, one can prove that the complex number

$$\sum_{k=1}^{n} \left(F_k(t_k) - F_k(t_{k-1})\right) \tag{A.29}$$

is independent of the choices of the partition of $[a,b]$, the balls, and the local primitives of f on these balls. Thus, the number in Equation A.29 is well-defined. Then, we may *define* the complex path integral of f along γ by

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{k=1}^{n} \left(F_k(t_k) - F_k(t_{k-1})\right)$$

(e.g., Lang, 1999, ch. III. §4).

There is also another way to define the complex path integral in this case. Since f is holomorphic, it has a local primitive G_0 on a neighbourhood of $\gamma(a)$ in Ω and another local primitive G_1 on a neighbourhood of $\gamma(b)$ obtained by so-called *analytic continuation of G_0 along γ*. In this case, the number

$$G_1(\gamma(b)) - G_0(\gamma(a))$$

is well-defined and agrees with the number in Equation A.29 (Jänich, 2004, ch. V; Lang, 1999, ch. XI).

Finally, if two paths in Ω are homotopic relative to initial and terminal points, then analytic continuations are unique in the sense of the *monodromy theorem* (e.g., Jänich, 2004, p. 59; Lang, 1999, p. 326):

Monodromy theorem

Theorem A.30 (Monodromy theorem)**.** Let γ and η be two paths in Ω homotopic relative to their initial point A and terminal point B. Let $f\colon K \longrightarrow \mathbb{C}$ be a holomorphic function in a neighbourhood of A in Ω. Assume further that $f_1\colon K' \longrightarrow \mathbb{C}$ is an analytical continuation of

f along γ and $f_2 : K' \longrightarrow \mathbb{C}$ is an analytical continuation of f along η for an open neighbourhood K' of B in Ω. Then, we have $f_1 = f_2$. ■

A.11 MORE PROPERTIES OF HOLOMORPHIC FUNCTIONS AND COMPLEX PATH INTEGRALS

We have seen fundamental differences between holomorphic functions and differentiable real functions such as the fact that holomorphic functions are automatically infinite differentiable and moreover they can be developed into power series on open balls around any point of their domain. There are many other differences. They have many other fascinating properties.

Continuous functions obtain their extreme points on compact sets. Of course, this remains true in complex analysis. In this case, the locations of these extreme points are always on the boundary of the compact set.

Proposition A.31 (Maximum modulus principle). If $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic and not constant, it does not possess maximal absolute values, that is there does not exist a $\xi \in \Omega$ such that $|f(\xi)| \geq |f(z)|$ for all $z \in \Omega$.

Furthermore, if f has a continuous extension to $\partial\Omega$, then there exists a $\xi \in \partial\Omega$ such that $|f(\xi)| \geq |f(z)|$ for all $z \in \overline{\Omega}$. ■ ■

A remarkable corollary of the residue theorem is the *argument principle* (Pöschel, 2015, pp. 335–336). We recall that a function f is meromorphic if it is of the form $\Omega \setminus A \longrightarrow \mathbb{C}$, where $A \subseteq \Omega$, and each point in A is a pole of f.

Corollary A.32 (Argument principle). Let $f : \Omega \longrightarrow \mathbb{C}$ be meromorphic, and γ a simple closed piecewise continuously differentiable path in Ω such that no singularity of f is on $\operatorname{tr}(\gamma)$. Then,

$$\operatorname*{Ind}_{f\circ\gamma}(0) = N_f - P_f, \tag{A.30}$$

where N_f is the sum of the orders of the zeros of f in $\operatorname{int}(\gamma)$ and P_f is the sum of the orders of the poles of f in $\operatorname{int}(\gamma)$. ■ ■

Equation A.30 is called *Rouché's formula*, named after Eugène Rouché (1832–1910). By definition of complex path integrals, this formula is equivalent to

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_f - P_f.$$

A corollary of the argument principle is Rouché's theorem (cf. Pöschel, 2015, p. 337):

Corollary A.33 (Rouché's theorem). Assume f and γ are as in Corollary A.32 and $g : \Omega \longrightarrow \mathbb{C}$ is another meromorphic functions without poles on $\operatorname{tr}(\gamma)$ such that $|f(z) - g(z)| < |f(z)|$ for all $z \in \operatorname{tr}(\gamma)$. Then,

$$N_f - P_f = N_g - P_g,$$

where N_g is the sum of orders of the zeros of g in $\operatorname{int}(\gamma)$ and P_g is the sum of orders of the poles of g in $\operatorname{int}(\gamma)$. ■ ■

The inequality in the last theorem can be strengthened to $|f(z) - g(z)| < |f(z)| + |g(z)|$ (e.g., Howell & Schrohe, 2017).

A complex version of Green's theorem

Lastly, we recall that Green's theorem (Theorem B.15) allows us to express a real path integral of second kind of a continuously differentiable vector field along a piecewise continuously differentiable Jordan path γ as a double integral.

Let γ be a piecewise continuously differentiable Jordan path in Ω, which surrounds its interior $\operatorname{int}(\gamma)$ counterclockwise, and let $f = u + iv$ be a complex function such that $\mathbf{f} = (u, v)^T$ continuously differentiable (e.g., this is the case when f is holomorphic). Then, Green's theorem yields

$$\int_\gamma u \,\mathrm{d}x + v \,\mathrm{d}y = \iint_{\operatorname{int}(\gamma)} (\partial_1 v - \partial_2 u) \,\mathrm{d}\mathcal{A}.$$

Note that the vector fields $(u, -v)^T$ and $(v, u)^T$ are continuously differentiable, too. Since one can show that

$$\int_\gamma f(z) \,\mathrm{d}z = \int_\gamma u \,\mathrm{d}x - v \,\mathrm{d}y + i \int_\gamma v \,\mathrm{d}x + u \,\mathrm{d}y$$

(see Chapter 8), an application of Green's theorem (Theorem B.15) yields

$$\int_\gamma f(z) \,\mathrm{d}z = -\iint_{\operatorname{int}(\gamma)} \partial_1 v + \partial_2 u \,\mathrm{d}\mathcal{A} + i \iint_{\operatorname{int}(\gamma)} \partial_1 u - \partial_2 v \,\mathrm{d}\mathcal{A}.$$

Green's formula for complex path integrals

This equation can also be rephrased as

$$\int_\gamma f(z) \,\mathrm{d}z = 2i \iint_{\operatorname{int}(\gamma)} \bar{\partial} f \,\mathrm{d}\mathcal{A} \tag{A.31}$$

(cf. Narasimhan & Nievergelt, 2001, p. 14). We call this formula *Green's formula for complex path integrals*. For the last equality, we set

$$\iint_{\operatorname{int}(\gamma)} \bar{\partial} f \,\mathrm{d}\mathcal{A} := \iint_{\operatorname{int}(\gamma)} \operatorname{Re}(\bar{\partial} f) \,\mathrm{d}\mathcal{A} + i \iint_{\operatorname{int}(\gamma)} \operatorname{Im}(\bar{\partial} f) \,\mathrm{d}\mathcal{A}.$$

Using slightly different notation and a partial differential operator other than $\bar{\partial}$, Trahan (1965) obtains a formula similar to Equation A.31 (see Section 8.1.5).

B

SUMMARY OF REAL ANALYSIS IN ONE AND TWO VARIABLES

This summary of definitions and propositions from real analysis in one and two variables is intended for the purpose of reference. The content can be revisited in any book on real analysis (e.g., Apostol, 1981; Forster, 2016; Königsberger, 2004a; Ross, 2013).

B.1 THE RIEMANN INTEGRAL

In this section, we review basic terminology and definitions in the context of Riemann integrals.

Let $[a, b]$ be an interval of real numbers and $f\colon [a, b] \longrightarrow \mathbb{R}$ be a bounded function.

Definition B.1. A *partition* P of $[a, b]$ is a finite and ascending set of real numbers $t_0, t_1, \ldots, t_n$ ($n \in \mathbb{N}$) in $[a, b]$, whose first and last entry agree with a and b. In this case, we write $P\colon a = t_1 < t_2 < \ldots < t_n = b$. n is called the *length of* P and $\rho_P = \max_{k=0,1,\ldots,n-1}\{t_{k+1} - t_k\}$ is called the *norm of* P. ■ *Partition*

Riemann integrals

We will now define the Riemann integral of a real-valued function on $[a, b]$. A similar definition dates back to Cauchy (1823).

Definition B.2 (Riemann sums and Riemann integrals). · Let $P\colon a = t_1 < t_2 < \ldots < t_n = b$ be a partition of $[a, b]$. For $k = 0, 1, \ldots, n-1$ let $\xi_\nu \in [t_\nu, t_{\nu+1}]$. We call $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$ a *tag vector for* P and each of its entries *tags for* P. The pair consisting of P and ξ is called a *tagged partition of* $[a, b]$. The sum *Riemann sum*

$$R(f, P, \xi) := \sum_{k=0}^{n-1} (t_{k+1} - t_k) f(\xi_k)$$

is called a *Riemann sum of f with respect to P and ξ*.

Riemann integral

- We call f *(Riemann) integrable* and $I \in \mathbb{R}$ the *Riemann integral of f* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and partitions P of $[a, b]$ of length n with norm $\rho_P \leq \delta$ and all tag vectors $\xi = (\xi_0, \xi_1, \ldots, \xi_{n-1})$ for P the inequality

$$|R(f, P, \xi) - I| < \varepsilon$$

holds. In this case, we write

$$\int_a^b f := \int_a^b f(x)\,\mathrm{d}x := I. \qquad \blacksquare$$

Clearly, if the Riemann integral of function f exists, then it is unique. Moreover, if the Riemann integral exists and there is a sequence of tagged partitions $\left(P^{(n)}, \xi^{(n)}\right)_{n \in \mathbb{N}}$ such that $\lim_{\left(P^{(n)}, \xi^{(n)}\right)} R\left(f, P^{(n)}, \xi^{(n)}\right)$ exists, where the limits is taken with respect to $n \longrightarrow \infty$ such that $\rho_{P^{(n)}} \longrightarrow 0$, then this limit is $\int_a^b f(x)\,\mathrm{d}x$.

A Riemann sum is illustrated in Figure B.1a. Vividly speaking, $\int_a^b f(x)\,\mathrm{d}x$ corresponds to the oriented area that is enclosed by the graph of f, the horizontal x-axis and the vertical axes at $x = a$ and $x = b$. Oriented means here that the area above the horizontal axis is weighted positively and the area below is weighted negatively (Figure B.1b). For example, every piecewise continuous function $[a, b] \longrightarrow \mathbb{R}$ is Riemann-integrable.

Darboux integrals

There is another approach to define the Riemann integral using so-called upper and lower sums. The integral defined this way is sometimes called *Darboux integral* instead of Riemann integral.

Let $P : a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a, b]$. The *upper sum of f with respect to P* is

$$U(f, P) := \sum_{k=0}^{n-1} \left(\sup_{x \in [t_k, t_{k+1}]} f(x) \right) (t_{k+1} - t_k);$$

the *lower sum of f with respect to P* is

$$L(f, P) := \sum_{k=0}^{n-1} \left(\inf_{x \in [t_k, t_{k+1}]} f(x) \right) (t_{k+1} - t_k).$$

We call f *Darboux integrable* if

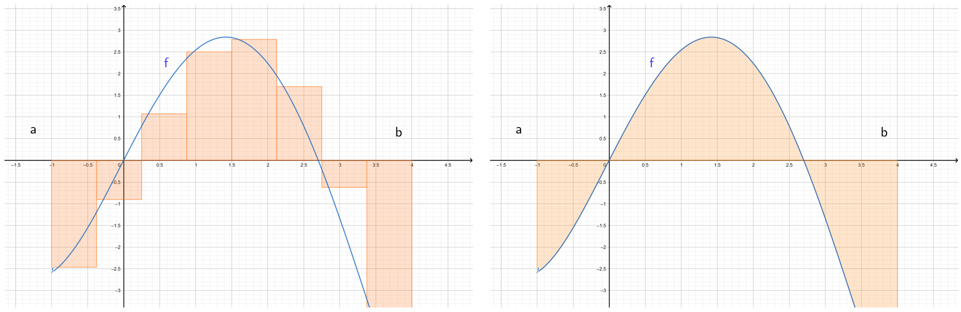
$$U := \sup \{L(f, P) \ : \ P \text{ partition of } [a, b]\}$$

and

$$L := \inf \{U(f, P) \ : \ P \text{ partition of } [a, b]\}$$

exist and are equal. In this case, the number $U = L$ is called the *Darboux integral of f*. It can be shown that the notions of Riemann integral and Darboux integral are in fact equivalent (Forster, 2016, §11; Ross, 2013, §32).

The approach to the Riemann integral via upper and lower sums is not applicable in complex analysis because the notion of suprema and infima is not applicable for complex-valued func-



(a) Riemann sum for an equidistant partition with eight elements.

(b) Riemann integral as an area.

Figure B.1: Riemann sum and Riemann integral.

tions directly. However, the definition via Riemann sums can be transferred to complex path integrals (see Section 8.1.1).

Fundamental theorem of calculus

One of the most important theorems in calculus / real analysis is the fundamental theorem of calculus. It relates Riemann integrals and primitive functions to each other.

Theorem B.3 (Fundamental theorem of calculus)**.** We assume that $f\colon [a, b] \longrightarrow \mathbb{R}$ is continuous and we fix a $x_0 \in [a, b]$. Then, we have the following:

Fundamental theorem of calculus

- The function $F\colon [a, b] \longrightarrow \mathbb{R}$ defined by

$$F(t) := \int_{x_0}^{t} f(x)\,\mathrm{d}x \tag{B.1}$$

  is a primitive function for f; that is F is differentiable and $F' = f$.

- For every primitive $G\colon [a, b] \longrightarrow \mathbb{R}$ of f and every $r, s \in [a, b]$, we have

$$\int_{r}^{s} f(x)\,\mathrm{d}x = G(s) - G(r). \tag{B.2}$$

■

There is an analogue proposition in complex analysis (see Proposition A.16).

B.2 PATH INTEGRALS IN REAL ANALYSIS OF TWO VARIABLES

The Riemann integral of a function of one real variable is generalised in different ways to the two-dimensional case, that is to $\mathbb{R}$- or $\mathbb{R}^2$-valued functions of two real variables. In the following, we define *real path integrals of first* and *second kind*. Here, integration is not carried out over an interval but rather along a *path* in $\mathbb{R}^2$.

We summarised the relevant definitions and properties of paths and domains in $\mathbb{C} \cong \mathbb{R}^2$ in Section A.4. Since we only deal with continuous functions in complex analysis in this thesis, we assume that all functions considered for the rest of this chapter are continuous, too.

In order to be able to define two variants of the real path integral of second kind for a vector field, we associate two vector fields to each (piecewise) continuously differentiable path, namely the *tangential field* and the *normal field*. Vividly speaking, the tangential field points into the direction induced by the path and the normal field is obtained by turning the tangential field by $\pi/2$ clockwise.

Tangential and normal field associated to γ

Definition B.4 (Tangential and normal vector field of a path)**.** Let $\gamma\colon [a, b] \longrightarrow \Omega$ be a piecewise continuously differentiable path. Let A be the finite subset of $[a, b]$ at which γ is not differentiable.

We set $\gamma'(t) := (0, 0)^T$ and thus extend $\gamma'\colon [a, b] \setminus A \longrightarrow \mathbb{R}^2$ to a piecewise continuous function $\gamma'\colon [a, b] \longrightarrow \mathbb{R}^2$ (for which we also use the symbol γ'). We call γ' the *tangential field* associated to γ and $\gamma'(t)$ the *tangential vector of γ at t* for each $t \in [a, b]$.

Additionally, we define $\mathbf{n}\colon [a, b] \longrightarrow \mathbb{R}^2$ by $\mathbf{n}(t) := \mathbf{J}^{-1}\gamma'(t)$. Here, $\mathbf{J}$ is the matrix, which corresponds to a rotation by $\pi/2$ counterclockwise. We call $\mathbf{n}$ the *normal field* associated to γ and $\mathbf{n}(t)$ the *normal vector of γ at t* for each $t \in [a, b]$. ■

From now on, we fix $\gamma\colon [a, b] \longrightarrow \Omega$ to be a (piecewise) continuously differentiable path in a domain $\Omega \subseteq \mathbb{R}^2$.

B.2.1 *Path integrals of first kind*

In this section, we define real path integrals of first kind. These are integrals for real-valued continuous functions defined on the traces of paths.

Real path integral of first kind

Definition B.5 (Real path integral of first kind)**.** Let $g\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{R}$ be a continuous real-valued function on the trace of γ. The *path integral of first kind of g along γ* is defined as

$$\int_\gamma g \,\mathrm{d}s := \int_a^b g(\gamma(t))|\gamma'(t)| \,\mathrm{d}t. \qquad ■$$

The path integral of first kind is invariant under a reparametrisation of the path.

Geometrical and physical interpretations of path integrals of first kind

Let $g\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{R}$ be a continuous function. Similar to the Riemann integral (Definition B.2) the real path integral of first kind $\int_\gamma g \,\mathrm{d}s$ can also be interpreted as the area under the graph of g measured with respect to sign. To make this precise, we set

$$A_{\gamma,g} := \left\{ (x, y, z)^T \in \mathbb{R}^3 : (x, y)^T \in \operatorname{tr}(\gamma), z \in \begin{cases} [0, g(x, y)], & g(x, y) \geq 0, \\ z \in [g(x, y), 0], & g(x, y) < 0 \end{cases} \right\}.$$

Then, $A_{\gamma,g}$ can be seen as the curved area enclosed by the graph of g and the trace of γ in $\mathbb{R}^3$. Its measure is $\int_\gamma g \,\mathrm{d}s$, where the parts of $A_{\gamma,g}$ above the (x, y)-plane are weighted positively and the parts below are weighted negatively (Schmid, 2018, pp. 78–80). Figure B.2 illustrates this: The graph of g is shown in red, the trace of γ is shown in green, and the set $A_{\gamma,g}$ is shaded grey. In this case, $A_{\gamma,g}$ lies entirely above the (x, y)-plane.

We may also interpret the real path integral of first kind physically, which corresponds to the accumulation interpretation for Riemann integrals (Section 2.2.3). For example, if g is the mass density of a thin body modelled by the set $\operatorname{tr}(\gamma)$, then $\int_\gamma g \,\mathrm{d}s$ can be interpreted as the mass of the body.

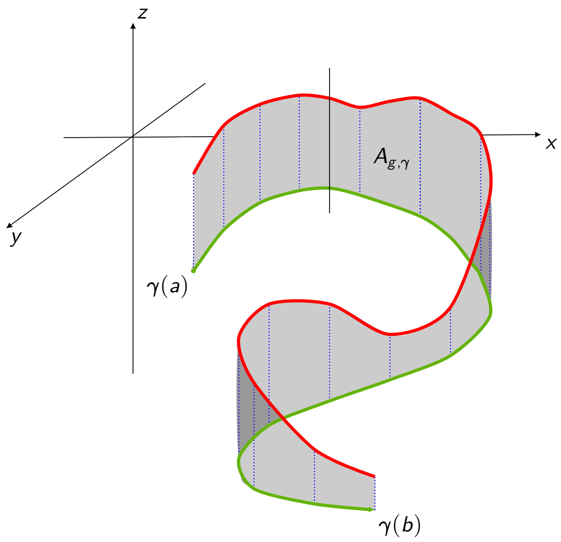


Figure B.2: Illustration of $A_{g,\gamma}$.

B.2.2 *Path integrals of second kind*

In this section, we define *real path integrals of second kind*. These are integrals for vector fields instead of real-valued functions defined on the trace of a path.

Let $\mathbf{F} = (P, Q)^T : \operatorname{tr}(\gamma) \longrightarrow \mathbb{R}^2$ be continuous.

Real path integral of second kind (with respect to the tangential field)

Definition B.6 (Real path integral of second kind)**.** The *real path integral of second of* $\mathbf{F}$ *along* γ is defined as

$$\int_\gamma \mathbf{F}\, \mathrm{d}\mathbf{T} := \int_\gamma P\, \mathrm{d}x + Q\, \mathrm{d}y := \int_a^b \langle \mathbf{F}(\gamma(t)), \gamma'(t)\rangle\, \mathrm{d}t. \tag{B.3}$$

■

Here, we use the letter $\mathbf{T}$ because the tangential field γ' appears at the right-hand side of Equation B.3. We also remark that the notation $\int_\gamma P\, \mathrm{d}x + Q\, \mathrm{d}y$ is to be understood in such a way that the first component of $\mathbf{F}$ (i.e., P) is placed in front of $\mathrm{d}x$ and the second component of $\mathbf{F}$ (i.e., Q) is placed in front of $\mathrm{d}y$. That being said, we may occasionally write $\int_\gamma Q\, \mathrm{d}y + P\, \mathrm{d}x$ for $\int_\gamma \mathbf{F}\, \mathrm{d}\mathbf{T}$ etc. We can memorise Equation B.3 by formally equating $\mathrm{d}\mathbf{T}$ with $\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$ and $\mathbf{F}\, \mathrm{d}\mathbf{T}$ with the scalar product $\langle \mathbf{F}, \mathrm{d}\mathbf{T}\rangle = P\, \mathrm{d}x + Q\, \mathrm{d}y$.

Replacing γ' with $\mathbf{n}$, we obtain a variant of the previously defined integral.

Real path integral of second kind (with respect to the tangential field)

Definition B.7 (Real path integral of second kind with respect to the normal field)**.** The *real path integral of second of* $\mathbf{F}$ *along* γ *with respect to the normal field* is defined as

$$\int_\gamma \mathbf{F}\, \mathrm{d}\mathbf{N} := \int_a^b \langle \mathbf{F}(\gamma(t)), \mathbf{n}(t)\rangle\, \mathrm{d}t. \tag{B.4}$$

■

Real path integrals of second kind can of course also be defined via Riemann sums. For the real path integral of second kind with respect to the tangential field, these sums have the form

$$\sum_{k=0}^{n-1} \langle \mathbf{F}(\gamma(\xi_k)), \gamma(t_{k+1}) - \gamma(t_k)\rangle\,, \tag{B.5}$$

where $a = t_0 < t_1 < \dots t_n = b$ is a partition of $[a, b]$ and $\xi \in [t_k, t_{k+1}]$ ($k = 0, \dots, n-1$) are tags. Similar sums could be derived for the path integral of second kind with respect to the normal field.

Real path integrals of second kind are invariant with respect to a reparametrisation of the path.

There is a simple relationship between the two notions of real path integral that we defined.

Remark B.8.

- We have

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N} = \int_\gamma P\,\mathrm{d}y - Q\,\mathrm{d}x = \int_\gamma \begin{pmatrix} -Q \\ P \end{pmatrix} \mathrm{d}\mathbf{T} = \int_\gamma \mathbf{JF}\,\mathrm{d}\mathbf{T}. \tag{B.6}$$

- Every path integral of second kind can also be written as a path integral of first kind. In order to do that, we define the *unit tangential vector field* related to γ by

$$\begin{aligned} \widetilde{\gamma}\colon [a, b] &\longrightarrow \mathbb{R}^2, \\ t &\longmapsto \begin{cases} \|\gamma'(t)\|^{-1}\gamma'(t), & \gamma'(t) \neq \mathbf{0}, \\ \mathbf{0}, & \text{else}. \end{cases} \end{aligned}$$

  Then, we have

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \int_\gamma \langle \mathbf{F} \circ \gamma, \widetilde{\gamma} \rangle\,\mathrm{d}s. \tag{B.7}$$

  where we understand $\langle \mathbf{F}, \widetilde{\gamma} \rangle$ to be the function, which is defined by $\langle \mathbf{F}(\gamma(t)), \widetilde{\gamma}(t) \rangle$ for $t \in [a, b]$.

◇

Interpretations of real path integrals of second kind

Work, flow, and flux

The real path integrals of second kind have allow for some physical interpretations.

We assume for the moment that $F \in \mathbb{R}^2$ represents a force acting on a particle traversing along a directed line segment represented by $s \in \mathbb{R}^2$. Then, $\langle F, s \rangle$ can be interpreted as the work done along s with respect to F. Hence, each addend in the sum in Equation B.5 can be interpreted as the *work* done by a particle in a force field represented by the function $\mathbf{F}$ on $\mathrm{tr}(\gamma)$. That is, the real path integral of second kind $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$ can be interpreted as the work done by a particle along γ.

Similarly, if $\mathbf{F}$ represents the velocity of a fluid, modelled in $\mathbb{R}^2$, then $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$ can also be interpreted as the total *flow* of that fluid along $\mathrm{tr}(\gamma)$ (directed according to γ). In this case, since $\mathbf{n}$ is obtained by turning γ' by $\pi/2$ clockwise, $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N}$ can be interpreted as the *flux* of the fluid across $\mathrm{tr}(\gamma)$ (directed according to γ).

Winsløw (2019) gives another interpretation based on the basic idea of area for Riemann integrals (Section 2.2.3): “One may even reach an interpretation of $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$ as a kind of ‘signed area of $\mathbf{F}$ along γ’ if considering the meaning of the vector product as a signed area of the parallelogram spanned by $\mathbf{F}(\gamma(\xi_k))$ and $\gamma(t_{k+1}) - \gamma(t_k)$” (Winsløw, 2019, p. 146, notation adapted).

B.2.3 *Gradient fields and the integrability condition*

In this section, we will review important integral theorems. On the one hand, we will partly generalise the fundamental theorem of calculus (Theorem B.3) and compute real path integrals of second kind in terms of the analogue of primitive functions, namely potentials. We will also discuss in which situations real path integrals do not depend on the chosen path between unless their initial and terminal points remain the same.

From now on, let $\mathbf{F}\colon \Omega \longrightarrow \mathbb{R}^2$ be a continuously partially differentiable vector field defined on a domain $\Omega \subseteq \mathbb{R}^2$.

Definition B.9. The vector field

Gradient, divergence, rotation

$$\operatorname{grad}(\mathbf{F}) := \nabla(\mathbf{F}) := \begin{pmatrix} \partial_1 P \\ \partial_2 Q \end{pmatrix} : \Omega \longrightarrow \mathbb{R}^2$$

is called the *gradient of* $\mathbf{F}$, the scalar field

$$\operatorname{div}(\mathbf{F}) := \partial_1 P + \partial_2 Q\colon \Omega \longrightarrow \mathbb{R}$$

is called the *divergence of* $\mathbf{F}$, and the scalar field

$$\operatorname{rot}(\mathbf{F}) := \partial_1 Q - \partial_2 P\colon \Omega \longrightarrow \mathbb{R}$$

is called the *rotation of* $\mathbf{F}$.[198] ■

According to Schwarz' theorem on the interchangeability of multiple partial derivatives (Forster, 2017a, p. 68), one immediately sees that

$$\operatorname{rot}\left(\operatorname{grad}\left(F\right)\right) = 0.$$

Definition B.10 (Gradient field; integrability condition)**.** The vector field $\mathbf{F}$ is called a *gradient field*, or *conservative*, if it is the gradient of a continuously differentiable function, that is, if there exists a continuously differentiable function $\varphi\colon \Omega \longrightarrow \mathbb{R}$ such that $\mathbf{F} = \nabla\varphi$. In this case, φ is called a *potential of* F.

Gradient field / conservative field and potential

Additionally, we say that $\mathbf{F}$ satisfies the *integrability condition on* Ω, or is *closed on* Ω, if

Integrability condition

$$\operatorname{rot}(\mathbf{F}) = \partial_1 Q - \partial_2 P = 0 \tag{B.8}$$

on Ω. ■

Potentials of a vector field are unique up to a constant, and we have seen before, that every gradient field fulfils the integrability condition since $\operatorname{rot}(\operatorname{grad}\varphi) = 0$ for all continuously differentiable scalar fields φ.

The next proposition is analogue to the first part of the fundamental theorem of calculus:

198 Oftentimes, the rotation is defined as a differential operator for three-dimensional vector fields. More precisely, for a continuously differentiable vector field $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3)\colon U \longrightarrow \mathbb{R}^3$ on an open subset $U \subseteq \mathbb{R}^3$ the rotation of $\mathbf{G}$ is defined as $\operatorname{rot}(\mathbf{G}) = (\partial_2\mathbf{G}_3 - \partial_3\mathbf{G}_2, \partial_3\mathbf{G}_1 - \partial_1\mathbf{G}_3, \partial_1\mathbf{G}_2 - \partial_2\mathbf{G}_1)^T$. The definition for the two-dimensional case we give here adapts this three-dimensional case omitting the third variable. Sometimes, the rotation is called *curl*, that is $\operatorname{rot}(\mathbf{F}) = \operatorname{curl}(\mathbf{F})$.

Proposition B.11. If $\mathbf{F}$ is a gradient field with potential φ, then

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \int_\gamma \nabla\varphi\,\mathrm{d}\mathbf{T} = \varphi(\gamma(b)) - \varphi(\gamma(a)).$$

■ ■

This follows directly from the fundamental theorem of calculus and the chain rule: $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \int_a^b \mathbf{F}(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (\varphi\circ\gamma)'(t)\,\mathrm{d}t$. In particular, in this situation, the real path integral of second kind does not depend on the chosen path between $\gamma(a)$ and γ but only of these two points. Consequently, if $\mathbf{F}$ is a gradient field and γ is closed, then

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = 0.$$

For the converse, we may ask under which circumstances a vector field is a gradient field. Essentially, this turns out to be equivalent to the satisfaction of the integrability condition as well as to the property that real path integrals of second kind do not depend on the chosen path between two points. More precisely, Theorem B.12 and Theorem B.13 characterise gradient fields in terms of the previously mentioned properties (see e.g., Galbis & Maestre, 2012, ch. 2; Hirn & Weiß, 2018, ch. 9).

Theorem B.12. The following conditions are equivalent:

1. For every closed piecewise continuously differentiable path γ in Ω, it holds that $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = 0$.
2. If γ and η are two piecewise continuously differentiable paths in Ω, whose initial points agree and whose terminal points agree, then $\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \int_\eta \mathbf{F}\,\mathrm{d}\mathbf{T}$.
3. $\mathbf{F}$ is a gradient field. ■

It remains to be asked how we can construct a potential for a vector field satisfying the integrability condition on Ω. There, let us assume that $\mathbf{F}$ satisfies the integrability conditions. Mimicking the second part of the fundamental theorem of calculus, we may wish to fix a point $\omega \in \Omega$ and define

$$\begin{aligned} \varphi\colon \Omega &\longrightarrow \mathbb{R}, \\ p &\longmapsto \int_{\gamma_p} \mathbf{F}\,\mathrm{d}\mathbf{T}, \end{aligned}$$

where γ_p is any piecewise continuously differentiable path in Ω, which links ω to p. However, this definition is not well-defined in general because the integrals here may depend on the actual choice of γ_p. For example, let $\Omega = \mathbb{R}^2 \setminus \{(0,0)^T\}$, $\mathbf{F}(x,y) = \big(-y/(x^2+y^2), x/(x^2++y^2)\big)$ for $(x,y)^T \in \Omega$, and $\gamma\colon [0,2\pi], t \longmapsto (\cos(t), \sin(t))$. Then,

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = 2\pi \neq 0,$$

and by Theorem B.12, there is no potential for $\mathbf{F}$ on Ω.

However, if Ω is *simply-connected*, the construction of φ is indeed well-defined. Consequently, vector fields satisfying the integrability condition on a simply-connected domain have a potential on this domain:

Theorem B.13 (Poincaré lemma for simply-connected domains)**.** Let Ω be simply-connected and $\mathbf{F}\colon \Omega \longrightarrow \mathbb{R}^2$ a continuously differentiable vector field. Then, $\mathbf{F}$ is a gradient field if and only $\mathbf{F}$ satisfies the integrability condition in Equation B.8.

In this case, a potential φ for $\mathbf{F}$ on Ω is given by $\varphi(p) := \int_{\gamma_p} \mathbf{F}\,\mathrm{d}\mathbf{T}$, where γ_p is any piecewise continuously differentiable path in Ω from a fixed point $\omega \in \Omega$ to $p \in \Omega$. ■

Since open disks are simply-connected domains, vector fields satisfying the integrability conditions locally possess potentials:

Corollary B.14 (Closed vector fields locally are gradient fields)**.** Suppose that $\mathbf{F}$ satisfies the integrability condition in Equation B.8, then $\mathbf{F}$ locally is a gradient field, that is, for every $(x, y)^T \in \Omega$ there is an open neighbourhood U of $(x, y)^T$ and a function $\varphi_U\colon U \longrightarrow \mathbb{R}$ such that $\mathbf{F} = \nabla\varphi_U$. ■

B.2.4 *Solenoidal and rotation-free vector fields*

Let $(x, y)^T \in \Omega$.

Solenoidal vector fields

Let us interpret the vector field $\mathbf{F}\colon \Omega \longrightarrow \mathbb{R}^2$ as the continuously differentiable velocity field of a fluid (idealised as a two-dimensional vector field in the sense that its third component is constant). Then, the *divergence of* $\mathbf{F}$ *at* $(x, y)^T$ is

$$\operatorname{div}(\mathbf{F})(x, y) = \lim_{k\to\infty} \frac{1}{\operatorname{vol}(Q_k)} \int_{\partial Q_k} \mathbf{F}\,\mathrm{d}\mathbf{N} \tag{B.9}$$

(e.g., Burg et al., 2012, pp. 130–131; Kemmer, 1977, ch. 9). Here, $(Q_k)_{k\in\mathbb{N}}$ is a sequence of rectangles with sides parallel to the coordinate axes entirely contained in Ω such that $\bigcap_{k\in\mathbb{N}} Q_k = \{(x, y)^T\}$ and $\operatorname{vol}(Q_k) \longrightarrow 0$ for $k \longrightarrow \infty$.[199] According to Equation B.9, the divergence of $\mathbf{F}$ at $(x, y)^T$ may be interpreted as the *source density* of the field $\mathbf{F}$ at this point (Burg et al., 2012, p. 130, own transl., emph. in orig.). Vividly speaking, the integrals $\int_{\partial Q_k} \mathbf{F}\,\mathrm{d}\mathbf{N}$ measure the net flow out of the rectangle Q_k and the right-hand side in Equation B.9 is the balance of this flow at $(x, y)^T$. If the $\operatorname{div}(\mathbf{F})(x, y) > 0$, then the fluid is described to have a source at $(x, y)^T$, if $\operatorname{div}(\mathbf{F})(x, y) < 0$, then it has a sink at $(x, y)^T$. If $\operatorname{div}(\mathbf{F})(x, y) = 0$, $\mathbf{F}$ is called *source- and sink-free at* $(x, y)^T$. If $\operatorname{div}(\mathbf{F}) \equiv 0$ on Ω, then $\mathbf{F}$ is called *source- and sink-free* or *solenoidal*. *Solenoidal vector field*

Rotation-free vector fields

For the rotation of a vector field, we have a similar result:

$$\operatorname{rot}(\mathbf{F})(x, y) = \lim_{k\to\infty} \frac{1}{\operatorname{vol}(Q_k)} \int_{\partial Q_k} \mathbf{F}\,\mathrm{d}\mathbf{T} \tag{B.10}$$

(e.g., Burg et al., 2012, pp. 149–152; Kemmer, 1977, ch. 8). We may interpret Equation B.10 in the sense that the rotation measures the *vorticity of* $\mathbf{F}$ *at* $(x, y)^T$ (Burg et al., 2012, p. 148, own transl., emph. in orig.). If $\operatorname{rot}(\mathbf{F}) \equiv 0$ on Ω, we call $\mathbf{F}$ *rotation-free* or *irrotational*. *Irrotational vector field*

199 ∂Q_k denotes a piecewise continuously differentiable simple-closed path along the boundary of Q_k in counterclockwise direction for each $k \in \mathbb{N}$.

B.3 GREEN'S AND GAUSS' THEOREMS

This is the final section of our summary of real analysis. It is about *Green's* and *Gauß' theorems*, which relate real path integrals of second kind along simple closed paths to double integrals over the area enclosed by the path.

Green's and Gauß' theorems

Let γ be a simple closed piecewise continuously differentiable path in Ω and let $D := \operatorname{int}(\gamma)$ be the interior of γ (Definition A.15). We say that γ *surrounds D positively* if the normal vector field of γ points into the exterior of γ (except for the points where γ is not differentiable). More precisely, we require that for all $t \in [a, b]$, for which γ is differentiable, there an $\varepsilon > 0$, such that the line segment from $\gamma(t)$ to $\gamma(t) + \varepsilon\mathbf{n}(t)$ without the point $\gamma(t)$ itself is contained completely in the exterior of γ.

The following theorem relates real path integrals of second kind along closed paths with double integrals.[200]

Green's theorem

Theorem B.15 (Green's theorem)**.** If $\mathbf{F} = (P, Q)^T : \Omega \longrightarrow \mathbb{R}^2$ is continuously differentiable, D is surrounded positively by γ, and $\overline{D} \subseteq \Omega$, then

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T} = \iint_D \operatorname{rot}(\mathbf{F})\,\mathrm{d}\mathcal{A}, \tag{B.11}$$

or in other words,

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D (\partial_1 Q - \partial_2 P)\,\mathrm{d}\mathcal{A}. \qquad \blacksquare$$

Replacing the real path integral of second kind with respect to the tangential field with the real path integral of second kind with respect to the normal field, we obtain *Gauß' integral theorem*, also called the *divergence theorem*.

Gauß' theorem

Theorem B.16 (Gauß' theorem)**.** Under the same conditions as in Green's theorem (Theorem B.15), we have

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N} = \iint_D \operatorname{div}(\mathbf{F})\,\mathrm{d}\mathcal{A},$$

or in other words,

$$\int_\gamma P\,\mathrm{d}y - Q\,\mathrm{d}x = \iint_D (\partial_1 P + \partial_2 Q)\,\mathrm{d}\mathcal{A}. \qquad \blacksquare$$

Green's theorem (Theorem B.15) and Gauß' theorem (Theorem B.16) are discussed in the literature with a varying set of prerequisites or from the point of view of integration on submanifolds (e.g., Akcoglu et al., 2009; Barney, 1914; Cohen, 1959; Cufí & Verdera, 2015; Forster, 2017b; Galbis & Maestre, 2012; Heuser, 2008; Königsberger, 2004b; Lovitt, 1915). They can also be seen as special cases of *Stokes' theorem* (e.g., Akcoglu et al., 2009; Forster, 2017b; Galbis & Maestre, 2012; Kemmer, 1977; Königsberger, 2004b).

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200 Note that is essential in these two theorems that D is surrounded positively by γ. If we replace γ with the its inverted path (i.e., $\gamma^- : [0, 1] \longrightarrow \Omega$, $t \longmapsto \gamma(b + t(a - b))$), then, both paths surround D but the corresponding real path integrals of second kind change their sign: $\int_{\gamma^-} \mathbf{F}\,\mathrm{d}\mathbf{T} = -\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$ and $\int_{\gamma^-} \mathbf{F}\,\mathrm{d}\mathbf{N} = -\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N}$. Roy (2013, p. 10) presumes that Green's theorem appeared first in a paper by Cauchy from 1846, which may have been inspired by a paper by George Green (1793–1841) from 1828.

C

VISUALISATIONS OF COMPLEX FUNCTIONS

In this chapter, we present examples for the visualisation of complex functions, which were referenced in the main body of the thesis, in particular in Section 5.2. Let $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ be an open set. We list five main possibilities:

(a) separate visualisations of subsets of Ω and their images under f (e.g., Freitag & Busam, 2006, pp. 53–56);

(b) plots of the graphs of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $\mathrm{Arg}(f)$, and the analytic landscape of f (i.e., the graph of $|f|$) (e.g., Freitag & Busam, 2006, pp. 53–56);

(c) plots of the vector field $\mathbf{f}$ or the Pólya vector field $\mathbf{w}_f = (u, -v)^T$ associated to f (e.g., Braden, 1985, 1987; Needham, 1997, chs. 10–11; Polya & Latta, 1974);

(d) phase plots or domain colourings (used synonymously in this thesis) (e.g., Farris, 2017; Poelke & Polthier, 2009; Ponce Campuzano, 2019a, 2019b, 2021; Wegert, 2012); and

(e) coloured analytic landscapes (e.g., Ponce Campuzano, 2019a; Wegert, 2012);

In order to illustrate these methods, we will use the following function:

$$f\colon \Omega = \mathbb{C} \setminus \{0\} \longrightarrow \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\},$$
$$z \longmapsto \frac{(z+1)^2(z-1)}{z}.$$

For this chapter, we agree that $f(z) = \infty$ if and only if f has a non-removable singularity at z. Note that f is holomorphic, has a simple root at 1, a double root at -1, and a simple pole at 0. A direct computation shows that the real part u, the imaginary part v, and the modulus m of f are given by

$$u(x,y) = \frac{x^4 + x^3 - x^2 - x + xy^2 - y^2 - y^4}{x^2 + y^2},$$

$$v(x,y) = \frac{2x^3y + x^2y + 2xy^3 + y^3 + y}{x^2 + y^2},$$

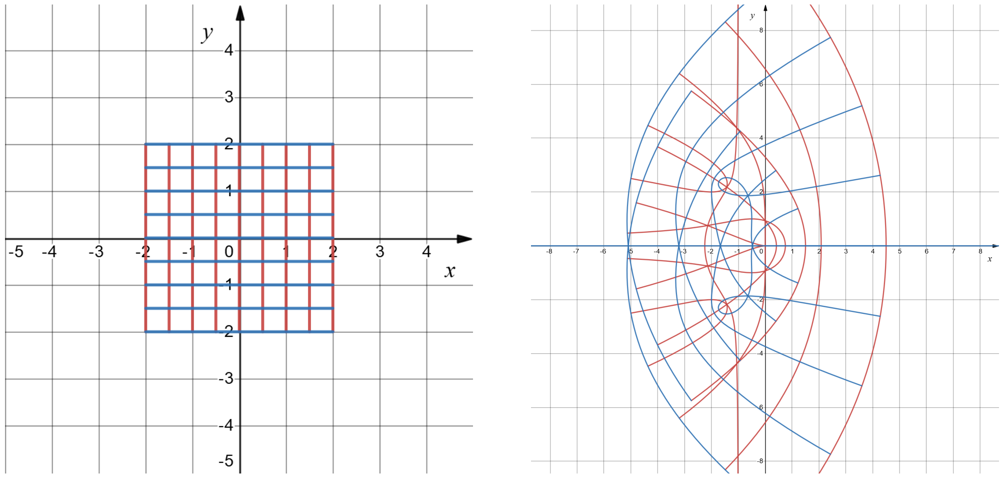
and

$$m(x,y) = \left(x^2 + 2x + y^2 + 1\right)\sqrt{\frac{(x-1)^2 + y^2}{x^2 + y^2}},$$

as a function of $(x,y)^T \in \mathbb{R}^2 \setminus \left\{(0,0)^T\right\} \cong \mathbb{C} \setminus \{0\}$.

(a) Visualising subsets and their images

For (a), one takes a subset Q of Ω, whose geometric realisation is a simple geometric shape and plots Q and $f(Q)$ in separate coordinate systems. For example, if Q is a grid in the plane, we call this method of visualisation a grid plot. Figure C.1 shows a square grid in one coordinate system (Figure C.1a)and the plot of image of that grid under f (Figure C.1b). The lines of the grid are coloured red and blue to help identify which line is mapped onto which curved line.



(a) A square grid. **(b)** The image of the grid under f.

Figure C.1: Square grid and its image under f (plotted with desmos calculator; retrieved 04/09/2021 from https://www.desmos.com/calculator).

(b) Plots of $\mathrm{Re}(f), \mathrm{Im}(f)$*, and* $|f|$

For (b), one uses 3D-plots of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $\mathrm{Arg}(f)$, and $|f|$, possibly also showing level sets. Figure C.2 shows plots of the graphs of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, and the analytic landscape of f.

(c) Plots of $\mathbf{f}$ *and* $\mathbf{w}_f$

For (c), we draw vector field plots for $\mathbf{f}$ or $\mathbf{w}_f$. That is, we choose a set of points $P \subseteq \Omega$ and attach the vector $\mathbf{f}(p)$ or $\mathbf{w}_f(p)$ to $p \in P$. In other words, the vector plot of $\mathbf{f}$ or $\mathbf{w}_f$ shows arrows that start at those $p \in P$ and end at $p + f(p)$ or $p + \bar{f}(p)$ in the complex plane. Figure C.3a shows an example for $\mathbf{f}$. For comparison, Figure C.3b shows the corresponding plot for the Pólya vector field $\mathbf{w}_f$.

(d) Phase plot and (e) coloured analytic landscape

The reader is referred to Wegert (2012) for a comprehensive treatment of domain colouring and analytic landscapes.

Let S^1 denote the unit circle. Let $\psi\colon \mathbb{C} \longrightarrow \widehat{S^1} := S^1\{0, \infty\}$ denote the *phase function*, that is, $\psi(z) := |z|^{-1}z$ for $z \in \mathbb{C} \setminus \{0\}$, $\psi(0) := 0$, and $\psi(\infty) := \infty$. Then, formally, the *phase plot of* f is the graph of the composition $\psi \circ f$, that is,

$$\{(z, (\psi \circ f)(z)) : z \in \Omega\} \subseteq \Omega \times \widehat{S^1},$$

and the *coloured analytic landscape of* f is the set

$$\{(z, |f(z)|, (\psi \circ f)(z)) : z \in \Omega\} \subseteq \widehat{\mathbb{C}} \times [0, \infty] \times \widehat{S^1}.$$

Choosing a colour wheel C, which assigns a colour to each point in $\widehat{S^1}$, the phase portrait of f can be drawn by colouring the points in Ω with the colour according to $\psi \circ f$ and the colour wheel C. Similarly, the coloured landscape of f can be drawn as a coloured version of the graph of $|f|$ (i.e., it is the analytic landscape, which is additionally coloured according to $\psi \circ f$ and C). Ideally, a colour wheel can be understood as a bijection from $\widehat{S^1}$ to a set of colours C distributed along the unit circle, including, by convention, black for 0 and white for ∞ (Wegert, 2012, ch. 2.4–2.5)

Figure C.4a shows phase plot of the identity function for a given colour wheel (e.g., the point 1 is coloured red, the point i is coloured yellow, and so on). Figure C.4b shows the same plot, but the level sets for arguments (lines) and moduli (circles) of complex numbers are shown, too. This points in this figure also shaded depending on the modulus of the function value; roughly speaking, complex numbers with smaller modulus are darker than those with larger modulus between two level sets arguments and moduli.

Figure C.5 shows a phase plot and analytic landscape for f using the colour wheel from Figure C.4. Note that the analytic landscape is the graph in Figure C.2c but additionally coloured. Essentially, the phase plot is a plot of $Arg(f)$, but not shown as surface in $\mathbb{R}^3$ but instead in terms of colours.

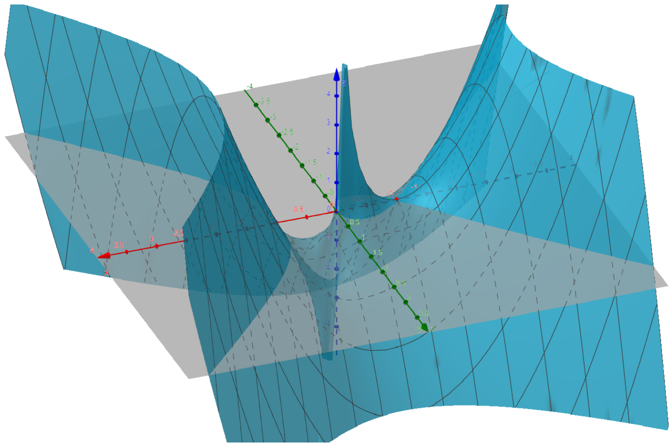
Interpreting phase plots or (coloured) analytic landscapes is a delicate issue (see Wegert, 2012). For instance, the cusp at -1 in Figure C.2c and Figure C.5b may incorrectly interpreted as a point where f is not differentiable at -1.

We finish this chapter with another example. Let

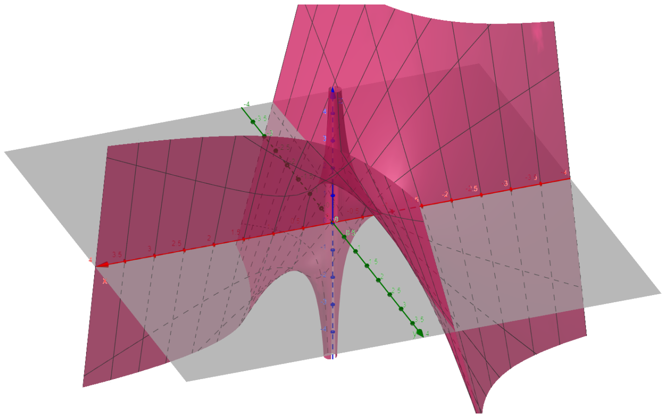
$$\begin{aligned} g\colon \mathbb{C}\setminus\{0\} &\longrightarrow \mathbb{C}, \\ z &\longmapsto \frac{(z-a)^3}{z-b}\exp\left(\frac{3}{(z-c)^2}\right) \end{aligned}$$

for $a, b, c \in \mathbb{C}$. Figure C.6 shows several domain colourings of g for $a \approx i$, $b \approx -i$, and $c \approx -1$. The colour wheel used is the same as in Figure C.4.[201] One can see the triple zero at a, simple pole at b, and essential singularity at c well: The colour wheel is wrapped around a three times respecting the order of colours, around b two times but with reversed order of colours, and infinitely many times around c (see Ponce Campuzano, 2019b, for the use of phase portraits to study singularities).

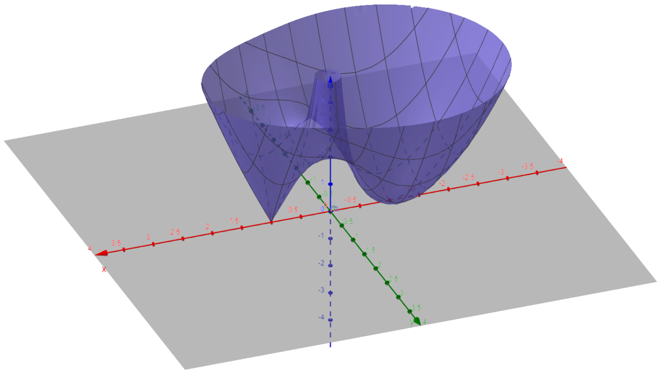
201 We do not state exact values for a, b, and c here the applet we used to create the plots in Figure C.6 allowed us to choose points with the mouse on the computer screen.



(a) Plot of $\mathrm{Re}(f)$.

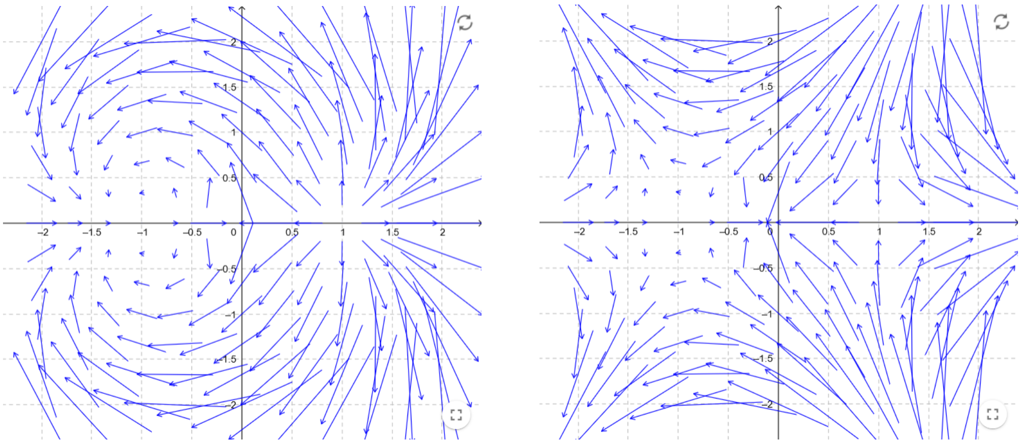


(b) Plot of $\mathrm{Im}(f)$.



(c) Analytic landscape of f.

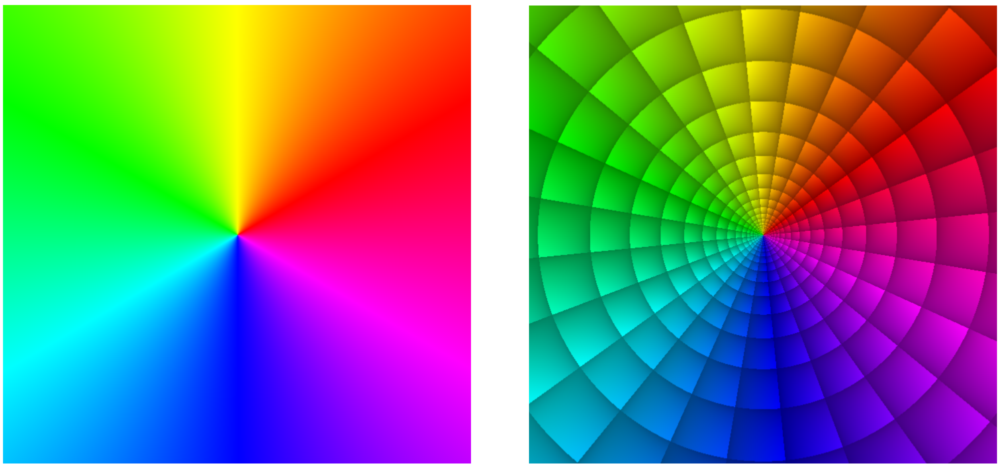
Figure C.2: Plots of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $|f|$ (the x-axis is red, pointing left, the y-axis green, pointing to the front, and the z-axis is blue, point upwards; plotted with GeoGebra), and **f**.



(a) Plot of vector field $\mathbf{f}$.

(b) Plot of Pólya vector field $\mathbf{w}_f$.

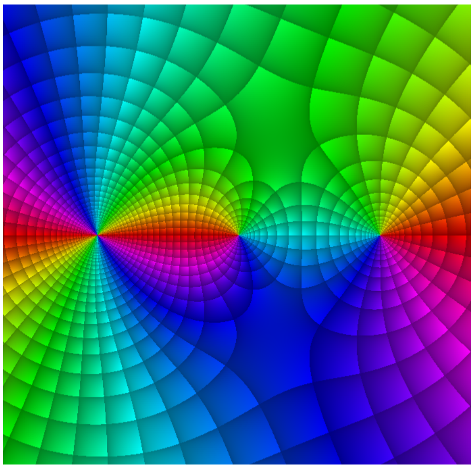
Figure C.3: Plots of the vector fields $\mathbf{f}$ and $\mathbf{w}_f$ (plotted with a GeoGebra applet by Juan Carlos Ponce Campuzano, based on an applet by Linda Fahlberg-Stojanovska; retrieved 04/09/2021 from https://www.geogebra.org/m/QPE4PaDZ)



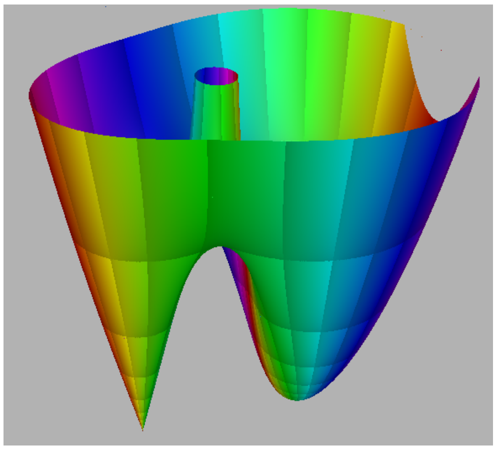
(a) Colour wheel.

(b) Colour wheel with level sets for the argument (lines) and modulus (circles).

Figure C.4: Colour wheel (plotted with CindyJS; retrieved 04/09/2021 from https://cindyjs.org/gallery/cindygl/ComplexExplorer/index.html).

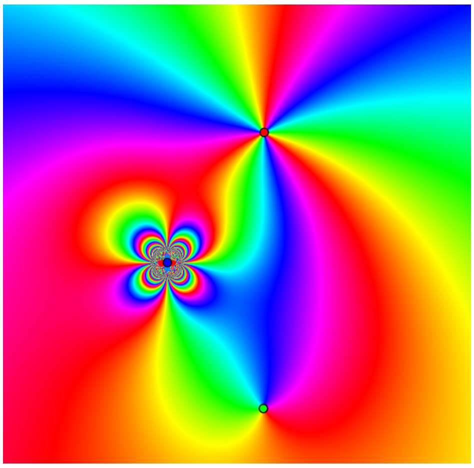


(a) Phase plot for f (plotted with CindyJS; retrieved 04/09/2021 from https://cindyjs.org/gallery/cindygl/ComplexExplorer/index.html).

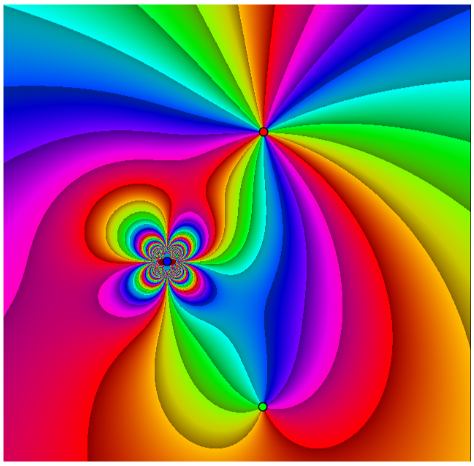


(b) Analytic landscape for f (plotted with the CindyJS widget by Aaron Montag; retrieved 04/09/2021 from https://cindyjs.org/gallery/cindygl/Landscape/index.html).

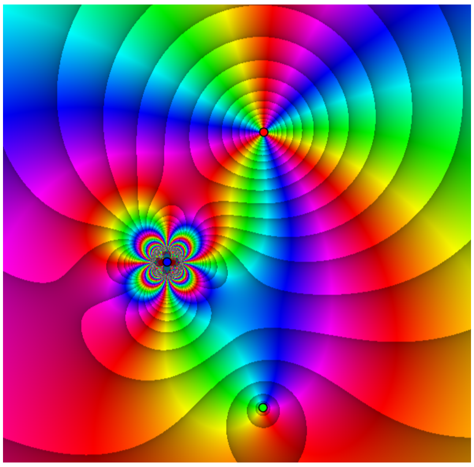
Figure C.5: Analytic landscape and phase plot for f.



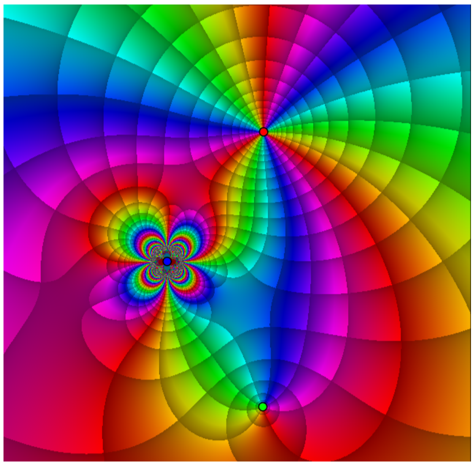
(a) Phase portrait for g.



(b) Phase portrait for g with level sets for the argument



(c) Phase portrait for g with level sets for the modulus.



(d) Phase portrait for g with level sets for the argument and modulus

Figure C.6: Phase portraits for g with $a \approx i$, $b \approx -i$, and $c \approx -1$ (plotted with CindyJS; retrieved 04/09/2021 from https://cindyjs.org/gallery/cindygl/ComplexExplorer/index.html).

D INTERVIEW GUIDELINE

This chapter contains the interview guideline used in all three interviews. The interview guideline was prepared in German but we will also provide the English translation.

Some of the German words, especially those involving the words "Vorstellung", "vorstellen" and "anschaulich" etc., could not be easily translated. We translate the first one with "mental image", as we did throughout the thesis, the second with "to imagine", and the third with "vivid" etc. We remark that other translations of "anschaulich" include the words "concrete, descriptive, graphic, demonstrative, ostensive".[202] As such, something is considered here as "vivid" if one finds it easily understandable or if it corresponds to what the interlocutor considers a mental image. It does not have to be visual though (see Chapter 2 and Chapter 4 for a discussion of these keywords and the conceptualisation of *intuitive mathematical discourse* we wish to elicit during the interviews).

The interview guideline consists of the following parts:

0. Introduction,
I. central contents of a first course in complex analysis,
II. mental images for holomorphicity and beliefs about the role of mental images in mathematics,
III. tasks on holomorphic functions,
IV. mental images about complex path integrals and intuitive explanations of integral theorems in complex analysis,
V. closing.

We describe the implementation in Section 11.1.2 and list it here in full. The original German wording is followed by the English translation separated with the arrows ⋙.

0. EINLEITUNG

⋙ *0. INTRODUCTION*

Ich freue mich, dass Sie sich bereit erklärt haben, an einem Gespräch über Funktionentheorie mit mir teilzunehmen. Es geht darum, wie Expert*innen in der Funktionentheorie, d. h. Forschende und Lehrende, einzelne Grundbegriffe verstehen und welche „Vorstellungen" sie dazu besitzen.

⋙ *I am happy that you agreed to participate in a conversation about complex analysis with me. It is about how experts in complex analysis, that is, researchers and lecturers, understand basic concepts and which "mental images" they have.*

202 Retrieved 05/26/2021, from https://dict.leo.org/englisch-deutsch/anschaulich.

Wenn ich „Vorstellungen“ sage, dann meine ich damit nicht bloß Bilder, die man “im Kopf haben” oder aufzeichnen könnte. „Vorstellungen“ sollen hier viel weiter gefasst sein. Es kann sich neben Bildern also etwa auch um Assoziationen, Symbole, Handlungen, Operationen, Formeln, Schemata, Modelle o. Ä. handeln, die für Sie von zentraler Bedeutung in Bezug auf einen Begriff sind.

⋙ *When I say “mental images”, then I do not only mean pictures you might “have in mind” or could draw. “Mental images” should rather be understood much more broadly here. In addition to pictures, it can also be about associations, symbols, actions, operations, formulas, schemes, models or the like that are of central significance for you with respect to a concept.*

Einerseits möchte ich herausfinden, welche Vorstellungen es dazu überhaupt gibt, welche sich in welchen Situationen zeigen, gegebenenfalls ob sich die Vorstellungen von Studierenden[203] mit denen von Expert*innen decken, sich bestimmte Gemeinsamkeiten oder Unterschiede zeigen etc. Mittelfristig könnte sich dann herauskristallisieren, welche Vorstellungen man entwickeln und möglicherweise unterstützen kann, vielleicht auch sollte.

⋙ *On the one hand, I would like to find out which mental images there are in the first place, which ones show up in which situations, whether students' and experts' mental images coincide, whether there are certain similarities or differences, etc. In the medium term, it could then emerge which mental can be developed or perhaps should be supported. In the medium term, it could then become clear which mental images can be developed and should be possibly supported.*

Insgesamt ist es aktuell wenig geklärt, worin Vorstellungen zur Mathematik auf Hochschulebene bestehen. Speziell trifft das auf Themen zu, die nicht in den Veranstaltungen aus dem ersten Studienjahr vorkommen.

⋙ *In general, it is a rather unsettled question what mental images for mathematics at university level may look like. Specifically, this is the case for topics that do not appear in courses from the first year of studies.*

Bitte beantworten Sie die Fragen so ausführlich wie möglich. Nutzen Sie beispielsweise auch Papier und Stift für Ihre Erklärungen. Es ist für mich sehr wichtig, einen Einblick in Ihre „Gedankenwelt“ zu bekommen, daher möchte ich Sie bitten, Ihre Gedanken, spontanen Einfälle etc. auszusprechen.

⋙ *Please answer the questions as thoroughly as possible. For instance, you may use paper and pencil for your explanations. It is very important for me to get insight into your “world of thoughts”, therefore I would like to ask you to express your thoughts, spontaneous ideas, etc. aloud.*

I. CENTRAL CONTENTS OF A FIRST COURSE IN COMPLEX ANALYSIS

1. Bitte beschreiben Sie zunächst einmal, was für Sie ganz persönlich die zentralen Inhalte in einer einführenden Vorlesung zur Funktionentheorie sind.

   ⋙ *First of all, please describe what you personally consider the central contents of an introductory lecture on complex analysis.*

203 In fact, we excluded the students' perspective from this study.

II. VORSTELLUNGEN ZUR HOLOMORPHIE UND ÜBERZEUGUNGEN ZUR ROLLE VON VORSTELLUNGEN IN DER MATHEMATIK

⋙ *II. MENTAL IMAGES FOR HOLOMORPHICITY AND BELIEFS ABOUT THE ROLE OF MENTAL IMAGES IN MATHEMATICS*

Vorstellungen zur Holomorphie und charakteristische Eigenschaften

⋙ *Mental images of holomorphicity and characteristic properties*

2. Beschreiben Sie bitte, wie Sie sich eine holomorphe Funktion bzw. eine komplex differenzierbare Funktion vorstellen.

   Haben Sie dazu bestimmte Repräsentationen (Bilder, Symbole, Handlungen, Sätze, ...) vor Augen?

   Impuls: An was denken Sie zuerst, wenn Sie das Wort „holomorph“ hören? (Verbinden Sie damit etwas anderes als zu „komplex differenzierbar“, oder „differenzierbar“ allgemein?)

   ⋙ *Please describe how you imagine a holomorphic function or a complex differentiable function, respectively.*

   Do you have some particular representations (pictures, symbols, actions, propositions, ...) in mind?

   Incentive: What do you think of first when you hear the word “holomorphic”? (Do you associate something else than for “complex differentiable” or “differentiable” in general?)

3. Welche Eigenschaften von holomorphen Funktionen sind für Sie persönlich besonders charakteristisch? An welche denken Sie unmittelbar bei dem Wort „Holomorphie“?

   Wie erklären Sie sich diese Eigenschaften (anschaulich, ohne Beweis)?

   ⋙ *Which properties of holomorphic functions are particularly characteristic for you personally? Which ones do you immediately think of when you hear the word* holomorphicity*?*

   How do you explain these properties (vividly, without proof)?

4. Was ist Ihrer Meinung nach die „Essenz“ bzw. der „Kern“ von Holomorphie?

   ⋙ *In your opinion, what is the “essence” or the “core” of holomorphicity?*

5. Betrachten wir ein konkretes Beispiel: Nehmen wir die Funktion $f : \mathbb{C} \longrightarrow \mathbb{C}, f(z) = z^2$. Was bedeutet es möglichst anschaulich, dass $f'(2) = 4$ ($f'(i) = 2i, f'(0) = 0$)?

   (Ggfs. bei Auftreten eines kalkülorientierten Vorgehens graphische Darstellungen aushändigen, Figure D.1a und Figure D.1b: Ist „Rechnenkönnen“ das Zentrale in Bezug auf dieses Beispiel bzw. zu Holomorphie allgemein? Wo wäre der Unterschied zu „etwas Anderem“?)

   ⋙ *Let us take at a concrete example: Let us consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}, f(z) = z^2$. What does it mean as vividly as possible that $f'(2) = 4$ ($f'(i) = 2i, f'(0) = 0$)?*

   (If necessary, hand out the graphical representations when a calculation-oriented procedure occurs, Figure D.1a and Figure D.1b: Is “being able to calculate / numeracy” the central aspect with

respect to this example or to holomorphicity in general? How would it differ from "something else"?)

Überzeugungen zur Rolle von Vorstellungen
⋙ *Beliefs about the role of mental images*

6. Wie wichtig sind Ihnen persönlich Vorstellungen zur Funktionentheorie?

   ⋙ *How important are mental images about complex analysis for you personally?*

7. Wie wichtig finden Sie es, dass Studierende, die Funktionentheorie hören, Vorstellungen zu den Begriffen und Sätzen aufbauen?

   ⋙ *How important do you think it is that students attending a lecture in complex analysis to build up mental images for the concepts and theorems?*

8. Unterscheidet sich die Funktionentheorie Ihrer Meinung nach von anderen Zweigen der Mathematik, was den Aufbau und die Nützlichkeit von Vorstellungen betrifft?

   ⋙ *In your opinion, does complex analysis differ from other branches of mathematics in terms of the structure and usefulness of mental images?*

9. Ganz allgemein gefragt: Ist es sinnvoll oder nützlich, wenn man zu verschiedenen Inhalten der Mathematik(-vorlesungen) Vorstellungen aufbaut oder mit ihnen umgehen kann?

   Jetzt gegebenenfalls den Rückbezug zur Funktionentheorie herstellen.

   ⋙ *Generally speaking: Is it meaningful or useful to build up mental images or to be able to work with them for different contents of mathematics (lectures)?*

   Now, if necessary, refer back to complex analysis.

Nachfragen zu Vorstellungen von Charakterisierungen von komplexer Differenzierbarkeit
⋙ *Follow-up questions about mental images of characterisations of complex differentiability*

10. Ein Szenario: Stellen Sie sich vor, in einer Funktionentheorievorlesung sagt eine Studentin oder ein Student, sie oder er hat sich bei der Ableitung einer Funktion an einer Stelle immer die Steigung des Graphen bzw. der Tangente an dem Graphen an dieser Stelle vorgestellt.

    Für wie anschlussfähig halten Sie diese Vorstellung für die Funktionentheorie?

    Was würden Sie dieser Person raten?

    ⋙ *A scenario: Imagine a student in a complex analysis course says that she or he always imagined the derivative of a function at one point as the slope of the graph or the tangent to the graph at that point.*

    To what extent do you think this mental image can be transferred to complex analysis?

What would you advise the student to do?

11. Wenn nicht schon vorher thematisiert:

    Lassen sich (andere) Vorstellungen für die Ableitung von reellwertigen Funktionen, die auf (offenen) Teilmengen von $\mathbb{R}$ definiert sind, auf die komplexe Ableitung übertragen?

    ⋙ *If not discussed earlier:*

    Can (other) mental images for the derivative of real-valued functions defined on (open) sets of $\mathbb{R}$ be transferred to the complex derivative?

12. Wenn nicht schon vorher thematisiert:

    Welche Beispiele würden Sie geben, um komplexe Differenzierbarkeit und Holomorphie zu erklären und wieso?

    Könnte die komplexe Konjugation hier eine Rolle spielen? Ist dies eine bedeutsame Abbildung in diesem Kontext?

    Wie kann man das anschaulich erklären, dass die komplexe Konjugation keine holomorphe Abbildung liefert?

    ⋙ *If not discussed earlier:*

    Which examples would you present in order to explain complex differentiability and holomorphicity and why?

    Could the complex conjugation play a role here? Is this a relevant function in this context?

    How can you explain vividly that the complex conjugation is no holomorphic function?

13. Wenn man nun komplexe Funktionen mit reellen Abbildungen, also solchen von offenen Teilmengen des $\mathbb{R}^2$ in den $\mathbb{R}^2$, vergleicht, was zeichnet die holomorphen Funktionen gegenüber den reell total differenzierbaren Funktionen aus? Wie kann man sich das vorstellen?

    ⋙ *When we compare complex functions to real functions, that is, functions from open sets of $\mathbb{R}^2$ to $\mathbb{R}^2$, what distinguishes the holomorphic functions from real totally differentiable functions? How can one imagine this?*

14. Cauchy-Riemannsche Differenzialgleichungen [CRDGLn] (falls nicht schon thematisiert):

    Welche Rolle spielen für Sie die CRDGLn?

    (Kann man sich die "Bedeutung" der CRDGLn anschaulich vorstellen?)

    ⋙ *Cauchy Riemann differential equations [CRDEs] (if not discussed earlier):*

    Which role do the CRDE play for you?

    (Can you imagine the "meaning" of the CRDEs vividly?)

15. Wenn nicht schon vorher thematisiert:

    Welche Definitionen von komplexer Differenzierbarkeit und Holomorphie geben Sie in Ihrer Vorlesung zur Funktionentheorie? Warum wählen Sie diese Definition?

    Gäbe es noch andere Möglichkeiten, mit komplexer Differenzierbarkeit oder Holomorphie zu starten (z. B. Analyzität, reell total differenzierbar + CRDGLn, Satz von Morera)?

    ⋙ *If not discussed earlier:*

    Which definitions of complex differentiability or holomorphicity do you present in your lecture on complex analysis? Why do you choose this definition?

    Are there other possibilities to start with complex differentiability or holomorphicity (e.g., analyticity, real totally differentiable + CRDEs, Morera's theorem)?

16. Ein Szenario: Manche Bücher über Funktionentheorie starten mit Potenzreihen und analytischen Funktionen als Ausgangspunkt für die Behandlung der Funktionentheorie.

    Wie sehen Sie dieses Vorgehen?

    Welchen Vorteil könnte dieses Vorgehen haben, welchen der Start mit der Definition von komplexer Differenzierbarkeit?

    Wie stellen Sie sich das Abbildungsverhalten einer analytischen Funktion vor?

    ⋙ *A scenario: Some books on complex analysis start with power series and analytic functions for the treatment of complex analysis.*

    What do you think about this approach?

    Which advantages could this procedure have; which advantages could the definition of complex differentiability have?

    How do you imagine the mapping properties of an analytic function?

III. AUFGABEN ZU HOLOMORPHEN FUNKTIONEN

⋙ *III. TASKS ON HOLOMORPHIC FUNCTIONS*

Wie schon angedeutet, habe ich mit Studierenden ein ähnliches Gespräch geführt. Dabei kamen auch Aufgaben zum Einsatz, bei denen zu entscheiden war, ob bestimmte Funktionen holomorph sind oder nicht.

⋙ *As already indicated, I conducted a similar conversation with students. There, I used some tasks to decide whether some functions were holomorphic or not.*

Ich möchte Ihnen jetzt einige Beispiele zeigen und bitte Sie, möglichst anschauliche Lösungen/Erklärungen zu geben. Etwa könnten Sie mir sagen, was für Sie „der richtige" Ansatz ist, um mit dem Beispiel umzugehen.

Hierzu die Aufgabenzettel austeilen.

⋙ *I would now like to show you some examples and ask you to give as vivid/descriptive solutions/explanations as possible. For instance, you could tell me which one is "the right" way for you to work with the example.*

Hand out the task sheets.

17. Es sei $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ mit $f(z) = (2+i)z$. Wie interpretieren Sie geometrisch, dass f' konstant ist mit Wert $2+i$?

    Ggfs. Bilder ausgeben (Figure D.1a und Figure D.1c)

    ⋙ *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = (2+i)z$. How do you interpret geometrically that f' is constant with value $2+i$?*

    Hand out images (Figure D.1a and Figure D.1c) if needed.

18. Wenn $f\colon \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ offen) eine beliebige komplexe Funktion ist, wie kann man einer graphischen Darstellung entnehmen – Sie können sich aussuchen, wie die aussehen kann –, ob die Funktion holomorph ist?

    ⋙ *If $f\colon \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ open) is an arbitrary complex function, how can one tell by a graphic representation—you can freely choose how it may look like—whether the function is holomorphic?*

19. Ist die Funktion $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ mit $f(z) = \exp(\overline{z})$ holomorph? Begründen Sie.

    Ggf. Zusatzfrage: Ist g holomorph, ist dann auch f mit $f(z) = g(\overline{z})$ holomorph?

    ⋙ *Is the function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \exp(\overline{z})$, holomorphic? Justify.*

    Potential further question: If g is holomorphic, is f with $f(z) = g(\overline{z})$ holomorphic, too?

20. Es sei $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ eine holomorphe Funktion. Ist die Funktion $g\colon \mathbb{C} \longrightarrow \mathbb{C}$ mit $g(z) = \overline{f(\overline{z})}$ holomorph? Begründen Sie.

    ⋙ *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Is the function $g\colon \mathbb{C} \longrightarrow \mathbb{C}$, $g(z) = \overline{f(\overline{z})}$, holomorphic? Justify.*

21. Es sei $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ eine holomorphe Funktion. Ist $(f+\overline{f})^2$ holomorph? Begründen Sie.

    ⋙ *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Is $(f+\overline{f})^2$ holomorphic? Justify.*

IV. VORSTELLUNGEN ZU WEGINTEGRALEN UND ANSCHAULICHE ERKLÄRUNGEN FÜR INTEGRALSÄTZE DER FUNKTIONENTHEORIE

⋙ *IV. MENTAL IMAGES ABOUT COMPLEX PATH INTEGRALS AND INTUITIVE EXPLANATIONS OF INTEGRAL THEOREMS IN COMPLEX ANALYSIS*

Jetzt möchte ich noch auf das komplexe Wegintegral zu sprechen kommen.

⋙ *Now I would like to talk about the complex path integral.*

22. In der reellen Analysis können wir das bestimmte Integral einer reellwertigen Funktion als Maß für die (vorzeichenbehaftete) Fläche unter dem Graphen interpretieren.

    Welche geometrische Bedeutung hat für Sie die komplexe Zahl

    $$\int_\gamma f(z)\,\mathrm{d}z$$

für einen (stückweise stetig differenzierbaren) Weg $\gamma\colon [a, b] \longrightarrow \Omega$ und eine stetige Funktion $f\colon \operatorname{im}\gamma \longrightarrow \mathbb{C}$.[204]

Dazu das Aufgabenblatt **A.** (siehe unten) ausgeben.

Dabei um vertiefende Erklärungen bitten und ggf. auch nach weiteren Veranschaulichungen fragen.

⋙ *In calculus / real analysis, we can interpret the definite integral of a real-valued function as the measure of the (signed) area under the graph.*

Which geometrical meaning does the complex number

$$\int_\gamma f(z)\,\mathrm{d}z$$

for a (piecewise continuously differentiable) path $\gamma\colon [a, b] \longrightarrow \Omega$ and a continuous function $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$ have for you?

*Hand out the task sheet **A.** (see below)*

Ask for more detailed explanations or other potential visual mediators.

23. Einer der zentralen Sätze, der ja ständig für weiterführende Sätze benötigt wird, ist der Cauchysche Integralsatz.

    Welche Fassung des Cauchyschen Integralsatzes bevorzugen Sie und wieso?[205]

    Wie könnte man anschaulich erklären, dass dieser Satz richtig ist?

    ⋙ *One of the central theorems, which is frequently needed for further theorems, is Cauchy's integral theorem.*

    Which version of Cauchy's integral theorem do you prefer and why?[206]

    How could one explain intuitively / vividly that this theorem is correct?

24. Eine Version der Cauchyschen Integralformel lautet

    $$f(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

    für $f\colon \Omega \longrightarrow \mathbb{C}$ holomorph, $z_0 \in \Omega$, $r > 0$ mit $B(z_0, r) \Subset \Omega$ und $z \in B(z_0, r)$.

    Wie stellen Sie sich die Aussage dieser Formel vor?

    Wie würden Sie anschaulich oder mit einer Ihrer Vorstellungen argumentieren, dass diese Formel gilt?

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204 Here, "im" is an abbreviation for "image". That is, $\operatorname{im}\gamma$ is the trace of γ, which we have usually signified with $\operatorname{tr}(\gamma)$ in this thesis. A similar prompt was reported by Hancock (2018), Oehrtman et al. (2019), Soto-Johnson et al. (2012), and Soto-Johnson et al. (2011).

205 Möglicher Vorschlag des Interviewers: „Ich würde sonst folgende Fassung vorschlagen: Ist $f\colon \Omega \longrightarrow \mathbb{C}$ eine holomorphe Funktion auf einem einfach zusammenhängenden Gebiet $\Omega \subseteq \mathbb{C}$, dann gilt für jeden stückweise stetig differenzierbaren, geschlossenen Weg γ in Ω, dass $\int_\gamma f(z)\,\mathrm{d}z = 0$."

206 Potential suggestion from the interviewer: "I would suggest the following version: If $f\colon \Omega \longrightarrow \mathbb{C}$ is a holomorphic function on a simply-connected domain $\Omega \subseteq \mathbb{C}$, then it holds that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every piecewise continuously differentiable path γ in Ω."

Dazu das Aufgabenblatt **B.** (siehe unten) ausgeben.

⋙ *One version of Cauchy's integral formula is*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for $f\colon \Omega \longrightarrow \mathbb{C}$ *holomorphic,* $z_0 \in \Omega$, $r > 0$ *such that* $B(z_0, r) \Subset \Omega$ *and* $z \in B(z_0, r)$.[207]

How do you imagine the assertion of this formula?

How would you argue vividly or with one of your mental images that this formula holds?

*Hand out the task sheet **B.** (see below).*

25. Stetige Funktionen $f\colon I \longrightarrow \mathbb{R}$ ($I \subseteq \mathbb{R}$ offenes Intervall) haben Stammfunktionen, wie man mit dem Hauptsatz der Differenzial- und Integralrechnung lernt.

    Wie könnte eine mögliche „analoge" Situation für komplexe Funktionen lauten?

    Warum hat nicht jede stetige Funktion $f\colon \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ Gebiet) eine Stammfunktion? Wo schlägt diese „analoge" Situation fehl?

    ⋙ *There are primitive functions for continuous functions* $f\colon I \longrightarrow \mathbb{R}$ ($I \subseteq \mathbb{R}$ *open interval) as one learns from the fundamental theorem of calculus.*

    What could an "analogous" situation for complex functions look like?

    Why isn't there a primitive function for every continuous function $f\colon \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ *domain)?*

    Where does this "analogue" situation fail?

V. ABSCHLUSS

⋙ *V. CLOSING*

26. Gibt es noch etwas, das Ihnen heute nicht ausführlich genug besprochen wurde? Zum Abschluss sollen Sie die Gelegenheit haben, nicht angesprochene Vorstellungen oder andere Themen, die Ihnen noch zu kurz gekommen sind, auszusprechen.

    ⋙ *Is there anything else that has not been discussed in enough detail today for you? Finally, you should have the opportunity to express any mental images or other topics that you feel have not been covered in enough detail so far.*

27. Vielen Dank!

    ⋙ *Thank you very much!*

207 Here, the utterance $B(z_0, r) \Subset \Omega$ is used to signify that the compact closure of the ball is contained in Ω.

TASKS IN INTERVIEWS

The following tasks were each printed out on a sheet of paper and handed to the interviewees at the appropriate places during the interview.

Ist die Funktion $f : \mathbb{C} \longrightarrow \mathbb{C}$ mit $f(z) = \exp(\bar{z})$ holomorph? Begründen Sie.

⋙ *Is the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \exp(\bar{z})$, *holomorphic? Justify.*

Es sei $f : \mathbb{C} \longrightarrow \mathbb{C}$ eine holomorphe Funktion.
Ist $(f + \bar{f})^2$ holomorph? Begründen Sie.

⋙ *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be a holomorphic function.*
Is $(f + \bar{f})^2$ *holomorphic? Justify.*

Es sei $f : \mathbb{C} \longrightarrow \mathbb{C}$ eine holomorphe Funktion.
Ist die Funktion $g : \mathbb{C} \longrightarrow \mathbb{C}$ mit $g(z) = \overline{f(\bar{z})}$ holomorph? Begründen Sie.

⋙ *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be a holomorphic function.*
Is the function $g : \mathbb{C} \longrightarrow \mathbb{C}$, $g(z) = \overline{f(\bar{z})}$ *holomorphic? Justify.*

Es sei $f : \mathbb{C} \longrightarrow \mathbb{C}$ mit $f(z) = (2 + i)z$.
Wie interpretieren Sie geometrisch, dass f' konstant ist mit Wert $2 + i$?

⋙ *Let* $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = (2 + i)z$.
How do you geometrically interpret that f' *is constant with value* $2 + i$*?*

Wenn $f : \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ offen) eine beliebige komplexe Funktion ist, wie kann man einer graphischen Darstellung entnehmen – Sie können sich aussuchen, wie die aussehen kann –, ob die Funktion holomorph ist?

⋙ *If* $f : \Omega \longrightarrow \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ *open) is an arbitrary complex function, how can one tell by a graphic representation—you can freely choose how it may look like—whether the function is holomorphic?*

A. Welche geometrische Bedeutung hat für Sie die komplexe Zahl

$$\int_\gamma f(z)\,\mathrm{d}z$$

für einen (stückweise stetig differenzierbaren) Weg $\gamma\colon [a,b] \longrightarrow \Omega$ und eine stetige Funktion $f\colon \operatorname{im}\gamma \longrightarrow \mathbb{C}$.

⋙ ***A.** Which geometrical meaning does the complex number*

$$\int_\gamma f(z)\,\mathrm{d}z$$

for a (piecewise continuously differentiable) path $\gamma\colon [a,b] \longrightarrow \Omega$ and a continuous function $f\colon \operatorname{tr}(\gamma) \longrightarrow \mathbb{C}$ have for you?

B. Eine Version der Cauchyschen Integralformel lautet

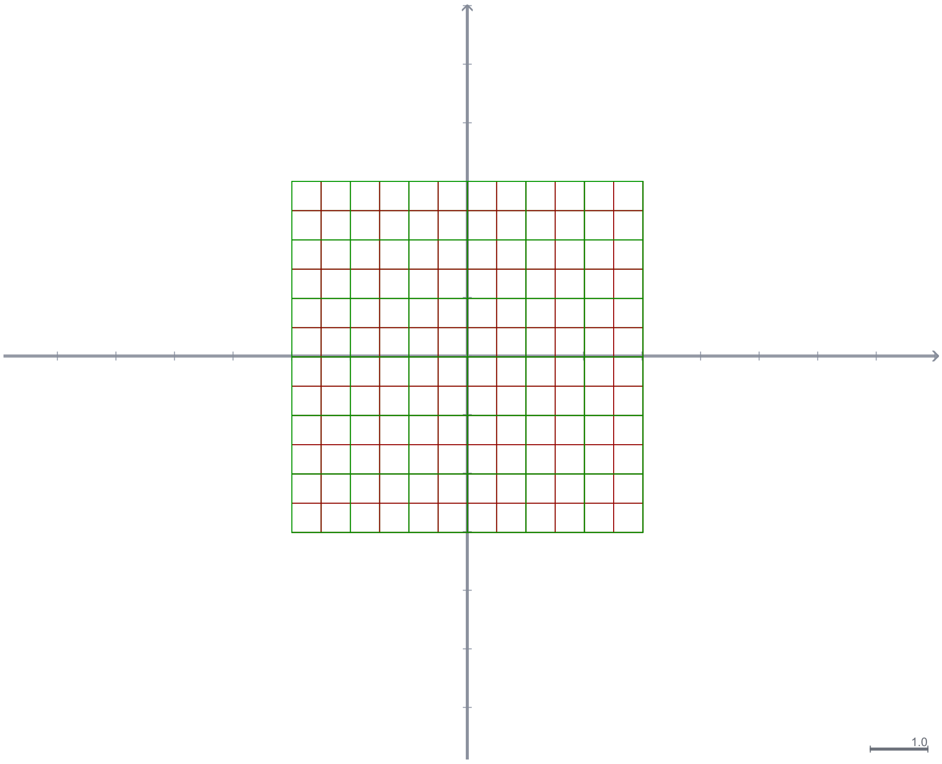
$$f(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

für $f\colon \Omega \longrightarrow \mathbb{C}$ holomorph, $\Omega \subseteq \mathbb{C}$ offen, $z_0 \in \Omega$, $r > 0$ mit $B(z_0,r) \Subset \Omega$ und $z \in B(z_0,r)$.
Wie stellen Sie sich die Aussage dieser Formel vor?
Wie würden Sie anschaulich oder mit einer Ihrer Vorstellungen argumentieren, dass diese Formel gilt?

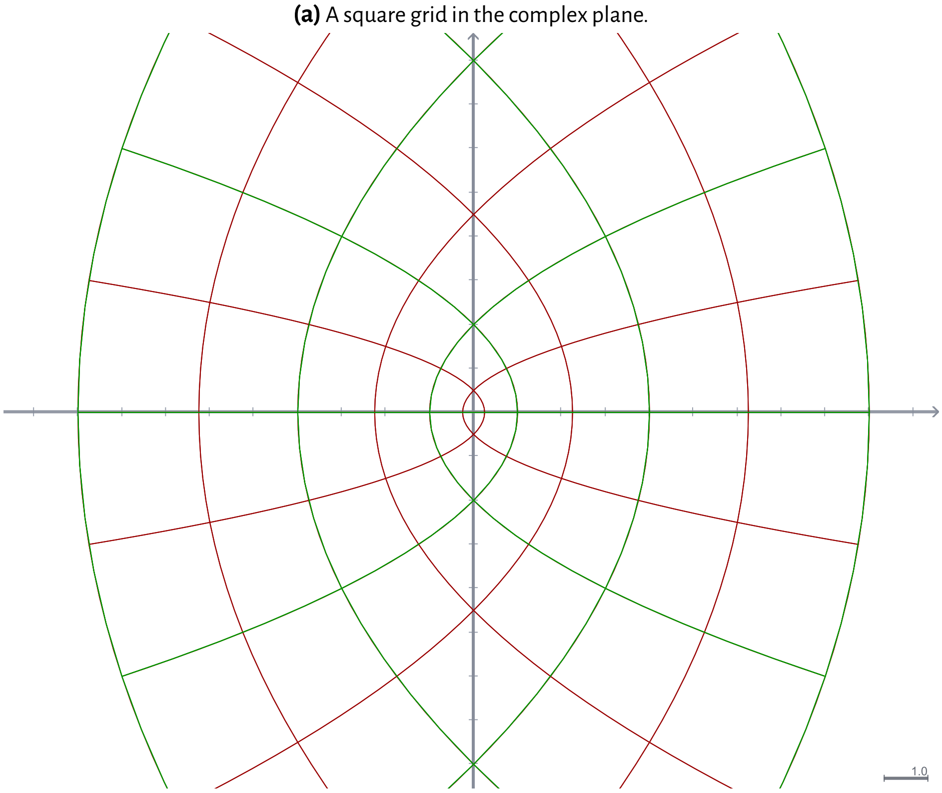
⋙ ***B.** One version of Cauchy's integral formula is*

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for $f\colon \Omega \longrightarrow \mathbb{C}$ holomorphic, $z_0 \in \Omega$, $r > 0$ such that $B(z_0,r) \Subset \Omega$ and $z \in B(z_0,r)$.
How do you imagine the assertion of this formula?
How would you argue vividly or with one of your mental images that this formula holds?

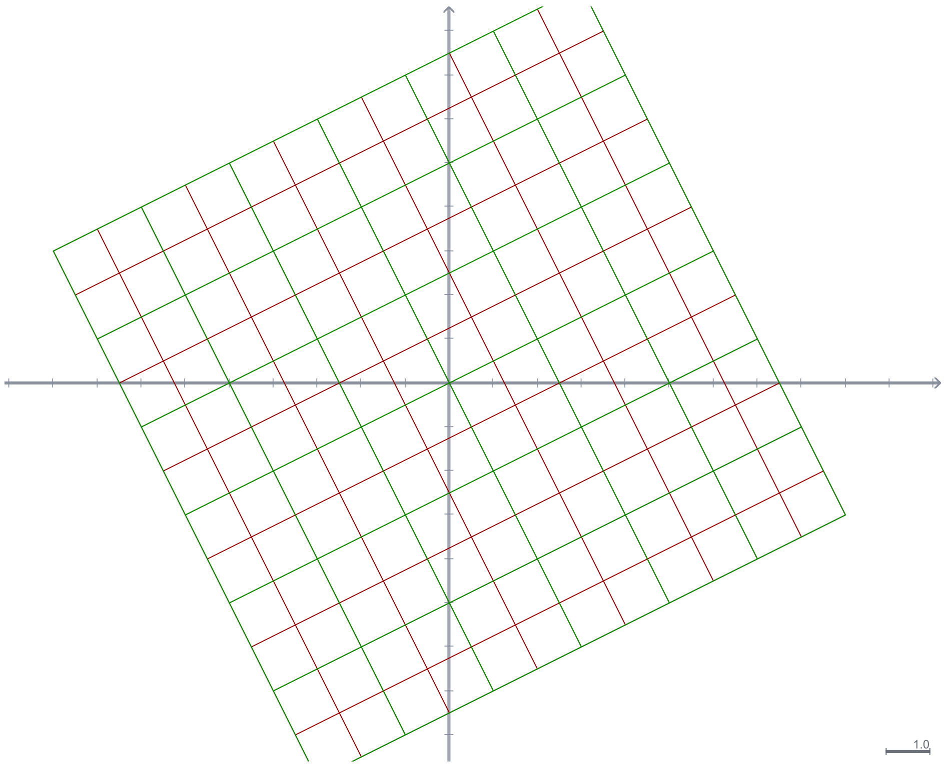


(a) A square grid in the complex plane.



(b) Image of the square grid in (a) under $z \longmapsto z^2$.

Figure D.1: A square grid and images of this grid under two mappings.



(c) Image of the square grid in (a) under $z \longmapsto (2+i)z$.

Figure D.1: A square grid and images of this grid under two mappings (cont.).

NOTATION

Table D.1 shows list of symbols and their use in this thesis. Furthermore, we use the signifier "complex function" for complex-valued functions defined on a subset of complex numbers. A "real function" denotes a real-valued functions defined on a subset of $\mathbb{R}$ or $\mathbb{R}^2$. The keyword "real integral" is used for integrals of real functions (i.e., we use it to denote Riemann integrals or real path integrals of first or second kind).

Table D.1: Table of symbols

Symbol	Explanation
$\mathbb{N}$	the set of natural numbers, i.e., $\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	the set of natural numbers including zero, i.e., $\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the set of integers, i.e., $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
Ω	usually an open subset of $\mathbb{C}$ or $\mathbb{R}^2$
$[a, b]$	closed interval of real numbers between $a, b \in \mathbb{R}$ such that $a < b$, i.e., $\{x \in \mathbb{R} : a \leq x \leq b\}$
(a, b)	open interval of real numbers between $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that $a < b$, i.e., $\{x \in \mathbb{R} : a < x < b\}$
$\#A$	the cardinality of the set A
$A^{\mathbb{N}}$	the set of sequences with elements in A
$B(z, r)$ or $B_r(z)$	the ball of radius r around z, i.e., $\{\omega \in \mathbb{C} : \lvert z - \omega \rvert < r\}$
$A \Subset B$	the topological closure of A is contained in B, i.e., $\overline{A} \subseteq B$
x, y, t	usually real variables
z, ξ, ζ	usually complex variables
$z = a + bi$	a complex number with real part $a = \mathrm{Re}(z)$ and imaginary part $b = \mathrm{Im}(z)$
$\overline{z}$	the complex conjugate of $z = z_1 + iz_2 \in \mathbb{C}$, i.e., $z_1 - iz_2$
$\lvert z \rvert$	the absolute value of $z = z_1 + iz_2 \in \mathbb{C}$, i.e., $\sqrt{z_1^2 + z_2^2}$
$\mathrm{Arg}(z)$	the argument of the complex number $z \neq 0$, i.e., $\theta \in [0, 2\pi)$ such that $z = \lvert z \rvert e^{i\theta}$

Symbol	Explanation
$\langle u, v\rangle$	the standard scalar product of $u = (u_1, u_2)^T, v = (v_1, v_2)^T \in \mathbb{R}^2$, i.e., $u_1 v_1 + u_2 v_2$
$f = u + vi$	a complex-valued function such that $u = \operatorname{Re}(f)$ and $v = \operatorname{Im}(f)$
$\Vert u\Vert$	the Euclidean norm of $u \in \mathbb{R}^2$, i.e., $\sqrt{\langle u, u\rangle}$
ι	the isomorphism of Banach spaces $\mathbb{C} \longrightarrow \mathbb{R}^2$, $z \longmapsto (\operatorname{Re}(z), \operatorname{Im}(z))^T$
κ	the isomorphism of fields $\mathbb{C} \longrightarrow \left\{\begin{pmatrix} a & -b \\ b & a\end{pmatrix} : a, b \in \mathbb{R}\right\}$, $z \longmapsto \begin{pmatrix}\operatorname{Re}(z) & -\operatorname{Im}(z) \\ \operatorname{Im}(z) & \operatorname{Re}(z)\end{pmatrix}$
$\mathbf{J}$	the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$
$J_{\mathbf{f}}$	the Jacobian of a totally differentiable or continuously differentiable vector field $\mathbf{f} = (u, v)^T$, i.e., $\begin{pmatrix}\partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v\end{pmatrix}$
γ	a path, i.e., a continuous function $[a, b] \longrightarrow \mathbb{C}$
$\operatorname{tr}(\gamma)$	the trace of the path γ; i.e., if γ is given as a function $[a, b] \longrightarrow \mathbb{C}$, then $\operatorname{tr}(\gamma) := \{\gamma(t) : t \in [a, b]\}$
$-\gamma$	the path γ with reversed parametrisation
$L(\gamma)$	the length of the rectifiable path γ
$\operatorname{int}(\gamma)$	the interior of the path γ
$\operatorname{int}(R)$	the interior of a rectangle R in the plane $\mathbb{C} \cong \mathbb{R}^2$
$\gamma \oplus \eta$	juxtaposition of the path γ with an end point that agrees with the start point of the path η
$\mathbf{f}$	vector field associated to a complex function $f = u + iv$, i.e., $\mathbf{f} = (u, v)^T$
$\mathbf{w}_f, \mathbf{w}$	the Pólya vector field associated to a complex function $f = u + iv$, i.e., $\mathbf{w} = (u, -v)^T$
$\int_a^b f(x)\,\mathrm{d}x$, $\int_a^b f(t)\,\mathrm{d}t$	the Riemann integral of f over $[a, b]$
$\int_\gamma f(z)\,\mathrm{d}z$, $\int_\gamma f(\zeta)\,\mathrm{d}\zeta$, $\int_\gamma f, \ldots$	the complex path integral of f along γ
$\int_\gamma g\,\mathrm{d}s$	the real path integral of first kind of g along γ

Symbol	Explanation
$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{T}$	the real path integral of second kind with respect to the tangential field of the vector field $\mathbf{F} = (P, Q)^T$ along γ, i.e., $\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y$
$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{N}$	the real path integral of second kind with respect to the normal field of the vector field $F = (P, Q)^T$ along γ, i.e., $\int_\gamma P\,\mathrm{d}y - Q\,\mathrm{d}x$
$\iint_D h\,\mathrm{d}\mathcal{A}$	the double integral of $h\colon D \longrightarrow \mathbb{C}$ over $D \subseteq \mathbb{R}^2 \cong \mathbb{C}$, i.e., $\iint_D h\,\mathrm{d}\mathcal{A} = \iint_D \mathrm{Re}(h)\,\mathrm{d}x\,\mathrm{d}y + i \iint_D \mathrm{Im}(h)\,\mathrm{d}x\,\mathrm{d}y$
∂_1, ∂_2	partial derivatives with respect to the first and second coordinate, i.e., $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$
$\partial, \bar{\partial}$	the Wirtinger differential operators, i.e., $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$
$\mathrm{d}\mathcal{A}, \mathrm{d}A$	the area elements $\mathrm{d}\mathcal{A} = \mathrm{d}x\,\mathrm{d}y$ and $\mathrm{d}A = i\,\mathrm{d}x\,\mathrm{d}y = i\,\mathrm{d}\mathcal{A}$
$\mathcal{H}(\Omega)$	the complex vector space of holomorphic functions on Ω
$\mathcal{P}(\Omega)$	the set of (piecewise continuously differentiable) paths in Ω

COLOPHON

This thesis was typeset by adopting the typographical look-and-feel classicthesis v4.6 developed by André Miede and Ivo Pletikosić (https://ctan.org/pkg/classicthesis), modified by Fabian Parsch (http://www.math.toronto.edu/fparsch/thesistemplate.html).
The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*".

Transcript excerpts were created with Gareth Walker's convtran.sty (https://gareth-walker.staff.shef.ac.uk/).

Figures 1.1, I, II, 8.1, 8.5, III, 13.1, 14.1, 15.1, A.2, A.3 and B.2 were created with Ti*k*Z (Tantau, n.d.) and partially with the assistance of Ti*k*Z editors (Hackenberger, n.d.; Willwacher & Ohrt, n.d.). Figures 4.1, 12.1, 16.1 and 16.2 were created with *Microsoft Word 2021* (Microsoft Corporation, 2021).

Apart from minor corrections, mostly on the linguistic level, this publication agrees with the version of the dissertation submitted for examination.