Dissertation

Regularization of ill-posed inverse problems with tolerances and sparsity in the parameter space

submitted by

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Georgia Sfakianaki: *Regularization of ill-posed inverse problems with tolerances and sparsity in the parameter space*

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Abstract

We consider the solution of ill-posed inverse problems using regularization with tolerances. In particular, we are interested in the reconstruction of solutions that lie within or close to an area outlined by a tolerance measure. To approximate the true solution of the problem in a stable way, we propose a Tikhonov functional with a tolerance function in the regularization term. The tolerances allow us to neglect errors in the penalty term up to a certain threshold. Our theoretical analysis proves that the proposed method complies with all the requirements of variational regularization methods. In addition, we establish convergence rates for the convergence of minimizers to the true solution.

Moreover, we are interested in obtaining sparse solutions. For this purpose, we extend the proposed approach with the idea of elastic net regularization by introducing an additional penalty term that promotes the sparsity of the solution. We establish theoretical results for this elastic net approach and give a convergence rate analysis for the minimizers. To confirm our analytical findings, we illustrate the effect of tolerances in the computed regularized solutions on some numerical examples.

Zusammenfassung

Wir beschäftigen uns mit der Lösung von schlecht gestellten inversen Problemen unter Verwendung von Regularisierung mit Toleranzen. Dabei interessieren wir uns insbesondere für die Rekonstruktion von Lösungen, die innerhalb oder in der Nähe eines durch ein Toleranzmaß umrissenen Bereichs liegen. Um die wahre Lösung des Problems auf stabile Weise zu approximieren, schlagen wir ein Tikhonov-Funktional mit einer Toleranzfunktion im Regularisierungsterm vor. Die Toleranzen erlauben es, Fehler im Strafterm bis zu einem bestimmten Schwellenwert zu vernachlässigen. Unsere theoretische Analyse beweist, dass die vorgeschlagene Methode alle erforderlichen Voraussetzungen für variationale Regularisierungsmethoden erfüllt. Insbesondere werden auch Konvergenzraten für die Konvergenz der Minimierer zur wahren Lösung bestimmt.

Außerdem interessiert es uns, sparse Lösungen zu erhalten. Dazu erweitern wir den vorgeschlagenen Ansatz um die Idee der elastischen Netzregularisierung, indem wir einen zusätzlichen Strafterm einführen, der die Sparsität der Lösung befördert. Wir präsentieren theoretische Ergebnisse für diesen Ansatz des elastischen Netzes und geben eine Konvergenzratenanalyse für die Minimierer. Um die analytischen Ergebnisse zu bestätigen, illustrieren wir den Effekt von Toleranzen in den regularisierten Lösungen an einigen numerischen Beispielen.

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1 Introduction

Solving ill-posed inverse problems is a very important task in science and industry as there exist many real-life applications in which such problems are encountered. An inverse problem is concerned with determining the cause for an observed (or desired) effect. Examples can be found in digital image and signal processing (e.g., image denoising), in medical applications (such as X-ray computerized tomography and nondestructive testing), in geophysics (e.g., seismic exploration), in geosciences (e.g., data assimilation methods for numerical weather prediction), in engineering and financial mathematics (e.g., parameter identification for PDEs), and in many more areas that are not mentioned here.

Solving an ill-posed inverse problem can be challenging depending on the form of ill-posedness, which usually is the lack of stability in the solution under measurement errors. Methods that tackle the ill-posedness of an inverse problem and lead to a good approximate solution, are the so-called *regularization methods*. These have been developed with the idea of incorporating available a priori information about the true solution of the problem, thereby offering a stabilization in the numerical treatment of the problem. In this dissertation, we will focus on *Tikhonov regularization*, a method that has been widely referenced in the literature, and on the so-called *elastic net regularization*, which is a combination of *ℓ^1-regularization* and Tikhonov, that offers the stable reconstruction of sparse solutions.

1.1 Motivation

Many engineering applications demand the identification of input parameters that will be used subsequently, for example, as settings in a machine or an experiment.

Such applications exist in the manufacturing industry, e.g., in the design and production of a piece of equipment or material. The design processes in these applications are often complex and may require a desired precision level or specific durability of the outcome materials. Our work is motivated by existing considerations in applications such as production processes for micro-components (micro machining) but also from the development of new structural materials.

In [GPR+18], the objective of the assumed micro machining problem is the identification of process parameters for a given output. Such a task requires model inversion that leads to an ill-posed inverse problem when the underlying model is nonlinear or not invertible. In their work, the authors propose altered Tikhonov functionals of the form

$$\mathcal{T}^{\delta}_{\alpha,\varepsilon}(u) = \|F(u) - v^{\delta}\|^{p}_{p,\varepsilon} + \alpha\|u\|^{q}_{q}, \tag{1.1}$$

so that the discrepancy term incorporates a *tolerance measure* ε. The tolerance measure in this case is the ε-insensitive function $|\cdot|_{\varepsilon} = \max\{|\cdot| - \varepsilon, 0\}$ and acts for neglecting deviations of a certain threshold while, at the same time, makes the problem of finding the desired parameters more stable.

A similar problem is treated in [OBPM18, OB20] for an application of neural networks for the solution of inverse problems that arise during the production of new structural materials. The objective of this work is the identification of a set of parameters such that the obtained material fulfills certain properties. These properties can, for example, consist of the chemical composition, density and hardness of the material, and are given in the form of a *performance profile*. In particular, in [OB20], the performance profile is defined by

$$\mathcal{B}(\tilde{y}, \varepsilon) := \{z \in \mathbb{R}^m : \tilde{y}_i - \varepsilon_i \leq z_i \leq \tilde{y}_i + \varepsilon_i \ \text{ for all } 1 \leq i \leq m\}$$

for given $\tilde{y} \in \mathbb{R}^m$ and $\varepsilon \in \mathbb{R}^m_+ \cup \{0\}$. For the identification of the unknown parameters $x \in \mathbb{R}^n$, the minimization of a Tikhonov functional

$$\frac{1}{2}\|\phi(x) - \hat{y}\|^2_{\varepsilon} + \alpha\mathcal{R}(x), \quad \alpha > 0$$

is considered, with an ill-posed nonlinear forward operator $\phi : \mathbb{R}^n \to \mathbb{R}^m$ that is obtained via the training of a neural network. A guess of the performance profile is used as noisy data $\hat{y}$ and the discrepancy between $\phi(x)$ and $\hat{y}$ is measured using the ε-insensitive function. The regularization term $\mathcal{R}(x)$ incorporates prior information about the true parameters which is related to their boundary conditions. The results in [OB20] show that the reconstructed parameters fit better the performance profile and the recovery process has a higher accuracy than the ε-free Tikhonov regularization. For more information on the setting of the problem and examples, we refer the reader to [OB20].

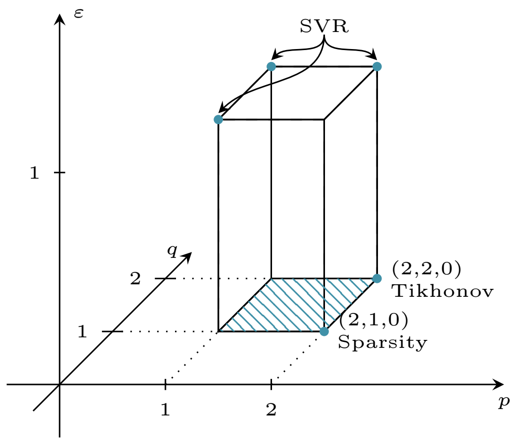


Figure 1.1: Existing theory for Tikhonov-type functionals as in (1.1) and the use of the ε-insensitive distance in the discrepancy term.

While the above approaches are more recent, the consideration of tolerances in the data-fitting term was introduced in the framework of statistical learning theory, in the so-called *support vector regression* (SVR) for classification tasks [Vap95, Kre11]. Figure 1.1 summarizes the existing analytical results for Tikhonov-type functionals as in (1.1). Each functional is characterized by (p, q, ε), where the axes p and q in the figure account for the norms of the discrepancy and the regularization term, respectively. The axis ε accounts for the tolerance assumption in the discrepancy term. For example, $(2, 2, 0)$ corresponds to the quadratic Tikhonov functional, whereas $(2, 2, 1)$ corresponds to a case of SVR. The shaded bottom face of the diagram represents the theory and analytical results of Tikhonov functionals with power-type norms between 1 and 2 which is well-studied, see for example [GHS08, SGG$^+$09].

Inspired by the existing work on the use of tolerances in the data fitting term, we wish to incorporate the ε-insensitive function in the regularization term of a Tikhonov functional. To the best of our knowledge, such an approach with tolerances in the regularization has not been considered somewhere else, and is compatible with applications in which the solution is encouraged to lie in a certain area. In addition, the tolerances used in combination with available *a priori* information of the true solution of the problem, can be perceived as additional confidence over the imposed information in a region dictated by the tolerances. This can be useful if, for example, such *a priori* information has been obtained in a previous experiment or when some known bounds on the true solution are available. A similar idea can be found in parameter identification problems, for example, when considering the identification of PDE coefficients which have to satisfy inequality constraints, see for example [HKR18]. However, in contrast to such inequality constraints, the tolerances

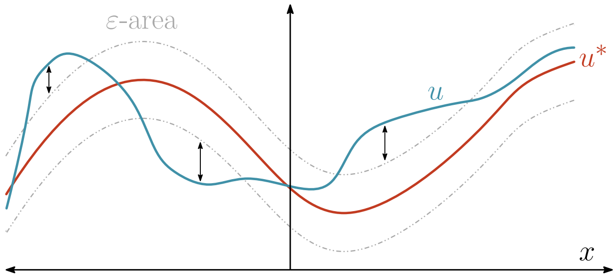


Figure 1.2: Illustration of the tolerance area defined around a reference solution u^*.

in the regularization do not oblige the solution to be within the bounds but rather encourage it to lie within the prescribed tolerance area.

1.2 Our work

Based on the previous motivation, for the solution of a problem $F(u) = v$, we propose a Tikhonov functional

$$\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) = \|F(u) - v^\delta\|_p^p + \alpha\|u - u^*\|^q_{q,\varepsilon} \tag{1.2}$$

with the ε-insensitive distance

$$|z|_\varepsilon = \begin{cases} |z| - \varepsilon, & |z| > \varepsilon \\ 0, & |z| \leq \varepsilon \end{cases} \quad \text{for } z \in \mathbb{R} \text{ and } \varepsilon > 0,$$

in the regularization term of $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$, and we study the analytical properties of its minimizers. In (1.2) u^* is a reference solution, which can be used for imposing known information about the true solution. Figure 1.2 serves as an illustration of what we wish to achieve by using the tolerances in the regularization term $\|u - u^*\|_{q,\varepsilon}$. By minimizing the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$, we seek for a regularized solution that is a good approximation to the true solution of the problem and is such that the difference between u and u^* outside the tolerance area (gray-shaded area in Figure 1.2) is small. Because of the ε-insensitive distance, the difference between u and u^* inside the tolerance area is always set to zero. Because this approach is new, we establish the existence, stability and weak convergence of minimizers of the functional in (1.2) and prove theoretical convergence rates in the Bregman distance.

In addition, the recovery of sparse solutions is a topic of great interest and sought in many applications of inverse problems, such as in electrical impedance tomography [GKL+12] and in imaging techniques for mass spectrometry [BTAM13]. In the course of our work, the desire for sparse solutions leads us to further consider the elastic net functional

$$\mathcal{J}^{\delta}_{\alpha,\beta,\varepsilon}(u) = \|Ku - v^{\delta}\|_V^2 + \alpha\|u\|^2_{\ell^2,\varepsilon} + \beta\|u\|_{\ell^1}, \tag{1.3}$$

which is modified by tolerances and allows the combination of both concepts, namely, tolerances and sparsity. The addition of the ℓ^1-penalty term is a natural next step because such a penalty term better promotes sparsity in the solution, [GHS08]. We investigate the regularization theory results for the modified by tolerances elastic net functional and prove convergence rates in the ℓ^2-norm.

Structure of the dissertation. Our work is structured as follows: in Chapter 2, we present the necessary tools and mathematical background and give an overview of the different approaches that motivate our work. Besides, it serves as the basis for understanding the core of the dissertation, namely, Chapters 3 and 4, in which we present our approach and theoretical findings. More specifically, Chapter 3 deals with the introduction of Tikhonov regularization with tolerances in the penalty term, for which we establish results in terms of the regularization theory. The main results of Chapter 3 have been included in [PKS20]. Chapter 4 deals with the theoretical analysis of the proposed elastic net functional with tolerances for obtaining sparse solutions. Chapter 5 is dedicated to a simple numerical consideration, for illustrating the effect of tolerances in the reconstructed solutions and is considered for confirming our theoretical results. In the same chapter, we also discuss ideas for the optimal parameter selection of regularization parameters when tolerances are considered. Chapter 6 concludes this dissertation with an outlook on our findings and possible future directions.

Chapter

2 Preliminaries

This chapter covers basic results from the existing theory in the field of nonlinear ill-posed inverse problems and Tikhonov regularization. Moreover, we discuss the existing theory that motivates the ideas presented in Chapters 3 and 4 of this dissertation. In Section 2.1 we introduce the basic notation and tools from functional analysis, calculus of variations and convex analysis. In Section 2.2 we proceed with the theory of inverse problems and introduce the idea of regularization methods with a particular focus on Tikhonov regularization. In Section 2.3 we discuss sparsity-promoting Tikhonov regularization, while in Section 2.4 we discuss numerical methods that are used for the minimization of such functionals. Then, Section 2.5 follows with an overview of the elastic net regularization. The chapter is concluded with Section 2.6 which offers a discussion on the use of tolerances in support vector regression and in Tikhonov regularization for solving inverse problems.

2.1 Notation and function properties

This section includes basic definitions and tools from functional analysis and convex analysis, which can be, for example, found in [Cla13] and [SGG+09]. We begin with the definition of the epigraph of a function and the definition of a proper function. Then, we continue with weak lower semi-continuity of a functional, an important property that will be used in the following chapters.

Definition 2.1. Let $f : X \to \mathbb{R} \cup \{\infty\}$.

1. The *effective domain* of f is defined as the set

$$\mathrm{dom}(f) = \{x \in X : f(x) < \infty\}.$$

2. The *epigraph* of f is defined as the set of points above the graph of f, that is,

$$\mathrm{epi}(f) = \{(x, \rho) \in X \times \mathbb{R} : f(x) \leq \rho\}.$$

3. The function f is called *proper* if its domain is non-empty, that is, $\mathrm{dom}(f) \neq \emptyset$.

Definition 2.2 (l.s.c. and w.l.s.c.)**.** A functional $f : X \to \mathbb{R}$ is said to be *lower semi-continuous* (abbreviated l.s.c.) at x_0 if, for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X with $x_k \to x_0$, there holds

$$f(x_0) \leq \liminf_{x_k \to x_0} f(x_k).$$

Similarly, $f : X \to \mathbb{R}$ is called *weakly lower semi-continuous* (abbreviated w.l.s.c.) at x_0 if, for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X with $x_k \rightharpoonup x_0$, there holds

$$f(x_0) \leq \liminf_{x_k \rightharpoonup x_0} f(x_k).$$

Now, we continue with the definition of the norm in L_q-spaces which will be considered later in our setting. We remind the reader of Fatou's lemma, which is useful in our proofs.

Notation 2.3. Let $\Omega \subset \mathbb{R}^n$ be bounded and $q \in [1, 2]$. The space $L_q(\Omega)$ consists of all q-integrable functions $f : \Omega \to \mathbb{R}$. For $f \in L_q(\Omega)$ we define the L_q-norm by

$$\|f\|_{L_q(\Omega)} = \left(\int_\Omega |f(x)|^q \, \mathrm{d}x \right)^{1/q}.$$

L_q spaces are an important class of Banach spaces, and L_2 is a Hilbert space with norm induced by the inner product

$$\langle f, g \rangle_{L_2(\Omega)} = \int_\Omega f(x) g(x) dx < \infty,$$

for real functions $f, g \in L_2(\Omega)$.

Lemma 2.4 (Fatou's lemma)**.** *Let $\{f_n\}$ be a sequence of nonnegative measurable functions $f_n : \Omega \to \mathbb{R}$ and let $f(x) := \liminf_{n\to\infty} f_n(x)$ for every $x \in \Omega$. Then, there holds*

$$\int_\Omega f(x) \, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega f_n(x) \, \mathrm{d}x. \tag{2.1}$$

The following definition is the Radon-Riesz property (also known as Kadets-Klee property) for normed spaces, which, given weak convergence and continuity of the norm, ensures convergence in the norm.

Definition 2.5 (Radon-Riesz property)**.** Let $(X, \|\cdot\|)$ be a normed space. The space X has the Radon-Riesz property when the following is fulfilled: for any sequence $\{x_k\}_{k\in\mathbb{N}} \in X$ and element $\tilde{x} \in X$ such that $x_k \rightharpoonup \tilde{x}$ and $\lim_{k\to\infty} \|x_k\| = \|\tilde{x}\|$ holds true, then $x_k \to \tilde{x}$, that is, $\|x_k - \tilde{x}\| \to 0$.

The Radon-Riesz property is fulfilled in any real Hilbert space, whereas for Banach spaces this is not always the case. In particular, the Radon-Riesz property is satisfied in uniformly convex Banach spaces. In addition, every uniformly convex Banach space X is *reflexive*, i.e., the bi-dual $X^{**} := (X^*)^*$ is identified with X, and therefore, reflexive Banach spaces often are considered in the analysis of nonlinear problems.

2.1.1 Differentiability and subdifferential calculus

The differentiability of a functional is an important concept in variational regularization methods. Naturally, Fréchet or Gâteaux differentiability of an operator is considered, therefore both types of differentiability are stated in the definition below.

Definition 2.6 (Differentiability, [SGG+09])**.** Let $F : X \to Y$ be an operator between normed spaces X and Y, and $x, h \in X$ be given.

1. The operator F has a *one-sided directional derivative* $F'(x; h) \in Y$ at $x \in X$ in the direction $h \in X$, if

$$F'(x; h) := \lim_{t\to 0^+} \frac{F(x + th) - F(x)}{t} \tag{2.2}$$

exists.

2. Assume $x \in X$ and that $F'(x; h)$ exists for all $h \in X$. The operator F is *Gâteaux differentiable* if there exists a bounded linear operator $F'(x) \in L(X, Y)$ such that the directional derivative fulfills

$$F'(x; h) = F'(x)h, \qquad \text{for every } h \in X.$$

Then, F' is called the Gâteaux derivative of F at x.

3. If additionally, there exists $F'(x) \in L(X, Y)$ such that

$$\lim_{\|h\|\to 0} \frac{\|F(x + h) - F(x) - F'(x)h\|_Y}{\|h\|_X} = 0,$$

then F is called *Fréchet differentiable* at x and F' is its Fréchet derivative.

When a function does not have a classical derivative, the notion of subdifferentiability becomes important. The following results are included in most books for convex analysis and variational calculus such as [ET99].

Definition 2.7. Let X be a normed space. A function $f : X \to \mathbb{R}$ is *convex* if for every two points $x, y \in X$ and $\lambda \in [0,1]$ there holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

The function f is *strictly convex* if the above is a strict inequality.

The following proposition is important for proving that a convex function is weakly lower semi-continuous using the closedness of its epigraph.

Proposition 2.8. *Let X be a normed space. A continuous and convex function $f : \Omega \subset X \to \mathbb{R}$ is lower semi-continuous if and only if its epigraph is closed. Moreover, every lower semi-continuous and convex function is weakly lower semi-continuous.*

Definition 2.9 (Subgradient and subdifferential of a convex function)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. The element $z \in \mathbb{R}$ is a subgradient of f at x if

$$f(y) \geq f(x) + z(y - x)$$

for all $y \in \mathbb{R}$. For a Banach/Hilbert space X with dual X^* and $f : X \to \mathbb{R}$ a convex functional, an element $z \in X^*$ is called a *subgradient* of f at $x \in \mathrm{dom}(f)$ if

$$f(y) \geq f(x) + \langle z, y - x \rangle_{X^* \times X}$$

for all $y \in X$. In addition, the set of all subgradients of f at x defined by

$$\partial f(x) := \{ z \in X^* \ : \ f(y) \geq f(x) + \langle z, y - x \rangle_{X^* \times X}, \ \text{ for all } y \in X \},$$

is called the *subdifferential* of f at x.

Remark 2.10. The inner product in the definition of the subgradient depends on the choice of the space X. If X is a Hilbert space, then, from Riesz representation theorem, $\langle \cdot , \cdot \rangle_X$ is the classical inner product in Hilbert spaces. If X is a Banach space, then $\langle \cdot , \cdot \rangle_{X^* \times X}$ is the dual pairing with X^* the dual space of X.

Example. The subdifferential of $|x| : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ is given by the *set-valued* sign function

$$\partial |x| = \mathrm{Sign}(x) := \begin{cases} \{1\}, & x > 0 \\ [-1, 1], & x = 0 \\ \{-1\}, & x < 0 \end{cases} .$$

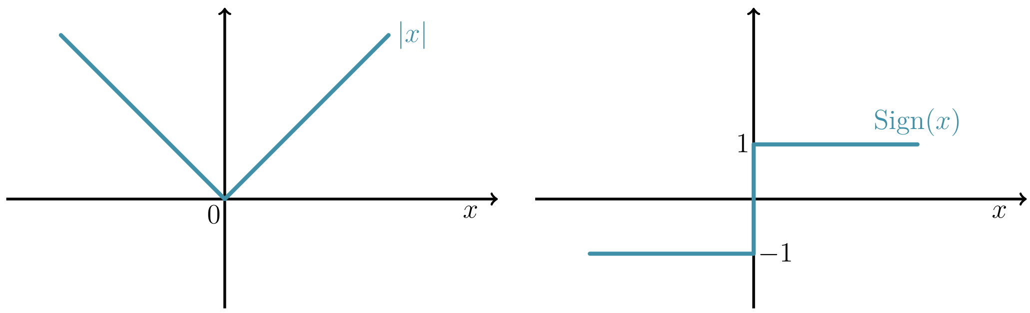


Figure 2.1: The absolute value function (left) and its subdifferential (right).

Figure 2.1 shows the graphs of the absolute value function and its subdifferential, the set-valued sign function. The graphical interpretation of the subdifferential at a point x is simply the set of all possible tangent slopes passing through x. To avoid any confusion, the standard (single-valued) sign function for an element x will be denoted by $\operatorname{sign}(x)$, whereas the set-valued function will be denoted by $\operatorname{Sign}(x)$.

From optimization theory we know that the minimum of a differentiable function is taken at points where the derivative of that function is zero (first order optimality condition) or at boundary points. When the classical derivative of a function does not exist, the notion of subgradients comes into play, as a subgradient always exists. The optimality condition for a minimizer of a convex function is given in the following proposition.

Proposition 2.11. *Let $f : X \to \mathbb{R}$ be convex. Then, $x^* \in X$ is a minimizer of f if and only if*

$$0 \in \partial f(x^*).$$

The following rules can be used to calculate the optimality conditions for our problem.

Lemma 2.12 (Rules for calculation of a subdifferential)**.** *Let $f, g : X \to \mathbb{R} \cup \{\infty\}$ be convex functions. For $\lambda > 0$, there holds*

$$\partial(\lambda f)(x) = \lambda \partial f(x), \quad x \in X.$$

If there exists a point in $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ where f is continuous, then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad \textit{for all } x \in X.$$

Lemma 2.13 (Subdifferential rule for compositions, [Zei85])**.** *Let $f : Y \to \mathbb{R} \cup \{\infty\}$ be a convex functional and $A : X \to Y$ a bounded linear operator. If f is continuous* *at a point in the range of A, then for all $x \in X$ there holds*

$$\partial(f \circ A)(x) = A^*(\partial f(Ax)).$$

2.1.2 Bregman Distance

In Banach spaces we often use the Bregman distance for measuring distances due to the geometrical properties of Banach spaces. The Bregman distance simply measures the gap between a functional and its linearization. It was first introduced by Bregman in [Bre67] and is a particularly useful distance measure for regularization methods in Banach space settings. Details on the properties of the Bregman distance can be found in [SGG+09] and [SKHK12]. In this work, we follow the general definition of the Bregman distance which exists in both references.

Definition 2.14 (Bregman distance)**.** Let $\mathcal{R} : X \to \mathbb{R} \cup \{\infty\}$ be a convex and proper functional with subdifferential $\partial\mathcal{R}$ and let $x, \tilde{x} \in X$. In addition, let $\xi \in \partial\mathcal{R}(x) \subset X^*$. Then, the Bregman distance of $\mathcal{R}$ at x and ξ is defined as

$$\mathcal{D}_\xi(\tilde{x}, x) := \mathcal{R}(\tilde{x}) - \mathcal{R}(x) - \langle \xi, \tilde{x} - x \rangle_{X^* \times X}. \tag{2.3}$$

The linearization of the convex functional $\mathcal{R}$ in this case is given by the slope of the tangent line passing from x. For this reason, it is understood that the Bregman distance is only defined at points where the subdifferential $\partial\mathcal{R}$ is not empty. Therefore, we define the *Bregman domain* as the set

$$\mathcal{D}_\mathrm{B}(\mathcal{R}) := \{x \in \mathrm{dom}(\mathcal{R}) \;:\; \partial\mathcal{R} \neq \emptyset\}. \tag{2.4}$$

Based on the definitions of the subdifferential and the Bregman distance, it follows that the Bregman distance is nonnegative with $D_\xi(x, x) = 0$ and therefore it makes sense to use it as a distance measure. Figure 2.2 provides a visual interpretation of the Bregman distance $\mathcal{D}_\xi(\tilde{x}, x)$ for a functional $\mathcal{R}$.

Remark 2.15. Considering the squared norm on a Hilbert space X, the functional $\mathcal{R}(x) = \|x - x^*\|_X^2$ is differentiable. Therefore, its subdifferential is a singleton and equals the derivative of $\mathcal{R}(x)$, i.e., $\xi = 2(x - x^*) = \partial\mathcal{R}(x)$. In this case, the usual norm can be obtained from the Bregman distance, $\mathcal{D}_\xi(\tilde{x}, x) = \|\tilde{x} - x^*\|_X^2 - \|x - x^*\|_X^2 - 2\langle x - x^*, \tilde{x} - x\rangle = \|\tilde{x} - x\|_X^2$, which is particularly useful for transferring estimates from the Bregman distance to estimates in the usual norm.

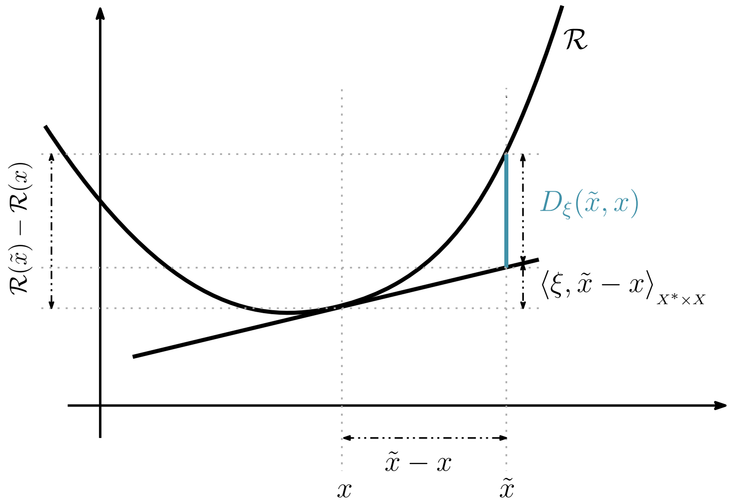


Figure 2.2: Geometrical interpretation of the Bregman distance of a convex functional $\mathcal{R}$, reproduced from [SGG$^+$09].

2.2 Inverse problems and Tikhonov regularization

In this section we discuss the basic theory of inverse problems using the books [Kir96, EHN96] as main references. In our work, we follow Hadamard's definition for the *well-posedness* of an inverse problem, which is given through the following statements (see [Had32, EHN96]):

For all admissible data, a solution exists. (2.5a)

For all admissible data, the solution is unique. (2.5b)

The solution depends continuously on the data. (2.5c)

If any of these properties is violated, the inverse problem is called *ill-posed*. In the next chapters, when referring to an inverse problem, we will always assume an ill-posed inverse problem.

Despite the fact that all three properties must be satisfied in order for an inverse problem to be well-posed, not all of them are difficult to deal with when not fulfilled. The existence of a solution is important but when it cannot be guaranteed, we can overcome this obstacle by considering a least-squares solution. An ill-posed problem which violates (2.5b) has multiple solutions that cause the same data. A common approach in linear problems is to choose the solution with minimal norm. The case of approximating the true solution of an inverse problem violating (2.5c) is the most difficult form of ill-posedness. What often happens is, that numerical methods used for solving an inverse problem become unstable as a small perturbation in the data leads to completely different or unreasonable solutions. This form of ill-posedness is related to the inverse of the forward operator, which either doesn't exist or, even if it exists, is unbounded.

A common strategy for solving ill-posed inverse problems is by using regularization methods. The idea of these methods is to incorporate additional information about the ground truth (e.g., *a priori* knowledge about the structure of the solution), which will allow for a stable recovery of an approximation of the true solution. When including additional information we, basically, decide how much we sacrifice accuracy over stability. The key-point of regularization methods is finding the optimal compromise between these two. In this thesis we consider Tikhonov regularization, a method first introduced by Tikhonov in his pioneering work [Tik63, TA77, TLY98] for the solution of integral equations. Since then, the classical Tikhonov regularization has become very popular in linear inverse problems because it uses the least-squares fidelity term. Thus, its minimizer can be written in a closed form, making its numerical implementation easy.

Let us now proceed with the mathematical definition of an inverse problem. We consider an operator equation of the form

$$F(u^\dagger) = v, \tag{2.6}$$

with $F : \mathrm{dom}(F) \subset U \to V$ a linear or nonlinear operator between Hilbert or (reflexive) Banach spaces U and V, and true solution $u^\dagger \in U$. Very often, instead of the true right hand side $v \in V$, only measured data

$$v^\delta := v + \eta(\delta) \in V$$

corrupted by additive noise $\eta(\delta)$ are available. For noise level $\delta > 0$, we assume that the noisy data v^δ are such that

$$\|v - v^\delta\|_V \leq \delta. \tag{2.7}$$

Given the operator F and noisy data v^δ, the aim is to find a good approximation of the true solution $u^\dagger \in U$ satisfying (2.6). We assume the inverse problem to be ill-posed and we consider Tikhonov regularization in order to approximate its true solution in a stable way. In its generalized form, Tikhonov regularization consists of minimizing a functional defined as

$$\mathcal{T}_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^p + \alpha \mathcal{R}_q(u), \quad \alpha > 0, \tag{2.8}$$

with the operator F as defined above, $u \in U$ and $v^\delta \in V$ satisfying (2.7) for $\delta > 0$. The first term in (2.8) is called *discrepancy* or *fidelity term* and measures the distance between the (noisy) data and the data that would be obtained by the estimate.

The second term is the so-called *regularization functional* or *penalty term* added for stabilizing the solution. The discrepancy is often measured in the p-power norm of the space V, while the regularization functional differs depending on the a priori information we take into account (e.g., sparse structure or other structural features such as edge preservation). Here, we consider a regularization term using the q-power norm of the parameter space U, i.e., we define the penalty term

$$\mathcal{R}_q(u) := \|u - u^*\|_U^q, \tag{2.9}$$

with u^* an *a priori guess* or *reference solution*. Note that for $p = q = 2$ we obtain the classical (quadratic) Tikhonov regularization.

As seen in (2.8), the penalty term is weighted by the positive constant α which is no other than the *regularization parameter* and whose value decides the amount of regularization. It is easy to understand that a large value of α gives reconstructions that are far from the true solution, whereas a small value will not give reasonable reconstructions due to the ill-posedness of the problem. Both are unwanted effects and therefore, the task of finding an optimal value for the regularization parameter, known as *parameter choice strategy or rule*, is considered to be very important. In the literature, there exist various parameter choice rules, some of which will be discussed in Subsection 2.2.2 of this chapter.

In order for the minimization of (2.8) to make sense, we consider that it happens only on the set $\mathcal{D} := \mathrm{dom}(F) \cap \mathrm{dom}(\mathcal{R}_q)$ which is further assumed to be non-empty. Moreover, we denote the minimizer of (2.8) by

$$u_\alpha^\delta := \arg\min\{\mathcal{T}_\alpha^\delta(u) : u \in \mathrm{dom}(F) \cap \mathrm{dom}(\mathcal{R}_q)\}. \tag{2.10}$$

2.2.1 Theoretical analysis

Inverse problems in the form of operator equations are divided in linear and nonlinear problems depending on whether a linear or nonlinear forward operator is used, respectively. The theoretical analysis of regularization methods is different for linear and nonlinear problems. This is mainly due to the fact that a singular value decomposition of the forward operator is, in general, not available for nonlinear operators. The analysis of nonlinear ill-posed inverse problems is instead based on variational regularization methods, see [SGG+09]. In the following, we focus on the regularization of nonlinear ill-posed problems but all results apply to linear inverse problems, as well. When discussing variational regularization, typically we are interested in the following analytical results:

1. *Existence* of minimizers of $\mathcal{T}_\alpha^\delta$ for fixed regularization parameter $\alpha > 0$ and any given noisy data v^δ.
2. *Stability*, that is, proving that for fixed $\alpha > 0$ the regularized solution u_α^δ depends continuously on the data v^δ.

3. *Convergence* which grants that for $\alpha \to 0$ and $\delta \to 0$ the regularized solution u_α^δ converges to a solution of $F(u^\dagger) = v$.

4. *Convergence rates* which provide estimates for the difference between the minimizer u_α^δ and the true solution $u^\dagger$.

In the classical theory, the spaces U and V are either assumed to be Hilbert spaces or reflexive Banach spaces. Common assumptions on the nonlinear operator F are continuity and weak (sequential) closedness with respect to the underlying topologies of U and V. The latter is important for proving the existence of minimizers of (2.8), while continuity is not essential for this result. Concerning the convergence of minimizers, the goal is to prove the convergence to a minimum norm solution $u^\dagger$ of (2.6) when the noise in the data vanishes. It is therefore understood that the regularization parameter $\alpha > 0$ should depend on the noise level δ. In addition, when working on weak topologies, convergence is proved in the weak sense. If the underlying spaces fulfill the Radon-Riesz property, one can transfer the convergence result in the norm.

The convergence rates, i.e., an estimate for $\|u_\alpha^\delta - u^\dagger\|_U$ is of most interest and normally, for obtaining such an estimate, additional assumptions are needed. Namely, these are: an assumption of Gâteaux or Fréchet differentiability of F, a restriction of its nonlinearity, as well as a source-wise representation (also called *source or range condition*) of the solution to be recovered. The source condition, basically, is a smoothness assumption of the solution with respect to the operator F, whose range is not closed. In [SKHK12, Section 3.2.2], the authors begin with the simplest source condition for linear ill-posed problems in Hilbert spaces, and generalize the concept to nonlinear problems and Banach space settings. In [SGG$^+$09, Table 3.1] a summary of the relations between different source conditions, for general penalty $\mathcal{R}$ and the classical $\mathcal{R}(u)$ as in (2.9), in combination with different assumptions on F is presented.

Returning to the convergence rates, in [JM12] the authors give the step-by-step process for obtaining classical convergence rates in Hilbert space settings and discuss the required modifications for Banach spaces. A common adaptation in Banach spaces is the use of the Bregman distance (see Definition 2.1.2) as the measure in which the convergence rates is calculated. The duality pairing in the definition of the Bregman distance is normally estimated via some variational inequalities, for details refer to [SKHK12].

Tikhonov regularization schemes have been studied by many authors and therefore, a long list of references exists. In the past years, mostly Hilbert spaces were considered, mainly due to the nice structure of inner product spaces. Classical results for Tikhonov regularization of linear and nonlinear inverse problems in Hilbert space settings can be found in [EHN96] and [SGG$^+$09]. Except these two references, a large collection of articles regarding Tikhonov-type regularization already exists, with the most relevant to this thesis being [EKN89, KNS08] and [Lor08]. However, in many applications it is more realistic to consider abstract Banach spaces. Examples of such applications are parameter identification problems in partial differential equations and inverse scattering problems. In [SKHK12] the authors present the standard theory of Tikhonov-type regularization in Banach spaces using different types of penalties. This book is the best general reference for a complete overview of the existing theory in Banach space settings but there also exist many other publications, for example [HKPS07, BKM+08, SHK12] as well as a chapter dedicated to variational methods in Banach spaces in [SGG+09]. Commonly used Banach spaces are the L_p function spaces which often arise in practice (eg. inverse scattering applications in L_p spaces [LKK13]), as well as the ℓ^p sequence spaces which promote the reconstruction of sparse solutions (cf. [GHS08]).

2.2.2 Parameter choice rules

The choice of the regularization parameter is a crucial task for obtaining the best regularized solution and for this reason, there exist various parameter choice rules designed for the optimal selection of α. These are divided into the *a priori*, *a posteriori* and *heuristic* strategies. In the a priori rules the regularization parameter depends on the noise level δ and not on the actual noisy data v^δ, i.e., $\alpha := \alpha(\delta)$. In a posteriori strategies, the choice of α depends both on the noise level and the noisy data, that is, $\alpha := \alpha(\delta, v^\delta)$, while the heuristic methods, also called *error-free* strategies, do not require an estimate of the noise level δ. In the following we introduce the idea of the L-curve criterion (heuristic rule) and Morozov's discrepancy principle (a posteriori rule), that will be used later in this thesis.

The L-curve method. The L-curve criterion is a parametric plot of the discrepancy norm $\|F(u_\alpha^\delta) - v^\delta\|_V$ against the norm of the regularized solution $\|u_\alpha^\delta - u^*\|_U$ created using different values of α. The L-curve shows the trade-off between the fit to the given data and the size of the regularized solution. It looks like an L-shaped curve, in which the optimal value of α is located near the point of maximum curvature. The use of logarithmic axes leads to a sharper L-curve, which makes the selection of the optimal value easier. The vertical part of the L-curve corresponds to the unregularized solutions (small values of α) while the horizontal part corresponds to the over-smoothed solutions obtained for large values of α. In [Han00], the L-curve method is presented for the optimal selection of α in the standard Tikhonov functional with a compact linear operator. For such a setting, the solution and residual norms used in the L-curve are written in terms of the singular value decomposition of the operator. In this case, the performance and main properties of the L-curve are associated with the decay rate of the singular values of the operator. Despite the simple implementation that is a big advantage of the L-curve, the reconstruction of very smooth exact solutions $u^\dagger$ as well as the asymptotic behavior (as the problem size increases) are known limitations of the method.

Morozov's discrepancy principle. An *a posteriori* parameter choice rule that is commonly used in Tikhonov regularization is the discrepancy principle, which was introduced by Morozov in [Mor66]. Given an estimate of the noise level $\delta > 0$, the idea of the discrepancy principle is to accept reconstructions which create measurements with a similar error as the one in the noisy data. This translates into choosing the maximum parameter $\alpha > 0$ such that

$$G(u_\alpha^\delta) := \|F(u_\alpha^\delta) - v^\delta\|_V \leq \tau\delta, \quad \text{for } \tau \geq 1, \tag{2.11}$$

where $u_\alpha^\delta := \arg\min \mathcal{T}_\alpha^\delta(u)$ defined in (2.8). In [IJZ11], the authors deal with Morozov's discrepancy principle for Tikhonov regularization of nonlinear problems with general convex penalties and show convergence rates results with respect to the Bregman distance. Choosing α based on the discrepancy principle leads to optimal convergence rates, which makes the discrepancy principle a popular choice rule for Tikhonov regularization, for more details refer to [Bon08, HKPS07].

2.3 Tikhonov regularization with sparsity constraints

The concept of sparsity has become very popular in many applications during recent years. Such an assumption requires the solution of the underlying inverse problem to be a linear combination of just a few elements of a given orthonormal basis. The basis is chosen depending on the structure of the solution that is to be reconstructed and, therefore, highly associated with the application in hand. For example, in image processing the Fourier basis or the orthogonal wavelet basis have been widely used for image deblurring. For examples and further reading on such applications, see, for example, [BB20] and [Mal99].

Assuming the equation $Ku = v$ with a linear operator $K : U \to V$ between Hilbert spaces U and V, the function $u \in U$ is called sparse with respect to a given basis $\{\phi_i\}_{i\in\Gamma}$ of U if there exists only a finite number of non-zero expansion coefficients $\langle u, \phi_i\rangle$. Regarding Tikhonov regularization with sparsity constraints, in their pioneering work [DDDM04] the authors showed that replacing the classical quadratic penalty term by a (weighted) ℓ^q-penalty, with $1 \leq q \leq 2$, on the coefficients of basis expansion yields a regularization method. They considered a functional of the form

$$\Psi_{\alpha,q}(u) := \frac{1}{2}\|Ku - v^\delta\|_V^2 + \alpha \sum_{i\in\Gamma} |\langle u, \phi_i\rangle|^q\,, \tag{2.12}$$

with strictly positive regularization parameter α. The authors focused on $1 \leq q \leq 2$, for which the functional is convex, weakly lower semi-continuous and coercive, thus the existence of minimizers is guaranteed, and they proved that sparsity is more pronounced for $q = 1$.

As noted in [JLS09] the case $q = 1$, with an abuse of notation where u denotes the sequence of expansion coefficients $\{u_i := \langle u, \phi_i \rangle\}$ and K the operator $u_i \mapsto K \sum_i u_i \phi_i$ mapping elements from ℓ^2 to V, the functional (2.12) is reformulated as

$$\Psi_\alpha(u) := \frac{1}{2}\|Ku - v^\delta\|_V^2 + \alpha\|u\|_{\ell^1} \tag{2.13}$$

and the minimization of Ψ_α is considered over all $u \in \ell^2$. Sparsity is most promoted using the ℓ^1-penalty because it decreases the penalization of large coefficients while small coefficients receive a higher penalization. As a result, the solution has only a few large components with respect to the chosen basis, i.e., it is sparse. It is worth mentioning that the best choice for sparsity would be an ℓ^0-penalty as it leads to the minimum number of non-zero coefficients. In this case, however, the functional is not convex anymore and therefore, ℓ^1-regularization is preferred as it is the closest convex norm.

In [JLS09], U was considered to be a general Hilbert space and V a reflexive Banach space. In other articles there exist results for abstract Banach spaces, see for instance [SHK12]. In addition, most often linear operator equations are considered but the analytical results also extend to nonlinear problems. For further reading, refer to the monograph [SKHK12]. In [GHS08], the authors considered a generalized Tikhonov functional for nonlinear operator equations using p-exponents for the discrepancy ($p \geq 1$) and sub-quadratic ℓ^q-penalty term with $1 \leq q \leq 2$. Under the premises that the true solution has a sparse representation in a given orthonormal basis for the space U and that the operator satisfies the so-called *finite basis injectivity* (FBI) property [BL08b], they show that for $q = 1$ and the choice of regularization parameter $\alpha \sim \delta^{p-1}$ the convergence rate is significantly improved from the usual rate $\mathcal{O}(\delta^{1/q})$ to $\mathcal{O}(\delta)$.

2.4 Minimization of Tikhonov functionals

Many efficient numerical algorithms have been developed for the minimization of Tikhonov functionals. In this section we discuss some of the iterative methods that are commonly used and we provide references to the literature for further reading. In the following we consider the minimization of a Tikhonov functional with sparsity constraints Ψ_α as given in (2.13) with linear operator $K : \ell^2 \to V$. Most iterative schemes are derived using the first order optimality condition for the minimizer of Ψ_α involving the first derivative of the functional. When the classical derivative is not defined, tools from subdifferential calculus (some were summarized in the first section of this chapter) are essential for the definition of the subdifferential $\partial\Psi_\alpha(u)$ which makes the numerical minimization of the functional possible. In our case, the first order optimality condition for Ψ_α reads as

$$0 \in \partial\Psi_\alpha(u) \implies -K^*(Ku - v^\delta) \in \alpha\,\mathrm{Sign}(u), \tag{2.14}$$

with $\mathrm{Sign}(u)$ the set-valued sign function. In the above calculation, the subdifferential of the penalty term is calculated as $\partial\|u\|_{\ell^1} := \mathrm{Sign}(u) \cap \ell^2$ via the Riesz representation theorem. In addition, due to the fact that the adjoint operator K^* already maps into ℓ^2, the intersection with ℓ^2 can be omitted. For further details refer to [Sch10].

With the optimality condition defined, our goal is the derivation of a fixed-point equation for the minimizer. In order to do so, we multiply the optimality condition by $\lambda \neq 0$ to obtain

$$-\lambda\, K^*(Ku - v^\delta) \in \alpha\lambda\,\mathrm{Sign}(u) \tag{2.15}$$

and then we add u to both sides, which yields

$$u - \lambda\, K^*(Ku - v^\delta) \in u + \alpha\lambda\,\mathrm{Sign}(u). \tag{2.16}$$

Equivalently, this can be written as

$$u - \lambda\, K^*(Ku - v^\delta) \in (\mathrm{id} + \alpha\lambda\,\mathrm{Sign})\,(u), \tag{2.17}$$

with the identity operator $\mathrm{id}(u) = u$. This expression is transformed into a fixed-point equation for the minimizer by inverting the operator on the right hand side

$$u := \mathcal{S}_{\alpha\lambda}(u - \lambda\, K^*(Ku - v^\delta)) \tag{2.18}$$

with the so-called *soft-thresholding operator*

$$\mathcal{S}_{\alpha\lambda} := (\mathrm{id} + \alpha\lambda\,\mathrm{Sign})^{-1}, \tag{2.19}$$

that was first introduced in [DDDM04]. $\mathcal{S}_{\alpha,\lambda}$ is a nonlinear operator given by

$$[\mathcal{S}_{\alpha\lambda}(t)]_i = \begin{cases} t_i + \alpha\lambda, & t_i < -\alpha\lambda \\ 0, & |t_i| \leq \alpha\lambda \\ t_i - \alpha\lambda, & t_i > \alpha\lambda \end{cases} \tag{2.20}$$

and, as seen in its graphical representation in Figure 2.3, it maps small values to zero. Equation (2.18) can then be converted into an iterative scheme with an initial guess u_0 and aim at creating a sequence of iterates $\{u^k\}_{k>0}$ given through the update step

$$u^{k+1} = u^k + s^k\, \partial\Psi_\alpha(u^k), \tag{2.21}$$

where u^k is the iterate of the previous step, s^k a suitable step size and $\partial\Psi_\alpha(u^k)$ the search direction. This iteration is based on the idea of gradient descent but it is

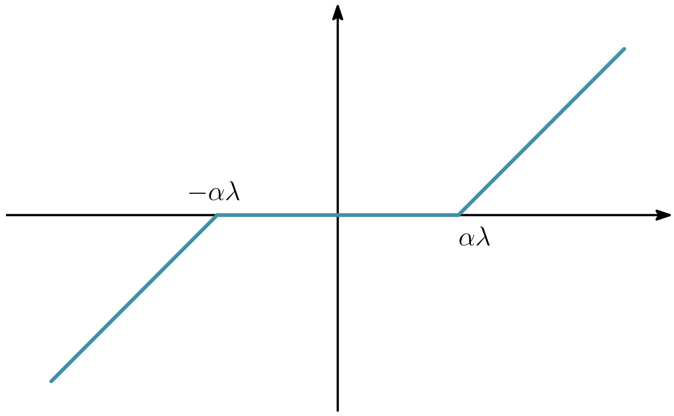


Figure 2.3: The 1D soft-thresholding operator $\mathcal{S}_{\alpha\lambda} := (\mathrm{id} + \alpha\lambda\,\mathrm{Sign})^{-1}$.

a generalization in the sense that most likely only the subdifferential of the functional can be computed. Note also that the above iteration differs from the classical Landweber [Lan51, Han91] as the nonlinear shrinkage operator is applied in each iteration. Such algorithmic schemes have been extensively used for the numerical treatment of inverse problems.

In [DDDM04] the so-called *iterative shrinkage-thresholding algorithm* (ISTA) was proposed for the minimization of Tikhonov functionals with sparsity constraints in the form of (2.12) with a linear operator. In this article, the authors proved that the sequence of iterates $\{u^k\}_{k>0}$ converges in norm to a minimizer of (2.12). Despite the efficiency of the algorithm, its convergence is rather slow (linear but with a constant close to 1). For this reason, an accelerated version called the *fast iterative shrinkage-thresholding algorithm* (FISTA) was proposed in [BT09], which has significantly better global convergence rate than ISTA. In the past years FISTA has been a widely-used optimization algorithm and further modifications where proposed for enhancing its practical performance, e.g., see [LLS18, CNXY19]. Lastly, another variation is the *iterated hard-thresholding algorithm* which has a slower convergence rate than ISTA but in practice is often faster, for further reading see [BL08a]. Regarding nonlinear operator equations, extensions of ISTA have been considered, for instance, in [RT06, BLM09].

Apart from the proximal-based algorithms, other methods exist, such as the *semi-smooth Newton* (SSN) [Ulb02], which follows an active-set strategy. A semi-smooth Newton method was used for the minimization of Tikhonov functionals with sparsity constraints in [GL08]. The theoretical analysis of this algorithm uses the notion of slant (Newton) differentiability and there, too, the thresholding operator was used. The main difference to the previous methods is that u^{k+1} depends on the previous iterate u^k only on the active-set. More information on this method is given in the next section as a regularized version of the SSN (RSSN) has been used for the minimization of the elastic net functional, which is a stable extension of the sparsity-promoting Tikhonov functional [JLS09, Sch10].

In the numerical examples that will be presented in Chapter 5, we use an iterative subgradient algorithm that has been proposed in [GMPK20] for the numerical minimization of convex but not necessarily smooth functionals. The characteristic of this method is the adaptive decreasing step size, which is chosen during the loop without any smoothness restriction on the functional. The theoretical analysis and a pseudo code of the algorithm are provided in [GMPK20].

2.5 Elastic net regularization

The elastic net is a regularization method originally used in the framework of statistics in [ZH05]. The goal of this approach is the same as in ℓ^1-regularization, i.e., retrieving sparse solutions. However, the motivation for its development comes from the observation that in ℓ^1-regularization (*lasso* in statistics) the functional in (2.13), fails to identify groups of highly correlated features and instead, it selects only one variable per group. The authors of [ZH05] proposed the addition of an ℓ^2-regularization term in the classical sparsity regularization functional for recovering the entire group (group selection). Their work proved that the elastic net outperforms the lasso while still recovering sparse solutions. Moreover, they showed that the elastic net is a stabilized version of the lasso, which motivates its use even more as the ℓ^1-regularization shows has numerical instabilities.

In [DMDVR09], the elastic net functional was investigated within the framework of statistical learning theory, more specifically, in supervised learning. In this work, nonparametric regression with random design was considered, with the assumption that the regression function has a sparse representation on the presumed dictionary (set of features). The authors proved that when the amount of available data increases, the elastic net estimator is consistent for prediction but also for feature selection. In addition to the statistical properties, an iterative thresholding algorithm for computing the solution of the elastic net functional was also derived in the same paper. For more details on the actual setting and results, refer to [DMDVR09]. Despite the fact that the present thesis does not fall into the area of statistical learning, we refer to this approach to acknowledge that the elastic net regularization originally was used in this field.

Returning to the viewpoint of regularization theory in inverse problems, we have already discussed in the previous section that the minimizer of the functional defined in (2.13) is sparse. Here, we consider a linear and continuous operator $K : U \to V$ between Hilbert spaces. The idea of the elastic net is to add an ℓ^2-penalty in Ψ for stabilization. This was done in [JLS09], where the authors investigated the theoretical properties of the elastic net functional

$$\Phi_{\alpha,\beta}(u) := \frac{1}{2}\|Ku - v^\delta\|^2 + \alpha\|u\|_{\ell^1} + \frac{\beta}{2}\|u\|_{\ell^2}^2, \tag{2.22}$$

and proved the stability and consistency of its minimizer $u_{\alpha,\beta}^\delta := \arg\min \Phi_{\alpha,\beta}(u)$. Moreover, they proved convergence rates for both a priori and a posteriori parameter choice rules under appropriate source conditions. Particularly for proving convergence rates, a linear coupling between α and β, namely, $\alpha = \eta\beta$ for $\eta > 0$ was assumed. However, it was shown that asymptotically linear coupling can be a sufficient assumption, too. The choice $\beta \sim \mathcal{O}(\delta)$ leads to the convergence rate $\|u_{\alpha,\beta}^\delta - u^\dagger\|_{\ell^2} = \mathcal{O}(\sqrt{\delta})$. If in addition, the operator K satisfies the FBI property introduced in [BL08b], the rate of convergence improves to $\mathcal{O}(\delta)$. The authors concluded that the elastic net simultaneously preserves the convergence rate of both the classical Tikhonov and that of ℓ^1-regularization.

In [JLS09], the authors write the optimality condition $-K^*(Ku - v^\delta) - \beta u \in \alpha\,\mathrm{Sign}(u)$ of $\Phi_{\alpha,\beta}$ in the equivalent form

$$\mathcal{F}(x) = \beta u - \mathcal{S}_a(-K^*(Ku - v^\delta)) = 0 \tag{2.23}$$

with the soft-thresholding operator $\mathcal{S}_\alpha$ from (2.20). The operator $\mathcal{S}_a$ is Newton differentiable and (2.23) is solved using a semi-smooth Newton method [CNQ00]. In connection to the previous section where we discussed numerical methods for the minimization of Tikhonov functionals, two adapted iterative algorithms for the minimization of (2.22) were developed in [JLS09]. These are the regularized semi-smooth Newton (RSSN) and the regularized feature sign search (RFSS). Both algorithms follow the idea of active set methods as they use an active set $A_u := \{i \in \mathbb{N} : \left|K^*(Ku - v^\delta)\right|_i > \alpha\}$ for specifying a Newton derivative of $\mathcal{F}$. A new active set is computed at each iteration, which means that regularization happens only on the active set each time. For further details on these algorithms, refer to [JLS09].

2.6 Tolerances: from SVR to inverse problems

In this section we discuss the idea of support vector regression (SVR) and its use for the solution of inverse problems. As will be seen in the following few pages, Vapnik's ε-insensitive distance used as a loss function in SVR and Tikhonov-type functionals, strongly motivates the main chapters of this thesis. Here we give an outline for the use of the ε-insensitive distance in SVR, but, in our analysis, we refer to it as the *tolerance function*.

2.6.1 The idea of SVR

Support vector regression originates from the support vector machines (SVM) introduced in the framework of statistical machine learning [Vap95]. In this context, it has been used for classification tasks, pattern recognition and optical character recognition, for further reading see [CV95, SBV95, Bur98, SS02]. The goal in machine learning is to predict an unknown function from a set of discrete noisy measurements (training data). The predicted function should be a good fit for the training data but should also be able to generalize on data that has not been seen yet [HTF09]. Compared to the classical (least-squares) regression, where the goal is to minimize the error of fitting the data, SVR offers more flexibility as it includes an error margin ε. This margin allows us to specify what error is acceptable in the model and then the task is to find an appropriate line or hyperplane that fits the data. The parameter ε is crucial to avoid over fitting to the noisy data, and for this reason, SVR is also often called ε-support vector regression (ε-SVR) [Vap95].

In [SS04], the authors present a tutorial for SVR using the following simple example: given training data $\{(x_1, y_1), \ldots, (x_\ell, y_\ell) \subset X \times \mathbb{R}\}$ with space of input patterns $X = \mathbb{R}^d$, determine a function $f(x)$ that has a deviation no bigger than ε from the actual points y_i for all training data and is as flat as possible. This is described as finding a linear function f of the form

$$f(x) = \langle w, x \rangle + b, \tag{2.24}$$

where $x \in X$ and $b \in \mathbb{R}$. The requirement for f to be as flat as possible can be translated into minimizing w which leads to the following convex minimization problem

$$\text{minimize} \quad \frac{1}{2}\|w\|^2 \tag{2.25}$$

$$\text{subject to} \quad \begin{cases} y_i - \langle w, x_i \rangle - b \leq \varepsilon \\ \langle w, x_i \rangle + b - y_i \leq \varepsilon \end{cases} \tag{2.26}$$

in which ε depicts the precision of the approximation. To ensure that the problem has a feasible solution, slack variables ξ_i, ξ_i^* are introduced and the problem is reformulated as

$$\text{minimize} \quad \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{\ell}(\xi_i + \xi_i^*) \tag{2.27}$$

$$\text{subject to} \quad \begin{cases} y_i - \langle w, x_i \rangle - b & \leq \varepsilon + \xi_i \\ \langle w, x_i \rangle + b - y_i & \leq \varepsilon + \xi_i^* \\ \xi_i, \xi_i^* & \geq 0 \end{cases} \tag{2.28}$$

with $C > 0$ controlling the desired flatness of f and deviations larger than ε. As stated in [SS04], this problem involves the ε-insensitive loss function (introduced by

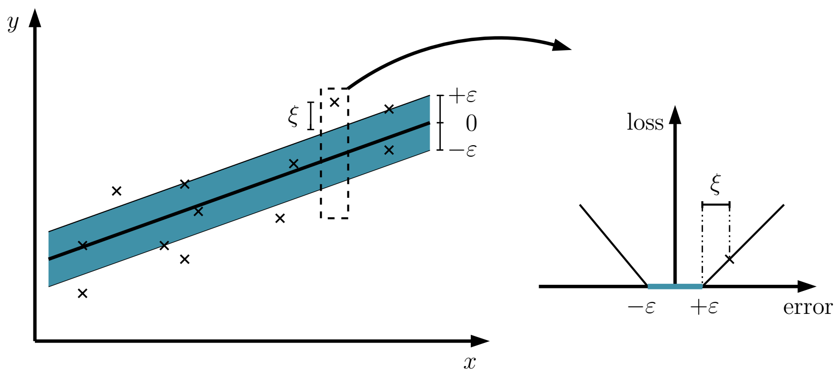


Figure 2.4: Graphic interpretation of the ε margin loss in linear SVR, reproduced from [SS04].

Vapnik in [Vap95]) given by

$$|\xi|_\varepsilon := \begin{cases} |\xi| - \varepsilon, & |\xi| > \varepsilon \\ 0 & |\xi| \leq \varepsilon \end{cases} \tag{2.29}$$

and Figure 2.4 illustrates the use of the ε-insensitive distance as a loss function and, as seen on the right part of the figure, errors inside the ε margin are ignored. In this section we do not continue with the details of this example as it is fully studied in the main reference. The basic idea is to write the dual formulation of the problem and solve it as a convex optimization problem. The important observation is that by writing the dual formulation (2.27), the so-called *support vector expansion* for w can be derived, meaning that it can be entirely described as a linear combination of the training patterns x_i. In addition, the support vector expansion is *sparse* with respect to x_i and the non-vanishing coefficients are the so-called *Support Vectors*.

As a conclusion to the main principles of SVR it is important to mention that the extension to nonlinear SVR is also possible. With regard to the above example such an extension could be to assume that w is indirectly given through a kernel function. Lastly, the use of the ε-insensitive loss is not the only possibility for SVR. In [Vap95], Vapnik suggests that the squared ε-insensitive function $|\cdot|_\varepsilon^2$ can be used. Further options can also be found in [SS04].

2.6.2 Solution of linear integral equations with SVR

The idea of support vector regression can also be applied in the framework of ill-posed inverse problems. Here, we focus on the work of Krebs in [Kre11], where he used SVR for the solution of linear integral equations. He considered a problem of the form $Ku = v$ with compact linear integral operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ defined as

$$Ku(x) = \int_{\Omega} k(x,t)\; u(t)\; \mathrm{d}t \tag{2.30}$$

over a bounded domain $\Omega \subset \mathbb{R}^d$ and Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. The choice of the integral kernel $k : \Omega \times \Omega \to \mathbb{R}$ can lead to an operator K with infinite-dimensional range and make the inverse problem ill-posed.

For the right hand side of the problem $Ku = v$, a semi-discrete setting was assumed in which the data v are given only on a discrete set of pairwise distinct data points $X = \{x_1, \dots, x_n\} \subset \Omega$. The data vector is assumed to be noisy with entries $v_j^\delta \approx v(x_j)$ satisfying $\left|v(x_j) - v_j^\delta\right| \le \delta$, for $j = 1, \dots, n$.

For regularizing the problem, Krebs considered a Tikhonov functional whose discrepancy term is measured using the ε-insensitive distance with a minimization problem given by

$$\min_{u \in \mathcal{H}_1} \left\{ \sum_{i=1}^{n} \left| Ku(x_i) - v_i^\delta \right|_\varepsilon + \alpha \|u\|_{\mathcal{H}_1}^2 \right\}, \tag{2.31}$$

for an operator K as in (2.30). Throughout [Kre11], $\mathcal{H}_1$ is assumed to be a reproducing kernel Hilbert space (RKHS) of continuous functions $u : \Omega \to \mathbb{R}$ with dual space $\mathcal{H}_1^*$. Roughly speaking, RKHS have the property that if two functions $u_1, u_2 \in \mathcal{H}_1$ are close in norm then they are also close in a pointwise sense. That is, if $\|u_1 - u_2\|_{\mathcal{H}_1}$ is small then $|u_1(x) - u_2(x)|$ is small too, for all $x \in \Omega$. The formal definition of a RKHS can be found in [Kre11, Definition 2.1].

For understanding why such a space setting is of interest, it is worth mentioning that RKHS are an important class of spaces in *statistical learning theory* because of the so-called *representer theorem* [SHSW00]. This states that the minimizer of a regularized empirical risk functional defined over an RKHS can be represented as a finite linear combination of kernel products evaluated on the input points in the training data. In other words, the minimization problem can be simplified from an infinite-dimensional to a finite-dimensional one and can be solved as a quadratic optimization problem. Following this idea with the additional assumptions that K is injective and that $v \in \mathrm{rg}(K) := \{Ku \mid u \in \mathcal{H}_1\}$, Krebs considered the semi-discrete operator

$$K_X : \mathcal{H}_1 \to \mathbb{R}^n, \qquad (K_X u)_i := Ku(x_i), \qquad 1 \le i \le n, \tag{2.32}$$

on the set of pairwise distinct data points X, for which the semi-discrete version of the integral equation (with exact data) is $K_X u = v_X$.

In this semi-discrete setting, the solution of the problem (2.31) is attained in a finite dimensional space and can be computed through the solution of a linear system. All theoretical results on error bounds and parameter selection as well as numerical examples, are found in [Kre10] and [Kre11], and will not be further analyzed here. The main conclusion is that the cut-off parameter ε ensures stability in a stronger way than the regularization parameter α. In addition, the error estimates presented by Krebs do not require a priori smoothness information of the solution and, for $\varepsilon \geq \delta$, the error behavior is the same both for exact and noisy data.

2.6.3 Tolerances in Tikhonov's discrepancy term

Except the semi-discrete setting for linear integral operators in RKHS used in Krebs' work, an extension to a continuous, nonlinear setting is possible. In [GPKM18] and [GMPK20] the authors consider an altered Tikhonov functional for the solution of nonlinear ill-posed inverse problems. The ε-insensitive distance is used in the same manner as by Krebs in [Kre11], i.e., for neglecting small deviations in the data fitting term of the functional. A nonlinear operator $F : U \to V$ is considered with U being a Hilbert space and $V = L_p(\Omega)$ for $p \in [1, 2]$ and $\Omega \subset \mathbb{R}^n$ a closed and bounded set, and the proposed Tikhonov functional is defined as

$$\mathcal{T}^{\delta}_{\alpha,\varepsilon}(u) := \|F(u) - v^{\delta}\|^p_{p,\varepsilon} + \alpha\mathcal{R}(u),$$

for general nonnegative convex regularization term $\mathcal{R}(u)$. The authors extend the definition of the ε-insensitive distance by introducing the $L_{p,\varepsilon}$-insensitive measure for measuring the data fitting error and present a set of useful properties and inequalities that this measure fulfills. Some of the proved inequalities are especially useful in our work and we will further refer to [GMPK20] in the next chapters when needed. The authors prove analytical results on the behavior of minimizers of $\mathcal{T}^{\delta}_{\alpha,\varepsilon}$ following the regularization theory of nonlinear inverse problems and also test their theory on numerical examples for deblurring and denoising of signals and compare their results to those of other existing methods. For further details we refer the reader to [GMPK20].

With the above, we have covered the existing theory that is relevant for our work and we are ready to continue with the main topic of our work, namely, the theoretical analysis of Tikhonov functionals with tolerances in the regularization term.

Chapter

3 Tikhonov regularization with tolerances

In this chapter, we consider an altered Tikhonov functional for the solution of an inverse problem. Motivated by the methods discussed in Section 2.6, we investigate the use of the ε-insensitive distance in the regularization term for tolerating small deviations in the solution within the area prescribed by the chosen *tolerance value* ε. In Section 3.1, we begin by defining the specific tolerance measure and examining its properties that are needed for the subsequent analysis. Then, in Section 3.2 we define the regularization term for our altered Tikhonov functional which is defined in Section 3.3 and focus on the theoretical analysis of minimizers for such a functional. In Section 3.4 we present results on the well-posedness, that is, the existence, stability and convergence of minimizers. In the last section of this chapter, we prove convergence rates for the Tikhonov functional with tolerances in the penalty term. In our theoretical analysis we distinguish between two cases for the assumed tolerances: a positive constant and positive vanishing tolerances in the limit.

3.1 Tolerance function

The ε-*insensitive distance* was first introduced by Cortes and Vapnik [CV95]. As discussed in Section 2.6, this measure has been used as a loss function in support vector regression [CV95] as well as for the solution of linear inverse problems in reproducing kernel Hilbert spaces [Kre11]. Here, we use the ε-insensitive distance in the regularization term of the proposed Tikhonov functional. We begin with the classical definition on the real numbers and then we extend it to L_q-function spaces for $1 \leq q \leq 2$, that we will use to analyze the inverse problem.

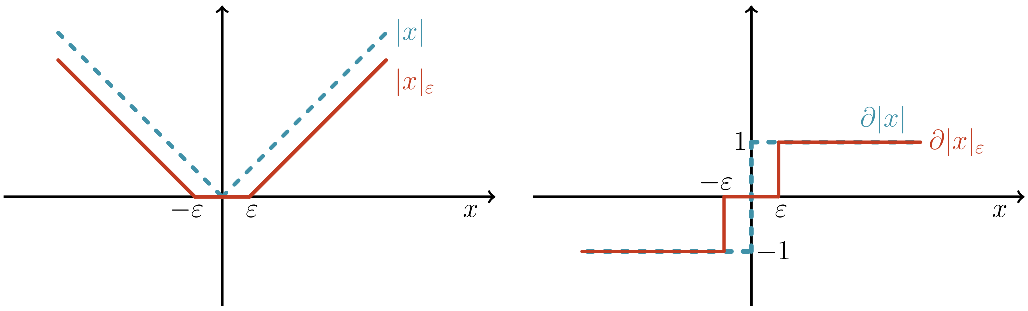


Figure 3.1: The tolerance function $|x|_\varepsilon$ in comparison to the absolute value $|x|$ (left) and their corresponding subdifferentials (right) for $x \in \mathbb{R}$ and $\varepsilon > 0$.

Notation. The ε-insensitive distance as introduced in [CV95] is denoted by $|\cdot|_\varepsilon$. For notational simplicity, we will denote it by $d_\varepsilon(\,\cdot\,)$ and refer to it as the *tolerance function*. Its formula was already introduced in (2.29) but for the sake of completeness of this chapter we include it in the following definition.

Definition 3.1 (ε-insensitive distance)**.** Let $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_0^+$. The ε-insensitive distance is defined as

$$d_\varepsilon(x) := |x|_\varepsilon = \max\{|x| - \varepsilon, 0\}. \tag{3.1}$$

Figure 3.1 shows the d_ε as given in (3.1) in comparison to the absolute value function (left), and their subdifferentials (right). As can be seen, the subdifferential of the ε-insensitive distance differs from that of the sign function only in that the zero value is achieved for all x in the interval $[-\varepsilon, \varepsilon]$. Apart from this classical definition, we can consider ε to be a sequence or a function defined in a corresponding sequence or function space. As in [GMPK20] we define the ε-modulus function in $\mathbb{R}^n$, as well as in L_q over $\mathbb{R}^n$ for $q \in [1, 2]$. Then, we further extend to the $L_{q,\varepsilon}$-insensitive measure which is the one to use throughout this thesis.

Definition 3.2 (ε-modulus function, [GMPK20])**.** For $0 < \varepsilon \in \mathbb{R}^n$ the ε-insensitive modulus $d_{\varepsilon,n} : \mathbb{R}^n \to \mathbb{R}^n$ is defined component-wise as

$$d_{\varepsilon,n}(x)_i := d_{\varepsilon_i}(x_i), \quad \text{for } i = 1, \ldots, n. \tag{3.2}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded set and $u \in L_q(\Omega)^n$. For $\varepsilon : \Omega \to \mathbb{R}^n$ with $0 < \varepsilon \in L_q(\Omega)^n$ the ε-insensitive modulus function $d_{\varepsilon,\Omega} : L_q(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ is given by

$$d_{\varepsilon,\Omega}(u)(\,\cdot\,) := d_{\varepsilon,n}(u(\,\cdot\,)). \tag{3.3}$$

The previous definition is well defined, since for $u, \varepsilon \in L_q(\Omega)^n$, there holds $d_\varepsilon(u) \in L_q(\Omega)^n$ and $d_\varepsilon(u_1)$, $d_\varepsilon(u_2)$ differ at most on a set of measure zero if u_1, u_2 belong to the same equivalence class $u \in L_q(\mathbb{R}^n)$. Moreover, we should note that (3.1) is evaluated component-wise and we can further use this definition (again component-wise) inside the L_q-induced norm. This enables the construction of a measure in $L_q(\Omega)$, which is given in the following definition.

Definition 3.3 ($L_{q,\varepsilon}$-insensitive measure [GMPK20]). Let $\Omega \subset \mathbb{R}^n$ be a bounded and closed set and $0 < \varepsilon \in L_q(\Omega)$. The $L_{q,\varepsilon}$-insensitive measure on $L_q(\Omega)$ is defined by

$$\|u\|_{L_{q,\varepsilon}} := \left\|u\big|_{L_q(\Omega)}\right\|_\varepsilon = \|u\|_{q,\varepsilon} = \left(\int_\Omega d_\varepsilon(u(x))^q \,\mathrm{d}x\right)^{\frac{1}{q}} < \infty. \tag{3.4}$$

For notational simplicity, the $L_{q,\varepsilon}$-insensitive measure will often be denoted by $\|\cdot\|_{q,\varepsilon}$ instead of $\|\cdot\|_{L_{q,\varepsilon}}$. In addition, the set Ω is always assumed to be bounded. In [GMPK20] the authors emphasize that d_ε does not satisfy the triangle inequality and therefore, it is not a distance function based on the classical definition of a distance. However, among other properties they prove the following inequalities that are useful in our analysis, too.

Proposition 3.4 (Proposition 1, [GMPK20]). *The $L_{q,\varepsilon}$-insensitive measure, for $\varepsilon \in L_q(\Omega)$, satisfies*

$$\|u\|_{q,\varepsilon} \leq \|u\|_q, \tag{3.5a}$$

$$\|u\|_q \leq \|u\|_{q,\varepsilon} + \|\varepsilon\|_q, \tag{3.5b}$$

$$\|u\|_q^q \leq \|u\|_{q,\varepsilon}^q + c\|\varepsilon\|_q^q, \quad \textit{with } c = q \max\left\{1, \|u\|_{q,\varepsilon}^{q-1}, \|\varepsilon\|_q^{q-1}\right\} \tag{3.5c}$$

for all $u \in L_q(\Omega)$ and $q \in [1, 2]$.

In addition, the $L_{q,\varepsilon}$-insensitive measure is continuous, convex for $q \geq 1$, and strictly convex for $q > 1$. From (3.5a) it follows easily that $d_\varepsilon(u) \in L_q(\Omega)$. Regarding the Definition 3.3 for L_q-spaces, in some of the proofs that follow, it is convenient to split the set Ω into two subsets, containing the elements of the nonzero or zero part of the insensitive function, when applied to an element $u \in L_q(\Omega)$.

Notation. Let $u \in L_q(\Omega)$ and $\varepsilon \in \mathbb{R}_0^+$. We divide Ω into two disjoint sets

$$\begin{aligned} \Omega_\varepsilon &:= \{x \in \Omega : |u(x)| > \varepsilon\}, \\ \Omega_0 &:= \{x \in \Omega : |u(x)| \leq \varepsilon\} \end{aligned} \tag{3.6}$$

such that $\Omega = \Omega_\varepsilon \cup \Omega_0$ and $\Omega_\varepsilon \cap \Omega_0 = \emptyset$. Note that the definition of these sets is not unique but rather depends on the chosen representative u out of the space of equivalence classes $L_q(\Omega)$.

Using the sets Ω_ε and Ω_0, we observe that

$$\|u\|_{q,\varepsilon}^q = \int_\Omega |u(x)|_\varepsilon^q \,\mathrm{d}x = \int_{\Omega_\varepsilon} |u(x)|_\varepsilon^q \,\mathrm{d}x + \int_{\Omega_0} |u(x)|_\varepsilon^q \,\mathrm{d}x, \tag{3.7}$$

from which we have that $\int_{\Omega_0} |u(x)|_\varepsilon^q \,\mathrm{d}x = 0$. In the results that follow we assume a constant tolerance $\varepsilon > 0$ if not stated otherwise. However, all proofs can be reproduced with ε being either a sequence or a function, as long as it is considered to be positive and bounded.

3.2 Regularization term with tolerances

Having gathered all necessary properties of the tolerance function, we continue with the definition of the regularization term $\mathcal{R}_{q,\varepsilon}$ that will be used in our Tikhonov functional and we prove two important properties for it.

Proposition 3.5. *We consider $u \in L_q(\Omega)$ for bounded $\Omega \subset \mathbb{R}^n$ and $1 \le q \le 2$, $\varepsilon > 0$ fixed. The regularization functional*

$$\mathcal{R}_{q,\varepsilon}(u) := \|u\|_{q,\varepsilon}^q = \int_\Omega (\max\{|u(x)| - \varepsilon, 0\})^q \,\mathrm{d}x \tag{3.8}$$

fulfills the following two properties:

a. $\mathcal{R}_{q,\varepsilon}$ is weakly lower semi-continuous.

b. $\mathcal{R}_{q,\varepsilon}$ is coercive.

Proof. We prove the first claim using Proposition 2.8, that is, we show that the epigraph of $\mathcal{R}_{q,\varepsilon}$

$$\mathrm{epi}(\mathcal{R}_{q,\varepsilon}) = \{(u,\lambda) \in L_q(\Omega) \times \mathbb{R} : \mathcal{R}_{q,\varepsilon}(u) \le \lambda \;\text{ for some } \lambda \in \mathbb{R}\}$$

is closed. Let $\{(u_k,\lambda_k)\}_{k\in\mathbb{N}} \subset \mathrm{epi}(\mathcal{R}_{q,\varepsilon})$ such that $\{(u_k,\lambda_k)\}_{k\in\mathbb{N}} \to (\bar{u},\bar{\lambda})$, that is, $u_k \to \bar{u}$ and $\lambda_k \to \bar{\lambda}$. Due to the choice of u_k and λ_k we have that

$$\bar{\lambda} = \lim_{k\to\infty} \lambda_k = \liminf_{k\to\infty} \lambda_k \ge \liminf_{k\to\infty} \mathcal{R}_{q,\varepsilon}(u_k). \tag{3.9}$$

Taking a closer look at the right hand side of (3.9) and by using the sets $\Omega_\varepsilon, \Omega_0$ as defined in (3.6), we obtain

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{R}_{q,\varepsilon}(u_k) &= \liminf_{k\to\infty} \int_\Omega (\max\{|u_k(x)| - \varepsilon, 0\})^q \,\mathrm{d}x \\ &= \liminf_{k\to\infty} \left[\int_{\Omega_\varepsilon} (|u_k(x)| - \varepsilon)^q \,\mathrm{d}x + \int_{\Omega_0} 0 \,\mathrm{d}x\right].\end{aligned} \tag{3.10}$$

Employing Fatou's lemma, we swap the limit inferior, which is super-additive, and the integrals, which yields

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{R}_{q,\varepsilon}(u_k) &\geq \int_{\Omega_\varepsilon} \liminf_{k\to\infty} (|u_k(x)| - \varepsilon)^q \,\mathrm{d}x + \int_{\Omega_0} 0 \,\mathrm{d}x \\ &= \int_{\Omega_\varepsilon} (|\bar{u}(x)| - \varepsilon)^q \mathrm{d}x + \int_{\Omega_0} 0 \,\mathrm{d}x \\ &= \int_{\Omega} |\bar{u}(x)|_\varepsilon^q \mathrm{d}x = \mathcal{R}_{q,\varepsilon}(\bar{u}). \end{aligned} \tag{3.11}$$

The expression (3.11) is obtained by using the fact that $u_k \to \bar{u}$ and by gathering again the individual integrals into one integral over the entire set Ω. Combining (3.9) and (3.11), we conclude that $\mathcal{R}_{q,\varepsilon}(\bar{u}) \leq \lambda$, which means that $(\bar{u}, \bar{\lambda}) \in \mathrm{epi}(\mathcal{R}_{q,\varepsilon})$. Since the initial sequence $\{(u_k, \lambda_k)\}_{k\in\mathbb{N}}$ was arbitrarily chosen, it follows that the epigraph of $\mathcal{R}_{q,\varepsilon}$ is closed. This is equivalent to saying that $\mathcal{R}_{q,\varepsilon}$ is lower semi-continuous. Together with the fact that $\mathcal{R}_{q,\varepsilon}$ is convex, we conclude that $\mathcal{R}_{q,\varepsilon}$ is weakly lower semi-continuous.

The coercivity of $\mathcal{R}_{q,\varepsilon}$ follows directly from the boundedness of Ω and the inequality (3.5b) by observing that $\lim_{\|u\|\to\infty} \mathcal{R}_{q,\varepsilon}(u) \to \infty$. ■

3.3 Tikhonov functional with tolerances in the penalty

With the basic properties of the regularization term proven, we now proceed with the main part of this thesis. We want to solve a problem of the form

$$F(u^\dagger) = v, \tag{3.12}$$

with nonlinear operator $F : \mathrm{dom}(F) \subset U \to V$, true solution $u^\dagger \in U$ and (exact) data $v \in V$. For the rest of this chapter we assume $U := L_q(\Omega)$ over a bounded set Ω with $1 \leq q \leq 2$ and V a reflexive Banach space. Moreover, we make the following assumption on the operator F.

Assumption 3.6 (On operator and minimization domain)**.** In the following we assume that:

(i) the operator $F : \mathrm{dom}(F) \subset U \to V$ is *weakly sequentially closed*. This means that for every sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{dom}(F)$ converging weakly to an element $u \in U$ such that $\{F(u_k)\}_{k\in\mathbb{N}}$ converges weakly to some $v \in V$, it holds that $u \in \mathrm{dom}(F)$ and $F(u) = v$.

(ii) the set $\mathcal{D} := \mathrm{dom}(F) \cap \mathrm{dom}(\mathcal{R}_{q,\varepsilon})$ is non-empty. This implies that $\mathcal{R}_{q,\varepsilon}$ is proper.

As discussed in Section 2.2, an ill-posed inverse problem can be solved using Tikhonov regularization. We aim at approximating the true solution of the problem by minimizing the functional

$$\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) := \|F(u) - v^\delta\|^p_V + \alpha \mathcal{R}_{q,\varepsilon}(u), \quad \alpha > 0 \tag{3.13}$$

with $\mathcal{R}_{q,\varepsilon}$ as in (3.8) for $\varepsilon > 0$ and noisy data $v^\delta \in V$ such that $\|v - v^\delta\|_V \leq \delta$. In the following, we consider the minimization of the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ only on $\mathcal{D}$, which means that $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) < \infty$ for $u \in \mathcal{D}$ and $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) = +\infty$ for $u \notin \mathcal{D}$.

As a remark we should mention here that in $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ as given in (3.13) one can assume a reference solution $u^* \in U$ in the regularization term, i.e., $\mathcal{R}_{q,\varepsilon}(u) = \|u - u^*\|^q_{q,\varepsilon}$. However, for the sake of simplicity, we consider u^* to be zero since it does not change our theoretical analysis. Moreover, in later results we use the notion of an $\mathcal{R}$-minimizing solution of $F(u) = v$. Depending on the regularization functional which is used in the proofs that follow, we refer to an $\mathcal{R}$-minimizing solution and denote it using the corresponding regularization functional.

Definition 3.7 ($\mathcal{R}$-minimizing solution)**.** An element $u^\dagger \in \mathcal{D}$ is called $\mathcal{R}$-minimizing solution, if $\mathcal{R}(u^\dagger) = \min\{\mathcal{R}(u) : F(u) = v, u \in \mathcal{D}\} < \infty$.

Following a similar approach as in [GHS08], the next lemma is crucial for obtaining results on the well-posedness of the minimizers of the Tikhonov functional defined in (3.13).

Lemma 3.8. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \operatorname{dom}(F)$. Assume that $\varepsilon > 0$ is fixed, $\{v_k\}_{k\in\mathbb{N}} \subset V$ is a bounded sequence and that there exist $\alpha > 0$ and $M > 0$ such that $\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u_k) < M$, for all $k \in \mathbb{N}$. Then, there exist $\tilde{u} \in \operatorname{dom}(F)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $u_{k_j} \rightharpoonup \tilde{u}$ and $F(u_{k_j}) \rightharpoonup F(\tilde{u})$.*

Proof. This proof is analogous to the proof of Lemma 4 found in [GHS08]. The coercivity of the regularization functional $\mathcal{R}_{q,\varepsilon}$ together with the estimate

$$\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u_k) = \|F(u_k) - v_k\|^p_V + \alpha \mathcal{R}_{q,\varepsilon}(u_k) \geq \alpha \mathcal{R}_{q,\varepsilon}(u_k). \tag{3.14}$$

implies that

$$\alpha \mathcal{R}_{q,\varepsilon}(u_k) \leq \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u_k) < M,$$

i.e., $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L_q(\Omega)$. Similarly, since $\{v_k\}_{k\in\mathbb{N}}$ is bounded in V it follows that the sequence $\{F(u_k)\}_{k\in\mathbb{N}}$ is bounded in V, too. Hence, there exist a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ and $\tilde{u} \in U$, $v \in V$ such that

$$u_{k_j} \rightharpoonup \tilde{u} \quad \text{and} \quad F(u_{k_j}) \rightharpoonup v.$$

Due to the fact that F is weakly sequentially closed, it follows that $\tilde{u} \in \operatorname{dom}(F)$ and $F(\tilde{u}) = v$. ∎

3.4 Well-posedness

In regularization theory usually we first study three essential results which are directly related to Hadamard's well-posedness. When considering the minimization of a functional, we begin by showing the existence of minimizers. Next, follows the stability of minimizers with respect to data perturbation. Moreover, we examine the convergence of the minimizer to the true solution of the problem when the noise level (in the data) tends to zero. In this section we prove these three important results for two different cases: for $\varepsilon > 0$ fixed and for $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{N}}$ considered to be a bounded sequence. In the following, we closely follow the concept in [GHS08, KNS08] as we investigate the behavior of the minimizers of $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$.

Theorem 3.9 (Existence of minimizers)**.** *Assume that $\varepsilon > 0$ is fixed. For $\alpha > 0$ and for every $v^\delta \in V$ the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ has a minimizer $u^\delta_{\alpha,\varepsilon}$ in $\mathcal{D} = \mathrm{dom}(F) \cap \mathrm{dom}(\mathcal{R}_{q,\varepsilon})$.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathrm{dom}(F)$ satisfy

$$\lim_{k\to\infty} \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u_k) = \inf\left\{\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) : u \in \mathcal{D}\right\}.$$

From Lemma 3.8, there exists a subsequence of u_k denoted by $\{u_{k_j}\}_{j\in\mathbb{N}}$, which is weakly convergent to some $\tilde{u} \in \mathrm{dom}(F)$ and is such that $F(u_{k_j}) \rightharpoonup F(\tilde{u})$. Since $\mathcal{R}_{q,\varepsilon}$ and the norm $\|\cdot\|^p_V$ are both weakly lower semi-continuous, and together with the fact that F is weakly sequentially closed, it follows that

$$\begin{aligned}\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(\tilde{u}) &= \|F(\tilde{u}) - v^\delta\|^p + \alpha\mathcal{R}_{q,\varepsilon}(\tilde{u}) \\ &\le \liminf_{j\to\infty} \|F(u_{k_j}) - v^\delta\|^p + \alpha \liminf_{j\to\infty} \mathcal{R}_{q,\varepsilon}(u_{k_j}).\end{aligned}$$

In the last term of the right hand side, we can transfer the regularization parameter α into the limit inferior. Moreover, by the super-additivity of the limit inferior, we conclude

$$\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(\tilde{u}) \le \liminf_{j\to\infty}\left\{\|F(u_{k_j}) - v^\delta\|^p + \alpha\mathcal{R}_{q,\varepsilon}(u_{k_j})\right\}.$$

Rewriting, we have

$$\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(\tilde{u}) \le \liminf_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u_{k_j}) \le \limsup_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u_{k_j}). \tag{3.15}$$

Now, we only need to observe that

$$\limsup_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u_{k_j}) \le \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u), \tag{3.16}$$

for any $u \in \mathrm{dom}(F)$. Combining equations (3.15) and (3.16) yields that $u^\delta_{\alpha,\varepsilon} := \tilde{u}$ is a minimizer of the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ in $\mathcal{D}$. ∎

Now that the existence of minimizers is settled, we discuss the uniqueness. In general, Tikhonov-type functionals associated with nonlinear operators are not (strictly) convex, and therefore, the uniqueness of its minimizers does not necessarily hold. In addition, (3.13) cannot be strictly convex also because of the regularization term, in which the norm has been relaxed by accepting more than one possible solutions within the prescribed tolerance area. Therefore, the uniqueness of minimizer of (3.13) is not guaranteed due to the lack of strict convexity.

Notation 3.10. For the sake of clarity, whenever v^δ is considered to be a sequence, the notation of the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ will be adjusted to include the respective sequence in its shorthand notation. For the sequence v_k, we will write $\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}$ for denoting the functional $\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u) := \|F(u) - v_k\|^p_V + \alpha\mathcal{R}_{q,\varepsilon}(u)$.

Theorem 3.11 (Stability for fixed $\varepsilon > 0$)**.** *Let $\alpha > 0$ and $\varepsilon > 0$ fixed. Assume that $\{v_k\}_{k\in\mathbb{N}} \subset V$ converges to some $v^\delta \in V$ and let $u_k \subset \mathcal{D}$ such that*

$$u_k \in \arg\min \left\{\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u) : u \in \mathcal{D}\right\}.$$

Then, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ which converges weakly to a minimizer $u^\delta_{\alpha,\varepsilon}$ of the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$. Moreover, we have that $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to \mathcal{R}_{q,\varepsilon}(u^\delta_{\alpha,\varepsilon})$.

Proof. Since u_k is a sequence of minimizers of $\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}$, it holds that

$$\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u_k) \le \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u), \quad \forall\, u \in \mathcal{D}.$$

From Lemma 3.8, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ that is weakly convergent to some $\tilde{u} \in \operatorname{dom}(F)$ such that $F(u_{k_j}) \rightharpoonup F(\tilde{u})$. Moreover, from the weak lower semi-continuity of $\|\cdot\|^p_V$ and $\mathcal{R}_{q,\varepsilon}$ there holds

$$\|F(\tilde{u}) - v^\delta\|^p_V \le \liminf_{j\to\infty} \|F(u_{k_j}) - v_{k_j}\|^p_V, \tag{3.17}$$

$$\mathcal{R}_{q,\varepsilon}(\tilde{u}) \le \liminf_{j\to\infty} \mathcal{R}_{q,\varepsilon}(u_{k_j}). \tag{3.18}$$

Combining the above, we obtain

$$\begin{aligned}\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(\tilde{u}) &= \|F(\tilde{u}) - v^\delta\|^p_V + \alpha\mathcal{R}_{q,\varepsilon}(\tilde{u})\\ &\le \liminf_{j\to\infty} \|F(u_{k_j}) - v_{k_j}\|^p_V + \liminf_{j\to\infty} \alpha\mathcal{R}_{q,\varepsilon}(u_{k_j}),\end{aligned}$$

and using the super-additivity of the limit inferior and that $v_k \to v^\delta$, we have

$$\begin{aligned}\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(\tilde{u}) &\le \liminf_{j\to\infty} \left\{\|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha\mathcal{R}_{q,\varepsilon}(u_{k_j})\right\}\\ &= \liminf_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,v_{k_j},\varepsilon}(u_{k_j})\\ &\le \limsup_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,v_{k_j},\varepsilon}(u_{k_j})\\ &\le \lim_{k\to\infty} \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon}(u) = \mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u), \quad u \in \mathcal{D},\end{aligned} \tag{3.19}$$

which implies that $u^\delta_{\alpha,\varepsilon} := \tilde{u}$ is a minimizer of $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$.

Since (3.19) holds for $u \in \mathcal{D}$, we can take $u := \tilde{u}$ on the right hand side, which gives

$$\begin{aligned}\|F(\tilde{u}) - v^\delta\|^p_V + \alpha\mathcal{R}_{q,\varepsilon}(\tilde{u}) &= \lim_{j\to\infty} \mathcal{J}^{p,q}_{\alpha,v_{k_j},\varepsilon}(u_{k_j}) \\ &= \lim_{j\to\infty} \left\{\|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha\mathcal{R}_{q,\varepsilon}(u_{k_j})\right\}. \end{aligned} \tag{3.20}$$

Assume that $\mathcal{R}_{q,\varepsilon}(u_{k_j})$ does not converge to $\mathcal{R}_{q,\varepsilon}(\tilde{u})$. Because $\mathcal{R}_{q,\varepsilon}$ is w.l.s.c. it follows that

$$\mathcal{R}_{q,\varepsilon}(\tilde{u}) < \limsup_{j\to\infty} \mathcal{R}_{q,\varepsilon}(u_{k_j}) =: c.$$

Now, we consider u_{k_j} to be a subsequence such that $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to c$. From (3.20) we have

$$\begin{aligned}\lim_{j\to\infty} \|F(u_{k_j}) - v_{k_j}\|^p_V &= \|F(\tilde{u}) - v^\delta\|^p_V + \alpha(\mathcal{R}_{q,\varepsilon}(\tilde{u}) - c) \\ &< \|F(\tilde{u}) - v^\delta\|^p_V,\end{aligned} \tag{3.21}$$

which, however, contradicts the w.l.s.c. of $\|\cdot\|^p_V$ in (3.17). Therefore, we obtain $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to \mathcal{R}_{q,\varepsilon}(u^\delta_{\alpha,\varepsilon})$. ■

Remark 3.12. In [GHS08, Proposition 6], the authors additionally to $\mathcal{R}_q(u_{k_j}) \to \mathcal{R}_q(u^\delta_\alpha)$ prove that $\mathcal{R}_q(u_{k_j} - u^\delta_\alpha) \to 0$ for their functional $\mathcal{R}_q$. In our case, such a result cannot be inferred as weak convergence is not preserved under the nonlinearity of d_ε. That is, assuming a subsequence u_{k_j} of the sequence $u_k \rightharpoonup u^\delta_{\alpha,\varepsilon}$, we can conclude $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to \mathcal{R}_{q,\varepsilon}(u^\delta_{\alpha,\varepsilon})$ as done in the previous result. However, we cannot prove that $\mathcal{R}_{q,\varepsilon}(u_{k_j} - u^\delta_{\alpha,\varepsilon}) \to 0$. In order to obtain this norm convergence, one can further assume $d_\varepsilon(u_k) \rightharpoonup d_\varepsilon(u^\delta_{\alpha,\varepsilon})$ and arrive to such a result. In our work, we choose not to make this additional assumption as it is quite restrictive.

Theorem 3.13 (Weak convergence for fixed $\varepsilon > 0$)**.** *Let $\varepsilon > 0$ be fixed. Assume that $F(u) = v$ attains a solution in $\mathrm{dom}(\mathcal{R}_{q,\varepsilon})$ and that $\alpha : (0,\infty) \to (0,\infty)$ satisfies*

$$\alpha(\delta) \to 0 \text{ and } \frac{\delta^p}{\alpha(\delta)} \to 0, \text{ as } \delta \to 0.$$

Let $\delta_k \to 0$ and let $v_k \in V$ satisfy $\|v - v_k\| \leq \delta_k$. Moreover, let $\alpha_k := \alpha(\delta_k)$ and

$$u_k \in \arg\min \left\{\mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon}(u) : u \in \mathcal{D}\right\}.$$

Then, there exist an $\mathcal{R}_{q,\varepsilon}$-minimizing solution $u^\dagger$ of $F(u) = v$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $u_{k_j} \rightharpoonup u^\dagger$ with $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to \mathcal{R}_{q,\varepsilon}(u^\dagger)$.

Proof. Let $\tilde{u} \in \mathrm{dom}(\mathcal{R}_{q,\varepsilon})$ be any solution of $F(\tilde{u}) = v$. From the definition of u_k it follows that

$$\begin{aligned}\mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon}(u_k) &= \|F(u_k) - v_k\|_V^p + \alpha_k \mathcal{R}_{q,\varepsilon}(u_k) \\ &\leq \|F(\tilde{u}) - v_k\|_V^p + \alpha_k \mathcal{R}_{q,\varepsilon}(\tilde{u})\end{aligned}$$

Now, using the fact that $F(\tilde{u}) \;= v$ for some $v \in V$, we can substitute v in the last expression and then, we get the following estimate

$$\mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon}(u_k) \leq \|v - v_k\|_V^p + \alpha_k \mathcal{R}_{q,\varepsilon}(\tilde{u}) \leq \delta_k^p + \alpha_k \mathcal{R}_{q,\varepsilon}(\tilde{u}).$$

It can be easily seen that $\|F(u_k) - v_k\|_V^p \leq \mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon}(u_k) \leq \delta_k^p + \alpha_k \mathcal{R}_{q,\varepsilon}(\tilde{u})$. Together with the assumptions on α_k and δ_k we conclude that $\|F(u_k) - v_k\|_V \to 0$ as $k \to \infty$.

Similarly, for the penalty term we have $\mathcal{R}_{q,\varepsilon}(u_k) \leq \frac{\delta_k^p}{\alpha_k} + \mathcal{R}_{q,\varepsilon}(\tilde{u})$, from which we deduce

$$\limsup_{k\to\infty} \mathcal{R}_{q,\varepsilon}(u_k) \leq \mathcal{R}_{q,\varepsilon}(\tilde{u}), \tag{3.22}$$

using the definition of limit superior. We consider $\alpha_{\max} := \max\{\alpha_k : k \in \mathbb{N}\}$. From (3.22) it follows that there exists $M > 0$ such that

$$\limsup_{k\to\infty} \left\{ \|F(u_k) - v_k\|_V^p + \alpha_{\max} \mathcal{R}_{q,\varepsilon}(u_k) \right\} \leq M < \infty, \; \forall k \in \mathbb{N}.$$

Therefore, Lemma 3.8 guarantees the existence of a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ and an element $u^\dagger \in \mathrm{dom}(F)$ such that $u_{k_j} \rightharpoonup u^\dagger$ and $F(u_{k_j}) \rightharpoonup F(u^\dagger)$. Since

$$\begin{aligned}\|F(u_{k_j}) - v\| &= \|F(u_{k_j}) - v_{k_j} + v_{k_j} - v\| \\ &\leq \|F(u_{k_j}) - v_{k_j}\| + \|v_{k_j} - v\| \to 0,\end{aligned}$$

it follows that $\|F(u^\dagger) - v\| = 0$, i.e., $F(u^\dagger) = v$. From the weak lower semi-continuity of $\mathcal{R}_{q,\varepsilon}$ and the fact that (3.22) holds for any $\tilde{u} \in \mathrm{dom}(\mathcal{R}_{q,\varepsilon})$ solving $F(\tilde{u}) = v$, we conclude that

$$\mathcal{R}_{q,\varepsilon}(u^\dagger) \leq \liminf_{j\to\infty} \mathcal{R}_{q,\varepsilon}(u_{k_j}) \leq \limsup_{j\to\infty} \mathcal{R}_{q,\varepsilon}(u_{k_j}) \leq \mathcal{R}_{q,\varepsilon}(\tilde{u}).$$

This confirms that $u^\dagger$ is an $\mathcal{R}_{q,\varepsilon}$-minimizing solution (recall Definition 3.7) of $F(u) = v$ and moreover, that $\mathcal{R}_{q,\varepsilon}(u_{k_j}) \to \mathcal{R}_{q,\varepsilon}(u^\dagger)$. ■

3.4.1 Stability and convergence for vanishing tolerances

Now, we treat the special case of a nonnegative bounded tolerance sequence $\varepsilon_k \to 0$. We begin with the remark below, which states that in the limit point $\varepsilon = 0$ we actually obtain the classical Tikhonov minimizers as in [GHS08].

Remark 3.14. In the case $\varepsilon = 0$ the tolerance function yields $d_0(u) = \max\{|u|, 0\} = |u|$ and therefore,

$$\mathcal{R}_{q,0}(u) = \int_\Omega |\max\{u(x), 0\}|^q \, \mathrm{d}x = \int_\Omega |u(x)|^q \, \mathrm{d}x = \mathcal{R}_q(u).$$

This means that for $\varepsilon = 0$ we obtain the regularization functional without tolerances. The minimizer of $\mathcal{J}^{p,q}_{\alpha,\delta}(u) := \mathcal{J}^{p,q}_{\alpha,\delta,0}(u)$ will be denoted by $u^\delta_\alpha := u^\delta_{\alpha,0}$.

Theorem 3.15 (Stability for $\varepsilon_k \to 0$)**.** *Assume $\alpha > 0$. Let $\{v_k\}_{k\in\mathbb{N}}$ converge to $v^\delta \in V$, $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a bounded sequence converging to 0, and let*

$$u_k \in \arg\min \left\{ \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon_k}(u) : u \in \mathcal{D} \right\}.$$

Then, there exist $\{(\varepsilon_{k_j}, u_{k_j})\}_{j\in\mathbb{N}}$ and a minimizer u^δ_α of the functional $\mathcal{J}^{p,q}_{\alpha,\delta}$ such that $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \to \mathcal{R}_q(u^\delta_\alpha)$.

Proof. Since u_k is a minimizing sequence of $\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon_k}$ there holds

$$\mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon_k}(u_k) \leq \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon_k}(u), \quad \forall\, u \in \mathcal{D}.$$

From Lemma 3.8, there exists a subsequence of u_k, denoted by $\{u_{k_j}\}_{j\in\mathbb{N}}$, which converges to an element $\tilde{u} \in \operatorname{dom}(F)$ and is such that $F(u_{k_j}) \rightharpoonup F(\tilde{u})$. From the weak lower semi-continuity of $\|\cdot\|^p_V$ and $\mathcal{R}_{q,\varepsilon_k}$ and the fact that $\varepsilon_k \to 0$ and $v_k \to v^\delta$, we have that

$$\begin{aligned}
\mathcal{J}^{p,q}_{\alpha,\delta,0}(\tilde{u}) &\leq \liminf_{j\to\infty} \|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha \liminf_{j\to\infty} \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \\
&\leq \liminf_{j\to\infty} \left\{ \|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \right\} \\
&\leq \limsup_{j\to\infty} \left\{ \|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \right\} \\
&\leq \lim_{k\to\infty} \mathcal{J}^{p,q}_{\alpha,v_k,\varepsilon_k}(u) = \mathcal{J}^{p,q}_{\alpha,\delta}(u), \quad u \in \mathcal{D}.
\end{aligned} \tag{3.23}$$

Therefore, $u^\delta_\alpha := \tilde{u}$ is a minimizer of $\mathcal{J}^{p,q}_{\alpha,\delta}$. Since (3.23) holds for $u \in \mathcal{D}$, we can as well take $u := \tilde{u}$ in the limit on the right hand side, which gives

$$\|F(\tilde{u}) - v^\delta\|^p_V + \alpha \mathcal{R}_q(\tilde{u}) = \lim_{j\to\infty} \left\{ \|F(u_{k_j}) - v_{k_j}\|^p_V + \alpha \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \right\}. \tag{3.24}$$

Now, let us assume that $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j})$ does not converge to $\mathcal{R}_q(\tilde{u})$. Since $\mathcal{R}_{q,\varepsilon_{k_j}}$ is weak lower semi-continuous, it follows that

$$\mathcal{R}_q(\tilde{u}) < \limsup_{j\to\infty} \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) =: c.$$

We further consider u_{k_j} to be a subsequence such that $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \to c$. From (3.24) we have

$$\begin{aligned}\lim_{j\to\infty} \|F(u_{k_j}) - v_{k_j}\|_V^p &= \|F(\tilde{u}) - v^\delta\|_V^p + \alpha(\mathcal{R}_q(\tilde{u}) - c)\\ &< \|F(\tilde{u}) - v^\delta\|_V^p. \end{aligned} \tag{3.25}$$

The last expression, however, contradicts the weak lower semi-continuity of $\|\cdot\|_V^p$, therefore we obtain $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \to \mathcal{R}_q(u_\alpha^\delta)$. ■

In the following theorem we prove the convergence of minimizers to an $\mathcal{R}_q$-minimizing solution of the problem when assuming a bounded sequence of tolerances $\varepsilon_k \to 0$.

Theorem 3.16 (Weak convergence for $\varepsilon_k \to 0$)**.** *Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a bounded sequence, convergent to* 0*. We assume that $F(u) = v$ attains a solution in* $\mathrm{dom}(\mathcal{R}_{q,\varepsilon_k})$ *and that $\alpha : (0,\infty) \to (0,\infty)$ satisfies*

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^p}{\alpha(\delta)} \to 0, \quad \text{as } \delta \to 0.$$

Let $\delta_k \to 0$ and $v_k \in V$ satisfy $\|v - v_k\| \le \delta_k$. Moreover, let $\alpha_k = \alpha(\delta_k)$ and

$$u_k \in \arg\min\left\{\mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon_k}(u) : u \in \mathcal{D}\right\}.$$

Then, there exist an $\mathcal{R}_q$-minimizing solution $u^\dagger$ of $F(u) = v$ and a subsequence of u_k denoted by $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \to \mathcal{R}_q(u^\dagger)$.

Proof. Let $\tilde{u} \in \mathrm{dom}(\mathcal{R}_{q,\varepsilon_k})$ be any solution of $F(\tilde{u}) = v$. The minimizing property of u_k implies

$$\mathcal{J}^{p,q}_{\alpha_k,v_k,\varepsilon_k}(u_k) \le \delta_k^p + \alpha_k \mathcal{R}_{q,\varepsilon_k}(\tilde{u})$$

Then, it follows that $\|F(u_k) - v_k\|_V^p \le \delta_k^p + \alpha_k\mathcal{R}_{q,\varepsilon_k}(\tilde{u})$ and $\|F(u_k) - v_k\|_V^p \to 0$ for $k \to \infty$. Furthermore, for the penalty term we obtain

$$\alpha_k \mathcal{R}_{q,\varepsilon_k}(u_k) \le \delta_k^p + \alpha_k \mathcal{R}_{q,\varepsilon_k}(\tilde{u}),$$

or more clearly, that $\mathcal{R}_{q,\varepsilon_k}(u_k) \le \frac{\delta_k^p}{\alpha_k} + \mathcal{R}_{q,\varepsilon_k}(\tilde{u})$. The limit superior for $k \to \infty$ yields

$$\limsup_{k\to\infty} \mathcal{R}_{q,\varepsilon_k}(u_k) \le \limsup_{k\to\infty}\left\{\frac{\delta_k^p}{\alpha_k} + \mathcal{R}_{q,\varepsilon_k}(\tilde{u})\right\} = \mathcal{R}_{q,0}(\tilde{u}), \tag{3.26}$$

which is true for any solution $\tilde{u}$ of $F(\tilde{u}) = v$. Now, we assume $\alpha_1 := \max\{\alpha_k : k \in \mathbb{N}\}$. Hence, there exists a constant $M > 0$ such that

$$\limsup_{k\to\infty}\left\{\|F(u_k) - v_k\|_V^p + \alpha_1 \mathcal{R}_{q,\varepsilon_k}(u_k)\right\} \le M < \infty \quad \forall k \in \mathbb{N}.$$

Then, from Lemma 3.8, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ weakly convergent to some $u^\dagger \in \mathrm{dom}(F)$ such that $F(u_{k_j}) \rightharpoonup F(u^\dagger)$. Since

$$\begin{aligned}\|F(u_{k_j}) - v\|^p &= \|F(u_{k_j}) - v_{k_j} + v_{k_j} - v\|^p \\ &\leq 2^{p-1}\left(\|F(u_{k_j}) - v_{k_j}\|^p + \|v_{k_j} - v\|^p\right) \to 0,\end{aligned} \tag{3.27}$$

we finally obtain that $\|F(u^\dagger) - v\| \to 0$ as $k \to \infty$, that is, $F(u^\dagger) = v$. From the weak lower semi-continuity of $\mathcal{R}_{q,\varepsilon_k}$, the fact that $\varepsilon_k \to 0$, and (3.26), we derive that

$$\mathcal{R}_{q,0}(u^\dagger) \leq \liminf_{j\to\infty} \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \leq \limsup_{j\to\infty} \mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \leq \mathcal{R}_{q,0}(\tilde{u})$$

for all $\tilde{u}$ such that $F(\tilde{u}) = v$. Using Remark 3.14, we conclude that $\mathcal{R}_q(u^\dagger) \leq \mathcal{R}_q(\tilde{u})$. Hence, $u^\dagger$ is an $\mathcal{R}_q$-minimizing solution of $F(u) = v$. Because of the w.l.s.c. of $\mathcal{R}_{q,\varepsilon_{k_j}}$ and $\|\cdot\|_V$, we have $\mathcal{R}_{q,\varepsilon_{k_j}}(u_{k_j}) \to \mathcal{R}_q(u^\dagger)$. ■

3.5 Convergence rates

In this section we present convergence rates results for the minimizers of the functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ defined in (3.13). We aim at computing an estimate for the distance between the regularized solution $u^\delta_{\alpha,\varepsilon}$ and the true solution $u^\dagger$. This estimate is made using the Bregman distance, which is a measure often adopted by many authors in Banach space settings and it basically measures the gap between a functional and its linearization. The most standard convergence rates results in Bregman distances can be found in [HKPS07, BRH07, GHS08, JM12], as well as in [SKHK12] where an overview of important properties of the Bregman distance and its use in the convergence rates analysis is presented. Regarding the estimate of the distance between $F(u^\delta_{\alpha,\varepsilon})$ and v^δ we use the usual norm of V.

We first state the subdifferential of the regularization term $\mathcal{R}_{q,\varepsilon}$ since it appears in the definition of the Bregman distance. Recalling Definition 2.9 the subdifferential of $\mathcal{R}_{q,\varepsilon} : L_q(\Omega) \to \mathbb{R}$ at an element $u \in L_q(\Omega)$ is given by

$$\begin{aligned}\partial\mathcal{R}_{q,\varepsilon}(u) = \Big\{ z \in L_q(\Omega)^* :\ &\text{for all } w \in L_q(\Omega), \\ &\mathcal{R}_{q,\varepsilon}(w) \geq \mathcal{R}_{q,\varepsilon}(u) + \langle z, w-u\rangle_{L_q(\Omega)^*\times L_q(\Omega)} \Big\},\end{aligned}$$

where $L_q(\Omega)^*$ denotes the dual space of $L_q(\Omega)$ and $\langle\cdot,\cdot\rangle_{L_q(\Omega)^*\times L_q(\Omega)}$ the dual pairing between $L_q(\Omega)^*$ and $L_q(\Omega)$. For $q = 1$ the subdifferential is

$$\partial\mathcal{R}_{1,\varepsilon}(u) = \partial\|d_\varepsilon(u)\|_1 = \partial d_\varepsilon(u) = \begin{cases} \{-1\}, & u(x) < -\varepsilon \\ [-1,0], & u(x) = -\varepsilon \\ \{0\}, & |u(x)| < \varepsilon \\ [0,1], & u(x) = \varepsilon \\ \{1\}, & u(x) > \varepsilon \end{cases} \tag{3.28}$$

and for $q = 2$ we have

$$\partial\mathcal{R}_{2,\varepsilon}(u) = \partial\|d_\varepsilon(u)\|_2^2 = 2d_\varepsilon(u)\partial d_\varepsilon(u) = \begin{cases} 2[u(x)+\varepsilon], & u(x) < -\varepsilon \\ 0, & |u(x)| \le \varepsilon \\ 2[u(x)-\varepsilon], & u(x) > \varepsilon \end{cases}. \tag{3.29}$$

Note that in the above, the tolerance function is always applied component-wise. The detailed computation of these subdifferentials for $q = 1, 2$ is found in Appendix A. The previous formulas are confirmed in the generalized form of the subdifferential for $1 \le q \le 2$

$$\partial\mathcal{R}_{q,\varepsilon}(u) = \partial\|d_\varepsilon(u)\|^q = q\|d_\varepsilon(u)\|^{q-1}\partial d_\varepsilon(u), \tag{3.30}$$

where $\partial d_\varepsilon(u)$ is determined by (3.28).

Now that we have clarified the formula of the subdifferential of $\mathcal{R}_{q,\varepsilon}$, we use the Definition 2.14 for writing the Bregman distance concerning the regularization $\mathcal{R}_{q,\varepsilon}$. Considering an element $\xi \in \partial\mathcal{R}_{q,\varepsilon}(u) \subset L_q(\Omega)^*$, the Bregman distance of $\mathcal{R}_{q,\varepsilon}$ at $u \in L_q(\Omega)$ is given by

$$D_\xi^\varepsilon(\tilde{u}, u) = \mathcal{R}_{q,\varepsilon}(\tilde{u}) - \mathcal{R}_{q,\varepsilon}(u) - \langle \xi, \tilde{u} - u\rangle_{L_q(\Omega)^* \times L_q(\Omega)}, \quad \tilde{u} \in L_q(\Omega). \tag{3.31}$$

Notation 3.17. (i) For the sake of simplicity, in the proofs that follow we denote the duality pairing using the usual inner product notation $\langle \cdot, \cdot \rangle$.

(ii) When using the notation $\alpha \sim \delta^s$ for $\alpha : (0, \infty) \to (0, \infty)$ and $s > 0$, we mean that there exist constants $C \ge c > 0$ and $\delta_0 > 0$ such that $c\delta^s \le \alpha(\delta) \le C\delta^s$ for $0 < \delta < \delta_0$.

The classical process for proving convergence rates requires additional assumptions on the smoothness of F, a restriction of its nonlinearity, as well as a source condition which allows the estimation of the duality pairing in the Bregman distance. These are included in the following assumption.

Assumption 3.18 (Smoothness of F and source condition)**.** Assume that the following hold:

(i) The operator $F : \mathrm{dom}(F) \subset L_q(\Omega) \to V$ is Gâteaux differentiable at $u^\dagger$ and F' denotes its Gâteaux derivative.

(ii) There exists a constant $\gamma > 0$, such that

$$\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\| \leq \gamma D_\xi^\varepsilon(u, u^\dagger)$$

for all $u \in \mathrm{dom}(F) \cap \mathcal{B}_\rho(u^\dagger)$, with a sufficiently large ρ.

(iii) There exists $w \in V$, such that $\xi = F'(u^\dagger)^* w$ with $\gamma\|w\| < 1$.

Theorem 3.19 (Convergence rates)**.** *Let Assumptions 3.6 and 3.18 hold. Moreover, let $\varepsilon > 0$, $1 \leq p, q \leq 2$ and consider the Bregman distance for $\mathcal{R}_{q,\varepsilon}$ as given by (3.31). Assume noisy data $v^\delta \in V$ such that $\|v - v^\delta\|_V \leq \delta$ and that there exists an $\mathcal{R}_{q,\varepsilon}$-minimizing solution $u^\dagger$ of (3.12) in the Bregman domain $\mathcal{D}_B^\varepsilon$. With positive constants $\beta_1 = \gamma\|w\| < 1$ and $\beta_2 = \|w\|$ and for the minimizer $u_{\alpha,\varepsilon}^\delta$ of $\mathcal{J}_{\alpha,\delta,\varepsilon}^{p,q}$, we have the following estimates:*

If $p = 1$ and $\alpha\beta_2 < 1$,

$$\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|_V \leq \frac{(1 + \alpha\beta_2)\delta}{1 - \alpha\beta_2} \quad \text{and} \quad D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) \leq \frac{\delta + \alpha\beta_2\delta}{\alpha(1 - \beta_1)}.$$

If $p > 1$,

$$\begin{aligned} \|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|_V^p &\leq p_*\delta^p + p_*\alpha\beta_2\delta + (\alpha\beta_2)^{p_*} \quad \text{and} \\ D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) &\leq \frac{\delta^p + \alpha\beta_2\delta + (\alpha\beta_2)^{p_*}/p_*}{\alpha(1 - \beta_1)}, \end{aligned}$$

with p_* being the conjugate of p such that $1/p + 1/p_* = 1$.

Moreover, we have:

For $p = 1$ and the choice $\alpha \sim \delta^s$ with fixed $0 < s < 1$

$$\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|_V = \mathcal{O}(\delta) \quad \text{and} \quad D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) = \mathcal{O}(\delta^{1-s}).$$

For $p > 1$ and the choice $\alpha \sim \delta^{p-1}$

$$\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|_V = \mathcal{O}(\delta) \quad \text{and} \quad D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) = \mathcal{O}(\delta).$$

Proof. We start by comparing the functional values $\mathcal{J}_{\alpha,\delta,\varepsilon}^{p,q}(u_{\alpha,\varepsilon}^\delta)$ and $\mathcal{J}_{\alpha,\delta,\varepsilon}^{p,q}(u^\dagger)$. Using the minimizing property of $u_{\alpha,\varepsilon}^\delta$, we obtain

$$\begin{aligned} \|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|^p + \alpha\mathcal{R}_{q,\varepsilon}(u_{\alpha,\varepsilon}^\delta) &\leq \|F(u^\dagger) - v^\delta\|^p + \alpha\mathcal{R}_{q,\varepsilon}(u^\dagger) \\ &\leq \delta^p + \alpha\mathcal{R}_{q,\varepsilon}(u^\dagger). \end{aligned} \tag{3.32}$$

By reordering and gathering some of the terms together, we introduce the Bregman distance $D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger)$ into the last inequality. This yields

$$\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|^p + \alpha D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) \leq \delta^p - \alpha\langle \xi, u_{\alpha,\varepsilon}^\delta - u^\dagger\rangle. \tag{3.33}$$

We employ the source condition (iii) of Assumption 3.18 for rewriting the duality mapping on the right hand side as

$$\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|^p + \alpha D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) \leq \delta^p - \alpha\langle w, F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle. \tag{3.34}$$

We now consider the last term separately. We have

$$\begin{aligned} -\langle w, F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle &= \langle w, -F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle \\ &\leq \|w\| \, \|F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\|. \end{aligned}$$

By adding and subtracting $F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)$ inside the term $\|F'(u_{\alpha,\varepsilon}^\delta)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\|$, and using the triangle inequality, we obtain

$$\begin{aligned} -\langle w, F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle \leq \|w\| \, &\|F(u_{\alpha,\varepsilon}^\delta) - F(u^\dagger) - F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\| \\ &+ \|w\| \, \|F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)\|. \end{aligned}$$

Now, we use the smoothness assumption for F as given in (ii) of Assumption 3.18 and write

$$-\langle w, F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle \leq \gamma\|w\| \, D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) + \|w\| \, \|F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)\|$$

and by defining constants $\beta_1, \beta_2 > 0$ such that $\beta_1 := \gamma\|w\| < 1$ and $\beta_2 := \|w\|$, we finally obtain

$$-\langle w, F'(u^\dagger)(u_{\alpha,\varepsilon}^\delta - u^\dagger)\rangle \leq \beta_1 D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) + \beta_2\|F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)\|. \tag{3.35}$$

In addition, we can estimate the term $\|F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)\|$. After adding and subtracting $v^\delta \in V$, we use the triangle inequality and conclude that

$$\|F(u^\dagger) - F(u_{\alpha,\varepsilon}^\delta)\| \leq \delta + \|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|. \tag{3.36}$$

Substituting the estimates (3.35), (3.36) into (3.34), we have

$$\begin{aligned} \|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|^p + \alpha D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) \leq \delta^p &+ \alpha\beta_1 D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) + \alpha\beta_2\delta \\ &+ \alpha\beta_2\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\|. \end{aligned} \tag{3.37}$$

For $p = 1$, rearranging (3.37) yields

$$(1 - \alpha\beta_2)\|F(u_{\alpha,\varepsilon}^\delta) - v^\delta\| + \alpha(1 - \beta_1)D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger) \leq \delta + \alpha\beta_2\delta.$$

For sufficiently small $\alpha > 0$ such that $\alpha\beta_2 < 1$, the first term is nonnegative. Moreover, the second term is also nonnegative by the assumption $\beta_1 < 1$. Therefore, we derive the following estimates

$$\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\| \le \frac{(1+\alpha\beta_2)\delta}{1-\alpha\beta_2}, \tag{3.38}$$

$$D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) \le \frac{\delta + \alpha\beta_2\delta}{\alpha(1-\beta_1)}. \tag{3.39}$$

Choosing $\alpha \sim \delta^s$ with fixed $0 < s < 1$, we obtain

$$\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\| = \mathcal{O}(\delta) \quad \text{and} \quad D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) = \mathcal{O}(\delta^{1-s}).$$

For $p > 1$, we have

$$\begin{aligned}\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\|^p - \alpha\beta_2\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\| + \alpha D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger)&\\ \le \delta^p + \alpha\beta_1 D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) + \alpha\beta_2\delta.&\end{aligned}$$

Considering $a = \|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\|$ and $b = \alpha\beta_2$, we apply Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^{p_*}}{p_*}, \qquad \text{with} \quad a, b \ge 0, \quad \text{and} \quad \frac{1}{p} + \frac{1}{p_*} = 1$$

and we have

$$\left(1 - \frac{1}{p}\right)\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\|^p + \alpha(1-\beta_1)D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) \le \delta^p + \alpha\beta_2\delta + \frac{(\alpha\beta_2)^{p_*}}{p_*}.$$

Both terms in the left hand side are nonnegative and we can conclude the following estimates by neglecting the other nonnegative term, respectively. Therefore,

$$\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\|^p \le p_*\delta^p + p_*\alpha\beta_2\delta + (\alpha\beta_2)^{p_*}, \tag{3.40}$$

$$D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) \le \frac{\delta^p + \alpha\beta_2\delta + (\alpha\beta_2)^{p_*}/p_*}{\alpha(1-\beta_1)}. \tag{3.41}$$

The choice $\alpha \sim \delta^{p-1}$ yields

$$\|F(u^\delta_{\alpha,\varepsilon}) - v^\delta\| = \mathcal{O}(\delta) \quad \text{and} \quad D^\varepsilon_\xi(u^\delta_{\alpha,\varepsilon}, u^\dagger) = \mathcal{O}(\delta).$$

■

Remark 3.20 (Bregman and norm estimates with ε and $q = 2$)**.** In the case of the classical penalty term $\mathcal{R}_2(u) = \|u\|_2^2$, i.e., without tolerances, it can be shown that the estimate can be transferred from the Bregman distance to the usual Hilbert space norm. This is due to $\mathcal{R}_2$ being differentiable with $\xi = \partial\mathcal{R}_2(u^\dagger) = 2u^\dagger$, which yields

$D_\xi(u_\alpha^\delta, u^\dagger) = \|u_\alpha^\delta - u^\dagger\|_2^2$. Regarding the $L_{2,\varepsilon}$-insensitive measure, which involves the nonlinear tolerance function in the computations, we don't have a similar equivalence between $\|u_{\alpha,\varepsilon}^\delta - u^\dagger\|_{2,\varepsilon}^2$ and $D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger)$. It is possible to derive the following estimate

$$\|u_{\alpha,\varepsilon}^\delta - u^\dagger\|_{2,\varepsilon}^2 \leq \|u_{\alpha,\varepsilon}^\delta - u^\dagger\|_2^2 = D_\xi(u_{\alpha,\varepsilon}^\delta, u^\dagger)$$

through the inequality (3.5a) but a relation between $D_\xi^\varepsilon(u_{\alpha,\varepsilon}^\delta, u^\dagger)$ and $\|u_{\alpha,\varepsilon}^\delta - u^\dagger\|_{2,\varepsilon}^2$ is not obvious.

Chapter

4 Elastic net approach for tolerances and sparsity

In this chapter we aim at recovering sparse solutions while maintaining the idea of the tolerances in the regularization that was proposed in the previous chapter. The natural approach for achieving this is by adopting the idea of elastic net regularization (introduced in Section 2.5) in which, more than one penalties are incorporated in the minimization functional with each one of them serving a different purpose. Inspired by this approach we consider a modified, with tolerances, elastic net functional consisting of the data fitting term, an ℓ^2-penalty term with tolerances and an ℓ^1-penalty term for enforcing sparsity.

We consider a problem in the form $Ku = v$ with a continuous linear operator $K : U \to V$ between Hilbert spaces U and V. We further consider an orthonormal basis $\{\phi_i \in U : i \in \mathbb{N}\}$ for U and we follow an established approach for sparsity results as in [DDDM04] and [GHS08]. Therefore, we examine sparsity in terms of the number of nonzero coefficients in the series expansion of the minimizer. Compared to the previous chapter, here we choose to work with a linear operator to facilitate the presentation. The nonlinear case can be handled in a similar way with some additional assumptions for the nonlinear operator, which are remarked in the end of this chapter.

With an abuse of the notation for u we denote the sequence of expansion coefficients $\{u_i = \langle u, \phi_i \rangle\}$ and consider the minimization of the *elastic net functional with tolerances* $\mathcal{J}^{\delta}_{\alpha,\beta,\varepsilon} : \ell^2 \to \mathbb{R} \cup \{\infty\}$ defined as

$$\mathcal{J}^{\delta}_{\alpha,\beta,\varepsilon}(u) := \frac{1}{2}\|Ku - v^{\delta}\|_V^2 + \frac{\alpha}{2}\|u\|^2_{\ell^2,\varepsilon} + \beta\|u\|_{\ell^1}, \tag{4.1}$$

in ℓ^2. Then, K denotes the operator $u_i \mapsto K\sum_i u_i\phi_i$ mapping from $\ell^2 \to V$. We denote the individual penalty terms by

$$\mathcal{R}_{2,\varepsilon}(u) = \frac{1}{2}\|u\|^2_{\ell^2,\varepsilon} \quad \text{and} \quad \mathcal{R}_{\mathrm{sp}}(u) = \|u\|_{\ell^1} \tag{4.2}$$

and because $\mathrm{dom}(\mathcal{R}_{\mathrm{sp}}) = \ell^1$, for elements $u \in \ell^2 \setminus \ell^1$ we assume that $\|u\|_{\ell^1} := \infty$, so that $\mathcal{R}_{\mathrm{sp}}$ is well defined. We consider noisy data $v^\delta \in V$ such that $\|v - v^\delta\|_V \leq \delta$, regularization parameters $\alpha, \beta > 0$ and a sequence of nonnegative tolerances $\varepsilon := \{\varepsilon_n\}_{n\in\mathbb{N}} \in \ell^2$.

The minimization of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ takes place only on

$$\hat{\mathcal{D}} = \mathrm{dom}(\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}) = \mathrm{dom}(K) \cap \mathrm{dom}(\mathcal{R}_{2,\varepsilon}) \cap \mathrm{dom}(\mathcal{R}_{\mathrm{sp}}) = \ell^1,$$

where $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ is proper. In addition, it is convex, coercive and weak lower semi-continuous as the sum of convex, coercive and weak lower semi-continuous functionals. For any element $u \in \ell^2$ there holds

$$\|u\|^2_{\ell^2,\varepsilon} = \|d_\varepsilon(u)\|^2_{\ell^2} = \sum_{i\in\mathbb{N}} |d_\varepsilon(u_i)|^2 \leq \sum_{i\in\mathbb{N}} |u_i|^2 < \infty.$$

With the functional $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ defined, we are ready to proceed with our theoretical analysis. In Section 4.1 we present results on the well-posedness of the minimizers of the elastic net functional with tolerances given in (4.1), while Section 4.2 is dedicated to the convergence rates analysis of the minimizers of its minimizers.

4.1 Well-posedness of minimizers

We begin with the existence of minimizers of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ which follows in the theorem below.

Theorem 4.1 (Existence of minimizers)**.** *Let $\alpha, \beta > 0$ and $\varepsilon := \{\varepsilon_n\}_{n\in\mathbb{N}} \in \ell^2$ be a nonnegative sequence. Then, for every $v^\delta \in V$ the functional $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ has a minimizer $u^\delta_{\alpha,\beta,\varepsilon}$ in ℓ^2.*

Proof. Since $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ is bounded from below by zero, its infimum is finite and therefore, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \in \hat{\mathcal{D}}$ that minimizes $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$, i.e,

$$\lim_{k\to\infty} \mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u_k) = \inf\{\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u) : u \in \hat{\mathcal{D}}\}.$$

Hence the sequence of functional values $\{\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u_k)\}$ is uniformly bounded. Since $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ is coercive, there exists a subsequence of u_k, also denoted by u_k, with weak limit $\tilde{u} \in \ell^2$. Moreover, the continuity of K implies weak continuity [Woj96, Part

II.A. 8, p. 29], which means that $Ku_k \rightharpoonup K\tilde{u}$. The weak lower semi-continuity of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ leads to

$$\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(\tilde{u}) \leq \liminf_{n\to\infty} \mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u_k) \leq \mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u) \quad \forall u \in \hat{\mathcal{D}},$$

which implies that $u^\delta_{\alpha,\beta,\varepsilon} := \tilde{u}$ is a minimizer of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ in $\hat{\mathcal{D}}$. ■

Remark 4.2 (On the uniqueness of minimizer)**.** In [JLS09] the authors discuss how the choice of α and β effects the uniqueness of minimizer. The classical elastic net functional has a unique minimizer as soon as the regularization parameter of the ℓ^2-penalty is positive, due to the strict convexity of the functional. However, as the authors in this work proved, even for vanishing α, whole sequence convergence can be achieved under the assumption that the sequence $\{(\alpha_k, \beta_k)\}_k$ satisfies

$$\lim_{k\to\infty} \alpha_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\beta_k - \beta}{\alpha_k} = \gamma \tag{4.3}$$

for some $\beta > 0$ and $\gamma \geq 0$. In that case, the minimizer of $\mathcal{J}^\delta_{\alpha,\beta}$ converges to the minimum $\left(\frac{1}{2}\|\cdot\|^2_{\ell^2} + \gamma\|\cdot\|_{\ell^1}\right)$ element of the set of minimizers of the functional $\mathcal{J}^\delta_{0,\beta}$ and, therefore, is unique.
With regard to our, modified by tolerances, elastic net functional, the uniqueness of minimizer is not guaranteed since the functional is not strictly convex. The lack of uniqueness mainly originates from the assumption of positive tolerances. Therefore, for positive tolerances only subsequential convergence can be expected. In the special event of vanishing tolerances, we obtain the results of the classical elastic net regularization. Moreover, by considering $\beta > 0$ and both α and ε convergent to zero, we fall in the previously described case (with similar as in (4.3) assumptions that involve ε, too), if ε vanishes faster than α. However, such a special case is not further considered because our main interest is the assumption of tolerances in the solution.

Next follows the stability of minimizers of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$ when the data $v^\delta \in V$ is slightly perturbed, i.e., we confirm that the minimizers depend continuously on the noisy data. For such perturbation in the data, the ill-posedness of the initial problem (without regularization) traditionally would lead to an inconsistent solution. In the stability theorem we consider two cases, namely, for positive and vanishing tolerances.

Notation. To simplify our proofs, we consider $\beta = \alpha\eta$, for $\eta \geq 0$ and we control both regularization parameters by η. For this reason, we consider the functional $\hat{\mathcal{R}}_{\eta,\varepsilon}$ defined by

$$\hat{\mathcal{R}}_{\eta,\varepsilon}(u) := \frac{1}{2}\|u\|^2_{\ell^2,\varepsilon} + \eta\|u\|_{\ell^1}. \tag{4.4}$$

Moreover, for simplicity, we will denote the functional $\hat{\mathcal{R}}_{\eta,0}$ by $\hat{\mathcal{R}}_\eta$ when $\varepsilon = 0$.

Theorem 4.3 (Stability of minimizers w.r.t. v^δ)**.** *Assume $\alpha, \beta > 0$ and let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of nonnegative tolerances that converges to some $\tilde{\varepsilon} \geq 0$. Moreover, consider $\{v_k\}_{k\in\mathbb{N}} \subset V$ to be a convergent sequence with limit $v^\delta \in V$ and u_k such that*

$$u_k \in \arg\min\{\mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u) : u \in \hat{\mathcal{D}}\}.$$

Then, there exists a subsequence of $\{u_k\}$, denoted again by $\{u_k\}$, which converges to a minimizer $u^\delta_{\alpha,\beta,\tilde{\varepsilon}}$ of the functional $\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}$. With $\beta = \alpha\eta$ and $\eta \geq 0$, we further obtain $\hat{\mathcal{R}}_{\eta,\tilde{\varepsilon}}(u_k) \to \hat{\mathcal{R}}_{\eta,\tilde{\varepsilon}}(u^\delta_{\alpha,\beta,\tilde{\varepsilon}})$.

Proof. From the minimizing property of u_k, there holds

$$\liminf_{k\to\infty} \mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u_k) \leq \mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u)$$

for all $u \in \hat{\mathcal{D}}$ and there exists a subsequence of u_k (again denoted by u_k) that converges weakly to some $\tilde{u} \in \ell^2$. The weak lower semi-continuity of all terms yields

$$\begin{aligned}
\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}(\tilde{u}) &\leq \liminf_{k\to\infty} \frac{1}{2}\|Ku_k - v_k\|^2_V + \liminf_{k\to\infty} \frac{\alpha}{2}\|u_k\|^2_{\ell^2,\varepsilon_k} + \liminf_{k\to\infty} \beta\|u_k\|_{\ell^1} \\
&\leq \liminf_{k\to\infty} \left(\frac{1}{2}\|Ku_k - v_k\|^2_V + \frac{\alpha}{2}\|u_k\|^2_{\ell^2,\varepsilon_k} + \beta\|u_k\|_{\ell^1}\right) \\
&= \liminf_{k\to\infty} \mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u_k).
\end{aligned}$$

On the other hand, for any $u \in \hat{\mathcal{D}}$, we have that

$$\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}(u) = \lim_{k\to\infty} \mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u) \geq \limsup_{k\to\infty} \mathcal{J}^{v_k}_{\alpha,\beta,\varepsilon_k}(u_k).$$

Therefore, for any $u \in \hat{\mathcal{D}}$ we conclude $\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}(\tilde{u}) \leq \mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}(u)$, which means that $\tilde{u} := u^\delta_{\alpha,\beta,\tilde{\varepsilon}}$ is a minimizer of $\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}$.

Introducing $\beta = \alpha\eta$, with $\eta \geq 0$ into the regularization term of $\mathcal{J}^\delta_{\alpha,\beta,\tilde{\varepsilon}}$, the weak lower semi-continuity of the functional implies that $\hat{\mathcal{R}}_{\eta,\tilde{\varepsilon}}(u_k) \to \hat{\mathcal{R}}_{\eta,\tilde{\varepsilon}}(u^\delta_{\alpha,\beta,\tilde{\varepsilon}})$. When $\tilde{\varepsilon} = 0$, we have $\hat{\mathcal{R}}_{\eta,\tilde{\varepsilon}}(u_k) \to \hat{\mathcal{R}}_\eta(u^\delta_{\alpha,\beta})$. ■

Our last result in this section regards the convergence of minimizers and uses the notion of an $\hat{\mathcal{R}}_{\eta,\varepsilon}$-minimizing solution, recall Definition 3.7 of the previous chapter.

Theorem 4.4 (Weak convergence for $\delta_k \to 0$)**.** *Let $\varepsilon := \{\varepsilon_n\}_{n\in\mathbb{N}} \in \ell^2$ be a nonnegative sequence and let $v \in V$ such that an $\hat{\mathcal{R}}_{\eta,\varepsilon}$-minimizing solution of $Ku = v$ in ℓ^2 exists. Moreover, assume regularization parameters $\alpha, \beta : (0,\infty) \to (0,\infty)$ that satisfy*

$$\alpha(\delta),\ \beta(\delta),\ \frac{\delta^2}{\alpha(\delta)},\ \frac{\delta^2}{\beta(\delta)} \to 0, \qquad \text{as } \delta \to 0, \tag{4.5}$$

and that there exists $\eta \geq 0$ *such that*

$$\lim_{\delta \to 0} \frac{\beta(\delta)}{\alpha(\delta)} = \eta. \tag{4.6}$$

Let $\{\delta_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ *be given with* $\delta_k \to 0$ *and* $v_k \in V$ *satisfy* $\|v - v_k\| \leq \delta_k$. *In addition, consider* $\alpha_k := \alpha(\delta_k)$, $\beta_k := \beta(\delta_k)$ *and*

$$u_k \in \arg\min\{\mathcal{J}^{v_k}_{\alpha_k,\beta_k,\varepsilon}(u) : u \in \ell^2\}.$$

Then, there exist a subsequence of u_k *(denoted again by* u_k*) such that* u_k *converges weakly to an* $\hat{\mathcal{R}}_{\eta,\varepsilon}$*-minimizing solution* $u^\dagger$ *and* $\hat{\mathcal{R}}_{\eta,\varepsilon}(u_k) \to \hat{\mathcal{R}}_{\eta,\varepsilon}(u^\dagger)$.

Proof. Let $u^\dagger$ be an $\hat{\mathcal{R}}_{\eta,\varepsilon}$-minimizing solution of $Ku = v$. The definition of u_k implies

$$\begin{aligned}
\frac{1}{2}\|Ku_k - v_k\|^2 &+ \frac{\alpha_k}{2}\|u_k\|^2_{\ell^2,\varepsilon} + \beta_k\|u_k\|_{\ell^1} \\
&\leq \frac{1}{2}\|Ku^\dagger - v_k\|^2 + \frac{\alpha_k}{2}\|u^\dagger\|^2_{\ell^2,\varepsilon} + \beta_k\|u^\dagger\|_{\ell^1} \\
&\leq \frac{{\delta_k}^2}{2} + \frac{\alpha_k}{2}\|u^\dagger\|^2_{\ell^2,\varepsilon} + \beta_k\|u^\dagger\|_{\ell^1}.
\end{aligned} \tag{4.7}$$

Hence, the assumptions on α_k, β_k as $\delta_k \to 0$ yield that the sequences $\{\|Ku_k - v_k\|\}$ and $\{\|u_k\|_{\ell^2,\varepsilon}\}_k$ are uniformly bounded. Therefore, there exist a subsequence of $\{u_k\}$, denoted again by $\{u_k\}$, and some element $\tilde{u} \in \ell^2$ such that $u_k \rightharpoonup \tilde{u}$. By the weak lower semi-continuity and triangle inequality, we have

$$\|K\tilde{u} - v\|^2 \leq 2 \liminf_{k\to\infty} \left(\|Ku_k - v_k\|^2 + \|v_k - v\|^2 \right).$$

The term $\|Ku_k - v_k\|^2$ can be bounded from above using (4.7) and together with the assumptions on α_k, β_k and $\|v - v_k\| \leq \delta_k$, we conclude

$$\|K\tilde{u} - v\|^2_V \leq 2 \liminf_{k\to\infty} \left(\delta_k^2 + \alpha_k\|u^\dagger\|^2_{\ell^2,\varepsilon} + 2\beta_k\|u^\dagger\|_{\ell^1} + \delta_k^2 \right) = 0,$$

which means that $K\tilde{u} = v$. In a similar way, we have

$$\begin{aligned}
\frac{1}{2}\|\tilde{u}\|^2_{\ell^2,\varepsilon} + \eta\|\tilde{u}\|_{\ell^1} &\leq \liminf_{k\to\infty} \left(\frac{1}{2}\|u_k\|^2_{\ell^2,\varepsilon} + \frac{\beta_k}{\alpha_k}\|u_k\|_{\ell^1} \right) \\
&\leq \liminf_{k\to\infty} \left(\frac{\delta_k^2}{2\alpha_k} + \frac{1}{2}\|u^\dagger\|^2_{\ell^2,\varepsilon} + \frac{\beta_k}{\alpha_k}\|u^\dagger\|_{\ell^1} \right) \\
&= \frac{1}{2}\|u^\dagger\|^2_{\ell^2,\varepsilon} + \eta\|u^\dagger\|_{\ell^1},
\end{aligned}$$

i.e., $\hat{\mathcal{R}}_{\eta,\varepsilon}(\tilde{u}) \leq \hat{\mathcal{R}}_{\eta,\varepsilon}(u^\dagger)$, from which we deduce that the subsequence u_k converges weakly to an $\hat{\mathcal{R}}_{\eta,\varepsilon}$-minimizing solution of $Ku = v$. Moreover, the weak lower semi-continuity of $\hat{\mathcal{R}}_{\eta,\varepsilon}$ implies $\hat{\mathcal{R}}_{\eta,\varepsilon}(u_k) \to \hat{\mathcal{R}}_{\eta,\varepsilon}(u^\dagger)$. ■

Remark 4.5. If the above theorem is modified by assuming a sequence of nonnegative tolerances ε_k such that $\varepsilon_k \to 0$ as $\delta_k \to 0$, we obtain $\hat{\mathcal{R}}_{\eta,\varepsilon_k}(u_k) \to \hat{\mathcal{R}}_{\eta}(u^\dagger)$.

4.2 Convergence rates

In this section we present the convergence rates results for the elastic net functional with tolerances in the ℓ^2-penalty. In the previous theorems we already considered $\lim_{\delta\to 0} \frac{\beta(\delta)}{\alpha(\delta)} = \eta$ for ensuring the convergence of minimizers $u^\delta_{\alpha,\beta,\varepsilon}$ when $\delta \to 0$. We follow the same principle here too, for proving the convergence rates for the minimizers $u^\delta_{\alpha,\beta,\varepsilon}$ with the assumption that there exists $\eta > 0$ such that $\beta = \alpha\eta$. In this way, we control both α and β by their fixed ratio given by $\eta > 0$. The following proposition contains the general source condition that we consider for obtaining convergence rates.

Assumption 4.6 (General source condition)**.** Let $\eta > 0$ be such that $u^\dagger$ fulfills the source condition

$$\text{there exists } w \in V \ : \ K^* w \in \partial \hat{\mathcal{R}}_{\eta,\varepsilon}(u^\dagger) \tag{4.8}$$

with $\hat{\mathcal{R}}_{\eta,\varepsilon}(u)$ as defined in (4.4) for $\beta = \alpha\eta$ and its subdifferential

$$\partial \hat{\mathcal{R}}_{\eta,\varepsilon}(u) = d_\varepsilon(u)\partial d_\varepsilon(u) + \eta \operatorname{Sign}(u).$$

The convergence rate of the discrepancy term will be computed in the norm of the Hilbert space V. In contrast to the previous chapter, here we compute the convergence rate of the distance between $u^\delta_{\alpha,\beta,\varepsilon}$ and $u^\dagger$ in the ℓ^2-norm. This is possible using the inequalities (3.5a) and (3.5c) from Proposition 3.4, which, for $q = 2$, can be transferred from the 2-norm to the ℓ^2-norm as the Hilbert space L_2 is isometric to ℓ^2. These inequalities are important for obtaining error estimates given the fact that the ε-insensitive measure itself violates the triangle inequality, which is used in the proof of the convergence rates.

Theorem 4.7. *Let $u^\dagger$ be the true solution of $Ku = v$ and $\|v - v^\delta\| \le \delta$. Moreover, consider that Assumption 4.6 holds and that $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ in ℓ^2 is a sequence of nonnegative tolerances with $\|\varepsilon\|_{\ell^2} = \mathcal{O}(\sqrt{\delta})$. Then, the minimizer $u^\delta_{\alpha,\beta,\varepsilon}$ of $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$, with $\beta = \alpha\eta$, fulfills*

$$\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| \le \delta - \alpha\|w\| + \sqrt{\alpha}\sqrt{c}\|\varepsilon\|_{\ell^2}$$

and

$$\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2} \le \frac{\delta}{\sqrt{\alpha}} - \sqrt{\alpha}\|w\| + \sqrt{c}\|\varepsilon\|_{\ell^2}.$$

Moreover, with a parameter choice $\alpha \sim \mathcal{O}(\delta)$ we obtain

$$\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| = \mathcal{O}(\delta) \quad \text{and} \quad \|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2} = \mathcal{O}(\delta^{1/2}).$$

Proof. The minimizing property of $u^\delta_{\alpha,\beta,\varepsilon}$ gives

$$\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \frac{\alpha}{2}\|u^\delta_{\alpha,\beta,\varepsilon}\|^2_{\ell^2,\varepsilon} + \beta\|u^\delta_{\alpha,\beta,\varepsilon}\|_{\ell^1} \\ \leq \frac{1}{2}\|Ku^\dagger - v^\delta\|^2 + \frac{\alpha}{2}\|u^\dagger\|^2_{\ell^2,\varepsilon} + \beta\|u^\dagger\|_{\ell^1}$$

and by gathering common terms, we have

$$\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \frac{\alpha}{2}\left(\|u^\delta_{\alpha,\beta,\varepsilon}\|^2_{\ell^2,\varepsilon} - \|u^\dagger\|^2_{\ell^2,\varepsilon}\right) + \beta\left(\|u^\delta_{\alpha,\beta,\varepsilon}\|_{\ell^1} - \|u^\dagger\|_{\ell^1}\right) \\ \leq \frac{1}{2}\|Ku^\dagger - v^\delta\|^2.$$

Using (3.5a) and (3.5c), we replace the terms of the first parenthesis by their estimates in the ℓ^2-norm and we have

$$\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \frac{\alpha}{2}\left(\|u^\delta_{\alpha,\beta,\varepsilon}\|^2_{\ell^2} - \|u^\dagger\|^2_{\ell^2}\right) - \frac{\alpha c}{2}\|\varepsilon\|^2_{\ell^2} + \beta\left(\|u^\delta_{\alpha,\beta,\varepsilon}\|_{\ell^1} - \|u^\dagger\|_{\ell^1}\right) \\ \leq \frac{1}{2}\|Ku^\dagger - v^\delta\|^2,$$

with $c = 2\max\{1, \|u^\delta_{\alpha,\beta,\varepsilon}\|_{\ell^2,\varepsilon}, \|\varepsilon\|_{\ell^2}\}$. Then, we use the identity $\|u^\delta_{\alpha,\beta,\varepsilon}\|^2_{\ell^2} - \|u^\dagger\|^2_{\ell^2} = \|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|^2_{\ell^2} + 2\langle u^\dagger, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle$ and we also add and subtract $\beta\langle\xi, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle$ in the left side, to obtain

$$\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \frac{\alpha}{2}\left(\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|^2_{\ell^2} + 2\langle u^\dagger, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle\right) - \frac{\alpha c}{2}\|\varepsilon\|^2_{\ell^2} \\ + \beta\left[\|u^\delta_{\alpha,\beta,\varepsilon}\|_{\ell^1} - \|u^\dagger\|_{\ell^1} - \langle\xi, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle\right] + \beta\langle\xi, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle \\ \leq \frac{1}{2}\|Ku^\dagger - v^\delta\|^2,$$

for any $\xi \in \partial\mathcal{R}_{\mathrm{sp}}(u^\dagger) = \mathrm{Sign}(u^\dagger)$. The expression in the square brackets is nothing more than the definition of a subgradient of $\mathcal{R}_{\mathrm{sp}}$ at $u^\dagger$, therefore, a nonnegative term that can be neglected for the rest of our computations. Subsequently, with $\beta = \alpha\eta$, we gather the inner products and we have

$$\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \frac{\alpha}{2}\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|^2_{\ell^2} + \alpha\langle u^\dagger + \eta\xi, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle - \frac{\alpha c}{2}\|\varepsilon\|^2_{\ell^2} \\ \leq \frac{1}{2}\|Ku^\dagger - v^\delta\|^2.$$

Since $u^\dagger + \eta\xi \in \partial\hat{\mathcal{R}}_{\eta,\varepsilon}$, the inner product in the last expression can be estimated via the source condition (4.8) to be

$$\begin{aligned}\langle u^\dagger + \eta\xi, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle &= \langle K^* w, u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\rangle = \langle w, K(u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger)\rangle \\ &= \langle w, Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta + v^\delta - Ku^\dagger\rangle \\ &\le \|w\| \left(\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| + \|Ku^\dagger - v^\delta\|\right).\end{aligned}$$

Combining the above into the previous estimate and rearranging the terms, we have

$$\begin{aligned}\frac{1}{2}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\|^2 + \alpha\|w\|\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| + \frac{\alpha}{2}\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|^2_{\ell^2} \\ \le \frac{1}{2}\|Ku^\dagger - v^\delta\|^2 - \alpha\|w\|\|Ku^\dagger - v^\delta\| + \frac{\alpha c}{2}\|\varepsilon\|^2_{\ell^2}.\end{aligned}$$

By adding $\frac{\alpha^2\|w\|^2}{2}$ on both sides, completing the squares, and estimating the term $\|Ku^\dagger - v^\delta\| \le \delta$, we get

$$\begin{aligned}\frac{1}{2}\Big(\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| + \alpha\|w\|\Big)^2 + \frac{\alpha}{2}\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|^2_{\ell^2} \\ \le \frac{1}{2}(\delta - \alpha\|w\|)^2 + \frac{\alpha c}{2}\|\varepsilon\|^2_{\ell^2}\end{aligned} \tag{4.9}$$

Each term on the left hand side of the inequality is nonnegative, therefore for estimating one of them we can forget the other one each time. For the first term we have

$$\Big(\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| + \alpha\|w\|\Big)^2 \le (\delta - \alpha\|w\|)^2 + \alpha c\|\varepsilon\|^2_{\ell^2}$$

which is equivalent to

$$\begin{aligned}\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| + \alpha\|w\| &\le \Big[(\delta - \alpha\|w\|)^2 + \alpha c\|\varepsilon\|^2_{\ell^2}\Big]^{1/2} \\ &\le \delta - \alpha\|w\| + \sqrt{\alpha}\sqrt{c}\|\varepsilon\|_{\ell^2},\end{aligned}$$

with the last inequality obtained by the subadditivity of the square root function. The final error estimate is

$$\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| \le \delta - 2\alpha\|w\| + \sqrt{\alpha}\sqrt{c}\|\varepsilon\|_{\ell^2}.$$

With similar steps we estimate the other term in (4.9) and obtain

$$\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2} \le \frac{\delta}{\sqrt{\alpha}} - \sqrt{\alpha}\|w\| + \sqrt{c}\|\varepsilon\|_{\ell^2}.$$

A choice $\alpha \sim \mathcal{O}(\delta)$ together with the assumption $\|\varepsilon\|_{\ell^2} = \mathcal{O}(\delta^{1/2})$ yields

$$\|Ku^\delta_{\alpha,\beta,\varepsilon} - v^\delta\| = \mathcal{O}(\delta) \quad \text{and} \quad \|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2} = \mathcal{O}(\delta^{1/2}).$$

■

Here, it is worth mentioning that the inequalities (3.5a)–(3.5c) are essential for establishing the above estimate for $\|u^{\delta}_{\alpha,\beta,\varepsilon} - u^{\dagger}\|_{\ell^2}$. Without them, it is still possible to obtain convergence rates using the Bregman distance, which is, however, a more general result. The use of the Bregman distance for measuring the difference between the minimizer $u^{\delta}_{\alpha,\beta,\varepsilon}$ and the true solution $u^{\dagger}$ and assuming a parameter choice $\alpha \sim \mathcal{O}(\delta)$ leads to a convergence rate $D^{\eta,\varepsilon}_{\xi}(u^{\delta}_{\alpha,\beta,\varepsilon}, u^{\dagger}) = \mathcal{O}(\delta)$.

Within this chapter, we assumed a bounded linear operator K. However, the use of a nonlinear operator is possible too, and requires some modifications or extra assumptions, which are summarized in the following remark.

Remark 4.8. (Modifications for nonlinear operators) With the assumption of a nonlinear operator $F : \mathrm{dom}(F) \subset U \to V$ the results for the elastic net functional with tolerances can be proved with a few more assumptions on the operator, like it was done in Chapter 3. These are:

1. *The weak sequential closedness of* F (recall (i) in Assumption 3.6). This allows the use of Lemma 3.8, in the proofs of existence and stability of minimizers, for extracting a subsequence that converges weakly to a minimizer of the functional.

For the convergence rates we need an assumption like Assumption 3.18, which consists of:

2. A smoothness assumption on the operator F (Gâteaux or Fréchet differentiability)

3. A restriction of the nonlinearity of F in a neighborhood of $u^{\dagger}$. This assumption can be like (ii) in Assumption 3.18 but it can also be differently formulated, depending on whether the convergence rate is to be computed in the norm or using the Bregman distance.

Lastly, in the case of a nonlinear operator F, the source condition (4.8) should also be reformulated such that includes the adjoint of the derivative of F, like, for example, (iii) in Assumption 3.18.

In the recent work [LW20], an elastic net approach is considered for sparsity regularization of nonlinear ill-posed inverse problems. The authors investigate the well-posedness of minimizers using the principles of regularization theory as we did in our analysis, too. For the detailed analysis and steps in a nonlinear regime, we refer the reader to [LW20].

Our results for convergence rates conclude the theoretical analysis of this chapter. In summary, we have proved the existence, stability, and weak convergence of the minimizers of the elastic net with tolerances. In addition, we have proved convergence rates for the minimizers of the elastic net functional with tolerances when the noise goes to 0.

Chapter

5 Numerical consideration

Having established our theoretical results, now we wish to examine our approach on a simple numerical example. The aim of this chapter is to illustrate the effect of tolerances in the reconstructed solutions. We also want show that good approximations of the true solution of the problem can be produced with appropriate selection of the regularization parameters. Moreover, we are interested in examining if a relation between the noise level δ and the tolerances ε emerges that leads to improved reconstructions. Therefore, this chapter serves for understanding what can be achieved in practice when considering the functionals proposed in Chapters 3 and 4 and for this reason, a simple example is considered.

We begin this chapter by giving some information on the algorithm that was used for the numerical minimization of the functionals and then we present the example of *noisy data differentiation*, which is considered. The numerical results are then divided into two sections: Section 5.1 is dedicated to the minimization of the Tikhonov functional with tolerances that was introduced in Chapter 3, and Section 5.2 concerns the minimization of the elastic net functional with tolerances that was introduced in Chapter 4 and the possibility of obtaining sparse solutions.

Subgradient algorithm. For the numerical minimization of a functional f we use a subgradient algorithm introduced in [GMPK20]. The suggested algorithm is an iterative subgradient method with adaptive step size. With an initial guess $x^0 \in \mathbb{R}^n$, the functional $f : \mathbb{R}^n \to \mathbb{R}$ is minimized iteratively via the update step

$$x^{k+1} = x^k - s_k \cdot g_k, \text{ with } g_k \in \partial f(x^k), \tag{5.1}$$

with diminishing step size $s_k > 0$ that is computed in every iteration step. In their work, the authors prove the stability and convergence of the algorithm and demonstrate its effectiveness by comparing their numerical results, on denoising and deblurring of 1D and 2D signals, to those obtained by existing methods.

Numerical differentiation of noisy data. In many applications (such as image reconstruction) one is interested in computing the derivative of a smooth function for which only discrete measurements are available. In practice the measured data will almost never be exact, but rather contain errors that can be modeled in the form of additive noise. The problem of differentiating noisy data is ill-posed [EHN96], meaning that small changes in the data (due to the noise) may produce large deviations in the computed derivative. Therefore, regularization is required for obtaining a stable approximation of the derivative.

The problem of numerical differentiation can be formulated as the operator equation

$$Ku(x) = \int_0^x u(s)\ \mathrm{d}s = v(x), \qquad u : [0,1] \ \rightarrow\ \mathbb{R} \tag{5.2}$$

with bounded, linear operator

$$K : L_2([0,1]) \rightarrow L_2([0,1]), \quad u \mapsto Ku.$$

Given noisy data

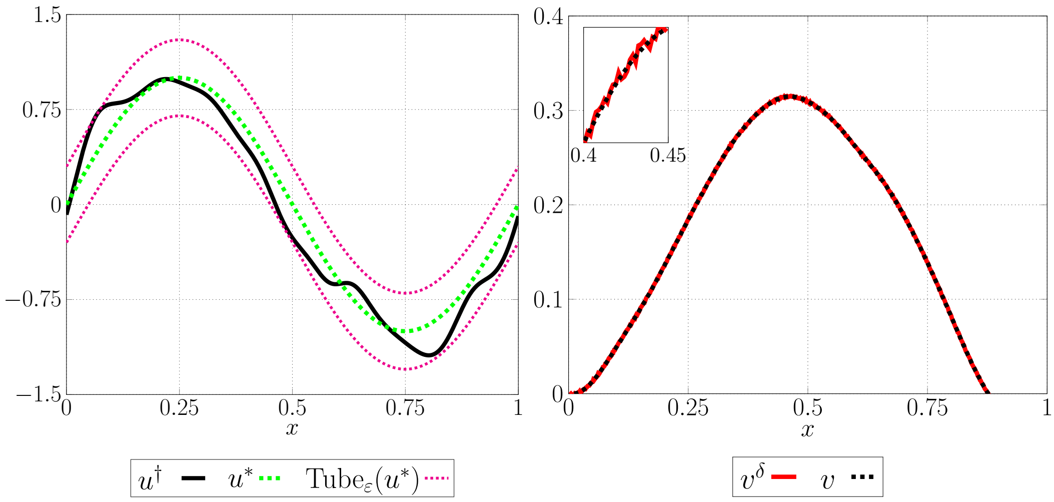
$$v^\delta = v + \hat{\eta}(\delta),$$

with additive Gaussian noise $\hat{\eta}(\delta)$ such that $\|v - v^\delta\| \leq \delta$, we wish to compute an approximation u to the first derivative of v.

In our numerical implementation we discretize the interval $[0,1]$ on the grid $x_i = (i - \frac{1}{2})h,\ i = 1, \dots, N$ with $h = \frac{1}{N}$ and $N = 600$ discretization points. Hence, u can be approximated in the interval $[(i-1)h, ih]$ by a piecewise constant function $\hat{u}$ with values $\hat{u}_i = \hat{u}(x_i)$, and Ku by a vector $v = [v_1, \dots, v_N]$ with elements $v_i = (K\hat{u})(x_i) = \int_0^{x_i} \hat{u}(s)\mathrm{d}s$. Moreover, the discretized operator is an $N \times N$ matrix with the following structure

$$K = h \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & \frac{1}{2} \end{pmatrix}.$$

In the following, we solve the discretized problem of approximating $u^\dagger$ by considering the minimization of the proposed functionals. In contrast to the previous chapters, here, we consider a reference solution u^* as *a priori* information on the true solution $u^\dagger$ and we will define it in the following when needed.



(a) tolerance value $\varepsilon = 0.3$

(b) $\|v - v^\delta\| = 10^{-2}$

Figure 5.1: The ground truth $u^\dagger$ and the reference solution u^* inside the tolerance area $\mathrm{Tube}_\varepsilon(u^*)$ in (a). The true data v and noisy data v^δ created in (b).

5.1 Minimization of the Tikhonov functional with tolerances in the regularization

We consider as reference solution the function $u^*(x) = \sin(2\pi x)$ for $x \in [0,1]$ and as true solution $u^\dagger$ a smooth perturbation of u^* that lies within the tolerance area

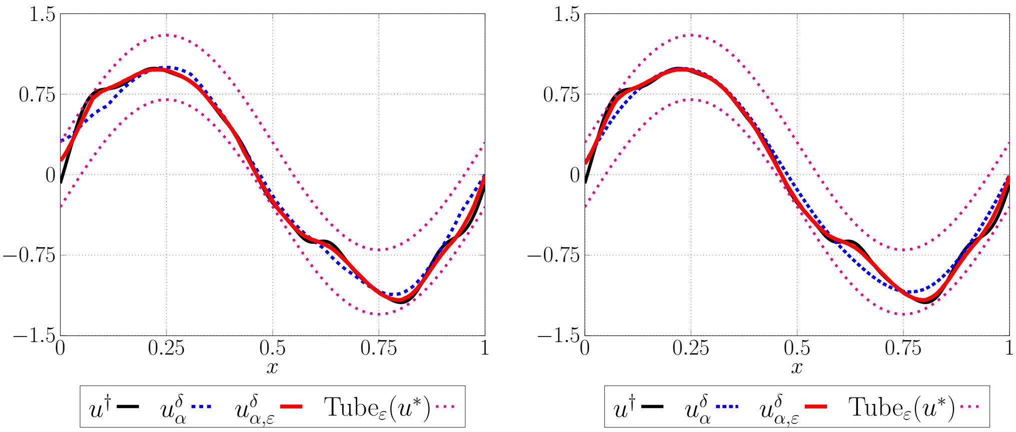
$$\mathrm{Tube}_\varepsilon(u^*) := \{(x,y) \in \Omega \times \mathbb{R} \mid y \in [u^*(x) - \varepsilon, u^*(x) + \varepsilon]\} \qquad \text{for } \varepsilon > 0, \tag{5.3}$$

defined around u^*. The noisy data are then created by $v^\delta = Ku^\dagger + \hat{\eta}(\delta)$, for a noise level $\delta > 0$. For approximating $u^\dagger$, we minimize the functional

$$\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}(u) = \|Ku - v^\delta\|^2 + \alpha\|u - u^*\|^q_{q,\varepsilon}, \quad \varepsilon \geq 0. \tag{5.4}$$

In the following we consider $p = 2$ and present results from the minimization of the functional with $q = 1$ and $q = 2$ in the penalty term. Moreover, we compare our results to the regularized solution $u^\delta_{\alpha,\varepsilon}$, obtained from the minimization of (5.4), to the solution $u^\delta_\alpha := \arg\min \mathcal{J}^\delta_{\alpha,0}$ of the generalized Tikhonov functional (without tolerances and p, q same as for the minimizer $u^\delta_{\alpha,\varepsilon}$). To get a glimpse of the effect of tolerances, we consider two cases for the reference solution u^* and the ground truth $u^\dagger$.

First, we consider that the true solution $u^\dagger$ lies entirely inside the $\mathrm{Tube}_\varepsilon(u^*)$, i.e., the reference solution $u^*(x) = \sin(2\pi x)$ is close to the ground truth and therefore a good source of *a priori* information. Figure 5.1a shows the ground truth $u^\dagger$, the reference solution u^* and the tolerance area around u^* for $\varepsilon = 0.3$ and Figure 5.1b

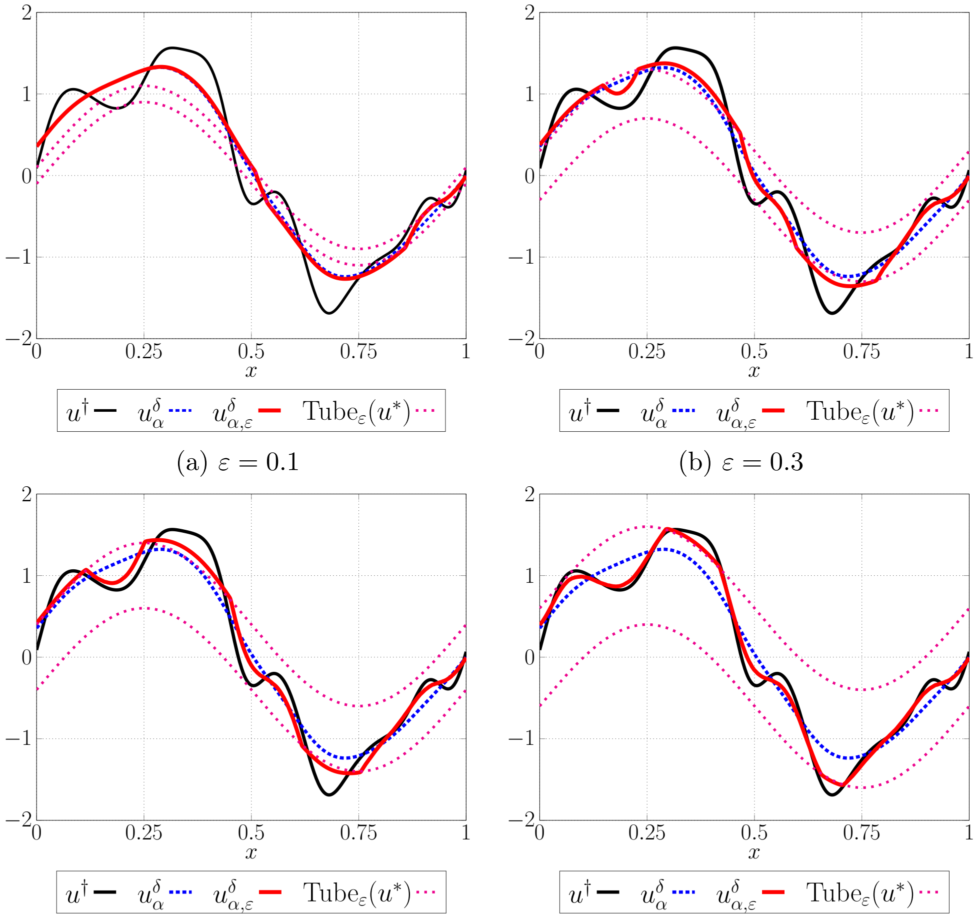


(a) $q = 1, \ \alpha = 10^{-3}$ (b) $q = 2, \ \alpha = 10^{-2}$

Figure 5.2: Reconstructions $u_{\alpha,\varepsilon}^\delta$, u_α^δ compared to the ground truth $u^\dagger$. In both cases $\delta = 10^{-2}$ and $\varepsilon = 0.3$.

the true and noisy data with $\delta = 10^{-2}$ for this setting. In Figure 5.2, we have the reconstructions $u_{\alpha,\varepsilon}^\delta$ and u_α^δ. In Figure 5.2a we have $q = 1$ and the chosen regularization parameter is $\alpha = 10^{-3}$ while in Figure 5.2b we have $q = 2$ and $\alpha = 10^{-2}$. In both cases $u_{\alpha,\varepsilon}^\delta$ has been computed with $\varepsilon = 0.3$ and compared to the solution u_α^δ we see that it is a better approximation of $u^\dagger$. The use of tolerances seems to be beneficial in this case but one needs to also investigate different settings and values of ε.

Now we turn to the second case, in which we consider that parts of the true solution lie outside the tolerance area $\mathrm{Tube}_\varepsilon(u^*)$. Figure 5.3 shows the approximations created with $q = 2$, $\delta = 10^{-2}$ and $\alpha = 10^{-2}$ and different tolerances for examining the behavior of $u_{\alpha,\varepsilon}^\delta$ based on ε. Note that here the true solution is different than the one considered before but we still use the same reference solution $u^*(x) = \sin(2\pi x)$. In Figures 5.3a–5.3d we see that for smaller ε the reconstruction $u_{\alpha,\varepsilon}^\delta$ is similar to u_α^δ, while once the tolerance gets larger, $u_{\alpha,\varepsilon}^\delta$ is improved. This is explained by the fact that the errors below the threshold ε are neglected. As a result $u_{\alpha,\varepsilon}^\delta$ is brought closer to the ground truth $u^\dagger$. What can also be observed in these figures is that the parts of $u^\dagger$ that are outside of the tolerance area are not well approximated, an effect that is more pronounced for smaller values of ε. This is an effect from the choice of the regularization parameter α, which in this case is not considered to be too small for the reason of incorporating the tolerances. This means that the regularization becomes stronger, which is something that, in general, would be a problem for the minimization of the discrepancy term. However, the presence of tolerances actually prohibits the regularization functional to take over, and enables the recovery of good approximations. Similar findings are observed in the case $q = 1$



(a) $\varepsilon = 0.1$

(b) $\varepsilon = 0.3$

(c) $\varepsilon = 0.4$

(d) $\varepsilon = 0.6$

Figure 5.3: Reconstructions $u_{\alpha,\varepsilon}^\delta$ computed with $q = 2$, $\delta = 10^{-2}$ and $\alpha = 10^{-2}$ and different tolerance values ε. The reconstruction $u_{\alpha,\varepsilon}^\delta$ is compared to u_α^δ and $u^\dagger$.

and thus are omitted here. These first results show the behavior of the solution with respect to the tolerance value and also, prove that good approximations of $u^\dagger$ can be produced when tolerances are incorporated in the regularization. The choice of α and ε should be further addressed because a very small α will not promote the tolerance assumption, whereas using a larger α can be more advisable. In addition, in Figure 5.3, we also observe that a larger ε leads to a better approximation but one has to be careful with its choice as it can make the regularized solution less smooth in points on the boundary of the tolerance area. This effect is more pronounced in Figure 5.3d. It is therefore clear that selecting appropriate values for ε and α is an important task. Except for the tuning of α and ε, we are also interested in the behavior of the approximation error when tolerances are taken into account.

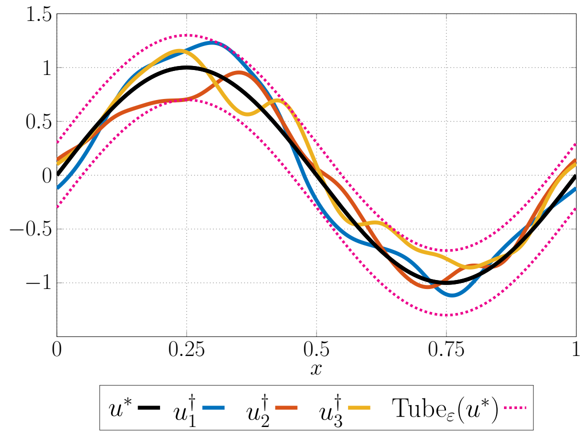


Figure 5.4: Example of u_i for $i = 1, 2, 3$ with $u^*(x) = \sin(2\pi x)$ and $\varepsilon = 0.3$.

5.1.1 Error behavior

We continue now by examining the behavior of the approximation error for different tolerances. We assume the same reference solution u^* and we use various values of $\varepsilon \in [0, 1.2]$. For each value of ε we define the corresponding tolerance area $\mathrm{Tube}_\varepsilon(u^*)$ as in (5.3) and we perform $M = 50$ simulations over which we calculate the mean approximation error $\|u^\delta_{\alpha,\varepsilon} - u^\dagger\|_{q,\varepsilon}$.

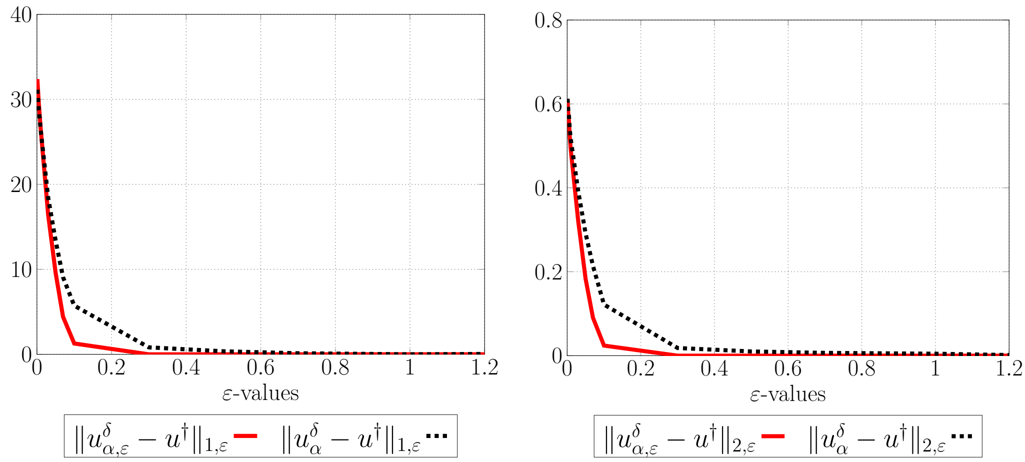
We denote the true solution by $u_i^\dagger$, where $i = 1, \ldots, M$ is the index of the i-th run. Each $u_i^\dagger$ is generated as a random and smooth perturbation of u^* inside the tolerance area $\mathrm{Tube}_\varepsilon(u^*)$. This is done by a convolution of a normally distributed random vector with the zero-mean Gaussian distribution with standard deviation $\sigma = 8 \cdot 10^{-2}$ (for smoothing). This random vector is further weighted by the tolerance value ε and then added to u^* as shown in (5.5). Therefore, the ground truth and the noisy data are computed by

$$u_i^\dagger = u^* + \varepsilon s_i \in \mathrm{Tube}_\varepsilon(u^*), \tag{5.5}$$

$$v_i^\delta = K u_i^\dagger + \hat{\eta}(\delta), \tag{5.6}$$

for a smooth random s_i with $i = 1, \ldots M$, and additive noise $\hat{\eta}(\delta)$ for noise level δ. The noise level $\delta = 7 \cdot 10^{-2}$ and the regularization parameter $\alpha = 10^{-3}$ are kept unchanged throughout all simulations since we examine the resulting error for different tolerances $\varepsilon \in [0, 1.2]$. Figure 5.4 is an illustration of the $u_i^\dagger$ created for $i = 1, 2, 3$.

Figure 5.5 shows the error between the true solution $u^\dagger$ and the reconstructions u^δ_α and $u^\delta_{\alpha,\varepsilon}$ computed in the ε-insensitive measure for different values of ε. Both cases for $q = 1$ and $q = 2$, reveal that the error obtained in our approach (red solid line) is smaller (or equal) than the error calculated for the generalized Tikhonov minimizers (black dashed line).



(a) $q = 1$ (b) $q = 2$

Figure 5.5: The mean approximation error over 50 runs using the ε-insensitive measure. In both cases, we consider $\alpha = 10^{-3}$, noise level $\delta = 7 \cdot 10^{-2}$ and tolerance values $\varepsilon \in [0, 1.2]$.

5.1.2 Choosing the regularization parameter α

The potential of obtaining better results when using tolerances takes us to the step of examining how we can increase the quality of our reconstructions. An important task in regularization is the choice of the regularization parameter $\alpha > 0$. In a certain application an indication for the appropriate size of tolerances may exists, meaning that ε cannot be arbitrarily large. In the following, we examine the parameter selection for α using the L-curve and the discrepancy principle and we test them for different values of tolerance. In Section 2.2.2 we introduced the idea for both parameter choice strategies.

In Figure 5.6 the L-curve for $q = 1$ (left) and $q = 2$ (right) using five different values of ε and for $\alpha \in [10^{-12}, 1]$ is shown. In order to compare the L-curves in a similar scale, we assume $u^*(x) = \sin(2\pi x)$ and the true solution is taken as a smooth perturbation inside the tolerance area for the fixed value $\varepsilon^\dagger = 0.5$ and is used in all simulations for all different values of ε. In this figure one can observe the different scaling (the L-curve is shifted down for larger ε) due to the value of the corresponding tolerance. Moreover, when ε and α both become large, the regularization term (y-axis of the L-curve) goes faster to zero. In both cases the L-curve is not as sharp as the one we normally obtain in the classical Tikhonov regularization. The nature of the tolerance function d_ε indicates that choosing optimally α and ε is not straightforward through the L-curve.

A different method, which is based on the noise level in the data, is the so-called Morozov's discrepancy principle [Bon08, Mor66]. Given an estimate of the

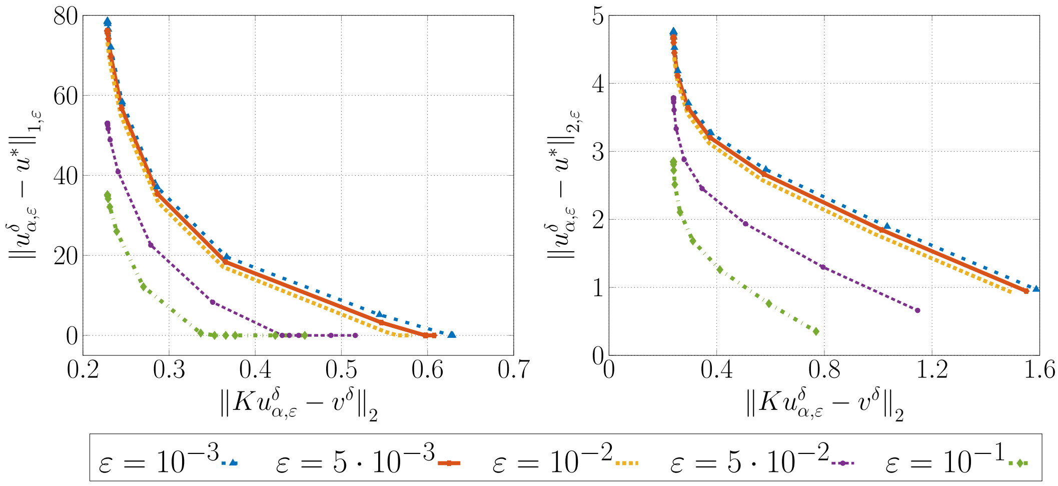


Figure 5.6: L-curve for different values of tolerance ε. On the left we have the results for $q = 1$, and on the right for $q = 2$. In both cases $p = 2$, $\delta = 0.015$, and $\alpha \in [10^{-12}, 1]$.

noise level, through the discrepancy principle, we accept reconstructions which create measurements with a similar error as the one in the noisy data. This translates into choosing the maximum $\alpha > 0$ such that

$$G(u^\delta_\alpha) := \|Ku^\delta_\alpha - v^\delta\| \leq \tau\delta, \qquad \text{for} \quad u^\delta_\alpha := \arg\min \mathcal{J}^{p,q}_{\alpha,\delta}(u), \tag{5.7}$$

with $\tau \geq 1$ and an estimate of the noise level $\delta > 0$. Note that here we implement the discrepancy principle for identifying the optimal regularization parameter α_{opt} when minimizing the generalized Tikhonov functional $\mathcal{J}^{p,q}_{\alpha,\delta}$ (no tolerance assumption). This optimal regularization parameter is subsequently used the regularization parameter in our functional $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$. That is, we compare the optimal reconstruction of the generalized Tikhonov to the minimizer obtained by our approach with tolerances in the regularization. In the following figures we show these results for the discrepancy principle in (5.7) with $\tau = 2$, and examine the cases $q = 1$ (in Figure 5.7) and $q = 2$ (in Figure 5.9) for the penalty norm.

In Figure 5.7 and Figure 5.9 we have the resulting approximations. In each of these figures our solution (in red) fits better the true solution (in black) than the one computed by the minimization of $\mathcal{J}^{p,q}_{\alpha,\delta}$ (in green). Moreover, in Figure 5.8 and Figure 5.10, respectively, we compare the absolute error of these reconstructions with respect to the true solution $u^\dagger$. Both cases show that tolerances can indeed advance the quality of the approximation but one has to further examine under which scenarios this happens.

The discrepancy principle often tends to select small α as the optimal one, which will not promote as much the use of tolerances. However, when the noise in the data is larger, the use of the discrepancy principle as the parameter choice rule makes the

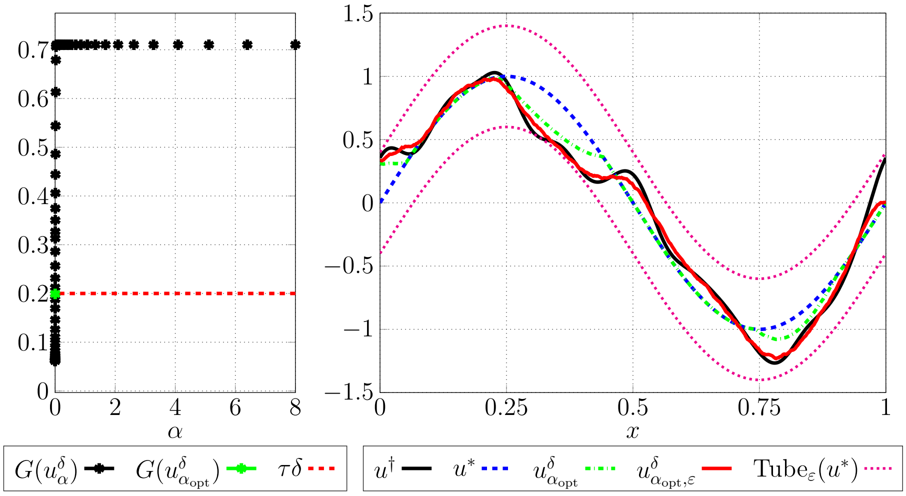


Figure 5.7: Morozov's discrepancy principle with $\delta = 10^{-1}$, $\tau = 2$, $\varepsilon = 0.4$ and $q = 1$. $G(u_\alpha^\delta)$ for various α and the optimal value $\alpha_{\text{opt}} = 10^{-3}$ (left). The reconstructions $u_{\alpha_{\text{opt}},\varepsilon}^\delta$, $u_{\alpha_{\text{opt}}}^\delta$ computed with α_{opt} and compared with $u^\dagger$ and u^* (right).

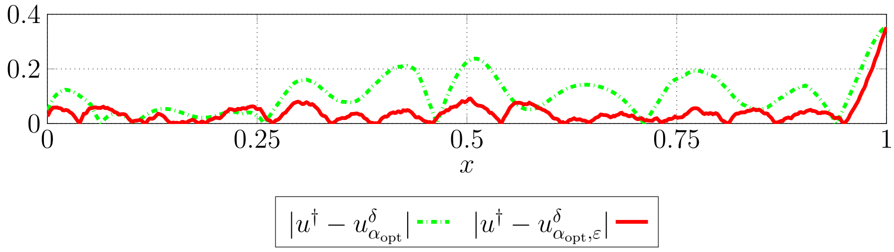


Figure 5.8: The absolute error of $u_{\alpha_{\text{opt}},\varepsilon}^\delta$ and $u_{\alpha_{\text{opt}}}^\delta$ that are shown in Figure 5.7.

regularization stronger. Especially in the case $\varepsilon \geq \delta \geq \alpha$, we obtain better results, as can be seen in both figures. In contrast to the classical regularization, here the tolerances do not allow the reconstruction to rely only on the reference solution.

The previous results were produced using the optimal regularization parameter for the generalized Tikhonov functional $\mathcal{J}_{\alpha,\delta}^{p,q}$. However, we can also use the discrepancy principle for finding the optimal regularization parameter for our functional $\mathcal{J}_{\alpha,\delta,\varepsilon}^{p,q}$ directly, i.e., implementing the discrepancy principle

$$G(u_{\alpha,\varepsilon}^\delta) := \|Ku_{\alpha,\varepsilon}^\delta - v^\delta\| \leq \tau\delta, \quad \text{with} \quad u_{\alpha,\varepsilon}^\delta := \arg\min \mathcal{J}_{\alpha,\delta,\varepsilon}^{p,q}(u).$$

This is shown in Figure 5.11 where the discrepancy principle was used for finding the optimal regularization parameter $\alpha_{\text{opt}} = 0.69$ for $\tau = 4$, $\varepsilon = 0.5$ and $q = 2$. On the right part of the figure we observe that the tolerances enhance indeed the quality of the reconstruction. In Figure 5.12 we compare the absolute error of the generalized

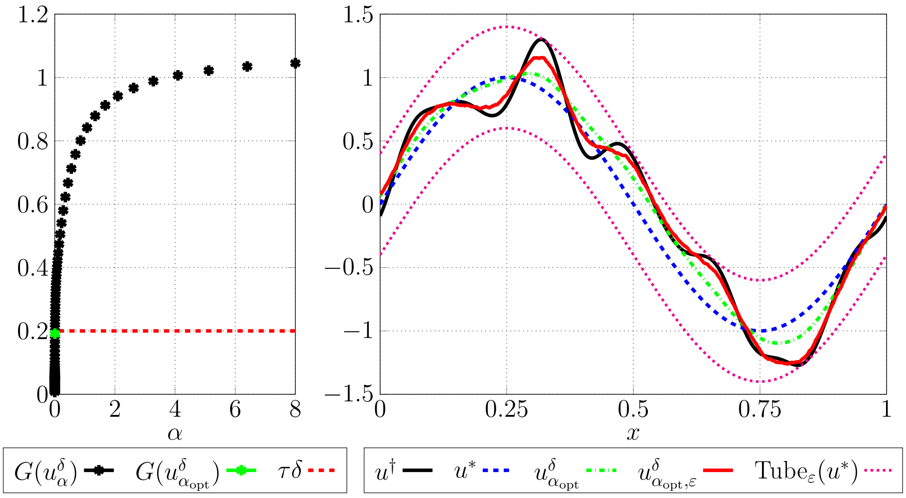


Figure 5.9: Morozov's discrepancy principle with $\delta = 10^{-1}$, $\tau = 2$, $\varepsilon = 0.4$ and $q = 2$. The discrepancy values for different α and for $\alpha_{\mathrm{opt}} = 0.0063$ (left). The reconstructed solutions $u^\delta_{\alpha_{\mathrm{opt}}}$ and $u^\delta_{\alpha_{\mathrm{opt}},\varepsilon}$ with α_{opt} compared with $u^\dagger$ and u^* (right).

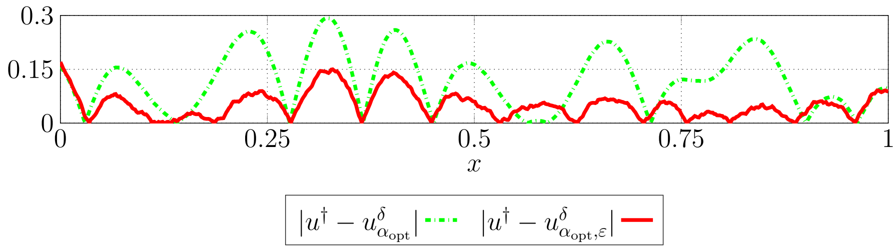


Figure 5.10: The absolute error of $u^\delta_{\alpha_{\mathrm{opt}},\varepsilon}$ and $u^\delta_{\alpha_{\mathrm{opt}}}$ that are shown in Figure 5.9.

Tikhonov minimizer and our solution with respect to the ground truth and confirm that our approximation is closer to $u^\dagger$.

In the left part of Figure 5.11, we also observe that the values of the discrepancy principle $G(u^\delta_{\alpha,\varepsilon})$ are not monotonically increasing, which means that the existence of an optimal α satisfying the discrepancy principle might not be guaranteed [AR09]. This phenomenon also indicates that τ should be larger (in Figure 5.11 we have used $\tau = 4$) in order for the discrepancy principle to be satisfied. These are only some first results on how to choose the regularization parameter when incorporating tolerances. Concerning the parameter choice rules, other strategies could be considered, for example, relaxation of the discrepancy principle as proposed in [AR09, Ram02], a generalization of the L-curve [BKM02] or other heuristic rules as proposed in [JL10].

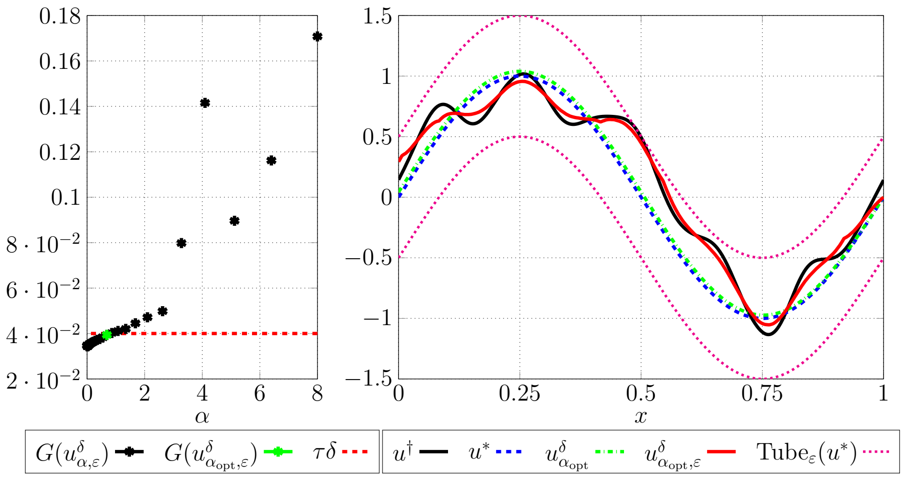


Figure 5.11: Morozov's discrepancy principle with $\delta = 10^{-1}$, $\tau = 4$, $\varepsilon = 0.5$ and $q = 2$. The discrepancy values for different α and the optimal parameter $\alpha_{\text{opt}} = 0.69$ (left). The reconstructed $u^\delta_{\alpha_{\text{opt}}}$ and $u^\delta_{\alpha_{\text{opt}},\varepsilon}$ for α_{opt} compared with $u^\dagger$ and u^* (right).

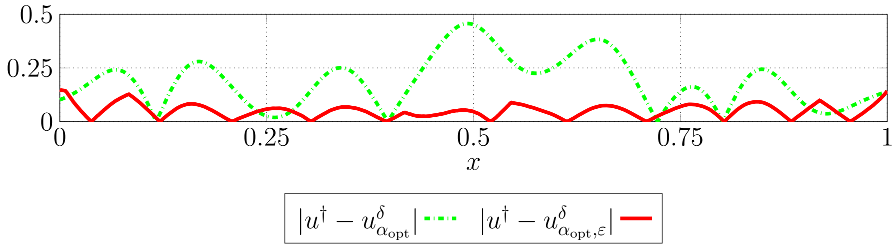


Figure 5.12: The absolute error of $u^\delta_{\alpha_{\text{opt}},\varepsilon}$ and $u^\delta_{\alpha_{\text{opt}}}$ that are shown in Figure 5.11.

5.2 Sparse solutions via the elastic net approach with tolerances

In this section we consider the same problem with the operator equation as in (5.2) and we address the sparsity of the reconstructed solutions. To this end, we consider the minimization of the elastic net functional $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u)$ which was proposed in Chapter 4 and includes an ℓ^2-penalty with the tolerance measure and an ℓ^1-penalty term for sparsity. Our goal here is to verify that the addition of the ℓ^1-penalty allows us to reconstruct sparser solutions compared to those obtained from the minimization of $\mathcal{J}^{2,2}_{\alpha,\delta,\varepsilon}(u)$.

We consider the discretized version of the numerical differentiation problem (5.2) with $N = 1000$ discretization points and as true solution $u^\dagger \in \mathbb{R}^N$ of the problem $Ku^\dagger = v$, with discretized operator $K : \mathbb{R}^N \to \mathbb{R}^N$ in (5.2), we assume the one shown in Figure 5.13. With this true solution, we examine sparsity with respect to the support of the function, i.e., the nodal basis $\{e_i\}_{i-1,\dots,N}$, which in this case is

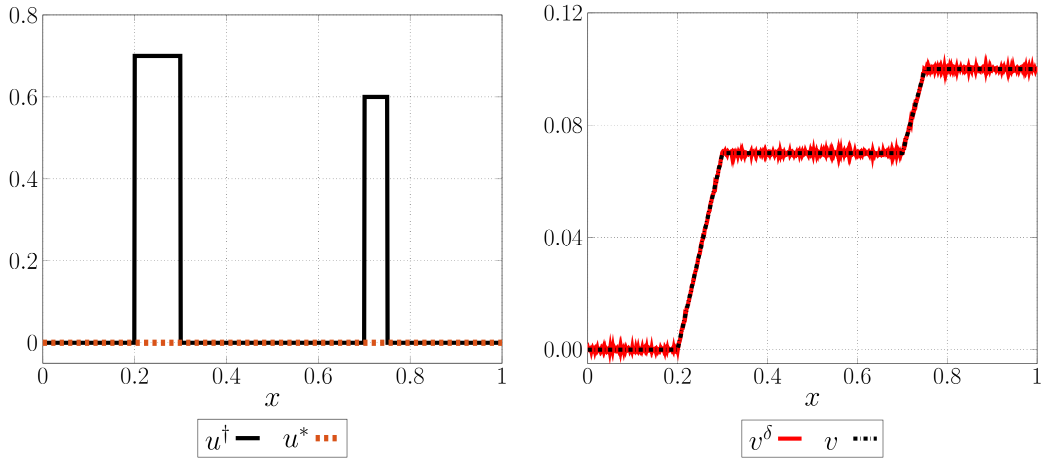


Figure 5.13: The ground truth $u^\dagger$ and the reference solution u^* (left). The true data v and the noisy data v^δ corrupted by 2% additive Gaussian noise (right).

sparse since there are only a few non-zero elements in $u^\dagger$. In addition, we consider as reference solution the zero function $u^*(x) = 0$ for $x \in [0, 1]$. The noisy data v^δ are created from the true data $v = Ku^\dagger$ with the addition of Gaussian noise with noise level $\delta = 2 \cdot 10^{-2}$. The true and the noisy data are shown in Figure 5.13.

As explained in our theoretical analysis but also in our previous numerical results, the choice of the regularization parameters is important for the quality of the reconstruction. Parameter choice rules, such as Morozov's discrepancy principle or a generalized L-curve [BKM02], can be used for the optimal selection of the regularization parameters α and β in the elastic net functional. A simple way of implementing the discrepancy principle is to consider $\beta = \alpha\eta$ with $\eta > 0$ and find the optimal η. This follows the idea that was used in our theoretical analysis in Chapter 4 for dealing with a single regularization parameter. Alternatively, one could fix β and focus on optimizing α (for example using the discrepancy principle), which is the parameter that influences the use of tolerances.

With a specific application in mind, the use of a parameter choice rule is highly recommended for appropriately selecting α and β. However, here we consider a simple example in which we are primarily interested in demonstrating the effect of tolerances in the solution while obtaining sparsity. For this reason we do not want to focus on further examining parameter choice strategies. Instead, we consider that α should not be too small in order for tolerances to be taken into account. In addition, it makes sense that β is not larger than α.

With that being clarified, in the following we consider $\alpha = 5 \cdot 10^{-4}$ and $\beta = 4 \cdot 10^{-5}$ and we further assume the tolerance value $\varepsilon = 0.2$. For this setting, we consider the minimization of the elastic net functional $\mathcal{J}^{\delta}_{\alpha,\beta,\varepsilon}$ and compare the regularized solution to that of our Tikhonov functional $\mathcal{J}^{2,2}_{\alpha,\delta,\varepsilon}$. In addition, we have computed the solutions of the standard ℓ^1-regularization and the classical elastic net approach.

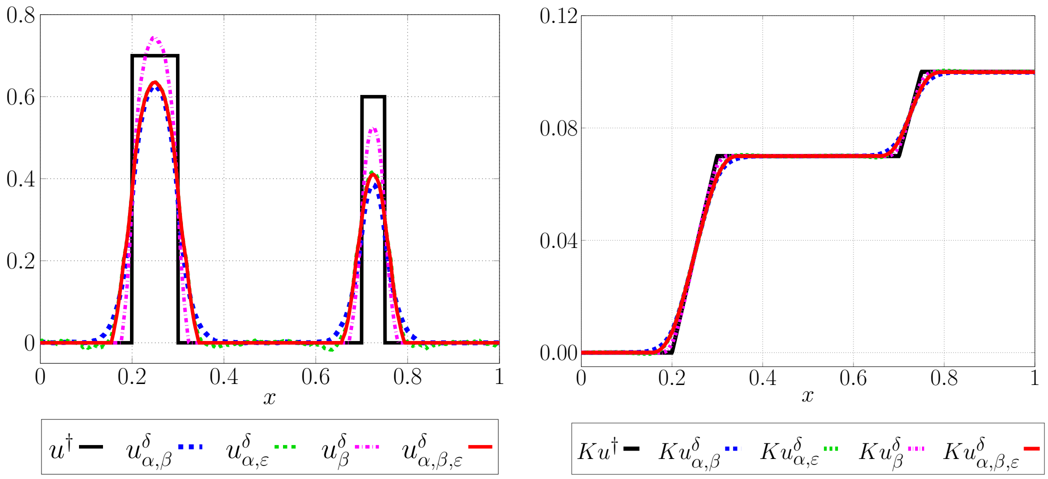


Figure 5.14: Reconstructed solutions (left) and reconstructed data (right) for regularization parameters $\alpha = 5 \cdot 10^{-4}$ and $\beta = 4 \cdot 10^{-5}$.

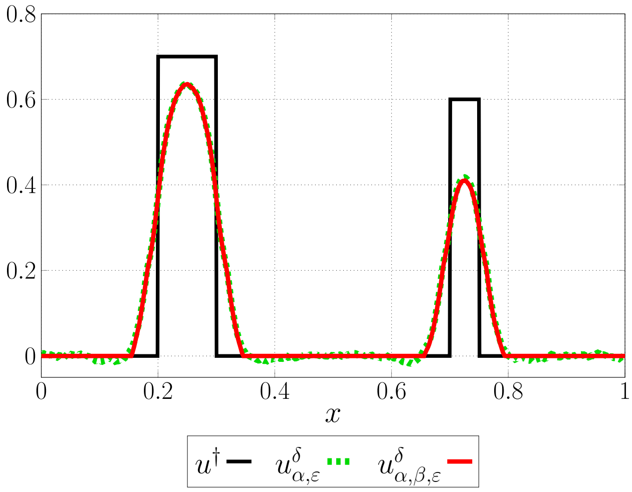


Figure 5.15: Comparison of the regularized solutions $u^\delta_{\alpha,\varepsilon}$ and $u^\delta_{\alpha,\beta,\varepsilon}$ to the true solution $u^\dagger$.

The regularized solution of the ℓ^1-regularization is denoted by $u^\delta_\beta := \arg\min \mathcal{J}^\delta_\beta(u)$ and that of the classical elastic-net by $u^\delta_{\alpha,\beta} := \arg\min \mathcal{J}^\delta_{\alpha,\beta}(u)$. All approximations have been produced using the same noisy data v^δ, tolerance value ε and regularization parameters α and β. For the numerical minimization of the functionals we have used the subgradient algorithm from [GMPK20] that was introduced in the beginning of the chapter. All reconstructions have been obtained after 3500 iterations of the algorithm.

In Figure 5.14 we show all the reconstructed solutions (on the left) and the corre-

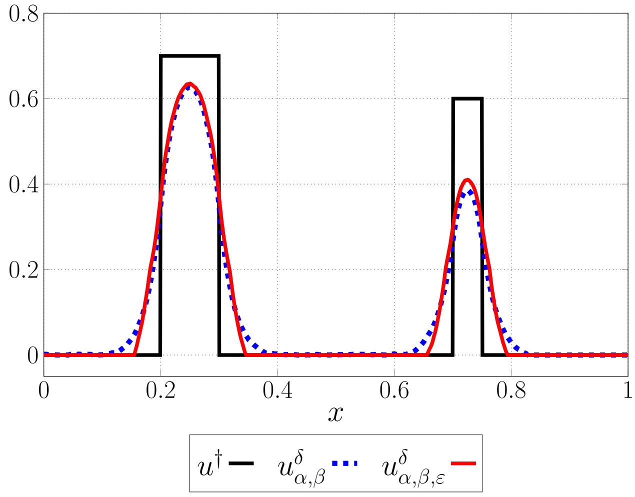


Figure 5.16: Comparison of the reconstructions $u^\delta_{\alpha,\beta}$ and $u^\delta_{\alpha,\beta,\varepsilon}$ obtained from minimization of the classical elastic net and the elastic net with tolerances, respectively.

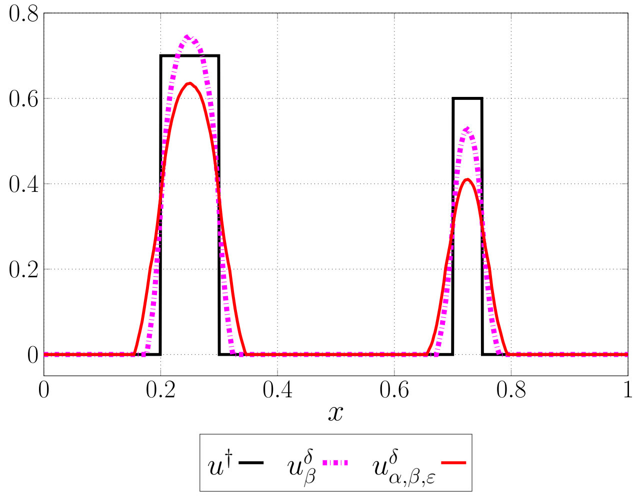


Figure 5.17: Comparison of the reconstructions u^δ_β and $u^\delta_{\alpha,\beta,\varepsilon}$ obtained from minimization of the ℓ^1-regularization functional and the elastic net with tolerances, respectively.

sponding reconstructed data (on the right). For the reader's convenience, we present separate figures as well: Figure 5.15 shows the reconstructed solutions $u^\delta_{\alpha,\varepsilon}$ and $u^\delta_{\alpha,\beta,\varepsilon}$ resulting from the minimization of our proposed functionals, namely the Tikhonov functional $\mathcal{J}^{2,2}_{\alpha,\delta,\varepsilon}$ and the elastic net functional $\mathcal{J}^\delta_{\alpha,\beta,\varepsilon}$, respectively. In the plot we see that $u^\delta_{\alpha,\beta,\varepsilon}$ is indeed more sparse compared to $u^\delta_{\alpha,\varepsilon}$ since the latter approximates poorly the sparse support of $u^\dagger$. With this result, our goal for promoting sparsity in

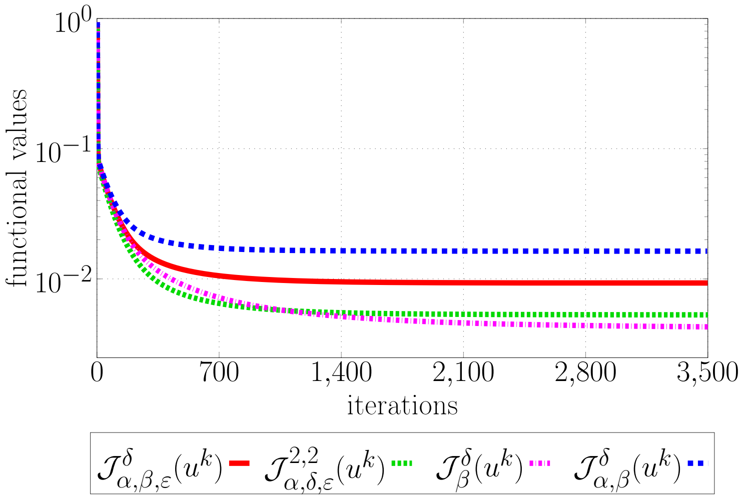


Figure 5.18: Minimization history of the functionals over 3500 iterations of the subgradient algorithm.

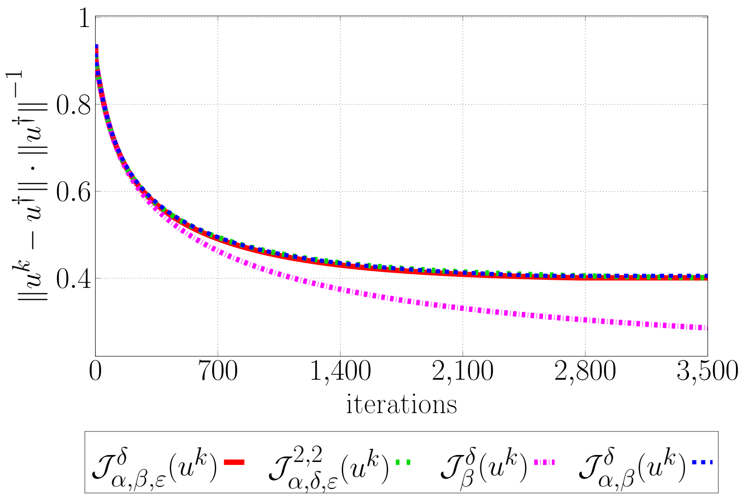


Figure 5.19: Relative error $\|u^k - u^\dagger\| \cdot \|u^\dagger\|^{-1}$ of the iterate u^k over the course of 3500 iterations for the minimization of the respective functionals.

the solution via the elastic net approach is confirmed.

Furthermore, comparing our solution $u^\delta_{\alpha,\beta,\varepsilon}$ to $u^\delta_{\alpha,\beta}$ in Figure 5.16, we find that our elastic net approach with tolerances has a more pronounced effect of sparsity than the classical elastic net solution. Together with our comment from the comparison of $u^\delta_{\alpha,\varepsilon}$ and $u^\delta_{\alpha,\beta,\varepsilon}$, this further indicates that while tolerances alone do not lead to sparsity, when they are combined in the elastic net approach they can further improve the sparsity effect in the reconstruction.

On the other hand, in Figure 5.17 we compare the solution $u^\delta_{\alpha,\beta,\varepsilon}$ to u^δ_β of the

ℓ^1-regularization and see that our approach does not offer as much localization as u^δ_β does around the nonzero part of the true solution. In addition, u^δ_β approximates the true solution better than $u^\delta_{\alpha,\beta,\varepsilon}$ for the same number of iterations of the subgradient algorithm. This indicates that the solution of the ℓ^1-regularization converges faster than $u^\delta_{\alpha,\beta,\varepsilon}$. This effect can also be observed in Figure 5.18 and Figure 5.19, where the minimization history of the functionals and the ℓ^2-norm relative error $\|u^k - u^\dagger\|_2/\|u^\dagger\|_2$ of the iterates u^k are shown over 3500 iterations of the algorithm.

Additional results with tolerance value $\varepsilon = 0.5$. After discussing the results obtained with tolerance $\varepsilon = 0.2$, we also include the comparison plots between $u^\delta_{\alpha,\beta,\varepsilon}$ and $u^\delta_{\alpha,\varepsilon}$, $u^\delta_{\alpha,\beta}$, u^δ_β using the same parameter setting but a larger tolerance value $\varepsilon = 0.5$. Our aim is to further explore the effect of tolerances on the computed reconstructions. Overall, the behavior of $u^\delta_{\alpha,\beta,\varepsilon}$ remains the same in comparison to $u^\delta_{\alpha,\beta}$ and $u^\delta_{\alpha,\varepsilon}$. However, what can further be observed is that a higher tolerance value corrects the localization issue (see Figure 5.22) that was pointed out in the previous comparison between u^δ_β and our solution $u^\delta_{\alpha,\beta,\varepsilon}$. This leads to a reconstruction that is closer to the solution of the ℓ^1-regularization and the approximation of the nonzero part of $u^\dagger$ is improved since $u^\delta_{\alpha,\beta,\varepsilon}$ now appears to follow better the structure of the true solution.

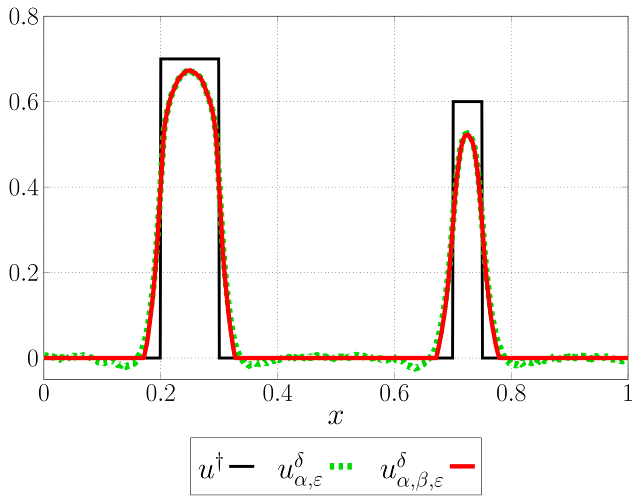


Figure 5.20: Comparison of the regularized solutions $u^\delta_{\alpha,\varepsilon}$ and $u^\delta_{\alpha,\beta,\varepsilon}$ with $\varepsilon = 0.5$ to the true solution $u^\dagger$.

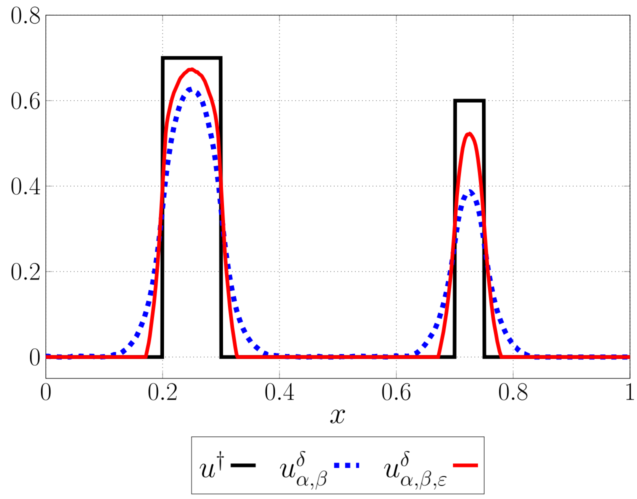


Figure 5.21: Comparison of the regularized solutions $u^\delta_{\alpha,\beta}$ and $u^\delta_{\alpha,\beta,\varepsilon}$ with $\varepsilon = 0.5$ to the true solution $u^\dagger$.

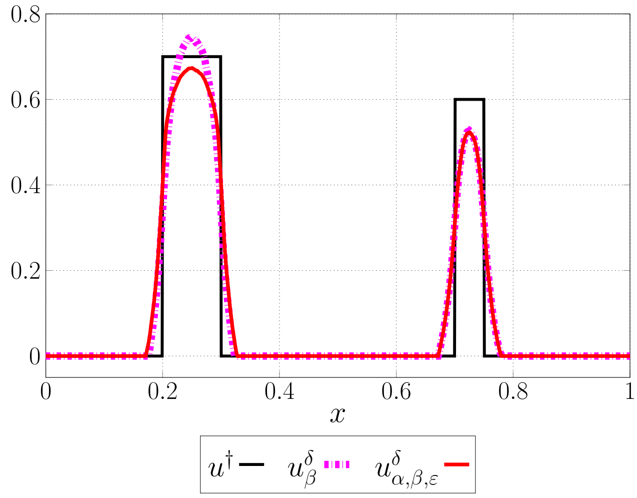


Figure 5.22: Comparison of the regularized solutions u^δ_{β} and $u^\delta_{\alpha,\beta,\varepsilon}$ with $\varepsilon = 0.5$ to the true solution $u^\dagger$.

5.2.1 Verification of the convergence rate

We conclude our numerical testing for the elastic net with tolerances with the verification of the convergence rate for the minimizer $u^\delta_{\alpha,\beta,\varepsilon} := \arg\min \mathcal{J}^\delta_{\alpha,\beta,\varepsilon}(u)$ that we proved in Theorem 4.7. This can be done by visualizing the approximation error $\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2}$ in a loglog plot with respect to different noise levels δ.

We consider $\delta = 10^i$ for $i \in \{-5, \ldots, 0\}$ and noisy data v^δ such that $\|v - v^\delta\| \leq \delta$. In addition, we assume $\alpha = \delta$ and $\varepsilon = \sqrt{\delta}$ to fulfill the assumptions of the

Theorem 4.7 and compute $u^\delta_{\alpha,\beta,\varepsilon}$ for $\beta = \alpha\eta$ with $\eta \in \{\frac{1}{4}, \frac{1}{2}, 1\}$. Figure 5.23 shows the approximation error for the different noise levels δ and we observe that indeed the rate is very close to the square-root-like estimate that we have proved in our analysis. For example, in the case $\beta = \alpha/2$ we have $\|u^\delta_{\alpha,\beta,\varepsilon} - u^\dagger\|_{\ell^2} \approx \tilde{c}\sqrt{\delta}$ with $\tilde{c} \approx 4.3$.

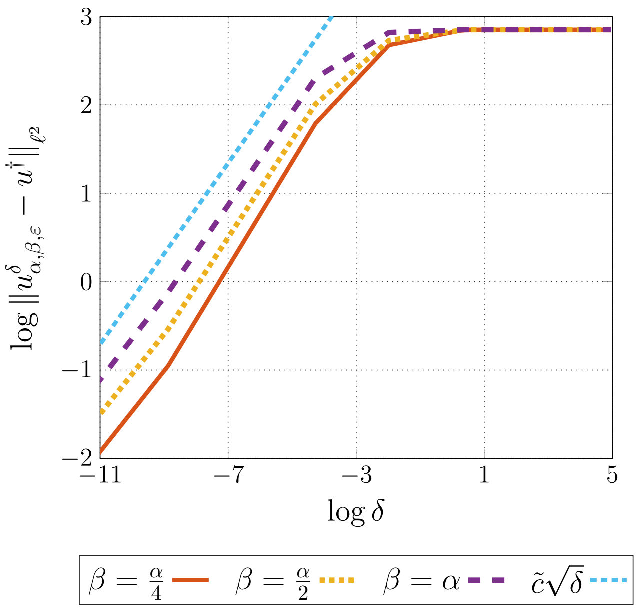


Figure 5.23: Reconstruction error with respect to different noise levels δ for regularization parameter $\alpha = \delta$ and tolerance $\varepsilon = \sqrt{\delta}$.

A last comment concerning the above convergence rates is that here we have used the assumption $\varepsilon = \sqrt{\delta}$, which is required in the convergence rates Theorem 4.7. This assumption on the tolerance has been made for applying the inequality (3.5c) that enables us to derive the error estimate for the distance between $u^\delta_{\alpha,\beta,\varepsilon}$ and $u^\dagger$ in the ℓ^2-norm. If we wish to consider a larger tolerance value (as we did in the previous examples) the theoretical error estimate can still be obtained using the Bregman distance instead, which leads to general error estimates. For the purpose of illustrating our theoretical results, we have chosen to show the particular choice $\varepsilon = \sqrt{\delta}$ for which the required inequality is then fulfilled.

Our numerical considerations in this chapter concerned the minimization of the functionals $\mathcal{J}^{p,q}_{\alpha,\delta,\varepsilon}$ and $\mathcal{J}^{\delta}_{\alpha,\beta,\varepsilon}$ that were proposed in Chapters 3 and 4, respectively. The aim of this chapter was to demonstrate the influence of tolerances in the regularized solutions and to discuss the quality of reconstructions compared to standard regularization methods such as the classical elastic net and the sparsity-constrained Tikhonov regularization. In addition, we compared the solutions obtained from each of our two functionals and proved that the adaptation to the elastic net approach is a viable approach for the acquisition of sparse reconstructions.

Chapter

6 Conclusion

In this work, we investigated the regularization of ill-posed inverse problems using tolerances. We proposed a Tikhonov-type functional with the ε-insensitive distance as the tolerance measure in the regularization term and analyzed the well-posedness of its minimizers. When considering the ε-insensitive measure we saw that, without further assumptions, only weak convergence can be expected. Moreover, we proved the rate of convergence for minimizers of the functional using the Bregman distance and concluded that the choice $\alpha \sim \delta^{p-1}$ leads to the rate $\mathcal{O}(\delta)$.

Furthermore, intending to promote sparsity in the regularized solution, we extended our approach with tolerances to the idea of elastic net regularization. After establishing the stability and weak convergence of minimizers, we proceeded with the convergence rates analysis. With a general source condition and the a priori parameter choice $\alpha \sim \delta$, we were able to conclude that the minimizers converge to the true solution with rate $\mathcal{O}(\delta^{1/2})$.

After our theoretical investigation, we demonstrated the effect of tolerances in the reconstructed solutions for both the Tikhonov- and the elastic net functionals on a few numerical examples. Our analytical and numerical examinations show that the assumption of tolerances in the regularization can be advantageous and leads to good reconstructions. Particularly in the presence of higher noise in the available data, the inclusion of tolerances can be useful because it provides more stability. Furthermore, it leads to a stronger regularization effect but without the unwanted effect of data underfitting. Concerning the numerical results of the elastic net with tolerances, we have successfully illustrated that sparsity is indeed promoted in the reconstructed solution and, therefore, such an approach can be useful in applications that require the reconstruction of sparse structures.

Further extension of our work is possible in different directions. For instance, one could investigate the connection between the noise level δ and the tolerance value

ε in more detail. As we saw in the numerical examples, when tolerances are larger than the noise level we obtain better reconstructions, which is a first indication of the relationship between δ and ε.

It is also important to invest in the optimal tuning of the regularization parameters and the tolerance ε. In our numerical testing, we considered ε as if it was given *a priori*, and used the discrepancy principle for selecting α. We also discussed why the L-curve might not be the best strategy for finding optimal parameters when the ε-insensitive loss function is used. Another idea is to choose ε simultaneously with the regularization parameters, which would imply that ε is viewed as such, too. This consideration would require further numerical testing and possibly the use of different parameter choice rules. Such experiments, however, should be done with an actual application in mind so that the tolerances have a physical meaning (instead of being arbitrarily chosen).

Lastly, it is even possible to use a different tolerance measure instead of the ε-insensitive measure. In that case, the new choice should belong to the same class of functions as the ε-insensitive distance, namely, to be a convex loss function that has some kind of an ε-insensitive area. Examples of such functions can be found in [SS04]. The interpretation of the tolerance area and the resulting solutions will naturally vary based on the choice of tolerance function but the theory will remain valid.

The initial inspiration of our work originates from the use of the ε-insensitive distance in the fidelity term of minimization functionals in the framework of support vector regression and inverse problems. Transferring the ε-insensitive measure from the fidelity to the regularization term of a Tikhonov functional, proves to be an interesting approach that allows us to reconstruct solutions with a specific *a priori* information about the region to which they belong to. Our theoretical analysis proves that the proposed method leads to minimizers that satisfy all important results required by the theory of variational regularization methods. Together with the above discussion on possible extensions, we are confident that our work on regularization with tolerances can be useful in applications that request such solutions.

Appendix

A Calculation of subdifferentials

In this appendix we include the detailed computation of the subdifferential formulas that were used in the previous chapters. Each formula is derived by the application of the sum, scalar multiplication and chain rules for subdifferential computation, which are found in Lemma 2.12. For ensuring that each claim is correct, we check that the suggested subdifferential of the convex functional f at a point u_0 satisfies the inequality

$$f(u) \geq f(u_0) + z(u - u_0) \tag{A.1}$$

for all $u \in \mathbb{R}$ and $z \in \partial f(u_0)$, recalling Definition 2.9.

Proposition A.1. *The subdifferential of* $|u|_\varepsilon = d_\varepsilon(u) = \max\{|u| - \varepsilon, 0\}$ *at a point* $u_0 \in \mathbb{R}$ *and for* $\varepsilon > 0$ *is given by*

$$\partial d_\varepsilon(u_0) = \begin{cases} -1 & u_0 < -\varepsilon \\ [-1, 0] & u_0 = -\varepsilon \\ 0 & |u_0| < \varepsilon \\ [0, 1] & u_0 = \varepsilon \\ 1 & u_0 > \varepsilon \end{cases}. \tag{A.2}$$

Proof. Each element of the subdifferential in (A.2) must satisfy the inequality (A.1) with $f(u) = d_\varepsilon(u)$, i.e., we check that

$$d_\varepsilon(u) \geq d_\varepsilon(u_0) + z(u - u_0). \tag{A.3}$$

To confirm the validity of this inequality, we consider all possible cases for $z \in \partial d_\varepsilon(u_0)$ as stated in (A.2) and for $d_\varepsilon(u)$ we look at the cases occurring in its definition, which are $|u| > \varepsilon$ and $|u| \leq \varepsilon$.

1. For $u_0 < -\varepsilon$ we have $z = -1$ and $d_\varepsilon(u_0) = |u_0| - \varepsilon$. The inequality (A.3) reads

$$d_\varepsilon(u) \geq -u_0 - \varepsilon - u + u_0.$$

a. If $u < -\varepsilon$ then $-u - \varepsilon \geq -u_0 - \varepsilon - u + u_0$, which is always true.

b. If $|u| \leq \varepsilon$ then $0 \geq -u_0 - \varepsilon - u + u_0$ implies $u + \varepsilon \geq 0$, which is true since $-\varepsilon \leq u \leq \varepsilon$.

c. If $u > \varepsilon$ then $u - \varepsilon \geq -u_0 - \varepsilon - u + u_0$ implies $u \geq 0$, which is true.

2. For $u_0 = -\varepsilon$ we have $z \in [-1, 0]$ and $d_\varepsilon(u_0) = 0$. The inequality (A.3) becomes

$$d_\varepsilon(u) \geq z(u + \varepsilon).$$

a. If $u < -\varepsilon$ then $-u - \varepsilon \geq z(u + \varepsilon)$ and we check the endpoints for z:

$$\begin{aligned} z = -1 \ &: \ -u - \varepsilon \geq -u - \varepsilon, \text{ which holds true.} \\ z = 0 \ &: \ -u - \varepsilon \geq 0, \text{ which is true since } u < -\varepsilon. \end{aligned}$$

b. If $|u| \leq \varepsilon$ the inequality becomes $0 \geq z(u+\varepsilon)$ and we check the endpoints for z:

$$\begin{aligned} z = -1 \ &: \ 0 \geq -u - \varepsilon, \text{ which is true for } |u| \leq \varepsilon. \\ z = 0 \ &: \ 0 \geq 0, \text{ which is always true.} \end{aligned}$$

c. If $u > \varepsilon$ then $u - \varepsilon \geq z(u+\varepsilon)$ and we validate that it holds for $z \in [-1, 0]$ by checking for the endpoints of the interval:

$$\begin{aligned} z = -1 \ &: \ u - \varepsilon \geq -u - \varepsilon \text{ implying that } u \geq 0, \\ z = 0 \ &: \ u - \varepsilon \geq 0, \end{aligned}$$

both of which are true since $u > \varepsilon$.

3. For $|u_0| < \varepsilon$ we have $z = 0$ and $d_\varepsilon(u_0) = 0$. The inequality (A.3) becomes

$$d_\varepsilon(u) \geq 0$$

and is always true as $d_\varepsilon(u)$ is a nonnegative function.

4. For $u_0 = \varepsilon$ we have $z \in [0, 1]$ and $d_\varepsilon(u_0) = 0$. The inequality (A.3) becomes

$$d_\varepsilon(u) \geq z(u - \varepsilon).$$

a. If $u < -\varepsilon$ then $-u - \varepsilon \geq z(u - \varepsilon)$ and we check the endpoints for z:

$$\begin{aligned} z = 0 \ &: \ -u - \varepsilon \geq 0, \\ z = 1 \ &: \ -u - \varepsilon \geq u - \varepsilon, \end{aligned}$$

both of which are true for $u < -\varepsilon$.

b. If $|u| \leq \varepsilon$ then $0 \geq z(u-\varepsilon)$ and we check the endpoints for z:

$$\begin{aligned} z = 0 &: \ 0 \geq 0, \ \text{which is always true.} \\ z = 1 &: \ 0 \geq u - \varepsilon, \ \text{which is true since } |u| \leq \varepsilon. \end{aligned}$$

c. If $u > \varepsilon$ then $u - \varepsilon \geq z(u-\varepsilon)$ and we check the endpoints for z:

$$\begin{aligned} z = 0 &: \ u - \varepsilon \geq 0, \\ z = 1 &: \ u - \varepsilon \geq u - \varepsilon, \end{aligned}$$

both of which are true.

5. For $u_0 > \varepsilon$ we have $z = 1$ and $d_\varepsilon(u_0) = |u_0| - \varepsilon$. The inequality (A.3) becomes

$$d_\varepsilon(u) \geq u_0 - \varepsilon + u - u_0.$$

a. If $u < -\varepsilon$ then $-u - \varepsilon \geq -\varepsilon + u$, which is true since $u < -\varepsilon < 0$.

b. If $|u| \leq \varepsilon$ then $0 \geq -\varepsilon + u$ which is true since $|u| \leq \varepsilon$.

c. If $u > \varepsilon$ then $u - \varepsilon \geq -\varepsilon + u$, which holds true.

We have checked all possible cases and therefore, our claim for the subdifferential in (A.2) is correct. ■

Proposition A.2. *The subdifferential of $|u|_\varepsilon^2 = (d_\varepsilon(u))^2$ at a point $u_0 \in \mathbb{R}$ is given by*

$$\partial d_\varepsilon(u_0)^2 = 2 d_\varepsilon(u_0)\, \partial d_\varepsilon(u_0) = \begin{cases} 2(u_0 + \varepsilon) & u_0 \ < -\varepsilon \\ 0 & |u_0| \leq \ \ \varepsilon \\ 2(u_0 - \varepsilon) & u_0 \ > \ \ \varepsilon \end{cases}. \tag{A.4}$$

Proof. For $z \in \partial d_\varepsilon(u_0)^2$ we confirm that the subgradient inequality

$$d_\varepsilon(u)^2 \geq d_\varepsilon(u_0)^2 + z(u - u_0) \tag{A.5}$$

is always fulfilled. We have the following cases:

1. For $u_0 < -\varepsilon$ we have $z = 2(u_0 + \varepsilon)$ and $d_\varepsilon(u_0) = -u_0 - \varepsilon$. The inequality (A.5) in that case is

$$d_\varepsilon(u)^2 \geq (-u_0 - \varepsilon)^2 + 2(u_0 + \varepsilon)(u - u_0)$$

a. If $u < -\varepsilon$ then

$$\begin{aligned} (-u - \varepsilon)^2 &\geq (-u_0 - \varepsilon)^2 + 2(u_0 + \varepsilon)(u - u_0) \\ u^2 + \varepsilon^2 + 2u\varepsilon &\geq u_0^2 + \varepsilon^2 + 2u_0\varepsilon + 2(u_0 + \varepsilon)(u - u_0) \\ u^2 + 2u\varepsilon - u_0^2 - 2u_0\varepsilon &\geq 2u_0 u - 2u_0^2 + 2\varepsilon u - 2\varepsilon u_0 \\ u^2 + u_0^2 - 2u_0 u &\geq 0 \\ (u - u_0)^2 &\geq 0, \end{aligned}$$

which is always true.

b. If $|u| \leq \varepsilon$ then $d_\varepsilon(u) = 0$ and we have

$$
\begin{aligned}
0 &\geq (-u_0 - \varepsilon)^2 + 2(u_0 + \varepsilon)(u - u_0) \\
0 &\geq u_0^2 + \varepsilon^2 + 2u_0\varepsilon + 2(u_0 + \varepsilon)(u - u_0) \\
0 &\geq u_0^2 + \varepsilon^2 + 2u_0\varepsilon + 2u_0 u - 2u_0^2 + 2u\varepsilon - 2\varepsilon u_0 \\
u_0^2 - \varepsilon^2 - 2u_0 u - 2u\varepsilon &\geq 0 \\
(u_0 - \varepsilon)(u_0 + \varepsilon) - 2u(u_0 + \varepsilon) &\geq 0 \\
(u_0 - \varepsilon - 2u)(u_0 + \varepsilon) &\geq 0.
\end{aligned}
$$

Since $u_0 + \varepsilon < 0$ the inequality is fulfilled if and only if $u_0 - \varepsilon - 2u \leq 0$. This is easily confirmed by the following

$$u_0 - 3\varepsilon \leq u_0 - \varepsilon - 2u \leq u_0 - \varepsilon < 0.$$

c. If $u > \varepsilon$

$$
\begin{aligned}
|u - \varepsilon|^2 &\geq |-u_0 - \varepsilon|^2 + 2(u_0 + \varepsilon)(u - u_0) \\
u^2 + \varepsilon^2 - 2u\varepsilon &\geq u_0^2 + \varepsilon^2 + 2u_0\varepsilon + 2(u_0 + \varepsilon)(u - u_0) \\
u^2 - u_0^2 - 2u\varepsilon - 2u_0\varepsilon &\geq 2u_0 u - 2u_0^2 + 2\varepsilon u - 2\varepsilon u_0 \\
u^2 + u_0^2 - 2u_0 u - 4u\varepsilon &\geq 0 \\
(u - u_0)^2 &\geq 4u\varepsilon.
\end{aligned}
$$

The last inequality is always fulfilled since $-\varepsilon \leq u_0 \leq \varepsilon$ and $u > \varepsilon$.

2. For $|u_0| \leq \varepsilon$ we have $z = 0$ and $d_\varepsilon(u_0) = 0$. The inequality (A.5) becomes

$$d_\varepsilon(u)^2 \geq 0$$

and is always true as $d_\varepsilon(u)^2$ is a nonnegative function.

3. For $u_0 > \varepsilon$ we have $z = 2(u_0 - \varepsilon)$, $d_\varepsilon(u_0) = u_0 - \varepsilon$ and the inequality (A.5) becomes

$$|d_\varepsilon(u)|^2 \geq |u_0 - \varepsilon|^2 + 2(u_0 - \varepsilon)(u - u_0).$$

a. If $u < -\varepsilon$ then $d_\varepsilon(u) = -u - \varepsilon$ and we have

$$
\begin{aligned}
(-u - \varepsilon)^2 &\geq (u_0 - \varepsilon)^2 + 2(u_0 - \varepsilon)(u - u_0) \\
u^2 + \varepsilon^2 + 2u\varepsilon &\geq u_0^2 + \varepsilon^2 - 2u_0\varepsilon + 2u_0 u - 2u_0^2 - 2\varepsilon u + 2u_0\varepsilon \\
u^2 + u_0^2 - 2u_0 u + 4u\varepsilon &\geq 0
\end{aligned}
$$

We examine if the last inequality holds true. Using $u_0 > \varepsilon$ we have

$$\begin{aligned} u^2 + u_0^2 - 2u_0 u + 4u\varepsilon &> u^2 + \varepsilon^2 - 2\varepsilon u + 4u\varepsilon \\ &= u^2 + \varepsilon^2 + 2\varepsilon u \\ &= (u+\varepsilon)^2. \end{aligned}$$

From $u < -\varepsilon$ we have that $u+\varepsilon < 0$ and its square is always positive. Therefore, the inequality $u^2 + u_0^2 - 2u_0 u + 4u\varepsilon > (u+\varepsilon)^2 > 0$ is indeed satisfied.

b. If $|u| \leq \varepsilon$ then $d_\varepsilon(u) = 0$ and we have

$$\begin{aligned} 0 &\geq u_0^2 + \varepsilon^2 - 2u_0\varepsilon + 2u_0 u - 2u_0^2 - 2\varepsilon u + 2\varepsilon u_0 \\ 0 &\geq -u_0^2 + \varepsilon^2 + 2(u_0 - \varepsilon)u \\ u_0^2 - \varepsilon^2 - 2(u_0 - \varepsilon)u &\geq 0 \\ (u_0 - \varepsilon)(u_0 + \varepsilon) - 2(u_0 - \varepsilon)u &\geq 0 \\ u_0 + \varepsilon - 2u &\geq 0. \end{aligned}$$

The last inequality is true because $u_0 + \varepsilon > 2\varepsilon \geq 2u$.

c. If $u > \varepsilon$ then $d_\varepsilon(u) = u - \varepsilon$ and the inequality is

$$\begin{aligned} (u - \varepsilon)^2 &\geq (u_0 - \varepsilon)^2 + 2(u_0 - \varepsilon)(u - u_0) \\ u^2 + \varepsilon^2 - 2u\varepsilon &\geq u_0^2 + \varepsilon^2 - 2u_0\varepsilon + 2u_0 u - 2u_0^2 - 2\varepsilon u + 2u_0\varepsilon \\ u^2 + u_0^2 - 2u_0 u \geq 0 &\iff (u - u_0)^2 \geq 0 \end{aligned}$$

which is always true for all $u, u_0 > \varepsilon$.

We have confirmed all cases and therefore, our claim in (A.4) is true. ■

In the case of $u : \mathbb{R}^n \to \mathbb{R}^m$, we have the following subdifferentials.

Proposition A.3 (Subdifferential of $\mathcal{R}_{1,\varepsilon}(u)$)**.** *The subdifferential of the functional $\mathcal{R}_{1,\varepsilon}(u) = \|u\|_{1,\varepsilon}$ is given by*

$$\partial \mathcal{R}_{1,\varepsilon}(u) = \sum_i^m \partial \left| d_\varepsilon(u_i) \right|$$

with $\partial \left| d_\varepsilon(u_i) \right|$ computed via (A.2).

Proof. The sum rule for the calculation of subdifferentials in Lemma 2.12 can be generalized for computing the subdifferential of the sum of more than two functionals, which gives

$$\partial \mathcal{R}_{1,\varepsilon}(u) = \partial \sum_i^m \left| d_\varepsilon(u_i) \right| = \sum_i^m \partial \left| d_\varepsilon(u_i) \right|$$

with the subdifferential $\partial d_\varepsilon(u_i)$ defined by (A.2). ■

Proposition A.4 (Subdifferential of $\mathcal{R}_{2,\varepsilon}(u)$)**.** *The subdifferential of the functional* $\mathcal{R}_{2,\varepsilon}(u) = \|u\|_{2,\varepsilon}^2$ *is given by*

$$\partial\mathcal{R}_{2,\varepsilon}(u) = \sum_i^m \partial \left|d_\varepsilon(u_i)\right|^2$$

with $\partial(d_\varepsilon(u_i))^2$ *computed via* (A.4)*.*

Proof. The result follows from the generalized sum rule for subdifferential computation

$$\partial\mathcal{R}_{2,\varepsilon}(u) = \partial\|u\|_{2,\varepsilon}^2 = \partial \sum_i^m \left|d_\varepsilon(u_i)\right|^2 = \sum_i^m \partial \left|d_\varepsilon(u_i)\right|^2$$

with the subdifferential $\partial(d_\varepsilon(u_i))^2$ given by (A.4). ∎

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