Toric Arrangements

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Zusammenfassung

Diese Dissertation befasst sich mit einigen grundlegenden Fragen über die Topologie der Komplemente torischer Arrangements. Sie wurde während meines Aufenthalts an der Universität Bremen verfasst und entsteht aus einer gemeinsamen Arbeit mit Emanuele Delucchi, zum Teil veröffentlicht in [12] und zum Teil zur Veröffentlichung eingereicht als [11].

Ein *torisches Arrangement* ist eine endliche Familie von Untertori des komplexen Torus $(\mathbb{C}^*)^d$:

$$\mathscr{A} = \{K_1, \ldots, K_n\},$$

wobei die K_i Niveaumengen von Charakteren sind (siehe Definition 3.1). Solche Objekte haben eine kombinatorische Seite, in Gestalt des Layer-Verband und der Face-Kategorie, und eine topologische Seite, in Gestalt des Komplement

$$M(\mathscr{A}) := (\mathbb{C}^*)^n \backslash (\cup_{K \in \mathscr{A}} K)$$

zusammen mit seinen topologischen Invarianten. Das Leitmotiv, sowohl in der Literatur über torischen Arrangements als auch in dieser Dissertation, ist die Darstellung von topologischen Eigenschaften des Komplements $M(\mathscr{A})$ mit Hilfe seiner kombinatorischen Pendants.

Das Thema "torische Arrangements" erwächst aus dem Gebiet "Hyperplane Arrangements". Hyperplane Arrangements haben eine umfangreiche Tradition und wurden in den letzten vierzig Jahren viel untersucht. In der Einleitung zum Kapitel 1 wird die Entwicklung dieses Themas auch aus einer historischen Perspektive erläutert.

Torische Arrangements wurden zum ersten Mal von Looijenga in seiner Arbeit [30] betrachtet. Dort wird die Kohomologie des Komplements eines torischen Arrangements berechnet, um die Kohomologie von gewissen Moduli-Räumen zu untersuchen.

Später hat die Arbeit [16] von De Concini und Procesi die Aufmerksamkeit der Forscher aus dem Gebiet "Hyperplane Arrangements" erregt. Die Verfasser dieser Arbeit kamen zu Looijengas Ergebnis mit anderen Methoden. Darüber hinaus machten sie den Zusammenhang zwischen der Topologie des Komplements und der Kombinatorik des Arrangements deutlich (siehe, unter anderem, Theorem 3.17).

Das Thema wurde dann von Moci weiter entwickelt, in seiner Doktorarbeit und in den Publikationen [31, 33, 32]. Zu seinen Ergebnissen zählen die Konstruktion eines "wonderful models" für Komplemente torischer Arrangements und die Entwicklung eines Tutte-Polynoms für torische Arrangements. Ein ausführlicher Bericht über die Entwicklung der Theorie der torischen Arrangements befindet sich in der Einleitung zu Kapitel 3.

In dieser Dissertation beweisen wir zwei zentrale Resultate über die Topologie von torischen Arrangements, die jeweils ein Ergebnis aus dem Gebiet "Hyperebene Arrangements" verallgemeinern. Zuerst definieren wir einen *Salvetti-Komplex* und beweisen, dass er homotopieäquivalent zum Komplement des entsprechenden Arrangements ist. Dann benutzen wir denselben Komplex, um zu beweisen, dass Komplemente von torischen Arrangements minimale Räume sind. Insbesonders haben sie torsionsfreie Homologie und Kohomologie.

In diesem Prozess benutzen wir etliche "Hilfsmittel" aus der kombinatorischen Topologie. Wir werden auch einige von diesen verallgemeinern müssen (siehe Kapitel 2).

Im Kapitel 1 führen wir die Theorie von Hyperebenen Arrangements ein. Dies wird sowohl einen logischen und historischen Kontext erbringen, als auch die notwendinge Grundlagen für unsere Argumente bereitstellen.

Kapitel 2 bietet eine Erläuterung der kombinatorischen Topologie, die wir in unseren Argumenten anwenden werden. Wir werden einige Konstruktionen der kombinatorischen Topologie an unsere Zwecken anpassen müssen. Insbesonders definieren wir Face-Kategorien von polyhedralen Komplexen und verallgemeinern die diskrete Morse-Theorie auf azyklischen Kategorien.

In Kapitel 3 befassen wir uns mit torischen Arrangements. Wir definieren den *Salvetti-Komplex* für torische Arrangements und beweisen, dass er homotopieäquivalent zum Komplement des Arrangements ist. Darüber hinaus benutzen wir den Salvetti-Komplex, um eine endliche Präsentation der Fundamentalgruppe des Komplements eines torischen Arrangements zu geben. Der Inhalt dieses Kapitels entspricht ungefähr der Veröffentlichung [12].

In Kapitel 4 beweisen wir die Minimalität der Komplemente von torischen Arrangements. Erst zeigen wir, wie der Salvetti-Komplex als Colimes von Salvetti-Komplexen von Hyperebenen Arrangements entsteht. Dann benutzen wir diese Konstruktion, um die Salvetti-Kategorie zu zerlegen, so dass jedes Stück als Face-Kategorie eines torischen Arrangements entsteht. Schließlich untersuchen wir die Face-Kategorien von torischen Arrangements und beweisen die Minimalität. Der Inhalt dieses Kapitels entspricht ungefähr der Veröffentlichung [11].

Introduction

This thesis addresses some fundamental questions on the topology of toric arrangement complements. It has been developed during my stay at the Mathematics Department of the University of Bremen and presents the results obtained in joint work with Emanuele Delucchi, partially published as [12] and partially submitted for publication as [11].

A *toric arrangement* is a finite family of subtori of the complex torus $(\mathbb{C}^*)^d$:

$$\mathscr{A} = \{K_1, \ldots, K_n\},$$

where the K_is are level sets of characters (see Definition 3.1). Such an object has a combinatorial side, represented by the *layer poset* (Definition 3.12) and the *face category* (Definition 3.22), and a topological side, represented by the complement

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus (\cup_{K \in \mathscr{A}} K)$$

and by its topological invariants. The leitmotif in the field of toric arrangements, as well as in this thesis, is to study the latter in terms of the former.

The theory of toric arrangements evolves, both chronologically and context-wise, from the study of hyperplane arrangements. Hyperplane arrangements have been studied in detail in the past half-century and it is a topic with a rich and extensive tradition. In the introduction to Chapter 1 we give an historical account on hyperplane arrangements.

One of the reasons why hyperplane arrangements fascinate so much, is the perfect symmetry between the topology of their complements and their combinatorics (represented by the *intersection poset* of Definition 1.4 and the *face poset* of Definition 1.17). An immediate question therefore arises: how much of this symmetry can be generalized to arrangements of objects of different type? Toric arrangements turn out to exhibit a similar interplay between combinatorics and topology as hyperplane arrangements and one of the purpose of this thesis is indeed to provide an evidence of that.

In this thesis we prove two main result about the topology of toric arrangements which generalize well known results about hyperplane arrangements. Namely, we define a *Salvetti complex* for toric arrangements and prove that it encodes the topology of the complement of the corresponding arrangement. Then we use the same complex to prove that complements of toric arrangements are minimal spaces and therefore have no torsion in homology and cohomology.

In doing this we use a number of combinatorial tools. In fact, we need to extend some of the usual notions of combinatorial topology, to adapt them to our purposes (cfr. Chapter 2).

In Chapter 1 we review the theory of hyperplane arrangements, providing thus a logical and historical context for toric arrangements and, at the same time, the foundations for the arguments of the following chapters. This chapter contains classical material, as well as specific material which will be needed for the arguments of Chapter 3 and Chapter 4.

In Chapter 2 we review the necessary tools of combinatorial topology. We also adapt some of them to our needs, in particular, we define face categories of polyhedral complexes and show how to generalize discrete Morse theory using acyclic categories.

In Chapter 3 we introduce toric arrangements, providing an account of the known results in this area. We then define the toric *Salvetti complex* and prove that it has the homotopy type of the complement of the arrangement. Finally, using the Salvetti complex, we give a finite presentation of the fundamental group of the complement of the arrangement. The content of this chapter corresponds roughly to the publication [12].

In Chapter 4 we prove minimality of toric arrangement complements. We use the Salvetti complex and show how to construct it "pasting together" local pieces, which are isomorphic to Salvetti complexes of hyperplane arrangements. Using this local constructions we are able to decompose the Salvetti complex of a toric arrangements into "strata", whose topology can be derived from the combinatorics of the face categories of some toric arrangements. Finally, studying these face categories, we prove minimality. The content of this chapter corresponds roughly to the publication [11].

Contents

CHAPTER 1

Arrangements of Hyperplanes

Even though the focus of this thesis is not Hyperplane Arrangements per se, we will rely on the theory of Hyperplane Arrangements as a foundation. There are several connections between Toric and Hyperplane Arrangements. The most evident one is, as a matter of fact, similarity. Researchers in the area of Toric Arrangements have been successfully trying to generalize known properties and results of Hyperplane Arrangements to Toric Arrangements. This is also the leitmotif of this work: once we identify the peculiarities of Toric Arrangements, we are able to generalize known arguments from the theory of Hyperplane Arrangements.

The other connection between Toric and Hyperplane Arrangements is more technical. We will see that, roughly speaking, the combinatorial objects associated to a toric arrangement can be constructed *gluing together* different combinatorial objects coming from Hyperplane Arrangements.

The last statement will become clear in Chapter 3, when we will treat diagrams and colimits of arrangements. For now it suffices to say that Hyperplane Arrangements are the foundation of our subject and therefore deserve to be introduced with the necessary honors. This is precisely the purpose of this Chapter.

For the material of this chapter one could cite many different references, we refer to [36, 35, 43].

1.1 Prologue

An affine hyperplane in a vector space V over a field $\mathbb{K}$ is a level set of a linear functional:

$$H = \{\boldsymbol{x} \in V : \quad f(\boldsymbol{x}) = a\} \text{ with } f \in V^*,\ a \in \mathbb{K}.$$

A hyperplane H is called *linear* if $a = 0$, i.e. if it is the kernel of a linear functional.

1.1 Definition
An (affine) hyperplane arrangement *in a vector space V over a field $\mathbb{K}$ is a set $\mathscr{A}$ of affine hyperplanes in V.*

The complement *of a hyperplane arrangement is the space*

$$M(\mathscr{A}) = V \setminus \bigcup_{H \in \mathscr{A}} H.$$

A hyperplane arrangement is called central *if all its hyperplanes are linear hyperplanes.*

1.2 Remark
Some authors define central arrangements as arrangements whose intersection $\cap_{H\in\mathscr{A}} H$ is not empty. This is obviously equivalent, up to translation, to arrangements of linear hyperplanes.

1.3 Definition
An affine hyperplane arrangement $\mathscr{A}$ is called locally finite *if for every point $p \in V$ the subarrangement*

$$\{H \in \mathscr{A} : \quad p \in H\}$$

is finite.

In the following we will only consider locally finite hyperplane arrangements.

Hyperplane arrangements first appeared -in disguise- with Arnold's paper [1] in the early seventies. At the time the interest was focused on *braid groups* and *knot theory*. Arnold was specifically interested in the cohomology of the pure braid group.

One possible definition of the pure braid group is as the fundamental group of the following configuration space

$$M(\mathscr{A}_{n-1}) = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n : \quad x_i \neq x_j \text{ for all } i < j\}.$$

This space is the complement of the union of a renowned arrangement: the *braid arrangement on n strings* also known as the *Weyl arrangement* *of type* A_{n-1}. Now, this is an Eilenberg-Mac Lane space, therefore its cohomology coincides with the group cohomology of its fundamental group (with constant coefficients $\mathbb{Z}$). The main result of Arnold's paper is a presentation of this cohomology algebra, which later lead to the general presentation of Orlik and Solomon for arbitrary hyperplane arrangements.

In [6], Brieskorn considered Artin groups as generalizations of braid groups. Given a root system Φ in a vector space V, one can consider the arrangement of hyperplanes orthogonal to the root system:

$$\mathscr{A}_\Phi = \{v^\perp : \quad v \in \Phi\}.$$

If Φ is an irreducible root system, we say the $\mathscr{A}_\Phi$ is the *Coxeter arrangement* of type Φ. For crystallographic root systems we also speak of *Weyl arrangements*.

Pure Artin groups are the fundamental groups of Coxeter arrangements. It turns out that, as for braid groups, the complements $M(\mathscr{A}_\Phi)$ are Eilenberg-Mac Lane spaces and therefore their cohomology coincides with the group cohomology of pure Artin groups. Following this line of thought Brieskorn computed the cohomology of the complement of Coxeter arrangements. His treatment is general and proves important results on the topology of the complements of arbitrary hyperplane arrangements.

The paper of Orlik and Solomon [35] represent a milestone in the study of hyperplane arrangements. Not only because it treats the topology of hyperplane arrangement complements in its full generality, but also because it makes the relation between the topology of the complement and the combinatorics of the arrangement explicit. From this point on, hyperplane arrangements are regarded as an area in mathematics between topology, algebra and combinatorics.

Today we know a lot about hyperplane arrangements. Without any claim of completeness, we'll sum up some important results in this field that are relevant for this thesis. For the particular class of *complexified affine arrangements* there is a simple description of the homotopy type of the complement $M(\mathscr{A})$ as a CW-Complex, due to Salvetti [40]. This complex is usually referred to as the *Salvetti complex*.

The Salvetti complex has been used in many applications, among others in the study of covering spaces of $M(\mathscr{A})$ and the $K(\pi,1)$-problem (see e.g. [37, 20]) and that of minimality of the complement (see e.g. [41, 19]). Furthermore, Salvetti was able in his seminal paper, using the Salvetti complex, to give a presentation of the fundamental group of the complement of a complexified arrangement $\pi_1(M(\mathscr{A}))$.

Another interesting property of hyperplane arrangement complements is *minimality*, i.e. these spaces have the homotopy type of a CW-complex, for which the number of k-cells equals the k-th Betti number, for every k. This was proved in 2004 by Dimca and Papadima [21] and Randell [38], using Morse theoretic arguments. Later Salvetti and Settepanella [41] reproved this result using the Salvetti complex and discrete Morse theory. Delucchi [19] used a similar argument, describing a discrete Morse function in terms of the combinatorics of the arrangement.

In the rest of this chapter we introduce those aspects of hyperplane arrangements that are essential for the treatment in this thesis.

1.2 Parodos

In the study of hyperplane arrangements we will concentrate on the specific case of *real* and *complex* arrangements (i.e. arrangements over real or complex vector spaces).

As already mentioned the study of hyperplane arrangements originates from topology and geometry and later spread itself in many different areas of mathematics. Combinatorialists are interested in the discrete structures associated to an arrangement, namely matroids, posets, polyhedral stratifications of $\mathbb{R}^n$. Algebraic geometers have been interested in singularities of the complement of a complex arrangement (this lead to the development of De Concini and Procesi's *wonderful models* [15]). Representation theorists have studied groups actions on the cohomology of arrangements' complements [29, 27, 28, 8, 24, 13]. The book [7] provides an introduction to modern topics in hyperplane arrangements.

In this thesis we will be mainly interested in the topology of the complement $M(\mathscr{A})$ of a complex arrangement $\mathscr{A}$ and in the combinatorics of real and complex arrangements. We now introduce the essential results in this area, which will be fundamental for the continuation of our exposition.

Combinatorics

1.4 Definition
Consider an hyperplane arrangement $\mathscr{A} = \{H_1, \dots, H_n\}$ in the vector space V, its intersection poset *is the poset*

$$\mathcal{L}(\mathscr{A}) = \left\{ \bigcap_{i \in I} H_i : \quad I \subseteq \{1, \dots, n\} \right\} \Big\backslash \{\emptyset\}$$

ordered by reverse inclusion.

Notice that the space V is an element of $L(\mathscr{A})$ (it is the intersection indexed by the empty set), whereas the empty set is not.

The poset $L(\mathscr{A})$ is a *meet-semilattice* and is a lattice if $\mathscr{A}$ is central. Furthermore, $L(\mathscr{A})$ is *geometric* (i.e. *atomic*, *ranked* and *semimodular*).

The intersection poset is generally considered as the "combinatorial side" of the arrangement. Indeed those properties that depends only on $\mathcal{L}(\mathscr{A})$ are usually called *combinatorial*.

1.5 Remark

Some authors associate to an arrangement the *matroid* of the dependencies of the hyperplanes in $\mathscr{A}$ and consider this as the "combinatorics" of the arrangement. The two approaches are clearly equivalent. In fact, on the one hand $\mathcal{L}(\mathscr{A})$ is the poset of flats of this matroid, on the other hand the rank function of $L(\mathscr{A})$ coincides with that of the matroid and therefore uniquely characterizes it.

1.6 Remark

An arrangement is, per se, just a set of hyperplanes. In particular it doesn't come with an ordering. However it is often necessary to order the hyperplanes in the arrangements.

In the following we will usually define *ordered hyperplane arrangements* by indexing the hyperplanes of an arrangements with natural numbers, as in $\mathscr{A} = \{H_1, \ldots, H_n\}$.

1.7 Definition

An arrangement $\mathscr{A}$ is called essential *if the maximal elements of $\mathcal{L}(\mathscr{A})$ are points. For central arrangements this is equivalent to $\bigcap_{H\in\mathscr{A}} H = \{0\}$.*

No broken circuits

Consider a *central* hyperplane arrangement $\mathscr{A}$. In this case the intersection poset $L(\mathscr{A})$ is a geometric lattice.

We now introduce some combinatorial objects associated to $L(\mathscr{A})$, which capture many topological properties of hyperplane arrangements.

Choose a total ordering on $\mathscr{A} = \{H_1, \ldots, H_n\}$.

1.8 Definition

A circuit *is a minimal dependent subset $C \subseteq \mathscr{A}$. A* broken circuit *is a subset of the form*

$$C \backslash \{min\, C\} \subseteq \mathscr{A}$$

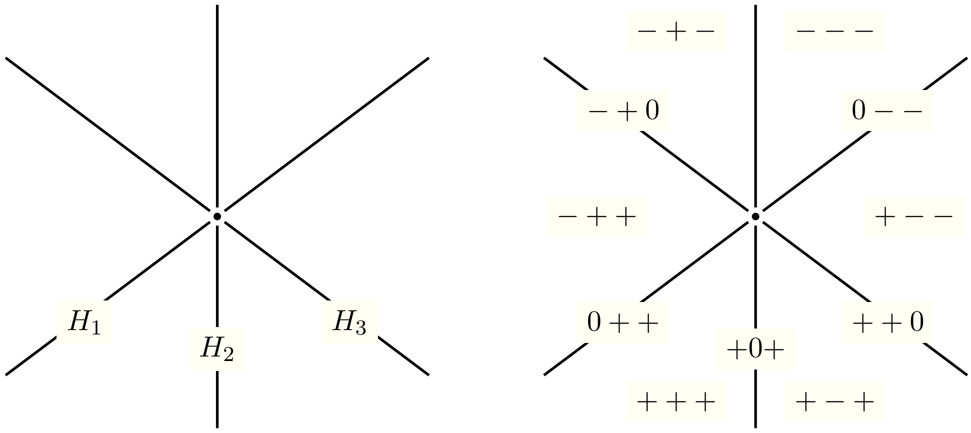
obtained from a circuit removing its least element. A no broken circuit set *is a subset $N \subseteq \mathscr{A}$ which does not contain any broken circuit.*

1.9 Remark

Alternatively, we can define no broken circuit sets as follows: A subset $N = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \mathscr{A}$ with $i_1 \leq \cdots \leq i_k$ is a *no broken circuit set* if it is independent and there is no $h \leq k$ and $j < i_h$ such that $\{H_j\} \cup \{H_{i_h}, \ldots, H_{i_k}\}$ is dependent.

1.10 Definition

We will write $\mathrm{nbc}(\mathscr{A})$ *for the set of no broken circuit sets (short:* nbcs*) of $\mathscr{A}$ and* $\mathrm{nbc}_k(\mathscr{A}) = \{N \in \mathrm{nbc}(\mathscr{A}) : |N| = k\}$ *for the set of all no broken circuit sets of cardinality k.*



(a) Arrangement of 3 lines (b) Sign vectors

Figure 1.1: An hyperplane arrangement.

1.11 Example
Consider the arrangement of three lines in $\mathbb{R}^2$ as in Figure 1.1a. The only circuit is $\{H_1, H_2, H_3\}$, therefore there are 6 no-broken-circuits:

$$\operatorname{nbc}(\mathscr{A}) = \{\emptyset, \{H_1\}, \{H_2\}, \{H_3\}, \{H_1, H_2\}, \{H_1, H_3\}\}$$

Notice that the arrangement has 6 chambers, as predicted by Theorem 1.27.

1.12 Remark
Definition 1.8 uses only the notion of dependency of hyperplanes. In other words, it depends only on the matroid structure associated to $\mathscr{A}$. Indeed circuits are one of the fundamental objects of matroid theory, and characterize the matroid uniquely.

Deletion and restriction

In the theory of hyperplane arrangements, as well as in the theory of matroids, many properties share a certain inductive structure, called of *deletion and restriction*. By this we mean that these properties can be inferred from the corresponding properties of “smaller” arrangements. This is a widely used technique and it will be central for our exposition too.

1.13 Definition
Consider a hyperplane arrangement $\mathscr{A}$ in the vector space V and an intersection $X \in \mathcal{L}(\mathscr{A})$. We associate to X two new arrangements:

$$\mathscr{A}_X = \{H \in \mathscr{A} \,:\, X \subseteq H\}, \quad \mathscr{A}^X = \{H \cap X \,:\, H \in \mathscr{A} \setminus \mathscr{A}_X\}.$$

$\mathscr{A}_X$ is called the deletion *of $\mathscr{A}$ w.r.t. X, while $\mathscr{A}^X$ is called the* restriction *of $\mathscr{A}$ on X.*

Notice that $\mathscr{A}_X$ is an arrangement inn V, while $\mathscr{A}^X$ is arrangement in X.

1.14 Remark

The terminology adopted in Definition 1.13 coincides with the one of [36], which is common in the literature on hyperplane arrangements. However, it can be confusing for the reader familiar with matroid theory.

If $\mathscr{A}$ is a central arrangement, then $\mathcal{L}(\mathscr{A})$ is the lattice of flats of a matroid on the set $\mathscr{A}$. In this regard the lattice associated to the *deletion* $\mathscr{A}_X$, $\mathcal{L}(\mathscr{A}_X)$, is the lattice of flats of a *restriction* of the matroid defined by $\mathcal{L}(\mathscr{A})$; on the other hand, the *restriction* $\mathscr{A}^X$ of the arrangement defines, via $\mathcal{L}(\mathscr{A}^X)$, a *contraction* of the matroid.

1.3 Real arrangements

Real arrangements are arrangements of hyperplanes in a finite dimensional real vector space V. Without loss of generality we can assume $V = \mathbb{R}^d$, for some d.

It is not difficult to verify that the complement $M(\mathscr{A})$ consist of several connected components, called the *chambers of* $\mathscr{A}$, each of which is contractible. More precisely, they are convex polyhedra. We will write $\mathcal{T}(\mathscr{A})$ for the set of chambers of $\mathscr{A}$.

Counting chambers

An interesting question is that of the *enumeration* of chambers. This question has been elegantly answered by Zaslavsky in [44].

1.15 Theorem

Let $\mathscr{A}$ be a real central hyperplane arrangement, the chambers of $\mathscr{A}$ can be counted as:

$$|\mathcal{T}(\mathscr{A})| = \sum_{X \in \mathcal{L}(\mathscr{A})} |\mu(X)|$$

where $\mu : L(\mathscr{A}) \to \mathbb{Z}$ is the Möbius function of the lattice $\mathcal{L}(\mathscr{A})$.

An alternative formula, more useful for our purposes, is the following.

1.16 Theorem

Consider a real central hyperplane arrangement $\mathscr{A}$, then the number of chambers of $\mathscr{A}$ is

$$|\mathcal{T}(\mathscr{A})| = |\operatorname{nbc}(\mathscr{A})|.$$

Faces

Consider now a real *affine locally finite* hyperplane arrangement $\mathscr{A}$. We can associate to $\mathscr{A}$ a stratification of the ambient space $\mathbb{R}^d$ as follows.

1.17 Definition

Let $\mathscr{A}$ be a real arrangement, the set of faces *of $\mathscr{A}$ is*

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X : \quad C \in \mathcal{T}(\mathscr{A}),\, X \in \mathcal{L}(\mathscr{A})\}.$$

The faces of an affine arrangement form a poset, ordered by inclusion. We usually speak of the *face poset* of $\mathscr{A}$.

As we will see in Section 1.5, the face poset contains the relevant information to determine the homotopy type of the complement of the complex arrangement corresponding to $\mathscr{A}$.

1.18 Remark

As we already mentioned the intersection poset is encoded in (in fact, is equivalent to)the matroid structure associated to the arrangement. Similarly, the face poset is encoded in (equivalent to) the *oriented matroid* associated to the arrangement. See [4] for more details on oriented matroids.

Taking sides

A peculiarity of real arrangements is that we can take account of orientations. Consider a real arrangement $\mathscr{A}$ on the vector space V and for every $H \in \mathscr{A}$ choose a linear functional $\alpha_H \in V^*$ such that $H = \{\boldsymbol{x} \in V : \alpha_H(\boldsymbol{x}) = a\}$. Then we can define for every hyperplane H its *positive* and *negative halfspace*:

$$H^+ = \{\boldsymbol{x} \in V : \alpha_H(\boldsymbol{x}) > 0\}, \qquad H^- = \{\boldsymbol{x} \in V : \alpha_H(\boldsymbol{x}) < 0\}.$$

Orienting the hyperplanes in $\mathscr{A}$ we obtain a description of the faces of the arrangements, as follows.

1.19 Definition

Consider a real (or complexified) locally finite arrangement $\mathscr{A}$ with any choice of orientations H^+ and H^- for every $H \in \mathscr{A}$. Consider a face $F \in \mathcal{F}(\mathscr{A})$, its sign vector *is the function $\gamma_F : \mathscr{A} \to \{-, 0, +\}$ defined as:*

$$\gamma_F(H) := \begin{cases} + & \textit{if } F \subseteq H^+ \\ 0 & \textit{if } F \subseteq H \\ - & \textit{if } F \subseteq H^- \end{cases}.$$

When the need will arise to specify the arrangement $\mathscr{A}$ to which the sign vector refers, we will write $\gamma[\mathscr{A}]_F(H)$ for $\gamma_F(H)$.

Notice that chambers are precisely those faces whose sign vector maps $\mathscr{A}$ to $\{-,+\}$.

1.20 Example

Figure 1.1b shows the faces of an arrangement of three lines in $\mathbb{R}^2$, together with their sign vectors.

1.21 Remark

The sign vectors $\{\gamma_F : F \in \mathcal{F}(\mathscr{A})\}$ satisfy the *covector axioms* of an oriented matroid, whose face poset is $\mathcal{F}(\mathscr{A})$. In this sense oriented matroids abstract and encapsulate the combinatorics of real arrangements. See [4] for more details.

1.22 Remark

A possible way of consistently choose an orientation for each hyperplane is the following. Choose a distinguished chamber $B \in \mathcal{T}(\mathscr{A})$, called the *base chamber*. For every hyperplane $H \in \mathscr{A}$ define the positive and negative halfspaces such that $B \subseteq H^+$. This way we have:

$$B = \bigcap_{H \in A} H^+.$$

Notice that the choice of a base chamber determines an orientation of the hyperplanes in $\mathscr{A}$, but the converse is generally false. Indeed the intersection $\cap_{H \in A} H^+$ could be empty.

Choosing a base chamber allows to define a partial ordering on the set of chambers of an arrangement, that will be of central importance in Chapter 4.

1.23 Definition

Let C_1 and $C_2 \in \mathcal{T}(\mathscr{A})$ be chambers of a real arrangement. We will write

$$S(C_1, C_2) := \{H \in \mathscr{A} : \quad \gamma_{C_1}(H) \neq \gamma_{C_2}(H)\}$$

for the set of hyperplanes of $\mathscr{A}$ which separate C_1 and C_2.
For all $C_1, C_2 \in \mathcal{T}(\mathscr{A})$ write

$$C_1 \leq C_2 \iff S(C_1, B) \subseteq S(C_2, B).$$

This turns $\mathcal{T}(\mathscr{A})$ into a poset $\mathcal{T}(\mathscr{A})_B$, the poset of regions *of the arrangement $\mathscr{A}$ with base chamber B.*

In chapter 4 we will be interested in linear extensions of this poset.

1.4 Complex arrangements

For an arrangement of hyperplanes $\mathscr{A}$ on a complex vector space V, the complement $M(\mathscr{A})$ is topologically non-trivial. It is therefore interesting to study its topological properties. This space has been investigated in detail. In this thesis we will be mainly interested in its singular cohomology and in its homotopy type. As general references on the topology of arrangement complements we cite [36, 43, 7].

1.24 Remark

When we're interested in the topology of the complement of an arrangement $M(\mathscr{A})$, we can assume without loss of generality that $\mathscr{A}$ is *essential* (see Definition 1.7). Indeed consider for every $H \in \mathscr{A}$ a normal vector $v_H \in H^\perp$. Let $X = \{v_H : H \in \mathscr{A}\}^\perp$.

The arrangement is essential if and only if $\dim X = 0$. Now consider the arrangement $\mathscr{B} = \{H/X : H \in \mathscr{A}\}$ on V/X. This is an essential arrangement and we have:

$$M(\mathscr{A}) \cong X \times M(\mathscr{B}) \simeq M(\mathscr{B}).$$

That is, the two complements are homotopy equivalent.

Poincaré Polynomial

We now describe the additive structure of $H^*(M(\mathscr{A}); \mathbb{Z})$.

The following formula dates back to Brieskorn [6].

1.25 Theorem

Let $\mathscr{A}$ be a finite affine complex hyperplane arrangement, then for every $p \in \mathbb{N}$, the following holds:

$$H^p(M(\mathscr{A}); \mathbb{Z}) \cong \bigoplus_{X \in \mathcal{L}(\mathscr{A})_p} H^p(M(\mathscr{A}_X); \mathbb{Z}),$$

where $L(\mathscr{A})_p = \{X \in \mathcal{L}(\mathscr{A}) : \operatorname{codim}(X) = p\}$.

1.26 Corollary (Brieskorn [6])

Let $\mathscr{A}$ be a finite affine hyperplane arrangement, then the cohomology modules $H^*(M(\mathscr{A}); \mathbb{Z})$ are free abelian groups of finite rank.

Proof. Consider a central arrangement $\mathscr{A} = \{H_1, \dots, H_n\}$ in $\mathbb{C}^k$, where $H_i = \ker \alpha_i$ for some linear functionals $\alpha_i : \mathbb{C}^k \to \mathbb{C}$. Define the arrangement $d\mathscr{A}$ on the subspace $\mathbb{C}^k \supseteq \alpha_1^{-1}(1) \cong \mathbb{C}^{k-1}$ as

$$d\mathscr{A} = \{H_i \cap \alpha_1^{-1}(1) : i = 2, \dots, n\}.$$

It is easy to see, that the map $\alpha_1 : M(\mathscr{A}) \to \mathbb{C}^*$ is a trivial fiber bundle with fiber $d\mathscr{A}$. In particular $M(\mathscr{A}) \cong \mathbb{C}^* \times M(d\mathscr{A})$.

Let now $\mathscr{A}$ be a finite affine hyperplane arrangement. We use an inductive argument on the dimension k of the ambient space, the base case being easy. For each $X \in \mathcal{L}(\mathscr{A})$ the arrangement $\mathscr{A}_X$ is central. Applying the previous argument we have

$$H^*(M(\mathscr{A}_X);\mathbb{Z}) \cong \mathbb{Z}[x]/\left\langle x^2\right\rangle \otimes H^*(M(d\mathscr{A}_X);\mathbb{Z}).$$

By inductive hypothesis $H^*(M(d\mathscr{A}_X);\mathbb{Z})$ is free abelian of finite rank, therefore also $H^*(M(\mathscr{A}_X);\mathbb{Z})$.

Finally applying Theorem 1.25 we get the conclusion. □

Another important result on the additive cohomology of $M(\mathscr{A})$ describes its Poincaré polynomial in terms of the combinatorics of $\mathcal{L}(\mathscr{A})$.

1.27 Theorem (Jambu and Terao [25])
Let $\mathscr{A}$ be a finite affine hyperplane arrangement, then:

$$P_{\mathscr{A}}(t) := \sum_{j=0}^{\infty} \left(\textit{rk}\ H^j(M(\mathscr{A});\mathbb{Z})\right)\, t^j = \sum_{j=0}^{\infty} |\operatorname{nbc}_j(\mathscr{A})|\, t^j.$$

1.28 Corollary
Let $\mathscr{A}$ be a finite affine complexified hyperplane arrangement and write $\mathscr{A}_{\mathbb{R}} = \{H \cap \mathbb{R}^k : H \in \mathscr{A}\}$ for its real part. Then

$$|\mathcal{T}(\mathscr{A}_{\mathbb{R}})| = P_{\mathscr{A}}(1).$$

1.29 Remark
Combining Theorem 1.25 with Theorem 1.27 we get the following formula for the Poincaré polynomial of the complement of an arbitrary finite affine complex arrangement:

$$P_{\mathscr{A}}(t) := \sum_{X \in \mathcal{L}(\mathscr{A})} |\operatorname{nbc}_{\operatorname{codim} X}(\mathscr{A}_X)|\, t^{\operatorname{codim} X}.$$

Cohomology algebra

In their pioneering work [35], Orlik and Terao computed the cohomology algebra of arrangement complements in terms of the combinatorics of the intersection poset $\mathcal{L}(\mathscr{A})$.

Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be a central hyperplane arrangement and consider the elements $e_1, \dots, e_n$. For a subset $S = \{i_1 < \cdots < i_h\} \subseteq \{1, \dots, n\}$ define the following formal products:

$$e_S = e_{i_1} e_{i_2} \cdots e_{i_h},$$

$$\partial e_S = \sum_{j=1}^{h} (-1)^j e_{S_j} \text{ where } S_j = S \backslash e_j.$$

1.30 Theorem (Orlik and Terao [35])
Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be a finite affine hyperplane arrangement in a finite dimensional complex vector space V. Then the cohomology algebra $H^*(M(\mathscr{A}); \mathbb{Z})$ is the skew symmetric algebra generated by the elements $\{e_1, \dots, e_n\}$ with the relations

$$\partial e_S = 0 \quad \forall S \subseteq \{1, \dots, n\} \text{ with } S \text{ dependent}, \tag{1.1}$$
$$e_S = 0 \quad \forall S \subseteq \{1, \dots, n\} \text{ with } \cap S = \emptyset. \tag{1.2}$$

Where $S \subseteq \{1, \dots, n\}$ is called dependent *if the hyperplanes $\{H_j : j \in S\}$ are dependent.*

This presentation is usually referred to as the Orlik-Solomon algebra *of the arrangement.*

Notice that for central hyperplane arrangements there are no relations as in (1.2), therefore the relations in (1.1) suffice to describe the Orlik-Solomon algebra.

1.31 Remark
Theorem 1.27 can also be formulated from the point of view of the Orlik-Solomon algebra.

Call a set $S \subseteq \{1, \dots, n\}$ a no-broken-circuit if the corresponding hyperplanes set is a no-broken-circuit (this is actually the correct notion of no-broken-circuit according to matroid theorists). Then the following set is a basis of $H^*(M(\mathscr{A}); \mathbb{C})$ (Jambu and Terao [25]):

$$\{e_S : S \in \mathrm{nbc}(\mathscr{A})\}.$$

1.5 Complexified arrangements

Complexified arrangements are interesting because we can relate the combinatorics of the real arrangement with the topology of the complex arrangement. Probably the most important result in this sense is that of Salvetti in [40] where he defines the cellular complex which bears his name.

1.32 Definition
Let $\mathscr{A}$ be an affine locally finite complexified arrangement and consider its face poset $\mathcal{F}(\mathscr{A})$. For a chamber $C \in \mathcal{T}(\mathscr{A})$ and a face $F \in \mathcal{F}(\mathscr{A})$ define the chamber $C_F \in \mathcal{F}(\mathscr{A})$ as the chamber with the following sign vector:

$$\gamma_{C_F}(H) = \begin{cases} \gamma_C(H) & \text{if } H \in \mathscr{A}_F \\ \gamma_F(H) & \text{if } H \notin \mathscr{A}_F. \end{cases}$$

1.5. Complexified arrangements

1.33 Definition
Let $\mathscr{A}$ be an affine locally finite complexified arrangement; its Salvetti poset *is the poset on the ground set*

$$\operatorname{Sal}\mathscr{A} = \{[F, C] : F \in \mathcal{F}(\mathscr{A}),\, C \in \mathcal{T}(\mathscr{A}),\, F \leq C\}$$

with the order relation

$$[F_1, C_1] \leq [F_2, C_2] \iff F_2 \leq F_1 \text{ and } (C_2)_{F_2} = C_1.$$

1.34 Definition
Let $\mathscr{A}$ be an affine locally finite complexified arrangement; its (simplicial) Salvetti complex *is*

$$\mathcal{S}(\mathscr{A}) = \Delta(\operatorname{Sal}\mathscr{A}),$$

i.e. the order complex of the Salvetti poset.

The importance of the Salvetti complex comes from the following.

1.35 Theorem (Salvetti [40])
Let $\mathscr{A}$ be an affine locally finite complexified arrangement. Then, there is an homotopy equivalence $|\mathcal{S}(\mathscr{A})| \simeq M(\mathscr{A})$.

More precisely, Salvetti in [40] constructs an embedding

$$\varphi : |\mathcal{S}(\mathscr{A})| \to M(\mathscr{A})$$

such that $\operatorname{Im}\varphi \subseteq M(\mathscr{A})$ is a strong deformation retract. The following lemma shows how the embedding is constructed.

1.36 Lemma (Salvetti [40])
Choose for every face $F \in \mathcal{F}(\mathscr{A})$ a point $w_F \in F$. The following vertex map

$$\varphi : [F, C] \mapsto w_F + i\,(w_{C_F} - w_F).$$

induces an embedding

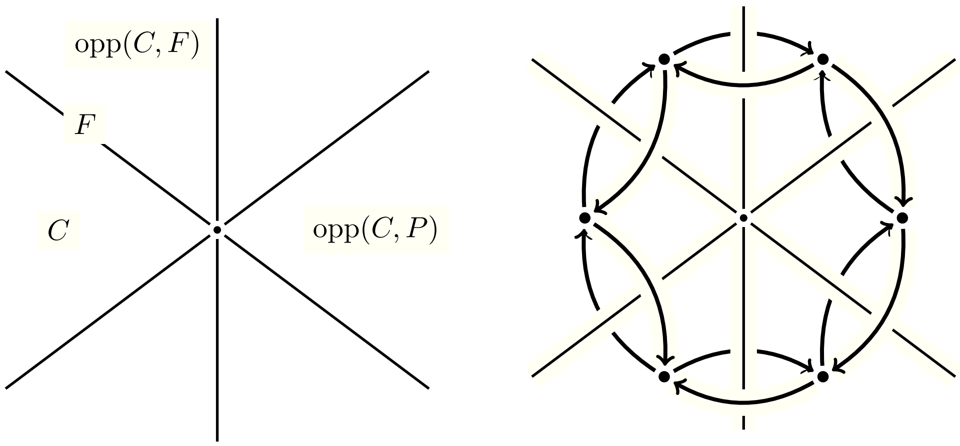
$$\varphi : |\mathcal{S}(\mathscr{A})| \to M(\mathscr{A})$$

Cellular Salvetti complex

The simplicial Salvetti Complex of Definition 1.34 is in fact the barycentric subdivision of a polyhedral complex, which we now describe.

1.37 Definition
Let $\mathscr{A}$ be an affine locally finite complexified arrangement. The arrangement graph $\mathcal{G}(\mathscr{A})$ is the oriented graph with a vertex for every chamber

$$\mathcal{V}(\mathcal{G}(\mathscr{A})) = \{e_C : \quad C \in \mathcal{T}(\mathscr{A})\}$$



(a) Opposite chambers (b) The arrangement graph

Figure 1.2: Arrangement of three lines

and a pair of edges for every couple of adjacent chambers

$$\mathcal{E}(\mathcal{G}(\mathscr{A})) = \bigcup_{C,K \in \mathcal{T}(\mathscr{A}):\, |S(C,K)|=1} \{(e_C, e_K), (e_K, e_C)\}.$$

The arrangement graph is also called *oriented system* in [37].

If $S(C, K) = \{H\}$ then the chambers C, K are adjacent and have a common face $F \in \mathcal{F}(\mathscr{A})$ of codimension 1 with $|F| = Y$. We say that the edges (e_C, e_K), (e_K, e_C) *cross* the hyperplane H.

Let $F \in \mathcal{F}(\mathscr{A})$ be a face and $C \in \mathcal{T}(\mathscr{A})$ a chamber with $F \leq C$. The *opposite chamber* $\mathrm{opp}(C, F)$ of C relative to F is defined as

$$\gamma_{\mathrm{opp}(C,F)}(H) = \begin{cases} -\gamma_C(H) & \text{if } H \in \mathscr{A}_F \\ \gamma_C(H) & \text{otherwise} \end{cases}.$$

1.38 Example

Figure 1.2a shows the opposite chamber of a chamber C with respect to a codimension 1 face F and to a codimension 2 face P in the arrangement of three lines of Figure 1.1a.

Figure 1.2b shows the arrangement graph $\mathcal{G}(\mathscr{A})$ of the same arrangement.

1.39 Definition

Let $\mathscr{A}$ be an affine locally finite complexified arrangement. The cellular Salvetti complex *of $\mathscr{A}$ is the regular polytopal complex*

(a) whose 1-skeleton is the realization of the graph $\mathcal{G}(\mathscr{A})$;

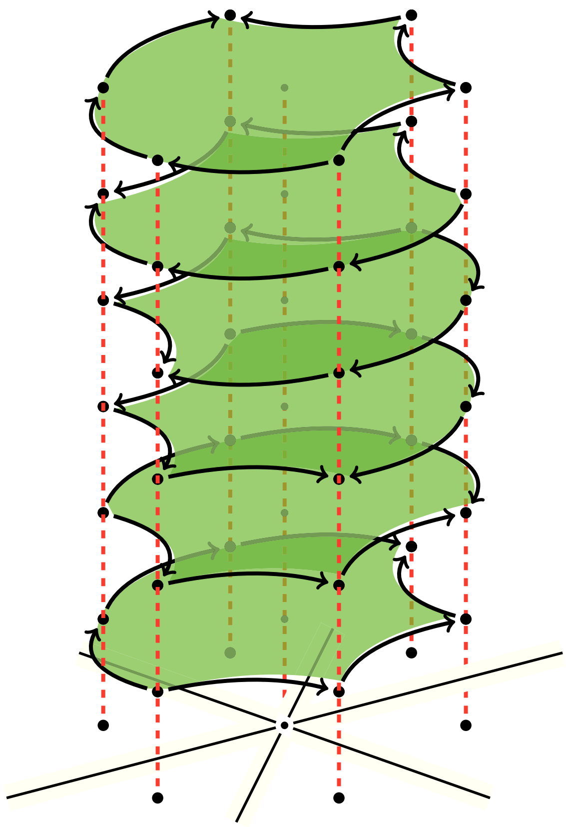


Figure 1.3: The Salvetti complex of the arrangement of three lines

(b) *whose k-cells $e_{[F,C]}$ correspond to the pairs $[F, C]$ with $F \in \mathcal{F}(\mathscr{A})$ a face of codimension k and $C \in \mathcal{T}(\mathscr{A})$ a chamber with $F \leq C$;*

(c) *where the 1-skeleton of a k-cell $e_{[F,C]}$ is attached along the* minimal paths *in $\mathcal{G}(\mathscr{A})$ from e_C to $e_{\mathrm{opp}(C,F)}$.*

1.40 Example

Figure 1.3 shows the cellular Salvetti Complex of the arrangement of three lines in Figure 1.1a.

The figure shows the six 2-dimensional cells and how they are attached to the 1-skeleton.

1.41 Lemma

Let ρ be a minimal path in $\mathcal{G}(\mathscr{A})$ between C and C' and consider and edge $(e_K, e_{K'}) \in \rho$. If $(e_K, e_{K'})$ crosses the hyperplane H at the face F, then $K = C_F$ and C and K lie on the same side of H.

Proof. From [40, Lemma 2] it follows that the path ρ decomposes as

$$\rho_1 \circ (e_K, e_{K'}) \circ \rho_2$$

where $\circ$ indicates path composition, ρ_1 is a minimal path between C and K and ρ_2 is a minimal path between K' and C'.

Again from [40, Lemma 2] it follows that ρ_1 doesn't cross H (otherwise ρ wouldn't be minimal), therefore C and K lie on the same side of H and $C_F = K$. $\square$

1.42 Proposition

Let $\mathscr{A}$ be an affine locally finite complexified arrangement. The Salvetti poset $\operatorname{Sal}\mathscr{A}$ is the face poset of the cellular Salvetti complex of $\mathscr{A}$. Hence the simplicial Salvetti complex is the barycentric subdivision of the cellular Salvetti complex.

Fundamental Group

We now review Salvetti's presentation of the fundamental group of the complement of a complexified arrangement $\pi_1(M(\mathscr{A}))$. In chapter 3 we will give a presentation of the fundamental group of the complement of a toric arrangement inspired by that of Salvetti.

1.43 Remark

Let $\mathscr{A}$ be a complexified arrangement and $\mathcal{F} := \mathcal{F}(\mathscr{A})$. It will be convenient for us to denote by $\mathcal{F}_j$ the subset of cells of $\mathcal{F}$ *codimension* j.

This is contrary to the usual convention for a cell complex K to denote by K_j the set of cells of K of *dimension* j. In this case, this little abuse of notation is justified by the fact that the cells of codimension j in $\mathcal{F}(\mathscr{A})$ index the cells of dimension j in $\mathcal{S}(\mathscr{A})$.

Choose - and from now on fix - a base chamber B of $\mathscr{A}$, and let x_0 be a generic point in B.

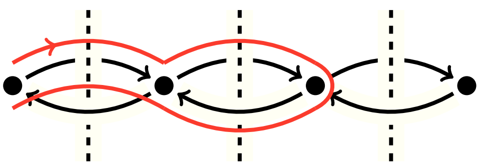


Figure 1.4: Generators, an example: $\beta_{F_2} = l_{F_1} l_{F_2}^2 l_{F_1}^{-1}$

1.5. Complexified arrangements

Generators Recall the arrangement graph $\mathcal{G} := \mathcal{G}(\mathscr{A})$ of Definition 1.37. Here we will adopt a useful notational convention inspired by [40]: we will write edges of $\mathcal{G}$ as indexed by the face of codimension 1 they cross, and in writing a path we will write l_F for a crossing of F 'along the direction of the edge', l_F^{-1} for a crossing 'against the direction' of the edge. By specifying the first vertex of the path then there is no confusion about which edge is used, and in which direction.

A *positive* path then is a path of the form

$$l_{F_1} l_{F_2} \dots l_{F_k}$$

for $F_1, \dots, F_k \in \mathcal{F}_1$. A path is *minimal* if it is of minimal length among all the paths on the same ends.

The following lemma will be fundamental in chapter 3 for most arguments on the fundamental group of the toric Salvetti complex. It is based on the observation that the border of a cell consists of minimal paths "around a face."

1.44 Lemma ([40, Lemma 11])
Let $\mathscr{A}$ be a complexified arrangement and $\mathcal{G}$ its arrangement graph. Any two positive minimal paths in $\mathcal{G}$ with same origin and same end are homotopic.

The following lemma is useful to determine when the composition of two minimal paths is minimal.

1.45 Lemma ([40, Lemma 2])
Let $\mathscr{A}$ be a complexified arrangement and $\mathcal{S}$ its (cellular) Salvetti complex.

A path $\nu \in \Omega(C, K)$ for two chambers $C, K \in \mathcal{T}(\mathscr{A})$ is minimal *if and only if*

$$\ell(\nu) = |S(C, K)|$$

that is, if and only if it crosses each hyperplane separating C and K exactly once.

Since any two positive minimal paths with same origin and same end are homotopic, given $C, C' \in \mathcal{F}_0$ we will sometimes write $(C \to C')$ for the (class of) positive minimal paths starting at C and ending at C'.

For every $F \in \mathcal{F}_1$ we define a path as follows:

$$\beta_F := (C_0 \to (C_0)_F) l_F^2 (C_0 \to (C_0)_F)^{-1}, \tag{1.3}$$

1.46 Lemma ([40, p. 616])
Let $\mathscr{A}$ be a complexified arrangement and $\mathcal{S}$ its (cellular) Salvetti complex. The group $\pi_1(\mathcal{S})$ is generated by the set $\{\beta_F | F \in \mathscr{F}_1\}$.

Given a positive path $\nu = l_{F_1} \cdots l_{F_k}$ define loops

$$\beta^{\nu}_{F_i} := l_{F_1} \cdots l_{F_{i-1}} l^2_{F_i} l^{-1}_{F_{i-1}} \cdots l^{-1}_{F_1}. \tag{1.4}$$

Moreover, let $F_{j_1}, \dots, F_{j_l}$ be the sequence obtained from $F_1, \dots, F_k$ by recursively deleting faces F_j that are supported on a hyperplane which supports an odd number of elements of $F_j, \dots, F_k$ (compare [40, p. 614]) and define

$$\Sigma(\nu) := (F_{j_l}, \dots, F_{j_l}). \tag{1.5}$$

1.47 Lemma (Lemma 12 in [40])
Given a positive path $\nu = l_{F_1}, \cdots, l_{F_k}$ starting in the chamber C and ending in C'. Then there is a homotopy

$$\nu \simeq \Big(\prod_{G \in \Sigma(\nu)} \beta^{\nu}_G \Big)(C \to C').$$

From this Lemma another useful result follows.

1.48 Lemma (Corollary 12 in [40])
Let F, G be two faces of codimension 1 that are supported on the same hyperplane. Then β_F is homotopic to

$$\Big(\prod_{i=1}^{h} \beta^{\nu}_{j_i}\Big)\beta_G\Big(\prod_{i=1}^{h} \beta^{\nu}_{j_i}\Big)^{-1},$$

where ν is a positive minimal path from C_0 to $(C_0)_G$, and $j_1, \dots, j_h$ are the indices of the edges in ν that cross a hyperplane that does not separate C_0 from $(C_0)_F$, in the order in which they appear in ν.

Relations For every face $G \in \mathcal{F}_2$ consider a chamber C with $G \leq C$ and let $C' = \mathrm{opp}(C, G)$ be its opposite chamber with respect to G. Consider a minimal positive path ω from C to C'. Let us then consider the set $h(G) := \{F_1, \dots F_k\}$ of the codimension 1 faces adjacent to G, indexed according to the order in which the positive minimal path ω 'crosses' them. Let now for $i = 1, \dots, k$ F_{i+k} be the facet opposite to F_i with respect to G. Define a path

$$\alpha_G(C) := l_{F_1} l_{F_2} \dots l_{F_{2k}}. \tag{1.6}$$

Salvetti introduces a set of relations associated with G:

$$R_G : \quad \beta_{F_k} \dots \beta_{F_1} = \beta_{F_1} \beta_{F_k} \dots \beta_{F_2} = \dots$$

stating the equality of all cyclic permutations of the product. In fact, for every cyclic permutation σ of $\{1, \dots, k\}$

$$\beta_{F_{\sigma(k)}} \cdots \beta_{F_{\sigma(1)}} \simeq (C_0 \to \widetilde{C})\alpha_G(\widetilde{C})(C_0 \to \widetilde{C})^{-1} \tag{1.7}$$

where $\widetilde{C} := (C_0)_G$ and $\simeq$ means homotopy.

1.5. Complexified arrangements

Presentation One of the results of [40] is that the fundamental group of $M(\widetilde{\mathscr{A}})$ can be presented as

$$\pi_1(\mathcal{S}) = \langle \beta_F,\, F \in \mathcal{F}_1 \mid R_G,\, G \in \mathcal{F}_2 \rangle.$$

CHAPTER 2

Methods of combinatorial algebraic topology

In this chapter we introduce the objects of combinatorial algebraic topology that are relevant for this thesis. Most of them are well-established notions, some of them though are small additions to the existing theory (e.g. the treatment of polyhedral complexes in Section 2.1, discrete Morse theory for acyclic categories in Section 2.3 and the treatment of face categories in Section 2.2).

In the first part of the chapter (Section 2.1) we introduce several types of complexes, which we will use in the forthcoming chapters. The most important complexes for our purposes are Δ-complexes and polyhedral complexes. Δ-complexes are a well-established notion. Polyhedral complexes are also a well-known tool, however we treat them more formally than usual. This degree of formality may seem confusing at first sight, but it helps to define the face category of Section 2.2.

In the second part (Section 2.2) we introduce one of the main characters of this thesis: acyclic categories. Acyclic categories can be thought of as a generalization of posets and turn out to provide the right combinatorial framework to treat the topology of polyhedral complexes. A comprehensive reference on acyclic categories is the book [26].

Finally in the third part (Section 2.3) we treat Discrete Morse Theory. We generalize the usual theory for regular cell complexes to non-regular polyhedral complexes. Acyclic categories play a key role in this process.

2.1 Complexes

Simplicial complexes are historically among the first objects of topology that have been studied in detail. They provide an abstract combinatorial description of a restricted class of topological spaces: the so-called *triangulable spaces*.

2.1 Definition
A simplicial complex *is a pair* (I, K)*, where* I *is a set and* $K \subseteq \mathcal{P}(I)$ *is a collection of subsets of* I *satisfying the following conditions:*

(a) $\emptyset \in K$*,*

(b) for every $A \in K$*,* $B \subseteq A$ *implies* $B \in K$*.*

The elements of I *are called* vertices*, while the elements of* K *are called* simplices*.*

Due to their simplicity, simplicial complexes allow to study topology without having to worry about the pathology of general topological spaces. However, simplicial complexes are too "rigid" for many applications, including ours. Therefore we will introduce a wider class of combinatorially defined complexes, which will suit better our purposes: the so-called *Δ-sets*.

Δ-sets

The main idea behind simplicial complexes is to define a space, first giving its vertices and then saying which sets of vertices form a simplex. The main drawback of this approach is that there can not be two different simplices, which share the same vertex set. In other words, simplicial complexes are *regular complexes*.

To overcome this drawback we can first give the simplices, and then say how they glue together. This is the main idea behind Δ-sets. A formal definition requires the language of category theory.

2.2 Definition
Define Δ *as the category on the object set*

$$\mathrm{Ob}\,\Delta = \{[n] : n \in \mathbb{N}\}, \textit{ with } [n] = \{0, \dots, n\}$$

whose morphisms are the identities and the following order-preserving injections

$$f : [m] \to [n] \textit{ such that } i < j \textit{ implies } f(i) < f(j).$$

2.3 Definition
A Δ-set *is a contravariant functor*

$$S : \Delta \to \mathit{Set}.$$

We will write $S_n := S([n])$*, the elements of* S_n *are called* n-dimensional simplices*.*

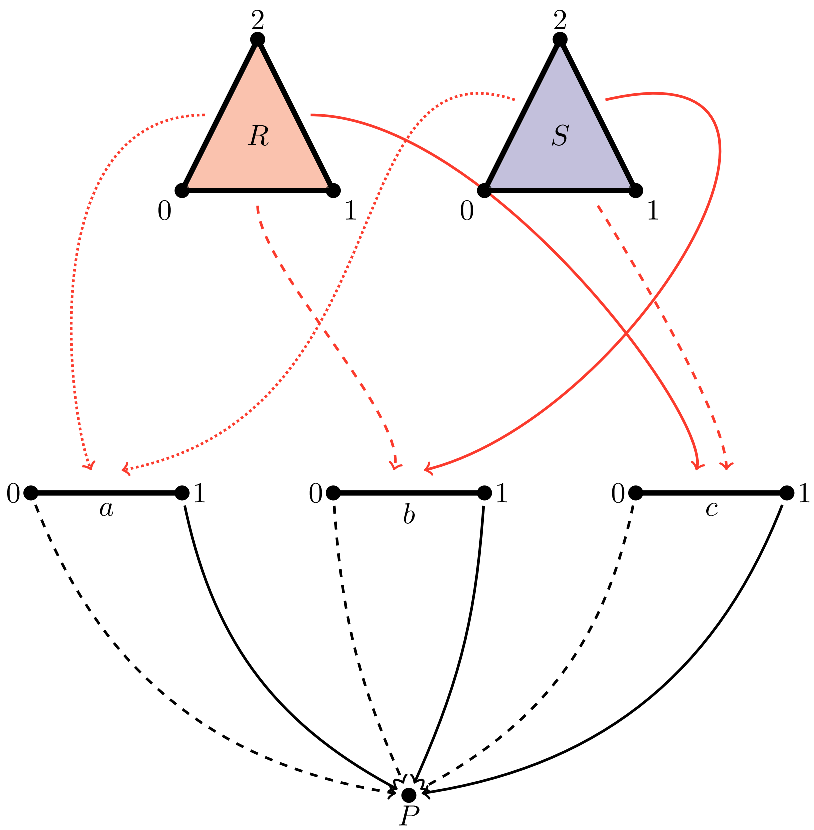


Figure 2.1: An example of Δ-set.

2.4 Example
Figure 2.1 shows a 2-dimensional Δ-set. There are one point $S_0 = \{P\}$, three edges $S_1 = \{a, b, c\}$ and two triangles $S_2 = \{R, S\}$. The arrows below represent the morphisms $S(j_i)$ for $j_i : [0] \to [1], 0 \mapsto i$ for $i = 0, 1$. The dashed lines represent $S(j_0)$, while the solid lines represent $S(j_1)$.

The arrows on top represent the images of morphism $[1] \to [2]$ in Δ_2. There are three of such morphisms h_i with $i = 1, 2, 3$, where h_i is the injection that "skips" i. In the picture the solid lines represent h_0, the dotted lines represent h_1 and the dashed lines represent h_2.

The geometric realization

Δ-sets provide a model for topological spaces in the following sense.

2.5 Definition
The standard n-dimensional simplex *is the convex hull of the stan-*

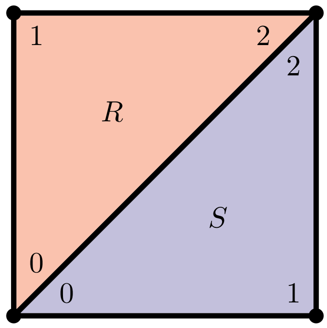


Figure 2.2: Geometric realization of the Δ-set of Figure 2.1.

dard basis of $\mathbb{R}^{n+1}$*:*

$$|\Delta_n| = \{\sum_{i=0}^{n} \lambda_i e_i : \quad \lambda_i \geq 0 \, \forall i, \quad \sum_{i=0}^{n} \lambda_i = 1\}.$$

Notice that we denote the standard basis of $\mathbb{R}^{n+1}$ *by* $\{e_0, \ldots, e_n\}$.

Given a Δ-Set S, each morphism $f : [m] \to [n]$ defines a map

$$|f| : |\Delta_m| \to |\Delta_n|.$$

The map $|f|$ is defined on the vertices e_i of the simplex $|\Delta_m|$ as $f(e_i) = e_{f(i)}$ and then extended by linearity.

2.6 Definition
Let $S : \Delta \to$ *Set be a* Δ*-set, its* geometric realization *is the space*

$$|S| = \left(\coprod_{n \in \mathbb{N}} |\Delta| \times S_n \right) \Big/ \sim .$$

where the relation $\sim$ *is defined by*

$$(x, S(f)(\rho)) \sim (|f|(x), \rho) \quad \forall (x, \rho) \in |\Delta_m| \times S_m, f : [m] \to [n] \in \mathrm{Hom}(\Delta_m, \Delta_n).$$

2.7 Example
Figure 2.2 shows the geometric realization of the Δ-set of Figure 2.1. The opposite side of the square are identified. In particular $|S|$ is homeomorphic to the 2-dimensional torus $(S^1)^2$.

2.8 Remark
The class of topological spaces, that are geometric realization of Δ-sets coincide with that of simplicial complexes. Indeed the barycentric subdivision of a Δ-set is always a simplicial complex. However, the combinatorics of Δ-sets is more powerful than that of simplicial complexes, as we will see later in this thesis.

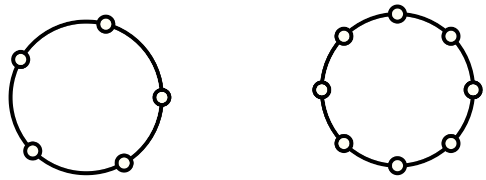


Figure 2.3: Two different polyhedra in $\mathbb{R}^2$

Polyhedral complexes

Even though Δ-sets describe the topology of their realizations in an elegant combinatorial way, they don't suffice for our purposes. We therefore introduce another class of complexes which doesn't share the combinatorial flavor of Δ-sets, but which has broader applications.

Our motivating examples are real hyperplane arrangements and, later in chapter 3, real toric arrangements. These arrangements define a decomposition of the ambient space $\mathbb{R}^k$ as a cell complex, which in general is not the geometric realization of a Δ-set.

Still, these complexes have a certain "rigidity" which we can exploit for our study, namely they are *polyhedral complexes*.

We will define polyhedral complexes as *embedded* in a topological space. One could define such complexes intrinsically, as we did in [12]. Anyway, the definition as embedded complex is simpler, more elegant, and suffices for our purposes.

We want to formalize the notion of space obtained "gluing polyhedra". We start clarifying what we mean with polyhedron, in a topological context. Let us fix a space X and consider polyhedra inside it. A subspace $P \subseteq X$ can usually be regarded as a polyhedron in many different way. For example a circle $S^1 \subseteq \mathbb{R}^2$ can be regarded as pentagon, as well as a heptagon (Figure 2.3). Therefore a topological notion of polyhedron should take into account the actual topological space and a subdivision thereof as well. We formalize this notion in the following.

Let X be a topological space and consider the following set of embedded polyhedra

$$\begin{aligned}\Xi_X = \{\chi : \widehat{P} \to X | \quad & \widehat{P} \subseteq \mathbb{R}^k \text{ } d\text{-dimensional convex polyhedron} \\ & \chi \text{ induces a homeomorphism } \operatorname{int}(\widehat{P}) \to \chi(\operatorname{int}(\widehat{P}))\},\end{aligned}$$

where $\operatorname{int}(\widehat{P})$ denotes the interior of $\widehat{P}$. Define the following equivalence relation on Ξ_X

$$\begin{aligned}&\text{for } \chi : \widehat{P} \to X, \xi : \widehat{Q} \to X \quad \text{with } \widehat{P} \subseteq \mathbb{R}^k, \widehat{Q} \subseteq \mathbb{R}^d \\ \chi \sim \xi \iff & \exists f : \mathbb{R}^k \to \mathbb{R}^d \text{ affine injection with } \xi = \chi \circ f_{|\widehat{P}} \text{ or vice versa}\end{aligned}$$

2.9 Definition

A d-dimensional topological polyhedron $(P, [\chi_P])$ *in a topological space X is a subset $P \subseteq X$, equipped with an equivalence class $[\chi_P] \in \Xi_X/_\sim$ such that $\chi_P(\widehat{P}) = P$.*

A face *of $(P, [\chi_P])$ is a topological polyhedron $(F, [\chi_F])$, where $F = \chi_P(\widehat{F})$ and $\chi_F \sim \chi_{P|\widehat{F}}$ for a representative $\chi_P \in \Xi$ of $[\chi_P]$ and a face $\widehat{F}$ of $\widehat{P}$.*

2.10 Remark

Let $(P, [\chi_P])$ be a topological polyhedron with $\chi_P : \widehat{P} \to X$ and $\chi' : \widehat{P}' \to X$ such that $\chi' \sim \chi$. We can assume, without loss of generality, that $\widehat{P}, \widehat{P}' \subseteq \mathbb{R}^k$.

Then there is an affine isomorphism $\mathbb{R}^k \to \mathbb{R}^k$ mapping $\widehat{P}$ to $\widehat{P}'$. This isomorphism induces an equivalence between the faces of $\widehat{P}$ and the faces of $\widehat{P}'$, showing that the definition of face doesn't depend on the chosen representative of $[\chi_P]$.

Since we do not require polyhedral complexes to be regular, we need a way to distinguish between different faces of a concrete polyhedron $\widehat{P}$ which are "glued" of the same face of a topological polyhedron $(P, [\chi_P])$. This distinction will be crucial to define the face category of a polyhedral complex.

Consider the following set:

$$\Theta = \left\{ (\chi_P, \widehat{F}) | \quad \chi_P : \widehat{P} \to X,\, \chi_P \in \Xi,\, \widehat{F} \text{ face of } \widehat{P} \right\}.$$

Define the following equivalence relation on Θ:

$$(\chi_P, \widehat{F}) \approx (\chi'_P . \widehat{F}') \iff \chi_P \sim \chi'_P \text{ and } \iota(\widehat{F}) = \widehat{F}'$$

where $\iota : \mathbb{R}^k \to \mathbb{R}^d$ is an affine injection with $\iota(\widehat{P}) = \widehat{P}'$ and $\chi'_P \circ \iota = \chi_P$.

2.11 Definition

Let $(P, [\chi_P])$ be a topological polyhedron and $(F, [\chi_F])$ one of its faces, an incidence relation *between $(P, [\chi_P])$ and $(F, [\chi_F])$ is an equivalence class $[(\chi_P, \widehat{F})] \in \Theta/_\approx$ such that $(\chi_P)_{|\widehat{F}} \sim \chi_F$.*

In order to keep the notation light we omit the double parenthesis when referring to incidence relations, i.e. we write $[\chi_P, \widehat{F}]$ for $[(\chi_P, \widehat{F})]$. When we fix a representative $\chi_P : \widehat{P} \to X$ we abbreviate the notation and write $\widehat{F}$ for the incidence relation $[\chi_P, \widehat{F}]$.

We can now define polyhedral complexes. In analogy with geometric simplicial complexes, we define polyhedral complexes as collection of polyhedra which intersect properly, i.e. the intersection of two polyhedra should be a common face of both. We formalize this notion in the following definition.

2.12 Definition
A polyhedral complex $\mathcal{P}$ *in a topological space* X *is a collection of topological polyhedra in* X*, such that:*

(i) for every $(P, [\chi_P]) \in \mathcal{P}$*, every face* $(F, [\chi_F])$ *of* P *is an element of* $\mathcal{P}$*,*

(ii) for every $(P_1, [\chi_{P_1}]), (P_2, [\chi_{P_2}]) \in \mathcal{P}$ *with* $\chi_{P_1} : \widehat{P_1} \to X, \chi_{P_2} : \widehat{P_2} \to X$ *let* $F = P_1 \cap P_2$*, then there exist* $\widehat{F}_1$*,* $\widehat{F}_2$ *faces of* $\widehat{P_1}$ *and* $\widehat{P_2}$ *respectively with* $\chi_{P_1}(\widehat{F}_1) = F = \chi_{P_2}(\widehat{F}_2)$ *and* $(\chi_{P_1})_{|\widehat{F}_1} \sim (\chi_{P_2})_{|\widehat{F}_2}$.

Notice that in point (ii) of the previous definition we have $(F, [(\chi_{P_1})_{|\widehat{F}_1}]) \in \mathcal{P}$.

A polyhedral complex is called *polytopal complex* if all its faces are polytopes.

2.13 Definition
Let $\mathcal{P}$ *be a polyhedral complex in* X*; its* geometric realization *is*

$$|\mathcal{P}| = \bigcup_{(P,[\chi_P]) \in \mathcal{P}} P \subseteq X.$$

Let $\mathcal{P}$ be a polytopal complex in the topological space X, we can associate to $\mathcal{P}$ a CW-complex whose cells are

$$\{\widehat{P} | \quad (P, [\chi_P]) \in \mathcal{P}\}$$

and whose attaching maps are the restriction of the maps χ_P to the border of $\widehat{P}$.

The following proposition can be easily proved and shows that the notion of polyhedral complex agrees with the more common notion of CW-complex.

2.14 Proposition
Let $\mathcal{P}$ *be a polytopal complex in the topological space* X*, let* $K_{\mathcal{P}}$ *be the CW-complex defined above. If* $\mathcal{P}$ *is finite dimensional, then there is a homeomorphism* $K_{\mathcal{P}} \cong |\mathcal{P}|$.

2.2 Acyclic categories

In the field of combinatorial topology usually the combinatorial information is encoded within posets. The first and probably most important example of this is the face poset of a cell complex.

However, since we are interested in non-regular complexes, posets do not suffice to fully encode the relevant topological information. Therefore we introduce the notion of *acyclic category*. This is the combinatorial side of Definition 2.11.

Our main general reference for this topic is Kozlov's book [26].

2.15 Definition

An acyclic category *is a small category C, such that:*

(i) the only morphisms that have inverses are the identities;

(ii) the only endomorphisms are the identities.

Morphisms of acyclic categories are functors.

Acyclic categories occur sometimes in the literature as "loop-free categories" or "scwol"s (small category without loops, cfr. [5]).

The connection between acyclic categories and topology is provided by the notion of *nerve*.

2.16 Definition

The nerve of an acyclic category $\mathcal{C}$ *is the* Δ*-set* $\Delta(\mathcal{C})$ *defined as follows:*

(a) the k-dimensional simplices are k-length chains of non trivial composable morphisms

$$\sigma = a_0 \overset{m_1}{\to} a_1 \overset{m_2}{\to} a_2 \overset{m_3}{\to} \cdots \overset{m_k}{\to} a_k,$$

(b) the boundary simplices of a simplex σ as above are defined as follows:

$$\begin{aligned}
\partial_0\sigma &= a_1 \overset{m_2}{\to} a_2 \overset{m_3}{\to} \cdots \overset{m_k}{\to} a_k \\
\partial_j\sigma &= a_0 \overset{m_1}{\to} \cdots \overset{m_{j-1}}{\to} a_{j-1} \overset{m_{j+1}\circ m_j}{\to} a_{j+1} \overset{m_{j+2}}{\to} \cdots \overset{m_k}{\to} a_k \\
\partial_k\sigma &= a_0 \overset{m_1}{\to} a_1 \overset{m_2}{\to} a_2 \overset{m_3}{\to} \cdots \overset{m_{k-1}}{\to} a_{k-1}
\end{aligned}$$

Where $\partial_i = S(j_i)$ and $j_i : [k-1] \to [k]$ is the injection that "skips" i.

2.17 Remark

Since every order preserving injection $[m] \to [n]$ can be decomposed as composition of "simpler" injections of the type j_i, Definition 2.16 suffices to define a Δ-set.

2.18 Remark

Notice that the hypothesis of acyclicity of $\mathcal{C}$ is required to ensure that $\Delta(\mathcal{C})$ is a Δ-set.

Colimits

In chapter 4 we will use diagrams and colimits of acyclic category as a tool to "glue" isomorphisms between "simple" categories in order to obtain an isomorphism between more complex categories.

As a preparation for chapter 4 we review the notion of diagram and colimit of acyclic categories. We also provide an explicit construction for the colimit of a restricted class of diagrams.

Recall the notion of a diagram in a category.

2.19 Definition
Let $\mathcal{I}$ be a small *index category and $\mathcal{C}$ a target category, a* diagram *of categories on $\mathcal{I}$ is a functor*

$$\mathcal{D} : \mathcal{I} \to \mathcal{C}.$$

A *co-cone* over $\mathcal{D}$ is a family of morphisms (C, γ) with

$$\gamma_X : X \to C \quad \text{for every } X \in \operatorname{Ob} \mathcal{I},$$

such that for every morphism $f : X \to Y$ in $\mathcal{I}$,

$$\gamma_X = \gamma_Y \circ \mathcal{D}(f).$$

A morphisms of co-cones between (C, γ) and (D, δ) is a morphism $\psi : C \to D$ in $\mathcal{C}$, such that for every $X \in \operatorname{Ob} \mathcal{I}$,

$$\psi \circ \gamma_X = \delta_X.$$

Colimits are objects defined by the following universal property.

2.20 Definition (Universal property of colimits)
Let $\mathcal{D} : \mathcal{I} \to \mathcal{C}$ be a diagram of acyclic categories; we say that a co-cone (C, γ) is the colimit *of $\mathcal{D}$ if for every co-cone (D, δ) over $\mathcal{D}$ there exists a unique morphism of co-cones*

$$\psi : (C, \gamma) \to (D, \delta).$$

It can be easily seen that the association of a colimit to a diagram is functorial. That is, morphisms of diagrams give morphisms between the respective colimits.

For many categories there are explicit constructions of colimits.

2.21 Example
Let $\mathcal{D} : \mathcal{I} \to$ Set be a diagram of sets, then $\operatorname{colim} \mathcal{D} = (C, \gamma)$ where

$$C = \left(\coprod_{X \in \operatorname{Ob} \mathcal{I}} \mathcal{D}(X) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by

$$x \sim y \text{ if } \exists f : X \to Y \text{ such that } \mathcal{D}(f)(x) = y.$$

and $\gamma(x) = \pi(x)\ \forall X \in \operatorname{Ob}\mathcal{I}, x \in X$, where $\pi : \coprod_{X \in \operatorname{Ob}\mathcal{I}} \mathcal{D}(X) \to \left(\coprod_{X \in \operatorname{Ob}\mathcal{I}} \mathcal{D}(X)\right)\big/\sim$ is the usual projection.

Colimits of acyclic categories

We will now see an explicit construction of the colimit of a diagram of acyclic categories, for a specific class of diagrams. Our hypothesis are restrictive, but sufficient for our purposes.

2.22 Definition
Let $\mathcal{D} : \mathcal{I} \to$ AC be a diagram of acyclic categories. We say that $\mathcal{D}$ is geometric *if*

(a) $\mathcal{I}$ is an acyclic category;

(b) for every $X \in \operatorname{Ob}\mathcal{I}$, $\mathcal{D}(X)$ is ranked and for every morphisms $f : X \to Y$ in $\mathcal{I}$, $D(f)$ preserves the ranks;

(c) for every $X \in \operatorname{Ob}\mathcal{I}$ and $x \in \operatorname{Mor}\mathcal{D}(X)$ there exist

(i) an index $\widehat{X} \in \operatorname{Mor}\mathcal{I}$,

(ii) a morphism $f : \widehat{X} \to X$ and

(iii) a morphism $\widehat{x} \in \operatorname{Mor}\mathcal{D}(\widehat{X})$ with $\mathcal{D}(f)(\widehat{x}) = x$;

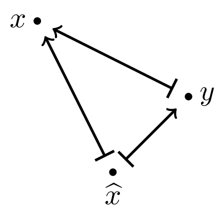
such that

(I) for every index $Y \in \operatorname{Ob}\mathcal{I}$,

(II) and morphism $g : Y \to X$ with $x = \mathcal{D}(g)(y)$ for some $y \in \operatorname{Mor}\mathcal{D}(Y)$

(III) there exists a morphism $\widehat{g} : \widehat{X} \to Y$, such that $\mathcal{D}(\widehat{g})(\widehat{x}) = y$.

The following diagram illustrates the definition.



Notice that from the acyclicity of $\mathcal{I}$ it follows that for every $x \in \mathcal{D}(X)$, $\widehat{x}$ is unique.

If $\mathcal{D}$ is a geometric diagram of acyclic categories, the properties of Definition 2.22 hold also for the set of objects $\operatorname{Ob}\mathcal{D}(X)$. This can be seen identifying the objects of $\mathcal{D}(X)$ with the identity morphisms.

2.23 Remark

Let $\mathcal{C}$ be a ranked acyclic category and $m : x \to y$ a morphism in $\mathcal{C}$. The *height* of m is defined as

$$\operatorname{ht}(m) = \mathbf{rk}\,(y) - \mathbf{rk}\,(x).$$

If $\mathcal{D} : \mathcal{I} \to \mathrm{AC}$ is a geometric diagram of acyclic categories, then for every morphism $f : X \to Y$ in $\mathcal{I}$, the map $\mathcal{D}(f) : \operatorname{Mor}\mathcal{D}(X) \to \operatorname{Mor}\mathcal{D}(Y)$ preserves the heights.

Define the following relation on the set of morphisms $\coprod_{X\in\operatorname{Ob}\mathcal{I}} \operatorname{Mor}\mathcal{D}(X)$:

2.24 Definition

Let $X, Y \in \operatorname{Ob}\mathcal{I}$, $x \in \operatorname{Mor}\mathcal{D}(X)$ and $y \in \operatorname{Mor}\mathcal{D}(Y)$. We write $x \sim y$ if there exist

(i) an index $Z \in \operatorname{Ob}\mathcal{I}$,

(ii) the morphisms $\xi : Z \to X$, $\zeta : Z \to Y$ and

(iii) a morphism $z \in \operatorname{Mor}\mathcal{D}(Z)$;

such that

$$\mathcal{D}(\xi)(z) = x \text{ and } \mathcal{D}(\zeta)(z) = y.$$

2.25 Proposition

The relation $\sim$ of Definition 2.24 is an equivalence relation.

Proof. The thesis follows immediately from the observation that $x \sim y \iff \widehat{x} = \widehat{y}$. □

Define the equivalence relation $\approx$ on $\coprod_{X\in\operatorname{Ob}\mathcal{I}} \operatorname{Mor}\mathcal{D}(X)$ analogously to Definition 2.24.

2.26 Proposition

Let $\mathcal{D} : \mathcal{I} \to AC$ be a geometric diagram of acyclic categories. Then the colimit $(\mathcal{C}, \gamma)$ of $\mathcal{D}$ exists and we have

$$\operatorname{Ob}\mathcal{C} = \left(\coprod_{X\in\operatorname{Ob}\mathcal{I}} \operatorname{Ob}\mathcal{D}(X)\right)\Big/\approx$$

$$\operatorname{Mor}\mathcal{C} = \left(\coprod_{X\in\operatorname{Ob}\mathcal{I}} \operatorname{Mor}\mathcal{D}(X)\right)\Big/\sim\ ,$$

where if $\nu : x \to y$*, then* $[\nu]_\sim : [x]_\approx \to [y]_\approx$*. Furthermore for every* $X \in \operatorname{Ob} I, x \in \operatorname{Ob} \mathcal{D}(X), \nu \in \operatorname{Mor} \mathcal{D}(X)$ *we have*

$$\gamma_X(x) = [x]_\approx, \quad \gamma_X(\nu) = [\nu]_\sim.$$

Proof. First of all, the category $\mathcal{C}$ is well defined, indeed if $\mu : x \to y$ and $\nu : x' \to y'$ are morphisms in $\mathcal{D}(X)$ with $\mu \sim \nu$, then $x \approx x'$ and $y \approx y'$. Furthermore Remark 2.23 ensures the acyclicity of $\mathcal{C}$.

Let $(\mathcal{E}, \varepsilon)$ be a co-cone over $\mathcal{D}$, we have to show that there exists a unique morphisms of co-cones $\Psi : (\mathcal{C}, \gamma) \to (\mathcal{E}, \varepsilon)$. Define for every $[x]_\sim \in \operatorname{Mor} \mathcal{C}$ and $[a] \in \operatorname{Ob} \mathcal{C}$

$$\Psi[x]_\sim = \varepsilon_X(x), \quad \Psi[a]_\approx = \varepsilon_X(a),$$

where $x \in \operatorname{Mor} \mathcal{D}(X)$ and $a \in \operatorname{Ob} \mathcal{D}(X)$.

The map Ψ is well-defined. Indeed, let x, y, z and ξ, ζ as in Definition 2.24, then

$$\varepsilon_X(x) = \varepsilon_Z(z) = \varepsilon_Y(y).$$

Analogously, equivalent objects give the same value of Ψ.

Uniqueness and functoriality of Ψ are easily shown. □

Face category

Our first motivation for considering acyclic categories is to encode the topology of a polyhedral complex.

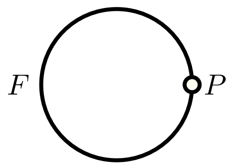
A well-know result on the topology of regular cell complexes states that the face poset completely describe the topology of the complex. That is, the order complex of the face poset is homeomorphic to the original complex (i.e. it is its barycentric subdivision).

This result, however, does not hold for non regular complexes. As shown in Example 2.27. If we focus on polyhedral complexes we can overcome this problem. To do this, we will need to encode the incidence relations between cells in an acyclic category instead of a poset.

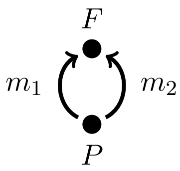
2.27 Example
Figure 2.4a shows a non regular CW-complex homeomorphic to a circle. Its face poset consists only of the two faces P and F and therefore its order complex is a segment, which is clearly not homeomorphic to the original complex.

The acyclic category of Figure 2.4b, on the other hand, encodes all information needed to reconstruct the topology of the complex. Indeed its nerve (Figure 2.4c) is still a circle.

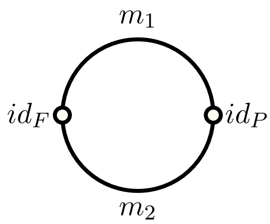
2.28 Remark
Let $\mathcal{P}$ be a polyhedral complex, in the following definition we will use the following conventions. Consider an element $(P, [\chi_P]) \in \mathcal{P}$



(a) A non regular CW-complex



(b) Face category



(c) Barycentric Subdivision

Figure 2.4: A non regular CW-complex and its face category

and choose a representative $\chi_P : \widehat{P} \to X$. For every face $\widehat{F}$ of $\widehat{P}$, call $F = \chi_P(\widehat{F})$ and $\chi_F = (\chi_P)_{|\widehat{F}}$ so that $(F, [\chi_F]) \in \mathcal{P}$.

Sometimes, for the sake of brevity, we will omit the $[\chi_P]$ and write $P \in \mathcal{P}$ for an element $(P, \chi_P) \in \mathcal{P}$

2.29 Definition
Let $\mathcal{P}$ be a polyhedral complex, the face category $\mathcal{F}(\mathcal{P})$ *of $\mathcal{P}$ is defined as follows.*

I. The objects *are the cells of the polyhedral complex*

$$\operatorname{Ob}\mathcal{F}(\mathcal{P}) = \mathcal{P}.$$

II. Let $(P, [\chi_P]), (F, [\chi_F]) \in \mathcal{P}$, then the morphisms $(F, [\chi_F]) \to (P, [\chi_P])$ correspond to the incidence relations between $(P, [\chi_P])$ and $(F, [\chi_F])$ (cfr. Definition 2.11). For every such incidence relation $[\chi_P, \widehat{F}]$ we write

$$m_{[\chi_P, \widehat{F}]} : (F, [\chi_F]) \to (P, [\chi_P]).$$

III. Consider the morphisms

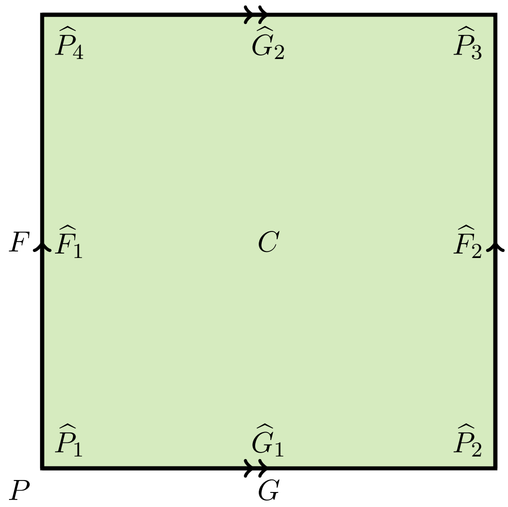
$$(G, [\chi_G]) \xrightarrow{m_{[\chi_F, \widehat{G}]}} (F, [\chi_F]) \xrightarrow{m_{[\chi_P, \widehat{F}]}} (P, [\chi_P])$$

Choose the representatives such that

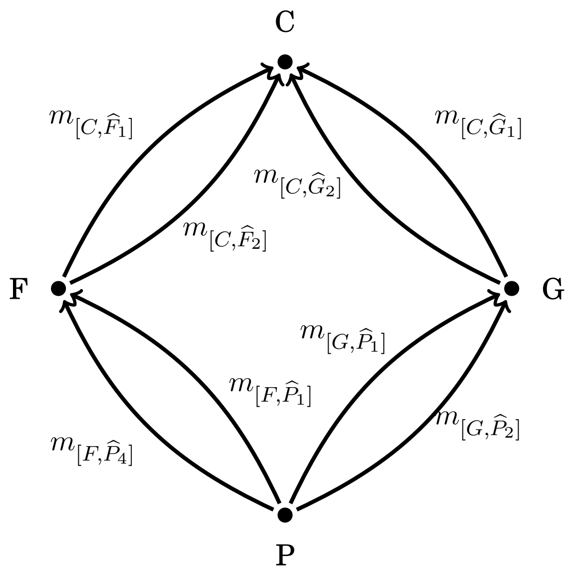
$$\chi_F = (\chi_P)_{|\widehat{F}} \text{ and } \chi_G = (\chi_F)_{|\widehat{G}},$$

then $\widehat{G}$ is a face of $\widehat{P}$ and the following composition law is well-defined:

$$m_{[\chi_P, \widehat{F}]} \circ m_{[\chi_F, \widehat{G}]} = m_{[\chi_P, \widehat{G}]}.$$



(a) A polyhedral complex



(b) Face Category

Figure 2.5: Face category of a polyhedral complex

2.30 Example

Figure 2.5a shows a polyhedral complex whose geometric realization is a 2-dimensional torus. The opposite edges of the square are identified. We have one dimension 0 face: the point P, two dimension 1 faces: F and G and one dimension 2 face: the square C.

Figure 2.5b shows an excerpt of the face category. In order to keep the notation light we write $[C, \widehat{P}_1]$ instead of $[\chi_C, \widehat{P}_1]$.

We have four incidence relations between P and C, that is four morphisms $P \to C$ in the face category:

$$m_{[C,\widehat{P}_1]},\ m_{[C,\widehat{P}_2]},\ m_{[C,\widehat{P}_3]},\ m_{[C,\widehat{P}_4]}$$

Between F and C there are two incidence relation, as well as between F and P. Notice that $[F, \widehat{P}_1] = [F, \widehat{P}_2]$ and $[F, \widehat{P}_3] = [F, \widehat{P}_4]$. Using the composition law of Definition 2.29 we can find out e.g. that

$$m_{[C,\widehat{F}_1]} \circ m_{[F,\widehat{P}_1]} = m_{[C,\widehat{P}_1]} = m_{[C,\widehat{G}_1]} \circ m_{[G,\widehat{P}_1]}.$$

2.31 Proposition

Let $\mathcal{P}$ be a polyhedral complex, then the category $\mathcal{F}(\mathcal{P})$ is acyclic.

Proof. Let $(P, [\chi_P]) \in \mathcal{P}$, the only morphism $(P, [\chi_P]) \to (P, [\chi_P])$ is $m_{[\chi_P,\widehat{P}]}$ which is the identity.

Furthermore for every morphism $m_{[\chi_P,\widehat{F}]} : (F, [\chi_F]) \to (P, [\chi_P])$ in $\mathcal{F}(\mathcal{P})$ it must be $\dim F \leq \dim P$. Therefore, since $(P, [\chi_P])$ is its only face of maximal dimension, the only invertible morphisms are endomorphisms and hence identities. □

2.32 Definition

Let $\mathcal{P}$ be a polyhedral complex, the barycentric subdivision *of $\mathcal{P}$ is defined as $\mathcal{B}(\mathcal{P}) = \Delta(\mathcal{F}(\mathcal{P}))$.*

The following proposition formalizes the fact that the face category encodes the topology of the polyhedral complex. It can be easily proved, but it is not relevant for our purposes.

2.33 Proposition

Let $\mathcal{P}$ be a polyhedral complex, then $|\mathcal{B}(\mathcal{P})| \cong |\mathcal{P}|$.

2.3 Discrete Morse Theory

In chapter 4 we will use Discrete Morse Theory as a tool to prove minimality of the toric Salvetti Complex. Discrete Morse Theory was introduced in [23] as a combinatorial analog of Morse Theory for cell complexes. In this thesis we will use the framework of [26].

Discrete Morse Theory allows to simplify a cell complex, based on some combinatorial data. Usually Discrete Morse Theory is stated for regular complexes. However, in chapter 4 we will need to apply it to a non-regular polyhedral complex. The framework of [26] extends with few minor modifications to this case.

In the following we will introduce Discrete Morse Theory as in [26] and show how it applies to non regular polyhedral complexes. Throughout this section we assume that the acyclic categories are of finite rank. In the case of face categories of polyhedral complexes this means that we consider only finite dimensional polyhedral complexes.

Recall the following basic fact on the topology of cell complexes.

2.34 Proposition ([26, Theorem 11.11])
Let X_1, X_2 be topological spaces and let $h : X_1 \to X_2$ be an homotopy equivalence. Let σ be a cell with attachment maps $f_1 : \partial\sigma \to X_1$, $f_2 : \partial\sigma \to X_2$ such that $h \circ f_1 \simeq f_2$. Then there is an homotopy equivalence

$$X_1 \sqcup_{f_1} \sigma \simeq X_2 \sqcup_{f_2} \sigma.$$

2.35 Definition
An indecomposable morphism *in an acyclic category is a morphism that cannot be written as the composition of two nontrivial morphisms.*

A linear extension $\leq$ *of an acyclic category is a total order on its set of objects, such that*

$$\mathrm{Mor}(x, y) \neq \emptyset \Longrightarrow x \leq y.$$

Let $m : x \to y$ be a morphism in an acyclic category, we will write $s(m) = x$ and $e(m) = y$ for the source and the target of m.

2.36 Definition
A matching *of an acyclic category $\mathcal{C}$ is a set $\mathfrak{M}$ of indecomposable morphisms such that, for every m, $m' \in \mathfrak{K}$, the sources and the targets of m and m' are four distinct objects of $\mathcal{C}$.*

2.37 Definition
A cycle *of a matching $\mathfrak{M}$ is an ordered sequence of morphisms*

$$a_1 b_1 a_2 b_2 \cdots a_n b_n$$

where

(1) For all i, $a_i \notin \mathfrak{M}$ and $b_i \in \mathfrak{M}$,

(2) For all i, $e(a_i) = e(b_i)$ and $s(a_{i+1}) = s(b_i)$, with the convention $s(a_1) = s(b_n)$.

A matching $\mathfrak{M}$ is called acyclic *if it has no cycles. A* critical element *of $\mathfrak{M}$ is an object of $\mathcal{C}$ that is neither source or target of any $m \in \mathfrak{M}$.*

2.38 Lemma
A matching $\mathfrak{M}$ of an acyclic category $\mathcal{C}$ is acyclic if and only if there is a linear extension of $\mathcal{C}$ such that

$$\forall m \in \mathfrak{M} \quad s(m) \lessdot e(m)$$

where $\lessdot$ denotes a covering relation.

Proof. The proof of [26, Theorem 11.1] shows the result for the case when $\mathcal{C}$ is a poset.

Consider the poset $\mathcal{C}_{\preceq}$ associated to $\mathcal{C}$, with the ordering relation defined by

$$x \preceq y \iff \mathrm{Mor}(x, y) \neq \emptyset.$$

The matching $\mathfrak{M}$ defines a matching on $\mathcal{C}_{\preceq}$

$$\mathfrak{M}_{\preceq} = \{(s(m), e(m))| \quad m \in \mathfrak{M}\}$$

and it is clear that $\mathfrak{M}$ is acyclic if and only if $\mathfrak{M}_{\preceq}$ is acyclic. □

A very handy tool for dealing with (and finding) acyclic matchings is the following result, the proof of which follows as an easy exercise by inspection of the definitions and comparison with [26, Theorem 11.10].

2.39 Lemma (Patchwork Lemma)
Consider a functor of acyclic categories

$$\varphi : \mathcal{C} \to \mathcal{C}'$$

and suppose that for each object $c \in \mathcal{C}'$ an acyclic matching $\mathfrak{M}_c$ of $\varphi^{-1}(c)$ is given.

Then the matching $\mathfrak{M} := \bigcup_{c \in \mathrm{Ob}\,\mathcal{C}'} \mathfrak{M}_c$ of $\mathcal{C}$ is acyclic.

Proof. Acyclicity of $\mathfrak{M}$ is proved via the linear extension of $\mathcal{C}$ obtained by concatenation of the linear extensions given by the $\mathfrak{M}_c$ on the categories $\varphi(c)$. □

Discrete Morse Theory uses acyclic matchings to simplify a cell complex. In the following we propose the part of the so-called *fundamental theorem of Discrete Morse Theory* that is relevant for our purposes.

2.40 Theorem
Let $\mathcal{P}$ be a polyhedral complex and $\mathcal{F}(\mathcal{P})$ its face category. Let $\mathfrak{M}$ be an acyclic matching on $\mathcal{F}(\mathcal{P})$. Then $|\mathcal{P}|$ is homotopy equivalent to a CW-complex X with one cell of dimension k for every critical element of $\mathfrak{M}$ of rank k.

Proof. We follow the proof of [26, Theorem 11.13 (b)]. We prove the theorem by induction on $\mathcal{F}(\mathcal{P})$, the base case being trivial.

Let $\mathcal{L}$ be a linear extension of $\mathcal{C}$ as in 2.38 and let $(P, \chi_P) = \max \mathcal{L}$. We distinguish two cases

I. If $(P, [\chi_P])$ is a *critical cell* we consider the polyhedral complex $\mathcal{P}' = \mathcal{P} \backslash \{(P, [\chi_P])\}$, so that $|\mathcal{P}'| = |\mathcal{P}| \backslash \operatorname{int} P$.

   Then $\mathfrak{M}$ is also an acyclic matching on $\mathcal{F}(\mathcal{P}')$ and by inductive hypothesis there exists an homotopy equivalence $h : |\mathcal{P}'| \to X'$ where X' is a CW-complex with one k-dimensional cell for each critical element of rank k in $\mathfrak{M}$ different from σ.

   Let $\iota : \partial P \to |\mathcal{P}|$ be the inclusion (i.e. the attaching map in the sense of Proposition 2.14). Applying Proposition 2.34 with $f_1 = \iota$ and $f_2 = h \circ \iota$ we get

$$|\mathcal{P}| \cong |\mathcal{P}'| \sqcup_\iota P \simeq X' \sqcup_{h \circ \iota} P =: X.$$

II. If $(P, [\chi_P])$ is not critical, i.e. there is a morphism $m \in \mathfrak{M}$ with $e(m) = (P, [\chi_P])$, let $(F, [\chi_F]) = s(m)$. Then in $\mathcal{L}$ we have $(F, [\chi_F]) \lessdot (P, [\chi_P])$. That is $(F, [\chi_F])$ is not a face of any other cell in $\mathcal{P}$ (otherwise these cells would be between $(F, [\chi_F])$ and $(P, [\chi_P])$). In particular

$$\mathcal{P}' = \mathcal{P} \backslash \{(F, [\chi_F]), (P, [\chi_P])\}$$

   is a polyhedral complex and $|\mathcal{P}'| = |\mathcal{P}| \backslash (\operatorname{int} P \cup \operatorname{int} F)$. Furthermore since F is on the border of $|\mathcal{P}|$ deleting F and P from $|\mathcal{P}|$ is a so-called *cellular collapse* and there is an homotopy equivalence (actually a strong retraction) $f : |\mathcal{P}'| \to |\mathcal{P}|$.

   Since $\mathfrak{M}' = \mathfrak{M} \backslash \{m\}$ is an acyclic matching on $\mathcal{F}(\mathcal{P}')$ with the same critical cells of $\mathfrak{M}$, the induction hypothesis gives

$$|\mathcal{P}| \simeq |\mathcal{P}'| \simeq X$$

   where X is a CW-complex as in the thesis. □

2.41 Remark

The actual fundamental theorem is stronger as the one proved here. In particular it also describes the incidence relations between the cells in X, thus allowing to compute (co)homology.

The theorem is usually stated and proved for *regular CW-complexes*, instead of polyhedral complexes. In that case the face poset suffices to encode the topology of the complex (in the sense of Proposition 2.33).

We needed to restrict ourselves to polyhedral complex to be able to define the face category.

CHAPTER 3

Toric arrangements

In this chapter we present the core topic of this thesis: *toric arrangements*. Toric arrangements are, roughly speaking, families of subtori in a complex torus $(\mathbb{C}^*)^n$ for which we want to develop a theory inspired by that of hyperplane arrangements.

Toric arrangements made their debut in Looijenga's paper [30], where he needed to compute the Betti numbers of the complement of a complex toric arrangement in order to study the cohomology of some moduli spaces.

Later the topic was brought to the attention of the arrangement community by the paper [16] of De Concini and Procesi. The authors reprove Looijenga's result (Theorem 3.17 below) using methods of algebraic geometry and outline the connection between the topology and the combinatorics of the arrangement. In particular they propose the *layer poset* as a "toric analog" of the intersection poset.

Ehrenborg, Readdy and Slone in [22] considered the combinatorics of toric arrangement on the "compact torus" $(S^1)^n$, solving the problem of enumerating the faces of the induced decomposition of the compact torus.

The topic was carried further by Moci with the work started in his PhD Thesis and in the papers [31, 33, 32]. In [31] he studied the toric arrangements arising naturally from Lie algebras and which are related to affine Weyl groups. In [33] he developed a theory of "wonderful models" of toric arrangement complements, inspired by that of De Concini and Procesi for hyperplane arrangements in [15]. While in [32] he introduces a two-variable polynomial that encodes enumerative invariants of many of the different objects populating the landscape outlined by De Concini and Procesi in [17]. The same author, in joint work with Settepanella [34], studied the homotopy type of the complement of a special class of toric arrangements (i.e. what they call *thick arrangements*).

Particular interest is raised by the problem of formalizing the combinatorics of toric arrangements. The most relevant attempt seems to be that of Moci and D'Adderio in [10] where they suggest a theory of arithmetic matroids as a "combinatorialization" of the essential algebraic data of toric arrangements. A theory of oriented matroids for toric arrangement is not yet known. This should ideally provide a combinatorial counterpart to our results on the toric Salvetti complex (cfr. §3.7).

In this chapter we introduce toric arrangements and apply the machinery of the first two chapters to work out a combinatorial model of the complement of a complexified toric arrangement in the spirit of Salvetti's [40]. This will involve using acyclic categories, polyhedral complexes and group actions on them.

After having presented the "toric Salvetti complex" we turn our attention to its fundamental group and give a finite presentation thereof.

3.1 Introduction

We have presented arrangements of hyperplanes in an affine space as families of level sets of linear forms. Now, we want to explain in which sense this idea has been generalized to a toric setting.

Our ambient spaces will be the *complex torus* $(\mathbb{C}^*)^d$ and the *compact* (or *real*) *torus* $(S^1)^d$, where we consider S^1 as the unit circle in $\mathbb{C}$.

3.1 Definition
A character *of the n-dimensional torus is a map $\chi : (\mathbb{C}^*)^d \to \mathbb{C}^*$ given by evaluations of Laurent monomials, that is:*

$$\chi(x_1, \ldots, x_d) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \textit{ for an } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d.$$

Characters form a lattice, which we denote by Λ, under pointwise multiplication. Notice that the assignment $\alpha \mapsto x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ provides an isomorphism $\mathbb{Z}^d \to \Lambda$.

We consider subtori defined as level sets of characters, that is hypersurfaces in $(\mathbb{C}^*)^d$ of the form

$$K = \{x \in (\mathbb{C}^*)^d \mid \chi(x) = a\} \text{ with } \chi \in \Lambda, a \in \mathbb{C}^*. \tag{3.1}$$

Notice that, if $a \in S^1$, the intersection $K \cap (S^1)^d$ is also a level set of a character (described by the same equation).

3.2 Definition
A (complex) toric arrangement $\mathscr{A}$ *in* $(\mathbb{C}^*)^d$ *is a finite set*

$$\mathscr{A} = \{K_1, \ldots, K_n\}$$

of hypersurfaces of the form (3.1) *in* $(\mathbb{C}^*)^d$

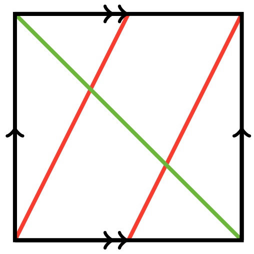


Figure 3.1: A toric arrangement on $(S^1)^2$

3.3 Definition
Let $\mathscr{A}$ be a toric arrangement in $(\mathbb{C}^*)^d$. Its complement *is*

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}.$$

3.4 Definition
A real toric arrangement *$\mathscr{A}$ in $(S^1)^d$ is a finite set*

$$\mathscr{A}^c = \{K_1^c, \ldots, K_n^c\}$$

of hypersurfaces in $(S^1)^d$ of the form (3.1) *with $a \in S^1$.*

If a complex toric arrangement restricts to a real toric arrangement on $(S^1)^d$ we will call $\mathscr{A}$ complexified.

We will often use this interplay between the complex and the 'real' hypersurfaces in the same vein that one exploits properties of the real part of complexified arrangements to gain insight into the complexification.

3.5 Example
Figure 3.1 shows the toric arrangement

$$\mathscr{A} = \big\{\{xy^{-1} = 1\}, \quad \{x^2y^{-1} = 1\}\big\}$$

on the two dimensional compact torus $(S^1)^2$. We will use this graphical representation through the rest of this thesis. The compact torus is pictured as a square where the opposite sides are identified, subtori are then periodic lines with rational slope.

3.2 An abstract approach

The definitions of §3.1, though being correct and sufficiently general, have the drawback of being dependent on a coordinate system. For many arguments it is convenient to reason about toric arrangements in a 'coordinate-free' way.

We will introduce an equivalent, but more abstract, approach to toric arrangements. In order to do this we will change point of view, taking as our basic object the character lattice rather than the torus.

3.6 Definition
Let $\Lambda \cong \mathbb{Z}^d$ a finite rank lattice. The corresponding complex torus *is*

$$T_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*).$$

The compact (or real) torus *corresponding to Λ is*

$$T^c_\Lambda = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1),$$

where, again, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

The choice of a basis $\{v_1, \dots, v_d\}$ of Λ gives isomorphisms

$$\begin{array}{cc} \Phi : T_\Lambda \to (\mathbb{C}^*)^d & \Phi^c : T^c_\Lambda \to (S^1)^d \\ \varphi \mapsto (\varphi(v_1), \dots, \varphi(v_d)) & \varphi \mapsto (\varphi(v_1), \dots, \varphi(v_d)) \end{array} \tag{3.2}$$

3.7 Remark
Consider a finite rank lattice Λ and the corresponding torus T_Λ. The *characters* of T_Λ are the functions

$$\chi_\lambda : T_\Lambda \to \mathbb{C}^*, \quad \chi_\lambda(\varphi) = \varphi(\lambda) \text{ with } \lambda \in \Lambda.$$

Characters form a lattice under pointwise multiplication, and this lattice is naturally isomorphic to Λ. Therefore in the following we will identify the character lattice of T_Λ with Λ.

Now, the 'abstract' definition of toric arrangements is the following.

3.8 Definition
Consider a finite rank lattice Λ, a toric arrangement *in T_Λ is a finite set of pairs*

$$\mathscr{A} = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times \mathbb{C}^*.$$

A toric arrangement $\mathscr{A}$ is called complexified *if $\mathscr{A} \subset \Lambda \times S^1$.*

3.9 Remark
The abstract definition is clearly equivalent to Definition 3.2 via the isomorphisms in (3.2) and by

$$K_i := \{x \in T_\Lambda \mid \chi_i(x) = a_i\}. \tag{3.3}$$

Accordingly, we have $M(\mathscr{A}) := T_\Lambda \setminus \bigcup\{K_1, \dots, K_n\}$.

3.10 Definition
Let Λ be a finite rank lattice. A real *toric arrangement in T^c_Λ is a finite set of pairs*

$$\mathscr{A}^c = \{(\chi_1, a_1), \dots (\chi_n, a_n)\} \subset \Lambda \times S^1.$$

3.2. An abstract approach

3.11 Remark

A complexified toric arrangement $\mathscr{A}$ in T_Λ induces a real toric arrangement $\mathscr{A}^c$ in T_Λ^c with

$$K_i^c := \{x \in T_\Lambda^c \mid \chi_i(x) = a_i\}.$$

Furthermore, embedding $T_\Lambda^c \hookrightarrow T_\Lambda$ in the obvious way, we have $K_i^c = K_i \cap T_\Lambda^c$ as in Definition 3.2 .

We now illustrate what has been proposed [16, 31] as the 'toric analogue' of the intersection poset.

3.12 Definition

Let $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ be a toric arrangement on T_Λ. A layer *of $\mathscr{A}$ is a connected component of a nonempty intersection of some of the subtori K_i (defined in Remark 3.9). The set of all layers of $\mathscr{A}$ ordered by reverse inclusion is the* poset of layers *of the toric arrangement, denoted by $\mathcal{C}(\mathscr{A})$.*

Notice that, as for hyperplane arrangements, the torus T_Λ itself is a layer, while the empty set is not.

3.13 Definition

Let Λ be a rank d lattice and let $\mathscr{A}$ be a toric arrangement on T_Λ. The rank *of $\mathscr{A}$ is rk $(\mathscr{A}) :=$ rk $\langle \chi \mid (\chi, a) \in \mathscr{A} \rangle$*

(a) A character $\chi \in \Lambda$ is called primitive *if, for all $\psi \in \Lambda$, $\chi = \psi^k$ only if $k \in \{-1, 1\}$.*

(b) The toric arrangement $\mathscr{A}$ is called primitive *if for each $(\chi, a) \in \mathscr{A}$, χ is primitive.*

(c) The toric arrangement $\mathscr{A}$ is called essential *if rk $(\mathscr{A}) = d$.*

3.14 Remark

For every non primitive arrangement there is a primitive arrangement which has the same complement. Furthermore, if $\mathscr{A}$ is a non essential arrangement, then there exist an essential arrangement $\mathscr{A}'$ such that

$$M(\mathscr{A}) \cong (\mathbb{C}^*)^{d-l} \times M(\mathscr{A}') \text{ where } l = \mathrm{rk}\,(\mathscr{A}').$$

Therefore the topology of $M(\mathscr{A})$ can be derived from the topology of $M(\mathscr{A}')$.

In view of Remark 3.14, our study of the topology of complements of toric arrangements will not loose in generality by stipulating the next assumption.

Assumption

From now on we assume every toric arrangement to be primitive *and* essential.

3.3 Combinatorics

As in the case of hyperplanes, one would like to describe the topology of the complement in terms of the combinatorics of the arrangement.

3.15 Definition ([16, 31])
Let $\mathscr{A}$ be a toric arrangement of rank d and let us fix a total ordering on $\mathscr{A}$. A local no broken circuit set *of $\mathscr{A}$ is a pair*

$$(X, N) \textit{ with } X \in \mathcal{C}(\mathscr{A}), N \in \mathrm{nbc}_k(\mathscr{A}(X)) \textit{ where } k = d - \dim X$$

We will write $\mathscr{N}$ for the set of local non broken circuits, and partition it into subsets

$$\mathscr{N}_j = \{(X, N) \in \mathscr{N} \mid \dim X = d - j\}.$$

3.16 Remark
Let $X \in \mathcal{C}(\mathscr{A})$ and $N \subseteq \mathscr{A}(X)$. If we consider the 'list' $\mathscr{X}$ of all pairs (χ_i, a_i) with $\chi_{i|X} \equiv a_i$, then the elements of N index a 'sublist' $\mathscr{X}_N$. Then, (X, N) is a local no broken circuit set if and only if $\mathscr{X}_N$ is a basis of $\mathscr{X}$ with no *local external activity* in the sense of d'Adderio and Moci [10, Section 5.3]

3.4 Cohomology

The cohomology (with complex coefficients) of the complements of toric arrangements was studied by Looijenga [30] and De Concini and Procesi [16].

3.17 Theorem ([16, Theorem 4.2])
Consider a toric arrangement $\mathscr{A}$. The Poincaré polynomial of $M(\mathscr{A})$ can be expressed as follows:

$$P_{\mathscr{A}}(t) = \sum_{j=0}^{\infty} \dim H^j(M(\mathscr{A}); \mathbb{C})\, t^j = \sum_{j=0}^{\infty} |\mathscr{N}_j|\, (t+1)^{k-j}\, t^j.$$

This result was reached in [16] by computing de Rham cohomology and in [30] via spectral sequence computations. In the special case of (totally) unimodular arrangements, De Concini and Procesi also determine the algebra structure of $H^*(M(\mathscr{A}), \mathbb{C})$ by formality of $M(\mathscr{A})$ [16, Section 5].

We will now proceed with the investigation of the topology of toric arrangement complements. Our arguments build on the ideas developed in [40] for complexified hyperplane arrangements. Therefore from now on we will consider only *complexified* toric arrangements.

3.5 Covering spaces

In order to connect the theory of toric arrangements to that of hyperplane arrangements, we will look at a particular *covering space* of a toric arrangement complement.

Consider the following covering map

$$\begin{aligned} p : \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) &\to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^*) \\ \varphi &\mapsto \exp \circ \varphi \end{aligned}$$

where $\exp : \mathbb{C} \to \mathbb{C}^*$ is the exponential map, i.e., $\exp : z \mapsto e^{2\pi i z}$. Notice through the isomorphism (3.2) p is just the universal covering map $\mathbb{C}^n \to (\mathbb{C}^*)^n$ given by

$$(t_1, \dots, t_n) \mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n}).$$

Furthermore, p restricts to a universal covering map

$$\mathbb{R}^n \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1) \cong (S^1)^n$$

of the compact torus, under which the preimage of a toric arrangement $\mathscr{A}$ is the (infinite) affine hyperplane arrangement

$$\mathscr{A}^{\upharpoonleft} = \{(\chi, a') \in \Lambda \times \mathbb{R} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\}, \tag{3.4}$$

or, in coordinates:

$$\mathscr{A}^{\upharpoonleft} = \{\langle \alpha, x\rangle = a' \mid (x^\alpha, e^{2\pi i a'}) \in \mathscr{A}\}.$$

Here $\alpha \in \mathbb{Z}^n$ and x^α is the associated character $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

For an element $\lambda \in \Lambda$ denote by $\chi_\lambda : \Lambda \to \mathbb{C}$ the corresponding functional as in Remark 3.7. To every character λ we can then associate the following homeomorphism:

$$\begin{aligned} g_\lambda : \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) &\to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \\ \varphi &\mapsto \varphi + \chi_\lambda \end{aligned}$$

The following proposition can be easily proved using isomorphism (3.2).

3.18 Proposition
Let $\mathscr{A}$ be a complexified toric arrangement on the complex torus T_Λ and consider the covering map $p : M(\mathscr{A}^{\upharpoonleft}) \to M(\mathscr{A})$ as above. Let G be the group of covering transformations of p, then the map

$$\begin{aligned} \Lambda &\to G \\ \lambda &\to g_\lambda \end{aligned}$$

is a natural isomorphism.

Proposition 3.18 provides us with a naturally defined action of Λ on the covering space $M(\mathscr{A}^{\upharpoonleft})$.

A polyhedral complex

Let $\mathscr{A}$ be a complexified toric arrangement and consider the arrangement $\mathscr{A}^{\upharpoonright}$ of (3.4). The faces of $\mathscr{A}^{\upharpoonright}$ (Definition 1.17) are polyhedra and build naturally a polyhedral complex

$$\mathcal{P} = \{(F, [id_F]) \mid \quad F \in \mathcal{F}(\mathscr{A}^{\upharpoonright})\}$$

whose face category coincides with the face poset $\mathcal{F}(\mathscr{A}^{\upharpoonright})$.

3.19 Definition

Let $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ be a complexified toric arrangement on the complex torus T_Λ.

A chamber *of $\mathscr{A}$ is a connected component of the space*

$$M(\mathscr{A}) \setminus \bigcup_{i=1}^{n} K_i.$$

We denote the set of chambers of $\mathscr{A}$ by $\mathcal{T}(\mathscr{A})$.

A face *of $\mathscr{A}$ is a subset $F \subseteq T_\Lambda^c$ of the compact torus such that*

$$F = \overline{C} \cap X \quad \textit{for} \quad C \in \mathcal{T}(\mathscr{A}), \quad X \in \mathcal{C}(\mathscr{A}).$$

The following lemma follows from the hypothesis of essentiality of $\mathscr{A}$ (indeed, it holds if and only if the arrangement is essential).

3.20 Lemma

Let $\mathscr{A}$ be a complexified toric arrangement on the complex torus T_Λ and consider the covering map $p : M(\mathscr{A}^{\upharpoonright}) \to M(\mathscr{A})$ of §3.5. For every face $F \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ the restriction

$$p_{|\operatorname{int}(F)} : \operatorname{int}(F) \to p(\operatorname{int}(F))$$

is an homeomorphism.

Lemma 3.20 ensures that the following is well-defined.

3.21 Definition

Let $\mathscr{A}$ be a complexified toric arrangement on the complex torus T_Λ, we associate to $\mathscr{A}$ a polyhedral complex as follows:

$$\mathcal{D}(\mathscr{A}) = \{(F, [p_{|F^{\upharpoonright}}]) \mid \quad F \textit{ face of } \mathscr{A}\},$$

where for a face F of $\mathscr{A}$ $F^{\upharpoonright}$ denotes an arbitrary lifting in the covering space.

Notice that for a different choice $G^{\upharpoonright}$ of a lifting of a face F of $\mathscr{A}$, the two embeddings $p_{|F^{\upharpoonright}}$ and $p_{|G^{\upharpoonright}}$ differ for the action of a character $\lambda \in \Lambda$ and therefore $[p_{|F^{\upharpoonright}}] = [p_{|G^{\upharpoonright}}]$.

3.22 Definition
Let $\mathscr{A}$ be a complexified toric arrangement on the complex torus T_Λ, its face category *is*

$$\mathcal{F}(\mathscr{A}) := \mathcal{F}(\mathcal{D}(\mathscr{A})),$$

i.e. the face category of its associated polyhedral complex.

3.23 Remark
In general, the homotopy type of a complexified toric arrangement $\mathscr{A}$ cannot be described in terms of the face poset of the induced decomposition $\mathcal{D}(\mathscr{A})$ of the compact torus. Indeed Moci and Settepanella in [34] characterize exactly the arrangements for which this poset describes the homotopy type of $M(\mathscr{A})$: these are the arrangements $\mathscr{A}$ for which $\mathcal{D}(\mathscr{A})$ is a regular cell-complex or, in the terminology of [34], *thick* arrangements.

Thick arrangements are precisely those arrangement for which the face category $\mathcal{F}(\mathscr{A})$ is a poset. For such arrangements the construction of the Salvetti complex in the affine case translates almost literally to the toric case (see [34] for the details).

Our construction of the Salvetti complex will be more general in the sense that it does not assume thickness and, moreover, in the thick case it specializes to the complex considered by Moci and Settepanella.

3.6 Deletion and restriction

We now consider the equivalent notions of those of §1.2 for toric arrangement. In particular, we will need deletion for defining the toric Salvetti complex and restriction for our proof of minimality in Chapter 4.

Deletion

The operation of passing to subarrangements, while intuitive and elementary in the case of hyperplane arrangements, needs some careful consideration in the toric case.

Let Γ be a subgroup of the lattice Λ. Then $T_\Gamma := \mathrm{Hom}_\mathbb{Z}(\Gamma, S^1)$ is a compact $(\mathrm{rk}\,\Gamma)$-torus and the inclusion $i_\Gamma : \Gamma \to \Lambda$ induces a map $\pi_\Gamma : T_\Lambda \to T_\Gamma$ given by restriction: $\pi_\Gamma(p) = p_{|\Gamma}$. Furthermore, if Γ is a direct factor of Λ, then the map π_Γ is surjective.

3.24 Definition
Given a subgroup $\Gamma \subseteq \Lambda$ and an arrangement $\mathscr{A}$ in T_Λ, we define the arrangement

$$\mathscr{A}_\Gamma = \{(\chi, a) \in \mathscr{A} : \chi \in \Gamma\}.$$

3.25 Proposition
The map $\pi_\Gamma : T_\Lambda \to T_\Gamma$ *induces a* cellular map $\pi_\Gamma^{cell} : \mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{A}_\Gamma)$.

Proof. We can choose a basis $x_1, \dots, x_n$ for Λ such that $\Gamma = \left\langle x_1^{k_1}, \dots, x_l^{k_l} \right\rangle$. The isomorphism $T_\Lambda \simeq \mathbb{C}^n$ is given by evaluation on the chosen basis: $p \mapsto (p(x_1), \dots p(x_n))$. Therefore the projection $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^l$ is given by the map $(y_1, \dots, y_n) \mapsto (y_1^{k_1}, \dots, y_l^{k_l})$. This map is continuous and maps hypersurfaces (of $\mathscr{A}_\Gamma \subseteq \mathscr{A}$ in $(\mathbb{C}^*)^n$) onto hypersurfaces (of $\mathscr{A}_\Gamma$ in $(\mathbb{C}^*)^l$), hence is cellular. □

The fact that π_Γ is cellular implies that π_Γ induces a morphism of acyclic categories $\pi_\Gamma : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_\Gamma)$.

3.26 Definition
Let $\mathscr{A}$ *be a toric arrangement and* $X \in \mathcal{C}(\mathscr{A})$ *a layer of* $\mathscr{A}$*. The* deletion *of* $\mathscr{A}$ *on* X *is the toric arrangement*

$$\mathscr{A}_X := \mathscr{A}_{\Gamma_X} \text{ on } T_{\Gamma_X}.$$

Given any face $F \in \mathcal{F}(\mathscr{A})$ we can let Γ be the lattice

$$\Lambda_F := \{\chi \in \Lambda \mid \chi \text{ is constant on } F\}.$$

Correspondingly, we obtain a toric subarrangement with an associated cellular map:

$$\mathscr{A}_F := \mathscr{A}_{\Lambda_F}, \qquad \pi_F := \pi_{\Lambda_F}^{\text{cell}} : \mathcal{D}(\mathscr{A}) \to \mathcal{D}(\mathscr{A}_F). \tag{3.5}$$

Restriction

Let $\mathscr{A}$ be a toric arrangement on T_Λ. Notice that for a layer $X \in \mathcal{C}(\mathscr{A})$ and an hypersurface K of $\mathscr{A}$, the intersection $K \cap X$ needs not to be connected.

In general $K \cap X$ consist of several connected components, each of which is a level set of a character in the torus X. In particular

$$\mathscr{A}^X := \{K_i \cap X \mid X \not\subseteq K_i\}$$

is a toric arrangement on the torus X, in the sense of Definition 3.8. The arrangement $\mathscr{A}^X$ is called the *restriction* of $\mathscr{A}$ to X.

3.7 The homotopy type of complexified toric arrangements

Let $\mathscr{A}$ be a complexified toric arrangement and recall the affine arrangement $\mathscr{A}^\upharpoonright$ of (3.4).

3.7. The homotopy type of complexified toric arrangements

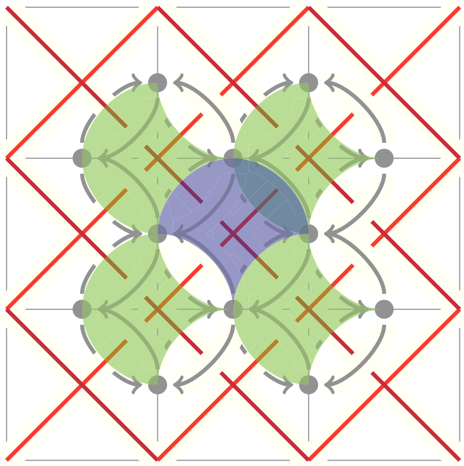


Figure 3.2: Salvetti Complex for $\mathscr{A}^{\upharpoonright}$

The arrangement $\mathscr{A}^{\upharpoonright}$ is a locally finite complexified affine hyperplane arrangement and therefore admits a Salvetti complex

$$\mathcal{S}^{\upharpoonright} = \mathcal{S}^{\upharpoonright}(\mathscr{A}) := \mathcal{S}(\mathscr{A}^{\upharpoonright}).$$

The character lattice Λ acts cellularly on $\mathcal{S}^{\upharpoonright}$ and continuously on the covering space $M(\mathscr{A})$. These two actions are compatible, meaning that the embedding $\mathcal{S}^{\upharpoonright} \to M(\mathscr{A}^{\upharpoonright})$ of Proposition 1.35 constructed in [40] is Λ-equivariant (more precisely, it can be so constructed).

3.27 Example
Figure 3.2 shows the Salvetti complex for the arrangement $\mathscr{A}^{\upharpoonright}$, with $\mathscr{A} = \{(ts, 1), (ts^{-1}, 1)\}$. The green cells belong to the same Λ-orbit.

With the previous constructions in mind, we can now restate a key result of [34].

3.28 Proposition ([34, Lemma 1.1])
Let $\mathscr{A}$ be an essential toric arrangement; the embedding $\mathcal{S}^{\upharpoonright} \to M(\mathscr{A}^{\upharpoonright})$ induces an embedding

$$\mathcal{S}^{\upharpoonright}/\Lambda \to M(\mathscr{A})$$

of the quotient $\mathcal{S}^{\upharpoonright}/\Lambda$ in the complement $M(\mathscr{A})$ as a deformation retract.

3.29 Remark
In the proof of Proposition 3.28 given in [34] the hypothesis of essentiality is required. Indeed the construction of the homotopy inverse $\psi : \mathcal{S}^{\upharpoonright}/\Lambda \to M(\mathscr{A})$ does not work for non-essential arrangements.

Toric Salvetti complex

We now head towards the first main result of this thesis, introducing the notion of Salvetti complex for general complexified toric arrangements.

3.30 Definition (Salvetti category)
Let $\mathscr{A}$ be a toric arrangement on $(\mathbb{C}^*)^n$. The Salvetti Category *of $\mathscr{A}$ is the acyclic category* $\mathrm{Sal} = \mathrm{Sal}(\mathscr{A})$ *defined as follows:*

(i) the objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers

$$\mathrm{Ob}(\mathrm{Sal}(\mathscr{A})) = \{m : F \to C : \quad m \in \mathcal{M}(\mathcal{F}(\mathscr{A})),\ C \textit{ chamber}\};$$

(ii) for every morphism $n : F_2 \to F_1$ in $\mathcal{F}(\mathscr{A})$, and for every pair $m_1 : F_1 \to C_1$, $m_2 : F_2 \to C_2$ in $\mathrm{Ob}(\zeta)$ there is a morphism $(n, m_1, m_2) : m_1 \to m_2$ if and only if

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2); \tag{3.6}$$

where π_{F_1} is the morphism of face categories induced by the cellular map in (3.5)*;*

(iii) let $m_i : F_i \to C_i$ for $i = 1, 2, 3$ be elements in $\mathcal{O}(\zeta)$, suppose the pairs (m_1, m_2) and (m_1, m_3) satisfy condition (3.6)*, then the pair (m_1, m_3) satisfies the same condition and we can define for morphisms $n : F_2 \to F_1$, $n' : F_3 \to F_2$ the composition*

$$(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3).$$

3.31 Definition
Let $\mathscr{A}$ be a toric arrangement; its Salvetti complex *is the nerve $\mathcal{S}(\mathscr{A}) :=$* $\Delta(\mathrm{Sal}(\mathscr{A}))$.

We can now state the main theorem of this chapter.

3.32 Theorem
Let Λ be a lattice and $\mathscr{A}$ be a complexified toric arrangement in T_Λ. The toric Salvetti complex $\mathcal{S}(\mathscr{A})$ embeds in $M(\mathscr{A})$ as a deformation retract.

3.33 Remark
Being the nerve of an acyclic category, $\mathcal{S}(\mathscr{A})$ is a regular Δ-set.

3.34 Remark
Recall from §3.7 the notion of cellular and simplicial Salvetti complex for affine arrangements of hyperplanes. In this section we

defined an analogue of the simplicial Salvetti complex for toric arrangement. This "toric simplicial Salvetti complex" is not, in general, a simplicial complex, but a Δ-set.

Our goal now is to describe a CW complex of which the complex $\mathcal{S}(\mathscr{A})$ is the barycentric subdivision. This complex will not be regular in general, but the resulting economy in terms of cells will come in handy in the following considerations.

Let then $\mathscr{A}$ denote a toric arrangement. Every cell of the cellular Salvetti complex of $\mathscr{A}^{\upharpoonright}$ corresponds to the topological closure of the star of a vertex $[F, C]$ of the simplicial complex. Because the projection $\mathrm{Sal}(\mathscr{A}^{\upharpoonright}) \to \mathrm{Sal}(\mathscr{A})$ is a covering of categories, the interior of the star of any vertex of $\mathcal{S}(\mathscr{A}^{\upharpoonright}))$ is mapped homeomorphically to the interior of the star of its image. This gives a canonical structure of polyhedral complex on $\mathcal{S}(\mathscr{A})$. The acyclic category $\mathcal{S}(\mathscr{A})$ is precisely the face category of the resulting complex.

In particular, the explicit determination of the boundary maps of this complex is now reduced to a straightforward computation.

Before we can get to the proof, some preparatory considerations are in order.

Deletion vs. covering

In order to proceed with the argument we still need to spend a few words on the quotient construction of Definition 3.26.

Let F be a face of $\mathcal{D}(\mathscr{A})$ and let Λ_F be the sublattice of characters in Λ that are constant on F. Choose a preimage $F^{\upharpoonright} \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ such that $q(F^{\upharpoonright}) = F$. We have the following isomorphismd of vector spaces

$$\mathbb{R}^d \big/ L \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \big/ L \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_F, \mathbb{R}) \tag{3.7}$$

where L is the linear subspace parallel to $|F^{\upharpoonright}|$.

Recall from (3.5) the arrangement

$$\mathscr{A}_F = \{(\chi, a) \in \mathscr{A} : \chi \in \Lambda_F\} \subseteq \mathscr{A}$$

in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_F, \mathbb{R})$ and the deletion $(\mathscr{A}^{\upharpoonright})_{F^{\upharpoonright}}$ in $\mathbb{R}^d \big/ L$.

The isomorphism (3.7) does not map the arrangement $(\mathscr{A}^{\upharpoonright})_{F^{\upharpoonright}}$ onto $(\mathscr{A}_F)^{\upharpoonright}$. Indeed $(\mathscr{A}_F)^{\upharpoonright}$ contains all the translates of the hyperplanes in $(\mathscr{A}^{\upharpoonright})_{F^{\upharpoonright}}$. That is

$$(\mathscr{A}^{\upharpoonright})_{F^{\upharpoonright}} \subseteq \mathscr{A}_F{}^{\upharpoonright} = \{(\chi, a + k) \mid (\chi, a) \in (\mathscr{A}^{\upharpoonright})_{F^{\upharpoonright}},\, k \in \mathbb{Z}\}$$

and therefore we have a natural cellular support map

$$s : \mathcal{D}(\mathscr{A}_F{}^{\upharpoonright}) \to \mathcal{D}(\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}})$$

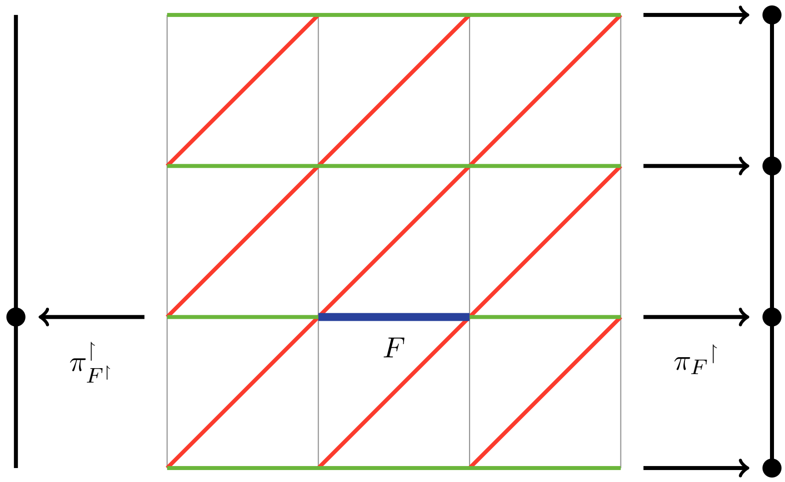


Figure 3.3: Restriction vs. Covering

The map π_F of (3.5) lifts (via p) to a map $\mathbb{R}^{rk\Lambda} \to \mathbb{R}^{rk\Lambda_F}$ which induces a cellular map

$$\pi_F{}^{\restriction} : \mathcal{D}(\mathscr{A}^{\restriction}) \to \mathcal{D}((\mathscr{A}_F)^{\restriction})$$

and the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}(\mathscr{A}^{\restriction}) & \xrightarrow{\pi_F{}^{\restriction}} & \mathcal{D}((\mathscr{A}_F)^{\restriction}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{D}(\mathscr{A}) & \xrightarrow[\pi_{F^{\restriction}}]{} & \mathcal{D}(\mathscr{A}_F) \end{array} \tag{3.8}$$

On the other hand, in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$ we have the projection $\pi^{\restriction}_{F^{\restriction}} : \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda_F, \mathbb{R})$ induces by (3.7) through isomorphism (3.2), which induces a cellular map

$$\pi^{\restriction}_{F^{\restriction}} : \mathcal{D}(\mathscr{A}^{\restriction}) \to \mathcal{D}((\mathscr{A}^{\restriction})_{F^{\restriction}})$$

and is related to $\pi_F{}^{\restriction}$ via

$$\pi^{\restriction}_{F^{\restriction}} = s \circ \pi_F{}^{\restriction}.$$

Figure 3.3 shows an example of projections $\pi^{\restriction}_F$ and $\pi_F{}^{\restriction}$.

3.35 Lemma

Let $F_1, F_2, C_1, C_2 \in \mathcal{F}(\mathscr{A}^{\restriction})$ with C_1, C_2 chambers, $F_1 \leq C_1$ and $F_1 \leq F_2 \leq C_2$. Then

$$\pi_{F_1}{}^{\restriction}(C_1) = \pi_{F_1}{}^{\restriction}(C_2) \iff \pi^{\restriction}_{F_1{}^{\restriction}}(C_1) = \pi^{\restriction}_{F_1{}^{\restriction}}(C_2).$$

Proof. The direction $\Rightarrow$ follows since $\pi^{\upharpoonleft}_{F^{\upharpoonleft}} = s \circ \pi_F{}^{\upharpoonleft}$. For $\Leftarrow$: if $\pi^{\upharpoonleft}_{F_1^{\upharpoonleft}}(C_1) = \pi^{\upharpoonleft}_{F_1^{\upharpoonleft}}(C_2)$, then $\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2 + \lambda)$, for some $\lambda \in \Lambda_F$. But since F_2 is a common face of C_1 and C_2, it has to be $\lambda = 0$. $\square$

3.36 Corollary
Let $[F_1, C_1]$, $[F_2, C_2]$ denote two elements of Sal $\mathscr{A}^{\upharpoonleft}$, the Salvetti poset of $\mathscr{A}^{\upharpoonleft}$. Then

$$[F_1, C_1] \leq [F_2, C_2] \iff F_1 \geq F_2 \text{ in } \mathcal{F}(\mathscr{A}) \text{ and } \pi_{q(F_1)}{}^{\upharpoonleft}(C_1) = \pi_{q(F_1)}{}^{\upharpoonleft}(C_2).$$

Quotients

Our strategy for the proof of Theorem 3.32 will be to prove that the toric Salvetti complex $\Delta(\mathrm{Sal}\,\mathscr{A})$ is the quotient of the action $\Lambda \curvearrowright \mathcal{S}^{\upharpoonleft}$ in the category of Δ-sets. For this, we need first to take care of some ground work.

3.37 Lemma
Let $\mathscr{A}$ be a complexified toric arrangement. Then there is a covering map $q : \mathcal{F}(\mathscr{A}^{\upharpoonleft}) \to \mathcal{F}(\mathscr{A})$ of acyclic categories with Galois group Λ and

$$\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}^{\upharpoonleft})/\Lambda$$

as a quotient of acyclic categories.

Proof. Let $F^{\upharpoonleft} \in \mathcal{D}(\mathscr{A}^{\upharpoonleft})$ be a face of the affine arrangement $\mathscr{A}^{\upharpoonleft}$. In particular $F^{\upharpoonleft}$ is a polytope and $(F, [\chi_F]) = (p(F^{\upharpoonleft}), [p_{|F^{\upharpoonleft}}]) \in \mathcal{D}(\mathscr{A})$ is a face of $\mathscr{A}$. Therefore for every face $G^{\upharpoonleft}$ of $F^{\upharpoonleft}$

$$m_{[F, G^{\upharpoonleft}]} : (G, [\chi_G]) \to (F, [\chi_F])$$

is a morphism in $\mathcal{F}(\mathscr{A})$ and we can define

$$q(F^{\upharpoonleft}) = F, \quad q(G^{\upharpoonleft} \leq F^{\upharpoonleft}) = m_{[F, G^{\upharpoonleft}]}.$$

This defines a functor $q : \mathcal{F}(\mathscr{A}^{\upharpoonleft}) \to \mathcal{F}(\mathscr{A})$. Furthermore q is a covering of categories in the sense of [5, Definition A.15] with Λ as automorphism group and Λ acts transitively on the fibers of q. It then follows that $\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^{\upharpoonleft})/\Lambda$. $\square$

In particular, we note the following consequence.

3.38 Corollary
The morphisms in $\mathcal{F}(\mathscr{A})$ correspond to the orbits $\{\Lambda(F_1 \leq F_2) \mid F_1, F_2 \in \mathcal{D}(\mathscr{A}^{\upharpoonleft})\}$.

The following lemma unveils our construction of the category $\mathrm{Sal}\,\mathscr{A}$.

3.39 Lemma

The category $\mathrm{Sal}(\mathscr{A})$ *is the quotient* $\mathrm{Sal}(\mathscr{A}^{\upharpoonleft})/\Lambda$ *in the category of acyclic categories.*

Proof. We first need to construct a projection, i.e., a functor $\Pi : \mathrm{Sal}(\mathscr{A}^{\upharpoonleft}) \to \mathrm{Sal}(\mathscr{A})$. Recall that the objects of $\mathrm{Sal}(\mathscr{A}^{\upharpoonleft})$ are of the form $[F, C]$ with $F, C \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$, $F \leq C$, and C a chamber of $\mathscr{A}^{\upharpoonleft}$. Also, from the proof of Lemma 3.37 we recall the projection $q : \mathcal{F}(\mathscr{A}^{\upharpoonleft}) \to \mathcal{F}(\mathscr{A})$. It is now possible to define Π on the objects as follows:

$$\Pi([F, C]) = q(F \leq C) : q(F) \to q(C).$$

According to Corollary 3.36, relations in $\mathcal{F}(\mathscr{A}^{\upharpoonleft})$ are of the form $[F_1, C_1] \leq [F_2, C_2]$ where $F_2 \leq F_1$ and $\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2)$.

On the other hand, morphisms in $\mathrm{Sal}(\mathscr{A})$ are given by triples (n, m_1, m_2) where $m_1 : F_1 \to C_2$, $m_2 : F_2 \to C_2$ are objects of $\mathrm{Sal}(\mathscr{A})$, $n : F_2 \to F_1$ is a morphism in $\mathcal{F}(\mathscr{A})$ and the following condition holds:

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2).$$

Therefore, in order to able to map a relation $[F_1, C_1] \leq [F_2, C_2]$ to the morphism $(q(F_2 \leq F_1), \Pi([F_1, C_1]), \Pi([F_2, C_2]))$ and for this map to be surjective, we need to verify the following condition:

$$\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2) \iff \pi_{q(F_1)}(\Pi([F_1, C_1])) = \pi_{q(F_1)}(\Pi([F_2, C_2])).$$

We go back to the diagram (3.8), and write the corresponding commutative diagram of face categories:

$$\begin{tikzcd} \mathcal{F}(\mathscr{A}^{\upharpoonleft}) \arrow[r, "\pi_{F_1}{}^{\upharpoonleft}"] \arrow[d, "q"'] & \mathcal{F}(\mathscr{A}_{F_1}{}^{\upharpoonleft}) \arrow[d, "q"] \\ \mathcal{F}(\mathscr{A}) \arrow[r, "\pi_{q(F_1)}"'] & F(\mathscr{A}_{q(F_1)}) \end{tikzcd}$$

Now $\pi_{F_1}{}^{\upharpoonleft}$ is a map of posets and since $\pi_{F_1}{}^{\upharpoonleft}(F_1) = \pi_{F_1}{}^{\upharpoonleft}(F_2)$ we have

$$\pi_{F_1}{}^{\upharpoonleft}(C_1) = \pi_{F_1}{}^{\upharpoonleft}(C_2) \iff \pi_{F_1}{}^{\upharpoonleft}(F_1 \leq C_1) = \pi_{F_1}{}^{\upharpoonleft}(F_2 \leq C_2).$$

Furthermore q is a covering of categories, in particular is injective on the morphisms incident on $\pi_{F_1}{}^{\upharpoonleft}(F_1)$. It then follows that

$$\begin{aligned} \pi_{F_1}{}^{\upharpoonleft}(F_1 \leq C_1) = \pi_{F_1}{}^{\upharpoonleft}(F_2 \leq C_2) &\Leftrightarrow q \circ \pi_{F_1}{}^{\upharpoonleft}(F_1 \leq C_1) = q \circ \pi_{F_1}{}^{\upharpoonleft}(F_2 \leq C_2) \\ &\Leftrightarrow \pi_{q(F_1)}(q(F_1 \leq C_1)) = \pi_{q(F_1)}(q(F_2 \leq C_2)). \end{aligned}$$

Concluding: the functor Π is well defined and it now follows easily from Lemma 3.37 that it is a Galois covering of acyclic categories with Λ as automorphism group. $\square$

We want to show that, in our particular case, the nerve construction commutes with the quotient. Babson and Kozlov in [2] give a necessary and sufficient condition for this:

3.40 Proposition ([2, Theorem 3.4])
Let $\mathcal{C}$ be an acyclic category equipped with a group action $G \curvearrowright \mathcal{C}$. A canonical isomorphism $\Delta(\mathcal{C})/G \cong \Delta(\mathcal{C}/G)$ exists if and only if the following condition is satisfied:

> *Let $t \geq 2$ and let $(m_1, \dots, m_{t-1}, m_a)$, $(m_1, \dots, m_{t-1}, m_b)$ composable morphism chains. Let $Gm_a = Gm_b$, then there exists some $g \in G$, such that $g(m_a) = m_b$ and $g(m_i) = m_i$, $\forall i \in \{1, \dots, t-1\}$.*

The next lemma ensures that we can apply the previous proposition to our case.

3.41 Lemma
Let $\mathcal{C}$ be an acyclic category and $G \curvearrowright \mathcal{C}$ act as the Galois group of a covering map. Then the condition of proposition 3.40 is satisfied.

Proof. Consider two composable morphism chains as in the condition of proposition 3.40. Since $t \geq 2$ and the chains are composable, m_a and m_b must have the same domain, $m_a : p \to q$, $m_b : p \to r$. Furthermore there is a $g \in G$, such that $m_b = gm_a$.

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a covering map with Galois group G. Then $\varphi(m_a) = \varphi(m_b) \Rightarrow m_a = m_b$ and the condition is trivially satisfied. $\square$

We finally get to the proof of Theorem 3.32, which now follows as an application of the previous considerations.

Proof of Theorem 3.32. According to proposition 3.28 the statement holds for the complex $S^{\upharpoonright}/\Lambda = \Delta(\mathrm{Sal}(\mathscr{A}^{\upharpoonright}))/\Lambda$. The lattice Λ acts on $S^{\upharpoonright}$ as the automorphism group of a covering map, in particular lemma 3.41 holds and we have:

$$S^{\upharpoonright}/\Lambda = \Delta(\mathrm{Sal}(\mathscr{A}^{\upharpoonright}))/\Lambda \cong \Delta(\mathrm{Sal}(\mathscr{A}^{\upharpoonright})/\Lambda) \cong \Delta(\mathrm{Sal}(\mathscr{A})). \qquad \square$$

3.8 Fundamental Group

In the second part of this chapter we compute a presentation of the fundamental group of the toric Salvetti complex. Our argument is inspired by that of Salvetti [40].

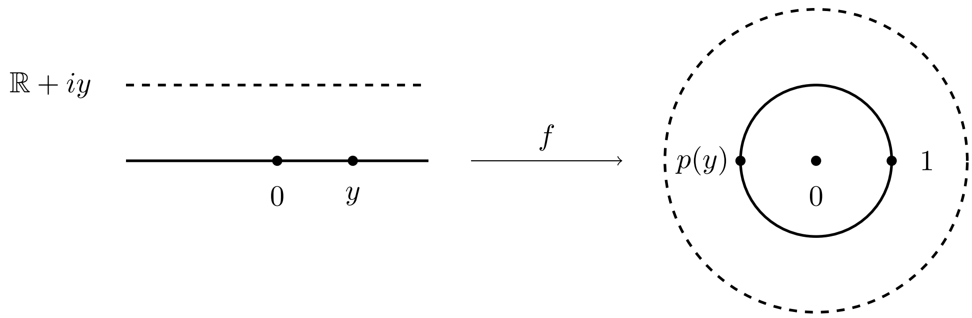


Figure 3.4: Construction of the map f in the proof of Lemma 3.42.

Product structure

A first step towards the computation of a presentation of the fundamental group of toric arrangement complements is the decomposition of this group as a semidirect product of two well known groups: the fundamental group of the torus and that of the hyperplane arrangement $\mathscr{A}^{\upharpoonright}$.

First, note that the inclusion $M(\mathscr{A}) \to T_\Lambda$ induces an epimorphism of groups

$$\varepsilon : \pi_1(M(\mathscr{A})) \to \pi_1(T_\Lambda) \simeq \mathbb{Z}^n.$$

3.42 Lemma

The map ε has a section ξ.

Proof. Choose a point $y \in \mathbb{R}^n$ in a chamber of $\mathscr{A}^{\upharpoonright}$. Then for all choices of $x \in \mathbb{R}^n$ we have

$$x + iy \in M(\mathscr{A}^{\upharpoonright}).$$

Accordingly, for every choice of arguments $\theta_1, \ldots, \theta_n \in \mathbb{R}$,

$$(\lambda_1 e^{2\pi i\theta_1}, \ldots, \lambda_n e^{2\pi i\theta_n}) \in M(\mathscr{A})$$

where, for all $j = 1, \ldots, n$, $\lambda_j := e^{-2\pi y_j}$ This defines a map

$$f : T_\Lambda \to M(\mathscr{A}),$$
$$z \mapsto (\lambda_1 e^{2\pi i \arg z_1}, \ldots, \lambda_n e^{2\pi i \arg z_n})$$

that induces a homomorphism

$$\xi : \pi_1(T_\Lambda) \to \pi_1(M(\mathscr{A})).$$

Since f is a homotopy (right-) inverse to the inclusion $M(\mathscr{A}) \to T_\Lambda$, $\varepsilon\xi = id$ and ξ is the required section.

Figure 3.4 shows the construction of the map f in the case of the 1-dimensional arrangement $\mathscr{A} = \{x = 1\}$. $\square$

3.8. Fundamental Group

3.43 Lemma
The sequence

$$0 \to p_*(\pi_1(\mathcal{S}^\upharpoonright)) \xrightarrow{\iota} \pi_1(M(\mathscr{A})) \xrightarrow{\varepsilon} \pi_1(T_\Lambda) \to 0$$

is split exact. Therefore

$$\pi_1(M(\mathscr{A})) \simeq \pi_1(\mathcal{S}^\upharpoonright) \rtimes \pi_1(T_\Lambda).$$

Proof. We already showed that the map ε has a section, we then need only to prove $\iota(p_*(\pi_1(\mathcal{S}^\upharpoonright))) = \ker\, \varepsilon$. It is clear that $\iota(p_*(\pi_1(\mathcal{S}^\upharpoonright))) \subseteq \ker\, \varepsilon$. For the opposite inclusion we consider the sequence

$$0 \to p_*(\pi_1(M(\mathscr{A}^\upharpoonright))) \to \pi_1(M(\mathscr{A})) \to \pi_1(T_\Lambda) \to 0$$

Let $[\gamma] \in \pi_1(M(\mathscr{A}))$ be an element of $\ker\, \varepsilon$. Let j be the inclusion of $M(\mathscr{A})$ in the ambient torus T_Λ. Then $j \circ \gamma$ is a null homotopic loop in T_Λ and lifts therefore to a closed path γ' in the universal cover $\mathbb{C}^n$. Let $\gamma^\upharpoonright$ be the lift of γ to $M(\mathscr{A}^\upharpoonright)$ with base point x, then $\gamma' = j^\upharpoonright \circ \gamma^\upharpoonright$ and $\gamma^\upharpoonright$ is also a closed path. That is, $[\gamma] = p_*[\gamma^\upharpoonright] \in p_*(\pi_1(M(\mathscr{A}^\upharpoonright))) \cong p_*(\pi_1(\mathcal{S}^\upharpoonright))$. □

Generators

Recall the arrangement $\mathscr{A}^\upharpoonright$ of (3.4) and the arrangement graph $\mathcal{G}^\upharpoonright := \mathcal{G}(\mathscr{A}^\upharpoonright)$ of Definition 1.37. The action of Λ on $\mathcal{M}(\mathscr{A}^\upharpoonright)$ of Proposition 3.18 induces an action of Λ on the arrangement graph $\mathcal{G}^\upharpoonright$ and therefore on the set of paths on $\mathcal{G}^\upharpoonright$. We denote the action of $u \in \Lambda$ on a path $\gamma \in \mathcal{G}^\upharpoonright$ by writing $u.\gamma$ for the path obtained by translation of γ with u.

Let us choose and fix a basis $u_1, \ldots, u_n$ of Λ such that no hypersurface of $\mathscr{A}$ is parallel to the direction of u_1. As in §1.5 choose - and from now fix - a chamber C_0 of $\mathscr{A}^\upharpoonright$, and let x_0 be a generic point in C_0 - i.e. such that for all $i = 1, \ldots, d$ the straight line segment s_i from x_0 to $u_i x_0$ meets only faces of codimension at most 1.

3.44 Definition
Let for $i = 1, \ldots, n$, $\omega_i = \omega_i^{(1)}$ be the positive minimal path of $\mathcal{G}^\upharpoonright$ from C_0 to $u_i C_0$ obtained by crossing the faces met by the straight line segment s_i (which connects from x_0 to $u_i x_0$). Also, for $k \geq 1$ let $\omega_i^{(k)} = \omega_i(u_i.\omega_i^{(k-1)})$. Similarly, let $\omega_i^{(-1)} := u_i^{-1}.\omega_i^{-1}$ and $\omega_i^{(-k)} := \omega_i^{(-1)}(u_i^{-1}.\omega_i^{(1-k)})$. Given any $u \in \Lambda$ write $u = u_1^{q_1} \cdots u_n^{q_n}$ and define

$$\omega_u := \omega_1^{(q_1)}\; u_1^{q_1}.\omega_2^{(q_2)}\; \cdots \big(\prod_{j=1}^{r-1} u_n^{q_n}\big).\omega_r^{(q_n)}. \tag{3.9}$$

Let then

$$\tau_i := p_*(\omega_i), \qquad \tau_u := p_*(\omega_u).$$

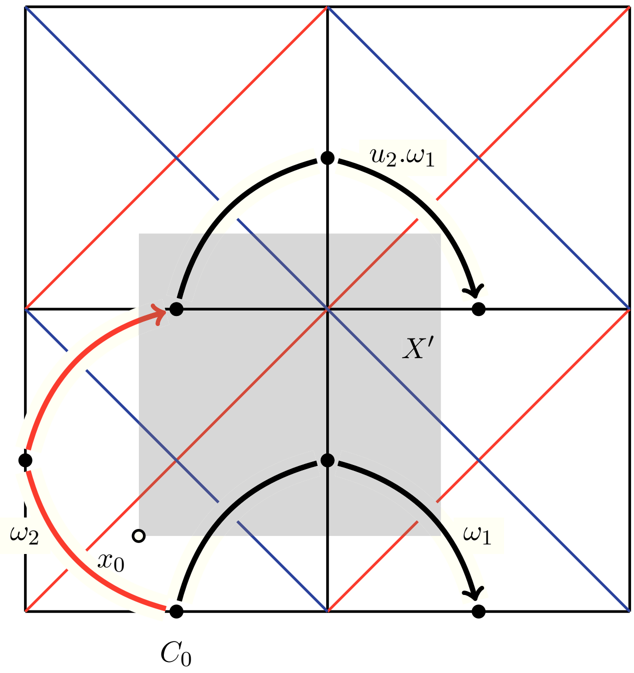


Figure 3.5: Constructions of Lemma 3.45

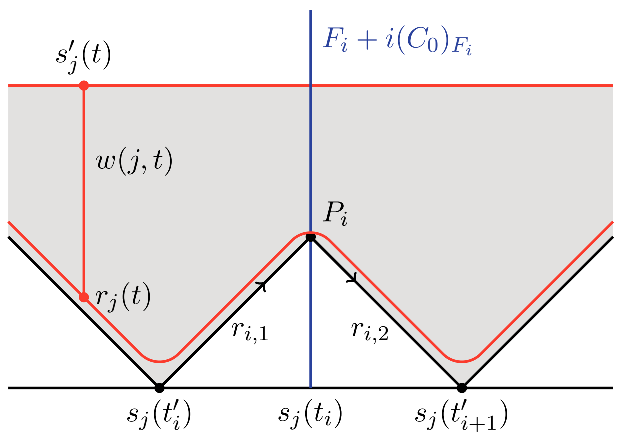


Figure 3.6: Construction for the proof of Lemma 3.45

Notice that a path ω_u needs not be minimal, nor positive. In fact, it is positive if and only if u has non-negative coordinates in Λ. Given i and k, the path $\omega_i^{(k)}$ is positive if and only if $k \geq 0$, and in this case it is also minimal.

3.45 Lemma
In $\pi_1(\mathcal{M}(\mathscr{A}))$, $p_*(\omega_i^{(k)}) = \tau_i^k$ and $\tau_i\tau_j = \tau_j\tau_i$ for all i, j. The $\varepsilon_*\tau_i$

generate $\pi_1(T_\Lambda)$.

Proof. Let $X = f(T_\Lambda)$ be the image of the map f in the proof of Lemma 3.42, where we now choose y to be a point of our base chamber C_0.

Let the straight line segment s_j be parametrized by

$$s_j(t) := (1-t)x_0 + tu_jx_0, \quad 0 \le t \le 1.$$

The Minkowski sum $X' := s_1 + \cdots + s_n \subset \mathbb{R}^n$ is a fundamental region for the action of Λ on $\mathbb{R}^n$. For $Y := X' + iy \subseteq M(\mathscr{A}^\restriction)$ we have $p(Y) = X$. In particular, the segments s_j map under ε to a system of generators of $\pi_1(T_\Lambda)$ - in fact, the one associated with the basis $u_1, \ldots, u_n$ of Λ.

We will next show that for all $j = 1, \ldots, n$ the path

$$s'_j(t) := s_j(t) + iy$$

is homotopic to the positive minimal path $\omega_j \in (C_0 \to u_jC_0)$.

Indeed, write $\omega_j = l_{F_1} \ldots l_{F_k}$ and let $t_1, \ldots t_k$ be such that $s_j(t_i) \in F_i$ for all $i = 1, \ldots, k$. Also, write C_i, C_{i+1} for the source and target chambers of l_{F_i} (note: $C_{k+1} = u_jC_0$) and for $i = 1, \ldots, k$ choose $t'_i \in]t_{i-1}, t_i[$, $t'_{k+1} := 1$, $t'_0 := 0$. Then $s_j(t'_i) \in C_i$ for all $i = 1, \ldots, k+1$.

Recall now that the subset of $M(\mathscr{A}^\restriction)$ with real part $x \in F$ consists of points with imaginary part belonging to the chambers of $A^\restriction_F$. In fact, the edge l_{F_i}, directed from C_i to C_{i+1}, is by construction ([40, p. 608]) the union of two segments, one from a point in $P'_i \in C_i + 0i$ to a point $P_i \in F + i(C_0)_F$, the other from P_i to a point $P'_{i+1} \in C_{i+1} + 0i$. We will parametrize these segments as $r_{i,1}(t)$, $t'_i \le t \le t_i$ and $r_{i,2}(t)$, $t_i \le t \le t'_{i+1}$. Together, they give a parametrization $r_j(t)$, $0 \le t \le 1$ of the positive minimal path ω_j.

The key observation is now that, having chosen $y \in C_0$, we have that

$$s'_j(t_h) \in F + i(C_0)_F \text{ for all } h = 1, \ldots, k.$$

Since chambers of arrangements are convex, for all $t \in [0,1]$ there is a straight line segment $w(j,t)$ joining $s'_j(t)$ and $r_j(t)$ in $M(\mathscr{A}^\restriction)$.

The (topological) disk $W_j := \bigcup_{t\in[0,1]} w(j,t)$ defines the desired homotopy between s'_j and ω_j.

Now fix $i, j \in \{1, \ldots, n\}$ clearly $s_i\, u_i.(s_j)$ is homotopic to $s_j\, u_j.(s_i)$, and in $\pi_1(M(\mathscr{A}))$ we thus have

$$\begin{aligned} \tau_i\tau_j = p_*([\omega_i\, u_i.\omega_j]) = p_*([s_i\, u_i.s_j]) \\ = p_*([s_j\, u_j.s_i]) = p_*([\omega_j\, u_j.\omega_i]) = \tau_j\tau_i. \end{aligned} \qquad \square$$

3.46 Definition
Let $\mathcal{Q}$ be the set of faces that intersect the fundamental region X' of the proof of Lemma 3.45. Then $\mathcal{Q}$ contains C_0 and x_0. Let $\mathcal{Q}_i := \mathcal{Q} \cap \mathcal{F}^{\upharpoonleft}_i$. In particular, $\mathcal{Q}_1$ contains the set of codimension-1 faces crossed by s_i, for all i.

Recall the parametrization $s_i(t)$ of the segments s_i, and call $\mathcal{B}$ the set of faces of the polyhedron X' which intersect the convex hull of $\{s_i([0,1[) \mid i \in I\}$ for some $I \subseteq \{1, \ldots, n\}$. Notice that every face of X' is a translate of some face in $\mathcal{B}$ by an element $u_1^{m_1} \cdots u_n^{m_n}$ with $m_1, \ldots, m_n \in \{0, 1\}$.

3.47 Definition
Let

$$\widehat{\mathcal{F}^{\upharpoonleft}} := \{F \in \mathcal{Q} \mid F \cap B = \emptyset \text{ for all } B \notin \mathcal{B}\}$$

The set $\widehat{\mathcal{F}^{\upharpoonleft}}$ is a set of representatives for the orbits of the action of Λ on $\mathcal{F}^{\upharpoonleft}$ however such representatives need not to be unique, as pointed out in [39]. We can choose a set of unique representatives $\overline{\mathcal{F}^{\upharpoonleft}} \subseteq \widehat{\mathcal{F}^{\upharpoonleft}}$ such that every face crossed by s_1 is contained in $\overline{\mathcal{F}^{\upharpoonleft}}$.

3.48 Definition
For any given $F \in \mathcal{F}^{\upharpoonleft}$ let $\overline{F}$ be the unique element of $\Lambda F \cap \overline{\mathcal{F}^{\upharpoonleft}}$. Then, call u_F the unique element of Λ such that $F = u_F \overline{F}$.
Define

$$\Gamma_F := \omega_{u_F}(u_F.\beta_{\overline{F}})\omega_{u_F}^{-1}$$

3.49 Remark
(1) For all $F \in \mathcal{F}^{\upharpoonleft}_1$ and all $u \in \Lambda$

$$p_*(\Gamma_{uF}) = \tau_u p_*(\Gamma_F)\tau_u^{-1}.$$

(2) If $F \in \overline{\mathcal{F}^{\upharpoonleft}}_1$, then $\Gamma_F = \beta_F$.

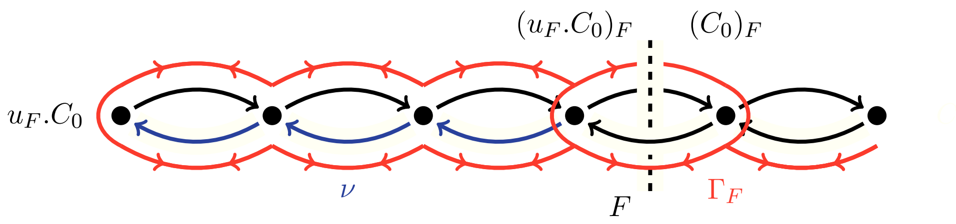
(3) If $F \in \mathcal{Q}$, then u_F has non-negative coordinates with respect to $u_1, \ldots, u_n$. (Recall the discussion before Definition 3.47.)

(4) Since X' is convex, $\mathcal{Q}_0$ contains the vertices of a positive minimal path between any two elements of $\mathcal{Q}_0$.

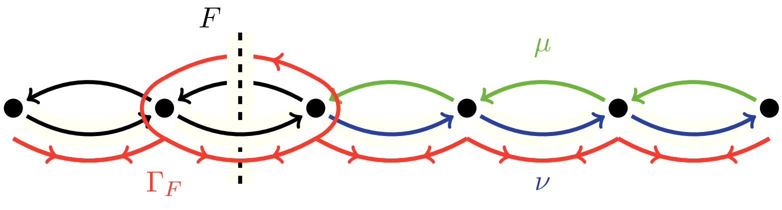
3.50 Definition
For $j = 1, \ldots, n$ let

$$\Omega_j := \{F \in \mathcal{F}^{\upharpoonleft}_1 : F \text{ is crossed by } \omega_j^{(k)} \text{ for some } k\},$$

And set $\Omega := \bigcup_j \Omega_j$.



(a) Case $F \in \Omega_1$, $k \geq 0$ and $(C_0)_F \neq (u_F.C_0)_F$



(b) Case $F \in \Omega_1$ and $k < 0$

Figure 3.7: Constructions for the proof of lemma 3.51

3.51 Lemma
For all $i = 1, \ldots, n$, the subgroup of $\pi_1(M(\mathscr{A}^{\upharpoonright}))$ generated by the elements β_F with $F \in \Omega_i$ is contained in the subgroup generated by the Γ_F, $F \in \Omega_i$.

Proof. Consider first $F \in \Omega_1$, by construction we have $F = u_1^k \overline{F}$ with $\overline{F} \in \overline{\mathcal{F}}_1 \cap \Omega_1$. If $k \geq 0$ and $(u_F.C_0)_F = (C_0)_F)$, then by construction we have $\Gamma_F = \beta_F$.

Suppose then $k \geq 0$ and $(u_F.C_0)_F \neq (C_0)_F$. Let $\nu = l_{F_1} l_{F_2} \ldots l_{F_r}$ be a positive minimal path between $(u_F.C_0)_F$ and $u_F.C_0$. We show the thesis by induction on r, the base case being trivial. Let μ be a positive minimal path between C_0 and $(C_0)_F$, then $\beta_F = \mu l_F^2 \mu^{-1}$ and

$$\Gamma_F = (\mu l_F \nu) u_F.\beta_{\overline{F}} (\mu l_F \nu)^{-1} \simeq \beta_{F_r} \cdots \beta_{F_1} \beta_F \beta_{F_1}^{-1} \cdots \beta_{F_r}^{-1}.$$

By induction hypothesis the β_{F_i}s belong to the subgroup generated by the Γ_Fs, therefore also β_F does.

Suppose then $k < 0$, and in this case $C' := (C_0)_F \neq (u_1^k C_0)_F$. Let ν denote the positive minimal path from C' to C_0 that follows the segments s_1. We argue by induction on the length $d(F)$ of ν: if $d(F) = 0$ we have in fact $\Gamma_F = \beta_F$.

Now let $d(F) > 0$. Then

$$\Gamma_F \simeq \nu^{-1} l_F^2 \nu; \quad \beta_F = \mu l_F^2 \mu^{-1}$$

where μ is the positive minimal path from C_0 to C' following s_1. Thus

$$\beta_F = \mu \nu \nu^{-1} l_F^2 \nu (\mu \nu)^{-1} = (\mu \nu) \Gamma_F (\mu \nu)^{-1}$$

where $\mu\nu$ is the product of all $\beta_{F'}$ with F' crossed by μ - therefore, with $F' \in \Omega_1$ and $d(F') < d(F)$. By induction, the claim follows.

Now consider $F \in \Omega_i$ with $i > 0$. If $\overline{F} \in \Omega_i$ proceed as in the case of $i = 1$, otherwise let G be the face in Ω_1 supported on the same hyperplane as F and use Lemma 1.48 to expresses β_F as product $\prod_{i=1}^{r} \beta_{F_i}$ with $F_i \in \Omega_1$. □

3.52 Lemma
The set $\{\Gamma_F \mid F \in \Omega\}$ *generates* $\pi_1(\mathcal{M}(\mathscr{A}^{\upharpoonright}))$.

Proof. Let $F \in \pi_1$, and let H the affine hyperplane supporting F.

By construction, there is $i \in \{1, \dots, d\}$ and $k \in \mathbb{Z}$ such that H is crossed by $\omega_i^{(k)}$ in, say, the face G ('every hyperplane is cut by the coordinate axes').

By Lemma 1.48, β_F is then product of β_G and other $\beta_{G'}^{\pm}$ with $G' \in \Omega$. These can be written in terms of the Γ_F by Lemma 3.51. □

Relations

We now turn to the study of the relations.

3.53 Lemma
Let $F \in \mathcal{Q}_1$. *Then there is a sequence* $F_1, \dots, F_k$ *of elements of* $\mathcal{Q}_1$ *such that* β_F *is homotopic to*

$$(\prod_{i=1}^{k} \Gamma_{F_i})^{-1} \Gamma_F (\prod_{i=1}^{k} \Gamma_{F_i})$$

with $F_i \in \Omega$. *In particular, the* F_i *are translates of elements of* $\Omega \cap \overline{\mathcal{F}^{\upharpoonright}}$.

Proof. By definition $\Gamma_F = \omega_{u_F} u_F.\beta_{\overline{F}} \omega_{u_F}^{-1}$. Writing μ for a positive minimal path $(u_F C_0 \to (u_F C_0)_F)$ we decompose this into

$$\Gamma_F = \omega_{u_F}\mu (l_F)^2 (\omega_{u_F}\mu)^{-1}.$$

With Remark 3.49.(3) we have that $\omega_{u_F}\mu$ is a positive path, and with Lemma 1.47 we write it as a product $\prod_j \beta_{G_j}^{\omega_{u_F}\mu}(C_0 \to (C_0)_F)$ where since μ is positive minimal, the G_j are crossed by ω_{u_F} and thus are translates of faces intersecting the segments s_i.

Now, using Lemma 3.51 we have

$$\beta_{G_j}^{\omega_{u_F}\mu} = \prod_{i=0}^{k_j} \Gamma_{F_i}.$$

Then, set

$$\Delta_F := \prod_j \beta_{G_j}^{\omega_{u_F}\mu}.$$

Therefore if $(C_0)_F = (u_F C_0)_F$ we are done with

$$\Gamma_F \simeq \Delta_F \beta_F \Delta_F^{-1}, \text{ and thus } \beta_F \simeq \Delta_F^{-1} \Gamma_F \Delta_F.$$

If $(C_0)_F \neq (u_F C_0)_F$, then we may choose a representant of $(C_0 \to (u_F C_0)_F)$ that ends with l_F, so its inverse begins with l_F^{-1} and we have the same relation as above. □

Keeping the notations of the Lemma we define, for every $F \in \mathcal{Q}_1$,

$$\Delta_F := \prod_{i=1}^{k} \Gamma_{F_i}; \qquad \Gamma_F^{\Delta} := \Delta_F^{-1} \Gamma_F \Delta_F. \tag{3.10}$$

Recall from §1.5 that to every face $G \in \mathcal{F}_2^{\restriction}$ we have an ordered set $h(G) = (F_1, \ldots, F_k)$ of incident codimension-1 faces, one for every hyperplane containing G. The relations associated with G assert the equality of

$$\beta_{F_{\sigma(1)}} \cdots \beta_{F_{\sigma(k)}} \tag{3.11}$$

where σ is a cyclic permutation, and we write β_i for β_{F_i}.

3.54 Lemma
Given $G \in \mathcal{F}_2^{\restriction}$ there is Δ_G such that, for all cyclic permutations σ, we have a homotopy of paths

$$\beta_{F_\sigma(k)} \ldots \beta_{F_\sigma(1)} \simeq \Delta_G \omega_{u_G} u_G.(\Gamma^{\Delta}_{u_G^{-1} F_{\sigma(k)}} \ldots \Gamma^{\Delta}_{u_G^{-1} F_{\sigma(1)}}) \omega_{u_G}^{-1} \Delta_G^{-1}.$$

Proof. Let us fix some notation and let $C' := (C_0)_G$, $C'' := (u_G.C_0)_G$, $\mu := (u_G C_0 \to C'')$, $\nu := (C'' \to C')$. By equation (1.7) we have the homotopy

$$\beta_{\sigma(k)} \ldots \beta_{\sigma(1)} \simeq (C_0 \to C') \alpha_G(C') (C_0 \to C')^{-1}$$

moreover, with Equation (1.6) we see

$$\alpha_G(C') \simeq \nu^{-1} \alpha_G(C'') \nu \simeq \nu^{-1} \mu^{-1} \omega_{u_G}^{-1} \omega_{u_G} \mu \alpha_G(C'') \mu^{-1} \omega_{u_G}^{-1} \omega_{u_G} \mu \nu$$

expanding $\mu \alpha_G(C'') \mu^{-1}$ according to Equation (1.7) and defining $\Delta_G := (C_0 \to C') \nu^{-1} \mu^{-1} \omega_{u_G}^{-1}$ we have the homotopy

$$\beta_{\sigma(k)} \ldots \beta_{\sigma(1)} \simeq \Delta_G \omega_{u_G} (u_G.\beta_{u_G^{-1} F_{\sigma(k)}}) \ldots (u_G.\beta_{u_G^{-1} F_{\sigma(1)}}) \omega_{u_G}^{-1} \Delta_G^{-1} \tag{3.12}$$

From which the claim follows by use of Lemma 3.53. □

3.55 Definition
For $F \in \mathcal{F}_1^{\restriction}$ let

$$\gamma_F := p(\Gamma_F).$$

Moreover, for $F \in \mathcal{Q}_1$ let

$$\delta_F := p(\Delta_F); \qquad \gamma_F^\delta := \delta_F^{-1}\gamma_F\delta_F$$

Given $G \in \overline{\mathcal{F}^\upharpoonleft}_2$ with $h(G) = (F_1, \dots, F_k)$, let $R_G^\downharpoonleft$ define the relation stating the equality of all words

$$\gamma^\delta_{F_{\sigma(k)}} \cdots \gamma^\delta_{F_{\sigma(1)}}$$

where σ ranges over all cyclic permutations.

3.56 Lemma
If $G \in \mathcal{F}_2^\upharpoonleft$ is a face of codimension 2, then the images under p_* of the associated relations of type (3.11) *are equivalent to $R_G^\downharpoonleft$*

Proof. Let $G \in \mathcal{F}_2^\upharpoonleft$. With Lemma 3.54 (and the notation thereof) we know that the image under p_* of the associated relations of type (3.11) states the equality of all

$$p_*(\Delta_G)p_*(\Gamma^\Delta_{F_{\sigma(k)}} \dots \Gamma^\Delta_{F_{\sigma(1)}})p_*(\Delta_G)^{-1},$$

where σ runs over all cyclic permutations. The middle term by Equation (3.12) is represented by the path

$$\omega_{u_G}(u_G.\beta_{u_G^{-1}F_{\sigma(k)}}) \dots (u_G.\beta_{u_G^{-1}F_{\sigma(1)}})\omega_{u_G}^{-1}$$

and thus its image under p_* is represented by the same path as

$$p_*(\omega_{u_G})p_*(\beta_{u_G^{-1}F_{\sigma(k)}} \dots \beta_{u_G^{-1}F_{\sigma(1)}})p_*(\omega_{u_G})^{-1}$$

Where $u_G^{-1}F_{\sigma(i)} \in \mathcal{Q}_1$ for all i. Now we apply Lemma 3.53. The element $\mu := p_*(\omega_{u_G}) \in \pi_1(T_\Lambda)$ is such that, for every cyclic permutation σ,

$$p_*(\Gamma^\Delta_{F_{\sigma(k)}} \dots \Gamma^\Delta_{F_{\sigma(1)}}) = \mu\, p_*(\Gamma^\Delta_{\overline{F}_{\sigma(k)}} \dots \Gamma^\Delta_{\overline{F}_{\sigma(1)}})\mu^{-1}$$

and therefore the image under p_* of the relations (3.11) associated to G is equivalent to the relations $R_G^\downharpoonleft$. □

Presentation

In this closing section we discuss presentations for $\pi_1(M(\mathscr{A}))$.

3.57 Lemma
For all $F \in \mathcal{Q}_1$ let $F_1, \dots, F_k$ as in Lemma 3.53. We have

$$\delta_F = \prod_{i=1}^k \tau_{u_{F_i}}\gamma_{\overline{F}_i}\tau_{u_{F_i}}^{-1}$$

and, in particular, γ_F^δ can be written as a word in the $\tau_1, \dots, \tau_n$ and γ_F with $F \in \overline{\mathcal{F}^\upharpoonleft}_1$.

Proof. This is an easy computation using Remark 3.49.(1). □

In Particular, the relations $R^{\downharpoonleft}_G$ can be written in terms of the τ_i and the γ_F with $F \in \overline{\mathcal{F}^{\upharpoonleft}}_1$. We have immediately

3.58 Theorem
The group $\pi_1(\mathcal{M}(\mathscr{A}))$ is presented as

$$\langle \tau_1, \ldots, \tau_n;\ \gamma_F,\ F \in \mathcal{F}_1 \mid \tau_i\tau_j = \tau_j\tau_i \textit{ for } i, j = 1, \ldots, n;\ R^{\downharpoonleft}_G,\ G \in \mathcal{F}_2\rangle,$$

where we identify $\mathcal{F}_1$ with $\overline{\mathcal{F}^{\upharpoonleft}}_1$ and $\mathcal{F}_2$ with $\overline{\mathcal{F}^{\upharpoonleft}}_2$.

This presentation, while not very economical in terms of generators, has the advantage that the relations can be described with an acceptable amount of complexity.

Using Lemma 3.52 and Remark 3.49.(1) we can let, for all $G \in \overline{\mathcal{F}^{\upharpoonleft}}_2$, $\widetilde{R}^{\downharpoonleft}_G$ denote the relations obtained from $R^{\downharpoonleft}_G$ by substituting every γ_F with the corresponding expression in terms of the generators $\tau_1, \ldots, \tau_d$ and $\gamma_{F'}$ with $F' \in \overline{\mathcal{F}^{\upharpoonleft}} \cap \Omega$. Under the identification of $\mathcal{F}_1$ with $\overline{\mathcal{F}^{\upharpoonleft}}$, these are the faces on the compact torus that are crossed by some fixed chosen representants of the generators $\tau_1, \ldots, \tau_d$.

3.59 Theorem
The group $\pi_1(\mathcal{M}(\mathscr{A}))$ is presented as

$$\langle \tau_1, \ldots, \tau_n;\ \gamma_F,\ F \in p(\Omega) \cap \mathcal{F}_1 \mid \tau_i\tau_j = \tau_j\tau_i \textit{ for } i, j = 1, \ldots, n;\ \widetilde{R}^{\downharpoonleft}_G,\ G \in \mathcal{F}_2\rangle.$$

CHAPTER 4

Minimality

This chapter is devoted to the minimality of arrangement complements. Consider a topological space X, which has the homotopy type of a CW-complex $\mathcal{K}$. Then the homology of X provides a lower bound on the number of cells of $\mathcal{K}$, in each dimension. This is easily seen using cellular homology.

For some spaces this lower bound is strict.

4.1 Definition
A CW-complex $\mathcal{K}$ is called minimal *if it has exactly $\beta_k = rk\, H^k(\mathcal{K};\mathbb{Z})$ cells in dimension k.*

A topological space is called minimal *if it has the homotopy type of a minimal CW-complex.*

Minimality has a number of consequences on the topology of the space. For example, the homology and cohomology with integer coefficients of minimal spaces is torsion-free (this is also easily seen using cellular homology). Minimality also gives informations on the cohomology of the space with some system of local coefficients (cfr. [14]).

4.2 Definition
A toric (or hyperplane) arrangement *is called* minimal *if $M(\mathscr{A})$ is a minimal space.*

Minimality of hyperplane arrangement complements was investigated first by Dimca and Papadima in [21] and by Randell in [38], with Morse theoretic arguments.

Once minimality was proved, arose the question of an explicit construction of such a minimal complex. This question was studied by Yoshinaga in [42] and by Salvetti and Settepanella in [41]. In particular, the strategy of Salvetti and Settepanella is to apply discrete Morse theory to the Salvetti complex.

In [19] Delucchi was able to define a more straightforward acyclic matching on the Salvetti complex of a central arrangements, based only on its combinatorics (i.e. on the corresponding oriented matroid).

In this chapter we will generalize Delucchi's method in order to prove the minimality of complements of toric arrangements. This gives as an interesting consequence the torsion-freeness of these spaces. This is a new result for toric arrangements and allows to derive the additive cohomology-structure of these spaces from Theorem 3.17.

4.1 Minimality of central hyperplane arrangements

Before proceeding with the minimality of toric arrangements, we review the methods of Delucchi [19] for minimality of central hyperplane arrangements. This will, on one hand, provide context for our arguments. On the other hand Delucchi's construction provides the "fundamental pieces" on which our proof is based.

4.3 Lemma ([19, Theorem 4.13])
Let $\mathscr{A}$ be a central arrangement of real hyperplanes, let $B \in \mathcal{T}(\mathscr{A})$ and let $\preceq$ be any linear extension of the poset $\mathcal{T}(\mathscr{A})_B$ of Definition 1.23.

The subset of $\mathcal{L}(\mathscr{A})$ given by all intersections X such that

$$S(C, C') \cap \mathscr{A}_X \neq \emptyset \quad \text{for all } C' \prec C$$

is an order ideal of $\mathcal{L}(\mathscr{A})$. In particular, it has a well defined and unique minimal element we will call X_C.

4.4 Remark
Note that X_C depends on the choice of B and of the linear extension of $\mathcal{T}(\mathscr{A})_B$.

4.5 Corollary
For all $C \in \mathcal{T}(\mathscr{A})$ we have

$$C = \min_{\preceq}\{K \in \mathcal{T}(\mathscr{A}) \mid K_{X_C} = C_{X_C}\},$$

where, for $Y \in \mathcal{L}(\mathscr{A})$ and $K \in \mathcal{T}(\mathscr{A})$, we define K_Y as the chamber in $\mathcal{T}(\mathscr{A}_Y)$ which contains K.

Recall the cellular Salvetti complex for hyperplane arrangements of Definition 1.39, whose maximal cells correspond to the pairs $[P, C]$ where P is a point and C is a chamber.

4.1. Minimality of central hyperplane arrangements

Since $\mathscr{A}$ is a central arrangement, the maximal cells correspond to the chambers in $\mathcal{T}(\mathscr{A})$. In this case we can stratify the Salvetti complex assigning to each chamber $C \in \mathcal{T}(\mathscr{A})$ the corresponding maximal cell of $\mathcal{S}(\mathscr{A})$, together with its faces. Let us make this precise.

4.6 Definition
Let $\mathscr{A}$ be a central complexified hyperplane arrangement and write $\min \mathcal{F}(\mathscr{A}) = \{P\}$. Define a stratification of the cellular Salvetti complex $\mathcal{S}(\mathscr{A}) = \bigcup_{C\in\mathcal{T}(\mathscr{A})} \mathcal{S}_C$ through

$$\mathcal{S}_C := \bigcup \{[F,K] \in \mathit{Sal}(\mathscr{A}) \mid [F,K] \leq [P,C]\}\,.$$

Given an arbitrary linear extension $(\mathcal{T}(\mathscr{A}), \preceq)$ of $\mathcal{T}(\mathscr{A})_B$, for all $C \in \mathcal{T}(\mathscr{A})$ define

$$\mathcal{N}_C := \mathcal{S}_C \setminus \left(\bigcup_{D \prec C} \mathcal{S}_D \right).$$

In particular the poset $\mathrm{Sal}(\mathscr{A})$ can be partitioned as

$$\mathrm{Sal}(\mathscr{A}) = \bigsqcup_{C\in\mathcal{T}(\mathscr{A})} \mathcal{N}_C(\mathscr{A}).$$

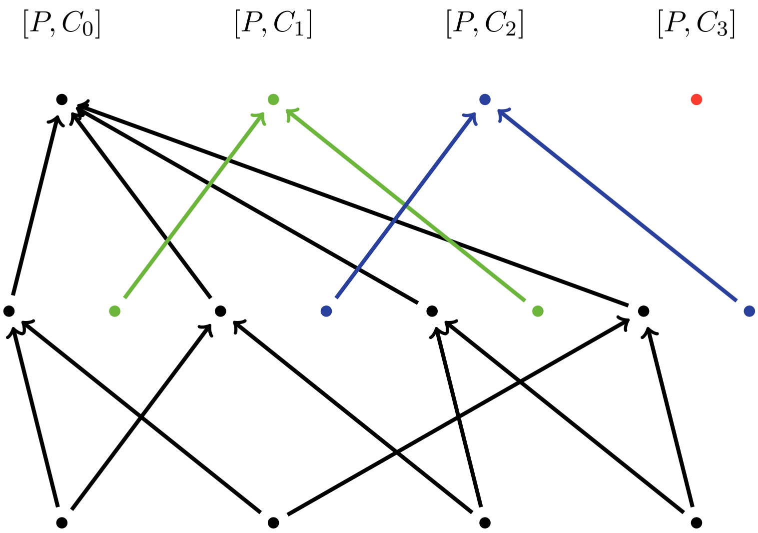
4.7 Theorem ([19, Lemma 4.18])
There is an isomorphism of posets

$$\mathcal{N}_C \cong \mathcal{F}(\mathscr{A}^{X_C})^{op}$$

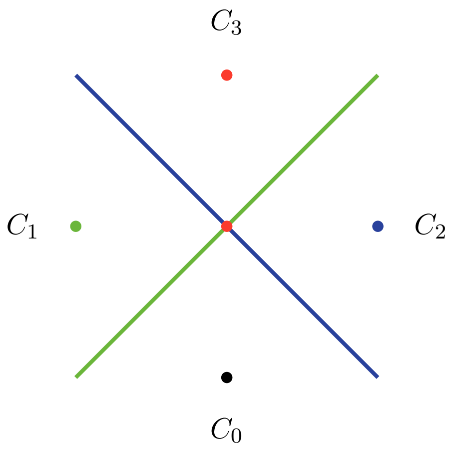
where X_C is the intersection defined via Lemma 4.3 by the same choice of base chamber and of linear extension of $\mathcal{T}(\mathscr{A})_B$ used to define the subposets $\mathcal{N}_C$.

4.8 Remark
The proof given in [19] of minimality of $M(\mathscr{A})$ for $\mathscr{A}$ a complexified central arrangement follows from Theorem 4.7 by an application of Discrete Morse Theory (see §2.3). Indeed, from a shelling order of $\mathcal{F}(\mathscr{A}^{X_C})$ one can construct a sequence of cellular collapses of the induced subcomplex of $\mathcal{S}_C$ that leaves only one ‘surviving’ cell. Via the Patchwork Lemma (Lemma 2.39) these sequences of collapses can be concatenated to give a sequence of collapses on the cell complex $\mathcal{S}(\mathscr{A})$. The resulting complex after the collapses has one cell for every $\mathcal{N}_C$, namely $|\operatorname{nbc}(\mathcal{A})| = P_{\mathscr{A}}(1)$ cells, and is thus minimal.

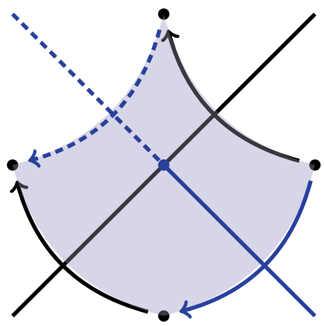
4.9 Example
Figure 4.1 shows the construction of the strata for the arrangement of two lines in $\mathbb{R}^2$.



(a) Stratification of $\mathcal{F}(\mathscr{A})$



(b) Correspondance $C \mapsto X_C$



(c) The layer X_{C_2} and the stratum $\mathcal{N}_{C_2}$

Figure 4.1: Example of stratification

Figure 4.1a shows the face poset $\mathcal{F}(\mathscr{A})$ decomposed in strata. The stratum $\mathcal{N}_{C_0} = \mathcal{S}_{C_0}$ is pictured in black, the strata $\mathcal{N}_{C_1}$, $\mathcal{N}_{C_2}$ and $\mathcal{N}_{C_3}$ are picture in green, blue and red respectively.

Figure 4.1b illustrates the correspondence $C \to X_C$ of Lemma 4.3. X_{C_0} is the whole plane and is not showed in the picture. The other correspondences are represented using the same colors of Figure 4.1a.

Figure 4.1c illustrates the isomorphism $\mathcal{F}(\mathscr{A}^{X_{C_2}})^{op} \cong \mathcal{N}_{C_2}$ of Theorem 4.7. The 2-dimensional cell of $\mathcal{N}_{C_2}$ corresponds to the point of $\mathcal{F}(\mathscr{A}^{X_{C_2}})$, the dashed line in $\mathcal{N}_{C_2}$ corresponds to the dashed line of $\mathcal{F}(\mathscr{A}^{X_{C_2}})$ and the solid line to the solid one.

Posets of regions

Fix a real hyperplane arrangement $\mathscr{A}$, a basis chamber $B \in \mathcal{T}(\mathscr{A})$ and recall the *poset of regions* of Definition 1.23.

4.10 Remark
Let $\mathscr{A}_0$ be a real arrangement and $B \in \mathcal{T}(\mathscr{A}_0)$. Given a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$, for every chamber $C \in \mathcal{T}(\mathscr{A}_0)$ there is a unique chamber $\widehat{C} \in \mathcal{T}(\mathscr{A}_1)$ with $C \subseteq \widehat{C}$. The correspondence $C \mapsto \widehat{C}$ defines a surjective map

$$\sigma_{\mathscr{A}_1} : \mathcal{T}(\mathscr{A}_0)_B \to \mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$$

such that $C \leq C'$ implies $\sigma_{\mathscr{A}_1}(C) \leq \sigma_{\mathscr{A}_1}(C')$ for all $C, C' \in \mathcal{T}(\mathscr{A}_0)$.

4.11 Definition
Let $\mathscr{A}_0$ be a real arrangement and let $\succ_0$ denote any total ordering of $\mathcal{T}(\mathscr{A}_0)$. Consider a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$. The section

$$\mu[\mathscr{A}_1, \mathscr{A}_0] : \mathcal{T}(\mathscr{A}_1) \to \mathcal{T}(\mathscr{A}_0), \quad C \mapsto \min_{\succ_0}\{K \in \mathcal{T}(\mathscr{A}_0) \mid K \subseteq C\}$$

of $\sigma_{\mathscr{A}_1}$ defines a total ordering $\succ_{0,1}$ on $\mathcal{T}(\mathscr{A}_1)$ by

$$C \succ_{0,1} D \iff \mu[\mathscr{A}_1, \mathscr{A}_0](C) \succ_0 \mu[\mathscr{A}_1, \mathscr{A}_0](D)$$

that we call induced by $\succ_0$.

4.12 Lemma
Consider real arrangements $\mathscr{A}_2 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_0$, a given total ordering $\succ_0$ of $\mathcal{T}(\mathscr{A}_0)$ and the induced total ordering $\succ_{0,1}$ of $\mathcal{T}(\mathscr{A}_1)$. Then

$$\mu[\mathscr{A}_1, \mathscr{A}_0] \circ \mu[\mathscr{A}_2, \mathscr{A}_1] = \mu[\mathscr{A}_2, \mathscr{A}_0].$$

Proof. Take any $C \in \mathcal{T}(\mathscr{A}_2)$ and define

$$\begin{aligned} &C_0 := \mu[\mathscr{A}_2, \mathscr{A}_0](C); \quad C_1 := \sigma_{\mathscr{A}_1}(C_0), \text{ so } \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0; \\ &C_2 := \mu[\mathscr{A}_2, \mathscr{A}_1](C); \quad C_3 := \mu[\mathscr{A}_1, \mathscr{A}_0](C_2). \end{aligned}$$

we have to show that $C_0 = C_3$.
First, notice that $C_0 \preceq_0 C_3$ because $C_3 \subseteq C_2 \subseteq C$. For the reverse inequality notice that we have $C_1, C_2 \subseteq C$, which implies $C_2 \preceq_{0,1} C_1$ and so, by definition of the induced ordering, $C_3 = \mu[\mathscr{A}_1, \mathscr{A}_0](C_2) \preceq_0 \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0$. □

4.13 Proposition
Let a base chamber B of $\mathscr{A}_0$ be chosen. If $\succ_0$ is a linear extension of $\mathcal{T}(\mathscr{A}_0)_B$, then $\succ_{0,1}$ is a linear extension of $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$.

Proof. We have to prove that for all $C, D \in \mathcal{T}(\mathscr{A}_1)$, $C \leq D$ in $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$ implies $C \preceq_{0,1} D$, i.e., $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$.

We argue by induction on $k := |\mathscr{A}_0 \setminus \mathscr{A}_1|$, the claim being evident when $k = 0$. Suppose then that $k > 0$, choose $H \in \mathscr{A}_0 \setminus \mathscr{A}_1$ and set $\mathscr{A}_0' := \mathscr{A}_0 \setminus \{H\}$. By induction hypothesis we have

$$\mu[\mathscr{A}_0', \mathscr{A}_1](C) \preceq_0' \mu[\mathscr{A}_0', \mathscr{A}_1](D).$$

which by definition means

$$\mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](C)) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](D))$$

and thus, via Lemma 4.12, $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$. □

4.2 Stratification of the toric Salvetti complex

We now work our way towards proving the minimality of complements of toric arrangements. We start by defining a stratification of the toric Salvetti Complex, in which each stratum corresponds to a local non broken circuit. Then, in the next Section, we will exploit the structure of this stratification to define a perfect acyclic matching on the Salvetti Category.

Local geometry of complexified toric arrangements

Consider a rank d complexified toric arrangement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ with $\chi_i(x) = x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^d$. As usual we write $K_i = \{x \in T_\Lambda \mid \chi_i(x) = a_i\}$.

We introduce some central hyperplane arrangements we will work with. Consider the arrangement

$$\mathscr{A}_0 = \{H_i = \ker \langle \alpha_i, \cdot \rangle \mid i = 1, \ldots, n\}$$

in $\mathbb{R}^d$ and, from now on, fix a chamber $B \in \mathcal{T}(\mathscr{A}_0)$ and a linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$.

4.2. Stratification of the toric Salvetti complex

4.14 Definition
For every face $F \in \mathcal{F}(\mathscr{A})$ define the arrangement

$$\mathscr{A}[F] = \{H_i \in \mathscr{A}_0 \mid \chi_i(F) = a_i\}.$$

If $Y \in \mathcal{C}(\mathscr{A})$ define

$$\mathscr{A}[Y] = \{H_i \in \mathscr{A}_0 \mid Y \subseteq K_i\}.$$

4.15 Remark
The linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$ induces as in Proposition 4.13 linear extensions $\prec_F$ of $\mathcal{T}(\mathscr{A}[F])_{B_F}$ and $\prec_Y$ of $\mathcal{T}(\mathscr{A}[Y])_{B_Y}$, for every $F \in \mathcal{F}(\mathscr{A})$ and every $Y \in \mathcal{C}(\mathscr{A})$.

Moreover, for $F \in \mathcal{F}(\mathscr{A})$ and $C, C' \in \mathcal{T}(\mathscr{A}[F])$ we denote by $S_F(C, C')$ the set of separating hyperplanes of the arrangement $\mathscr{A}[F]$, as introduced in Definition 1.23.

4.16 Definition
Given $X \in \mathcal{C}(\mathscr{A})$ let $\widetilde{X} \in \mathcal{L}(\mathscr{A}_0)$ be defined as

$$\widetilde{X} := \bigcap_{X \subseteq K_i} H_i.$$

4.17 Definition
Let $Y \in \mathcal{C}(\mathscr{A})$ be a layer of $\mathscr{A}$. For $C \in \mathcal{T}(\mathscr{A}[Y])$ let $X(Y, C) \supseteq Y$ be the layer determined by the intersection defined by Lemma 4.3 from $\prec_Y$. Analogously, for $C \in \mathcal{T}(\mathscr{A}[F])$ let $X(F, C)$ be defined with respect to $\prec_F$.

We write $\widetilde{X}(Y, C)$ and $\widetilde{X}(F, C)$ for the corresponding elements of $\mathcal{L}(\mathscr{A}[Y])$ and $\mathcal{L}(\mathscr{A}[F])$.

4.18 Definition
Let

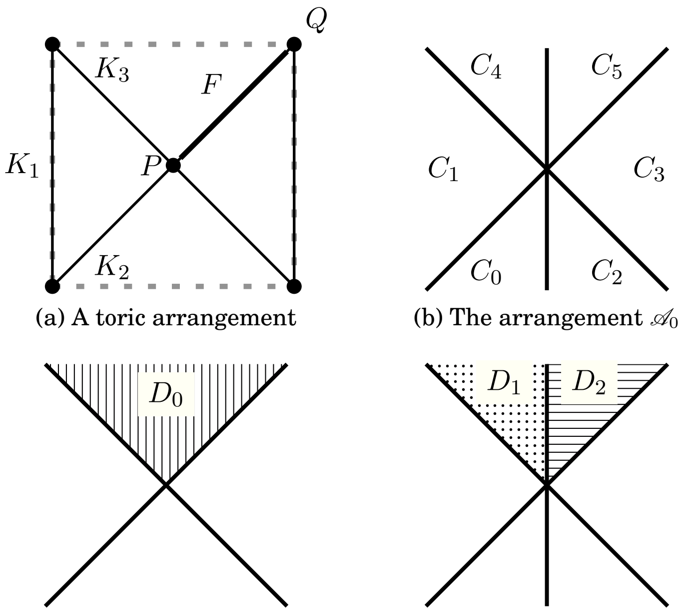
$$\mathscr{Y} := \{(Y, C) \mid Y \in \mathcal{C}(\mathscr{A}),\ C \in \mathcal{T}(\mathscr{A}[Y]),\ X(Y, C) = Y\}.$$

For $i = 0, \ldots, d$ let $\mathscr{Y}_i := \{(Y, C) \in \mathscr{Y} \mid \dim(Y) = i\}$.

4.19 Example
Consider the toric arrangement $\mathscr{A} = \{(x, 1), (xy^{-1}, 1), (xy, 1)\}$ of Figure 4.2a. In this and in the following pictures we consider the compact torus $(S^1)^2$ as a quotient of the square. Therefore we draw toric arrangements in a square (pictured with a dashed line), where the opposite sides are identified.

The layer poset consists of the following elements

$$\mathcal{C}(\mathscr{A}) = \{P, Q, K_1, K_2, K_3, (\mathbb{C}^*)^2\}.$$



(a) A toric arrangement
(b) The arrangement $\mathscr{A}_0$
(c) The arrangement $\mathscr{A}[P]$
(d) The arrangement $\mathscr{A}[Q]$

Figure 4.2: A toric arrangement and its associated hyperplane arrangements

Figures 4.2c and 4.2d show respectively the arrangements $\mathscr{A}[P]$ and $\mathscr{A}[Q] = \mathscr{A}_0$.

Let $\mathscr{Y}$ as in Definition 4.18. There is one element $(P, D_0) \in \mathscr{Y}$ and two elements $(Q, D_1), (Q, D_2) \in \mathscr{Y}$. Furthermore we have an element for each 1-dimensional Layer $(K_i, C_{K_i}) \in \mathscr{Y}$.

4.20 Lemma
Let $\mathscr{A}$ be a rank d toric arrangement. For all $i = 0, \dots, d$, we have $|\mathscr{Y}_i| = |\mathscr{N}_i|$.

Proof. This follows because for every $i = 0, \dots, d$,

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\operatorname{nbc}_i(\mathscr{A}[Y])|$$

Every summand on the right hand side counts the number of generators in top degree cohomology or - equivalently - the number of top dimensional cells of a minimal CW-model of the complement of the complexification of $\mathscr{A}[Y]$. By [19, Lemma 4.18 and Proposition 2] these top dimensional cells correspond bijectively to chambers $C \in \mathcal{T}(\mathscr{A}[Y])$ with

$X(Y,C) = Y$. Therefore

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y,C) = Y\}| = |\mathscr{Y}_i|.$$

□

4.21 Definition
Recall Definition 4.11. The assignment $(Y,C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}_0](C)$ *defines a function* $\xi_0 : \mathscr{Y} \to \mathcal{T}(\mathscr{A}_0)_B$*. Choose, and fix, a total order* $\dashv$ *on* $\mathscr{Y}$ *that makes this function order preserving.*

4.22 Remark
For $y_1, y_2 \in \mathscr{Y}$, by definition $\xi_0(y_1) \prec_0 \xi_0(y_2)$ implies $y_1 \dashv y_2$.

4.23 Example
Consider the toric arrangement $\mathscr{A}$ of Figure 4.2a and Example 4.19. The corresponding arrangement $\mathscr{A}_0$ is pictured in Figure 4.2b, where the chambers in $\mathcal{T}(\mathscr{A}_0)$ are ordered according to their indexes.

The map ξ_0 evaluates on $\mathscr{Y}$ as follows:

$$\begin{array}{ccc} ((\mathbb{C}^*)^2, \mathbb{C}^2) \mapsto C_0 & (K_1, C_{K_1}) \mapsto C_2 & (K_2, C_{K_2}) \mapsto C_1 \\ (K_3, C_{K_3}) \mapsto C_3 & (P, D_0) \mapsto C_4 & (Q, D_1) \mapsto C_4 \\ (Q, D_2) \mapsto C_5. & & \end{array}$$

A possible total order on $\mathscr{Y}$ is given by:

$$((\mathbb{C}^*)^2, \mathbb{C}^2) \dashv (K_2, C_{K_2}) \dashv (K_1, C_{K_1}) \dashv (K_3, C_{K_3}) \dashv (P, D_0) \dashv$$
$$(Q, D_1) \dashv (Q, D_5).$$

We now examine the local properties of the ordering $\dashv$.

4.24 Definition
For $F \in \mathcal{F}(\mathscr{A})$ *let* $\mathscr{Y}_F := \{(Y,C) \in \mathscr{Y} \mid F \subseteq Y\}$.

Since $F \subseteq Y$ *implies* $\mathscr{A}[Y] \subseteq \mathscr{A}[F]$*, we can define a function* $\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F])$ *by* $(Y,C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}[F]](C)$.

4.25 Remark
By Lemma 4.12, $\mu[\mathscr{A}[F], \mathscr{A}_0] \circ \xi_F = \xi_0$ on $\mathscr{Y}_F$. Therefore, for $y_1, y_2 \in \mathscr{Y}_F$, $\xi_F(y_1) \prec_F \xi_F(y_2)$ implies $\xi_0(y_1) \prec_0 \xi_0(y_2)$, and thus $y_1 \dashv y_2$.

4.26 Proposition
For all $F \in \mathcal{F}(\mathscr{A})$ *and every* $y = (Y,C) \in \mathscr{Y}_F$*,*

$$X(F, \xi_F(y)) = Y.$$

Proof. We will use the lattice isomorphisms $\mathcal{L}(\mathscr{A}[F])_{\leq \widetilde{Y}} \simeq \mathcal{L}(\mathscr{A}[Y]) \simeq \mathcal{C}(\mathscr{A})_{\leq Y}$. By definition we have that

$$\xi_F(y) = \mu[\mathscr{A}[Y], \mathscr{A}[F]](C) = \min_{\prec_F}\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K \subseteq C\}$$

and therefore $\mathscr{A}[F]_{\widetilde{Y}} \cap S_F(\xi_F(y), C_1) \neq \emptyset$ for all $C_1 \prec_F \xi_F(y)$, which shows that $\widetilde{Y} \geq \widetilde{X}(F, \xi_F(y))$ in $\mathcal{L}(\mathscr{A}[F])$ and thus $Y \geq X(F, \xi_F(y))$ in $\mathcal{C}(\mathscr{A})$. Now, for every layer Z with $Z < Y$ we have that $\mathscr{A}[Z] \subseteq \mathscr{A}[Y]$. Because by definition $Y = X(Y, C)$, we have $\widetilde{Z} < \widetilde{Y} = \widetilde{X}(Y, C)$ in $\mathcal{L}(\mathscr{A}[Y])$ and so there is $C_2 \prec_Y C$ with $S_Y(C_2, C) \cap A[Y]_{\widetilde{Z}} = \emptyset$.

Let $C_3 := \mu[\mathscr{A}[Y], \mathscr{A}[F]](C_2)$. We have $C_3 \subseteq C_2$ and $\xi_F(y) \subseteq C$, therefore $S_F(C_3, \xi_F(y)) \cap \mathrm{supp}(\widetilde{Z}) = \emptyset$, and $C_3 \prec_F \xi_F(y)$ by $C_2 \prec_Y C$. This means $Z \not\geq X(F, \xi_F(y))$, and the claim follows. □

4.27 Lemma

For $F \in \mathcal{F}(\mathscr{A})$ and $C \in \mathcal{T}(\mathscr{A}[F])$ we have

$$\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) = C$$

In particular $\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F])$ is a bijection.

Proof. Using the definition of ξ_F and Corollary 4.5 we have

$$\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) = \mu[\mathscr{A}[X_C], \mathscr{A}[F]](\sigma_{\mathscr{A}[X_C]}(C))$$

$$= \min\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K_{X_C} = C_{X_C}\} = C.$$

Letting $\beta_F : \mathcal{T}(\mathscr{A}[F]) \to \mathscr{Y}_F$ be defined by $C \mapsto (X_C, \sigma_{\mathscr{A}[X_C]}(C))$, the above means $\xi_F \circ \beta_F = id$, therefore the map ξ_F is surjective. Injectivity of ξ_F amounts now to proving $\beta_F \circ \xi_F = id$, which is an easy check of the definitions. □

4.28 Corollary

For $y_1, y_2 \in \mathscr{Y}_F$, $y_1 \dashv y_2$ if and only if $\xi_F(y_1) \preceq_F \xi_F(y_2)$.

From global to local

We now relate our local constructions to the covering $\mathscr{A}^\upharpoonright$ of $\mathscr{A}$ defined in §3.5.

4.29 Definition

Consider a toric arrangement $\mathscr{A}$ on $T_\Lambda \cong (\mathbb{C}^*)^k$ and a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$. We associate to m a face $F_m \in \mathcal{F}(\mathscr{A}[F])$ as follows.

(a) Fix an $F^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright) = F$.

(b) From Lemma 3.37 and from the freenees of the action of Λ *it follows that there is a unique* $G^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ *such that*

$$q(F^{\upharpoonleft} \leq G^{\upharpoonleft}) = m.$$

(c) Consider the arrangement

$$\mathscr{A}^{\upharpoonleft}_{F^{\upharpoonleft}} = \{H \in \mathscr{A}^{\upharpoonleft} : \quad F^{\upharpoonleft} \in H\}.$$

Clearly, up to translation, $\mathscr{A}^{\upharpoonleft}_{F^{\upharpoonleft}} = \mathscr{A}[F]$ *and we can identify the two arrangements.*

(d) Define F_m *as the face of* $\mathscr{A}[F]$ *which contains* $G^{\upharpoonleft}$*. That is, in terms of sign vectors and identifying each* $H \in \mathscr{A}[F]$ *with its unique translate which contains* $G^{\upharpoonleft}$*:*

$$\gamma_{F_m} = \gamma_{G^{\upharpoonleft}|\mathscr{A}[F]}.$$

In particular, if G *is a chamber, then* F_m *also is.*

4.30 Remark
In order to keep the notation transparent we will often identify a face $F \in \mathcal{F}(\mathscr{A})$, with the corresponding minimal face $F_{id} \in \mathcal{F}(\mathscr{A}[F])$.

4.31 Remark
Consider a face $F \in \mathcal{F}(\mathscr{A})$ and an element $G^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}[F])$. Then there is a unique face $G \in \mathcal{F}(\mathscr{A})$ and a unique morphism $m : F \to G$ such that $G^{\upharpoonleft} = i_m(G_{id})$.

4.32 Lemma
If $m_1 : F_1 \to C_1$ *and* $m_2 : F_2 \to C_2$ *are elements of* $\mathrm{Sal}\,\mathscr{A}$ *and if there is* $l : F_2 \to F_1$*, then*

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2) \text{ if and only if } S_{F_2}(F_{l\circ m_1}, F_{m_2}) \cap \mathscr{A}[F_1] = \emptyset$$

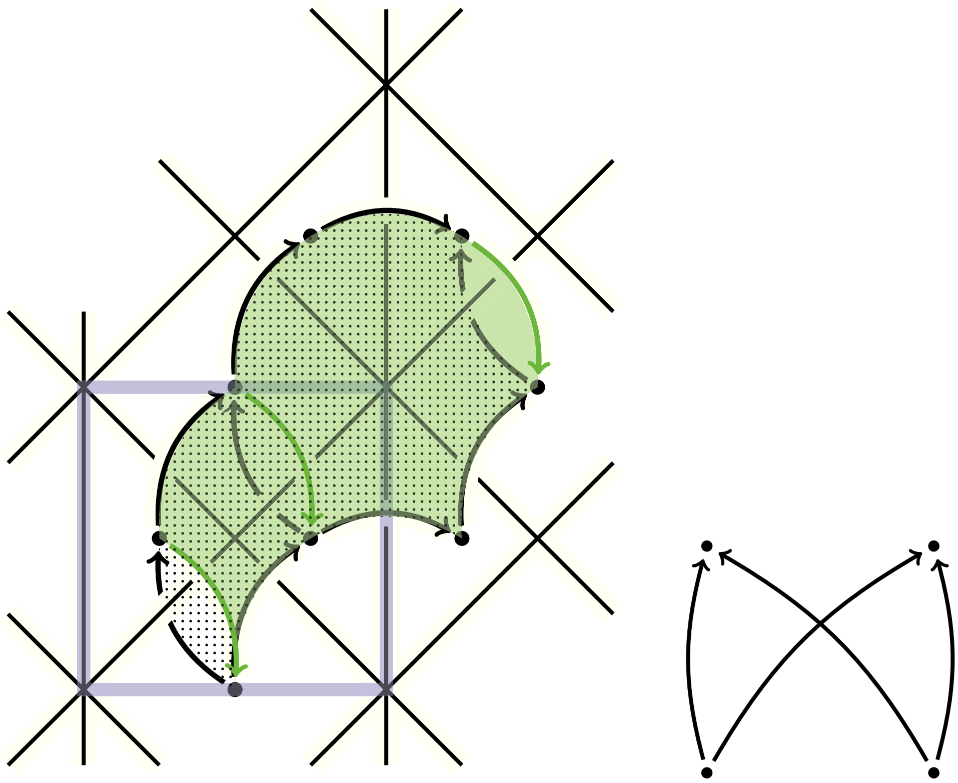
Proof. This is a rephrasing of the definitions. $\square$

Definition of the strata

4.33 Definition
Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$ *as follows*

$$\theta : (m : F \to C) \mapsto (X(F, F_m),\ \sigma_{\mathscr{A}[X(F,F_m)]}(F_m))$$

4.34 Remark
For every object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ we have $\xi_F(\theta(m)) = F_m$.



(a) Stratification of the toric Salvetti complex

(b) $\mathcal{N}_{(K_2,D_2)} = \mathcal{F}(\mathscr{A}^{K_2})^{op} = \mathcal{F}(\mathscr{A}^{K_2})$

Figure 4.3: Stratification of the toric Salvetti Complex

4.35 Lemma

For $m : G \to C, m' : G \to C' \in \zeta$*, if* $\theta(m) \dashv \theta(m')$ *then* $F_m \prec_G F_{m'}$.

Proof. If $\theta(m) \dashv \theta(m')$, then with Remark 4.34 and Corollary 4.28, $F_m = \xi_G(\theta(m)) \prec_G \xi_G(\theta(m')) = F_{m'}$. □

4.36 Definition

Given a complexified toric arrangement $\mathscr{A}$ *on* $(\mathbb{C}^*)^d$*, we consider the following stratification of* $\mathrm{Sal}(\mathscr{A})$ *indexed by* $\mathscr{Y}$*:* $\mathrm{Sal}(\mathscr{A}) = \cup_{(Y,C)\in\mathscr{Y}}\mathcal{S}_{(Y,C)}$ *where*

$$\mathcal{S}_{(Y,C)} = \{m \in \boldsymbol{Sal}(\mathscr{A}) \mid \exists (m \to n) \in \boldsymbol{Mor}(\boldsymbol{Sal}(\mathscr{A})),\, n \in \theta^{-1}(Y,C)\}.$$

Moreover, recall the total ordering $\vdash$ *on* $\mathscr{Y}$ *and define*

$$\mathcal{N}_y = \mathcal{S}_y \setminus \bigcup_{y' \dashv y} \mathcal{S}_{y'}.$$

4.37 Example

Consider the toric arrangement $\mathscr{A}$ of Figure 4.2a, Example 4.19 and Example 4.23. Figure 4.3a shows two strata of the stratification on $\mathrm{Sal}\,\mathscr{A}$ of Definition 4.36.

The stratum $\mathcal{S}_{((\mathbb{C}^*)^2,D)}$ is pictured in dotted black, while the stratum $\mathcal{N}_{(K_2,D_2)}$ is pictured in solid green. Thus $\mathcal{N}_{(K_2,D_2)}$ consists of two 1-dimensional layers and two 2-dimensional layers. The category $\mathcal{N}_{(K_2,D_2)}$ is showed in Figure 4.3b and it is isomorphic to $\mathcal{F}(\mathscr{A}^{K_2})$ (which is self-dual).

4.3 The topology of the Strata

We now want to show that, for $y \in \mathscr{Y}$, the category $\mathcal{N}_y$ is isomorphic to the face category of a complexified toric arrangement. The main result of this section is the following.

4.38 Theorem

Consider a complexified toric arrangement $\mathscr{A}$ and for $y = (Y, C) \in \mathscr{Y}$ let $\mathcal{N}_y$ be as in Definition 4.36. Then there is an isomorphism of acyclic categories

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op}$$

The main idea for proving this theorem is to use the 'local' combinatorics of the (hyperplane) arrangements $\mathscr{A}[F]$ to understand the 'global' structure of the strata in $\mathrm{Sal}(\mathscr{A})$. We carry out this 'local-to-global' approach by using the language of diagrams.

4.39 Definition

Let $\mathscr{A}$ be a complexified toric arrangement. Consider the following diagram of acyclic categories.

$$\begin{gathered}\mathscr{F} : \mathcal{F}(\mathscr{A})^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{F}(\mathscr{A}[F]);\\ (m : F \to G) \mapsto (i_m : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F]))\end{gathered}$$

where for $G' \in \mathcal{F}(\mathscr{A}[G])$ the face $i_m(G') \in \mathcal{F}(\mathscr{A}[F])$ is defined by the following sign vector

$$\gamma_{i_m(G')}(H) = \begin{cases} \gamma_{F_m}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{G'}(H) & \text{if } H \in \mathscr{A}[G] \end{cases}$$

Notice that $\mathscr{F}$ is a *geometric* diagram in the sense of Definition 2.22 (cfr. also Remark 4.31).

4.40 Example

Consider the arrangement $\mathscr{A}$ of Figure 4.2. Figure 4.4 illustrates the maps i_m and i_n for the morphisms $m : P \to F$ and $n : Q \to F$.

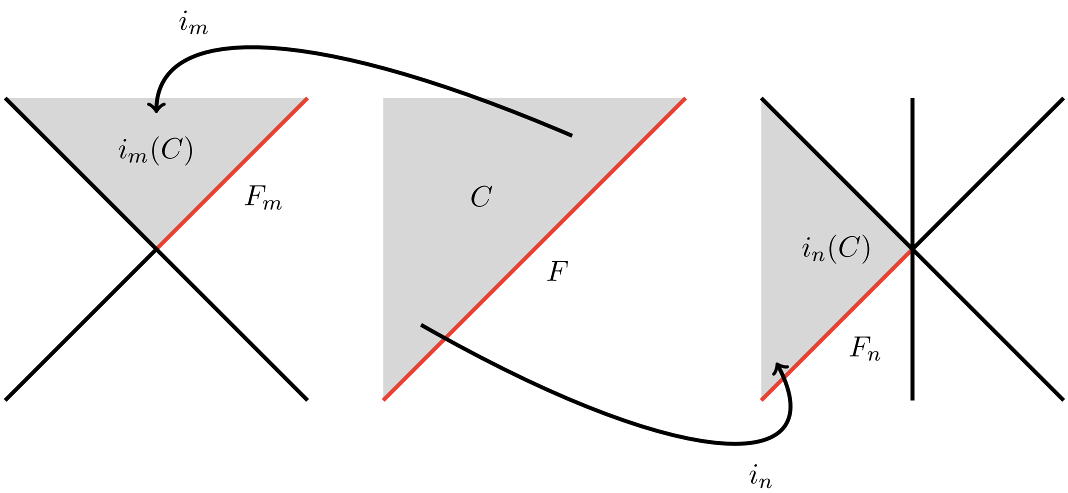


Figure 4.4: F_m and the map i_m

4.41 Lemma
Consider the composable morphisms $F \xrightarrow{m} G \xrightarrow{n} K$. Then, with the notation of Definition 4.29,

$$i_m(G_n) = F_{n \circ m}.$$

Proof. Choose a lift $F^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ such that $q(F^{\upharpoonleft}) = F$ and let $G^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ the unique face of $\mathscr{A}^{\upharpoonleft}$ such that $q(F^{\upharpoonleft} \leq G^{\upharpoonleft}) = m$. Then $q(G^{\upharpoonleft}) = G$ and there exists a unique $K^{\upharpoonleft} \in \mathcal{F}(\mathscr{A}^{\upharpoonleft})$ such that $q(G^{\upharpoonleft} \leq K^{\upharpoonleft}) = n$. Furthermore $q(F^{\upharpoonleft} \leq K^{\upharpoonleft}) = n \circ m$. According to Definition 4.29 we have:

$$\gamma_{i_m(G_n)}(H) = \begin{cases} \gamma_{F_m}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{G_n}(H) & \text{if } H \in \mathscr{A}[G] \end{cases} = \begin{cases} \gamma_{G^{\upharpoonleft}}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{K^{\upharpoonleft}}(H) & \text{if } H \in \mathscr{A}[G] \end{cases} \tag{4.1}$$

In terms of sign vectors, the property $G^{\upharpoonleft} \leq K^{\upharpoonleft}$ translates to the following.

$$\text{For all } H \in \mathscr{A}^{\upharpoonleft}: \quad \gamma_{G^{\upharpoonleft}}(H) \neq 0 \Longrightarrow \gamma_{G^{\upharpoonleft}}(H) = \gamma_{K^{\upharpoonleft}}(H).$$

In particular $H \notin \mathscr{A}[G]$ implies $\gamma_{G^{\upharpoonleft}}(H) = \gamma_{K^{\upharpoonleft}}(H)$ and therefore from Equation (4.1) we get

$$\gamma_{i_m(G_n)}(H) = \gamma_{K^{\upharpoonleft}}(H) \quad \forall H \in \mathscr{A}[F], \quad \text{which means} \quad i_m(G_n) = F_{mon}.$$

□

4.42 Lemma

$$\operatorname{colim} \mathscr{F} = \mathcal{F}(\mathscr{A})$$

Proof. Since $\mathscr{F}$ is a geometric diagram, Proposition 2.26 applies and we can describe the objects and the morphisms of $\operatorname{colim}\mathscr{F}$ through the usual equivalence relations.

Equivalence classes with respect to these equivalence relations will be denoted by $[\![\cdot]\!]$, to avoid confusion with the square brackets used to identify elements of the Salvetti complex.

We construct an isomorphism $\Phi : \mathcal{F}(\mathscr{A}) \to \operatorname{colim}\mathscr{F}$. Consider an object $F \in \mathcal{F}(\mathscr{A})$ and define $\Phi(F) = [\![F_{id}]\!]$, where F_{id} is a face in $\mathcal{F}(\mathscr{A}[F])$. Consider now a morphism $m : F \to G$ in $\mathcal{F}(\mathscr{A})$ and define

$$\Phi(m) = [\![F_{id} \leq F_m]\!].$$

The bijectivity of Φ is easily seen. We only need to show the functoriality of Φ. Consider the composable morphisms $F \overset{m}{\to} G \overset{n}{\to} H$. Using Lemma 4.41 we get

$$\begin{aligned}\Phi(n) \circ \Phi(m) &= [\![G_{id} \leq G_n]\!] \circ [\![F_{id} \leq F_m]\!] \\ &= [\![\mathscr{F}(m)(G_{id} \leq G_n)]\!] \circ [\![F_{id} \leq F_m]\!] = [\![i_m(G_{id}) \leq i_m(G_n)]\!] \circ [\![F_{id} \leq F_m]\!] \\ &= [\![F_m \leq F_{n\circ m}]\!] \circ [\![F \leq F_m]\!] = [\![F \leq F_{n\circ m}]\!] = \Phi(n \circ m).\end{aligned}$$ □

Next we construct the Salvetti category as a colimit of Salvetti posets.

4.43 Definition

$$\begin{aligned}\mathscr{D} = \mathscr{D}(\mathscr{A}) : \mathcal{F}(\mathscr{A})^{op} &\to \boldsymbol{AC}; \\ F &\mapsto \operatorname{Sal}(\mathscr{A}[F]); \\ (m : F \to G) &\mapsto j_m : \operatorname{Sal}(\mathscr{A}[G]) \hookrightarrow \operatorname{Sal}(\mathscr{A}[F])\end{aligned}$$

where $j_m([G, C]) = [i_m(G), i_m(C)]$.

4.44 Lemma

$$\operatorname{colim}\mathscr{D}(\mathscr{A}) = \operatorname{Sal}(\mathscr{A})$$

4.45 Remark

Using Remark 4.31 we have that every element $\varepsilon \in \operatorname{colim}\mathscr{D}(\mathscr{A})$ has a (unique) representant $[F, C] \in \operatorname{Sal}(\mathscr{A}[F])$ such that for every other representant $[G, K]$ with $\varepsilon = [\![G, K]\!]$ there is a unique morphism $m : F \to G^{\downarrow}$ with $[G, K] = [F_m, i_m(C)]$.

In particular $\mathscr{D}$ is also a geometric diagram.

Proof of Lemma 4.44. The proof follows the outline of the proof of Lemma 4.42, the isomorphism $\Psi : \operatorname{Sal}(\mathscr{A}) \to \operatorname{colim}\mathscr{D}$ being defined as follows.

For an object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ (i.e. a morphism of $\mathcal{F}(\mathscr{A})$) define $\Psi(m) = \Phi(m) = [\![F_{id}, F_m]\!]$. For a morphism (n, m_1, m_2) of $\mathrm{Sal}(\mathscr{A})$ with $m_i : F_i \to C_i$ and $n : F_2 \to F_1$ define

$$\Psi(n, m_1, m_2) = [\![\mathscr{D}(n)([(F_1)_{id}, F_{m_1}]) \leq [(F_2)_{id}, F_{m_2}]]\!] =$$
$$[\![[i_n((F_1)_{id}), i_n(F_{m_1})] \leq [(F_2)_{id}, F_{m_2}]]\!] = [\![[F_n, F_{m_1 \circ n}] \leq [(F_2)_{id}, F_{m_2}]]\!],$$

where in the last equality we used Lemma 4.41. □

4.46 Remark

Note that, given any chamber C of $\mathscr{A}[G]$, $j_m : \mathcal{S}(\mathscr{A}[G])_C \hookrightarrow \mathcal{S}(\mathscr{A}[F])_{C'} \subseteq \mathcal{S}(\mathscr{A}[F])$ if and only if $S(i_m(C), C') \cap \mathscr{A}[G] = \emptyset$.

Proof. We have $j_m(\mathcal{S}(\mathscr{A}[G])) \subseteq S(\mathscr{A}[F])_{C'}$ if and only if

$$j_m([G_{id}, C]) = [F_m, i_m(C)] \leq [F_{id}, C'].$$

Since $F_{id} \subseteq F_m$ by definition, this is equivalent to $i_m(C)_{F_m} = C'_{F_m}$, which in turns means

$$\forall H \in \mathscr{A}[F] : \quad F_m \subseteq H \Longrightarrow \gamma_{i_m(C)}(H) = \gamma_{C'}(H),$$

that is $\forall H \in \mathscr{A}[G] \quad H \notin S(i_m(C), C')$. □

4.47 Lemma

Let $m : F \to G$ be a morphism of $\mathcal{F}(\mathscr{A})$ and consider an $(Y, C) \in \mathscr{Y}_F$. Then the inclusion $j_m : \mathrm{Sal}(\mathscr{A}[G]) \to \mathrm{Sal}(\mathscr{A}[F])$ restricts to an inclusion

$$j_m : \mathcal{S}_{\xi_G(Y,C)} \to \mathcal{S}_{\xi_F(Y,C)}.$$

Proof. We only need to show that $S(i_m(\xi_G(Y, C)), \xi_F(Y, C)) \cap \mathscr{A}[G] = \emptyset$. Let $H \in \mathscr{A}[G]$, then

$$\gamma_{i_m(\xi_G(Y,C))}(H) = \gamma_{\xi_G(Y,C)}(H) = \gamma_{\xi_F(Y,C)}(H) \Longrightarrow H \notin S(i_m(\xi_G(Y, C)), \xi_F(Y, C))$$

where the last equality follows from the fact that $\xi_F(Y, C) \subseteq \xi_G(Y, C)$. □

Lemma 4.47 allows us to state the following definition.

4.48 Definition

For any $(Y, C) \in \mathscr{Y}$ let

$$\mathscr{E}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{S}(\mathscr{A}[F])_{\xi_F(Y,C)}; \quad (m : F \to G) \mapsto (j_m)_{|\mathscr{E}_{(Y,C)}(G)}$$

4.49 Lemma

Let $(Y, C) \in \mathscr{Y}$, then

$$\operatorname{colim} \mathscr{E}_{(Y,C)} = \mathcal{S}_{(Y,C)}$$

Proof. We consider the isomorphism $\Psi : \mathrm{Sal}(\mathscr{A}) \to \operatorname{colim} \mathscr{D}$ of Lemma 4.44. We want to show that $\Psi(\mathcal{S}_{(Y,C)}) = \operatorname{colim} \mathscr{E}_{(Y,C)}$.

Let $[\![G, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)}$, then (recall Remark 4.45) there is a morphism of $\mathcal{F}(\mathscr{A})$ $m : F \to G$ such that $[F_m, i_m(K)] \in \mathcal{S}_{\xi_F(Y,C)} \subseteq \mathrm{Sal}(\mathscr{A}[F])$, i.e.

$$[F_m, i_m(K)] \leq [F, \xi_F(Y, C)].$$

Taking the preimage through Ψ of this relation we get a morphism

$$\Psi^{-1}([\![G, K]\!]) \to \Psi^{-1}([\![F, \xi_F(Y, C)]\!]) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})).$$

Now, using Proposition 4.26 we have

$$\begin{aligned}\theta(\Psi^{-1}([\![F, \xi_F(Y, C)]\!])) &= (X(F, \xi_F(Y, C)), \sigma_{\mathscr{A}[Y]}\xi_F(Y, C)) \\ &= (Y, \sigma_{\mathscr{A}[Y]}\mu[\mathscr{A}[Y], \mathscr{A}[F]]C) = (Y, C).\end{aligned}$$

Therefore $\Psi^{-1}([\![G, K]\!]) \in \mathcal{S}_{(Y,C)}$, so $[\![G, K]\!] \in \Psi\left(\mathcal{S}_{(Y,C)}\right)$ and we have proved that $\operatorname{colim} \mathscr{E}_{(Y,C)} \subseteq \Psi(\mathcal{S}_{(Y,C)})$.

To prove the converse inclusion, let $(m : G \to K) \in \mathcal{S}_{(Y,C)}$. Then there is a morphism $(h, m, n) : m \to n \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A}))$ with $n : F \to K'$, $h : F \to G$ and $\theta(n) = (Y, C)$. In particular, in view of Remark 4.34, we get $F_n = \xi_F(\theta(n)) = \xi_F(Y, C)$.

Applying Ψ to the morphism (h, m, n), in $\mathrm{Sal}(\mathscr{A}[F])$ we obtain

$$j_n([G, G_m]) \leq [F, F_n] = [F, \xi_F(Y, C)], \text{ thus } j_n([G, G_m]) \in \mathcal{S}_{\xi_F(Y,C)},$$

and we conclude that

$$\Psi(m) = [\![G, G_m]\!] = [\![j_n([G, G_m])]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)},$$

proving $\Psi(\mathcal{S}_{(Y,C)}) \subseteq \operatorname{colim} \mathscr{E}_{(Y,C)}$. □

4.50 Definition

$$\mathscr{G}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{N}_{\xi_F(Y,C)}; \quad (m : F \to G) \mapsto (j_m)_{|\mathscr{G}_{(Y,C)}(G)}$$

4.51 Remark

To prove that the diagram $\mathscr{G}_{(Y,C)}$ is well defined, we have to show that for every morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A}^Y)$ holds:

$$j_m(\mathcal{N}_{\xi_G(Y,C)}) \subseteq \mathcal{N}_{\xi_F(Y,C)}. \tag{4.2}$$

This follows because by Proposition 4.26 we have $X(F, \xi_F(Y, C)) = Y$, and thus with [19, Lemma 4.18] we can rewrite

$$\mathcal{N}_{\xi_F(Y,C)} = \{[G, K] \in \mathrm{Sal}(\mathscr{A}[F]) \mid G \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}}),\ K_G = \xi_F(Y, C)_G\}.$$

Now let $[G', C'] \in \mathcal{N}_{\xi_G(Y,C)}$. Then since $G' \subseteq \widetilde{Y}$ we have $i_m(G') \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})$, and from $\xi_F(Y,C) \subseteq \xi_G(Y,C)$ we conclude $i_m(C')_{G'} = \xi_F(Y,C)_{G'}$. Therefore $j_m([G',C']) = [i_m(G'), i_m(C')] \in \mathcal{N}_{\xi_F(Y,C)}$, and the inclusion (4.2) is proved.

4.52 Lemma

$$\operatorname{colim} \mathscr{G}_{(Y,C)} = \mathcal{N}_{(Y,C)}$$

Proof. First, we prove that $\operatorname{colim} \mathscr{G}_{(Y,C)} \subseteq \mathcal{N}_{(Y,C)}$. For this, let $[\![F,K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$ and suppose $[\![F,K]\!] \notin \mathcal{N}_{(Y,C)}$. Then $[\![F,K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ for some $(Y',C') < (Y,C)$. Now, since $[\![F,K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$ there exist a point $P \in \mathcal{F}(\mathscr{A})$ and a morphism $m : P \to F$ with $[P_m, i_m(K)] \in \mathcal{N}_{\xi_P(Y,C)}$. Therefore, in $\mathscr{A}[P]$ we have $[P_m, i_m(K)] \leq [P, \xi_P(Y,C)]$, which implies $K_{P_m} = \xi_P(Y,C)_{P_m}$, and thus $K = \sigma_{\mathscr{A}[F]}(K_{P_m}) = \xi_F(Y,C)$.

Similarly, since $[\![F,K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ there is a point $Q \in \mathcal{F}(\mathscr{A})$ and a morphism $n : Q \to F$ with $[Q_n, i_n(K)] \in \mathcal{S}_{\xi_Q(Y',C')}$. Then, as above, $K = \xi_F(Y',C')$

From the bijectivity proven in Lemma 4.27 we conclude $(Y,C) = (Y',C')$, which contradicts $(Y',C') < (Y,C)$, proving that $[\![F,K]\!] \in \mathcal{N}_{(Y,C)}$, as desired.

The other inclusion is easier. Suppose $[F,K] \in \mathcal{N}_{(Y,C)} \setminus \operatorname{colim} \mathscr{G}_{(Y,C)}$. Then $[F,K] \in \mathcal{S}_{\xi_P(Y',C')}$ for some point $P \in \mathcal{F}(\mathscr{A})$ and some $(Y',C') < (Y,C)$. But then $[F,K] \in \operatorname{colim} \mathscr{E}_{(Y',C')} \Rightarrow [F,K] \notin \mathcal{N}_{(Y,C)}$. □

4.53 Lemma
There is an equivalence of diagrams

$$\mathscr{G}_{(Y,C)} \cong \mathscr{F}(\mathscr{A}^Y)^{op}$$

Proof. For each $F \in \mathcal{F}(\mathscr{A}^Y)$ define the isomorphisms $\mathscr{G}_{(Y,C)}(F) \to \mathscr{F}(\mathscr{A}^Y)^{op}(F)$ as follows

$$\mathscr{G}_{(Y,C)}(F) = \mathcal{N}_{\xi_F(Y,C)} \cong \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})^{op} = \mathcal{F}(\mathscr{A}^Y[F])^{op} = \mathscr{F}(\mathscr{A}^Y)^{op}(F).$$

Where the isomorphism in the middle comes from Theorem 4.7.

It can be easily checked that these isomorphisms are indeed morphisms of diagrams. □

As a consequence of Lemma 4.53 we can write the following.

Proof of Theorem 4.38.

$$\mathcal{N}_{(Y,C)} = \operatorname{colim} \mathscr{G}_{(Y,C)} \cong \operatorname{colim} \mathscr{F}(\mathscr{A}^Y)^{op} = \mathcal{F}(\mathscr{A}^Y)^{op}.$$ □

4.4 Minimality of toric arrangements

In this section we will construct a perfect acyclic matching of the Salvetti category of a complexified toric arrangement. This implies minimality.

Perfect matchings for the compact torus

Let $\mathscr{A}$ be a complexified toric arrangement in T_Λ and choose a point $P \in \max \mathcal{C}(\mathscr{A})$. Up to a biholomorphic transformation we may suppose that P is the origin of the torus.

Let then $(\chi_1, 1), \ldots, (\chi_d, 1) \in \mathscr{A}$ be such that $\alpha_1, \ldots, \alpha_d$ are ($\mathbb{Q}$-) linearly independent. For $i = 1, \ldots, d$ let H_i^1 denote the hyperplane of $\mathscr{A}^\upharpoonright$ lifting K_i at the origin of $\mathrm{Hom}(\Lambda, \mathbb{R}) \simeq \mathbb{R}^d$. We identify for ease of notation $\Lambda \simeq \mathbb{Z}^d \subseteq \mathbb{R}^d$, and in particular think of α_i as the normal vector to H_i^1.

For $j \in [d]$ we consider the rank $j-1$ lattice

$$\Lambda_j := \mathbb{Z}^d \cap \bigcap_{i \geq j} H_i^1$$

4.54 Lemma
There is a basis $u_1, \ldots, u_d$ of Λ such that for all $i = 1, \ldots, d$, the elements $u_1, \ldots, u_{i-1}$ are a basis of Λ_i.

Proof. The proof is by repeated application of the Invariant Factor Theorem, e.g. [9, Theorem 16.18], to the free $\mathbb{Z}$-submodule Λ_j of Λ_{j-1}. □

The lattice Λ acts on $\mathbb{R}^d$ by translations. Given $u \in \Lambda$, let the corresponding translation be

$$t_u : \mathbb{R}^d \to \mathbb{R}^d; \qquad x \mapsto t_u(x) := x + u.$$

Let $(H_i^1)^+ := \{x \in \mathbb{R}^d \mid \langle x, \alpha_i \rangle \geq 0\}$.

4.55 Remark
In particular, $u_i \notin H_i^1$, hence $t_{u_i}(H_i^1) \neq H_i^1$. Moreover, without loss of generality we may suppose $u_i \in (H_i^1)^+$.

4.56 Corollary
For all $x \in \mathbb{R}^d$ and all $i < j \in [d]$, $\langle t_{u_i}(x), \alpha_j \rangle = \langle x, \alpha_j \rangle$.

Proof. We have $u_i \in \Lambda_j \subseteq H_j^1$, therefore $\langle u_i, \alpha_j \rangle = 0$ and thus

$$\langle t_{u_i}(x), \alpha_j \rangle = \langle x + u_i, \alpha_j \rangle = \langle x, \alpha_j \rangle + \langle u_i, \alpha_j \rangle = \langle x, \alpha_j \rangle + 0. \qquad \square$$

For $i = 1, \ldots, d$ let $(H_i^2)^+ := t_{u_i}((H_i^1)^+)$, and define

$$Q := \bigcap_{i=1}^{d} [(H_i^1)^+ \setminus (H_i^2)^+].$$

4.57 Lemma

The region Q is a fundamental region for the action of Λ on $\mathbb{R}^d$.

Proof. For $i = 1, \ldots, d$, write

$$l_i := \langle u_i, \alpha_i \rangle.$$

Then, $Q = \{x \in \mathbb{R}^d \mid 0 \leq \langle x, \alpha_i \rangle < l_i \text{ for all } i = 1, \ldots, d\}$. It is clear that Q can contain at most one point for each orbit of the action of Λ.

Now choose and fix an $x \in \mathbb{R}^d$. We want to construct an $y \in Q$ such that $x \in y + \Lambda$.

To this purpose write $x_0 := x$ and let $\lambda_d := \lfloor \langle x_0, \alpha_d \rangle / l_d \rfloor$. Then let

$$x_1 := x_0 - \lambda_d u_d, \text{ thus } 0 \leq \langle x_1, \alpha_d \rangle < l_d$$

For every $i \in \{1, \ldots d-1\}$ let now $\lambda_{d-i} := \lfloor \langle x_i, \alpha_{d-i} \rangle / l_{d-i} \rfloor$.

Then set $x_{i+1} := x_i - \lambda_{d-i} u_{d-i}$, so that

$$0 \leq \langle x_{i+1}, \alpha_{d-i} \rangle < l_{d-i}$$

and so, by Corollary 4.56, for every $j < i$:

$$\langle x_{i+1}, \alpha_{d-j} \rangle = \langle t_{u_{d-i}}^{-\lambda_{d-i}} \cdots t_{u_{d-j-1}}^{-\lambda_{d-j-1}}(x_{j+1}), \alpha_{d-j} \rangle = \langle x_{j+1}, \alpha_{d-j} \rangle \in [0, l_{d-j}[.$$

After d steps, we will have reached x_d, with

$$0 \leq \langle x_d, \alpha_i \rangle < l_i \text{ for all } i = 1, \ldots, d.$$

Hence $y := x_d \in Q$ is the required point because, putting $u := \sum_{i=1}^d \lambda_i u_i$, we have by construction $x_d = t_{-u}(x)$ and so $x = t_u(y) \in y + \Lambda$. $\square$

4.58 Definition

Let $\mathscr{A}$ be a rank d toric arrangement, and let $\mathcal{B}_d$ be the 'Boolean poset on d elements', i.e. the acyclic category on the subsets of $[d]$ with the inclusion morphisms. Since $\mathcal{B}_d$ is a poset, the function

$$\mathrm{Ob}(\mathcal{F}(\mathscr{A})) \to \mathrm{Ob}(\mathcal{B}_d), \qquad F \mapsto \{i \in [d] \mid F \subseteq K_i\},$$

induces a well defined functor of acyclic categories

$$\mathcal{I} : \mathcal{F}(\mathscr{A}) \to \mathcal{B}_d^{op}.$$

For every $I \subseteq [d]$ define the category

$$\mathcal{F}_I := \mathcal{I}^{-1}(I)$$

4.59 Lemma

For all $I \subseteq [d]$, the subcategory $\mathcal{F}_I$ is a poset admitting an acyclic matching with only one critical element (in top rank).

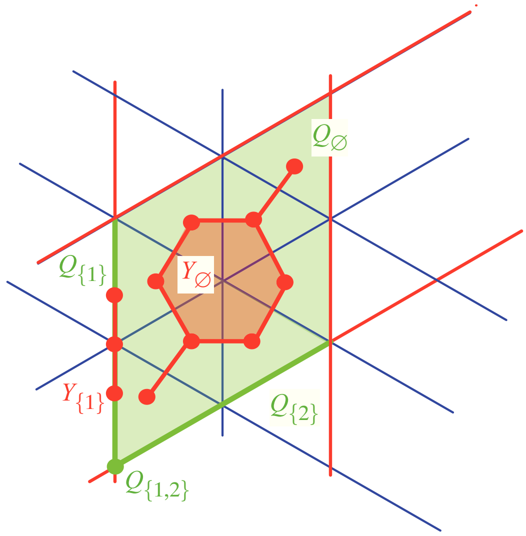


Figure 4.5: The case of the toric Weyl arrangement of Type A_2

Fix $I \subset [d]$, let $k := |I|$.
We consider

$$Q_I := Q \cap \big(\bigcap_{i \in I} H_i^1\big) \setminus \bigcup_{j \notin I} (H_j^1 \cup H_j^2).$$

The set $\mathscr{B} := \{H \cap X \mid H \in \mathscr{A}^{\upharpoonright}, H \cap Q \neq \emptyset\}$ is a finite arrangement of affine hyperplanes in the affine hull X of Q_I. This arrangement determines a (regular) polyhedral decomposition $\mathcal{D}(\mathscr{B})$ of $\mathbb{R}^{d-k}$ that coincides with $\mathcal{D}(\mathscr{A}^{\upharpoonright}{}_X)$ on Q.

The exponential covering of Section 3.5 maps Q_I homeomorphically to its image, hence $\mathcal{F}_I$ is the face category of the set of cells of the decomposition of Q_I by $\mathcal{D}(\mathscr{B})$. Regularity of $\mathcal{D}(\mathscr{B})$ implies that $\mathcal{F}_I$ is a poset. Indeed, if $\mathcal{D}(\mathscr{B})^\vee$ is the (regular) CW-decomposition dual to the one induced by $\mathscr{B}$, then $\mathcal{F}_I^{op}$ is the poset of cells of a subcomplex Y_I that is entirely contained in Q_I.

Let $\mathcal{Q}$ be the subdivision of the closure $\overline{Q_I}$ induced by $\mathscr{B}$.

4.60 Lemma

The complex $\mathcal{Q}$ is shellable.

Proof. Coning the arrangement $\mathscr{B}$ (as in [36, Definition 1.15]) we obtain a central arrangement $\widehat{\mathscr{B}} = \{\widehat{H} \mid H \in \mathscr{B}\}$ which subdivides the unit sphere into a regular cell complex $\mathcal{K}$. Then, $\mathcal{Q}$ is isomorphic to the subcomplex of $\mathcal{K}$ given by

$$\bigcap_{i \notin I} \widehat{H_i^1}^{+} \cap \bigcap_{i \notin I} \widehat{H_i^2}^{-}$$

which, by [4, Proposition 4.2.6 (c)], is shellable. □

Proof of Lemma 4.59. The pseudomanifold $\mathcal{Q}$ is constructible because it is shellable. With [3, Theorem 4.1], it is also endo-collapsible, i.e., it admits an acyclic matching where the critical cells are precisely the cells on the boundary plus one single cell in the interior of $\mathcal{Q}$. But this restricts to an acyclic matching of the subposet $\mathcal{F}_I \subseteq \mathcal{F}(\mathcal{Q})$ with exactly one critical cell.

In turn this gives an acyclic matching of $\mathcal{F}_I^{op}$ with exactly one critical cell. Since $\mathcal{F}_I^{op}$ is the face poset of the CW-complex Y_I, the critical cell must be in bottom rank - thus in top rank of $\mathcal{F}_I$, as required. □

4.61 Proposition
For any complexified toric arrangement $\mathscr{A}$, the acyclic category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching.

Proof. Let $\mathscr{A}$ be of rank d. The proof is a straightforward application of the Patchwork Lemma 2.39 in order to merge the 2^d acyclic matchings described in Lemma 4.59 along the map $\mathcal{I}$ of Definition 4.58. The resulting 'global' acyclic matching has 2^d critical elements and is thus perfect. □

Perfect matchings for the toric Salvetti complex

Let $\mathscr{A}$ be a (complexified) toric arrangement.

4.62 Proposition
The Salvetti Category $\operatorname{Sal}\mathscr{A}$ *admits a perfect acyclic matching.*

Proof. Let P denote the acyclic category given by the $|\mathscr{Y}|$-chain. We define a functor of acyclic categories

$$\varphi : \operatorname{Sal}\mathscr{A} \to P; \quad m \mapsto (Y, C) \text{ for } m \in \mathcal{N}_{(Y,C)}$$

and we have an isomorphism of acyclic categories $\varphi^{-1}((Y,C)) = \mathcal{N}_{(Y,C)} \simeq \mathcal{F}(\mathscr{A}_Y)^{op}$. Then, by Proposition 4.61, $\varphi^{-1}((Y,C))$ has an acyclic matching with $2^{d-\mathbf{rk}\,X}$ critical cells.

An application of the Patchwork Lemma 2.39 gives then an acyclic matching on $\operatorname{Sal}(\mathscr{A})$ with

$$\sum_j |\mathscr{Y}_j| 2^{d-j} = \sum_j |\mathcal{N}_j| 2^{d-j} = P_{\mathscr{A}}(1)$$

critical cells, where the first equality is given by Lemma 4.20. This matching is thus perfect. □

4.63 Corollary
The complement $M(\mathscr{A})$ is a minimal space.

4.64 Corollary
The homology and cohomology groups $H_k(M(\mathscr{A}), \mathbb{Z})$, $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion free for all k.

4.5 Application: minimality of affine arrangements

After the existence proofs of Dimca and Papadima in [21] and of Randell in [38], the first step towards an explicit characterization of the minimal model was taken by Yoshinaga [42] who, for complexified arrangements, identified the cells of the minimal complex using their incidence with a general position flag in real space and studied their incidence and boundary maps. Salvetti and Settepanella [41] obtained a complete description of the minimal complex by using a 'polar ordering' determined by a general position flag to define a perfect acyclic matching on the Salvetti complex.

In this section we use our techniques to extend to affine complexified hyperplane arrangements the idea of [19]. We thus obtain a minimal complex that is defined only in terms of the arrangement's (affine) oriented matroid.

Consider a finite affine complexified arrangement $\mathscr{A} = \{K_1, \dots, K_n\}$. Define the central arrangements $\mathscr{A}_0$ and $\mathscr{A}[F]$ for $F \in \mathcal{F}(\mathscr{A})$ in analogy to those of Section 4.2. Choose a base chamber $B \in \mathcal{T}(\mathscr{A}_0)$, fix a total ordering $\prec_0$ on $\mathscr{A}_0$ and define $\prec_F, \prec_Y$ for $F \in \mathcal{F}(\mathscr{A}), Y \in \mathcal{L}(\mathscr{A})$ as in Section 4.2. Moreover, let $\mathscr{Y}$ be as in Definition 4.18.

4.65 Remark

Notice that, given the affine oriented matroid of $\mathscr{A}$, the oriented matroid of $\mathscr{A}_0$ can be recovered without referring to the geometry. For instance, the tope poset of $\mathscr{A}_0$ can be defined in terms of the tope poset of $\mathscr{A}$ based at any unbounded chamber.

4.66 Lemma

Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement, and $\mathscr{Y}$ as above, then

$$|\mathscr{Y}| = \sum_{k\in\mathbb{N}} \textbf{\textit{rk}}\, H^k(M(\mathscr{A});\mathbb{Z})$$

Proof. As in Lemma 4.20, applying [19, Lemma 4.18 and Proposition 2] we have

$$|\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y,C) = Y\}| = \mathrm{rk}\, H^{\operatorname{codim} Y}(M(\mathscr{A}_Y);\mathbb{Z}) \quad \forall Y \in \mathcal{L}(\mathscr{A}).$$

The claim follows with Theorem 1.25. □

We now define the analogue of the map θ of Definition 4.33.

4.67 Definition

Let $F, G \in \mathcal{F}(\mathscr{A})$ with $F \subseteq G$ and identify

$$\mathscr{A}[F] = \mathscr{A}_F = \{H \in \mathscr{A} \mid F \subseteq H\},$$

in particular we have an inclusion $\mathscr{A}[G] \subseteq \mathscr{A}[F]$. *Define the map* $i_{F\leq G} : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F])$ *as follows*

$$\gamma_{i_{F\leq G}(J)}(H) = \begin{cases} \gamma_G(H) & \text{if } H \in \mathscr{A}[G] \\ \gamma_F(H) & \text{if } H \notin \mathscr{A}[G], \end{cases} \qquad \forall J \in \mathcal{F}(\mathscr{A}[G]).$$

As above, the map $i_{F\leq G}$ induces a function $j_{F\leq G} : \mathrm{Sal}(\mathscr{A}[F]) \to \mathrm{Sal}(\mathscr{A}[G])$.

4.68 Theorem (Lemma 3.2.8 and Theorem 4.2.1 of [18])
The assignment $\mathscr{E} : \mathcal{F}(\mathscr{A}) \to AC^{op}$, $\mathscr{E}(F) := \mathrm{Sal}(\mathscr{A}[F])$, $\mathscr{E}(F \leq G) = j_{F\leq G}$ *defines a diagram of posets such that* $\operatorname{colim} \mathscr{E}$ *is poset isomorphic to* $\mathrm{Sal}(\mathscr{A})$.

The stratification of $\mathrm{Sal}(\mathscr{A})$ is also defined along the lines of the preceding sections.

4.69 Definition
Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$ *as follows*

$$\theta([F,C]) = (X(F, i_{F\leq G}(G)), \sigma_{\mathscr{A}[X(F,i_{F\leq G}(G))]}(G)).$$

where we identified $G = \min \mathcal{L}(\mathscr{A}[G])$.

4.70 Definition
Let $\mathscr{A}$ *be a finite complexified affine hyperplane arrangement and define a total ordering* $\dashv$ *on* $\mathscr{Y}$ *as in Definition 4.21. Define:*

$$\mathcal{S}_{(Y,C)} = \left\{ [F,C] \in \mathrm{Sal}(\mathscr{A}) \,\middle|\, \begin{array}{c} \text{there is } [G,K] \in \mathrm{Sal}(\mathscr{A}) \text{ with} \\ [F,C] \leq [G,K] \text{ and } \theta([G,K]) = (Y,C) \end{array} \right\}$$

$$\mathcal{N}_{(Y,C)} = \mathcal{S}_{(Y,C)} \setminus \bigcup_{(Y',C') \dashv (Y,C)} \mathcal{S}_{(Y',C')}.$$

The arguments of Section 4.3 can now be adapted to the affine case, obtaining the following analog of Theorem 4.38.

4.71 Theorem
Let $\mathscr{A}$ *be a finite complexified affine hyperplane arrangement. There is an isomorphism of posets*

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op} \qquad \text{for all } (Y,C) \in \mathscr{Y}.$$

The analog of Proposition 4.61 is proved in [4, Theorem 4.5.7 and Corollary 4.5.8], from which it follows that the poset $\mathcal{N}^{op}_{(Y,C)}$ is shellable, and therefore $\mathcal{N}_{(Y,C)}$ admits an acyclic matching with one critical cell in top dimension. Applying the Patchwork Lemma as in Proposition 4.62 we obtain a perfect acyclic matching $\mathfrak{M}$ of $\mathrm{Sal}(\mathscr{A})$. We summarize.

4.72 Proposition
Let $\mathscr{A}$ *be a finite complexified affine hyperplane arrangement. The oriented matroid data of* $\mathscr{A}$ *define a discrete Morse function on* $\mathrm{Sal}(\mathscr{A})$ *that collapses the Salvetti complex to a minimal complex.*

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