GÖDEL'S INCOMPLETENESS THEOREMS WITH CONCATENATION INSTEAD OF ADDITION AND MULTIPLICATION

by Michael Deutsch (Nov. 22nd 2007)

In [6] we have replaced the axiom system of Gödel with its use of $0,1,+,\times$ by a simple system which uses as its only free variable the following predicate E:

$xEy \Leftrightarrow$ y has the digit 1 in the x^{th} position in the binary representation
(Counting the positions starts with position "0" from the righthand side.)

Nearer at hand than E is the use of the concatenation $\upsilon^\circ = \lambda xy\ xy$ (for the usual binary representations) and the two binary successor functions
$f_0{}^\circ = \lambda x\ x0$ and $f_1{}^\circ = \lambda x\ x1$.

We exclude 0 from the natural numbers because otherwise the concatenation function would not be associative. For example (10)0 would be 100, but 1(00) would be 10, if we identify 0...0 as usual with 0. The set of natural numbers without 0 is called $\mathbb{N}_1$. Like in [6] we do not use an induction scheme in the axiom system.

Notice, we use the usual binary representation, not the representation of [4].

"$\mathbb{V}$" means "either...or" . If we use it several times like in the first axiom, we mean that exactly one case is true.

Because of optical reasons we write **x0** instead of $\mathbf{f_0(x)}$ in the axiom system. But in fact **0** is not a variable. The free variables of the axiom systems are **1**, υ and f_0, where **1** is a individual variable, υ is a two-place function variable and f_0 is a one-place function variable. Therefore, the cases in A_2, A_3, A_4 that appear to be special cases for **0** are necessary.

AXIOM SYSTEM $\mathbb{M}^*$:

A_1 $\forall x\ (x=1\ \mathbb{V}\ \exists u\ x=u0\ \mathbb{V}\ \exists u\ x=u1)$

A_2 $\forall xyz\ (xz = yz \vee zx = zy \vee x0 = y0 \rightarrow x = y)$

A_3 $\forall xy\ (xy \neq x \wedge x0 \neq x)$

A_4 $\forall xyz\ x(yz) = (xy)z \wedge \forall xy\ x(y0) = (xy)0$

A_5 $\forall xyab\ (xy = ab \rightarrow y\text{END}b \vee b\text{END}y)$

xENDy is meaning $x=y \vee \exists z\ y=zx$.

In this case we call x an end of y. Please notice that the single end of 1000 (in binary representation) is 1000 itself and the only ends of 1001 are 1 and 1001.

DEFINITION.

1\. A first order formula $\mathcal{B}$ is called C-formula (concatenation formula) iff it contains as free variables only individual variables and υ and f_0.

We write **xy** sometimes instead of $\upsilon(x,y)$ and $f_1(x)$ instead of $\upsilon(x,1)$.

If we talk about variables we mean from now on only individual variables different from **1**. As variables we use **I**, **II**, **III**,.... . The number of strokes is called index. A C-formula is called n-place iff it contains exactly n different variables free.

2\. If the n-place C-formula $\mathcal{B}$ has as <u>distinct</u> free variables $x_1,...,x_n$ (ordered according to growing index), we write $\mathcal{B}[x_1,...,x_n]$ instead of $\mathcal{B}$. The denotation $\mathcal{B}(\tau_1,...,\tau_n)$ is used for the formula arising from $\mathcal{B}[x_1,...,x_n]$ by substituting τ_m for x_m $(1 \le m \le n)$ simultaneously (for any terms $\tau_1,...,\tau_n$).

3\. A term which can be generated from **1** by a finite number of transitions from τ

to $f_i(\tau)$ (i=0,1) is called binary representation. For any n of $\mathbb{N}_1$ we have exactly one binary representation **n°**.

4. In any model I on D the interpretations of **1**, $\mathbf{f_0}$, $\boldsymbol{\upsilon}$ (in bold letters) are denoted by 1, f_0, υ (in usual letters). Instead of $\upsilon(x,1)$ we write sometimes $f_1(x)$. Let $I°$ be the standard model on $\mathbb{N}_1$ and 1°, $f_0°$, $\upsilon°$ the interpretations of **1**, $\mathbf{f_0}$, $\boldsymbol{\upsilon}$ by $I°$. We write also $f_1°(x)$ instead of $\upsilon°(x,1°)$, and xy for $\upsilon°(x,y)$ resp. $\upsilon(x,y)$. For interpreted formulas we use usual print instead of bold print.

5. The n-place predicate P on $\mathbb{N}_1$ is called semirepresented in $\mathcal{M}^\wedge$ by the C-formula A iff A is n-place and for all natural numbers $i_1,\dots,i_n$

(a) in case of $Pi_1\dots i_n$ the formula $A(\mathbf{i_1°},\dots,\mathbf{i_n°})$ can be proved in $\mathcal{M}^\wedge$,

(b) the standard interpretation $I°$ on $\mathbb{N}_1$ is a model of $A(\mathbf{i_1°},\dots,\mathbf{i_n°})$ iff $Pi_1\dots i_n$.

6. A C-formula is called ∀-bounded iff it is generated from equations and negated equations by a finite number of the following steps:

a) from A,B to $(A\wedge B)$ or $(A\vee B)$,

b) from A to $\exists\mathbf{x}\, A$,

c) from A to $\forall\mathbf{x}\, (\neg\mathbf{xENDy} \vee A)$, also written $\forall\mathbf{x}_{\ \mathbf{xENDy}}\, A$.

LEMMA 1. In $\mathcal{M}^\wedge$ the following formulas are provable:

1. $\forall\mathbf{xy}\ \mathbf{xy} \neq \mathbf{1}$
2. $\forall\mathbf{xy}\ \mathbf{xy} \neq \mathbf{y}$
3. $\forall\mathbf{xyz}\ (\mathbf{xy} = \mathbf{f_0(z)} \rightarrow \exists\mathbf{u}\ (\mathbf{y=f_0(u)} \wedge \mathbf{xu = z}))$
4. $\forall\mathbf{xyz}\ (\mathbf{xy} = \mathbf{z1} \rightarrow \mathbf{y=1} \wedge \mathbf{x=z} \vee \exists\mathbf{u}\ (\mathbf{y=u1} \wedge \mathbf{xu = z}))$

PROOF.

We prove the validity of the formulas in any model I on D.

1. According to A_1 we have

$y=1 \veebar \exists u\ y=f_0(u) \veebar \exists u\ y=f_1(u)$.

With A_4 we get

$xy = f_i(u)$ for a certain u and i=0 or i=1, hence with A_1 the proposition.

2. Let be xy = y, thus

xy = x(xy), thus with A_4

xy = (xx)y, thus with A_2

x = xx , contradiction to A_3.

3. Let be $xy = f_0(z)$. With A_1 we get

$y=1 \veebar \exists u\ y=f_0(u) \veebar \exists u\ y=f_1(u)$, thus with A_4 and A_1

$y=f_0(u)$ for some u, thus with A_4

$xy = f_0(xu)$,

thus with A_2

xu = z.

4. Let be xy = z1.

In case of y=1 we have with A_2

x=z.

In case of y = u1 we have with A_4

xy = (xu)1, thus with A_2

xu = z.

In case of $y = f_0(u)$ we have with A_4

$xy = f_0(xu)$.

Contradiction to A_1.

LEMMA 2.
1. For any x,y of $\mathbb{N}_1$ with x=y resp. ¬x=y the formulas $x°=y°$ resp. $¬x°=y°$ are provable in $\mathcal{M}^*$.
2. In any model I on D of the axioms we write D_0 for the intersection of all subdomains of D containing 1 and for any x also $f_0(x)$ and $f_1(x)$.
Then all axioms are valid on D_0. The restriction of I to D_0 is isomorphic to the standard model $I°$ on $\mathbb{N}_1$. A closed ∀-bounded formula is valid for I on D, if it is valid for the restriction of I to D_0.

PROOF.
1. We have to prove the validity of the formulas in any model I on D. For x=y the proposition is evident. For x≠y the assumption $x°=y°$ leads to a contradiction because of the first two axioms.
2. For the restriction of I to D_0 the following second order axiom is valid:
A_6 $\forall M\ (M1 \wedge \forall x(Mx \rightarrow Mf_0(x) \wedge Mf_1(x)) \rightarrow \forall x\ Mx)$.
Furthermore we use the following axiom instead of A_2:
A_2* $\forall xy\ (f_i(x) = f_i(y) \rightarrow x = y)$ i=0,1
We show first that the axiom system $\{A_1, A_2^*, A_6\}$ is monomorphic by defining an isomorphism ϕ relative to $1, f_0, f_1$ from the standard interpretation $I°$ on $\mathbb{N}_1$ to the interpretation I restricted to D_0. Let
$\phi(1°) = 1$
$\phi(f_0°(x)) = f_0(\phi(x))$
$\phi(f_1°(x) = f_1(\phi(x))$.
Because of the definition we have just to show that ϕ is a one-to-one mapping onto D_0.
1) By induction on x we show
$x \neq y \rightarrow \phi(x) \neq \phi(y)$.
a) For x=1° and $y=f_i°(u)$ for some u and 0≤i≤1 we have $\phi(x)=1$ and
$\phi(y) = f_i(\phi(u))$, thus $\phi(y) \neq 1$ using A_1.
b) Let $x=f_i°(u)$ for some u and x≠y. For y=1° compare a). Now let $y=f_k°(v)$ for some v and some k with 0≤k≤1. Because of x≠y we have i≠k or i=k ∧ u≠v.
For i≠k we get $\phi(x) \neq \phi(y)$ with A_1.
For i=k ∧ u≠v we assume $\phi(x)=\phi(y)$. With A_2* we get $\phi(u)=\phi(v)$, hence with the induction hypothesis u=v. Contradiction.
2) Because of A_6 the set of values of ϕ is D_0.
3) Because of A_4 the values of the function υ are in D_0 if the arguments belong to D_0. ϕ is also an isomorphism from $I°$ on $\mathbb{N}_1$ to I restricted to D_0 relative to υ, that means: For any x,y of $\mathbb{N}_1$
$\phi(\upsilon°(x,y)) = \upsilon(\phi(x),\phi(y))$.
This is easily shown by induction on y.
Therefore the axioms are also valid for the restriction of I to D_0.
4) We prove for any x of D by induction on z:
$z \in D_0 \wedge x\mathrm{END}z \rightarrow x \in D_0$.
a) For z=1 we get the proposition with Lemma 1.1.

b) Induction step from z to $f_i(z)$ for i=0,1.
For $x\mathrm{END}f_i(z)$ we get $x=f_i(z)$ (thus $x \in D_0$) or
$yx = f_i(z)$ for some y.
For i=0 we have $x=f_0(u)$ for some u according to Lemma 1.3 and
$yu = z$, because of the induction hypothesis $u \in D_0$ and therefore $x \in D_0$.
For i=1 we get with Lemma 1.4
$x=1 \land y=z$, thus $x \in D_0$ or
$x = u1$ for some u and
$yu = z$,
thus because of the induction hypothesis $u \in D_0$ and therefore $x \in D_0$.

DEFINITION.
1) We write $\underline{x}$ instead of $x_1 \ldots x_n$ resp. $x_1, \ldots, x_n$.
2) Let be $\mathbb{N}_{SR}$ the set of natural numbers of $\mathbb{N}_1$, having only the digit 1 in the binary representation (series of strokes).
3) A predicate on $\mathbb{N}_1$ belongs to RE^ iff it can be generated from equations and negated equations by a finte number of the following steps:
a) the composition of predicates with $\land$ or $\lor$,
b) the use of an unbounded $\exists$-quantifier,
c) the use of a bounded $\forall$-quantifier, that means the transition from Q to
$P b\underline{x} \Leftrightarrow \forall a_{\, a\mathrm{END}^\circ b}\, Q\underline{x}a$
where
$x\mathrm{END}^\circ y \Leftrightarrow x=y \lor \exists z\, y=\upsilon^\circ(z,x)$,
d) permutation or identification of variables.
In the equations and negated equations function terms are permitted only if they consist just of variables, 1°, $f_0{}^\circ$ and υ°.

ABBREVIATIONS.

xBEGy	for $\mathbf{x = y \lor \exists z\, xz = y}$	(x is a beginning of y)
10	for $\mathbf{f_0(1)}$	
100	for $\mathbf{f_0(f_0(1))}$	
101	for $\mathbf{f_1(f_0(1))}$	
¬100BEGz	for **z=1 ∨ z=10 ∨ 11BEGz ∨ 101BEGz**	
SRz	for $\mathbf{1ENDz \land \forall u_{\,uENDz}\, \exists v\, (z=vu \land 1ENDv)}$	(z is a series of strokes)
xPWy	for $\mathbf{xBEGy \lor xENDy \lor \exists uv\; y = uxv}$	(x is part of y)
<x,y>	for $\mathbf{f_0(x)f_0(y)}$	

For the corresponding abbreviations in an arbitrary model I on D we choose usual letters instead of bold letters. If I is the standard model I° on $\mathbb{N}_1$, we add sometimes the symbol ° in order to distinguish two models.

THEOREM 1.
Every recursively enumerable predicate on $\mathbb{N}_{SR}$ is semirepresented by a $\forall$-bounded formula in $\mathbb{M}$^.

PROOF.
We have to show that the graph of any primitive recursive function on $\mathbb{N}_{SR}$ belongs to RE^. The demand 5(a) in the definition ahead of Lemma 1 is a consequence of 5(b) according to Lemma 2.2 because we use $\forall$-bounded formulas.
We omit here the symbol °.

1) $y=1$ belongs to RE^*.
2) The graph of the successor function on $\mathbb{N}_{SR}$ has on $\mathbb{N}_1$ the representation
$SRx \wedge y = f_1(x)$.
3) The graph of $\lambda x_1...x_n\ x_i$ on $\mathbb{N}_{SR}$ has on $\mathbb{N}_1$ the representation
$SRx_1 \wedge ... \wedge SRx_n \wedge y=x_i$.
4) The graph of $f(\underline{x}) = h(g_1(\underline{x}),...,g_k(\underline{x}))$ on $\mathbb{N}_{SR}$ has the representation
$z = f(\underline{x}) \leftrightarrow \exists y_1...y_m\ (z = h(y_1,...,y_m) \wedge y_1 = g_1(\underline{x}) \wedge ... \wedge y_m = g_m(\underline{x}))$.
5) Let $f(\underline{x},1) = g(\underline{x})$, $f(\underline{x},y') = h(\underline{x},y,f(\underline{x},y))$ on $\mathbb{N}_{SR}$.
The graph of f has on $\mathbb{N}_1$ the representation

$$
\begin{aligned}
& z=f(\underline{x},y) \leftrightarrow \\
& \exists a\ (100<y,z>BEGa \\
& \quad \wedge \forall b_{\ bENDa}\ (\neg 100BEGb \\
& \qquad\qquad \vee \exists cd\ (100<c,d>BEGb \\
& \qquad\qquad\quad \wedge (c=1 \wedge d=g(\underline{x}) \\
& \qquad\qquad\qquad \vee \exists uvw\ (c=f_1(u) \wedge d=h(\underline{x},u,v) \wedge wENDb \wedge 100<u,v>BEGw)))))
\end{aligned}
$$

"$\rightarrow$":
We choose $a = 100<y,f(\underline{x},y)>100<y-1,f(\underline{x},y-1>100...100<2,f(\underline{x},2)>100<1,f(\underline{x},1)>$.
"$\leftarrow$":
We show by induction on the length of the binary representation of b
$bENDa \wedge 100<p,q>BEGb \rightarrow SRp \wedge SRq \wedge q=f(\underline{x},p)$.
Let
$bENDa \wedge 100<p,q>BEGb.$ (*)
Then there are c,d with $100<c,d>BEGb$, hence $c=p$ and $d=q$.
α) For the shortest b with (*) we have
$c=1 \wedge d=g(\underline{x})$, hence $d=f(\underline{x},1)$.
β) Induction step:
For $c=1$ we get again $d=g(\underline{x})$. Otherwise we have for some u,v,w
$c=f_1(u) \wedge d=h(\underline{x},u,v) \wedge wENDb \wedge 100<u,v>BEGw$.
Because of $100<u,v>BEGw$ we have $w \neq b$. According to the induction hypothesis we have
$SRu \wedge SRv \wedge v=f(\underline{x},u)$.
Thus we get
$SRc \wedge SRd \wedge d=f(\underline{x},c)$. q.e.d.

For the proof of Gödel's two incompleteness theorems we assume any Gödelnumbering of the C-formulas with Gödelnumbers in $\mathbb{N}_{SR}$. Furthermore we define a Gödelnumbering of all proofs of C-formulas from a set U of C-formulas.

We choose a correct and complete first order calculus where any rule has not more than two premises and where the following two predicates H and K are recursive:
H is the set of Gödelnumbers of C-formulas beeing logical axioms of the calculus.
$Kxyz \leftrightarrow$
x,y,z are Gödelnumbers of C-formulas and the C-formula of z is gained from the C-formulas of x and y by a rule of the calculus.
Let G be the set of Gödelnumbers of C-formulas.
Proofs of C-formulas in (from) a set U of C-formulas are finite series $\mathcal{C}_1,...,\mathcal{C}_k$ of C-formulas where for any i with $1 \leq i \leq k$:
$\mathcal{C}_i \in U$, or $\mathcal{C}_i$ is a logical axiom, or there are r,s with $1 \leq r,s \leq i-1$, such that $\mathcal{C}_i$ is gained from $\mathcal{C}_r$ and $\mathcal{C}_s$ by a rule of the calculus.

As Gödelnumber of the proof $C_1,\ldots,C_k$ we choose the natural numbar
$100f_0°(a_k)100f_0°(a_{k-1})100\ldots\ldots\ldots\ldots.100f_0°(a_1)1$,
where $a_1,\ldots,a_k$ are the Gödelnumbers of $C_1,\ldots,C_k$ in $\mathbb{N}_{SR}$.

The following predicates S,F are decidable and therefore recursive according to Church's thesis. If we want to prove this without using Church's thesis, we have to specify the Gödelnumbering of the C-formulas. We omit these details.

Let M be the set of Gödelnumbers of C-formulas belonging to U.

DEFINITION.
1) $S_M ac$ ⇔ a is the Gödelnumber of a C-formula and
c is the Gödelnumber of a proof in U of the formula belonging to a
We use this predicate only for a recursive M.
In case of U=$\mathbb{M}$^, we write Sac instead of $S_M ac$.
Let $\mathcal{S}$[x,y] be a ∀-bounded formula semirepresenting the predicate S. (Because the Gödelnumbers of a proof do not belong to $\mathbb{N}_{SR}$ we can not use Theorem 1 for getting such a formula. But later on we will define this formula.)
2) Bac ⇔ a is the Gödelnumber of a C-formula and
c is the Gödelnumber of a proof in $\mathbb{M}$^ of the diagonal formula of a
The diagonal formula of a one-place C-formula $\mathcal{A}$[x] with the Gödelnumber a is the formula $\mathcal{A}$(a°). The diagonal formula of any other formula is the formula itself.
3) If x is the Gödelnumber of a one-place C-formula, let f(x) be the Gödelnumber of the diagonal formula. Otherwise we choose f(x)=x.
Fxy :⇔ y = f(x).
Let $\mathcal{F}$[x,w] be a ∀-bounded formula semirepresenting the predicate F.
4) $\mathcal{B}$[x,y] ≡ ∀w ($\mathcal{F}$[x,w] → $\mathcal{S}$(w,y))
The formula $\mathcal{B}$ is <u>not</u> ∀-bounded. But in the standard model $I°$ on $\mathbb{N}$ we have:
$\mathcal{B}$[x,y] is valid for $I°$ iff $BI°(x)I°(y)$.
However, we have not a semirepresentation.
5) We call b the Gödelnumber of the formula ¬∃y $\mathcal{B}$[x,y] and d the Gödelnumber of the diagonal formula ¬∃y $\mathcal{B}$(b°,y) .
6) We call e the Gödelnumber of the formula ∃x ¬x=x .

THEOREM 2 (GÖDEL'S FIRST INCOMPLETENESS THEOREM FOR $\mathbb{M}$^):
If $\mathbb{M}$^ is consistent, there is no proof in $\mathbb{M}$^ of the formula ¬∃y $\mathcal{B}$(b°,y).
However, in the standard model $I°$ on $\mathbb{N}_1$ the formula ¬∃y $\mathcal{B}$(b°,y) is valid, i.e.
¬∃y Bby.

PROOF. For any proof of ¬∃y $\mathcal{B}$(b°,y) we get ∃y Bby. But the formula is true in $I°$ when it is provable, i.e ¬∃y Bby. (According to [1] and Lemma 2.2 $\mathbb{M}$^ is valid in $I°$ if it is consistent.)

¬∃y Sey means that there is no proof of ∃x ¬x=x.. This is a very natural formulation of the consistency of $\mathbb{M}$^.
Gödel's second incompleteness theorem asserts for a natural choice of the C-formula $\mathcal{S}$ that also ¬∃y $\mathcal{S}$(e°,y) is not provable in $\mathbb{M}$^ (if $\mathbb{M}$^ is consistent), that means a formula expressing the consistency in the standard model on $\mathbb{N}_1$ in a natural way. For the second incompleteness theorem we need a proof in $\mathbb{M}$^ of
¬∃y $\mathcal{S}$(e°,y) → ¬∃y $\mathcal{B}$(b°,y) .
We first show that there is a proof in $\mathbb{M}$^ of
¬∃y $\mathcal{S}$(d°,y) → ¬∃y $\mathcal{B}$(b°,y) .

For otherwise we would have in a model I both $\neg\exists y\ \mathcal{S}(d^\circ,y)$ and $\exists y\ \mathcal{B}(b^\circ,y)$. Because of the semirepresentation of F by $\mathcal{F}$ we have $\mathcal{F}(b^\circ,d^\circ)$ in I, because of $\exists y\ \mathcal{B}(b^\circ,y)$ therefore $\exists y\ \mathcal{S}(d^\circ,y)$. Contradiction.
Thus we have to prove in $\mathbb{M}^\wedge$

$\neg\exists y\ \mathcal{S}(e^\circ,y) \rightarrow \neg\exists y\ \mathcal{S}(d^\circ,y)$, i.e.

$\exists y\ \mathcal{S}(d^\circ,y) \rightarrow \exists y\ \mathcal{S}(e^\circ,y)$. (°)

Because of the First Incompleteness Theorem we have $\neg\exists y$ Sey $\rightarrow \neg\exists y$ Sdy, i.e.

$\exists y$ Sdy $\rightarrow \exists y$ Sey . (°°)

To prove (°) we have so show the validity of (°) in any model of $\mathbb{M}^\wedge$, whereas (°°) asserts the validity only in the standard model.

LEMMA 3. If I is the standard model I° on $\mathbb{N}_1$, we have:

$S_M$ac $\leftrightarrow$ Ga $\wedge$ $100f_0(a)$BEGc
$\wedge\ \forall b$ bENDc $\exists$upq ($\neg$100BEGb
$\vee$ b $=100f_0(u)1 \wedge$ (Mu $\vee$ Hu)
$\vee$ b $=100f_0(u)100f_0(p)$q
$\wedge$ (Mu $\vee$ Hu
$\vee\ \exists$vw($100f_0(v)$PW$100f_0(p)$q $\wedge$ $100f_0(w)$PW$100f_0(p)$q $\wedge$ Kwvu)))

PROOF.
"$\rightarrow$": Because of $S_M$ac we have

c = $100f_0(a_k)100f_0(a_{k-1})100$..............$100f_0(a_1)1$, where

$a_k,\dots,a_1$ is a series of Gödelnumbers of C-formulas with a_k=a and for $0\le i\le k$:
$a_i \in M$, or Ha_i, or there are r,s with $1\le r,s\le i-1$ and $Ka_ra_sa_i$.
The proposition is verified easily.
"$\leftarrow$":
Let
bENDc $\wedge$ 100BEGb. (1)
Then there are u,p,q with
b $=100f_0(u)1 \wedge$ (Mu $\vee$ Hu)
$\vee$ b $=100f_0(u)100f_0(p)$q
$\wedge$ (Mu $\vee$ Hu $\vee\ \exists$vw($100f_0(v)$PW$100f_0(p)$q $\wedge$ $100f_0(w)$PW$100f_0(p)$q $\wedge$ Kwvu))
By induction on the length (number of digits) of b we show
$S_M$ub. (2)
α) For the shortest b with (1) we have
b $=100f_0(u)1 \wedge$ (Mu $\vee$ Hu),
thus (2).
β) Induction step: We have b $=100f_0(u)100f_0(p)$q and
Mu $\vee$ Hu $\vee\ \exists$vw($100f_0(v)$PW$100f_0(p)$q $\wedge$ $100f_0(w)$PW$100f_0(p)$q $\wedge$ Kwvu) .
With the induction hypothesis we get the proposition.

LEMMA 4. In $\mathbb{M}^\wedge$ the following formulas are provable:
1. **$\forall$abcd (ab = cd $\rightarrow$ aBEGc $\wedge$ dENDb $\vee$ cBEGa $\wedge$ bENDd)**
2. **$\forall$ab ab $\neq$ 100**

PROOF. We show that the formulas are valid in any model of $\mathbb{M}^\wedge$ by using $\mathcal{A}_5$.
1. a) Let ab=cd $\wedge$ dENDb.
For b=d we get with $\mathcal{A}_2$ a=c, thus aBEGc.
For b$\neq$d we get for a certain u

b=ud, thus
aud = cd, hence with A_2
au = c, thus again aBEGc.
b) Let ab=cd ∧ ¬dENDb, thus according to A_5
bENDd ∧ b≠d, thus for some u
d = ub, thus
ab = cub, thus with A_2
a = cu, thus cBEGa.
2. With ab = $f_0(f_0(1))$ we get for some u
b = $f_0(u)$, thus
au = $f_0(1)$, thus for some v
av = 1, contradicting Lemma 1.1.

LEMMA 5. In $\mathcal{M}^{\wedge}$ the following formula is provable:
bENDy_1y_2 ∧ 100BEGb
→ bENDy_2 ∨ ∃p (b=py_2 ∧ pENDy_1 ∧ 100BEGp)

PROOF. We show the validity of the formula in any model of $\mathcal{M}^{\wedge}$.
Case 1: Let b=y_1y_2.
We choose p=y_1 and show 100BEGp:
Because of 100BEGy_1y_2 and Lemma 4.2 we have for some v
100v = y_1y_2.
With Lemma 4.1 we get
1.1 100BEGy_1 ∧ y_2ENDv or
1.2 y_1BEG100 ∧ vENDy_2.
Case 1.1: 100BEGp is evident.
Case 1.2: According to Lemma 4.2 we have y_1=100, thus again 100BEGp.
Case 2: Let ub = y_1y_2.
According to Lemma 4.1 we get
uBEGy_1 ∧ y_2ENDb ∨ y_1BEGu ∧ bENDy_2 .
For bENDy_2 the proposition is proved. Therefore we can assume
uBEGy_1 ∧ b=py_2 for some p.
Because of 100BEGb and Lemma 4.2 we have
py_2 = 100w.
Like in case 1 we get
100BEGp.
With ub=y_1y_2 and b=py_2 we get also
upy_2 = y_1y_2, thus with A_2
up = y_1, thus
pENDy_1. q.e.d.

Now we choose ∀-bounded formulas *G, M, H, K* semirepresenting the predicates G, M, H, K (compare Theorem 1).

$S_M[x,y] :\equiv G(x) \wedge 100f_0(x)BEGy$
$\wedge \forall b_{\ bENDy}\ \exists upq\ (\neg 100BEGb$
$\vee\ b = 100f_0(u)1 \wedge (M(u) \vee H(u))$
$\vee\ b = 100f_0(u)100f_0(p)q$
$\wedge\ (M(u) \vee H(u)$
$\vee\ \exists vw(100f_0(v)PW100f_0(p)q \wedge 100f_0(w)PW100f_0(p)q \wedge K(w,v,u))))$

According to Lemma 3 the predicate S_Mac is semirepresented by the ∀-bounded formula $S_M[x,y]$. Special cases:

1) M is the set M° of Gödelnumbers of the formulas of $\mathcal{M}^*$. Then we have $S_Mac \Leftrightarrow Sac$. Instead of $S_{M^\circ}[x,y]$ we write $S[x,y]$.

2) M is the set $M^{\circ\circ} = M^\circ \cup \{d\}$; as $M[p]$ we chose $M^\circ[p] \vee p = d^\circ$, where the ∀-bounded formula $M^\circ[p]$ is semirepresenting the set M°.

THEOREM 3 (GÖDEL'S SECOND INCOMPLETENESS THEOREM FOR $\mathcal{M}^*$):
If $\mathcal{M}^*$ is consistent, there is no proof of $\neg\exists y\ S(e^\circ,y)$ in $\mathcal{M}^*$.

PROOF. The denotations I,D,D_0 are used as in Lemma 2.2.
We have to show the validity of (°) for I. Because of (°°) we have $S_{M^{\circ\circ}}(e^\circ,y)$ in the standard model I° on $\mathbb{N}_1$ for a certain choice of $I^\circ(y)$, therefore also for the restriction of I to D_0 with a certain choice of $I(y)=y_1$ (compare Lemma 2.2), therefore also for I on D, because the formula is ∀-bounded. Furthermore we assume the validity of $S(d^\circ,y)$ in I for a certain choice of $I(y)=y_2$.
Let $y_3 = y_1y_2$.
Proposition: $S(e^\circ,y)$ is valid in I for $I(y)=y_3$. Obviously e is Gödelnumber of a C-formula, and we have $100f_0(e^\circ)BEGy_1y_2$ because of $100f_0(e^\circ)BEGy_1$.
Therefore we have just to prove for any b with $bENDy_1y_2 \wedge 100BEGb$ for some u,p,q:

$b = 100f_0(u)1 \wedge (M^\circ(u) \vee H(u))$ (1)
$\vee\ b = 100f_0(u)100f_0(p)q$ (2)
$\wedge(M^\circ(u) \vee H(u) \vee \exists vw(100f_0(v)PW100f_0(p)q \wedge 100f_0(w)PW100f_0(p)q \wedge K(w,v,u)))$

According to Lemma 5 we have one of the following two cases:
1) $bENDy_2$
2) $b=py_2 \wedge pENDy_1 \wedge 100BEGp$ for some p.
In the first case we get (1)∨(2) from the validity of $S(d^\circ,y)$ in I for $I(y)=y_2$.
In the second case we get from the validity of $S_{M^{\circ\circ}}(e^\circ,y)$ in I for $I(y)=y_1$ some r,s,t with
$p = 100f_0(t)1 \wedge (M^\circ(t) \vee t=d^\circ \vee H(t))$ or
$p = 100f_0(t)100f_0(r)s \wedge (M^\circ(t) \vee t=d^\circ \vee H(t)$
$\vee\ \exists vw(100f_0(v)PW100f_0(r)s \wedge 100f_0(w)PW100f_0(r)s \wedge K(w,v,u)))$
For u = t we get (2) because of $b=py_2$, using in case of $t=d^\circ$ again the validity of $S(d^\circ,y)$ in I for $I(y)=y_2$.

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