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A SIMPLER PROOF AND A REFINING OF GÖDEL'S SECOND INCOMPLETENESS THEOREM

We want to replace Gödel's symbols 0,1,+,× by a single predicate symbol refering to the binary representation of natural numbers:

xEy :⇔ y has the digit 1 in the x^{th} position in the binary representation
(Counting the positions starts with position "0" from the righthand side.)

Advantages of the new formulation of the incompleteness theorems:

1. an easy proof of the second incompleteness theorem,
2. very simple axioms which allow also a direct set theoretical interpretation.

The proofs of both incompleteness theorems are also valid for any stronger system.

SYSTEM $\mathbb{M}$:

A_1 ∀xy (∀z (zEx ⇔ zEy) → x=y)
A_2 ∃z ∀u ¬uEz
A_3 ∀x∃z∀u (uEz ⇔ u=x)
A_4 ∀xy∃z∀u (uEz ⇔ uEx ∨ uEy)

DEFINITION. 1. Let $\mathbb{H}$ be the domain of hereditary finite sets, that means the sets generated in the following way:

a) Ø is a member of $\mathbb{H}$,
b) with all x,y also {x} and x∪y are members of $\mathbb{H}$.

2. As one-to-one mapping of $\mathbb{H}$ onto $\mathbb{N}$ (Gödel numbering) we use the following Φ:
Φ(Ø) = 0,
$\Phi(\{x\}) = 2^{\Phi(x)}$,
Φ(x∪y) = Φ(x)+Φ(y) if x∩y=Ø.
Then we have for all members x,y of $\mathbb{H}$:
x∈y iff Φ(x)EΦ(y).

3. A first order formula *A* is called E-formula iff it does not contain function symbols and at most E as predicate symbol. We call it n-place formula iff it contains exactly n different variables free. As variables we use **I**, **II**, **III**,... . The number of lines is called index. Formulas of the type **xEy** and **x=y** are called prime formulas.

4. E-formulas are called ∀-bounded iff they are generated from prime formulas and negated prime formulas by a finite number of the following steps:
a) from *A*,*B* to (*A*∧*B*) or (*A*∨*B*),
b) from *A* to ∃x *A*,
c) from *A* to ∀x (¬xEy ∨ *A*) ($\forall x_{\,xEy}$ *A*).

5. For any i∈$\mathbb{N}$ and any variable y we define a ∀-bounded formula **y = i°** as follows:
y = 0° ≡ $\forall u_{\,uEy}$ ¬uEy
For mEi ⇔ $m=m_1$ ∨...∨ $m=m_k$ we define
y = i° ≡
∀u (¬uEy ∨ $u=m_1$° ∨...∨ $u=m_k$°) ∧ ∃u ($u=m_1$° ∧ uEy) ∧...∧ ∃u ($u=m_k$° ∧ uEy).
For all m,n∈$\mathbb{N}$ we write m° = n° instead of ∃y (y = m° ∧ y = j°).

6. If the n-place E-formula *B* has as <u>distinct</u> free variables $x_1,...,x_n$ (ordered according to growing index), we write *B*[$x_1,...,x_n$] instead of *B*. The denotation *B*($y_1,...,y_n$) is used for the formula arising from *B*[$x_1,...,x_n$] by substituting y_m for x_m ($1 \le m \le n$) simultaneously (for <u>all</u> variables $y_1,...,y_n$). For any m with $1 \le m \le n$ and any i∈$\mathbb{N}$ the denotation *B*($x_1,...,x_{m-1}$, i°, $x_{m+1},...,x_n$) is used for the formula

$\exists x_m(x_m=i^\circ \wedge \mathcal{B}[x_1,...,x_n])$.
We use analogously repeated replacements, for example for natural numbers i_1,i_2,i_3 and any variables $y_4,...,y_n$ the denotation $\mathcal{B}(i_1{}^\circ,i_2{}^\circ,i_3{}^\circ,y_4,...,y_n)$.
7. The n-place predicate P on $\mathbb{N}$ is called semirepresented in $\mathbb{M}$ by the E-formula $\mathcal{A}$ iff $\mathcal{A}$ is n-place and for all natural numbers $i_1,...,i_n$
a) in case of $Pi_1...i_n$ the formula $\mathcal{A}(i_1{}^\circ,...,i_n{}^\circ)$ can be proved in $\mathbb{M}$,
b) the standard interpretation I° on $\mathbb{N}$, which interprets E by E, is a model of $\mathcal{A}(i_1{}^\circ,...,i_n{}^\circ)$ iff $Pi_1...i_n$.
8. Let AL be the class of predicates on $\mathbb{H}$, which can be generated by $\neg,\wedge$ from equations containing only functions on $\mathbb{H}$ obtained from $\lambda\emptyset$, $\lambda x\ \{x\}$, $\lambda xy\ x\cup y$ and $\lambda x_1...x_n\ x_i$ $(1\le i\le n)$ by simultaneous substitution.

LEMMA: For all $x,y\in\mathbb{N}$ with $x=y$ resp. $\neg x=y$ we can prove in $\mathbb{M}$ the formula $x^\circ=y^\circ$ resp. $\neg x^\circ=y^\circ$.

PROOF: We have to show that the mentioned formulas are true in any model I of $\mathbb{M}$ on D. We write $\mathbb{E}$ for $I(E)$ and D_0 for the intersection of all subdomains of D where the axioms $\mathcal{A}_1,...,\mathcal{A}_4$ are valid.
α) $0^\wedge$ designates the $x\in D_0$ with $\forall u\ \neg u\mathbb{E}x$ (compare $\mathcal{A}_1,\mathcal{A}_2$).
β) Let be $i\neq 0$ and $mEi \leftrightarrow m=m_1 \vee ... \vee m=m_k$. Then $i^\wedge$ designates the $x\in D_0$ with $\forall u\ (u\mathbb{E}x \leftrightarrow u = m_1{}^\wedge \vee ... \vee u = m_k{}^\wedge)$, compare $\mathcal{A}_1,\mathcal{A}_3,\mathcal{A}_4$.
The models $(\in;\mathbb{H})$, $(E;\mathbb{N})$ and $(\mathbb{E};D_0)$ of $\mathbb{M}$ are isomorphic ([4], page 131). The mapping ψ from $(E;\mathbb{N})$ onto $(\mathbb{E};D_0)$ with $\psi(i) = i^\wedge$ is an isomorphism.
a) For $x=y$ we have obviously $x^\circ=y^\circ$ in our model I on D, that means $\exists u\ (u = x^\circ \wedge u = y^\circ)$.
b) For $x\neq y$ we have to prove $\neg\exists u\ (u=x^\circ \wedge u=y^\circ)$ in our model I on D. But the formula $u=x^\circ$ is valid in I iff $I(u)=x^\wedge$ is true, and because of $x\neq y$ we have $x^\wedge\neq y^\wedge$.

NOTICE: Any closed $\forall$-bounded formula which is valid on D_0 is also valid on D because of $x\in D_0 \wedge u\mathbb{E}x \rightarrow u\in D_0$.

THEOREM 1: Any recursively enumerable predicate on $\mathbb{N}$ can be semirepresented by a $\forall$-bounded formula in $\mathbb{M}$.

PROOF: Let P be an n-place ϕ-recursively enumerable predicate on $\mathbb{H}$. According to [5], page 152, there is an $r\in\mathbb{N}$ and an (n+r+2)-place predicate Q of AL with
$Px_1...x_n \leftrightarrow \exists a\ \forall b_{\ b\underline{\in} a}\ \exists c_1...\exists c_r\ Qabc_1...c_rx_1...x_n$.
The predicate Q can be written
$Qabc_1...c_rx_1...x_n \leftrightarrow \exists d_1...d_s\ Rabc_1...c_rd_1...d_sx_1...x_n$,
where R... is a conjunction of graphs of the functions $\lambda\emptyset$, $\lambda x\ \{x\}$, $\lambda xy\ x\cup y$ and a combination by $\wedge,\vee$ of equations $x=y$ and negated equations $x\neq y$. The mentioned graphs can be represented as follows:

$z = \emptyset \leftrightarrow \forall u_{\ u\in z}\ \neg u\in z$

$z = \{x\} \leftrightarrow x\in z \wedge \forall u_{\ u\in z}\ u=x$

$z = x\cup y \leftrightarrow \forall u_{\ u\in z}\ (u\in x \vee u\in y) \wedge \forall u_{\ u\in x}\ u\in z \wedge \forall u_{\ u\in y}\ u\in z$

Of course we also have
$\forall b_{\ b\underline{\in} a}\ \exists c_1...\exists c_r\ Qab... \leftrightarrow \forall b_{\ b\in a}\ \exists c_1...\exists c_r\ Qab... \wedge \exists c_1...\exists c_r\ Qaa...$.
If we substitute E for $\in$ we get a standard form of all recursively enumerable

predicates on $\mathbb{N}$. If we write everything in bold print we get a semirepresenting formula for the predicate. For the requirement 7a) remember that we have a $\forall$-bounded formula.

We refer now to any Gödel numbering of all E-formulas and any Gödel numbering of all proofs of E-formulas in $\mathbb{M}$ in any consistent and complete first order calculus. The following predicate B and the function $\underline{f}$ are for any of these Gödel numberings decidable resp. computable. According to Church's thesis they are recursive. If we want to prove this without using Church's thesis we have to specify the Gödel numberings. We don't carry out these tedious details.

DEFINITION:

1. Bac $\Leftrightarrow$ a is the Gödel number of an E-formula and
   c is the Gödel number of a proof in $\mathbb{M}$ of the diagonal formula of the formula belonging to a

The diagonal formula of a one-place E-formula $A[\mathbf{x}]$ with the Gödel number a is the formula $A(a^\circ)$. The diagonal formula of an E-formula A that is not one-place is A itself.

2. Sac $\Leftrightarrow$ a is the Gödel number of an E-formula and
   c is the Gödel number of a proof in $\mathbb{M}$ of the formula belonging to a

Let $S[\mathbf{x},\mathbf{y}]$ be a $\forall$-bounded formula semirepresenting the predicate S.

3. $\underline{f}(x)$ is the Gödel number of the diagonal formula if x is the Gödel number of a one-place E-formula. In all other cases we define $\underline{f}(x) = x$.

Fxy :$\Leftrightarrow$ y = $\underline{f}(x)$.

Let $F[\mathbf{x},\mathbf{w}]$ be a $\forall$-bounded formula semirepresenting the predicate F.

4. $B[\mathbf{x},\mathbf{y}] \equiv \forall \mathbf{w}\ (F[\mathbf{x},\mathbf{w}] \rightarrow S(\mathbf{w},\mathbf{y}))$

The formula B is not $\forall$-bounded. In the standard model I° on $\mathbb{N}$ we have:

$B[\mathbf{x},\mathbf{y}]$ is valid in I° iff $BI^\circ(x)I^\circ(y)$.

However B is not semirepresented by B !

5. The Gödel number of the formula $\neg\exists \mathbf{y}\ B[\mathbf{x},\mathbf{y}]$ is called b, the Gödel number of the diagonal formula $\neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$ is called d.
6. The Gödel number of the formula $\exists \mathbf{x}\ \neg \mathbf{x}=\mathbf{x}$ is called e.

THEOREM 2 (GÖDEL'S FIRST INCOMPLETENESS THEOREM FOR $\mathbb{M}$):
If $\mathbb{M}$ is consistent, there is no proof in $\mathbb{M}$ of the formula $\neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$.

However in the standard interpretation I° on $\mathbb{N}$ the formula $\neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$ is true, that means $\neg\exists$y Bby.

PROOF: For any proof of $\neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$ we get $\exists$y Bby. On the other hand the formula is valid in I° when it is provable, that means $\neg\exists$y Bby. (According to [1] and the proof of the Lemma $\mathbb{M}$ is valid in I° if it is consistent.)

$\neg\exists$y Sey means that there is no proof of $\exists \mathbf{x}\ \neg\mathbf{x}=\mathbf{x}$. This is a very natural formulation of the consistency of $\mathbb{M}$

Gödel's second incompleteness theorem asserts <u>for a special natural choice of the E-formula</u> S, that also $\neg\exists \mathbf{y}\ S(e^\circ,\mathbf{y})$ is not provable in $\mathbb{M}$ (if $\mathbb{M}$ is consistent), that means a formula expressing the consistency in the standard model I° on $\mathbb{N}$ in a natural way.

For the second incompleteness theorem we need a proof in $\mathbb{M}$ of

$\neg\exists \mathbf{y}\ S(e^\circ,\mathbf{y}) \rightarrow \neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$.

We show first that there is a proof in $\mathbb{M}$ of

$\neg\exists \mathbf{y}\ S(d^\circ,\mathbf{y}) \rightarrow \neg\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$.

Argument: Otherwise in a model I both $\neg\exists \mathbf{y}\ S(d^\circ,\mathbf{y})$ and $\exists \mathbf{y}\ B(b^\circ,\mathbf{y})$ would be valid.

Because of the semirepresentation of F by $\mathcal{F}$ we have $\mathcal{F}[b°,d°]$ in I, because of $\exists y\ \mathcal{B}(b°,y)$ therefore $\exists y\ \mathcal{S}(d°,y)$. Contradiction.
Thus we have to prove in $\mathbb{M}$
$\neg\exists y\ \mathcal{S}(e°,y) \rightarrow \neg\exists y\ \mathcal{S}(d°,y)$, or equivalently
$\exists y\ \mathcal{S}(d°,y) \rightarrow \exists y\ \mathcal{S}(e°,y)$. (°)
Because of the first incompleteness theorem we have
$\neg\exists y$ Sey $\rightarrow \neg\exists y$ Sdy, or equivalently
$\exists y$ Sdy $\rightarrow \exists y$ Sey . (°°)
To prove (°) we have to show the validity of (°) in <u>any</u> model of $\mathbb{M}$, whereas (°°) asserts the validity only in the standard model.
We have now to specify partially the formula $\mathcal{S}$.

In any model I on D let <x,y> be the member z of D with
$\exists uv\forall w\ ((w\mathbb{E}z \leftrightarrow w=u \vee w=v) \wedge (w\mathbb{E}u \leftrightarrow w=x) \wedge (w\mathbb{E}v \leftrightarrow w=x \vee w=y))$.
The function λxy <x,y> is a one-to-one mapping.
We choose a correct and complete first order calculus where any rule has not more than two premises. The set H of Gödel numbers of E-formulas being logical axioms of the calculus is recursive. Then we have a three-place recursive predicate K on $\mathbb{N}$ with: Kxyz $\leftrightarrow$
x,y,z are Gödel numbers of E-formulas and the E-formula of z can be obtained from the E-formulas of x and y by a rule of the calculus .
The set of Gödel numbers of all E-formulas is called G. Let M be any recursive set of natural numbers.
$S_M ac \leftrightarrow Ga \wedge c$ is a natural number, having in the binary representation for certain $a_0,..,a_k$ with $a_k=a$ exactly in the positions $\langle a_0,0\rangle,...,\langle a_k,k\rangle$ the digit 1 and:
For any i with $0\le i\le k$ we have Ma_i or Ha_i or there are r,s with
$rEi \wedge sEi \wedge Ka_r a_s a_i$.
Now we choose $\forall$-bounded formulas $\mathcal{G}[x]$, $\mathcal{M}[p]$, $\mathcal{H}[p]$, $\mathcal{K}[p_1,p_2,p]$ semirepresenting the predicates G, M, H, K.
z = <x,y> is an abbreviation for
$\exists uv\ (uEz \wedge vEz \wedge \forall w_{wEz}\ (w=u \vee w=v) \wedge xEu \wedge \forall w_{wEu}\ w=x \wedge xEv \wedge yEv$
$\wedge \forall w_{wEv}\ (w=x \vee w=y))$.
$\mathcal{S}_M[x,y]$ is an abbreviation for
$\exists uv\ (u = \langle x,v\rangle \wedge uEy \wedge \mathcal{G}[x])$
$\wedge \forall u_{\ uEy}\ \exists pq\ (u = \langle p,q\rangle \wedge (\mathcal{M}[p] \vee \mathcal{H}[p]$
$\vee \exists q_1 q_2 p_1 p_2 u_1 u_2\ (q_1 Eq \wedge q_2 Eq \wedge u_1=\langle p_1,q_1\rangle \wedge u_2=\langle p_1,q_2\rangle$
$\wedge u_1 Ey \wedge u_2 Ey \wedge \mathcal{K}[p_1,p_2,p]))$.
S_M is semirepresented by this $\forall$-bounded formula. Special cases:
1. M is the set M° of Gödel numbers of the formulas of $\mathbb{M}$. Then we have $S_M ac \leftrightarrow$ Sac. Instead of $\mathcal{S}_{M°}[x,y]$ we write $\mathcal{S}[x,y]$.
2. M is the set M°° = M°∪{d}; as $\mathcal{M}[p]$ we choose $\mathcal{M}°[p] \vee p = d°$, where $\mathcal{M}°[p]$ is semirepresenting the set M°.

THEOREM 3 (GÖDEL'S SECOND INCOMPLETENESS THEOREM FOR $\mathbb{M}$):
If $\mathbb{M}$ is consistent, there is no proof of $\neg\exists y\ \mathcal{S}(e°,y)$ in $\mathbb{M}$.

PROOF: The denotations I,D,D_0 are used as in the proof of the Lemma.
But we write now E instead of $I°$(E) and instead of I(E). For interpreted formulas we use usual print instead of bold print. On D let 0 be the only z with $\forall u\ \neg uEz$. For all $x_1,...,x_n$ of D let $\{x_1,...,x_n\}$ be the only z with $\forall u\ (uEz \leftrightarrow u=x_1 \vee ... \vee u=x_n)$.

We must show that (°) is valid in I. Because of (°°) $S_{M^{\circ\circ}}(e^\circ,y)$ is valid in the standard model I° on $\mathbb{N}$ for some choice of $I^\circ(y)=y_1$. Hence we have on $\mathbb{N}$

$$\exists w\, (<e,w>Ey_1 \wedge G(e)) \qquad (*)$$
$$\wedge\ \forall u\, (\neg uEy_1 \vee \exists pq\, (u=<p,q> \wedge (M^\circ(p) \vee p=d \wedge q=0 \vee H(p)$$
$$\vee\ \exists q_1q_2p_1p_2\, (q_1Eq \wedge q_2Eq \wedge <p_1,q_1>Ey_1 \wedge <p_2,q_2>Ey_1 \wedge K(p_1,p_2,p)))))$$

Because we can start the proof with the formula belonging to d we can add "q=0" in the formulation.

(*) is valid also in I restricted to D_0 for some y_1 and (because this formula is $\forall$-bounded) also in I on D. Then e and d are denoting the corresponding elements of D_0. Let $\exists y\, S(d^\circ,y)$ be valid in I on D, hence we have for some v,y_2 of D

$$<d,v>Ey_2 \wedge G(d) \qquad (**)$$
$$\wedge\ \forall u\, (\neg uEy_2 \vee \exists pq\, (u=<p,q> \wedge (M^\circ(p) \vee H(p)$$
$$\vee\ \exists q_1q_2p_1p_2(q_1Eq \wedge q_2Eq \wedge <p_1,q_1>Ey_2 \wedge <p_2,q_2>Ey_2 \wedge K(p_1,p_2,p)))))$$

We want to use the union of y_1 and y_2 as "proof" for the formula of e, but there is a small difficulty: In the first "proof" y_1 we have d together with 0 as a pair of y_1, but in the second "proof" y_2 we have d together with v. Therefore we define a function $\wp$ from D_0 to D by induction.

$$\wp(0) = v$$
$$\wp(\{x_1,\ldots,x_n\}) = \{\wp(x_1),\ldots,\wp(x_n)\}.$$

For any u,v of D_0 we have

$$uEv \text{ iff } \wp(u)E\wp(v) \ . \qquad (^{\circ\circ\circ})$$

We change from

$$y_1 = \{<d,0>, <a_1,1>,\ldots,<a_{k-1},k-1>,<e,k>\}$$

to

$$y_1^\circ = \{<d,\wp(0)>, <a_1,\wp(1)>,\ldots,<a_{k-1},\wp(k-1)>,<e,\wp(k)>\}.$$

Because of (°°°) we have

$$\exists w\, (<e,w>Ey_1^\circ \wedge G(e))$$
$$\wedge\ \forall u\, (\neg uEy_1^\circ \vee \exists pq\, (u=<p,q> \wedge (M^\circ(p) \vee p=d \wedge q=v \vee H(p)$$
$$\vee\ \exists q_1q_2p_1p_2\, (q_1Eq \wedge q_2Eq \wedge <p_1,q_1>Ey_1 \wedge <p_2,q_2>Ey_1 \wedge K(p_1,p_2,p)))))$$

Let y_3 be the member of D with

$$\forall u\, (uEy_3 \leftrightarrow uEy_1^\circ \vee uEy_2).$$

Then we get

$$\exists w\, (<e,w>Ey_3 \wedge G(e))$$
$$\wedge\ \forall u\, (\neg uEy_3 \vee \exists pq\, (u=<p,q> \wedge (M^\circ(p) \vee H(p)$$
$$\vee\ \exists q_1q_2p_1p_2\, (q_1Eq \wedge q_2Eq \wedge <p_1,q_1>Ey_3 \wedge <p_2,q_2>Ey_3 \wedge K(p_1,p_2,p))))).$$

Thus $\exists y\, S(e^\circ,y)$ is valid in I on D. Q.e.d.

The incompleteness theorems are valid for any axiom system of the theory of natural numbers where the predicate E can be defined and the formulas of $\mathbb{M}$ can be proved for this predicate. So we get the original incompleteness theorems of Gödel very easily.

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