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Non-Renewable Resources in a Ramsey Economy with Subsistence Consumption, Human and Physical Capital Accumulation: A Full Characterization

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Abstract

We investigate the question on how to use a non-renewable resources efficiently in the presence of a minimum subsistence level of consumption. In our model, households are characterized by Stone-Geary preferences and output is Cobb-Douglas using physical and human capital as well as resources as input factors. This setup gives rise to a six dimensional dynamic system with three control and three state variables. Despite this complexity, it is shown that a closed form solution exists in terms of the Gaussian hypergeometric function. The closed form solution allows us to calibrate the model to the situation of 108 countries using data from the World Bank on countries' endowments with physical capital and natural resources. We are able to quantify the implications of observed capital stocks for the growth perspective of each country. In particular, we analyze whether a level of subsistence consumption equivalent to the World Bank's poverty lines can be accomplished. Our calibration results also shed some light on what has been termed the "resource curse".

Keywords

Non-renewable resources, subsistence consumption, closed form solution

JEL Classifications

Q32, E21, O44

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1 Introduction

We consider the continuous time Dasgupta-Heal-Solow-Stiglitz (Dasgupta and Heal 1974, Solow 1974 and Stiglitz 1974, DHSS from here on) model extended to include human capital accumulation besides the accumulation of reproducible physical capital. We do so by taking account of an additional human capital sector following the well known Uzawa-Lucas endogenous growth model (Uzawa 1965 and Lucas 1988) together with a non-renewable resource that is essential for production of final output. Physical and human capital are subject to depreciation whereas population of the economy grows at a constant exponential rate. The model is therefore an extension of a frequently used set-up in resource economics where man-made capital and resource extraction are essential input factors as in e.g. Mitra et al. (2013) and many others.

We solve the problem of a benevolent social planer that aims at maximizing a utilitarian criterion reflecting CRRA preferences in consumption in excess of some subsistence level. As such, we introduce Stone-Geary (Stone (1954) and Geary (1950)) type of preferences into a resource allocation problem as in (Antony and Klarl 2019 a,b).

The model considered in this contribution is not completely new. It has been analyzed without subsistence consumption together with further extensions by Schou (2000) which is based on the contribution of Robson (1980). Their analysis, however, is restricted to an investigation of the steady state growth behavior only. The novelty of our contribution is to offer a full characterization of the entire unique saddle-path which the economy follows on its way towards its steady state. We offer a closed form solution to a six dimensional problem with three control and three state variables. This closed form can be found by making use of the integral representation of the Gaussian hypergeometric function.

We contribute further to the existing literature in the following ways. First, we provide a technical contribution regarding the use of special functions in analyzing economic dynamics. Second, we fully analyze the global dynamics of an augmented DHSS model allowing for endogenous growth of the Uzawa-Lucas type taking account of subsistence needs of households. Third, as we can derive the global dynamics of the economy in optimum, we are able to identify the initial conditions required for a solution of our problem to exist. Typically, complex dynamic problems are solved by linearizing the dynamics of the economy around steady state which prevents one to identify the necessary initial conditions for a solution to the problem to exist. These conditions are boiling down to the question whether initial endowments with physical and human capital together with initial resource stocks are sufficient to allow at least for realizing permanently the subsistence level consumption. As we can pin-down these conditions, we can calibrate our model to the current situation of particular countries. We do so by using data mainly from the Worldbank (2018) which allows us to assess the growth perspective of 108 economies confronted with subsistence consumption defined by the World Bank's the poverty lines.

Our findings are as follows. Only 98 out of 108 countries are equipped with sufficient initial endowments with resources, physical and/or human capital. Low income countries do suffer in particular from insufficient endowments. We find further, that 91 out of these 98 economies qualify for positive long-run growth while 7 converge to a zero growth scenario where households can just afford minimum subsistence consumption. We quantify the deficits in the different stocks of capital for those countries with initial endowments too low. Furthermore, we find a typical pattern in our calibration results that are comparable to what is known as the "resource curse" and identify the underlying mechanism that is relevant in our model set-up.

The plan of the paper is as follows. The next section reviews literature relevant to our contribution. Section 3 lays out the economic problem that we aim to solve and Section 4 presents the solution and elaborates on the solution's existence properties. We provide a calibration of our model and discus the results in Section 5. Finally, Section 6 concludes.

2 Review of Literature

Subsistence consumption, which enables individuals to meet their minimum basic needs of life, has repeatedly discussed in an economic growth context. Two papers closely related to this one are Steger (2000) and Strulik (2010). Both solve a utility maximization problem with Stone-Geary preferences but with a standard AK-type production technology. Their models are nested in ours if one is setting the output elasticity of the resource equal to zero and taking no account of human capital accumulation. The general lesson we can learn from these studies is that the requirement of subsistence consumption represents an important mechanism of β divergence. However, these settings leave out the fact that many developing low income countries are resource rich (Barbier 2005), which facing substantial development needs (see Araujo et al. 2016 and grow less rapidly Gaitan and Roe 2012).

Another strand of literature analyzing the DHSS framework sofar didn't discuss the implication of minimum consumption. The Cobb Douglas constant returns to scale production structure with reproducible man-made capital and resource input has been employed by Benchekroun and Withagen (2011), Asheim and Buchholz (2004) and others. Mitra et al. (2013) employ a general constant returns to scale technology with reproducible man-made capital and resource input[1]. With the notable extension of Antony and Klarl (2019, a,b), this strand of the literature however has completely left out the possibility of a minimum subsistence level of consumption. Antony and Klarl (2019a) introduce a minimum subsistence level of consumption in an utilitarian approach into the DHSS model without capital depreciation and technical change while Antony and Klarl (2019b) allow for both.

Our approach is related to the nexus between resources on the one and growth as well as development on the other hand. First, as inter alia argued by Collier et al. (2010) and van der Ploeg and Venables (2011), because of human as well as physical capital scarcity, many resource rich developing countries should use resource rent windfalls to speed up development by accumulating

[1] See Antony and Klarl (2019, a,b) for a more detailed review of this literature.

capital. They argue that capital scarcity implies higher return on domestic capital. Hence, it might be beneficial to invest in human and physical capital than investing abroad. Empirical findings in Venables (2016) seem to suggest that this is not happening, however. We add to this literature by asking the questions whether initial endowments with physical and human capital together with initial resource stocks are sufficient to allow at least for realizing permanently the subsistence level consumption. If not, we provide model predictions of the corresponding shortfalls.

Second, there is by now a considerable literature on what has been termed the "resource curse". The hypotheses dates back at least to Auty (1993) and Sachs and Warner (1995) and postulates that we observe in general a negative relationship between resource dependence of countries and their economic growth. Different arguments have been put forward which might be able to explain this observation. One of them is related to the actual deficit in domestic investments in physical or human capital. For a more detailed review of the literature and also opposing critical opinions see e.g. van der Ploeg (2011) and Smith (2015). We also add to this literature as our model predicts some type of resource curse via the transmission channel of physical and human capital accumulation.

From the technical point of view we add to the literature using special functions in solving dynamic problems. This involves the Gaussian hypergeometric function which has been found to be useful by other economists as well. Lucas type of models have been analyzed by Boucekkine and Ruiz-Tamarit (2008), Boucekkine et al. (2008), Ruiz-Tamarit (2008) and Hiraguchi (2009). Guerrini (2010) uses the Gaussian hypergeometric function to solve the problem of an AK Ramsey economy with logistic population growth. Hiraguchi (2014) solves a Ramsey problem with leisure as one argument of the utility function. Regarding problems related to environmental economics, Perez-Barahona (2011) solve an AK Ramsey problem involving natural resources.

3 Subsistence Consumption in the DHSS Model

In this section, we lay out the intertemporal utilitarian problem that we aim to solve. Preliminary calculations are presented that are helpful in finding a solution to the problem together with necessary conditions for its existence.

3.1 The Optimization Problem

The economy is populated by a mass 1 of infinitively living representative households with the following Stone-Geary intertemporal utility function

$$U_t = \int_0^\infty \frac{(c_t - \underline{c})^{1-\eta} - 1}{1-\eta} L_t e^{-\rho t} dt, \tag{1}$$

where c_t is consumption per capita at time t, $\underline{c}$ is the minimum subsistence level of consumption, $\eta > 0$ and $\rho > 0$ is the rate of time preference. $L_t = L_0 e^{nt}$ is household size at time t which is growing at rate n. We will refer to $c_t - \underline{c}$ as excess consumption in the sense that is taking place in excess of a subsistence level $\underline{c}$.

We consider a social planer to maximize households' lifetime utility given the relevant budget constraints. These constraints are given by the accumulation of reproducible physical capital, the accumulation of human capital, and by the use of a non-renewable resource that is necessary for production.

We assume that production is given by the aggregate Cobb-Douglas production technology

$$Y_t = AK_t^{\alpha}(H_t u_t L_t)^{\beta} R_t^{\gamma}, \tag{2}$$

where K_t denotes the stock of physical capital and H_t is the level of human capital. Each household member supplies inelastically one unit of raw labor of which the fraction u_t is employed in final goods production. Total effective labor input into final goods production is therefore $H_t u_t L_t$. R_t is the use and extraction of the resource. We assume constant returns to scale, i.e. $\alpha + \beta + \gamma = 1$, and $0 < \alpha, \beta, \gamma < 1$. A denotes a constant level of total factor productivity. (2) shows the potential of long-run growth in case human and physical capital accumulation occurs fast enough to compensate for the scarcity problem reflected by the presence of the non-renewable resource.

Physical capital is produced from foregone final output with unit productivity and depreciates at a rate $\delta_1 > 0$. The net increase in the stock of reproducible capital is therefore

$$\frac{\partial K_t}{\partial t} = \dot{K}_t = Y_t - C_t - \delta_1 K_t. \tag{3}$$

Human capital is accumulated by foregone labor supply in production of final output

$$\dot{H}_t = B(1 - u_t)H_t - \delta_2 H_t, \tag{4}$$

where $B > 0$ is a constant productivity parameter, $\delta_2 \geq 0$ is the constant rate of depreciation of human capital and $(1 - u_t)$ is the fraction of labor supply not used in final goods production but spent on learning and accumulating human capital.

Production requires the use of R_t units of a non-renewable resource at time t. The stock S_t of the resource develops according to

$$\dot{S}_t = -R_t \tag{5}$$

The present value Hamiltonian for the representative household therefore reads as

$$\mathscr{H}_t = \frac{(c_t - \underline{c})^{1-\eta} - 1}{1-\eta} e^{-\rho t} L_t \quad + \lambda_{1,t}[Y_t - c_t L_t - \delta_1 K_t] \quad + \lambda_{2,t}[B(1-u_t)H_t - \delta_2 H_t] \quad + \lambda_{3,t}[-R_t], \tag{6}$$

where $\lambda_{i,t}$, $i = 1,2,3$ are the co-state variables associated with the constraints of the dynamic problem. The first order conditions for a maximum read as

$$\frac{\partial \mathscr{H}_t}{\partial c_t} = (c_t - \underline{c})^{-\eta} e^{-\rho t} L_t - \lambda_{1,t} L_t = 0, \tag{7}$$

$$-\frac{\partial \mathscr{H}_t}{\partial K_t} = \dot{\lambda}_{1,t} = -\lambda_{1,t} \frac{\partial Y_t}{\partial K_t} + \lambda_{1,t}\delta_1, \tag{8}$$

$$\frac{\partial \mathscr{H}_t}{\partial u_t} = \lambda_{1,t} \frac{\partial Y_t}{\partial u_t} - \lambda_{2,t} B H_t = 0, \tag{9}$$

$$-\frac{\partial \mathscr{H}_t}{\partial H_t} = \dot{\lambda}_{2,t} = -\lambda_{1,t} \frac{\partial Y_t}{\partial H_t} - \lambda_{2,t} B(1-u_t) + \lambda_{2,t}\delta_2, \tag{10}$$

$$\frac{\partial \mathscr{H}_t}{\partial R_t} = \lambda_{1,t} \frac{\partial Y_t}{\partial R_t} - \lambda_{3,t} = 0, \tag{11}$$

$$-\frac{\partial \mathscr{H}_t}{\partial S_t} = \dot{\lambda}_{3,t} = 0. \tag{12}$$

The corresponding transversality conditions read as

$$\lim_{t\to\infty} \lambda_{1,t} K_t = 0, \tag{13}$$

$$\lim_{t\to\infty} \lambda_{2,t} H_t = 0, \tag{14}$$

$$\lim_{t\to\infty} \lambda_{3,t} S_t = 0. \tag{15}$$

3.2 Preliminary Calculations

Given the first order conditions, we now take the first steps in solving the model. The primary aim of this section is to solve for the time paths of the stock variables K_t, H_t and S_t as well as consumption c_t and the co-states $\lambda_{i,t}$, $i = 1,2,3$. This is necessary for pinning down the implications of the transversality conditions (13) through (15) for the initial values of the co-states. Additionally, some results are derived that will prove to be useful in the remainder of our analysis.

We start in reverse order and note that (12) directly impliest $\lambda_{3,t} = \lambda_{3,0}$ where the latter is simply the initial value for the resource' shadow value.

Proceding with $\lambda_{2,t}$, we find conditions (9) and (11 to imply

$$\lambda_{1,t}\beta AK_t^{\alpha}(H_tu_tL_t)^{\beta-1}R_t^{\gamma}H_tL_t = \lambda_{2,t}BH_t, \tag{16}$$
$$\lambda_{1,t}\gamma AK_t^{\alpha}(H_tu_tL_t)^{\beta}R_t^{\gamma-1} = \lambda_{3,t}. \tag{17}$$

From (8) and (10) we know that

$$\dot{\lambda}_{1,t} = -\lambda_{1,t}\alpha AK_t^{\alpha-1}(H_tu_tL_t)^{\beta}R_t^{\gamma}+\lambda_{1,t}\delta_1, \tag{18}$$
$$\dot{\lambda}_{2,t} = -\lambda_{1,t}\beta AK_t^{\alpha}(H_tu_tL_t)^{\beta-1}R_t^{\gamma}u_tL_t-\lambda_{2,t}B(1-u_t)+\lambda_{2,t}\delta_2. \tag{19}$$

Using (16) in (19) gives

$$\dot{\lambda}_{2,t} = -\lambda_{2,t}Bu_t-\lambda_{2,t}B(1-u_t)+\lambda_{2,t}\delta_2 = -\lambda_{2,t}(B-\delta_2),$$

which directly implies

$$\lambda_{2,t} = \lambda_{2,0}e^{-(B-\delta_2)t}, \tag{20}$$

where $\lambda_{2,0}$ is the initial value of the co-state variable $\lambda_{2,t}$ at time $t=0$. It is this astonishing simple time path for the evolution of human capital's shadow price that allows for a closed form solution of the above problem. (20) takes such a simple form because human capital creation is linear in its own stock given u_t and is not directly depending on K_t, S_t or R_t.

Solving for the path of $\lambda_{1,t}$ is a bit more complex. Dividing both sides of (16) by (17) by each other gives

$$\frac{R_t}{BH_tu_t} = \frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}. \tag{21}$$

Rearranging (16) and using (21) yields

$$
\begin{aligned}
\lambda_{1,t}\beta A K_t^{\alpha}(H_t u_t L_t)^{\beta} R_t^{\gamma} &= \lambda_{2,t} B H_t u_t, \\
\beta A \left(\frac{K_t}{BH_t u_t}\right)^{\alpha}\left(\frac{R_t}{BH_t u_t}\right)^{\gamma}\left(\frac{L_t}{B}\right)^{\beta} &= \frac{\lambda_{2,t}}{\lambda_{1,t}} \\
\beta A \left(\frac{K_t}{BH_t u_t}\right)^{\alpha}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}\right)^{\gamma} &= \frac{\lambda_{2,t}}{\lambda_{1,t}}\left(\frac{L_t}{B}\right)^{-\beta} \\
\frac{K_t}{BH_t u_t} &= \left(\frac{\lambda_{2,t}}{\lambda_{1,t}}\frac{1}{\beta A}\right)^{\frac{1}{\alpha}}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{-\frac{\beta}{\alpha}} \\
&= A^{-\frac{1}{\alpha}}\lambda_{1,t}^{-\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{\frac{1-\gamma}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{-\frac{\beta}{\alpha}}
\end{aligned}
\tag{22}
$$

At this point it is helpful to introduce additional variables that simplify the notation and are useful to solve the model. Define

$$
\begin{aligned}
\varphi_1 &= A^{-\frac{1}{\alpha}}\lambda_{1,0}^{\frac{\alpha-1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{\alpha}}, \\
\varphi_2 &= \frac{1-\alpha}{\psi}, \\
\zeta &= \frac{\varphi_2-\varphi_1}{\varphi_2}, \\
x_t &= e^{-\psi t}, \\
\psi &= \frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}.
\end{aligned}
\tag{23}
$$

We will elaborate on the economic intuition underlying ζ a little bit further down below. At the moment we note that while t runs from 0 to ∞, x_t develops from 1 to 0. This property of the variable x_t will make it convenient to solve the model using the Gaussian hypergeometric function. With results (18), (21) and (22) at hand, we are now able to trace the behavior of the co-state $\lambda_{1,t}$ over time as $\lambda_{1,t}^{1-\frac{1}{\alpha}}$ is governed by a Bernoulli equation (see Appendix A at the end of the paper for details)

$$
\lambda_{1,t} = \lambda_{1,0}e^{\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{\frac{\alpha}{1-\alpha}}. \tag{24}
$$

And thus, as a first intermediate result, we are able to trace per capita consumption over time by using (24) in (7)

$$\begin{aligned} c_t &= \underline{c} + e^{-\frac{\rho}{\eta}t}\lambda_{1,t}^{-\frac{1}{\eta}} \\ &= \underline{c} + \lambda_{1,0}^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}+\frac{\delta_1}{\eta}\right)t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{\alpha}{(1-\alpha)\eta}}. \end{aligned} \tag{25}$$

(25) paths the way to find a closed form for the development of the physical and human capital stock. The physical capital stock K_t behaves according to

$$\dot{K}_t = Y_t - c_t L_t - \delta_1 K_t = K_t A \left(\frac{H_t u_t L_t}{K_t}\right)^{\beta} \left(\frac{R_t}{K_t}\right)^{\gamma} - c_t L_t - \delta_1 K_t.$$

Making use of the relative factor intensities in (21), (22), the development of $\lambda_{2,t}$ in (20) and $L_t = L_0 e^{nt}$ gives

$$\begin{aligned} \dot{K}_t &= A\left(\frac{BH_t u_t}{K_t}\right)^{\beta}\left(\frac{R_t}{BH_t u_t}\frac{BH_t u_t}{K_t}\right)^{\gamma}\left(\frac{L_t}{B}\right)^{\beta} K_t - c_t L_t - \delta_1 K_t, \\ &= A\left(\frac{K_t}{BH_t u_t}\right)^{-\beta-\gamma}\left(\frac{R_t}{BH_t u_t}\right)^{\gamma}\left(\frac{L_t}{B}\right)^{\beta} K_t - \delta_1 K_t - c_t L_t, \\ &= A\left[\left(\frac{\lambda_{2,t}}{\lambda_{1,t}}\frac{1}{\beta A}\right)^{\frac{1}{\alpha}}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,t}}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{-\frac{\beta}{\alpha}}\right]^{\alpha-1}\left(\frac{\gamma}{\beta}\frac{\lambda_{2,t}}{\lambda_{3,0}}\right)^{\gamma}\left(\frac{L_t}{B}\right)^{\beta} K_t - \delta_1 K_t - c_t L_t, \\ &= A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}}\lambda_{1,t}^{\frac{1-\alpha}{\alpha}} K_t - \delta_1 K_t - c_t L_t, \\ &= A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\lambda_{1,t}^{\frac{1-\alpha}{\alpha}} e^{\frac{\beta(B-\delta_2+n)}{\alpha}t} K_t - \delta_1 K_t - c_t L_0 e^{nt}. \end{aligned} \tag{26}$$

Inspecting the representation of $\lambda_{1,t}$ in (24) reveals that

$$\begin{aligned} & A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\lambda_{1,t}^{\frac{1-\alpha}{\alpha}} e^{\frac{\beta(B-\delta_2+n)}{\alpha}t} \\ = & A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\left(\lambda_{1,t}e^{-\delta_1 t}\right)^{\frac{1-\alpha}{\alpha}} e^{\frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}t}, \\ = & \left[\varphi_1 + \varphi_2(e^{\psi t}-1)\right]^{-1} e^{\psi t} = \left[\varphi_1 x_t + \varphi_2(1-x_t)\right]^{-1}. \end{aligned}$$

We can now continue at (26) with solving for the time path of K_t

$$\dot{K}_t = \left[(\varphi_1 x_t + \varphi_2(1-x_t))^{-1} - \delta_1\right] K_t - c_t L_0 e^{nt}.$$

This differential equation can be solved in a quite straightforward manner. To do so, we reformulate this differential equation by making use of (25) into a standard text book form used for finding the solution

$$\dot{K}_t + f_1(t)K_t = g_1(t), \tag{27}$$

with

$$f_1(t) = -\left[(\varphi_1 x_t + \varphi_2(1-x_t))^{-1} - \delta_1\right],$$

$$g_1(t) = -(c_t - \underline{c})L_t - \underline{c}L_t = -\lambda_{1,t}^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}-n\right)t} L_0 - \underline{c}e^{nt}L_0, \tag{28}$$

where $-f(t)$ is the net return on physical capital at time t. We denote the initial stock of capital at $t=0$ by K_0. The solution to the differential equation (27) is given by

$$K_t = K_0 e^{-\int_0^t f_1(z)dz} + \int_0^t g_1(z) e^{-\int_z^t f_1(s)ds} dz. \tag{29}$$

Building the integral $\int_z^t f_1(s)ds$ and using $x_z = e^{-\psi z}$ gives

$$\begin{aligned}
-\int_z^t f_1(s)ds &= -\delta(t-z) + \int_z^t \left[\varphi_1 e^{-\psi s} + \varphi_2\left(1-e^{-\psi s}\right)\right]^{-1} ds \\
&= -\delta(t-z) + \frac{1}{1-\alpha} \ln\left[\frac{\varphi_1 + \varphi_2\left(e^{\psi t}-1\right)}{\varphi_1 + \varphi_2\left(e^{\psi z}-1\right)}\right] \\
&= -\delta_1(t-z) + \frac{1}{1-\alpha} \ln\left[\frac{\varphi_1 + \varphi_2\left(x_t^{-1}-1\right)}{\varphi_1 + \varphi_2\left(x_z^{-1}-1\right)}\right].
\end{aligned} \tag{30}$$

Using (28), (30), (7) and (24) in (29) gives the stock of capital K_t as (see Appendix B)

$$\begin{aligned}
K_t = {} & K_0 e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\
& -e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{(1-\alpha)\eta}} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \frac{L_0}{\psi}\int_{x_t}^1 x_z^{\frac{1}{\psi}\left(\frac{\rho}{\eta}+\psi\frac{\eta-\alpha}{(1-\alpha)\eta}-\delta_1+\frac{\delta_1}{\eta}-n\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}} dxz \\
& -e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \underline{c}\frac{L_0}{\psi}\int_{x_t}^1 x_z^{\frac{1}{\psi}\left(\frac{\psi}{1-\alpha}-n-\delta_1\right)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}} dx_z
\end{aligned} \tag{31}$$

with

$$\zeta = \frac{\varphi_2-\varphi_1}{\varphi_2} = \frac{\frac{1}{\varphi_1}-\frac{1}{\varphi_2}}{\frac{1}{\varphi_1}}.$$

Inspecting the derivations in (26), it can easily be verified that $\frac{1}{\varphi_1}$ is equal to the initial capital productivity $\frac{Y_0}{K_0}$ at time $t=0$. As will become clear further down below, $\frac{1}{\varphi_2}$ is equal to the capital productivity in steady-state as $t \to \infty$, i.e. $\lim_{t\to\infty} \frac{Y_t}{K_t} = \frac{\psi}{1-\alpha} = \frac{1}{\varphi_2}$. ζ therefore measures the relative distance of the limiting from the initial capital productivity. If it were by chance that $\zeta = 0$, we would encounter an economy that starts in steady-state right away. Unsurprisingly, the expressions in (31) would simplify a great deal if this case prevails. As the first order conditions (7), (9) and (11) imply $\lambda_{i,0} > 0$, $i = 1,2,3$, we necessarily find $\zeta < 1$.

Appendix B at the end of the paper demonstrates that the integrals in (31) - as long as they converge - can be evaluated using the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ which has in general the integral representation

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{32} $$

This integral representation is valid for $R(c) > R(b) > 0$ where $R(\cdot)$ denotes the real part of the argument and $\Gamma(\cdot)$ the Gamma function (Abramowitz and Stegun, 1964, 15.3.1). In general, ${}_2F_1(a,b;c;z)$ defined as a Gauss series (Abramowitz and Stegun, 1964, 15.1.1) converges if $|z| < 1$. It also converges if additionally $R(c-b-a) > 0$ for $|z| \leq 1$ and if $-1 < R(c-b-a) \leq 0$ for $|z| \leq 1$ but $z \neq 1$. Comparing the integral on the right hand side of (32) with the integrals in (31) reveals that the present case can be seen as a special case with $c-b-1 = 0$ or equivalent $c = b+1$. And hence, $R(c) > R(b)$ holds. We will see shortly that $R(b) > 0$ poses no problem for the model's parametrization.

If we apply the representation (32) to our problem, ζ will play the role of z. We already saw above that $\zeta < 1$ holds. If $\lambda_{1,0}$ is sufficiently small and/or $\lambda_{2,0}$ or $\lambda_{3,0}$ are sufficiently large, it might turn out that $\zeta \leq -1$. In this case, one has to take care about how to compute the integrals in (31) or other integrals of the same type that appear further down below. This is because the integral representation (32) is an analytic continuation of the Gaussian hypergeometric function defined by a Gauss series (Abramowitz and Stegun 1972, 15.3.1). Only for the restrictions on z and $R(c-b-a)$ laid out above, both are identical. In general, for $z \leq -1$ and $R(c) > R(b) > 0$, the integral (32) exists but the Gauss series that defines the hypergeometric function is not converging and, hence, it is not identical to the integrals that we aim to compute. In such cases, it is necessary to use analytic continuation formulas for ${}_2F_1(a,b;c;z)$ (see Abramowitz and Stegun 1972, 15.3.3 through 15.3.9).[2]

We can therefore make use of

$$ \begin{aligned} {}_2F_1(a,b;b+1;z) &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \\ &= \frac{\Gamma(b+1)}{\Gamma(b)\Gamma(1)} \int_0^1 t^{b-1}(1-zt)^{-a}dt \\ &= b\int_0^1 t^{b-1}(1-zt)^{-a}dt, \end{aligned} $$

[2] For a general discussion about this situation see Section 3.1 in Boucekkine and Ruiz-Tamarit (2008).

where we applied the gamma function's continuation $\Gamma(b+1) = b\Gamma(b)$ and the fact that $\Gamma(1) = 1$ (Abramowitz and Stegun, 1964, 6.1.15). Note that we need to keep in mind that $z \leq -1$ needs special attention. Inspecting (31) shows that we can apply this special case of the Gaussian hypergeometric function to both integrals. Through a suitable change in the variable of integration, the integrals ranging from x_t to 1 can be split up into two separate integrals each running from 0 to 1 and each representable by the hypergeometric function. K_t is then given by

$$\begin{aligned}
K_t \quad &= K_0 e^{-\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \\
&- e^{-\delta_1 t} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \frac{1}{\psi}\frac{1}{\tilde{b}_1} L_0 \left[{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta) - x_t^{\tilde{b}_1} {}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1+1; \zeta x_t)\right] \\
&- \underline{c} e^{-\delta_1 t} \left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}} \frac{1}{\psi}\frac{1}{\tilde{b}_2} L_0 \left[{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta) - x_t^{\tilde{b}_2} {}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2+1; \zeta x_t)\right],
\end{aligned} \tag{33}$$

with

$$\begin{aligned}
\tilde{a}_1 &= \frac{\eta-\alpha}{\eta(1-\alpha)}, \\
\tilde{b}_1 &= \frac{1}{\psi}\left(\frac{\rho}{\eta} + \frac{\eta-\alpha}{(1-\alpha)\eta}\psi + \frac{1-\eta}{\eta}\delta_1 - n\right) = 1 + \frac{\alpha}{1-\alpha}\frac{1}{\eta}\frac{(1-\alpha)(\rho-n) + (\eta-1)\left[\beta(B-\delta_2) - \gamma n\right]}{\beta(B-\delta_2+n) + (1-\alpha)\delta_1} \\
\tilde{a}_2 &= \frac{1}{1-\alpha} > 1, \\
\tilde{b}_2 &= \frac{1}{1-\alpha} - \frac{\delta_1}{\psi} - \frac{n}{\psi} = 1 + \frac{\alpha}{1-\alpha}\frac{\beta(B-\delta_2) - \gamma n}{\beta(B-\delta_2+n) + (1-\alpha)\delta_1}.
\end{aligned}$$

A few words on the admissible space for the model's parameters might be in order at this point. For the integrals in (31) to converge and to be represented by the Gaussian hypergeometric function, we need both, $\tilde{b}_1$ and $\tilde{b}_2$ to be strictly positive. As we will see further down below, dealing with finite resources will impose even tighter restrictions. In particular, $\tilde{b}_1, \tilde{b}_2 > 1$ will need to be satisfied. This boils down in the two conditions

$$(1-\alpha)(\rho-n) + (\eta-1)\left[\beta(B-\delta_2) - \gamma n\right] \quad > \quad 0, \tag{34}$$

$$\beta(B-\delta_2) - \gamma n \quad > \quad 0. \tag{35}$$

If at least one of the above conditions were not satisfied, we would witness an economy characterized by parameters that don't allow for a proper solution to the problem. This could be either an intertemporal utility that is unbounded or a production structure requiring infinite resources. As it is reasonable to assume $B - \delta_2 > 0$, the above conditions imply in general that γ and n don't need to be too large and η doesn't need to be too small.

Given the development of the physical capital stock in (33), it is now straightforward to infer the time path of the human capital stock H_t and resource use R_t. We refer to Appendix C and D for the details and report only the results here in the main text.

$$\begin{aligned} H_t &= e^{(B-\delta_2)t}H_0 - e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\left\{K_0\frac{1}{\psi}(1-x_t^{-1})\right. \\ &\quad -\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)(1-x_t^{-1}) \\ &\quad +\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] \\ &\quad -\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2}L_0{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)(1-x_t^{-1}) \\ &\quad \left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}, \end{aligned} \tag{36}$$

Effective human capital $L_tH_tu_t$ employed in final goods production is given by

$$\begin{aligned} L_tH_tu_t &= e^{[(B-\delta_2)+n]t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{L_0}{B}x_t^{-1}\times \\ &\times\left\{K_0-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right]\right. \\ &\quad\left.-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\right\}. \end{aligned} \tag{37}$$

It is, of course, also possible to compute the propensity to spend labor into final production u_t as $\frac{L_tH_tu_t}{L_tH_t}$ with $L_tH_tu_t$ given by (37), $L_t = L_0e^{nt}$ and H_t given by (36).

Resource use R_t is given by (see Appendix D)

$$\begin{aligned} R_t &= \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}x_t^{-1}\left\{K_0\right. \\ &\quad -\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\ &\quad\left.-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\right\}. \end{aligned} \tag{38}$$

The only stock variable left is the stock of the resource $S_t = S_0 - \int_0^t R_s ds$ to which we turn now. To do so, we need to integrate over R_t given by (38) which leads to

$$
\begin{aligned}
\int_0^t R_s ds \quad = \quad & \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1} \times \\
& \times \left\{ -K_0 \frac{1}{\psi}\left[1 - x_t^{-1}\right] \right. \\
& + \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}} \frac{1}{\psi^2}\frac{1}{\tilde{b}_1} L_{02}F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\left[1-x_t^{-1}\right] \\
& + \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}} \frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)} L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right] \\
& + \underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}} \frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2} L_{02}F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\left[1-x_t^{-1}\right] \\
& \left. + \underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}} \frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)} L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right] \right\}.
\end{aligned} \tag{39}
$$

This completes our preliminary calculations and we can proceed by solving for the initial conditions, i.e. the initial values for the three co-state variables in the next section.

4 Solving the Model

To pin down a particular solution to our dynamic system that comprises the economy, we have to pin down the initial values for the three co-state variables of the model. Given these initial values, we can then proceed by tracing the full dynamics of the model's variables.

4.1 Initial Co-States

Assuming given initial values for the three state variables, i.e. the stock of human and physical capital together with the stock of the resource, the three transversality conditions (13) through (15) are serving as the mathematical basis. Appendix D (derivation of equations 84, 87 and 89) at the end of the paper shows that these conditions imply

$$
\begin{aligned}
K_0 &= \frac{L_0}{\psi}\left[\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1}+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}\right] && (40)\\
H_0 &= \left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{L_0}{\psi^2}\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. && (41)\\
&\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\},\\
S_0 &= \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\frac{L_0}{\psi^2}\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. && (42)\\
&\left.+\,\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}.
\end{aligned}
$$

The interpretation of this system of equations is straightforward. The left hand side gives the initial endowment of the economy at time $t=0$ in terms of capital stocks and resources. The right hand side gives the demand for initial endowments of the economy given that it follows an optimal behavior in the sense of the above utilitarian criterion. This optimality based demand is reflected by the initial co-states $\lambda_{1,0}, \lambda_{2,0}$ and $\lambda_{3,0}$. We note that $\lambda_{1,0}$ is contained - besides the model's parameters - in φ_1.

Using the definition of φ_1 from above (23) and dividing (41) by (42) gives

$$
\frac{H_0}{S_0}=\frac{\beta}{\gamma}\frac{\lambda_{3,0}}{\lambda_{2,0}}. \tag{43}
$$

It follows from the definitions of φ_1, φ_2 and ζ in (23) together with (43) that

$$
\frac{\lambda_{1,0}}{\lambda_{2,0}}\beta = A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{\alpha}{1-\alpha}}(1-\zeta)^{-\frac{\alpha}{1-\alpha}}. \tag{44}
$$

Using this again in the definition of φ_1 gives

$$
\frac{\lambda_{1,0}}{\varphi_1}\frac{\beta}{\lambda_{2,0}} = A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}. \tag{45}
$$

The results (43), (44) and (45) used in the system of transversality conditions (41) through (42) give

$$K_0 = \frac{L_0}{\psi}\left[\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1}+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}\right] \tag{46}$$

$$\begin{aligned} H_0 = {} & A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\times \\ & \left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. \\ & \left.+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}, \end{aligned} \tag{47}$$

$$\begin{aligned} S_0 = {} & A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\times \\ & \left\{\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. \\ & \left.+\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}. \end{aligned} \tag{48}$$

Note that the system of reformulated transversality conditions (46) through (48) is now a system in just two variables, i.e. ζ and $\lambda_{1,0}$, that condenses the initial conditions $\lambda_{i,0}, i = 1,2,3$. We note further that (48) together with (43) implies (47), and hence, that is sufficient to concentrate either on (46) and (47) or (46) and (48) in solving for ζ in place of $\lambda_{i,0}, i = 1,2,3$.

To arrive at a system of equations that summarizes initial conditions in only one variable, i.e. ζ, we need to define some additional quantities. We split up each state variable, i.e. each capital stock, into a component used to cover subsistence consumption and a second component available for excess consumption. A solution can only exist if available stocks are able to cover both components.

$$
\begin{aligned}
K_0^{+} &= \frac{L_0}{\psi}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\,\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1},\\
\underline{K}_0 &= K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\,\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}\\
&= K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\tilde{a}_2}\,\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2},
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
H_0^{+} &= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\,\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1(\tilde{b}_1-1)}\\
&= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{-\alpha}{(1-\alpha)\eta}}\,\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1(\tilde{b}_1-1)},\\
\underline{H}_0 &= H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}(1-\zeta)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\,\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)}\\
&= H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)},
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
S_0^{+} &= A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\lambda_{1,0}^{-\frac{1}{\eta}}(1-\zeta)^{\frac{-\alpha}{(1-\alpha)\eta}}\,\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1(\tilde{b}_1-1)},\\
\underline{S}_0 &= S_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)}.
\end{aligned}
\tag{51}
$$

These six quantities have a straightforward economic interpretation. K_0^+, H_0^+ and S_0^+ are the parts of the initial capital stocks that are required to allow for future consumption in excess of $\underline{c}$. $\underline{K}_0$,$\underline{H}_0$ and $\underline{S}_0$ are the parts of the initial capital stocks left for excess consumption after covering the needs for subsistence consumption.

Taking ratios gives

$$
\frac{K_0^{+}}{H_0^{+}} = A^{\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{1-\alpha}}\varphi_2^{\frac{1}{1-\alpha}}\psi(\tilde{b}_1-1)(1-\zeta)^{\frac{1}{1-\alpha}}\,\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)}, \tag{52}
$$

$$
\frac{\underline{K}_0}{\underline{H}_0} = \frac{K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}}{H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)}{\tilde{b}_2(\tilde{b}_2-1)}}. \tag{53}
$$

In equilibrium, (52) and (53) have to equal each other, i.e. $\frac{K_0^+}{H_0^+}=\frac{\underline{K}_0}{\underline{H}_0}$. We note that this defines one non-linear equation in ζ given initial values for K_0, H_0 and S_0. Once a solution ζ^* for this equation is found, it will pin down the soltion $\lambda_{1,0}^*$ through e.g. (46). This pins down $\lambda_{2,0}^*$ via (44). $\lambda_{3,0}^*$ can then be computed via e.g. (43).

Appendix F at the end of the paper proves that if a solution ζ^* exists it is unique. The Appendix further elaborates on the necessary conditions for such a solution to exist. Here, we focus on the economic intuition behind these conditions. Basically, the conditions demand the initial capital stocks K_0, H_0 and S_0 to be large enough to guarantee at least the subsistence level of consumption $\underline{c}$ for all times. Mathematically, we need

$$\underline{K}_0 > 0,\ \underline{H}_0 > 0 \text{ and } \underline{S}_0 > 0, \tag{54}$$

where $\underline{K}_0$, $\underline{H}_0$ and $\underline{S}_0$ are given by (49), (50) and (51). If one of the initial stocks fails to satisfy its condition, the problem has no solution at all. It is worth noting at this point that H_0 and S_0, i.e. the initial stocks of human and natural capital can be substituted against each other as long as both are larger than zero. Multiplying (51) by $\frac{H_0}{S_0}$ gives $\underline{S}_0\frac{H_0}{S_0} = \underline{H}_0$. This implies that if a solution exists, both, $\underline{H}_0 > 0$ and $\underline{S}_0 > 0$ will be satisfied automatically (if no solution exists, both won't be satisfied). It is therefore sufficient to focus e.g. on $\underline{K}_0 > 0$ and $\underline{H}_0 > 0$ to pin down the implications for a solution to exist.

Appendix F at the end of the paper shows that any ζ^* needs to satisfy $\underline{\zeta} < \zeta^* < \bar{\zeta}$. $\underline{\zeta}$ is a lower bound for ζ^* that exactly satisfis $\underline{K}_0 = 0$, i.e.

$$K_0 \quad = \quad \frac{L_0}{\psi}\underline{c}\left(1-\underline{\zeta}\right)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\underline{\zeta})}{\tilde{b}_2}.$$

Any $\zeta < \underline{\zeta}$ would characterize a situation where the initial physical stock of capital is insufficiently low to allow for subsistence consumption.

$\bar{\zeta}$ is an upper bound for ζ^* that is implicitly defined by

$$\bar{\zeta} =_{\zeta^* \leq 1} \left| H_0 - A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)}\right|.$$

Any $\zeta > \bar{\zeta}$ for $\bar{\zeta} < 1$ would characterize a situation in which initial human capital is not capable of allowing for subsistence consumption. If $\bar{\zeta} = 1$, we encounter a situation in which initial human capital is that large that it never can cause a scarcity problem to the economy.

Taken together, only for $\zeta > \underline{\zeta}$ we find the initial physical capital stock K_0 sufficiently large to cover subsistence consumption. And only for $\zeta < \bar{\zeta}$, the initial stocks of human and natural capital are sufficient to do so.[3] It can easily be verified that $\underline{c} \to 0$ implies $\underline{\zeta} \to -\infty$ and $\bar{\zeta} \to 1$. In case of zero subsistence consumption a solution always exists. Once ζ^* satisfying $\underline{\zeta} < \zeta^* < \bar{\zeta}$ is found, the initial co-states $\lambda^*_{i,0}, i = 1,2,3$ can be uniquely computed as explained above.

[3]The borderline case $\underline{\zeta} = \zeta^* = \bar{\zeta}$ implies that the economy is in equilibrium right from the beginning at $t = 0$.

4.2 Dynamics

As we know that the transversality conditions pin down the solution (if it exists) uniquely, we can now proceed by using the transversality conditions to trace the model's variables over time starting at $t = 0$.

We already gave the consumption path in (25) which can now be rewritten by using the definition of ψ and $x_t = e^{-\psi t}$ as

$$\begin{aligned} c_t &= \underline{c} + (\lambda_{1,0}^*)^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}+\frac{\delta_1}{\eta}\right)t} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} \left(\frac{x_t}{1-\zeta^* x_t}\right)^{-\frac{\alpha}{(1-\alpha)\eta}}, \\ &= \underline{c} + (\lambda_{1,0}^*)^{-\frac{1}{\eta}} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} (1-\zeta^* x_t)^{\frac{\alpha}{(1-\alpha)\eta}} e^{\frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta}t} \end{aligned} \tag{55}$$

As $t \to \infty$ and $x_t \to 0$ we find in the limit

$$\lim_{t\to\infty} c_t = \begin{cases} \underline{c} & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n) < 0, \\ \underline{c} + (\lambda_{1,0}^*)^{-\frac{1}{\eta}} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n) = 0, \\ \underline{c} + (\lambda_{1,0}^*)^{-\frac{1}{\eta}} (1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}} \times & \\ \quad \lim_{t\to\infty} e^{\frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta}t} = \infty & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n) > 0. \end{cases} \tag{56}$$

Differentiating (55) with respect to time gives the growth rate of consumption in excess of $\underline{c}$ as

$$\frac{\dot{c}_t}{c_t - \underline{c}} = \frac{1}{\eta}\left(\frac{\alpha}{1-\alpha}\frac{\psi}{1-\zeta^* x_t} - \rho - \delta_1\right). \tag{57}$$

In the limit as $t \to \infty$ and $x_t \to 0$, the growth rate of excess consumption approaches

$$\begin{aligned} \lim_{t\to\infty} \frac{\dot{c}_t}{c_t - \underline{c}} &= \frac{1}{\eta}\left(\frac{\alpha}{1-\alpha}\psi - \rho - \delta_1\right) \\ &= \frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta}. \end{aligned}$$

It is obvious that the growth rate of excess consumption can be negative or positive, depending on the model's parameters and the economy's position during adjustment. Negative growth rates are likely to whenever ζ^* is small and in particular if it is negative. This is straightforward as e.g. a negative ζ^* implies the economy initially to have a lower capital productivity compared with steady state. This leads the economy to use parts of its initial physical capital stock for excess consumption. As the capital stock declines this excess consumption has to be gradually reduced and, hence, excess consumption declines in such a situation. In general, positive growth rates in excess consumption are to be expected whenever the economy has a high capital productivity and the incentive for accumulating physical capital is high.

Whether excess consumption growth is positive in the limit depends on whether $\frac{\alpha}{1-\alpha}\psi-\rho-\delta_1>0$. Using the definition of ψ, it turns out that this is the case if $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0$. The condition for the model's parameters (34) implies after some manipulations that $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)<\eta\,[\beta(B-\delta_2)-\gamma n]$ where the right hand side needs to be positive due to the second condition for the parameters (35). Therefore, we find that a positive, zero or negative growth rate for excess consumption is possible in the limit.

(57) is nothing else than a Keynes-Ramsey rule for the case of non-zero subsistence consumption as $\frac{\alpha}{1-\alpha}\frac{\psi}{1-\zeta^* x_t}$ is equal to the gross rate of return to investments into physical capital as will be shown further down below.

Turn to the dynamic behavior of the input factors to final goods production. We focus on per capita values to eliminate the effects of pure population growth.

Using the preliminary result (33) for the physical capital stock, the corresponding transversality condition (46), $L_t=e^{nt}L_0$ and the definition $x_t=e^{-\psi t}$ results in $k_t=\frac{K_t}{L_t}$ as[4]

$$\begin{aligned} k_t &= e^{-\frac{(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]}{(1-\alpha)\eta}t}(1-\zeta^* x_t)^{\frac{1}{1-\alpha}}\frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{\psi}(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^* x_t)}{\tilde{b}_1} \\ &\quad +(1-\zeta^* x_t)^{\frac{1}{1-\alpha}}\frac{c}{\psi}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^* x_t)}{\tilde{b}_2}. \end{aligned} \tag{58}$$

The behavior of the physical capital stock is in general non-monotonic.

In the limit, $t\to\infty$ and $x_t\to 0$, $\lim_{t\to\infty}k_t$ can behave in different ways depending on the models parameters[5]

$$\lim_{t\to\infty}k_t = \begin{cases} \frac{c}{\psi\tilde{b}_2} & \text{for } [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)<0, \\ \frac{c}{\psi\tilde{b}_2}+\frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{\psi}\frac{(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1} & \text{for } [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)=0, \\ \frac{c}{\psi\tilde{b}_2}+\frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{\psi}\frac{(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1}\times & \\ \quad \lim_{t\to\infty}e^{\frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta}t}=\infty & \text{for } [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0. \end{cases} \tag{59}$$

Human capital employed in final goods production is $H_t u_t$. Using (37), the initial condition (45), $L_t=e^{nt}L_0$, $\varphi_2=\frac{1-\alpha}{\psi}$ and $x_t=e^{-\psi t}$, we find

$$H_t u_t = e^{\frac{\gamma}{1-\alpha}(B-\delta_2+n)t}A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{1-\alpha}{\psi}\right)^{-\frac{1}{1-\alpha}}(1-\zeta^* x_t)^{-\frac{1}{1-\alpha}}k_t. \tag{60}$$

[4]For the details see the derivation of equation (85) in Appendix E.
[5]Note that ${}_2F_1(\tilde{a},\tilde{b};\tilde{b}+1;0)=1$.

It is usefull to note that H_t is the economy wide level of human capital from which each member of the household benefits. As effective human capital in production depends linearly on k_t, it shares the non-monotonic behavior of the capital-labor ratio in general. Its behavior in the limit as $t \to \infty$ is to some extend different. As it's reasonable to assume that $B - \delta_2 + n > 0$, we find $H_t u_t$ to tend to infinity in any case provided the condition for the model's parameters (34) is met. To see that note that $\lim_{t\to\infty} k_t > 0$ in all cases considered above. As $e^{\frac{\gamma}{1-\alpha}(B-\delta_2+n)t}$ dominates $1 - \zeta^* x_t$ for large t, $H_t u_t$ tends to infinity as $t \to \infty$.

Turning to the per capita resource use $r_t = \frac{R_t}{L_t}$, using (38), (60) and the initial condition (43) gives

$$r_t = e^{-\frac{\beta}{1-\alpha}(B-\delta_2+n)t} A^{-\frac{1}{1-\alpha}} \left(\frac{H_0}{S_0}\right)^{-\frac{\beta}{1-\alpha}} \left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}} \left(\frac{1-\alpha}{\psi}\right)^{-\frac{1}{1-\alpha}} (1-\zeta^* x_t)^{-\frac{1}{1-\alpha}} k_t \qquad (61)$$

It is obvious that per capita resource use can behave non-monotonic as well. In the long run as $t \to \infty$, $\frac{R_t}{L_t}$ declines and approaches zero if condition (34) is met. This is true because $e^{-\frac{\beta}{1-\alpha}(B-\delta_2+n)t} \times e^{-\frac{(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]}{(1-\alpha)\eta}t} = e^{-\frac{(1-\alpha)(\rho-n)+(\eta-1)[\beta(B-\delta_2)-\gamma n]}{(1-\alpha)\eta}t-nt}$ where $e^{-\frac{(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]}{(1-\alpha)\eta}t}$ is the function in time governing the behavior of $\lim_{t\to\infty}\frac{K_t}{L_t}$. From this last expression it is clear that also $R_t \to 0$ as $t \to \infty$.

Given the results (60) and (61), we can now turn to per capita production $y_t = \frac{Y_t}{L_t}$ next

$$y_t = A k_t^{\alpha} (H_t u_t)^{\beta} r_t^{\gamma} = \frac{\psi}{1-\alpha}(1-\zeta^* x_t)^{-1} k_t. \qquad (62)$$

The limiting behavior of per capita production can easily be derived by making use of the dynamics of the per capita stock of physical capital given in (59)

$$\lim_{t\to\infty} y_t = \begin{cases} \frac{c}{(1-\alpha)\tilde{b}_2} & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)<0, \\ \frac{c}{(1-\alpha)\tilde{b}_2} + \frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{1-\alpha}\frac{(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1} & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)=0, \\ \frac{c}{(1-\alpha)\tilde{b}_2} + \frac{(\lambda_{1,0}^*)^{-\frac{1}{\eta}}}{1-\alpha}\frac{(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}}{\tilde{b}_1} \times & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0. \\ \quad \lim_{t\to\infty} e^{\frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta}t} = \infty & \end{cases} \qquad (63)$$

Again, we observe three cases where only one is characterized by long-run positive growth.

From (62), we can infer also that physical capital productivity is given by $\frac{y_t}{k_t} = \frac{\psi}{1-\alpha}(1-\zeta^* x_t)^{-1}$ and the gross rate of return to physical capital is $i_t = \alpha\frac{Y_t}{K_t} = \frac{\alpha}{1-\alpha}\psi(1-\zeta^* x_t)^{-1}$. In the limit, $\frac{y_t}{k_t} \to \frac{\psi}{1-\alpha} = \frac{1}{\varphi_2}$ and $i_t \to \frac{\alpha}{1-\alpha}\psi = \frac{\alpha}{\varphi_2}$ as $t \to \infty$. It also becomes clear now that $\frac{1}{\varphi_1}$ is equal to the capital productivity at time 0. Evaluating (62) at $t = 0$ ($x_0 = 1$) gives $\frac{Y_0}{K_0} = \frac{\psi}{1-\alpha}(1-\zeta^*)^{-1} = \frac{1}{\varphi_2}\frac{\varphi_1}{\varphi_2} = \frac{1}{\varphi_1}$.

Using the previous results, we find I human capital as[6]

$$\begin{aligned} H_t &= e^{(B-\delta_2)t}A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\left(\frac{1-\alpha}{\psi}\right)^{-\frac{1}{1-\alpha}}L_0\times \\ &\left[(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}x_t^{\tilde{b}_1-1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*x_t)}{\tilde{b}_1(\tilde{b}_1-1)}\right. \\ &\left.+\underline{c}\frac{1}{\psi^2}x_t^{\tilde{b}_2-1}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*x_t)}{\tilde{b}_2(\tilde{b}_2-1)}\right]. \end{aligned}$$

H_t tend always to infinity as $t\to\infty$. This is because of the second term in curly brackets in (64). It is easy to verify that $e^{(B-\delta_2)t}x_t^{\tilde{b}_2-1}=e^{\frac{\gamma}{1-\alpha}(B-\delta_2+n)t}$.[7] Hence, human capital necessary to cover subsistence consumption grows asymptotically at a positive rate. The first term in curly brackets represents human capital necessary to cover excess consumption. This part of human capital might tend to zero or infinity depending on the model's parameters. The responsible term $e^{(B-\delta_2)t}x_t^{\tilde{b}_1-1}=e^{-\frac{1}{(1-\alpha)\eta}\{(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]-\gamma\eta(B-\delta_2+n)\}t}$ tends to infinity as long as $(1-\alpha)(\rho-n)-[\beta(B-\delta_2)-\gamma n]<\gamma\eta(B-\delta_2+n)$. If this condition is not met, this part of individual human capital tends to zero. However, total individual human capital will always grow without bounds.

While H_t tends to infinity as $t\to\infty$, it is not surprising that S_t will be depleted asymptotically as this just reflects the transversality for the resource stock. S_t can be found by using (39), the transversality condition (46) together with (44) and (45)[8]

$$\begin{aligned} S_t &= S_0-\int_0^t R_s ds \\ &= S_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{-\beta}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{-\beta}{1-\alpha}}\left(\frac{1-\alpha}{\psi}\right)^{-\frac{1}{1-\alpha}}\frac{1}{\psi^2}L_0\times \\ &\left[(\lambda_{1,0}^*)^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{x_t^{\tilde{b}_1-1}}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*x_t)\right. \\ &\left.+\underline{c}\frac{x_t^{\tilde{b}_2-1}}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*x_t)\right]. \end{aligned} \tag{64}$$

As the transversality conditions for the model parameters demand $\tilde{b}_1,\tilde{b}_2>1$, we find the terms inside brackets to shrink down to zero over time. Hence, $\lim_{t\to\infty}S_t=0$.

Taken together, the expressions derived in this section are fully characterizing the solution of the problem posed in the beginning of the paper. As such, these equations are fully characterizing the unique saddle-path which is followed by the economy.

[6] For the details see the derivation of equation (90) in Appendix E.
[7] Note that $\lim_{x_t\to 0}{}_2F_1(a,b-1;b+1;\zeta^*x_t)=b(b-1)\frac{1}{6}$.
[8] For the details see the derivation of equation (88) in Appendix E.

Summarizing this section, we find that the economy might tend towards three possible states in the long run. In case $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0$, the economy grows in the long-run. Consumption and production per capita grow without bound and tend to infinity. If $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)<0$, the economy is characterized by subsistence consumption and production per capita; there is no long-run growth. By change and if it happens to be that $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)=0$, the economy is tending to a long-run zero growth scenario in per capita consumption and production above the subsistence level. Which case is relevant in a particular case is depending on the parameters of the model. Below, we will calibrate the model to the situations of particular countries and find some countries with zero and some with positive long-run growth. To highlight our conclusions on growth, we focus on the economy's steady-state growth in the next section.

4.3 Steady-State Growth

Up to now, we developed the representations of the model's variables in levels. This section turns to the steady state of our economy focusing on the growth rates of the model's variables as time tends to infinity. Obviously, the preceding section is helpful as only have to inspect the asymptotic behavior of the variables' levels as $t\to\infty$.

We already elaborated on the growth rate of per capita consumption in (57) and the gross rate of return on physical capital. Rearranging the results by using the definition of ψ gives

$$\begin{aligned}\lim_{t\to\infty}\frac{\dot c_t}{c_t} &= \lim_{t\to\infty}\frac{c_t-\underline{c}}{c_t}\frac{1}{\eta}\left(\frac{\alpha}{1-\alpha}\psi-\rho-\delta_1\right),\\ &= \begin{cases} 0 & \text{for}\quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)\le 0\\ \frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta} & \text{for}\quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0\end{cases}.\end{aligned}\tag{65}$$

In case the model's parameters satisfy $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0$, we observe per capita to grow asymptotically in steady state at a constant positive rate. Consequently, $\frac{c_t-\underline{c}}{c_t}\to 1$ as $t\to\infty$. If, however, $[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)\le 0$ prevails, per capita consumption shrinks towards zero growth over time. Comparing with the growth rate of per capita excess consumption reveals differences. The latter grows asymptotically at a constant rate which might by smaller, equal or greater than zero.

Looking at the parameter constellation responsible for the asymptotic growth rate of consumption reveals that this is exactly the same condition qualifying for positive or zero long-run growth in the per capita physical capital stock (see equation 59).

The rate of return on physical capital is given by

$$\begin{aligned}\lim_{t\to\infty} i_t &= \lim_{t\to\infty}\frac{\partial Y_t}{\partial K_t}=\frac{\alpha}{1-\alpha}\psi \\ &= \frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{1-\alpha},\end{aligned} \tag{66}$$

which is positive in any case if $B-\delta_2>0$.

Next, we turn to the asymptotic growth rate of the capital stocks. The dynamic behavior of k_t is given in (59). It is straightforward that the corresponding growth rate is given by

$$\lim_{t\to\infty}\frac{\dot{k}_t}{k_t} = \begin{cases} 0 & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)\leq 0, \\ \frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta} & \text{for} \quad [\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)>0. \end{cases} \tag{67}$$

Inspecting the behavior of $H_t u_t$ and r_t given by (60) and (61) and taking account of the production function $Y_t=AK_t^\alpha(L_tH_tu_t)^\beta R_t^\gamma$ shows that the asymptotic growth rate of per capita production is identical to the one for physical capital, $\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}$. This is the case because growth in human capital and resource use in production exactly cancel in the limit. While $H_t u_t$ grows in the long-run, r_t shrinks. In the limit, we find

$$\beta\lim_{t\to\infty}\frac{(\dot{H_t u_t})}{H_t u_t} = \frac{\beta\gamma}{1-\alpha}(B-\delta_2+n)+\beta\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}, \tag{68}$$

$$\gamma\lim_{t\to\infty}\frac{\dot{r}_t}{r_t} = -\frac{\beta\gamma}{1-\alpha}(B-\delta_2+n)+\gamma\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}. \tag{69}$$

The source for this asymptotically positive rate of return can be found in the limiting behavior of the human capital stock which grows without bound.

5 Calibration

This section utilizes the above findings to analyze the full adjustment path of the model economy calibrated to the situation of different country groups. Given that we can pin down the initial conditions for the solution of the problem, we can calibrate the model using recent World Bank data on endowments with different types of capital.

5.1 Preliminaries

Before starting our calibration of the above model, some words on the units of measurement are in order. As usual in theoretical models, all the quantities in our model are denominated in real units. The data we are using in the below standing sections will be denominated in US $ of 2014. This requires us to transform all the model's quantities into a common unit that we calibrate to match this currency. Due to this, the implicit relative prices in the model become relevant.

From the maximization it is obvious that optimal relative prices are given by the ratios of the Lagrange multipliers. The price of one unit of human capital H_t in terms of physical capital at time t is given by $\frac{\lambda_{2,t}}{\lambda_{1,t}}$ and the corresponding resource' relative price by $\frac{\lambda_{3,0}}{\lambda_{1,t}}$ as $\lambda_{3,t}$ is constant over time. We note further that in our model consumption c_t and output y_t carry the same units as the physical capital stock k_t.

Our calibration further down below will match a country's actually realized output and stock of reproducible capital per capita in the base year $t = 0$, i.e. $\tilde{y}_0$ and $\tilde{k}_0$, with the model's predicted output at $t = 0$. From here on, we denote the currency denominated counterpart of model's real valued quantity X_t by $\tilde{X}_t$. We chose the year 2014 as the base year and will trace the models quantities thereafter. 2014 is chosen as the most recent data are available for 2014. The other quantities of interest that we trace over time will be consumption $\tilde{c}_t$, resource use $\tilde{r}_t = \frac{\lambda_{3,t}}{\lambda_{1,t}} r_t$ and its corresponding stock $\tilde{s}_t = \frac{\lambda_{3,t}}{\lambda_{1,t}} s_t$ as well as the stock of human capital $\tilde{H}_t = \frac{\lambda_{2,t}}{\lambda_{1,t}} H_t$.

Some additional calculations are required as we proceed here in a different way compared with the model's solution in Section 4. This is necessary as we don't calibrate on initial stocks alone but on the available stocks together with matching countries' actual output in the base year 2014. The latter turns the solution of the model somehow upside down as we start by calibrating ζ^* first by noting that (see the discussion below equation 63)

$$1 - \zeta^* = \frac{\psi}{1-\alpha} \frac{\tilde{k}_0}{\tilde{y}_0}.$$

Given ζ^*, we have to solve for $\lambda_{1,0}^*$ next. This is done by solving the transversality condition (46) which gives

$$\lambda_{1,0}^* = \left(\psi \tilde{K}_0 - \underline{\tilde{c}}(1-\zeta^*)^{\frac{1}{1-\alpha}} \frac{{}_2F_1(\tilde{a}_2, \tilde{b}_2; \tilde{b}_2 + 1; \zeta^*)}{\tilde{b}_2} \right)^{-\eta} s \left((1-\zeta^*)^{\frac{\eta-\alpha}{(1-\alpha)\eta}} \frac{{}_2F_1(\tilde{a}_1, \tilde{b}_1; \tilde{b}_1 + 1; \zeta^*)}{\tilde{b}_1} \right)^{\eta}.$$

Note that this introduces the chosen currency units into the shadow price $\lambda_{1,0}^*$. With ζ^* and $\lambda_{1,0}^*$ we are already able to trace consumption, the stock of reproducible capital and output over time by using (55), (58) and (62).

The remaining interesting quantities are $\tilde{r}_t$, $\tilde{s}_t$ and $\tilde{H}_t$. By using the development of r_t (61), $\lambda_{1,t}$ (24), the optimal ratio $\frac{H_0}{S_0}$ (43) and the definition of φ_1 we find

$$\tilde{r}_t = \frac{\lambda_{3,t}}{\lambda_{1,t}} r_t = \frac{\gamma}{1-\alpha} \psi (1 - \zeta^* x_t) \tilde{k}_t.$$

Using the development of S_t (64), $\lambda_{1,t}$ (24), the optimal ratio $\frac{H_0}{S_0}$ (43) and the definition of φ_1 we find

$$
\begin{aligned}
\tilde{s}_t = \frac{\lambda_{3,t}}{\lambda_{1,t}}\frac{S_t}{L_t} \quad = \quad & \frac{\tilde{S}_0}{L_0} e^{-(\delta_1+n)t}(1-\zeta^*)^{-\frac{\alpha}{1-\alpha}}\left(\frac{x_t}{1-\zeta^* x_t}\right)^{-\frac{\alpha}{1-\alpha}} - \frac{\gamma}{1-\alpha}\frac{1}{\psi}e^{-nt}\times \\
& \left[(\lambda^*_{1,0})^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{x_t^{\tilde{b}_1-1}}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^* x_t)\right. \qquad (70) \\
& \left. + \underline{\tilde{c}}\frac{x_t^{\tilde{b}_2-1}}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^* x_t)\right].
\end{aligned}
$$

Finally, using the development of H_t (64), $\lambda_{2,t}$ (20), $\lambda_{1,t}$ (24), the optimal ratio $\frac{H_0}{S_0}$ (43) and the definition of φ_1 delivers

$$
\begin{aligned}
\tilde{H}_t = \frac{\lambda_{2,t}}{\lambda_{1,t}}H_t \quad = \quad & e^{-\delta_1 t}\frac{\beta}{1-\alpha}\psi\left(\frac{x_t}{1-\zeta^* x_t}\right)^{-\frac{\alpha}{1-\alpha}} L_0 \times \\
& \left[(\lambda^*_{1,0})^{-\frac{1}{\eta}}(1-\zeta^*)^{-\frac{\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}x_t^{\tilde{b}_1-1}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^* x_t)}{\tilde{b}_1(\tilde{b}_1-1)}\right. \\
& \left. + \underline{\tilde{c}}\frac{1}{\psi^2}x_t^{\tilde{b}_2-1}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^* x_t)}{\tilde{b}_2(\tilde{b}_2-1)}\right].
\end{aligned}
$$

We note that we don't have to use an initial value for human capital in the transformed equations on the variables' optimal paths. This is not a deficit of the approach taken but simply a reflection of the optimal ratio of $\frac{H_0}{S_0}$ given in (43) that implies an initial stock of human capital given an initial stock of resources. This can be see by rewriting (43) as $\frac{\lambda_{2,0}H_0}{\lambda_{3,0}S_0} = \frac{\tilde{H}_0}{\tilde{S}_0} = \frac{\beta}{\gamma}$.

Finally, we have to care about the existence of a solution to our problem. For this, we reformulate conditions (54) in nominal terms. Using the transversality conditions for K_t, H_t and S_t given by (40), (41) and (42) together with the definition of $1-\zeta^*$ gives

$$
\underline{\tilde{k}}_0 \quad = \quad \frac{\underline{\tilde{K}}_0}{L_0} = \frac{\tilde{K}_0}{L_0} - \frac{\underline{\tilde{c}}}{\psi}(1-\zeta^*)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2} > 0, \qquad (71)
$$

$$
\underline{\tilde{s}}_0 \quad = \quad \frac{\underline{\tilde{S}}_0}{L_0} = \frac{\tilde{S}_0}{L_0} - \frac{\gamma}{1-\alpha}\frac{\underline{\tilde{c}}}{\psi}(1-\zeta^*)^{\frac{\alpha}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)} > 0, \qquad (72)
$$

$$
\underline{\tilde{h}}_0 \quad = \quad \frac{\underline{\tilde{H}}_0}{L_0} = \frac{\beta}{\gamma}\underline{\tilde{s}}_0 > 0. \qquad (73)
$$

$\underline{\tilde{h}}_0$ is a per capita value that is not to be interpreted as per capita human capital as human capital is benefitting all household members as a positive externality. Rather, it shoud be interpreted as a minimum investment per capita in human capital that should have been done (net of depreciation) in the past up to $t = 0$.

5.2 Data on Initial Endowments

In the course of calibration, reasonable numbers for the initial stocks of natural resources and reproducible capital have to be found. The World Bank (2018) provides estimates for stocks of produced, natural and human capital up to 2014 in US $ at current prices. This is part of a quite comprehensive cross country data base on what the World Bank terms “The Wealth of Nations”. We chose 2014 as the starting year for our calibration in order to make use of the most recent data available.

Although it is clear that such a data base provides estimates only, the data are the best available and can be of use for the present purpose Table 1 gives a summary of the data for 2014 in per capita terms for income based groupings of countries.[9]

World Bank data

	no. countries	prod. capital	nat. capital (incl. land)	nat. capital (excl. land)	net for. assets	GNI
Low-income	24	1,967	6,421	1,236	-322	789
Lower-middle income	37	6,531	6,949	2,187	-650	2,035
Upper-middle income	36	28,527	18,960	8,339	-432	8,563
High-income (non-OECD)	15	59,069	80,104	74,243	14,005	
High-income (OECD)	29	195,929	19,525	12,877	-5,464	
High-income	44	166,438	32,579	26,100	-1269	43,351
World	141	44,760	5,841	8,810	-676	10,987

Table 1: Capital stocks and GNI per capita in 2014 US$

Note: World Bank (2018, Appendix B) estimates for stocks of different types of capital and net foreign assets per capita in 2014 US $. High-income values are averaged values (weighted by population) for OECD and non-OECD high-income countries reported in World Bank (2018, p. 233). Produced capital: machinery, equipment, structures, urban land; natural capital (incl. land): energy resources (oil, natural gas, hard coal, lignite), mineral resources (bauxite, copper, gold, iron, lead, nickel, phosphate, silver, tin, zinc), timber resources, nontimber forest resources, crop land, pasture land, protected areas. natural capital (excl. land): natural capital (incl. land) less of crop land, pasture land, protected areas. Human capital estimated from expected presented value of labor income. Population in mln. people. GNI for 2014 in US $ taken from the World Bank data base `https://data.worldbank.org/indicator/NY.GNP.PCAP.CD`

Table 1 also contains data on the country groups' net foreign assets. For calibration of initial stocks of physical capital, we will add these to the stock of produced capital in order to arrive at the capital which is actually owned by the economies. Consequently, we are investigating the domestic economy which is motivated as our model above focuses on the closed economy.

Regarding natural capital, we will draw on the data on natural capital excluding land in our calibration. The reason for doing so will become clear further down below where we elaborate on the models parameters. In case of resources, natural capital excluding land corresponds quite well with other data used in calibration of the resources output elasticity.

The numbers in Table 1 are reflecting country groups' averages. However, our calibration can be done for any single country where we have no missing values in the data base. World Bank (2018) also provides us with estimates of the stock of human capital. However, we are not using them in our calibration. The reasons for this are twofold. First, the World Bank data on human capital don't match with the model's stock of human capital. The World Bank estimates the stocks

[9] Income groups according to the World Bank's thresholds on countries GNI. Details are available from the World Bank's permanent URL `http://go.worldbank.org/L547EEP5C0`.

by computing an expected present value of labor income. Labor income in 2014 is thereby largely determined by the labor share in GDP taken from the Penn World Tables (PWT, Feenstra et al. 2015). The expected present value is computed assuming the economy is in a steady-state where growth is constantly exceeding the discount rate by 1.5% p.a. The expectation is reflecting countries demographic characteristics regarding life expectancy. This concept is not reflecting our intention of calibrating the model for economies potentially starting in the base year off the steady-state and adjusting to a balanced growth path over time. Second, we don't need to pick a value for initial human capital as the nominal value of human capital is implicitly calibrated as explained in the preceding section.[10]

In addition to the data taken from World Bank (2018), we are using World Bank data on the countries GNI for calibrating initial output. We chose for GNI instead of GDP following the argument in Asheim and Buchholz (2004) who favor national income over domestic production in relation to the DHSS model. Thus, we capture output produced using the production factors owned by the economy. This squares well with correcting produced capital using the net foreign asset position of the economies.

Besides the above World Bank data, we are drawing on the PWT 9.1[11] as we need additional information on countries' labor share in GDP and the depreciation rate of physical capital. Furthermore, we are using additional World Bank data on mortality to calibrate human capital depreciation. We postpone discussion of these data to the section where we elaborate on the model's parameters.

5.3 Calibration Values Country Groups

Regarding households' preferences, ρ, η, L_0, n and $\underline{c}$ need to be specified. The rate of time preference is a parameter that is frequently calibrated. We feel that an extensive discussion on this parameter's value is not necessary. We will chose $\rho = 0.03$ which seems to be a common choice also used in e.g. Benchekroun and Withagen (2011).

There exist some contributions to the literature that calibrate the type of Stone-Geary utility function that is used in the present context. Achury et al. (2012) calibrate an intertemporal utility function identical to the present one in (1) for the US and use $\eta = \frac{1}{0.23}$ which is roughly equal to 4.3. They refer to their choice of η as a standard choice in the portfolio literature. Ogaki et al. (1996) provide estimates for $\frac{1}{\eta}$ ranging from 0.569 up to 0.646 corresponding to η decreasing from about 1.68 down to 1.55. Alavarez-Pelaez and Diaz (2005) are calibrating η in a range from 1.5 up to 2.5 in their application of Stone-Geary preferences. Ravn et al. (2006, 2008) analyze the influence of subsistence points such as subsistence consumption on the dynamics of macroeconomic development in

[10] Note that we are unable to identify real human capital this way. For this we would require data on human capital measured in terms of real output of the economy. Such data are unavailable to the best of our knowledge. We can only trace human capital valued at its optimal price where real human capital and its price substitute with unitary elasticity (see condition 43).

[11] Available at `https://www.rug.nl/ggdc/productivity/pwt/`.

general. Despite this, their specification for intertemporal utility is in accordance with the present situation. During calibration of their models they use a value of 2 for η. Regarding the choices for η, we follow Ravn et al. (2006, 2008) with a value of 2. This is an intermediate value that is in between what has been used in Alavarez-Pelaez and Diaz (2005) and Achury et al. (2012).

We are calibrating our model on a per capita basis and, hence, normalize L_0 to 1. The population growth rate n is taken from the World Bank. Its value across different groups of countries during 2014 together with the crude mortality rate across all age groups is given in Table 2 below. The mortality rate will be used later on for calibrating human capital depreciation δ_2 (see equation 4).

	no. countries	pop. growth	crude mortality	resource rents' share in GDP	no. countries	labor income's share in GDP	no. countries	capital depreciation
Low-income	34	2.6	0.9	12.57	15	51.30	24	4.99
Lower-middle income	47	1.5	0.8	5.57	26	52.87	34	4.58
Upper-middle income	56	0.8	0.7	5.83	37	47.94	35	5.00
High-income	79	0.6	0.8	2.00	55	52.79	44	4.40
World	216	1.2	0.8	3.38	133	51.29	137	4.70

Table 2: Demographics, GDP shares and Capital Depreciation 2014 in %

Note: Population growth and mortality in % p.a. from the World Bank's data base: https://data.worldbank.org/indicator/SP.POP.GROW and https://data.worldbank.org/indicator/sp.dyn.cdrt.in. Averages of resource rents in % of GDP calculated using data from https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. Labor income share and depreciation rates on physical capital averages computed using the Penn World Tables 9.1 (variable labsh and delta, https://www.rug.nl/ggdc/productivity/pwt/); country classification in accordance with the World Bank's classification scheme available at https://datahelpdesk.worldbank.org/knowledgebase/articles/906519.

For the level of subsistence consumption $\underline{c}$, we consider the poverty lines used be the World bank.[12] As of today, the threshold for extreme absolute poverty is set at 1.90 US $ at 2011 prices and at PPP a day available to an individual for covering basic needs (Ferreira et al. 2016). By now, this is considered to apply to low-income countries. The World Bank recently has introduced two additional poverty lines applying to lower- and upper- middle-income countries at 3.20 US $ and 5.50 US $ per day at 2011 prices and PPP. For the calculation behind these numbers see Joliffe and Prydz (2016) who furthermore provide an absolute poverty level for high-income countries at 21.70 US $ per day at 2011 prices and PPP. We convert this numbers into yearly values at prices of 2014 in US $ using the PPP exchange rate. This gives a poverty line of 1,833 (3,631; 3,793; 8,675) US $ using the PPP exchange rate for low (lower-middle, upper-middle, high) income countries.[13]

[12] Values for subsistence consumption have also been proposed in Koulovatianos et al. (2007) and Atkeson and Ogaki (1996) which have been used also in Achury et al. (2012) and Ogaki et al. (1996). These numbers, however, reflect very specific countries which doesn't seem to be in accordance with our analysis. Additionally, investigating poverty lines in this context is interesting as they influences economic policy initiatives especially in low-income countries (see e.g. the United Nation's Sustainable Development Goal on poverty, https://www.un.org/sustainabledevelopment/).

[13] Price changes are taken account by using the implicit GDP deflator obtained by dividing the time series for GDP at PPP valued at constant and current prices for low-income countries available at https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.KD and https://data.worldbank.org/indicator/NY.GDP.PCAP.PP.CD. This results in a growth in prices of 5.34% between 2011 and 2014. PPP exchange rates are implemented by using the implicit exchange rate between GNI per capita in 2014 in int. % (https://data.worldbank.org/indicator/NY.GNP.PCAP.PP.CD) and current US $ (https://data.worldbank.org/indicator/NY.GNP.PCAP.CD).

We turn now to the parameters governing production. The output elasticity γ of resource use R_t is, given the Cobb-Douglas production technology (2), set equal to the share of natural resource rents in GDP. Data on this share is available from the World Bank.[14] Table 2 provides a summary of the data for different groups of countries classified according to the country's level of income. It is clearly visible that the resource dependence increases as income decreases. Resources seem to be most important for the low-income countries.

We use the labor income share in GDP for calibrating the output elasticity of effective human capital in production β. Numbers for the labor income share in GDP in 2014 were taken from the Penn World Tables 9.1 and are provided in Table 2. For the labor share we cannot observe a clear pattern and observe values on average around 0.5 with only moderate variation.[15] Given the assumption of constant returns to scale in production, β and γ, the capital's share $\alpha = 1 - \beta - \gamma$ follows as a remainder.

Produced capital published in World Bank (2018) and discussed above originates largely from the Penn World Tables. It is estimated employing the perpetual inventory method using country and capital good specific rates of depreciation. The country specific rates vary between 3 and 8% per annum. Table 2 gives the average depreciation rates for the country groups under consideration.

Further, we need to find appropriate values for the parameters governing the creation of human capital. Our specification (4) is similar to the specification originally proposed and calibrated by Lucas (1988). In his specification, depreciation of human capital was excluded, i.e. δ_2 was set equal to zero. Lucas (1988) calibrated B at a value of 0.05 which also has been used e.g. in Funke and Strulik (2000). Chen and Funke (2013) used a somewhat higher value of 0.095 for a calibration concerning the Chinese economy. We decided to use 0.05 as a conservative value that is not too optimistic about human capital formation. Regarding δ_2, we choose for the crude mortality rate across all age groups as the unconditional probability for individual human capital ceasing to exist.

Table 3 summarizes our calibration scenario for the different country groups.

5.4 Calibration Results Country Groups

Proceeding as explained above and using the calibration values of the last section, we find that for all country groups the parameter restrictions (34) and (35) are fulfilled. This means that the problem is properly defined and a solution can potentially exist. The second question is then whether such a solution actually exists, i.e. whether initial endowments with physical, natural and human capital are sufficiently large enough (conditions 71, 72 and 73). If not, we would like to find out by how much

This results in an adjustment factor of 2.51 (2.95; 1.79; 1.04) for low (lower-middle, upper-middle, high) income countries.

[14] Data are available from the World Bank Data Base at https://data.worldbank.org/indicator/NY.GDP.TOTL.RT.ZS. For the details on how the numbers are derived see World Bank (2011). Natural resources rents are the sum of oil, natural gas, coal (hard and soft), mineral, and forest rents.

[15] The labor shares reported in Table 2 are low compared with e.g. the traditional $\frac{2}{3}$ that is frequently used. See e.g. the discussion in Karabarbounis and Neiman (2014) on the recently decreasing development of the labor income share.

	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{\tilde{c}}$	η	ρ	n
Low-income	1,980	3,953.1	2,970.2	304	2	0.03	0.026
Lower-middle income	6,001	17,482.5	6,501.3	414	2	0.03	0.015
Upper-middle income	15,358	50,474.3	14,981.5	1,177	2	0.03	0.008
High-income	45,327	173,039.0	27,344.0	7,964	2	0.03	0.006
	L_0	δ_1	B	δ_2	α	β	γ
Low-income	21.892	0.0499	0.05	0.009	0.3613	0.5130	0.1257
Lower-middle income	73.659	0.0458	0.05	0.008	0.4156	0.5287	0.0557
Upper-middle income	61.022	0.0500	0.05	0.007	0.4376	0.4794	0.0583
High-income	30.320	0.0440	0.05	0.008	0.4521	0.5279	0.0200

Table 3: Calibration values

Note: Calibration values as explained in the main text. All values corresponding to nominal variables are measured in US $ at prices of 2014 per capita. Population L_0 as of 2014 in million people.

initial endowments fall short of their minimum requirements. Table 4 provides the results.

	cond. (34) fulfilled	cond. (35) fulfilled	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$
low income	yes	yes	3,003	2,047	8,354	625
lower-middle income	yes	yes	15,704	6,092	57,827	3,848
upper-middle income	yes	yes	44,748	13,755	113,106	8,082
high income	yes	yes	129,982	24,851	655,930	32,006

Table 4: Calibration results country groups

Note: Results using calibration values from Table 3. All quantities in 2014 US $ per capita.

We see that a solution actually exists for all country groups. Initial endowments with physical and natural capital are sufficiently large to guarantee for subsistence consumption for all times. Per capita endowments, $\underline{\tilde{k}}_0$, $\underline{\tilde{s}}_0$ and $\underline{\tilde{h}}_0$, available for consumption in excess of its subsistence level are all positive, i.e. endowments are sufficient to allow for excess consumption.

It is interesting to investigate what the maximum subsistence consumption would be that could be afforded by the country groups. To find out about this quantity, $\underline{\tilde{c}}^{\max}$, we are searching the values for $\underline{\tilde{c}}$ that solve at least one of the conditions (71), (72) and (73) with equality while the others are not violated. The model's predictions are optimistic with $\underline{\tilde{c}}^{\max}$ quite above the poverty lines defined by the World Bank.

Looking at the long-run behavior implied by the calibration values, we find that all countries qualify for positive steady-state growth in per capita quantities. Using the results from Section 4.3, we find these growth rates to vary moderately around 1% p.a. and a rate of interest net of depreciation between 4.3 and 5.4% p.a. Table 5 reports the corresponding findings. Results for low income countries are reported although we know that initial endowments are insufficient.

	$\lim_{t\to\infty} \frac{\dot{c}_t}{c_t} = \lim_{t\to\infty} \frac{\dot{y}_t}{y_t} = \lim_{t\to\infty} \frac{\dot{k}_t}{k_t}$	$\lim_{t\to\infty} i_t - \delta_1$
Low-income	1.19	5.38
Lower-middle income	1.08	5.16
Upper-middle income	0.69	4.35
High-income	0.81	4.62

Table 5: Steady-state growth and interst rates in % p.a.
Note: Growth rates computed according to (65) and (67). Interest rate computed according to (66).

5.5 Calibration Results Individual Countries

For calibration of individual country cases, we proceeded exactly the same way as in case of country groups before by using the same data sources. The calibration values can be found in Appendix G at the end of the paper which lists all 108 countries for which all the required data are available.[16]

We note that the problem we analyze is not properly defined for 4 of the countries: Iraq, Kuwait, Oman and Qatar. They are all subject to a quite high resource rents' share in GDP γ which leads to a violation of conditions (34) and (35). Production in these economies is simply too dependent on resources and no finite initial endowment could ever fulfill their total resource consumption over time given their initial output.

One main purpose of this section is to see whether initial endowments are sufficient to cover at least the subsistence level of consumption. As in case of the analyzed country groups, we compute $\underline{\tilde{k}}_0$, $\underline{\tilde{s}}_0$ and $\underline{\tilde{h}}_0$ defined by (71), (72) and (73). Appendix G reports on the results for all countries where we were able to assemble the full data set. We find several countries with insufficient endowments in physical and natural capital. Table 6 reports them together with the per capita gap in the initial endowment that prevents countries from realizing at least subsistence consumption. From the total of 108 countries for which we have complete set of data, 98 are equipped with sufficient initial endowments, 6 have insufficient endowments and 4 have a parameter constellation preventing a solution to the problem.

We note in particular that low income countries in our data sample suffer from insufficient endowments with initial capital stocks. Somewhat surprisingly, Saudi Arabia as a high income country is in such an initial position as well. It is important to remember, that we calibrated an initial situation matching actual GNI during 2014. Whenever we find a country with insufficient endowments, it is a combination of reasons behind this finding. Initial production and subsistence consumption is too high combined with the dependence on capital stocks implied by output elasticities. Again, we compute the maximum subsitence consumption $\underline{\tilde{c}}^{\max}$ affordable at the initial and actual GNI during 2014.

[16] In total 33 countries were excluded from the World Bank (2018) database. Mostly, this was due to missing data on GNI in the World Bank data and the labor share in the PWT 9.1. One country (Togo) was excluded due to inconsistent ouptut shares, i.e. the resource and labor share in GDP added up to more than 1. Malta was excluded as being the only country with a resource share in GDP of exactly zero which is not covered by the model's formulation above.

Country	Income group	$\tilde{k}_0$	$\tilde{s}_0$	$\tilde{h}_0$	$\tilde{c}$	$\tilde{c}^{max}$
Burundi	low	-119	-1,259	-4,482	247	28
Central African Rep.	low	-2,068	8,742	11,621	437	233
Mozambique	low	-34	1,543	4,719	399	294
Niger	low	-317	727	2,283	329	256
Saudi Arabia	high	82,029	-440,228	-298,461	3,884	975
Sierra Leone	low	470	-5,314	-7,945	289	95

Table 6: Countries with insufficient initial endowments
Note: Countries with insufficient initial endowments in 2014. Negative numbers in columns 2 to 4 indicate the additional needs of physical, natural or human capital per capita to guarantee subsistence consumption given in column 5. Column 6 gives the maximum subsistence consumption feasible with the given initial endowments. All quantities in 2014 US$.

In most of the cases in Table 6 it is quite below the poverty lines defined by the World Bank

Our calibration excercise allows us to trace the full dynamics of each of the economies. Here, we focus on output growth, i.e. the growth of GNI per capita. Figure 1 displays the growth behavior of the 98 countries with sufficient initial endowments during adjustment to staedy-state growth. The growth rates are plotted against the economies' initial resource share in GDP at for distinct points in time (initial growth at $t = 0$ and growths at $t = 15, 30, 100$). Additionally, a least squares projection of growth on the resource share is provided.

We observe growth rates' variation to decrease over time and a tendency towards a negative relationship between growth and initial resource share in GDP.

Contrary to the growth rates during transition, steady-state growth given by (65) and (67) can be computed for all 108 countries. It is pinned down by the parameters of the model alone. We have, of course, to keep in mind that countries contained in Table 6 would only be able to approach steady-state if their initial endowments would be augmented by some type of transfer of capital. Appendix E reports on the steady-state per capita growth rate and the net rate of interest. We note that most countries qualify by their parameter constellation for positive steady-state growth. The only exceptions are Latvia, Saudi Arabia, Azerbaijan, Bulgaria, Bosnia and Herzegovina, Ukraine and the Central African Republic. The negative relation between resource share in GDP and growth prevails in steady-state with a correlation of -0.31. We discus the finding in the following section.

5.6 Discussion

The predictions of our model support the recent finings in Venables (2016) that in particular lower income and resource depending economies lack investments in physical and human capital. Venables (2016) reports findings that net adjusted savings in low income countries are negatively related to countries' dependence on resource rents.

We find such a relationship on average for all countries in our data sample. Figure 2 plots the net savings' share in output, i.e. $(\tilde{y}_t - \tilde{c}_t - \tilde{r}_t)/\tilde{y}_t$ for $t = 0$ against countries' resource share in GDP γ.[17] As we can see, the model clearly predicts (on average) a negative relationship. It is, given our

[17] The negative relationship holds for all t during adjustment in our data sample.

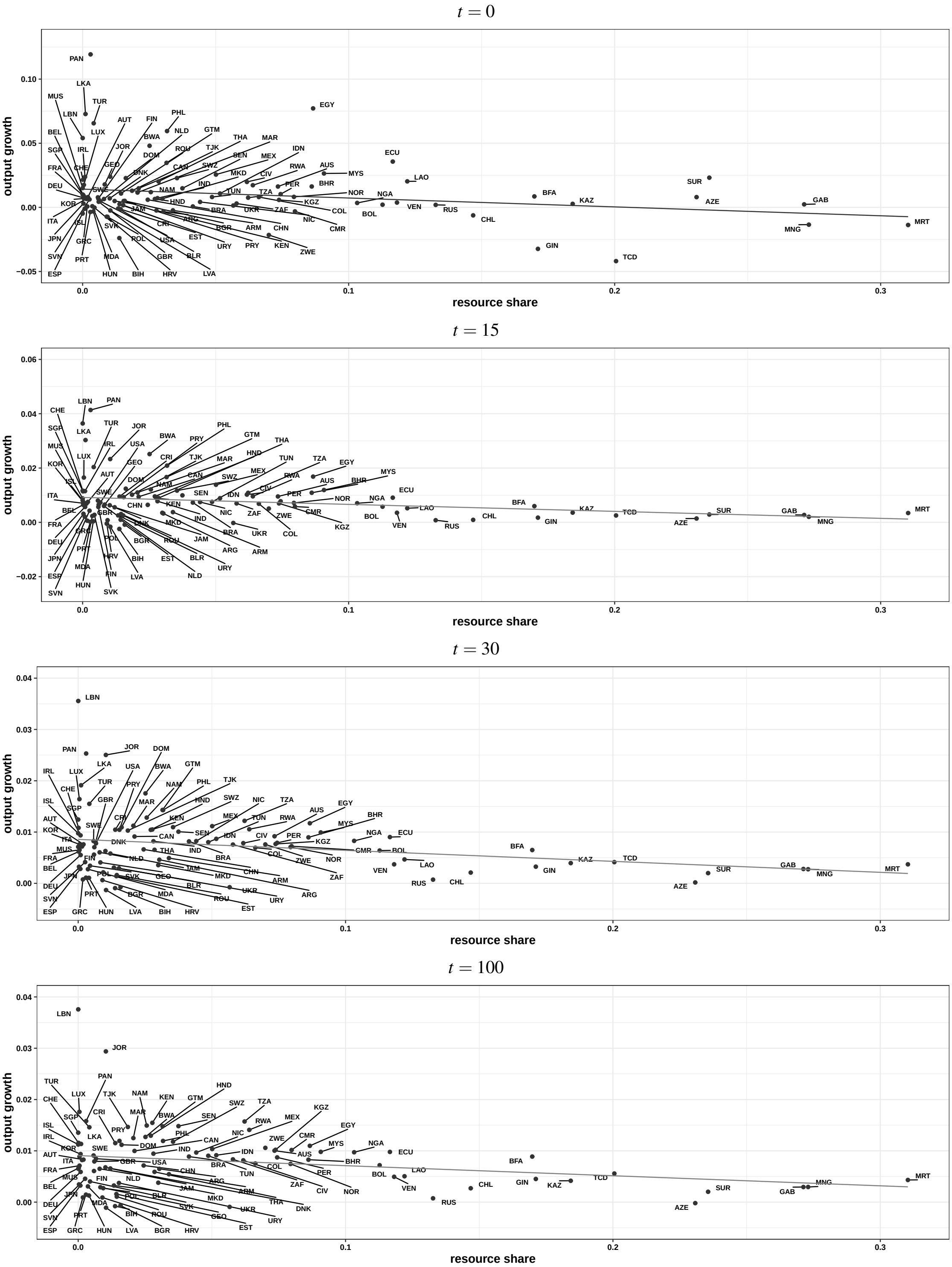

Figure 1: Output per capita growth and resource share in GDP

Note: Growth rates of per capita output at $t = 0$, $t = 15$ and $t = 30$. Grey line projects growth linearly onto resource share at $t = 0$ by least squares.

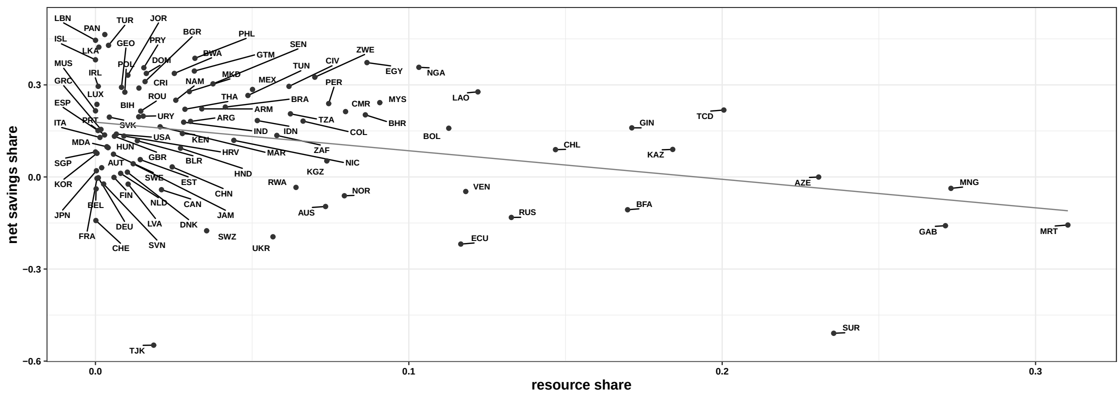

Figure 2: Output per capita growth and resource share in GDP

Note: Net savings as a share of output and the resource share in GPD at $t = 0$ (2014). Grey line projects net savings share onto resource share at $t = 0$ by least squares.

optimization problem, optimal to reduce net savings with increasing resource dependence.

Also the cases reported in Table 6 suffer from too low investments in accumulated capital stocks (physical and/or human) in the past. However, the situation here is different as these cases can't be seen as the result of optimal behavior in the past. This stresses further the necessity of properly managing resource rents in these countries in particular (van der Ploeg and Venables 2011 and Collier et al. 2010).

The debate on the existence of a resource curse is still open and critical arguments against the hypothesis can be found e.g. in Smith (2015). However, several explanations underlying such a resource curse have been put forward by now. Some are related to international trade, others to the link with (political) institutions. Both arguments are not suitable in our case as the model is not taking account of international relations or institutions. An alternative explanation is based on the accumulation of physical and human capital (Bravo-Ortega and De Gregorio 2005, Atkinson and Hamilton 2004, Papyrakis and Gerlagh 2004 and Gylfason 2001). It is this transmission channel that is behind the findings in the preceding section.

Inspecting the steady-state growth rate (65 and 67) immediately reveals that the growth rate - if positive - decreases, ceteris paribus, if the resource share γ increases. To see this, rewrite the steady-state growth in two alternative ways

$$\frac{[\beta(B-\delta_2)-\gamma n]-(1-\alpha)(\rho-n)}{(1-\alpha)\eta} = \frac{\beta(B-\delta_2+n)-(\beta+\gamma)(\rho)}{(\beta+\gamma)\eta} \tag{74}$$

$$= \frac{(1-\alpha-\gamma)(B-\delta_2+n)-(1-\alpha)(\rho)}{(1-\alpha)\eta} \tag{75}$$

It is obvious form (74) and (75) that an increasing γ lowers steady-state growth. In case of (74) we would hold β constant and any increase in γ would occur at the expense of the physical capital's share α while it would be at the expense of the share of effective human capital β in case of (75). This is simply because we assumed constant return to scale in production. As most of the countries converge towards this steady-state growth, the negative relationship between γ and growth should be more visible as economies proceed in adjustment which seems to be the case in Figure 1. Of course, steady-state growth depends on other model parameters as well. However, this partial relationship dominates on average in the variation of calibration values in our country sample.

This points us towards the underlying mechanism behind the negative relationship found in the preceding section. The model contains two engines of long-run growth. The first is human capital accumulation. As explained before, H_t tends to infinity regardless of the model's parameter values. The second engine is physical capital accumulation which might be weaker as $\frac{K_t}{L_t}$ not always grows without bound. Increasing γ automatically reduces the influence of human and/or physical capital accumulation on growth. Furthermore, it is also reducing the incentive to accumulate the corresponding stocks as their marginal product declines - ceteris paribus - with an increasing γ.

6 Conclusion

We summarize by highlighting the two major contributions made in this paper. First, it is a technical one on solving complex dynamic problems using special functions. Second, and probably more interesting from an economic point of view, the solution method allows us to calibrate the model to individual countries' endowment situation. We are able to shed some light quantitatively on deficits in human and physical capital accumulation.

Furthermore, we find in our model increasing resource dependence as an obstacle to growth. This might be interpreted as some type of resource curse. It might indeed be a “curse” as it occurs even in the optimal solution of a benevolent planer confronted with a particular contribution of resource rents in GDP. In such a case, the optimal solution for steady-state growth is negatively related to the importance of resource rents. The same holds for net savings which also decrease with increasing resource dependence.

7 Appendix

A: Co-state $\lambda_{1,t}$: Using (21) and (22) in (18) gives

$$\begin{aligned}
\dot{\lambda}_{1,t} &= -\lambda_{1,t}\alpha A\left(\frac{K_t}{L_tH_tu_t}\right)^{\alpha-1}\left(\frac{R_t}{L_tH_tu_t}\right)^{\gamma}+\lambda_{1,t}\delta_1, \\
&= -\lambda_{1,t}\alpha A\left(\frac{L_t}{B}\right)^{\beta}\left(\frac{K_t}{BH_tu_t}\right)^{\alpha-1}\left(\frac{R_t}{BH_tu_t}\right)^{\gamma}+\lambda_{1,t}\delta_1, \\
&= -\alpha A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,t}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}}+\lambda_{1,t}\delta_1, \\
&= -\alpha A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}e^{\frac{\beta(B-\delta_2+n)}{\alpha}t}+\lambda_{1,t}\delta_1.
\end{aligned} \tag{76}$$

(76) takes the form of a Bernoulli equation for $\lambda_{1,t}$. Defining $m_t=\lambda_{1,t}^{1-\frac{1}{\alpha}}$ implies

$$\begin{aligned}
\dot{m}_t &= \frac{\alpha-1}{\alpha}\lambda_{1,t}^{-\frac{1}{\alpha}}\dot{\lambda}_{1,t} \\
&= -\frac{1-\alpha}{\alpha}\delta_1 m_t+(1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}e^{\frac{\beta(B-\delta_2+n)}{\alpha}t},
\end{aligned}$$

which has the solution

$$\begin{aligned}
m_t &= e^{-\frac{1-\alpha}{\alpha}\delta_1 t}\left[m_0+(1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\int_0^t e^{\frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}z}dz\right], \\
&= e^{-\frac{1-\alpha}{\alpha}\delta_1 t}\left[m_0+(1-\alpha)A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\frac{e^{\psi t}-1}{\psi}\right],
\end{aligned}$$

with

$$\psi=\frac{\beta(B-\delta_2+n)+(1-\alpha)\delta_1}{\alpha}.$$

Introducing

$$\varphi_1 = A^{-\frac{1}{\alpha}}\lambda_{1,0}^{\frac{\alpha-1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{\frac{\beta}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{\alpha}},\quad \varphi_2=\frac{1-\alpha}{\psi},\quad \zeta=\frac{\varphi_2-\varphi_1}{\varphi_2},\quad x_t=e^{-\psi t},$$

gives

$$\begin{aligned}\lambda_{1,t} &= e^{\delta_1 t} A^{-\frac{1}{1-\alpha}} \left(\frac{\lambda_{2,0}}{\beta}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\lambda_{3,0}}{\gamma}\right)^{\frac{\gamma}{1-\alpha}} \left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}} \left[\varphi_1 + \varphi_2(e^{\psi t}-1)\right]^{\frac{\alpha}{\alpha-1}} \\ , &= e^{\delta_1 t} \left(\frac{\varphi_1}{\lambda_{1,0}^{\frac{\alpha-1}{\alpha}}}\right)^{\frac{\alpha}{1-\alpha}} \varphi_2^{\frac{\alpha}{\alpha-1}} e^{-\frac{\alpha}{1-\alpha}\psi t} (1-\zeta x_t)^{\frac{\alpha}{\alpha-1}} \\ &= \lambda_{1,0} e^{\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{x_t}{1-\zeta x_t}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned} \quad (77)$$

B: Derivation of K_t: Using (30) in (27) gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t} \left(\frac{\varphi_1 + \varphi_2(x_t^{-1}-1)}{\varphi_1}\right)^{\frac{1}{1-\alpha}} - \int_0^t (c_z - \underline{c}) L_z e^{-\delta_1 (t-z)} \left(\frac{\varphi_1 + \varphi_2(x_t^{-1}-1)}{\varphi_1 + \varphi_2(x_z^{-1}-1)}\right)^{\frac{1}{1-\alpha}} dz \\ &- \int_0^t \underline{c} L_z e^{-\delta_1 (t-z)} \left(\frac{\varphi_1 + \varphi_2(x_t^{-1}-1)}{\varphi_1 + \varphi_2(x_z^{-1}-1)}\right)^{\frac{1}{1-\alpha}} dz. \end{aligned}$$

Inserting (7) and rearranging delivers

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_0^t \lambda_{1,z}^{-\frac{1}{\eta}} e^{-\left(\frac{\rho}{\eta}-n\right)z} L_0 e^{-\delta_1 (t-z)} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz \\ &- \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \int_0^t \underline{c} e^{nz} L_0 e^{-\delta_1 (t-z)} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz. \end{aligned}$$

Using (24) and rearranging gives

$$\begin{aligned} K_t &= K_0 e^{-\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\ &- e^{-\delta_1 t} \left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{(1-\alpha)\eta}} \lambda_{1,0}^{-\frac{1}{\eta}} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} L_0 \int_0^t e^{-\left(\frac{\rho}{\eta}-n-\delta_1+\frac{\delta_1}{\eta}\right)z} x_z^{\frac{\eta-\alpha}{(1-\alpha)\eta}} (1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}} dz \\ &- e^{-\delta_1 t} \left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \underline{c} L_0 \int_0^t e^{(n+\delta_1)z} x_z^{\frac{1}{1-\alpha}} (1-\zeta x_z)^{-\frac{1}{1-\alpha}} dz \end{aligned}$$

Using $x_z = e^{-\psi z}$ gives

$$
\begin{aligned}
K_t &= K_0 e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\
&\quad -e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{(1-\alpha)\eta}}\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} L_0\int_0^t x_z^{\frac{1}{\psi}\left(\frac{\rho}{\eta}+\psi\frac{\eta-\alpha}{(1-\alpha)\eta}-\delta_1+\frac{\delta_1}{\eta}-n\right)}(1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}}dz \\
&\quad -e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}}\underline{c}L_0\int_0^t x_z^{\frac{1}{\psi}(\frac{\psi}{1-\alpha}-n-\delta_1)}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}dz.
\end{aligned}
$$

Changing the domain of integration from z to dx_z with $dz=-\frac{1}{\psi}x_z^{-1}dx_z$ and integrating from x_t to 1 instead of 0 to t gives

$$
\begin{aligned}
K_t &= K_0 e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{1}{1-\alpha}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}} \\
&\quad -e^{-\delta_1 t}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{(1-\alpha)\eta}}\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi}\int_{x_t}^1 x_z^{\frac{1}{\psi}\left(\frac{\rho}{\eta}+\psi\frac{\eta-\alpha}{(1-\alpha)\eta}-\delta_1+\frac{\delta_1}{\eta}-n\right)-1}(1-\zeta x_z)^{\frac{\alpha-\eta}{(1-\alpha)\eta}}dxz \\
&\quad -e^{-\delta_1 t}\left(\frac{x_t}{1-\zeta x_t}\right)^{-\frac{1}{1-\alpha}}\underline{c}\frac{L_0}{\psi}\int_{x_t}^1 x_z^{\frac{1}{\psi}(\frac{\psi}{1-\alpha}-n-\delta_1)-1}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}dx_z
\end{aligned}
$$

which is identical to (31) in the main text.

C: Derivations related to H_t: Using (22) and (24) gives

$$
\begin{aligned}
BH_t u_t &= A^{\frac{1}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,t}}{\beta}\right)^{-\frac{1-\gamma}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_t}{B}\right)^{\frac{\beta}{\alpha}}K_t \\
&= A^{\frac{1}{\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-\frac{1-\gamma}{\alpha}}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-\frac{\gamma}{\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{\alpha}}\lambda_{1,t}^{\frac{1}{\alpha}}e^{\frac{(1-\gamma)(B-\delta_2)+\beta n}{\alpha}t}K_t \\
&= e^{[(B-\delta_2)+\delta_1]t}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}x_t^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)^{-\frac{1}{1-\alpha}}K_t. \qquad (78)
\end{aligned}
$$

(4) implies $\dot{H}_t$=$B(1-u_t)H_t-\delta_2 H_t=(B-\delta_2)H_t-Bu_tH_t$. Proceding analogous to (29) gives

$$
\begin{aligned}
H_t &= H_0 e^{-\int_0^t f_2(z)dz}+\int_0^t g_2(z)e^{-\int_z^t f_2(s)ds}dz, \\
&\quad \text{with} \\
f_2(z) &= -(B-\delta_2), \quad g_2(z)=-BH_zu_z.
\end{aligned}
$$

This delivers H_t as $H_t=H_0e^{(B-\delta_2)t}-\int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz$. Using (78) gives

$$
\int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\int_0^t e^{\delta_1 z}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}x_z^{\frac{\alpha}{1-\alpha}}(1-\zeta x_z)^{-\frac{1}{1-\alpha}}K_zdz.
$$

Inserting (33) for the physical capital stock yields

$$\int_0^t Bu_z \quad H_z \quad e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\left\{K_0\int_0^t x_z^{-1}dz \right.$$
$$-\int_0^t \lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0x_z^{-1}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_z^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)\right]dz$$
$$\left.-\int_0^t \underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0x_z^{-1}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_z^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)\right]dz\right\}.$$

Using $\zeta x_z = \zeta e^{-\psi z}$, $d\zeta x_z = -\zeta\psi e^{-\psi z}dz$, the integration rule

$$\int z^{b-2}{}_2F_1(a,b;c;z)dz = \frac{z^{b-1}}{b-1}{}_2F_1(a,b-1;c;z)+\text{constant} \tag{79}$$

and adjusting the the direction of integration delivers

$$\int_0^t Bu_z \quad H_z \quad e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\left\{\int_{\zeta x_t}^{\zeta} K_0\frac{\zeta}{\psi}(\zeta x_z)^{-2}d\zeta x_z\right.$$
$$-\int_{x_t}^{1}\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0(\zeta x_z)^{-2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)d\zeta x_z$$
$$+\int_{\zeta x_t}^{\zeta}\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0(\zeta x_z)^{\tilde{b}_1-2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_z)d\zeta x_z$$
$$-\int_{\zeta x_t}^{\zeta}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_1}}{\psi^2}\frac{1}{\tilde{b}_2}L_0(\zeta x_z)^{-2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)d\zeta x_z$$
$$\left.+\int_{\zeta x_t}^{\zeta}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_2}}{\psi^2}\frac{1}{\tilde{b}_2}L_0(\zeta x_z)^{\tilde{b}_2-2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_z)d\zeta x_z\right\},$$
$$= e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\left\{K_0\frac{1}{\psi}(1-x_t^{-1})\right. \tag{80}$$
$$-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)(1-x_t^{-1})$$
$$+\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right]$$
$$-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2}L_0{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)(1-x_t^{-1})$$
$$\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\},$$

Using (80), H_t can now be computed as $H_t = e^{(B-\delta_2)t}H_0 - \int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz$

$$
\begin{aligned}
H_t \;=\;& e^{(B-\delta_2)t}H_0 - e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\left\{K_0\frac{1}{\psi}(1-x_t^{-1})\right.\\
&-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0\,{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)(1-x_t^{-1})\\
&+\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right]\\
&-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2}L_0\,{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)(1-x_t^{-1})\\
&\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}.
\end{aligned}
$$

Effective human capital $L_tH_tu_t$ employed in final goods production follows next. Multiplying both sides of (78) by $\frac{L_t}{B}=\frac{L_0}{B}e^{nt}$ gives

$$
L_tH_tu_t \;=\; e^{[(B-\delta_2)+n+\delta_1]t}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}\frac{L_0}{B}x_t^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)^{-\frac{1}{1-\alpha}}K_t, \tag{81}
$$

inserting (33) for K_t gives (37) in the main text.

D: Derivations involving R_t and S_t: Reformulating (21) using $L_t=L_0e^{nt}$ and the time path for the co-state $\lambda_{2,t}$ in (20) gives

$$
R_t \;=\; L_tH_tu_t\frac{\lambda_{2,0}}{\beta}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left(\frac{L_0}{B}\right)^{-1}e^{-(B-\delta_2+n)t},
$$

which yields together with (81)

$$
\begin{aligned}
R_t \;=\;& \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}x_t^{-1}\left\{K_0\right.\\
&-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right]\\
&\left.-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right]\right\}.
\end{aligned} \tag{82}
$$

We turn to $S_t=S_0-\int_0^t R_s ds$. Integration over R_t given by (82) gives

$$
\begin{aligned}
\int_0^t R_s ds \quad = \quad & \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left\{K_0\int_0^t x_s^{-1}ds\right.\\
&-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\int_0^t x_s^{-1}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_s^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_s)\right]ds\\
&\left.-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\int_0^t x_s^{-1}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_s^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_s)\right]ds\right\}.
\end{aligned}
$$

Using again the integration rule (79), $\zeta x_s = \zeta e^{-\psi s}$ and $d\zeta x_s = -\zeta\psi e^{-\psi s}ds = -\psi\zeta x_s ds$ and hence $ds = -\frac{1}{\psi}\frac{1}{\zeta x_s}d\zeta x_s$ and adjusting the direction of integration delivers

$$
\begin{aligned}
\int_0^t R_s ds \quad = \quad & \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left\{K_0\frac{\zeta}{\psi}\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s\right.\\
&-\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_1}L_0{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s\\
&+\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{\zeta^{1-\tilde{b}_1}}{\psi^2}\frac{1}{\tilde{b}_1}L_0\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{\tilde{b}_1-2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_s)d\zeta x_s\\
&-\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{-2}d\zeta x_s\\
&\left.+\,\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta^{1-\tilde{b}_2}}{\psi^2}\frac{1}{\tilde{b}_2}L_0\int_{\zeta x_t}^{\zeta}(\zeta x_s)^{\tilde{b}_2-2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_s)d\zeta x_s\right\},\\
= \quad & \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left\{-K_0\frac{1}{\psi}\left[1-x_t^{-1}\right]\right.\\
&+\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\left[1-x_t^{-1}\right]\\
&+\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right]\\
&+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\left[1-x_t^{-1}\right]\\
&\left.+\,\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}.
\end{aligned} \tag{83}
$$

E: Transversality conditions

Transversality condition K_t: We have to show that $\lim_{t\to\infty}\lambda_{1,t}K_t = 0$.

Using $\lambda_{1,t}$ given by (24) and K_t given by (33), $\lambda_{1,t}K_t$ reads as

$$
\begin{aligned}
\lambda_{1,t}K_t &= \lambda_{1,0}K_0\left(\frac{\varphi_1}{\varphi_2}\right)^{-1}\frac{1-\zeta x_t}{x_t} \\
&-\lambda_{1,0}^{1-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}}\frac{1-\zeta x_t}{x_t}\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\
&-\lambda_{1,0}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}\frac{1-\zeta x_t}{x_t}\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right],
\end{aligned}
$$

As $t\to\infty$ we see $x_t\to 0.$ and $x_t^{-1}\to\infty$. Rewriting $\lambda_t K_t$ as $\frac{x_t\lambda_t K_t}{x_t}$ and applying L'Hospital's rule as $x_t\to 0$ requires $\lim_{x_t\to 0}\frac{\partial x_t\lambda_t K_t}{\partial x_t}=0$. $\frac{\partial x_t\lambda_t K_t}{\partial x_t}$ using the above expression is given by

$$
\begin{aligned}
\frac{\partial x_t\lambda_t K_t}{\partial x_t} &= -\lambda_{1,0}K_0\left(\frac{\varphi_1}{\varphi_2}\right)^{-1}\zeta \\
&+\lambda_{1,0}^{1-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}}\zeta\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\
&+\lambda_{1,0}^{1-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}}(1-\zeta x_t)\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[\tilde{b}_1x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)\right] \\
&+\lambda_{1,0}^{1-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}}(1-\zeta x_t)\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[x_t^{\tilde{b}_1}\frac{\partial{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)}{\partial x_t}\right] \\
&+\lambda_{1,0}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}\zeta\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right] \\
&+\lambda_{1,0}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[\tilde{b}_2x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)\right] \\
&+\lambda_{1,0}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}(1-\zeta x_t)\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[x_t^{\tilde{b}_2}\frac{\partial{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)}{\partial x_t}\right].
\end{aligned}
$$

Evaluating $\frac{\partial x_t\lambda_t K_t}{\partial x_t}$ at $x_t=0$ gives as long as $\tilde{b}_1-1>0$ and $\tilde{b}_2-1>0$ and because $\frac{{}_2F_1(a,b;b+1;0)}{b}=1$

$$
\begin{aligned}
\left.\frac{\partial x_t\lambda_t K_t}{\partial x_t}\right|_{x_t=0} &= -\lambda_{1,0}K_0\left(\frac{\varphi_1}{\varphi_2}\right)^{-1}\zeta \\
&+\lambda_{1,0}^{1-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha(\eta-1)}{(1-\alpha)\eta}}\zeta\frac{1}{\psi}\frac{1}{\tilde{b}_1}L_0\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)\right] \\
&+\lambda_{1,0}\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\alpha}{1-\alpha}}\zeta\frac{1}{\psi}\frac{1}{\tilde{b}_2}L_0\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\right]
\end{aligned}
$$

For transversality to hold, it is then required additionally that $\left.\frac{\partial x_t\lambda_t K_t}{\partial x_t}\right|_{x_t=0}=0$ which implies

$$
K_0 = \frac{L_0}{\psi}\left[\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta)}{\tilde{b}_1}+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)}{\tilde{b}_2}\right] \tag{84}
$$

Inserting this into (33) gives K_t as

$$K_t = e^{-\delta_1 t}\left(\frac{1-\zeta x_t}{x_t}\right)^{\frac{1}{1-\alpha}}\frac{L_0}{\psi}\left[\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{-\frac{\alpha}{1-\alpha}\frac{1}{\eta}}\frac{x_t^{\tilde{b}_1}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta x_t)}{\tilde{b}_1} + \underline{c}\frac{x_t^{\tilde{b}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta x_t)}{\tilde{b}_2}\right]. \quad (85)$$

Transversality condition S_t: Transversality demands that $\lim_{t\to\infty}\lambda_{3,t}S_t = 0$. As $\lambda_{3,t} = \lambda_{3,0}$, this is equivalent to $\lim_{t\to\infty}S_t = 0$ or $\int_0^t R_s ds = S_0$. Rearranging (83) yields

$$\begin{aligned}\int_0^t R_s ds = {} & \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\left[1-x_t^{-1}\right]\Bigg\{-K_0\frac{1}{\psi} \\ & +\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1}L_0{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta) \\ & +\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{\zeta}{\psi^2}\frac{1}{\tilde{b}_2}L_0{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta)\Bigg\} \\ & +\frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}L_0\Big[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) \\ & -x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\Big] \\ & +\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}L_0\Big[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) \\ & -x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\Big]\Bigg\},\end{aligned} \quad (86)$$

where we note that the first term in curly brackets is zero due to the transversality condition for K_t given by (84). As both, $\tilde{b}_1$ and $\tilde{b}_2$ are larger than one $x_t \to 0$ for $t\to\infty$, we find[18]

$$\lim_{t\to\infty}\int_0^t R_s ds = S_0 = \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\frac{1}{\psi^2}L_0\Bigg\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) + \underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\Bigg\}, \quad (87)$$

as the transversality condition for S_t.

Inserting (84) and (87) into (86) gives S_t as

[18] It helpful to note that $\lim_{z\to 0}{}_2F_1(a,b-1;b+1,z) = \frac{2}{(b+1)b}$ is finite.

$$\begin{aligned} S_t &= S_0 - \int_0^t R_s ds \\ &= S_0 - \frac{\lambda_{1,0}}{\varphi_1}\left(\frac{\lambda_{3,0}}{\gamma}\right)^{-1}\frac{1}{\psi^2}L_0\left[\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right. \\ &\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]. \end{aligned} \tag{88}$$

Transversality condition H_t: Transversality demands $\lim_{t\to\infty}\lambda_{2,t}H_t = 0$.

$$\begin{aligned} \lambda_{2,t}H_t &= \lambda_{2,0}e^{-(B-\delta_2)t}H_0e^{(B-\delta_2)t} - \lambda_{2,0}e^{-(B-\delta_2)t}\int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz, \\ &= \lambda_{2,0}\left(H_0 - \int_0^t Bu_zH_ze^{-(B-\delta_2)z}dz\right). \end{aligned}$$

Using $\int_0^t Bu_zH_zdz$ given in (78) together with the transversality condition for K_t in (84) gives

$$\begin{aligned} \lambda_{2,t}H_t &= \lambda_{2,0}H_0 - \lambda_{2,0}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0\times \\ &\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta) - x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right]\right. \\ &\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta) - x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}. \end{aligned}$$

As $t\to\infty$, $x_t\to 0$. With $\tilde{b}_1,\tilde{b}_2 > 1$ and by noting that $\lim_{z\to 0}{}_2F_1(\tilde{a},b-1;b+1;z)$ is finite, we find

$$\begin{aligned} \lim_{x_t\to 0}\lambda_{2,t}H_t &= \lambda_{2,0}H_0 - \lambda_{2,0}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0\times \\ &\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. \\ &\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}. \end{aligned}$$

Transversality consequently demands

$$\begin{aligned} H_0 &= \left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)\right. \\ &\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)\right\}. \end{aligned} \tag{89}$$

Imposing the transversality condition for K_t (84) onto (80) yields

$$\int_0^t Bu_z \quad H_z \quad e^{(B-\delta_2)(t-z)}dz = e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times$$
$$\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}\left[{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta)-x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right]\right.$$
$$\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}\left[{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta)-x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right]\right\}.$$

Inserting (89) into this expression gives

$$\int_0^t Bu_z \quad H_z \quad e^{(B-\delta_2)(t-z)}dz = H_0e^{(B-\delta_2)t}-e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times$$
$$\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right.$$
$$\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\},$$

and finally

$$\begin{aligned} H_t &= H_0e^{(B-\delta_2)t}-\int_0^t Bu_zH_ze^{(B-\delta_2)(t-z)}dz \\ &= e^{(B-\delta_2)t}\left(\frac{\lambda_{2,0}}{\beta}\right)^{-1}\frac{\lambda_{1,0}}{\varphi_1}L_0 \times \\ &\quad\times\left\{\lambda_{1,0}^{-\frac{1}{\eta}}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{\eta-\alpha}{(1-\alpha)\eta}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_1(\tilde{b}_1-1)}x_t^{\tilde{b}_1-1}{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta x_t)\right. \\ &\quad\left.+\underline{c}\left(\frac{\varphi_1}{\varphi_2}\right)^{\frac{1}{1-\alpha}}\frac{1}{\psi^2}\frac{1}{\tilde{b}_2(\tilde{b}_2-1)}x_t^{\tilde{b}_2-1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta x_t)\right\}. \end{aligned} \tag{90}$$

F: Uniqueness of the solution ζ^*: The equilibrium value ζ^* satisfies $\frac{K_0^+}{H_0^+}=\frac{K_0}{H_0}$, with

$$\frac{K_0^+}{H_0^+} = A^{\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{-\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{\frac{\beta}{1-\alpha}}\varphi_2^{\frac{1}{1-\alpha}}\psi(\tilde{b}_1-1)(1-\zeta^*)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*)}, \tag{91}$$

$$\frac{K_0}{H_0} = \frac{K_0-\frac{L_0}{\psi}\underline{c}(1-\zeta^*)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2}}{H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)}}. \tag{92}$$

We first notice that (91) and (92) demand $\zeta<1$.
We show first that $\frac{K_0^+}{S_0^+}$ given by (91) is decreasing in ζ^*. Second, we show that $\frac{K_0}{S_0}$ given by (92) is

increasing in ζ^*. This implies that there can be at most one solution to $\frac{K_0^+}{H_0^+} = \frac{K_0}{H_0}$.

Investigating $\frac{K_0^+}{S_0^+}$, we have to distinguish three cases, i.e. $\tilde{a}_1 < 0, \tilde{a}_1 = 0, \tilde{a}_1 > 0$.

Case 1: $\tilde{a}_1 < 0$: Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that $\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*)}$ is decreasing in ζ^* in case $\tilde{a}_1 < 0$. It is obvious that $(1-\zeta^*)^{\tilde{a}_2}$ is decreasing in ζ^* as well because $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$. Therefore, $\frac{K_0^+}{S_0^+}$ is in this case decreasing in ζ^*.

Case 2: $\tilde{a}_1 = 0$: This case prevails if it happens to be that $\eta = \alpha$. Lemma 1 in Boucekkine and Ruiz-Tamarit (2008) shows that in this case $\frac{\partial \frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*)}}{\partial \zeta^*} = 0$ applies. As $(1-\zeta^*)^{\tilde{a}_2}$ is decreasing in ζ^*, $\frac{K_0^+}{S_0^+}$ is in this case again decreasing in ζ^*.

Case 3: $\tilde{a}_1 > 0$: The denominator in $\frac{K_0^+}{S_0^+}$ is increasing in ζ^* as $\frac{\partial {}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*)}{\partial \zeta^*} = \frac{\tilde{a}_1(\tilde{b}_1-1)}{\tilde{b}_1+1} {}_2F_1((\tilde{a}_1+1,\tilde{b}_1;\tilde{b}_1+2;\zeta^*) > 0$ (Abramowitz and Stegun 1972, 15.2.1) because $\tilde{b}_1 - 1 > 0$ is required by the transversality conditions (34) and (35). There are opposing forces at work in the nominator as ${}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)$ increases and $(1-\zeta^*)^{\tilde{a}_2}$ decreases in ζ^*. To find out which is stronger, we define $h(\zeta^*)$ as

$$
\begin{aligned}
h(\zeta^*) \quad &= \quad (1-\zeta^*)^{\tilde{a}_2} {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*) = (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*) \\
& \quad\ \ \text{with} \\
& \quad\ \ \tilde{a}_2 - \tilde{a}_1 = \frac{1}{1-\alpha} - \frac{\eta-\alpha}{\eta(1-\alpha)} = \frac{\alpha}{\eta(1-\alpha} > 0, \\
& \quad\ \ {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*) = \tilde{b}_1 \int_0^1 x^{\tilde{b}_1-1}(1-\zeta^* x)^{-\tilde{a}_1} dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial h(\zeta^*)}{\partial \zeta^*} &= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} - \tilde{a}_1 \frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \frac{\partial {}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{\partial z} \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} - \tilde{a}_1 \frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 x^{\tilde{b}_1}(1-\zeta^* x)^{-\tilde{a}_1-1} dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1}(1-\zeta^* x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1}\frac{(1-\zeta^* x)^{-\tilde{a}_1}}{1-\zeta^*} \right) dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1}(1-\zeta^* x)^{-\tilde{a}_1-1} - x^{\tilde{b}_1-1}\frac{(1-\zeta^* x)^{-\tilde{a}_1}}{1-\zeta^*} \right) dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1-1}(1-\zeta^* x)^{-\tilde{a}_1-1} \left(x - \frac{1-\zeta^* x}{1-\zeta^*} \right) \right) dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} + (1-\zeta^*)^{\tilde{a}_2-\tilde{a}_1}(1-\zeta^*)^{\tilde{a}_1} \tilde{a}_1 \tilde{b}_1 \int_0^1 \left(x^{\tilde{b}_1-1}(1-\zeta^* x)^{-\tilde{a}_1-1} \frac{x-1}{1-\zeta^*} \right) dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} - \tilde{a}_1(1-\zeta^*)^{\tilde{a}_2-1} \tilde{b}_1 \int_0^1 x^{\tilde{b}_1-1}(1-x)(1-\zeta^* x)^{-\tilde{a}_1-1} dx \\
&= -(\tilde{a}_2-\tilde{a}_1)\frac{h(\zeta^*)}{1-\zeta^*} - \tilde{a}_1(1-\zeta^*)^{\tilde{a}_2-1} \frac{{}_2F_1(\tilde{a}_1+1,\tilde{b}_1;\tilde{b}_1+2;\zeta^*)}{\tilde{b}_1+1}.
\end{aligned}
$$

As $\tilde{a}_2 - \tilde{a}_1 > 0$ and $\tilde{a}_1 > 0$ in this case, we find $\frac{\partial h(\zeta^*)}{\partial \zeta^*} < 0$. Summing up case 3, the denominator in $\frac{K_0^+}{H_0^+}$

is increasing while the nominator is decreasing in ζ^*. Hence, $\frac{K_0^+}{S_0^+}$ is again decreasing in ζ^*.

We turn to $\frac{K_0}{H_0}$ given by (92). Its denominator is obviously decreasing in ζ^* as $\tilde{a}_2 = \frac{1}{1-\alpha} > 0$ and $\frac{\partial_2 F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\partial \zeta^*} = \frac{\tilde{a}_2(\tilde{b}_2-1)}{\tilde{b}_2+1} {}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta^*)$ with $\tilde{b}_2 - 1 > 0$ due to the transversality condition (35).

The nominator in $\frac{K_0}{H_0}$ is increasing in ζ^*. To see this, define

$$k(\zeta^*) = (1-\zeta^*)^{\tilde{a}_2} {}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*) = (1-\zeta^*)^{\tilde{a}_2}\tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1}(1-\zeta^* x)^{-\tilde{a}_2} dx.$$

Therefore,

$$\begin{aligned}
\frac{\partial k(\zeta^*)}{\partial \zeta^*} &= \tilde{a}_2(1-\zeta^*)^{\tilde{a}_2}\tilde{b}_2 \left[\int_0^1 x^{\tilde{b}2}(1-\zeta^* x)^{-\tilde{a}_2-1}dx - \int_0^1 x^{\tilde{b}2-1}\frac{(1-\zeta^* x)^{-\tilde{a}_2}}{1-\zeta^*}dx\right] \\
&= \tilde{a}_2(1-\zeta^*)^{\tilde{a}_2}\tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1}(1-\zeta^* x)^{-\tilde{a}_2-1}\left[x - \frac{1-\zeta^* x}{1-\zeta^*}\right]dx \\
&= -\tilde{a}_2(1-\zeta^*)^{\tilde{a}_2-1}\tilde{b}_2 \int_0^1 x^{\tilde{b}_2-1}(1-x)(1-\zeta^* x)^{-\tilde{a}_2-1}dx \\
&= -\tilde{a}_2(1-\zeta^*)^{\tilde{a}_2-1}\frac{{}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta^*)}{\tilde{b}_2+1}
\end{aligned}$$

which is negative for $\zeta^* < 1$.

Summing up, we have shown that $\frac{K_0}{H_0}$ is increasing while $\frac{K_0^+}{H_0^+}$ is decreasing in ζ. If an equilibrium $\frac{K_0^+}{H_0^+} = \frac{K_0}{H_0}$ exists, it is unique.

Properties of $\frac{K_0^+}{S_0^+}$: To work out conditions for existence, we focus first on $\frac{K_0^+}{S_0^+}$ given by (91). Any solution ζ^* needs to fulfill $\zeta^* < 1$; we know that $\frac{K_0^+}{S_0^+}$ is decreasing in ζ^*. We show first that $\frac{K_0^+}{S_0^+}$ is unbounded from above for $\zeta^* \to -\infty$. Let ε_1 be an arbitrarily large but finite real number. The critical term in $\frac{K_0^+}{S_0^+}$ is $(1-\zeta^*)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}$. Now suppose that

$$\lim_{\zeta^* \to -\infty} (1-\zeta^*)^{\tilde{a}_2}\frac{{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*)}{{}_2F_1(\tilde{a}_1,\tilde{b}_1-1;\tilde{b}_1+1;\zeta^*)} < \varepsilon_1$$

would be true. As $\frac{K_0^+}{S_0^+}$ decreases with ζ^*. This would imply that for any finite $\zeta^* < 1$ and for $\zeta^* \to -\infty$ it would be true that

$$\begin{aligned}
{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*) &< \varepsilon_1(1-\zeta^*)^{-\tilde{a}_2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*),\\
\tilde{b}_1\int_0^1 x^{\tilde{b}_1-1}(1-\zeta^*x)^{-\tilde{a}_1}dx &- \varepsilon_1(1-\zeta^*)^{-\tilde{a}_2}\tilde{b}_1(\tilde{b}_1-1)\int_0^1 x^{\tilde{b}_1-2}(1-x)(1-\zeta^*x)^{-\tilde{a}_1}dx<0,\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta^*x)^{-\tilde{a}_1} & \left[x-\varepsilon_1(\tilde{b}_1-1)(1-\zeta^*)^{-\tilde{a}_2}(1-x)\right]dx<0,\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta^*x)^{-\tilde{a}_1} & \kappa(x;\varepsilon_1)dx<0, \qquad (93)\\
\text{with} & \\
\kappa_1(x;\varepsilon_1) &= \left[x-\varepsilon_1(\tilde{b}_1-1)(1-\zeta^*)^{-\tilde{a}_2}(1-x)\right],
\end{aligned}$$

where $\kappa(x;\varepsilon)$ is an affine function of x. $\kappa(x;\varepsilon)$ is zero for $x=\bar{x}(\varepsilon_1)$ with

$$\bar{x}(\varepsilon_1)=\frac{\varepsilon_1(\tilde{b}_1-1)(1-\zeta^*)^{-\tilde{a}_2}}{1+\varepsilon_1(\tilde{b}_1-1)(1-\zeta^*)^{-\tilde{a}_2}}.$$

Therefore, $\kappa(x;\varepsilon_1)<0$ for $x<\bar{x}(\varepsilon_1)$ and $\kappa(x;\varepsilon_1)>0$ for $x>\bar{x}(\varepsilon_1)$. For any finite ε_1, $\bar{x}(\varepsilon_1)\to 0$ for $\zeta^*\to-\infty$ as $\tilde{a}_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_1)$ becomes positive for $0\le x\le 1$ as $\zeta^*\to -\infty$ and inequality (93) cannot be fulfilled. Hence, $\frac{K_0^+}{H_0^+}$ cannot be bounded from above as $\zeta^*\to-\infty$ and $\lim_{\zeta^*\to-\infty}\frac{K_0^+}{H_0^+}=\infty$.

Next, turn to the case $\zeta^*\to 1$. Suppose that $\frac{K_0^+}{H_0^+}$ would be bounded from below by some $\varepsilon_2>0$. By the same logic as above, this would imply for any $\zeta^*<1$ and $\zeta^*\to 1$ that

$$\begin{aligned}
{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*) &> \varepsilon_2(1-\zeta^*)^{-\tilde{a}_2}{}_2F_1(\tilde{a}_1,\tilde{b}_1;\tilde{b}_1+1;\zeta^*),\\
\int_0^1 x^{\tilde{b}_1-2}(1-\zeta^*x)^{-\tilde{a}_1} & \kappa(x;\varepsilon_2)dx>0. \qquad (94)
\end{aligned}$$

For any finite $\varepsilon_2>0$, $\bar{x}(\varepsilon_2)\to 1$ for $\zeta^*\to 1$ as $\tilde{a}_2=\frac{1}{1-\alpha}>0$. As we integrate from 0 to 1, $\kappa(x;\varepsilon_2)$ becomes negative for $0\le x\le 1$ as $\zeta^*\to 1$ and inequality (94) cannot be fulfilled. Hence, $\frac{K_0^+}{H_0^+}$ cannot be bounded from below by any finite $\varepsilon_2>0$ and $\lim_{\zeta^*\to 1}\frac{K_0^+}{H_0^+}=0$.

Properties of $\frac{K_0}{H_0}$: We turn to $\frac{K_0}{H_0}$ which we know is increasing in ζ^* for $\zeta^*<1$. If a maximum exists, it must be reached as $\zeta^*\to 1$. The critical term in the nominator is $(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)$ which can be written as $(1-\zeta^*)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde{a}_2}}$. We are interested in

$$\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde{a}_2}}$$

as $\lim_{\zeta^*\to 1}(1-\zeta^*)$ is finite and equal to zero, we can rewrite this expression as

$$\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde{a}_2}}=\left[\lim_{\zeta^*\to 1}(1-\zeta^*)\right]\left[\lim_{\zeta^*\to 1}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde{a}_2}}\right]$$

if the second limit on the right hand side in the above equation is finite. 15.4.23 in DLMF (URL) states that

$$\lim_{\zeta^*\to 1}\frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}}=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

if $R(c-a-b)<0$. Applied to our case, $c-a-b=1+\tilde{b}_2-\tilde{a}_2-\tilde{b}_2=1-\tilde{a}_2=-\frac{\alpha}{1-\alpha}<0$. Furthermore, $\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}=\frac{\Gamma(\tilde{b}_2+1)\Gamma(\tilde{a}_2-1)}{\Gamma(\tilde{a}_2)\Gamma(\tilde{b}_2)}=\frac{\tilde{b}_2}{\tilde{a}_2-1}$ which is finite. Hence, $\lim_{\zeta^*\to 1}(1-\zeta^*)\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)}{(1-\zeta^*)^{1-\tilde{a}_2}}=0$ and $\lim_{\zeta^*\to 1}\underline{K}_0=K_0$.

The critical term in the denominator of $\frac{K_0}{H_0}$ is ${}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)$. As $\frac{\partial\,{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\partial\zeta^*}=\frac{\tilde{a}_2(\tilde{b}_2-1)}{\tilde{b}_2+1}{}_2F_1(\tilde{a}_2+1,\tilde{b}_2;\tilde{b}_2+2;\zeta^*)>0$ for $\zeta^*<1$, $\underline{H}_0$ declines with ζ^* in this range. 15.3.6 in Abramowitz and Stegun (1972) implies that $\lim_{\zeta^*\to 1}{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)=\frac{\Gamma(\tilde{b}_2+1)\Gamma(2-\tilde{a}_2)}{\Gamma(\tilde{b}_2+1-\tilde{a}_2)\Gamma(2)}$ if $2-\tilde{a}_2=\frac{1-2\alpha}{1-\alpha}>0$ which is the case for $\alpha<\frac{1}{2}$. In case $\alpha>\frac{1}{2}$ we find ${}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)\to\infty$ as $\zeta^*\to 1$. In both cases, it is possible that $\underline{H}_0$ turns negative as ζ^* grows for $\zeta^*<1$. Define $\bar{\zeta}$ as

$$\bar{\zeta}=_{\zeta^*\leq 1}|H_0-A^{-\frac{1}{1-\alpha}}\left(\frac{H_0}{S_0}\right)^{\frac{\gamma}{1-\alpha}}\left(\frac{L_0}{B}\right)^{-\frac{\beta}{1-\alpha}}\varphi_2^{-\frac{1}{1-\alpha}}\frac{L_0}{\psi^2}\underline{c}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2-1;\tilde{b}_2+1;\zeta^*)}{\tilde{b}_2(\tilde{b}_2-1)}|, \tag{95}$$

As $\underline{H}_0$ is decreasing in ζ^* for $\zeta^*<1$, the admissible range for a solution to the present problem has the upper bound $\bar{\zeta}$. Therefore, if $\bar{\zeta}<1$ ($\bar{\zeta}=1$) we find $\underline{H}_0|_{\zeta^*=\bar{\zeta}}=0$ ($\underline{H}_0|_{\zeta^*=\bar{\zeta}}\geq 0$).

Lastly, we turn to $\frac{K_0}{H_0}$ as $\zeta^*\to-\infty$. Again, we start with the nominator $\underline{K}_0$We know already that $(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)$ is decreasing in ζ^* for $\zeta^*<1$. Obviously, $\underline{K}_0$ then declines as $\zeta^*\to-\infty$. 15.3.4 in Abramowitz and Stegun (1972) states that

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1})$$

which implies for the present case

$$(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)={}_2F_1(\tilde{a}_2,1;\tilde{b}_2+1;\frac{\zeta^*}{\zeta^*-1}).$$

As $\tilde{a}_2,\tilde{b}_2+1>0$ and $\lim_{\zeta^*\to-\infty}\frac{\zeta^*}{\zeta^*-1}=1$, $\lim_{\zeta^*\to-\infty}(1-\zeta^*)^{\tilde{a}_2}{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\zeta^*)=\infty$. This implies that $\underline{K}_0$ becomes necessarily negative if ζ^* becomes too small. The range for admissible values for ζ^* is therefore bounded from below at $\underline{\zeta}$ which satisfies the condition

$$K_0=\frac{L_0}{\psi}\underline{c}\left(1-\underline{\zeta}\right)^{\frac{1}{1-\alpha}}\frac{{}_2F_1(\tilde{a}_2,\tilde{b}_2;\tilde{b}_2+1;\underline{\zeta})}{\tilde{b}_2}. \tag{96}$$

We observe $\lim_{\zeta^*\to\underline{\zeta}}\frac{K_0}{H_0}=0$.

Taken together, if $\underline{\zeta}<\bar{\zeta}$ and $\bar{\zeta}<1$, $\lim_{\zeta^*\to\bar{\zeta}}\frac{K_0}{H_0}\to\infty$. If $\underline{\zeta}<\bar{\zeta}$, $\bar{\zeta}=1$,$\lim_{\zeta^*\to\bar{\zeta}}\frac{K_0}{H_0}$ either diverges to infinity or a strictly positive constant. The latter occurs if $\underline{H}_0\neq 0$ for $\zeta^*\leq 1$. In all possible cases we therefore observe $\lim_{\zeta^*\to\bar{\zeta}}\frac{K_0}{H_0}>\lim_{\zeta^*\to\bar{\zeta}}\frac{K_0^+}{H_0^+}$.

Furthermore, if $\underline{\zeta}<\bar{\zeta}$ we know that $\lim_{\zeta^*\to\underline{\zeta}}\frac{K_0}{H_0}=0$ and $\lim_{\zeta^*\to\underline{\zeta}}\frac{K_0^+}{h_0^+}>0$ as $\frac{K_0^+}{h_0^+}$ is decreasing in ζ^* for $\zeta^*<1$ and approaches 0 as $\zeta^*\to 1$.

If it happens that $\underline{\zeta}=\bar{\zeta}$, this value is the unique solution to the initial value problem. If we find $\underline{\zeta}>\bar{\zeta}$, there is no solution to the initial value problem because initial endowments K_0,H_0 are too low to allow for subsistence consumption $\underline{c}$.

This proves that a unique solution always exits if and only if $\underline{\zeta} \leq \bar{\zeta} < 1$.

Appendix G: Calibration values for individual countries 2014

country	(34), (35) fulfilled	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$
							high income countries									
ARG	yes	42.98	11,994	38,901	7,033	5,144	0.01033	0.03769	0.00758	0.54428	0.03035	14,947	4,619	82,837	8,353	0.00998
AUS	yes	23.48	60,767	280,077	168,785	11,125	0.01492	0.03775	0.00660	0.59662	0.07343	221,057	148,928	1,210,077	52,795	0.01096
AUT	yes	8.55	51,786	256,361	10,353	8,842	0.00782	0.04317	0.00920	0.57805	0.00198	200,370	10,015	2,930,020	40,485	0.00922
BEL	yes	11.21	47,682	236,425	1,398	8,858	0.00444	0.04191	0.00930	0.63210	0.00020	184,458	1,362	4,271,091	40,300	0.00756
BHR	yes	1.34	20,745	100,488	13,722	4,500	0.01583	0.03467	0.00234	0.29356	0.08612	65,592	9,078	30,944	5,049	0.00954
CAN	yes	35.54	49,813	233,383	44,420	9,281	0.01084	0.03563	0.00730	0.64179	0.02107	183,759	40,301	1,227,602	43,647	0.01092
CHE	yes	8.19	86,978	435,276	3,366	11,673	0.01220	0.05146	0.00780	0.64420	0.00013	369,774	3,331	16,018,929	77,570	0.01219
CHL	yes	17.61	14,280	43,453	51,354	5,372	0.00860	0.04304	0.00602	0.43962	0.14681	18,721	35,694	106,885	9,438	0.00471
DEU	yes	80.98	49,085	253,695	3,906	8,512	0.00417	0.03731	0.01070	0.61892	0.00092	200,532	3,747	2,532,906	40,618	0.00670
DNK	yes	5.64	64,824	299,123	9,402	10,895	0.00507	0.04022	0.00910	0.61373	0.01019	237,215	7,207	434,275	29,115	0.00761
ESP	yes	46.48	29,525	116,579	5,164	7,332	-0.00299	0.03736	0.00850	0.57561	0.00080	77,318	5,078	3,640,052	21,771	0.00423
EST	yes	1.31	19,671	83,770	15,487	5,832	-0.00262	0.04548	0.01180	0.58462	0.01429	50,749	13,904	568,762	14,796	0.00236
FIN	yes	5.46	50,368	248,303	13,111	10,042	0.00414	0.04041	0.00960	0.60866	0.00591	187,585	11,929	1,229,367	41,068	0.00705
FRA	yes	66.32	43,941	214,595	5,326	8,939	0.00480	0.03725	0.00850	0.63269	0.00049	162,802	5,241	6,766,450	37,039	0.00813
GBR	yes	64.61	46,004	182,879	4,625	9,589	0.00753	0.03838	0.00880	0.58624	0.00604	132,009	3,630	352,220	26,387	0.00912
GRC	yes	10.89	21,908	109,716	5,611	6,765	-0.00666	0.03005	0.01040	0.49296	0.00176	63,510	5,469	1,534,619	16,063	0.00141
HRV	yes	4.24	13,337	47,597	6,719	5,142	-0.00407	0.04342	0.01200	0.59600	0.00906	22,189	5,914	388,835	9,633	0.00171
HUN	yes	9.87	13,606	56,339	3,013	4,642	-0.00269	0.04276	0.01280	0.59265	0.00403	30,426	2,669	392,506	10,093	0.00214
IRL	yes	4.66	46,979	136,491	2,996	9,066	0.00731	0.05303	0.00630	0.43200	0.00088	95,124	2,925	1,428,838	29,915	0.01045
ISL	yes	0.33	51,612	83,266	3,390	9,900	0.01113	0.03901	0.00630	0.60821	0.00000	57,702	3,389	590,192,538	32,245	0.01241
ITA	yes	60.79	35,398	177,497	3,803	8,188	0.00918	0.03643	0.00980	0.52226	0.00144	121,296	3,609	1,312,410	25,858	0.00962
JPN	yes	127.28	39,481	202,798	2,286	8,116	-0.00133	0.04480	0.01010	0.56206	0.00030	148,485	2,242	4,183,485	30,303	0.00428
KOR	yes	50.75	27,899	127,476	1,161	6,909	0.00628	0.05101	0.00530	0.51756	0.00049	83,891	1,108	1,177,813	20,206	0.01047
KWT	no	3.78	47,131	258,269	585,432	4,793	0.04989	0.04751	0.00263	0.24949	0.55145	-	-	-	-	0.00015
LUX	yes	0.56	78,931	398,040	5,011	9,787	0.02357	0.04873	0.00690	0.36746	0.00046	316,481	4,952	3,938,495	47,765	0.01829
LVA	yes	1.99	15,660	104,696	13,948	5,508	-0.00942	0.03278	0.01430	0.54459	0.01042	58,363	12,911	674,992	12,446	0.00000
NLD	yes	16.87	52,699	266,325	4,713	8,953	0.00360	0.03952	0.00830	0.58774	0.00803	210,091	3,369	246,719	18,707	0.00734
NOR	yes	5.14	100,268	563,923	98,381	12,285	0.01128	0.03857	0.00790	0.50830	0.07943	476,107	75,103	480,618	30,514	0.00808
OMN	no	3.96	19,361	61,021	98,895	4,004	0.06505	0.05356	0.00262	0.30267	0.39661	-	-	-	-	0.00933
POL	yes	38.01	13,858	31,173	7,185	4,673	-0.00075	0.05005	0.00990	0.55816	0.00937	14,795	6,631	394,877	8,894	0.00435

country	(34), (35) fulfilled	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$
								high income countries								
PRT	yes	10.40	21,700	93,101	5,079	6,408	-0.00539	0.03309	0.01010	0.57859	0.00291	56,355	4,798	955,193	16,235	0.00217
QAT	no	2.37	82,936	342,282	603,424	5,674	0.05361	0.10040	0.00150	0.17672	0.30554	-	-	-	-	0.00371
RUS	yes	143.82	13,876	50,982	43,517	4,569	0.00218	0.03347	0.01310	0.63030	0.13272	31,865	25,807	122,564	8,566	0.00114
SAU	yes	30.78	25,016	104,184	250,438	3,884	0.02741	0.05542	0.00353	0.27962	0.41243	82,029	-440,228	-298,461	975	0.00000
SGP	yes	5.47	55,597	309,021	22	5,559	0.01298	0.05306	0.00470	0.43947	0.00001	263,184	22	1,777,791	37,476	0.01414
SVK	yes	5.42	18,395	58,445	5,205	5,373	0.00097	0.04920	0.00950	0.54965	0.00443	33,458	4,860	602,647	12,568	0.00557
SVN	yes	2.06	23,975	112,045	11,317	6,545	0.00098	0.04175	0.00920	0.63842	0.00259	75,521	10,978	2,701,517	20,077	0.00581
SWE	yes	9.70	60,506	281,718	23,802	10,613	0.00992	0.04313	0.00920	0.56164	0.00575	216,264	22,660	2,214,136	45,679	0.01010
URY	yes	3.42	16,172	61,290	4,756	6,665	0.00338	0.03861	0.00933	0.47415	0.01524	23,937	3,514	109,312	10,937	0.00634
USA	yes	318.39	56,731	193,502	16,766	8,343	0.00734	0.04784	0.00824	0.59442	0.00666	154,690	15,805	1,410,912	41,598	0.00928
VEN	yes	30.74	15,516	74,968	36,313	7,400	0.01378	0.03645	0.00553	0.48818	0.11817	28,385	15,108	62,414	7,683	0.00845
								upper-middle income countries								
AZE	yes	9.54	7,665	27,445	38,001	945	0.01248	0.06869	0.00580	0.25055	0.23078	21,415	30,400	33,005	2,273	0.00000
BGR	yes	7.22	7,797	17,602	10,331	949	-0.00568	0.05315	0.01510	0.53085	0.01582	14,149	10,136	340,123	4,836	0.00000
BIH	yes	3.57	5,245	12,044	4,495	986	-0.01088	0.05002	0.01072	0.67110	0.01380	8,818	4,256	206,980	3,681	0.00000
BLR	yes	9.47	8,064	28,800	5,935	928	0.00090	0.04652	0.01280	0.60588	0.01336	24,281	5,692	258,211	5,917	0.00364
BRA	yes	204.21	11,789	28,466	24,911	1,566	0.00888	0.04683	0.00603	0.56810	0.04143	22,879	23,912	327,860	7,978	0.00963
BWA	yes	2.17	7,334	23,105	16,519	963	0.01865	0.05741	0.00741	0.27796	0.02516	17,998	16,371	180,841	4,355	0.01308
CHN	yes	1364.27	7,693	29,670	5,953	1,205	0.00506	0.05457	0.00716	0.57801	0.02448	23,547	5,391	127,279	5,840	0.00798
COL	yes	47.79	7,712	25,682	9,618	1,260	0.00944	0.04321	0.00588	0.46960	0.06625	19,416	8,352	59,201	5,130	0.00847
CRI	yes	4.76	10,180	20,186	8,596	1,480	0.01081	0.05333	0.00482	0.58774	0.01387	15,712	8,322	352,547	6,680	0.01235
DOM	yes	10.41	6,037	17,984	2,931	1,003	0.01204	0.03062	0.00606	0.45351	0.01623	13,437	2,781	77,717	3,968	0.01202
ECU	yes	15.90	6,299	19,748	23,419	1,174	0.01531	0.04199	0.00514	0.66696	0.11659	15,990	19,750	112,986	5,873	0.01061
GAB	yes	1.88	9,188	37,977	75,725	1,163	0.03165	0.07821	0.00805	0.27539	0.27124	29,557	17,560	17,830	828	0.00354
IRQ	no	35.01	6,629	14,824	69,087	904	0.03260	0.05120	0.00518	0.29631	0.45630	-	-	-	-	0.00024
JAM	yes	2.86	4,712	22,442	2,432	1,202	0.00360	0.03253	0.00689	0.60594	0.01203	15,441	2,177	109,608	3,854	0.00790
JOR	yes	8.81	4,020	11,027	2,150	944	0.04598	0.03683	0.00383	0.49429	0.01035	7,004	2,022	96,549	2,587	0.03013
KAZ	yes	17.29	11,498	37,691	53,914	1,087	0.01473	0.04043	0.00765	0.40222	0.18421	32,177	48,671	106,271	6,553	0.00457
LBN	yes	5.60	8,557	18,894	124	1,253	0.06016	0.03939	0.00458	0.44452	0.00002	14,287	123	2,476,275	5,140	0.03779
MEX	yes	124.22	10,337	36,850	9,050	1,280	0.01366	0.03660	0.00480	0.37419	0.05012	29,745	8,408	62,774	6,638	0.01096
MKD	yes	2.08	5,367	16,009	4,932	865	0.00085	0.03440	0.00970	0.50208	0.02996	12,111	4,624	77,501	3,553	0.00442

country	(34), (35) fulfilled	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$
							upper-middle income countries									
MNG	yes	2.92	3,842	8,282	27,593	737	0.01892	0.06082	0.00632	0.40758	0.27300	5,874	19,176	28,629	1,645	0.00374
MUS	yes	1.26	9,897	42,519	137	1,113	0.00181	0.04530	0.00770	0.42614	0.00001	35,379	137	4,874,670	6,629	0.00705
MYS	yes	30.23	10,814	29,495	20,053	925	0.01740	0.05739	0.00481	0.38044	0.09073	25,309	18,886	79,193	6,519	0.01027
NAM	yes	2.37	5,382	13,620	10,615	1,122	0.02324	0.05561	0.00778	0.52139	0.02563	9,223	10,168	206,836	3,476	0.01620
PAN	yes	3.90	11,522	19,916	8,125	1,278	0.01693	0.04855	0.00498	0.29566	0.00298	16,551	8,116	806,021	7,567	0.01567
PER	yes	30.97	6,199	17,515	19,892	1,125	0.01325	0.03944	0.00564	0.46026	0.07443	12,597	18,716	115,746	4,006	0.00980
PRY	yes	6.55	5,935	10,190	3,630	1,124	0.01334	0.04533	0.00565	0.56489	0.01544	7,099	3,432	125,580	3,704	0.01308
ROU	yes	19.91	9,900	35,742	6,736	1,028	-0.00375	0.05165	0.01280	0.45684	0.01444	30,030	6,551	207,220	6,433	0.00121
SUR	yes	0.55	9,426	43,553	80,516	1,249	0.00988	0.07574	0.00720	0.45216	0.23559	37,065	67,512	129,574	5,319	0.00232
THA	yes	68.42	5,647	18,691	2,414	802	0.00401	0.06271	0.00764	0.39281	0.02855	14,413	2,167	29,817	3,302	0.00662
TUN	yes	11.14	4,121	10,164	4,126	794	0.01168	0.04425	0.00636	0.50206	0.04865	7,124	3,610	37,255	2,657	0.01021
TUR	yes	77.03	12,021	21,197	1,655	1,067	0.01627	0.05573	0.00581	0.42765	0.00416	18,105	1,627	167,080	7,316	0.01494
ZAF	yes	54.54	6,257	18,550	8,673	1,038	0.01426	0.05200	0.01053	0.55234	0.05788	14,120	7,411	70,728	4,349	0.00932
							lower-middle income countries									
ARM	yes	2.91	4,181	13,305	5,222	584	0.00438	0.02959	0.00971	0.57967	0.03394	10,673	4,905	83,768	2,950	0.00610
BOL	yes	10.56	2,963	6,903	9,924	575	0.01545	0.06089	0.00742	0.47238	0.11274	4,795	8,629	36,155	1,883	0.00842
CIV	yes	22.53	1,528	4,409	3,402	598	0.02540	0.03778	0.01286	0.33192	0.06175	1,481	3,023	16,250	900	0.01136
CMR	yes	22.24	1,558	3,509	5,591	572	0.02662	0.05142	0.01062	0.50266	0.07981	1,454	4,666	29,387	977	0.01348
EGY	yes	91.81	3,249	4,537	5,210	392	0.02208	0.06946	0.00602	0.35369	0.08664	3,576	4,944	20,183	1,853	0.01153
GEO	yes	3.72	4,398	15,764	2,504	590	0.00047	0.03775	0.01322	0.43352	0.00831	12,536	2,455	128,122	2,879	0.00327
GTM	yes	15.92	3,600	8,743	3,689	601	0.02077	0.04595	0.00490	0.41637	0.03155	6,380	3,516	46,407	2,223	0.01562
HND	yes	8.81	2,059	6,153	4,591	617	0.01734	0.05597	0.00481	0.59647	0.02715	3,675	4,280	94,048	1,532	0.01490
IDN	yes	255.13	3,375	13,738	4,823	406	0.01222	0.03685	0.00708	0.46375	0.05165	11,318	4,495	40,365	2,307	0.00981
IND	yes	1293.86	1,557	4,722	1,276	341	0.01189	0.05745	0.00731	0.51626	0.02814	3,176	1,124	20,613	1,041	0.01088
KEN	yes	46.02	1,316	3,007	1,202	565	0.02636	0.05277	0.00600	0.63845	0.02773	1,254	900	20,729	968	0.01872
KGZ	yes	5.84	1,227	5,130	2,583	468	0.02006	0.03629	0.00610	0.52781	0.07382	2,523	1,832	13,097	922	0.01306
LAO	yes	6.58	1,929	3,005	14,887	432	0.01252	0.06235	0.00685	0.39763	0.12204	1,834	14,243	46,404	1,109	0.00630
LKA	yes	20.78	3,732	9,309	971	418	0.00933	0.03744	0.00674	0.31044	0.00108	7,750	969	279,360	2,493	0.01121
MAR	yes	34.32	3,133	11,492	5,245	520	0.01448	0.05203	0.00517	0.49735	0.02063	8,699	5,080	122,453	2,139	0.01348
MDA	yes	3.56	2,477	12,630	415	549	-0.00061	0.03093	0.01141	0.57133	0.00364	9,000	381	59,749	1,909	0.00387
MRT	yes	4.06	1,264	2,381	10,270	423	0.02940	0.06129	0.00805	0.45416	0.31032	1,310	1,051	1,539	360	0.00619

country	(34), (35) fulfilled	L_0	$\tilde{y}_0$	$\tilde{k}_0$	$\tilde{s}_0$	$\underline{c}$	n	δ_1	δ_2	β	γ	$\underline{\tilde{k}}_0$	$\underline{\tilde{s}}_0$	$\underline{\tilde{h}}_0$	$\underline{\tilde{c}}^{\max}$	$\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}$
								upper-middle income countries								
NGA	yes	176.46	3,114	3,531	8,423	660	0.02660	0.04960	0.01307	0.48877	0.10321	2,175	7,326	34,692	1,718	0.01122
NIC	yes	6.01	1,923	6,882	6,763	481	0.01141	0.03975	0.00477	0.55262	0.04416	4,545	6,401	80,100	1,416	0.01123
PHL	yes	100.10	3,445	7,318	1,788	503	0.01633	0.04890	0.00646	0.35654	0.03173	5,579	1,694	19,031	2,115	0.01249
SEN	yes	14.55	1,333	3,068	1,840	560	0.02971	0.04528	0.00627	0.40446	0.03751	928	1,639	17,676	802	0.01861
SWZ	yes	1.30	3,388	18,359	1,112	500	0.01842	0.04704	0.01024	0.61208	0.03548	15,319	626	10,798	741	0.01250
TJK	yes	8.36	1,366	29,601	2,407	505	0.02236	0.02373	0.00524	0.46346	0.01857	18,279	2,074	51,747	1,320	0.01727
UKR	yes	45.27	2,914	23,756	5,972	439	-0.00480	0.02811	0.01470	0.55837	0.05664	19,718	5,376	52,992	2,580	0.00000
								low income countries								
BDI	yes	9.89	273	415	219	247	0.02992	0.03845	0.01128	0.60622	0.17033	-119	-1,259	-4,482	28	0.01179
BFA	yes	17.59	683	1,599	2,394	308	0.02962	0.05959	0.00910	0.57301	0.16982	710	412	1,391	276	0.01220
CAF	yes	4.52	379	2,374	9,271	437	0.00349	0.02989	0.01451	0.16432	0.12361	-2,068	8,742	11,621	233	0.00000
GIN	yes	11.81	719	1,069	2,451	296	0.02304	0.05999	0.00990	0.48346	0.17116	337	1,094	3,091	351	0.00832
MOZ	yes	27.21	616	95	1,752	399	0.02901	0.06237	0.01069	0.41484	0.13564	-34	1,543	4,719	294	0.01074
NER	yes	19.15	422	2,088	2,701	329	0.03843	0.03633	0.01022	0.43218	0.13771	-317	727	2,283	256	0.01466
RWA	yes	11.35	691	1,320	751	299	0.02501	0.04574	0.00633	0.74101	0.06399	683	283	3,282	387	0.01661
SLE	yes	7.08	692	769	2,991	289	0.02244	0.07837	0.01336	0.54505	0.36456	470	-5,314	-7,945	95	0.00270
TCD	yes	13.57	982	1,006	5,390	342	0.03265	0.06349	0.01352	0.45241	0.20052	365	2,944	6,642	491	0.00895
TZA	yes	52.23	941	2,684	3,112	281	0.03108	0.04500	0.00732	0.49800	0.06222	1,459	2,778	22,232	615	0.01778
ZWE	yes	15.41	1,145	1,675	3,763	396	0.02345	0.03573	0.00886	0.55074	0.07000	702	3,347	26,338	681	0.01365

Note: Calibration values and results as explained in Section 5.5. Parameters $\rho = 0.03$, $\eta = 2$ and $B = 0.05$ uniformly calibrated for all countries. All nominal quantities in 2014 US$ per capita except population L_0 for 2014 in mln. Asymptotic per capita growth for $\frac{\dot{c}_t}{c_t}$ in decimals.

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