Block Numerical Ranges

von Markus Wagenhofer

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Gutachter: Prof. Dr. Christiane Tretter (Universität Bern)
Prof. Dr. Heinz Langer (TU Wien)

Contents

Introduction

Linear equations involving block operator matrices play an important role in many applications. For instance, they occur as saddle point problems arising in the discretization process of systems of partial differential equations with constraints in fluid dynamics or linear elasticity. An impressive list of more fields providing such problems (including references) is given in [BGL05].

One method for solving such problems is the preconditioning of the original problem, that is, its transformation into an equivalent problem with better spectral properties (see [BGL05, Section 10]). Knowledge about the spectral properties of a matrix or an operator is essential in the construction of effective preconditioners for linear systems. In the context of block operator matrices, it is additionally highly desirable to exploit as much information as possible from the block structure. However, even in the finite dimensional case very little is known about the spectral properties of block operator matrices.

The numerical range of an operator is a well-known and effective tool when studying spectral properties of operators. However, it does not respect the block structure of block operator matrices, thus destroying information which could be of subsequent use. To address this problem, in [LT98] the *quadratic numerical range* of a block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{0.1}$$

with respect to a decomposition $\mathcal{H} = H_1 \times H_2$ was defined by

$$W^2(\mathcal{A}) := \bigcup \left\{ \sigma_{\mathrm{p}} \left(\begin{pmatrix} (Ax,x) & (By,x) \\ (Cy,x) & (Dy,y) \end{pmatrix} \right) \,:\, x \in H_1,\ y \in H_2,\ \|x\| = \|y\| = 1 \right\}.$$

It was used there as a tool to locate the spectrum of non selfadjoint block operator matrices $\mathcal{A}$ with possibly unbounded entries A and D. More properties of the quadratic numerical range, often similar to properties of the *numerical range*

$$W(\mathcal{A}) := \{(\mathcal{A}f, f) \,:\, f \in \mathcal{H},\ \|f\| = 1\},$$

were shown in [LMMT01], [LMT01], including the inclusions

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}, \quad W^2(\mathcal{A}) \subset W(\mathcal{A}),$$

estimates of the resolvent, and the fact that corners of $W^2(\mathcal{A})$ lie in the spectrum of $\mathcal{A}$. Moreover, if $\overline{W^2(\mathcal{A})}$ consists of two connected components, the existence of certain $\mathcal{A}$-invariant subspaces and corresponding solutions of Riccati equations related to $\mathcal{A}$ was proved; additionally, as a generalization of the well-known fact that a 2×2 matrix with two distinct eigenvalues is diagonalizable, $\mathcal{A}$ was shown to be block-diagonalizable in this case.

The definition of the quadratic numerical range was generalized to decompositions $\mathcal{H} = H_1 \times \cdots \times H_n$ in the obvious way in [Wag00] and [TW03]: Given a block operator matrix $\mathcal{A}$ with respect to the decomposition $\mathcal{H}$,

$$\mathcal{A} = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix},$$

its *block numerical range* with respect to this decomposition was defined by

$$W_{\mathcal{H}}(\mathcal{A}) := \bigcup \Big\{ \sigma_{\mathrm{p}}(\mathcal{A}_x) \, : \, x = (x_1, \dots, x_n) \in \mathcal{H}, \; \|x_i\| = 1, \; i = 1, \dots, n \Big\},$$

where

$$\mathcal{A}_x := \begin{pmatrix} (A_{11}x_1, x_1) & \dots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots \\ (A_{n1}x_1, x_n) & \dots & (A_{nn}x_n, x_n) \end{pmatrix}, \quad x = (x_1, \dots, x_n) \in \mathcal{H}. \tag{0.2}$$

It was shown that $W_{\widetilde{\mathcal{H}}}(\mathcal{A}) \subset W_{\mathcal{H}}(\mathcal{A})$ if $\widetilde{\mathcal{H}}$ is a refined decomposition of $\mathcal{H}$, while the spectral inclusion $\sigma(\mathcal{A}) \subset \overline{W_{\mathcal{H}}(\mathcal{A})}$ continues to hold. These two properties alone justify a further investigation of the block numerical range.

Aims

It is the aim of this thesis to prove generalizations of theorems known for the numerical range of operators and the quadratic numerical range of 2×2 block operator matrices and, more generally, block operator functions, including

- the block diagonalization of block operator matrices if the closure of the block numerical range consists of n connected components (which is, in some sense, a generalization of the well-known fact that a complex $n \times n$ matrix with n distinct eigenvalues is diagonalizable),
- the fact that corners of the block numerical range of a block operator matrix are contained in its spectrum,
- the definition and properties of the block numerical range of block operator functions (generalizing [Tre06]), and, in particular, block operator polynomials.

Overview

In **Chapter 1**, the concept and main properties of block numerical ranges of bounded operators are presented. In particular, Chapter 1 contains the most important results from [Wag00] and [TW03] (Section 1.2), including, e. g.,

(1) the spectral inclusions $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W_{\mathcal{H}}(\mathcal{A})$, $\sigma(\mathcal{A}) \subset \overline{W_{\mathcal{H}}(\mathcal{A})}$,

(2) the resolvent estimate

$$\|(\mathcal{A}-z)^{-1}\| \leq \frac{(\|\mathcal{A}\|+|z|)^{n-1}}{\operatorname{dist}(z, W_{\mathcal{H}}(\mathcal{A}))^n}, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(\mathcal{A})}, \tag{0.3}$$

(3) the inclusion $W_{\widetilde{\mathcal{H}}}(\mathcal{A}) \subset W_{\mathcal{H}}(\mathcal{A})$ for a decomposition $\widetilde{\mathcal{H}}$ which is finer than $\mathcal{H}$.

Section 1.3 addresses continuity properties of mappings connected with the block numerical range. For example, it is shown, that the closure $\overline{W}_{\mathcal{H}}$ of the numerical range depends continuously on the operator with respect to the Hausdorff metric on $\mathbb{C}$. It is known that the block numerical range is, in contrast to the numerical range, not connected anymore. If the closure of the quadratic numerical range of a block operator matrix $\mathcal{A}$ consists of two connected components, it was shown in [LMMT01] that there are invariant subspaces of $\mathcal{A}$ given by the graph subspaces of solutions of certain Riccati equations related to the block entries of $\mathcal{A}$, and that $\mathcal{A}$ allows a block diagonalization. To prove corresponding statements in the general block numerical case (Chapter 4), the results of Section 1.4, where the connected components of block numerical ranges are examined, are of major interest. In particular, the notion of (strongly) $\mathcal{H}$-separated connected components of $W_{\mathcal{H}}(A)$ is introduced (Definition 1.22) and a Gershgorin type condition on the block entries of $\mathcal{A}$ for strong $\mathcal{H}$-separateness (Proposition 1.27) is presented. Finally, Section 1.5 shows relations between properties of the block numerical range of a block operator $\mathcal{A}$ and its block determinant set $D_{\mathcal{H}}(\mathcal{A}) = \{\det \mathcal{A}_x : \|x_i\| = 1,\ i = 1, \ldots, n\}$ with respect to $\mathcal{H}$.

In **Chapter 2** the concept of the block numerical range is extended to block operator functions $\mathcal{F} : \Omega \to L(\mathcal{H})$ by the definition

$$W_{\mathcal{H}}(\mathcal{F}) := \{z \in \Omega : 0 \in W_{\mathcal{H}}(\mathcal{F}(z))\},$$

which is in accordance with the definition $W(\mathcal{F}) = \{z \in \Omega : 0 \in W(\mathcal{F}(z))\}$ of the numerical range (see [Mar88, § 26.3]) and of the quadratic numerical range for 2×2 block operator matrix functions (see [Tre06]). The most important part of this chapter is Section 2.2 where, for analytic block operator functions $\mathcal{F}$, the spectral inclusions $\sigma_{\mathrm{p}}(\mathcal{F}) \subset W_{\mathcal{H}}(\mathcal{F})$ and $\sigma(\mathcal{F}) \subset \overline{W_{\mathcal{H}}(\mathcal{F})}$ are shown under an additional condition on $\mathcal{F}$ generalizing a well-known condition for the numerical range (see [Mar88, Equation (26.5)]), namely

$$0 \notin \overline{W_{\mathcal{H}}(F(z_0))} \text{ for some } z_0 \in \Omega. \tag{0.4}$$

Moreover, in Section 2.3, an estimate of the resolvent norm $\|\mathcal{F}^{-1}\|$ similar to (0.3) is proved,

$$\|F^{-1}(z)\| \leq \frac{\gamma}{\operatorname{dist}(z, C)^{\nu}}, \quad z \in \overline{U} \setminus \overline{C},$$

for certain connected components C of $W_{\mathcal{H}}(\mathcal{F})$ and compact neighborhoods $\overline{U}$ of them. This estimate, in turn, allows to give upper bounds for the lengths of Jordan chains corresponding to eigenvalues of $\mathcal{F}$ on the boundary of $W_{\mathcal{H}}(F)$ having the exterior cone property with respect to $W_{\mathcal{H}}(\mathcal{F})$.

Operator polynomials represent an important class of operator functions; their spectral properties are of interest in many classical areas of mathematical physics, such as differential equations, boundary value problems and hydrodynamics. Properties of the block numerical range of operator polynomials are studied in **Chapter 3**. Criteria on the boundedness of $W_{\mathcal{H}}(\mathcal{P})$ for an operator polynomial $\mathcal{P}$ are proved in Section 3.1, the estimate of the resolvent norm $\|\mathcal{P}^{-1}\|$, already known from the general block operator function case, is improved in Section 3.2 (making it look rather like (0.3)), and the inclusion of $W_{\mathcal{H}}(\mathcal{P})$ in the block numerical range of its companion polynomial with respect to a (canonical) decomposition are shown in Section 3.3.

Chapter 4 deals with the generalization of the following result of [LMMT01]: if the closure $\overline{W^2(\mathcal{A})}$ of the quadratic numerical range of the block operator matrix $\mathcal{A}$ in (0.1) consists of two connected components, the Riccati equations

$$K_1BK_1 + K_1A - DK_1 - C = 0, \quad K_2CK_2 + K_2D - AK_2 - B = 0,$$

have solutions $K_1 \in L(H_1, H_2)$ and $K_2 \in L(H_2, H_1)$ respectively, and hence $\mathcal{A}$ is block diagonalizable,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & K_2 \\ K_1 & 1 \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \begin{pmatrix} 1 & K_2 \\ K_1 & 1 \end{pmatrix}^{-1}.$$

As already mentioned above, it is the aim of **Chapter 4** to prove these statements also in the general case of the block numerical range, at least under some additional assumptions. These assumptions are satisfied by large classes of block operator matrices and are reasonable in most practical situations. It is shown that, provided that the dimension condition

$$\dim H_i \geq n, \quad i = 1, \dots, n, \tag{0.5}$$

is fulfilled, the results of [LMMT01] have generalizations to $n \times n$ block operator matrices, if one of the following holds:

(1) the operator $\mathcal{A}$ is selfadjoint,

(2) all Hilbert spaces H_i are infinite dimensional,

(3) the connected components of $W_{\mathcal{H}}(\mathcal{A})$ are strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ (see Definition 1.22).

As in [LMMT01], a factorization of the Schur complements of $\mathcal{A}$ with respect to certain diagonal entries of $\mathcal{A}$ (Section 4.1) is the key to obtain solutions of the corresponding Riccati equations. In Section 4.2, several conditions for the existence of such factorizations are given. The corresponding theorems on the solutions of Riccati equations and the block diagonalization of $\mathcal{A}$ are established in Section 4.3.

Chapter 5 contains a collection of theorems concerning corners of block numerical ranges of both block operator matrices and block operator functions. It is well-known that corners of the numerical range of an operator belong to its (point) spectrum. In [LMT01], several generalizations of this statement were proved:

(1) If $F : \Omega \to \mathbb{C}$ is an analytic operator function and $\lambda_0 \in W(F)$ is a corner of $W(F)$, then λ_0 is an eigenvalue of F if an additional smallness assumption on the angle of $W(F)$ in λ_0 is fulfilled.

(2) If λ_0 is a corner of $W^2(\mathcal{A})$, where $\mathcal{A}$ is the block operator in (0.1), then λ_0 lies in the spectrum $\mathcal{A}$ itself or in the spectrum of one of its diagonal entries A and D. If, additionally, $\lambda_0 \in W^2(A)$, then it is even an eigenvalue of one of these operators.

After the rather technical Section 5.1 on analytic perturbations of matrices, it is shown in Section 5.2 that, provided that the dimension condition (0.5) holds, for every corner λ_0 of $W_{\mathcal{H}}(\mathcal{A})$ belonging to $W_{\mathcal{H}}(\mathcal{A})$, there is a subspace $\mathcal{H}'$ of $\mathcal{H}$ of the form $\mathcal{H}' = H_{i_1} \times \cdots \times H_{i_m}$, where $i_1 < i_2 < \cdots < i_m$, such that $\lambda_0 \in \sigma_{\mathrm{p}}(P_{\mathcal{H}'} A|_{\mathcal{H}'})$. If $\lambda_0 \notin W_{\mathcal{H}}(\mathcal{A})$, then σ_{p} must be replaced by σ (Section 5.3). Finally, in Section 5.4 we will see that the corresponding assertions even hold for corners of the block numerical range of analytic block operator functions satisfying (0.4). Note that even in the latter case no condition on the size of the angle of the corner has to be imposed, thus improving the results in [LMT01, Theorems 2.7 and 2.9].

Finally, **Appendix A** contains the description of algorithms which allow the visualization of block numerical ranges.

Notation

Throughout the thesis, H denotes a complex Hilbert space, equipped with a scalar product $(\cdot\,,\cdot)$ such that $(\alpha x, \beta y) = \alpha\bar{\beta}(x,y)$, $x, y \in H$, $\alpha, \beta \in \mathbb{C}$. The term 'operator' will always mean 'linear operator' and all operators are assumed to be bounded. For two Hilbert spaces H, H', the space of all bounded operators from H to H' is denoted by $L(H, H')$ and the short form $L(H) = L(H,H)$ is used for $H = H'$. For the identity operator on H we write 1_H or simply 1 if the underlying Hilbert space is clear from the context. Moreover, the following notations and conventions are used:

- The symbols $\mathbb{R}$ and $\mathbb{C}$ have the standard meaning, $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.
- For $n \in \mathbb{N}$, the space of all complex $n \times n$ matrices is denoted by $M_n(\mathbb{C})$. As usual, it is identified with the space $L(\mathbb{C}^n)$.
- The resolvent set $\rho(A)$ of an operator $A \in L(H)$ is the set of all complex numbers λ such that $A - \lambda$ is invertible, that is, has an inverse in $L(H)$, the spectrum is then the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$; $\sigma_{\mathrm{p}}(A)$ is the set of eigenvalues of A.
- $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ for a metric space X, $x \in X$ and $\varepsilon > 0$.

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Chapter 1

The block numerical range of bounded operators

In [LMMT01] the *quadratic numerical range* of a block operator matrix was introduced as a tool to locate its spectrum: Let $\mathcal{H} = H_1 \times H_2$ be the product of two Hilbert spaces and let the operator $\mathcal{A} \in L(\mathcal{H})$ have the block operator representation

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = H_1 \times H_2$. Then the quadratic numerical range of $\mathcal{A}$ with respect to this decomposition is the set

$$W^2(\mathcal{A}) := \bigcup \left\{ \sigma_{\mathrm{p}} \left(\begin{pmatrix} (Ax,x) & (By,x) \\ (Cx,y) & (Dy,y) \end{pmatrix} \right) \,:\, x \in H_1,\ y \in H_2,\ \|x\| = \|y\| = 1 \right\}.$$

This definition has been generalized to decompositions $\mathcal{H} = H_1 \times \cdots \times H_n$ in [Wag00]; see also [TW03]. During the work on this thesis it turned out that it is often necessary to deal with different decompositions of the Hilbert space $\mathcal{H}$. Therefore, it seems to be more convenient not to define $\mathcal{H}$ as a product of Hilbert spaces initially. A new viewpoint, where a Hilbert space H is the starting point of the considerations, is presented in the following section.

1.1 About decompositions

Let H be a Hilbert space. If $M \subset H$ is a closed subspace of H, let $P_M \in L(H, M)$ denote the orthogonal projection of H onto M and let $x_M := P_M x \in M$, $x \in H$. For $A \in L(H)$ and closed subspaces $M, N \subset H$ let

$$A_{MN} := P_M A|_N \in L(N, M).$$

For brevity, write $A_M := A_{MM}$ in the case $M = N$.

A non-empty, finite set $\mathcal{H}$ of closed subspaces $\neq \{0\}$ of H such that $M \perp N$, $M, N \in \mathcal{H}$, $M \neq N$, and

$$H = \bigoplus \mathcal{H} := \Big\{ \sum_{M \in \mathcal{H}} x_M \,:\, x_M \in M,\ M \in \mathcal{H} \Big\}$$

is called a (finite orthogonal) *decomposition of* H. The collection of all decompositions of H is denoted by $\mathcal{Z}(H)$. Further, let

$$\mathcal{Z}'(H) := \bigcup \{\mathcal{Z}(H') \,:\, H' \subset H \text{ is a closed subspace}\},$$

that is, $\mathcal{Z}'(H)$ is the set of all decompositions of all closed subspaces of H. For $\mathcal{H} \in \mathcal{Z}'(H)$, let

$$\begin{aligned} \mathcal{H}^* &:= \{x \in H \,:\, \forall\, M \in \mathcal{H} \quad P_M x \neq 0\}, \\ \mathcal{H}^\square &:= \{x \in H \,:\, \forall\, M \in \mathcal{H} \quad \|P_M x\| = 1\}. \end{aligned}$$

and denote the set of all bijections $\{1, \dots, |\mathcal{H}|\} \to \mathcal{H}$ by $\Pi(\mathcal{H})$. (Here $|\mathcal{H}|$ denotes the number of elements of $\mathcal{H}$.)

Let now $\mathcal{H} \in \mathcal{Z}'(H)$, $n := |\mathcal{H}|$ and $A \in L(H)$. For $\pi \in \Pi(\mathcal{H})$ and $x \in H$, we set

$$M_n(\mathbb{C}) \ni A_x^\pi := A_{x,\pi} := \big((A_{\pi(i)\pi(j)} x_{\pi(j)}, x_{\pi(i)})\big)_{i,j=1}^n.$$

Note that for $\pi, \tilde{\pi} \in \Pi(\mathcal{H})$ we have

$$\det A_x^\pi = \det A_x^{\tilde{\pi}}, \quad \sigma_{\mathrm{p}}(A_x^\pi) = \sigma_{\mathrm{p}}(A_x^{\tilde{\pi}}), \quad x \in H.$$

In this sense we will write $A_x^{\mathcal{H}}$ (or $A_{x,\mathcal{H}}$) whenever the choice of $\pi \in \Pi(\mathcal{H})$ is clear or not essential, or even A_x if it is clear from the context which decomposition $\mathcal{H}$ is used, e. g.,

$$\begin{aligned} \det A_x = \det A_{x,\mathcal{H}} = \det A_x^{\mathcal{H}} = \det A_x^\pi, \\ \sigma_{\mathrm{p}}(A_x) = \sigma_{\mathrm{p}}(A_{x,\mathcal{H}}) = \sigma_{\mathrm{p}}(A_x^{\mathcal{H}}) = \sigma_p(A_x^\pi). \end{aligned}$$

Example 1.1. Assume that $H_1, \dots, H_n$ are non-trivial Hilbert spaces and let

$$H := H_1 \times \cdots \times H_n. \tag{1.1}$$

For $k = 1, \dots, n$ let M_k be the image of the inclusion mapping $H_k \hookrightarrow H$, and let $\mathcal{H} := \{M_1, \dots, M_n\} \in \mathcal{Z}(H)$. Moreover, define $\pi \in \Pi(\mathcal{H})$ by $k \mapsto M_k$, $k = 1, \dots, n$. If $\mathcal{A} \in L(H)$ is a block operator matrix with respect to (1.1), given by $\mathcal{A} = (A_{ij})_{i,j=1}^n$, where $A_{ij} \in L(H_j, H_i)$, $i, j = 1, \dots, n$, then

$$\mathcal{A}_x^\pi = \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix} = \mathcal{A}_x, \quad x \in H,$$

as defined previously (see (0.2)). This example leads to the following convention.

Convention. Within this thesis, calligraphic letters like $\mathcal{H}$ are used for both products of Hilbert spaces and decompositions of a given Hilbert space in the following sense:

(1) If $\mathcal{H}$ is defined to be the product of Hilbert spaces, it is also referred to as a decomposition. Then $M \in \mathcal{H}$ refers to a subspace of $\mathcal{H}$ induced by one of the factors appearing in the definition of $\mathcal{H}$ (not the factor itself).

(2) If $\mathcal{H} \in \mathcal{Z}'(H)$ is used in situations where a Hilbert space is expected, we identify $\mathcal{H}$ with $H' := \bigoplus \mathcal{H} \subset H$; for instance, we write $A_{\mathcal{H}} := A_{H'} = P_{H'}A|_{H'}$.

Additionally, whenever we write $\mathcal{H} = \{M_1, \ldots, M_n\}$ for some $\mathcal{H} \in \mathcal{Z}(H)$, the following abbreviations will subsequently be used without any further comment:

$$P_i := P_{M_i}, \quad x_i := x_{M_i}, \quad A_{ij} := A_{M_iM_j}, \qquad i, j \in \{1, \ldots, n\},\ x \in H.$$

1.2 The definition and known results

We may now define the block numerical range of operators $A \in L(H)$ with respect to a decomposition $\mathcal{H} \in \mathcal{Z}'(H)$.

Definition 1.2. Let H be a Hilbert space, $\mathcal{H} \in \mathcal{Z}'(H)$, and $A \in L(H)$. Then the *block numerical range of A with respect to $\mathcal{H}$* (or *$\mathcal{H}$-numerical range of A*) is defined to be the set

$$W_{\mathcal{H}}(A) := \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}(A_x).$$

Note that, if $H' := \bigoplus \mathcal{H}$, and $A' := A_{H'} \in L(H')$, then $\mathcal{H} \in \mathcal{Z}(H')$ and

$$W_{\mathcal{H}}(A') = W_{\mathcal{H}}(A).$$

For future reference, we now quote some results which were obtained in [Wag00] and [TW03].

Proposition 1.3. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$.*

(0) *If $\mathcal{H} = \{H\}$, then $W_{\mathcal{H}}(A) = W(A)$.*

(1) *$W_{\mathcal{H}}(A)$ is bounded. If H is finite dimensional, then $W_{\mathcal{H}}(A)$ is compact.*

(2) *For $\alpha, \beta \in \mathbb{C}$ we have $W_{\mathcal{H}}(\alpha A + \beta) = \alpha W_{\mathcal{H}}(A) + \beta$.*

(3) *If H is finite dimensional and $|\mathcal{H}| = \dim H$, then $W_{\mathcal{H}}(A) = \sigma_{\mathrm{p}}(A)$.*

(4) *The equality $W(A^*) = W(A)^*$ holds*[1].

(5) *If A is selfadjoint, then $W_{\mathcal{H}}(A) \subset \mathbb{R}$. The opposite direction is, in general, only true for $|\mathcal{H}| = 1$.*

[1]To avoid confusion with the closure of a complex set, the notation $C^* = \{z : \bar{z} \in C\}$ is used instead of $\overline{C}$ for the complex conjugation of a set $C \subset \mathbb{C}$ throughout this thesis.

Theorem 1.4. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$. Then the spectral inclusions*

$$\sigma_{\mathrm{p}}(A) \subset W_{\mathcal{H}}(A), \qquad \sigma(A) \subset \overline{W_{\mathcal{H}}(A)}$$

hold. (See [Wag00, Satz 4.6] or [TW03, Theorem 2.5].)

Theorem 1.5. *For $\mathcal{H} \in \mathcal{Z}(H)$, $n := |\mathcal{H}|$ and $A \in L(H)$ the following estimate of the resolvent of A holds:*

$$\|(A - z)^{-1}\| \le \frac{\|A - z\|^{n-1}}{\operatorname{dist}(z, W_{\mathcal{H}}(A))^n}, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}. \tag{1.2}$$

(See [TW03, Theorem 4.2]; note that this theorem is also a special case of Theorem 3.10.)

Definition 1.6. Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{Z}(H)$. Then $\mathcal{H}_1$ is called a *refinement of* $\mathcal{H}_2$ if

$$\forall\, M \in \mathcal{H}_2\ \exists\, \mathcal{H}_M \subset \mathcal{H}_1\ \ \mathcal{H}_M \in \mathcal{Z}(M).$$

That is, $\mathcal{H}_1$ is a finer decomposition than $\mathcal{H}_2$ in the sense that each element of $\mathcal{H}_2$ is again decomposed by appropriate elements of $\mathcal{H}_1$. In this case we write $\mathcal{H}_1 \le \mathcal{H}_2$.

Theorem 1.7. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{Z}(H)$ such that $\mathcal{H}_1 \le \mathcal{H}_2$. Then the inclusion*

$$W_{\mathcal{H}_1}(A) \subset W_{\mathcal{H}_2}(A)$$

holds for any $A \in L(H)$. (See [Wag00, Satz 4.7] or [TW03, Theorem 3.5].)

As an example for Theorem 1.4 and Theorem 1.7 consider the matrix

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}. \tag{1.3}$$

The block numerical ranges of A with respect to two different decompositions are shown in Figure 1.1.

To see that refined decompositions do not always result in smaller block numerical ranges, consider the following example. Note, however, that this example is trivial in the sense that already the numerical range itself gives an (almost) optimal approximation of the spectrum. Additionally, this example shows that $W_{\mathcal{H}}(A) = \overline{W_{\mathcal{H}}(A)}$ and $\sigma(A) \subset W_{\mathcal{H}}(A)$ do not hold in general.

Example 1.8. Consider the left shift $S \in L(l^2(\mathbb{N}))$ given by $S(\delta_{1k})_{k=1}^{\infty} = 0$ and $S(\delta_{nk})_{k=1}^{\infty} = (\delta_{n-1,k})_{k=1}^{\infty}$, $n \ge 2$. An easy exercise shows that $\sigma_{\mathrm{p}}(S) = B_1(0)$ and $\sigma(S) = \overline{B_1(0)}$. Moreover, $W(S) = B_1(0)$ according to [GR97, Example 1-1]. For any decomposition $\mathcal{H} \in \mathcal{Z}(l^2(\mathbb{N}))$, Theorem 1.4 and Theorem 1.7 yield

$$B_1(0) = \sigma_{\mathrm{p}}(S) \subset W_{\mathcal{H}}(A) \subset W(A) = B_1(0),$$

thus, $W_{\mathcal{H}}(S) = W(S) = B_1(0)$.

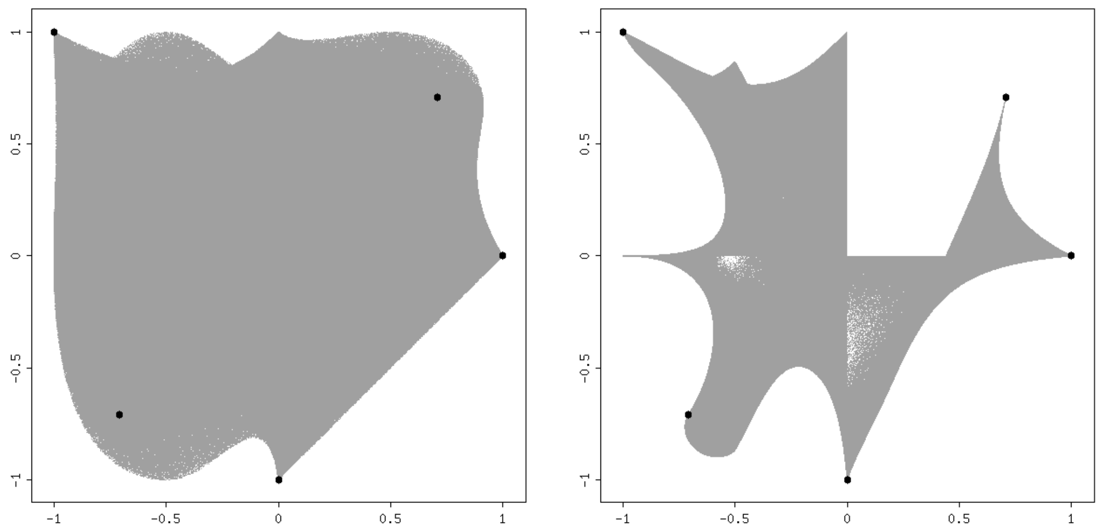


Figure 1.1: The block numerical range of the matrix (1.3) with respect to the decompositions $\mathcal{H}_1 = \mathbb{C}^3 \times \mathbb{C}^2$ and $\mathcal{H}_2 = \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}^2$. The circles (•) denote the eigenvalues of the matrix.

Proposition 1.9. *Let* $\mathcal{H} \in \mathcal{Z}(H)$ *and* $A \in L(H)$. *Assume that* $\mathcal{H} = \mathcal{H}_1 \dot\cup \ldots \dot\cup \mathcal{H}_k$ *such that the block operator representation of* A *with respect to the decomposition* $H = \bigoplus \mathcal{H}_1 \times \cdots \times \bigoplus \mathcal{H}_k$ *is upper or lower triangular. Then*

$$W_{\mathcal{H}}(A) = \bigcup_{j=1}^{k} W_{\mathcal{H}_j}(A).$$

Proof. Obviously, it is sufficient to consider the case $\mathcal{H} = \mathcal{H}_1 \dot\cup \mathcal{H}_2$ only. Write $\mathcal{H}_1 = \{M_1, \ldots, M_\nu\}$, $\mathcal{H}_2 = \{M_{\nu+1}, \ldots, M_n\}$ and $H_1 := \bigoplus \mathcal{H}_1$, $H_2 := \bigoplus \mathcal{H}_2$. Assume that A is upper triangular with respect to the decomposition $H = H_1 \times H_2$, that is, $A_{H_2 H_1} = 0$ or, equivalently, $A_{ij} = 0$ for $i \in \{\nu+1, \ldots, n\}$ and $j \in \{1, \ldots, \nu\}$. Thus, for every $x \in \mathcal{H}^\square$ the matrix $A_x^{\mathcal{H}}$ is of the form

$$A_x^{\mathcal{H}} = \begin{pmatrix} (A_{11}x_1, x_1) & \cdots & (A_{1\nu}x_\nu, x_1) & (A_{1,\nu+1}x_{\nu+1}, x_1) & \cdots & (A_{1n}x_n, x_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ (A_{\nu 1}x_1, x_\nu) & \cdots & (A_{\nu\nu}x_\nu, x_\nu) & (A_{\nu,\nu+1}x_{\nu+1}, x_\nu) & \cdots & (A_{\nu n}x_n, x_\nu) \\ 0 & \cdots & 0 & (A_{\nu+1,\nu+1}x_{\nu+1}, x_{\nu+1}) & \cdots & (A_{\nu+1,n}x_n, x_{\nu+1}) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & (A_{n,\nu+1}x_{\nu+1}, x_n) & \cdots & (A_{nn}x_n, x_n) \end{pmatrix}$$
$$= \begin{pmatrix} A_x^{\mathcal{H}_1} & * \\ 0 & A_x^{\mathcal{H}_2} \end{pmatrix}.$$

It follows that $\sigma_{\mathrm{p}}(A_x^{\mathcal{H}}) = \sigma_{\mathrm{p}}(A_x^{\mathcal{H}_1}) \cup \sigma_{\mathrm{p}}(A_x^{\mathcal{H}_2})$, and therefore

$$W_{\mathcal{H}}(A) = \bigcup_{x \in \mathcal{H}^\square} \sigma_{\mathrm{p}}(A_x^{\mathcal{H}}) = \bigcup_{x \in \mathcal{H}^\square} \left(\sigma_{\mathrm{p}}(A_x^{\mathcal{H}_1}) \cup \sigma_{\mathrm{p}}(A_x^{\mathcal{H}_2})\right) = W_{\mathcal{H}_1}(A) \cup W_{\mathcal{H}_2}(A). \qquad \square$$

1.3 Continuity properties of $\overline{W_{\mathcal{H}}}$

In this section, the continuity of some mappings concerning the block numerical range with respect to a decomposition $\mathcal{H} \in \mathcal{Z}(H)$ are studied. In particular, we are interested in the mappings

$$\lambda_A : \mathcal{H}^\square \to \mathbb{P}(\mathbb{C}), \quad x \mapsto \sigma(A_x), \tag{1.4}$$

and

$$\overline{W_{\mathcal{H}}} : L(H) \to \mathbb{P}(\mathbb{C}), \quad A \mapsto \overline{W_{\mathcal{H}}(A)}, \tag{1.5}$$

where $\mathbb{P}(M)$ denotes the power set of a set M.

To this end, first recall the definition of the *Hausdorff metric* on the set of compact subsets of a metric space (see e. g. [HP05, Section 4.1.1]): Let (X, d) be a metric space and

$$\mathfrak{K}(X) := \{K \subset X \,:\, K \text{ compact}, K \neq \emptyset\} \subset \mathbb{P}(X).$$

Then $d_{\mathrm{H}} : \mathfrak{K}(X) \times \mathfrak{K}(X) \to [0, \infty)$,

$$d_{\mathrm{H}}(K_1, K_2) := \max\{\max_{x_1 \in K_1} \operatorname{dist}(x_1, K_2), \max_{x_2 \in K_2} \operatorname{dist}(x_2, K_1)\},$$

where $\operatorname{dist}(x, K) := \min\{d(x, y) \,:\, y \in K\}$ for $K \in \mathfrak{K}(X)$ and $x \in X$, defines a metric on $\mathfrak{K}(X)$. For $\varepsilon > 0$ and $K_1, K_2 \in \mathfrak{K}(X)$ we have the equivalence

$$d_{\mathrm{H}}(K_1, K_2) < \varepsilon \iff K_1 \subset B_\varepsilon(K_2) \;\wedge\; K_2 \subset B_\varepsilon(K_1),$$

where, for $\varepsilon > 0$ and $M \subset X$,

$$B_\varepsilon(M) := \bigcup_{x \in M} B_\varepsilon(x) = \{x \in X \,:\, \operatorname{dist}(x, M) < \varepsilon\}.$$

The following proposition contains the continuity properties of the spectrum of matrices which are of importance for our purposes.

Proposition 1.10. *Let $n \in \mathbb{N}$ and*

$$\sigma(\mathcal{M}) := \bigcup_{A \in \mathcal{M}} \sigma(A), \quad \mathcal{M} \subset M_n(\mathbb{C}).$$

Then the following assertions hold:

(1) *The set $\sigma(\mathcal{K}) \subset \mathbb{C}$ is compact for each compact $\mathcal{K} \subset M_n(\mathbb{C})$ and the spectrum*

$$\sigma : \mathfrak{K}(M_n(\mathbb{C})) \to \mathfrak{K}(\mathbb{C}), \quad \mathcal{K} \mapsto \sigma(\mathcal{K}),$$

is continuous with respect to the respective Hausdorff metrics.

(2) *If $\mathcal{M} \subset M_n(\mathbb{C})$ is bounded, then $\sigma(\overline{\mathcal{M}}) = \overline{\sigma(\mathcal{M})}$.*

(3) *If $\mathcal{M} \subset M_n(\mathbb{C})$ is connected, then $\sigma(\mathcal{M})$ consists of at most n connected components. For each such connected component $C \subset \sigma(\mathcal{M})$, the number $\nu_C(A)$ of eigenvalues of $A \in \mathcal{M}$ in C (counting multiplicities) does not depend on A, that is, $\nu_C(A) = \nu_C(\mathcal{M})$, $A \in \mathcal{M}$, for suitable numbers $\nu_C(\mathcal{M})$. Moreover, for every connected component C of $\sigma(\mathcal{M})$, the mapping*

$$\sigma_C : \mathcal{M} \to \mathfrak{K}(\mathbb{C}), \quad \sigma_C(A) := \sigma(A) \cap C,$$

is continuous.

Proof. Assertion (1) is [HP05, Lemma 4.2.2]. To prove (2), let $A \in \overline{\mathcal{M}}$, $z_0 \in \sigma(A)$ and $\varepsilon > 0$. From the continuity of σ and $A \in \overline{\mathcal{M}}$ it follows that there exists a $B \in \mathcal{M}$ such that $d_{\mathrm{H}}(\sigma(A), \sigma(B)) < \varepsilon$. In particular, there is a $z \in \sigma(B)$ such that $|z - z_0| < \varepsilon$. As $\sigma(B) \subset \sigma(\mathcal{M})$, we have $B_\varepsilon(z_0) \cap \sigma(\mathcal{M}) \neq \emptyset$. This implies that $z_0 \in \overline{\sigma(\mathcal{M})}$ because $\varepsilon > 0$ was arbitrary. It follows that $\sigma(\overline{\mathcal{M}}) \subset \overline{\sigma(\mathcal{M})}$. On the other hand, we have $\sigma(\overline{\mathcal{M}}) = \overline{\sigma(\overline{\mathcal{M}})} \supset \overline{\sigma(\mathcal{M})}$.

For a proof of the first two assertions of (3) see [Wag00, Appendix B]. The continuity of the mappings σ_C follows easily from the continuity of $A \mapsto \sigma(A)$. □

Theorem 1.11. *Let $\mathcal{H} \in \mathcal{Z}(H)$. Then the mapping*

$$\Lambda : \mathfrak{K}(\mathcal{H}^\square \times L(H)) \to \mathfrak{K}(\mathbb{C}), \quad \Lambda(\mathcal{K}) := \bigcup_{(x,A)\in\mathcal{K}} \sigma(A_x),$$

is continuous. In particular, for fixed $A \in L(H)$, and a connected component C of $W_\mathcal{H}(A)$, the mappings

$$\lambda_A, \lambda_A^C : \mathcal{H}^\square \to \mathfrak{K}(\mathbb{C}), \qquad \lambda_A(x) := \sigma(A_x), \quad \lambda_A^C(x) := \sigma(A_x) \cap C,$$

are also continuous.

Proof. Let $n := |\mathcal{H}|$. The mapping

$$\chi : \mathcal{H}^\square \times L(H) \to M_n(\mathbb{C}), \quad (x, A) \mapsto A_x,$$

is continuous. Thus, also the induced mapping

$$\hat{\chi} : \mathfrak{K}(\mathcal{H}^\square \times L(H)) \to \mathfrak{K}(M_n(\mathbb{C})), \quad \mathcal{K} \mapsto \hat{\chi}(\mathcal{K}) := \{A_x \,:\, (x, A) \in \mathcal{K}\},$$

is continuous with respect to the Hausdorff metrics (see, e. g., [HP05, Lemma 4.1.9]). According to Proposition 1.10 the spectrum $\sigma : \mathfrak{K}(M_n(\mathbb{C})) \to \mathfrak{K}(\mathbb{C})$ is continuous, so, from $\Lambda = \sigma \circ \hat{\chi}$ the claim follows. □

Thus we have shown the continuity of the mapping mentioned in (1.4). It remains to show that of (1.5)

Lemma 1.12. *Let X, Y, Z be metric spaces and $f : X \times Y \to Z$ fulfill the Lipschitz condition*

$$d(f(x, y_1), f(x, y_2)) \le L\, d(y_1, y_2), \quad x \in X, \ y_1, y_2 \in Y,$$

for some $L > 0$*. Moreover, let* $\overline{f(X \times K)}$ *be compact for every* $K \in \mathfrak{K}(Y)$*. Then the mapping*

$$\tilde{f} : \mathfrak{K}(Y) \to \mathfrak{K}(Z), \quad \tilde{f}(K) := \overline{f(X \times K)},$$

is Lipschitz continuous with Lipschitz constant L*.*

Proof. Let $K_1, K_2 \in \mathfrak{K}(Y)$ and $z_1 \in f(X \times K_1)$, $z_1 = f(x_1, y_1)$ with $(x_1, y_1) \in X \times K_1$. Then

$$\begin{aligned}\operatorname{dist}(z_1, \tilde{f}(K_2)) &= \operatorname{dist}(f(x_1, y_1), f(X \times K_2)) \\ &= \inf\{d(f(x_1, y_1), f(x_2, y_2)) \,:\, (x_2, y_2) \in X \times K_2\} \\ &\le \inf\{d(f(x_1, y_1), f(x_1, y_2)) \,:\, y_2 \in K_2\} \\ &\le L \inf\{d(y_1, y_2) \,:\, y_2 \in K_2\} \\ &= L \operatorname{dist}(y_1, K_2) \le L d_{\mathrm{H}}(K_1, K_2).\end{aligned}$$

As $\operatorname{dist}(\,\cdot\,, \tilde{f}(K_2))$ is continuous on Z, this inequality also holds for all $z_1 \in \tilde{f}(K_1)$. By symmetry it follows that $\operatorname{dist}(z_2, \tilde{f}(K_1)) \le L d_{\mathrm{H}}(K_1, K_2)$ for all $z_2 \in \tilde{f}(K_2)$. Therefore,

$$d(\tilde{f}(K_1), \tilde{f}(K_2)) = \max\{\max_{z_1 \in \tilde{f}(K_1)} \operatorname{dist}(z_1, \tilde{f}(K_2)), \max_{z_2 \in \tilde{f}(K_2)} \operatorname{dist}(z_2, \tilde{f}(K_1))\} \le L d_{\mathrm{H}}(K_1, K_2).$$

□

Theorem 1.13. *Let* $\mathcal{H} \in \mathcal{Z}(H)$*. Then the mapping*

$$\overline{W}_{\mathcal{H}} : \mathfrak{K}(L(H)) \to \mathfrak{K}(\mathbb{C}), \quad \mathcal{K} \mapsto \overline{W}_{\mathcal{H}}(\mathcal{K}) := \overline{\bigcup_{A \in \mathcal{K}} W_{\mathcal{H}}(A)},$$

is continuous. In particular, the mapping

$$\overline{W}_{\mathcal{H}} : L(H) \to \mathfrak{K}(\mathbb{C}), \quad A \mapsto \overline{W_{\mathcal{H}}(A)},$$

is also continuous.

Proof. Let $n := |\mathcal{H}|$. The mapping $\chi : \mathcal{H}^{\square} \times L(H) \to M_n(\mathbb{C})$, $\chi(x, A) = A_x$, fulfills the Lipschitz condition

$$\|\chi(x, A) - \chi(x, B)\| = \|A_x - B_x\| = \|(A - B)_x\| \le \|A - B\|, \quad x \in \mathcal{H}^{\square},\ A, B \in L(H),$$

and, obviously, $\chi(\mathcal{H}^{\square} \times \mathcal{K}) \subset M_n(\mathbb{C})$ is bounded for each $\mathcal{K} \in \mathfrak{K}(L(H))$. Thus, by Lemma 1.12, the induced mapping

$$\tilde{\chi} : \mathfrak{K}(L(H)) \to \mathfrak{K}(M_n(\mathbb{C})), \quad \tilde{\chi}(\mathcal{K}) := \overline{\chi(\mathcal{H}^{\square} \times \mathcal{K})},$$

is continuous. Using Proposition 1.10 (2), we get

$$\overline{W}_{\mathcal{H}}(\mathcal{K}) = \overline{W_{\mathcal{H}}(\mathcal{K})} = \overline{\sigma(\chi(\mathcal{H}^{\square} \times \mathcal{K}))} = \sigma(\overline{\chi(\mathcal{H}^{\square} \times \mathcal{K})}) = \sigma(\tilde{\chi}(\mathcal{K})), \quad \mathcal{K} \in \mathfrak{K}(L(H)),$$

thus $\overline{W}_{\mathcal{H}} = \sigma \circ \tilde{\chi}$ is the composition of continuous mappings. □

1.4 Connected components of $W_{\mathcal{H}}(A)$

Some of the results presented in this section are already known from [Wag00]. However, we obtain more detailed information on the connected components of the block numerical range here which will be useful later. In particular, we will see that there is a correspondence $M \mapsto \kappa(M)$ between the subspaces $M \in \mathcal{H}$ and the connected components of $W_{\mathcal{H}}(A)$ such that a removal of certain subspaces from $\mathcal{H}$ will result in a removal of the corresponding connected components from $W_{\mathcal{H}}(A)$ (see Proposition 1.17).

Definition 1.14. Let $\mathcal{H} \in \mathcal{Z}'(H)$ and $A \in L(H)$. Then the set of the connected components of $W_{\mathcal{H}}(A)$ is denoted by $\mathcal{C}_{\mathcal{H}}(A)$. Similarly, the set of connected components of $\overline{W_{\mathcal{H}}(A)}$ is denoted by $\overline{\mathcal{C}}_{\mathcal{H}}(A)$.

Statements about the (number of) zeros of complex valued functions and also eigenvalues of matrices will frequently make use of the following notations.

Notation. Let $f : \Omega \to \mathbb{C}$ be an analytic function and $U \subset \Omega$. Then the set of zeros of f in U is denoted by $N_U(f)$, and their number, counted according to their multiplicities, by $\nu_U(f)$. If $U = \mathbb{C}$, then we will also write $N(f) = N_{\mathbb{C}}(f)$ and $\nu_{\mathbb{C}}(f) = \nu(f)$, respectively.

For a matrix $A \in M_n(\mathbb{C})$ we write $\nu_U(A) := \nu_U(f_A)$, where $f_A : \mathbb{C} \to \mathbb{C}$, $f_A(z) := \det(A - z)$. That is, $\nu_U(A)$ is the number of eigenvalues of A in U, counting multiplicities.

Proposition 1.15. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$. Then the following assertions hold:*

(1) *The block numerical range $W_{\mathcal{H}}(A)$ consists of at most $|\mathcal{H}|$ connected components.*

(2) *For every $C \in \mathcal{C}_{\mathcal{H}}(A)$ there is a number $\nu_C \in \mathbb{N}$ such that for every $x \in \mathcal{H}^\square$ the matrix A_x has exactly ν_C eigenvalues (counting multiplicities) in C.*

(3) *The mapping $L(H) \to \mathbb{N}$, $B \mapsto |\overline{\mathcal{C}}_{\mathcal{H}}(B)|$, is lower semi-continuous, that is, if $\overline{W_{\mathcal{H}}(A)}$ consists of k connected components, then, for $\|B - A\|$ small enough, $\overline{W_{\mathcal{H}}(B)}$ consists of at least k connected components.*

Proof. (1) and (2) are immediate consequences of Proposition 1.10. To prove (3), assume that $\overline{W_{\mathcal{H}}(A)}$ consists of k connected components and let $\delta > 0$ such that $B_\delta(C) \cap B_\delta(C') = \emptyset$, $C, C' \in \overline{\mathcal{C}}_{\mathcal{H}}(A)$, $C \neq C'$. By the continuity of the mapping $\overline{W}_{\mathcal{H}}$ (see Theorem 1.13) there is an $\varepsilon > 0$ such that for all $B \in L(H)$ for which $\|B - A\| < \varepsilon$, we have $\overline{W_{\mathcal{H}}(B)} \subset B_\delta(\overline{W_{\mathcal{H}}(A)})$ and $\overline{W_{\mathcal{H}}(A)} \subset B_\delta(\overline{W_{\mathcal{H}}(B)})$. The latter inclusion shows that $\overline{W_{\mathcal{H}}(B)}$ intersects every set $B_\delta(C)$, $C \in \overline{\mathcal{C}}_{\mathcal{H}}(A)$. As these k sets are connected and pairwise disjoint, $\overline{W_{\mathcal{H}}(B)}$ consists of at least k connected components. □

The following lemma is a more detailed version of [Wag00, Corollary 4.3].

Lemma 1.16. *Let $\mathcal{H} \in \mathcal{Z}(H)$ such that the dimension condition*

$$\dim M \geq |\mathcal{H}|, \quad M \in \mathcal{H}, \tag{1.6}$$

is fulfilled. Then, for $A \in L(H)$, the following assertions hold:

(1) *Let $\mathcal{H}' \subset \mathcal{H}$ and $\mathcal{H}'' := \mathcal{H} \setminus \mathcal{H}'$. Then for any $x' \in (\mathcal{H}')^\square$ there exists an $x'' \in (\mathcal{H}'')^\square$ such that, writing $x := x' + x''$ the equation*

$$\det A_x = \det A_{x'} \prod_{M \in \mathcal{H}''} (A_M x''_M, x''_M) \tag{1.7}$$

holds. In particular[2]*,*

$$\sigma_{\mathrm{p}}(A_x) = \sigma_{\mathrm{p}}(A_{x'}) \cup \{(A_M x''_M, x''_M) \,:\, M \in \mathcal{H}''\}.$$

(2) *For every $M \in \mathcal{H}$ there is a connected component $\kappa(M) \in \mathcal{C}_{\mathcal{H}}(A)$ of $W_{\mathcal{H}}(A)$ such that $W(A_M) \subset \kappa(M)$. (That is, the numerical range of every diagonal entry of A is contained in some connected component of the block numerical range of A.)*

(3) *Setting $\mathcal{H}_C := \kappa^{-1}(C) = \{M \in \mathcal{H} \,:\, \kappa(M) = C\}$ for $C \in \mathcal{C}_{\mathcal{H}}(A)$, there is an $x \in \mathcal{H}^\square$ such that*

$$\det A_x = \prod_{C \in \mathcal{C}_{\mathcal{H}}(A)} \prod_{M \in \mathcal{H}_C} (A_M x_M, x_M). \tag{1.8}$$

In particular, all the matrices A_ξ, $\xi \in \mathcal{H}^\square$, have exactly $|\mathcal{H}_C|$ eigenvalues in C counting multiplicities.

Proof. (1) Write $\mathcal{H}' = \{M_1, \dots, M_m\}$ and $\mathcal{H}'' = \{M_{m+1}, \dots, M_n\}$. As $\dim M_i \geq n > |\{M_1, \dots, M_{i-1}\}|$, $i = m+1, \dots, n$, we recursively find normed vectors $x_i \in M_i$, $i = m+1, \dots, n$, such that

$$(A_{ij} x_j, x_i) = 0, \quad j = 1, \dots, i-1.$$

Setting $x'' := x_{m+1} + \dots + x_n \in (\mathcal{H}'')^\square$ and $x := x' + x''$, the matrix A_x is of the form

$$A_x = \begin{pmatrix} A_{x'} & * \\ 0 & A_{x''} \end{pmatrix}.$$

Moreover, $A_{x''}$ is an upper triangular matrix and it follows that

$$\det A_x = \det A_{x'} \det A_{x''} = \det A_{x'} \prod_{i=m+1}^{n} (A_{ii} x_i, x_i).$$

Thus, (1) is shown. For the special case $\mathcal{H}' = \{M'\}$, where $M' \in \mathcal{H}$ is arbitrary, and any normed vector $x' \in M'$, it follows that there is an $x \in \mathcal{H}^\square$ such that

$$\det A_x = \prod_{M \in \mathcal{H}} (A_M x_M, x_M). \tag{1.9}$$

[2] Replace A by $A - z$, $z \in \mathbb{C}$, in Equation (1.7) and note that $(A - z)_x = A_x - z$ for $x \in \mathcal{H}^\square$.

From
$$\{(A_M x_M, x_M) \,:\, M \in \mathcal{H}\} = \sigma_{\mathrm{p}}(A_x) \subset W_{\mathcal{H}}(A)$$
it follows that for every $M \in \mathcal{H}$ there is a connected component $\kappa(M) \in \mathcal{C}_{\mathcal{H}}(A)$ such that $W(A_M) \cap \kappa(M) \neq \emptyset$. As the numerical range of an operator is a convex and, in particular, connected set, this yields $W(A_M) \subset \kappa(M)$ for $M \in \mathcal{H}$, which has been claimed in (2). Obviously, $\{\mathcal{H}_C \,:\, C \in \mathcal{C}_{\mathcal{H}}(A)\}$ is a partition of $\mathcal{H}$ and therefore equation (1.8) immediately follows from (1.9). □

We have seen that the numerical ranges of the diagonal entries of A are contained in connected components of $W_{\mathcal{H}}(A)$, provided that (1.6) holds. In this context, a natural question arises: Is it possible that there exists a connected component of $W_{\mathcal{H}}(A)$ which does not contain the numerical range of some diagonal entry of A? The following proposition gives a negative answer.

Proposition 1.17. *Let $\mathcal{H} \in \mathcal{Z}(H)$ fulfill the dimension condition* (1.6)*. Then for any decomposition $\mathcal{H}' \subset \mathcal{H}$ the inclusion*

$$W_{\mathcal{H}'}(A) \subset \bigcup_{M \in \mathcal{H}'} \kappa(M) \subset W_{\mathcal{H}}(A) \tag{1.10}$$

holds. In particular,

$$W_{\mathcal{H} \setminus \mathcal{H}_C}(A) \subset W_{\mathcal{H}}(A) \setminus C, \quad C \in \mathcal{C}_{\mathcal{H}}(\mathcal{A}), \tag{1.11}$$

where $\mathcal{H}_C$ is defined as in Lemma 1.16 (3)*.*

Proof. Let $\mathcal{H}' \subset \mathcal{H}$, $\mathcal{H}'' := \mathcal{H} \setminus \mathcal{H}'$ and $C \in \mathcal{C}_{\mathcal{H}}(A)$ such that $\kappa(M) \neq C$, $M \in \mathcal{H}'$, i. e., $\mathcal{H}_C \subset \mathcal{H}''$. We have to show that $W_{\mathcal{H}'}(A) \subset W_{\mathcal{H}}(A) \setminus C$. To this end, let $\lambda_0 \in W_{\mathcal{H}'}(A)$ and $x' \in (\mathcal{H}')^{\square}$ such that $\det(A_{x'} - \lambda_0) = 0$, that is,

$$\det(A_{x'} - z) = p(z)(z - \lambda), \quad z \in \mathbb{C}.$$

According to Lemma 1.16 (1) there is an $x \in \mathcal{H}^{\square}$ such that

$$\begin{aligned} \det(A_x - z) &= \det(A_{x'} - z) \prod_{M \in \mathcal{H}''} ((A_M x_M, x_M) - z) \\ &= p(z)(z - \lambda_0) \prod_{M \in \mathcal{H}'' \setminus \mathcal{H}_C} ((A_M x_M, x_M) - z) \prod_{M \in \mathcal{H}_C} ((A_M x_M, x_M) - z) \end{aligned} \tag{1.12}$$

for $z \in \mathbb{C}$. This implies $\lambda_0 \in W_{\mathcal{H}}(A)$. By Lemma 1.16 (3), A_x has $|\mathcal{H}_C|$ eigenvalues in C counting multiplicities. By (1.12) and the definition of $\mathcal{H}_C$, these are given by $(A_M x_M, x_M)$, $M \in \mathcal{H}_C$. Therefore, $\lambda_0 \notin C$, again by (1.12). □

Remark 1.18. The dimension condition (1.6) can be weakened in many cases:

(1) If $\mathcal{H} = \mathcal{H}' \,\dot{\cup}\, \mathcal{H}'' \in \mathcal{Z}(H)$, then the condition $\dim M \geq |\mathcal{H}|$ for all $M \in \mathcal{H}''$ already suffices for the inclusion $W_{\mathcal{H}'}(A) \subset W_{\mathcal{H}}(A)$. (See the corresponding part in the beginning of the proof of Lemma 1.16).

(2) The inclusion $W(A_M) \subset W_{\mathcal{H}}(A)$ already holds if $\mathcal{H} = \{M, M_2, \dots, M_n\}$ and $\dim M_j \geq n - j + 2$, $j = 2, \dots, n$.

Nevertheless, in most cases we are only interested in decompositions $\mathcal{H}$ where inclusions like (1.10) and (1.11) above hold for *all* sub-decompositions $\mathcal{H}' \subset \mathcal{H}$.

Remark 1.19. Without any dimension assumptions on the decomposition $\mathcal{H}$ the inclusion $W(A_M) \subset W_{\mathcal{H}}(A)$ may fail. This is easily seen by a trivial example where $H = \mathbb{C} \times \mathbb{C}$ and $A \in M_2(\mathbb{C})$ is a matrix whose diagonal entries are not eigenvalues of A. To have a less trivial example, consider

$$A = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 2 & 0 \end{array}\right), \quad \mathcal{H} = \mathbb{C}^2 \times \mathbb{C}. \tag{1.13}$$

It is easy to see that

$$\sigma_{\mathrm{p}}(A_x) = \Big\{ \frac{t_x \pm \sqrt{t_x^2 + 8(1 - t_x)}}{2} \Big\},$$

where $t_x = |x_{11}|^2 \in [0, 1]$ for $x = ((x_{11}, x_{12}), x_2) \in \mathcal{H}^{\square}$, and

$$W_{\mathcal{H}}(A) = [-\sqrt{2}, 0] \cup [1, \sqrt{2}].$$

However, $W(A_{11}) = [0, 1]$ in this case. Additionally note, that $\sigma(A_{11}) \cap \sigma(A_{22}) = \{0\} \neq \emptyset$, and $W(A_{11})$ at least intersects $W_{\mathcal{H}}(A)$ (even both of its connected components). The latter observation assumes a more general form in the following proposition.

Proposition 1.20. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $M \in \mathcal{H}$ such that* $\dim M < \dim N$ *for every $N \in \mathcal{H} \setminus \{M\}$. Then, for all $A \in L(H)$,*

$$W(A_M) \subset W_{\mathcal{H}}(A), \qquad W_{\mathcal{H} \setminus \{M\}}(A) \cap W_{\mathcal{H}}(A) \neq \emptyset.$$

Proof. Write $\mathcal{H} = \{M_1, \dots, M_n\}$ such that $M = M_1$. As $\dim M_1 < \dim M_i$, $i > 1$, there exist normed vectors $x_i \in M_i$ such that $A_{1i} x_i = 0$, $i = 2, \dots, n$. For an arbitrary $x_1 \in M_1$, $\|x_1\| = 1$, and $x' := x_2 + \dots + x_n \in (\mathcal{H} \setminus \{M_1\})^{\square}$, $x := x_1 + x' \in \mathcal{H}^{\square}$, we get

$$A_x = \begin{pmatrix} (A_{11} x_1, x_1) & 0 \\ * & A_{x'} \end{pmatrix},$$

thus, $\{(A_{11} x_1, x_1)\} \cup \sigma_{\mathrm{p}}(A_{x'}) = \sigma_{\mathrm{p}}(A_x) \subset W_{\mathcal{H}}(A)$. Hence, $W_{\mathcal{H} \setminus \{M_1\}}(A) \cap W_{\mathcal{H}}(A) \neq \emptyset$, and, as $x_1 \in M_1$ was arbitrary, $W(A_{11}) \subset W_{\mathcal{H}}(A)$. □

Remark 1.21. It is still an open question if, provided that $\overline{W_{\mathcal{H}}(A)}$ consists of $|\mathcal{H}|$ connected components, these connected components are simply connected[3]. The

[3]It is clear that, without the additional assumption that $|\overline{\mathcal{C}}_{\mathcal{H}}(A)| = |\mathcal{H}|$, the connected components of $\overline{W_{\mathcal{H}}(A)}$ are not simply connected in general. If, e. g., $A := \operatorname{diag}(1, \mathrm{i}, \mathrm{i}, -1, -1, 1) \in M_6(\mathbb{C})$, then, by Proposition 1.9, $W_{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}(A)$ is the boundary of the triangle with the corners $1, \mathrm{i}, -1$.

concept of the proof of the diagonalizability of operators with respect to a decomposition $\mathcal{H}$ presented in Chapter 4 is to encircle each connected component of the block numerical range by a Jordan curve whose interior does not intersect any other connected component. If now a connected component of the block numerical range has a hole which contains another connected component, this separation of its connected components is not possible anymore. In the case $|\mathcal{H}| = 2$ this problem has been solved in [LMMT01] by extending the calculations to the sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the separation of the connected components by a single Jordan curve is still possible. However, this method is not applicable anymore if $|\mathcal{H}| \geq 3$. The following definition, frequently used in Chapter 4, helps to avoid these problems right from the beginning.

Notation. For $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$, let

$$\operatorname{diag}_{\mathcal{H}}(A) := \sum_{M \in \mathcal{H}} A_M P_M.$$

The operator A is said to be $\mathcal{H}$-diagonal if $A = \operatorname{diag}_{\mathcal{H}}(A)$.

Definition 1.22. Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $C \in \mathcal{C}_{\mathcal{H}}(A)$ be a connected component of $W_{\mathcal{H}}(A)$.

(1) C is called *$\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$* if the following two conditions hold:

  (a) There is a Jordan curve $\Gamma \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$ such that $\operatorname{int} \Gamma \cap W_{\mathcal{H}}(A) = C$.

  (b) For some (and thus for any) $x \in \mathcal{H}^{\square}$, the matrix A_x has exactly one eigenvalue in C.

  A Jordan curve Γ as above will be called *$\mathcal{H}$-separating C in $W_{\mathcal{H}}(A)$*.

(2) Let the operator function $\widetilde{A}$ be given by

$$\widetilde{A} : [0,1] \to L(H), \quad \widetilde{A}(t) := \operatorname{diag}_{\mathcal{H}} A + t(A - \operatorname{diag}_{\mathcal{H}} A).$$

  Then C is called *strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$* if there exist a Jordan curve $\Gamma \subset \mathbb{C}$ and, for every $t \in [0,1]$, an $\mathcal{H}$-separated connected component $\widetilde{C}(t)$ of $W_{\mathcal{H}}(\widetilde{A}(t))$ such that $\widetilde{C}(1) = C$ and Γ is $\mathcal{H}$-separating $\widetilde{C}(t)$ in $W_{\mathcal{H}}(\widetilde{A}(t))$ for all $t \in [0,1]$. In this case, we say that Γ *strongly $\mathcal{H}$-separates C in $W_{\mathcal{H}}(A)$*.

The block numerical range $W_{\mathcal{H}}(A)$ itself is called (*strongly*) *$\mathcal{H}$-separated* if all its connected components are (strongly) $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$.

Remark 1.23. (1) It is easy to see that (strongly) $\mathcal{H}$-separating Jordan curves always can be chosen to be piecewise smooth.

(2) If $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, then $\overline{W_{\mathcal{H}}(A)}$ consists of the maximal number $|\mathcal{H}|$ of connected components.

(3) If the dimension condition (1.6) holds, condition (b) in Definition 1.22 (1) may be replaced by

  (b') There exists exactly one $M \in \mathcal{H}$ such that $\kappa(M) = C$.

Here $\kappa(M)$ denotes the connected component of $W_{\mathcal{H}}(A)$ containing $W(A_M)$ (see Lemma 1.16 (2)).

Example 1.24. If A and $\mathcal{H}$ are as in (1.13), then $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated but not strongly $\mathcal{H}$-separated. The latter already fails because $W(A_{11}) \cap W(A_{22}) \neq \emptyset$. This, however, can not happen if the dimension condition (1.6) is fulfilled, and leads to the following question:

Open Question. Is it true that, provided that the dimension condition (1.6) holds, an $\mathcal{H}$-separated connected component of $W_{\mathcal{H}}(A)$ is strongly $\mathcal{H}$-separated?

Note that a positive answer to this question would greatly simplify the considerations in Chapter 4. We mention immediate consequences of Theorem 1.11 and Proposition 1.17 for $\mathcal{H}$-separated connected components.

Corollary 1.25. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $C \in \mathcal{C}_{\mathcal{H}}(A)$ be an $\mathcal{H}$-separated connected component of $W_{\mathcal{H}}(A)$. Then the following assertions hold:*

(1) *There is a continuous mapping $\lambda_C : \mathcal{H}^{\square} \to C$ such that, for $x \in \mathcal{H}^{\square}$, $\lambda_C(x)$ is the eigenvalue of A_x in C.*

(2) *If (1.6) holds, then there is exactly one $M \in \mathcal{H}$ such that $\kappa(M) = C$ and the inclusion*

$$W_{\mathcal{H}\setminus\{M\}}(A) \subset W_{\mathcal{H}}(A) \setminus \kappa(M). \tag{1.14}$$

holds.

Proposition 1.26. *Let $\dim H < \infty$, $\mathcal{H} \in \mathcal{Z}(H)$ such that (1.6) holds, $A \in L(H)$ and $\kappa(M)$ be strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$. Then A has $\dim M$ eigenvalues in $\kappa(M)$ counting multiplicities.*

Proof. Let Γ, $\widetilde{A}$ and $\widetilde{C}$ as in Definition 1.22 (2). From $W_{\mathcal{H}}(\widetilde{A}(0)) = \bigcup_{N\in\mathcal{H}} W(A_N)$ (by Proposition 1.9) and $W(A_M) \subset \kappa(M) \subset \operatorname{int}\Gamma$ it follows that $\widetilde{C}(0) = W(A_M)$. Now, $\sigma_{\mathrm{p}}(\widetilde{A}(0)) = \bigcup_{N\in\mathcal{H}} \sigma_{\mathrm{p}}(A_N)$ implies that $\widetilde{A}(0)$ has $\dim M$ eigenvalues in $\operatorname{int}\Gamma$ counting multiplicities. Moreover, as $[0,1]$ is connected and $\widetilde{A}$ is continuous, the image $\mathcal{M} := \{\widetilde{A}(t) : t \in [0,1]\} \subset L(H)$ of $\widetilde{A}$ is connected. Because $\operatorname{int}\Gamma$ is connected and $\sigma(\mathcal{M}) \cap \Gamma = \emptyset$, it follows from Proposition 1.10 (3) that the number of eigenvalues of $\widetilde{A}(t)$ in $\operatorname{int}\Gamma$ counting multiplicities does not depend on $t \in [0,1]$. In particular, also $A = \widetilde{A}(1)$ has $\dim M$ eigenvalues in $\operatorname{int}\Gamma$ counting multiplicities, thus in $\kappa(M)$. □

Gershgorin's Theorem enables us to give a very simple sufficient condition on A for the existence of strongly $\mathcal{H}$-separated connected components of $W_{\mathcal{H}}(A)$ involving the numerical ranges of the diagonal entries of its block operator representation with respect to $\mathcal{H}$ and the norms of the off-diagonal entries.

Proposition 1.27. *Let $\mathcal{H} \in \mathcal{Z}(H)$ such that (1.6) holds, $A \in L(H)$ and $M \in \mathcal{H}$.*

If there exist numbers $p_N > 0$, $N \in \mathcal{H}$, such that

$$p_M^{-1} \sum_{K\in\mathcal{H}\setminus\{M\}} p_K \|A_{MK}\| + p_N^{-1} \sum_{K\in\mathcal{H}\setminus\{N\}} p_K \|A_{NK}\| < \operatorname{dist}(W(A_M), W(A_N)), \quad N \in \mathcal{H} \setminus \{M\}, \tag{1.15}$$

then $\kappa(M)$ is strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$. In particular, if $\dim H < \infty$, then the matrix A has exactly $\dim M$ eigenvalues in $\kappa(M)$, counting multiplicities.

Proof. Let, for $N \in \mathcal{H}$,

$$R_N := p_N^{-1} \sum_{K\in\mathcal{H}\setminus\{N\}} p_K \|A_{NK}\|, \qquad r_N(x) := p_N^{-1} \sum_{K\in\mathcal{H}\setminus\{N\}} p_K |(A_{NK} x_K, x_N)|, \quad x \in \mathcal{H}^\square.$$

Then, by Gershgorin's Theorem (see, e. g., [HP05, Theorem 4.2.19]),

$$\sigma_{\mathrm{p}}(A_x) \subset \bigcup_{N\in\mathcal{H}} \overline{B_{r_N(x)}((A_N x_N, x_N))} \subset \bigcup_{N\in\mathcal{H}} \overline{B_{R_N}(W(A_N))} =: \bigcup_{N\in\mathcal{H}} G_N, \quad x \in \mathcal{H}^\square.$$

Thus,

$$W_{\mathcal{H}}(A) = \bigcup_{x\in\mathcal{H}^\square} \sigma_{\mathrm{p}}(A_x) \subset \bigcup_{N\in\mathcal{H}} G_N =: G. \tag{1.16}$$

From $W(A_N) \subset G_N$, $W(A_N) \subset \kappa(N) \subset W_{\mathcal{H}}(A) \subset G$ and the connectedness of $\kappa(N)$ and G_N it follows that $\kappa(N) \subset G_N$ for all $N \in \mathcal{H}$. Now, by (1.15), we have the inequality $R_M + R_N < \operatorname{dist}(W(A_M), W(A_N))$ and therefore $G_M \cap \kappa(N) \subset G_M \cap G_N = \emptyset$ for all $N \in \mathcal{H} \setminus \{M\}$. It follows that M is the only element of $\mathcal{H}$ such that $\kappa(M) \subset G_M$ and, in particular, A_x has exactly one eigenvalue in $\kappa(M)$, $x \in \mathcal{H}^\square$, by Lemma 1.16 (3). Moreover, as G_M and $G \setminus G_M$ are closed and G_M is simply connected (even convex), it is clear that there exists a Jordan curve $\Gamma \subset \mathbb{C} \setminus G$ such that $\operatorname{int}\Gamma \cap G = G_M$. Clearly, Γ is $\mathcal{H}$-separating $\kappa(M)$ in $W_{\mathcal{H}}(A)$. Let now $\widetilde{A}$ be as in Definition 1.22. From

$$\|\widetilde{A}(t)_{NK}\| = t\|A_{NK}\| \le \|A_{NK}\|, \quad t \in [0,1],$$

for $N, K \in \mathcal{H}$, $N \ne K$, it follows, that (1.16) continues to hold if A is replaced by $\widetilde{A}(t)$ for some $t \in [0,1]$. Hence, as we have seen above, Γ also $\mathcal{H}$-separates the connected component of $W_{\mathcal{H}}(\widetilde{A}(t))$ containing $W(A_M)$. □

It has been shown in [LMMT01] that, if $|\mathcal{H}| = 2$, the assumptions on A may be weakened: If $\mathcal{H} = \{M, N\}$ such that

$$\sqrt{\|A_{MN}\|\|A_{NM}\|} < \frac{1}{2} \operatorname{dist}(W(A_M), W(A_N)),$$

then $\overline{W_{\mathcal{H}}(A)}$ consists of two connected components. In this case, (1.15) reads

$$\frac{p_N}{p_M}\|A_{MN}\| + \frac{p_M}{p_N}\|A_{NM}\| < \operatorname{dist}(W(A_M), W(A_N)),$$

which is a stronger assumption by the inequality of arithmetic and geometric means.

Corollary 1.28. *Let $\mathcal{H} \in \mathcal{Z}(H)$ such that* (1.6) *holds and $A \in L(H)$. If for every $M \in \mathcal{H}$ there exist numbers p_N, $N \in \mathcal{H}$, such that* (1.15) *holds, then $W_{\mathcal{H}}(A)$ is strongly $\mathcal{H}$-separated. This is, for example, the case if*

$$\sum_{\substack{M,N\in\mathcal{H}\\ M\neq N}} \|A_{MN}\| < \min\{\operatorname{dist}(W(A_M), W(A_N)) \,:\, M,N \in \mathcal{H},\ M \neq N\}.$$

1.5 The block determinant set

The following definition of the block determinant set of an operator $A \in L(H)$ with respect to a decomposition $\mathcal{H} \in \mathcal{Z}(H)$ is motivated by the equivalence

$$z \in W_{\mathcal{H}}(A) \iff \exists\, x \in \mathcal{H}^{\square} \quad \det(A-z)_x = 0, \qquad z \in \mathbb{C}.$$

It will serve as the main tool in determining points belonging to the closure of the block numerical range (see Theorem 1.34), frequently used in the remaining part of the thesis.

Definition 1.29. Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$. Then the *block determinant set of A with respect to $\mathcal{H}$* is defined by

$$D_{\mathcal{H}}(A) := \{\det A_x \,:\, x \in \mathcal{H}^{\square}\}.$$

Remark 1.30. (1) $W(A) = D_{\{H\}}(A)$.

(2) For $z \in \mathbb{C}$ the equivalence

$$z \in W_{\mathcal{H}}(A) \iff 0 \in D_{\mathcal{H}}(A-z)$$

holds.

Lemma 1.31. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $K \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$ be compact. Then*

$$\inf\{|\det(A-z)_x| \,:\, z \in K,\ x \in \mathcal{H}^{\square}\} > 0.$$

Proof. Consider the mapping $\chi : \mathcal{H}^{\square} \to M_n(\mathbb{C})$, $\chi(x) := A_x$. As $\|A_x\| \leq \|A\|$ for all $x \in \mathcal{H}^{\square}$, its image $\mathcal{M} := \operatorname{im}(\chi) \subset M_n(\mathbb{C})$ is bounded, thus $\overline{\mathcal{M}} \subset M_n(\mathbb{C})$ is compact and $\sigma(\overline{\mathcal{M}}) = \overline{\sigma(\mathcal{M})}$ by Proposition 1.10 (2). Define

$$\Delta : \overline{\mathcal{M}} \times K \to \mathbb{R}_+, \quad \Delta(B,z) := \det(B-z).$$

Then Δ does not have zeros: $0 = \Delta(B_0, z_0)$ would imply $z_0 \in \sigma(B_0) \subset \sigma(\overline{\mathcal{M}}) = \overline{\sigma(\mathcal{M})} = \overline{W_{\mathcal{H}}(A)}$, a contradiction to $z_0 \in K \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$. As Δ is continuous and $\overline{\mathcal{M}} \times K$ compact, this leads to

$$0 < \min_{B\in\overline{\mathcal{M}}, z\in K} |\Delta(B,z)| = \inf_{B\in\mathcal{M}, z\in K} |\det(B-z)| = \inf_{x\in\mathcal{H}^{\square}, z\in K} |\det(A_x - z)|,$$

which has been claimed. □

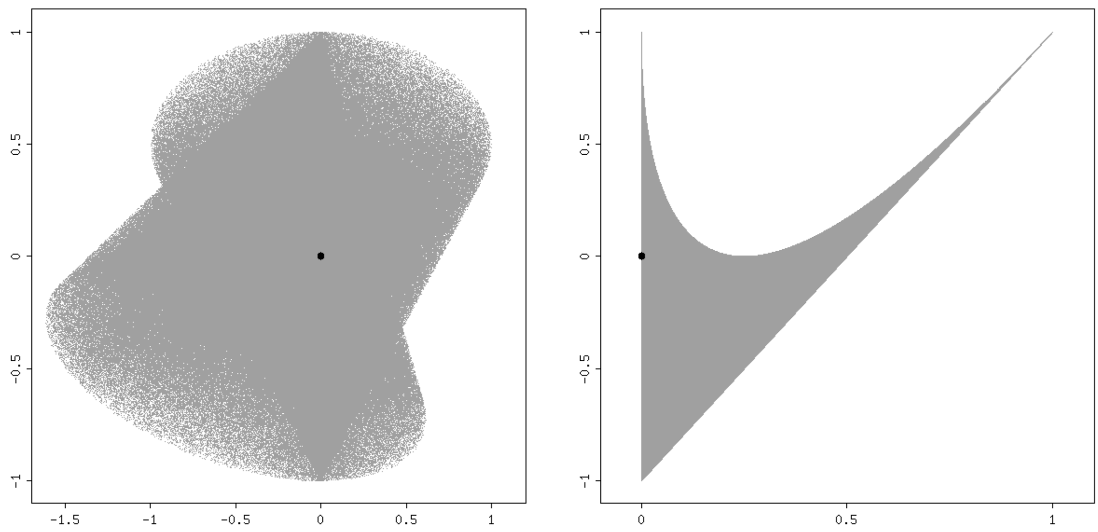


Figure 1.2: To have an impression of the shape of block determinant sets, consider the matrix A from (1.3) again. The pictures show $D_{\mathbb{C}^3\times\mathbb{C}^2}(A)$ and $D_{\mathbb{C}\times\mathbb{C}^2\times\mathbb{C}^2}(A)$. (The circles mark the point $0 \in \mathbb{C}$.)

Corollary 1.32. *If* $0 \in \overline{D_{\mathcal{H}}(A-z)}$*, then* $z \in \overline{W_{\mathcal{H}}(A)}$.

Proof. Let $z \notin \overline{W_{\mathcal{H}}(A)}$. Applying Lemma 1.31 to $K := \{z\} \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$ yields $\inf\{|\det(A-z)_x| \,:\, x \in \mathcal{H}^{\square}\} > 0$; thus, $0 \notin \overline{D_{\mathcal{H}}(A-z)}$. □

To prove the converse of the previous corollary, the following simple lemma is useful.

Lemma 1.33. (1) *Let* X *and* Y *be metric spaces and* $f : X \to Y$ *uniformly continuous. If* $(a_k)_1^\infty$, $(b_k)_1^\infty \subset X$ *are sequences such that* $(f(a_k))_1^\infty$ *converges and* $d(a_k, b_k) \to 0$, $k \to \infty$, *then* $\lim_{k\to\infty} f(b_k) = \lim_{k\to\infty} f(a_k)$.

(2) *If* $(A_k)_1^\infty, (B_k)_1^\infty \subset M_n(\mathbb{C})$ *are bounded sequences such that* $(\det A_k)_1^\infty$ *converges and* $A_k - B_k \to 0$, $k \to \infty$, *then* $\lim_{k\to\infty} \det B_k = \lim_{k\to\infty} \det A_k$.

Proof. (1) As f is uniformly continuous, we have $d(f(a_k), f(b_k)) \to 0$, $k \to \infty$. Defining $y := \lim_{k\to\infty} f(a_k)$, the claim follows from

$$d(y, f(b_k)) \le d(y, f(a_k)) + d(f(a_k), f(b_k)), \quad k \in \mathbb{N}.$$

(2) is an immediate consequence of (1) because the sequences are bounded and det is uniformly continuous on compact subsets of $M_n(\mathbb{C})$. □

Maybe more interesting than the proof of Lemma 1.33 (2) is an example why the boundedness assumption is essential: For $A_k = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$, $B_k = \begin{pmatrix} k & 0 \\ 0 & k^{-2} \end{pmatrix}$, we have $A_k - B_k = \begin{pmatrix} 0 & 0 \\ 0 & k^{-1}-k^{-2} \end{pmatrix} \to 0$, $k \to \infty$, but $\det A_k = 1$, $k \in \mathbb{N}$, and $\det B_k = 1/k \to 0$, $k \to \infty$.

Theorem 1.34. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$. Then the equivalence*

$$z \in \overline{W_{\mathcal{H}}(A)} \iff 0 \in \overline{D_{\mathcal{H}}(A-z)}, \qquad z \in \mathbb{C},$$

holds. In particular,

$$0 \in \overline{W_{\mathcal{H}}(A)} \iff 0 \in \overline{D_{\mathcal{H}}(A)}.$$

Proof. If $\lambda_0 \in \overline{W_{\mathcal{H}}(A)}$, then there exist sequences $(x_k)_1^\infty \subset \mathcal{H}^\square$ and $(\lambda_k)_1^\infty \subset \mathbb{C}$ such that $\det(A_{x_k} - \lambda_k) = 0$, $k \in \mathbb{N}$, and $\lambda_k \to \lambda_0$, $k \to \infty$. Noting that

$$(A_{x_k} - \lambda_k) - (A_{x_k} - \lambda_0) = \lambda_0 - \lambda_k \to 0, \quad k \to \infty,$$

we have $\det(A - \lambda_0)_{x_k} = \det(A_{x_k} - \lambda_0) \to 0$, $k \to \infty$, by Lemma 1.33 (2). Therefore, $0 \in \overline{D_{\mathcal{H}}(A - \lambda_0)}$. The other implication has been shown in Corollary 1.32. □

Corollary 1.35. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $n := |\mathcal{H}|$ and $A \in L(H)$. If $0 \notin \overline{D_{\mathcal{H}}(A-z)}$ for some $z \in \mathbb{C}$, then $A - z$ is invertible and*

$$\|(A-z)^{-1}\| \leq \frac{\|A-z\|^{n-1}}{\inf |D_{\mathcal{H}}(A-z)|}. \tag{1.17}$$

Proof. By assumption we have $0 \notin \overline{D_{\mathcal{H}}(A-z)}$, thus $z \notin \overline{W_{\mathcal{H}}(A)}$ by Theorem 1.34, and therefore $z \notin \sigma(A)$ by Theorem 1.4. Moreover, it follows from [Kat95, Chapter I, (4.12)], that

$$\|(A-z)_x^{-1}\| \leq \frac{\|(A-z)_x\|^{n-1}}{|\det(A-z)_x|} \leq \frac{\|A-z\|^{n-1}}{\inf |D_{\mathcal{H}}(A-z)|}, \quad x \in \mathcal{H}^\square,$$

thus, by [TW03, Lemma 4.1],

$$\|(A-z)^{-1}\| \leq \frac{\|A-z\|^{n-1}}{\inf |D_{\mathcal{H}}(A-z)|}.$$ □

Note that the estimates (1.2) and (1.17) are equivalent for the trivial decomposition $\mathcal{H} = \{H\}$. In fact,

$$\operatorname{dist}(z, W(A)) = \operatorname{dist}(0, W(A-z)) = \inf |W(A-z)| = \inf |D_{\{H\}}(A-z)|, \quad z \in \mathbb{C}.$$

Open Question. Is it true that, for $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$,

$$z \in \partial W_{\mathcal{H}}(A) \iff 0 \in \partial D_{\mathcal{H}}(A-z)?$$

We will see later (see Remark 2.15), that this equivalence does not hold for the block numerical range of operator functions. However, no counter-example is known by now for the operator case. (The equivalence is, by the way, trivial for the numerical range case $|\mathcal{H}| = 1$.)

Chapter 2

The block numerical range of operator functions

In what follows, $\Omega \subset \mathbb{C}$ will always denote a domain (i. e., Ω is open and connected) and H a Hilbert space. First of all, we recall some well-known definitions concerning operator functions (see, e. g. [Mar88, § 11.2 and § 26.3]).

For an operator function $F : \Omega \to L(H)$ the *resolvent set*, the *spectrum*, the *point spectrum*, the *approximate point spectrum* and the *numerical range* are defined by

$$\begin{aligned} \rho(F) &:= \{z \in \Omega \,:\, 0 \in \rho(F(z))\}, \\ \sigma(F) &:= \{z \in \Omega \,:\, 0 \in \sigma(F(z))\}, \\ \sigma_{\mathrm{p}}(F) &:= \{z \in \Omega \,:\, 0 \in \sigma_{\mathrm{p}}(F(z))\}, \\ \sigma_{\mathrm{app}}(F) &:= \{z \in \Omega \,:\, 0 \in \sigma_{\mathrm{app}}(F(z))\}, \\ W(F) &:= \{z \in \Omega \,:\, 0 \in W(F(z))\}. \end{aligned} \tag{2.1}$$

On $\rho(F)$, the inverse operator function F^{-1} is defined by $F^{-1}(z) := F(z)^{-1}$. If $\lambda_0 \in \sigma_{\mathrm{p}}(F)$, a vector $x_0 \in H \setminus \{0\}$ such that $F(\lambda_0)x_0 = 0$ is called an *eigenvector of F in λ_0*. In this situation, vectors $x_0, \dots, x_{m-1} \in H$ are called a *Jordan chain of F in λ_0* of length m if

$$\sum_{j=0}^{k} \frac{1}{j!} F^{(j)}(\lambda_0) x_{k-j} = 0, \quad k = 0, \dots, m-1.$$

Moreover, recall that for an operator $A \in L(H)$ the *approximate point spectrum* $\sigma_{\mathrm{app}}(A)$ is the set of all $z \in \mathbb{C}$ for which there exists a sequence $(x_k)_1^\infty \subset H$ of normed vectors such that $(A - z)x_k \to 0$, $k \to \infty$.

Remark 2.1. If, for an operator $A \in L(H)$, we define the operator polynomial P_A by $P_A : \mathbb{C} \to L(H)$, $P_A(z) := A - z$, it is obvious that

$$\sigma(A) = \sigma(P_A), \quad \sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{p}}(P_A), \quad \sigma_{\mathrm{app}}(A) = \sigma_{\mathrm{app}}(P_A), \quad W(A) = W(P_A).$$

Under an additional condition, just as in the case of the numerical range of an operator, spectral inclusion holds (cf. [Mar88, Theorem 26.6]). This theorem will be proved later for the more general block numerical case (see Theorem 2.14).

Theorem 2.2. *For an analytic operator function $F : \Omega \to L(H)$ the spectral inclusion*

$$\sigma(F) \subset \overline{W(F)}$$

holds, provided there exists a $z_0 \in \Omega$ such that $0 \notin \overline{W(F(z_0))}$.

2.1 The definition and simple properties

Definition 2.3. Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be an operator function. The *block numerical range of F with respect to the decomposition $\mathcal{H}$* is the set

$$W_{\mathcal{H}}(F) := \{z \in \Omega \,:\, 0 \in W_{\mathcal{H}}(F(z))\},$$

that is,

$$z \in W_{\mathcal{H}}(F) \iff 0 \in W_{\mathcal{H}}(F(z)), \qquad z \in \Omega, \tag{2.2}$$

or,

$$z \in W_{\mathcal{H}}(F) \iff \exists\, x \in \mathcal{H}^{\square} \quad \det F_x(z) = 0, \qquad z \in \Omega,$$

where the matrix functions F_x are defined by

$$F_x : \Omega \to M_{|\mathcal{H}|}(\mathbb{C}), \quad F_x(z) := F(z)_x, \qquad x \in \mathcal{H}.$$

Remark 2.4. (1) Let $A \in L(H)$, $\mathcal{H} = \{M_1, \dots, M_n\}$ and assume that $\det A_x = 0$ for some $x \in \mathcal{H}^*$ (i. e. $x_i \neq 0$, $i \in \{1, \dots, n\}$). Define $\hat{x} \in \mathcal{H}^{\square}$ by $\hat{x}_i := \|x_i\|^{-1} x_i$, $i \in \{1, \dots, n\}$. Then

$$0 = \det A_x = \|x_1\|^2 \cdots \|x_n\|^2 \det A_{\hat{x}},$$

hence $0 \in W_{\mathcal{H}}(A)$ by definition. This implies the equivalence

$$0 \in W_{\mathcal{H}}(A) \iff \exists\, x \in \mathcal{H}^* \;\; \det A_x = 0.$$

(2) From (1) and the definition of $W_{\mathcal{H}}(F)$ it follows that

$$W_{\mathcal{H}}(F) = \{z \in \Omega \,:\, \exists\, x \in \mathcal{H}^* \;\; \det F_x(z) = 0\}.$$

Using definition (2.1) of the point spectrum for matrix valued functions, this yields the representation

$$W_{\mathcal{H}}(F) = \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}(F_x) = \bigcup_{x \in \mathcal{H}^*} \sigma_{\mathrm{p}}(F_x). \tag{2.3}$$

(3) For an operator $A \in L(H)$ let $P_A(z) = A - z$, $z \in \mathbb{C}$. Then

$$W_{\mathcal{H}}(A) = W_{\mathcal{H}}(P_A).$$

Indeed, we have $P_{A_x} = A_x - \lambda = (A - \lambda)_x = (P_A)_x$ for $x \in \mathcal{H}^{\square}$, and thus

$$W_{\mathcal{H}}(A) = \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}(A_x) = \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}(P_{A_x}) = \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}((P_A)_x) = W_{\mathcal{H}}(P_A).$$

Example 2.5. As a first, rather trivial example, consider an operator function F of the form $F(z) = A - f(z)$, $z \in \Omega$, where $f : \Omega \to \mathbb{C}$. Then, for $z \in \Omega$, we have

$$\begin{aligned} z \in W_{\mathcal{H}}(F) &\iff 0 \in W_{\mathcal{H}}(F(z)) = W_{\mathcal{H}}(A - f(z)) = W_{\mathcal{H}}(A) - f(z) \\ &\iff f(z) \in W_{\mathcal{H}}(A), \end{aligned}$$

that is, $W_{\mathcal{H}}(F) = f^{-1}(W_{\mathcal{H}}(A))$. Thus, with $\mathcal{H} = \mathbb{C} \times \mathbb{C}^2 \times \mathbb{C}$,

$$A = \left(\begin{array}{c|cc|c} 1 & 3+\mathrm{i} & 2 & \mathrm{i} \\ \hline 3+\mathrm{i} & 1 & \mathrm{i} & 2 \\ -2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \hline \mathrm{i} & -2 & 3+\mathrm{i} & 1 \end{array}\right)$$

and $f(z) = z^2$, we get the square root of the celebrated "kissing fish" from [Wag00] in Figure 2.1.

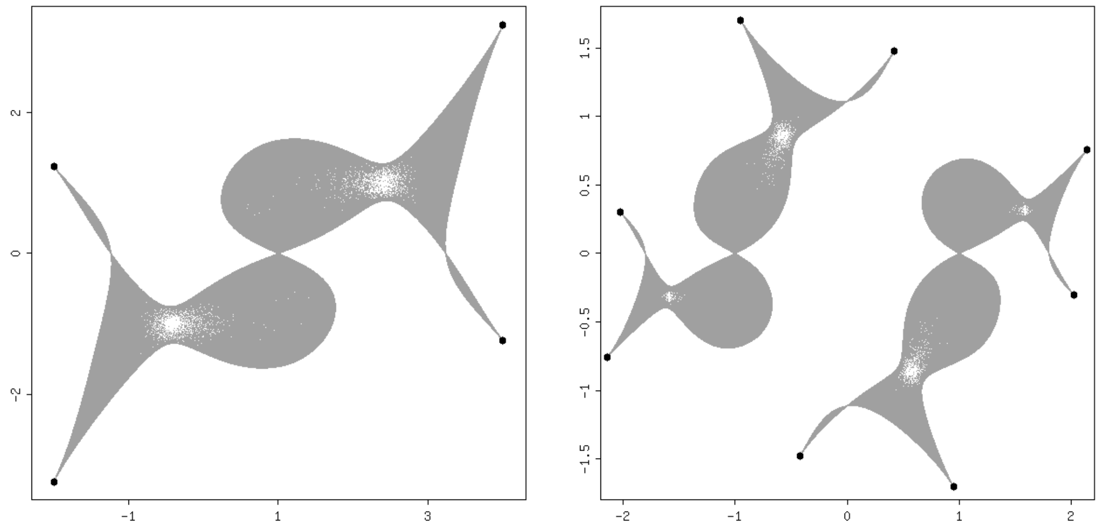


Figure 2.1: The "kissing fish" and their square root.

The assertions of the following proposition are immediate consequences of the definition of the block numerical range of operator functions and theorems known from the operator case.

Proposition 2.6. *Let $F : \Omega \to L(H)$ be an operator function and $\mathcal{H} \in L(H)$. Then the following assertions hold:*

(1) *If $\dim H < \infty$ and $|\mathcal{H}| = \dim H$, then $W_{\mathcal{H}}(F) = \sigma_{\mathrm{p}}(F)$.*

(2) *If $\widetilde{\mathcal{H}} \in \mathcal{Z}(H)$ and $\widetilde{\mathcal{H}} \le \mathcal{H}$ (i. e., $\widetilde{\mathcal{H}}$ is a refinement of $\mathcal{H}$; see Definition 1.6), then $W_{\widetilde{\mathcal{H}}}(F) \subset W_{\mathcal{H}}(F)$.*

(3) *If $\dim M \ge |\mathcal{H}|$ for all $M \in \mathcal{H}$, then $W_{\mathcal{H}'}(F) \subset W_{\mathcal{H}}(F)$ for all $\mathcal{H}' \subset \mathcal{H}$.*

Definition 2.7. Let $F : \Omega \to L(H)$ be an operator function. Then the *adjoint operator function F^* of F* is defined by

$$F^* : \Omega^* \to L(H), \quad F^*(z) := F(\bar{z})^*,$$

where $\Omega^* = \{z \in \mathbb{C} : \bar{z} \in \Omega\}$. The operator function F is called *selfadjoint* if $F^* = F$. (Note that this implies that Ω is symmetric with respect to the real axis.)

Proposition 2.8. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be an operator function. Then*

$$W_{\mathcal{H}}(F^*) = W_{\mathcal{H}}(F)^*.$$

In particular, $W_{\mathcal{H}}(F)$ is symmetric with respect to the real axis for selfadjoint operator functions F.

Proof. For any $z \in \Omega$, by definition and Proposition 1.3 we have

$$\begin{aligned} z \in W_{\mathcal{H}}(F^*) &\iff 0 \in W_{\mathcal{H}}(F^*(z)) \iff 0 \in W_{\mathcal{H}}(F(\bar{z})^*) \iff 0 \in W_{\mathcal{H}}(F(\bar{z}))^* \\ &\iff 0 \in W_{\mathcal{H}}(F(\bar{z})) \iff \bar{z} \in W_{\mathcal{H}}(F) \iff z \in W_{\mathcal{H}}(F)^*. \end{aligned}$$ □

In Theorem 1.34, we have seen that for an operator $A \in L(H)$ the equivalence

$$z \in \overline{W_{\mathcal{H}}(A)} \iff 0 \in \overline{D_{\mathcal{H}}(A - z)}$$

holds. Only the implication "⇒" continues to hold under very moderate assumptions on F in the general case. The proof is essentially the same as the one of Theorem 1.34.

Proposition 2.9. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be continuous in $z_0 \in \Omega$. Then the implication*

$$z_0 \in \overline{W_{\mathcal{H}}(F)} \implies 0 \in \overline{D_{\mathcal{H}}(F(z_0))}$$

holds. In particular, if $0 \notin \overline{W_{\mathcal{H}}(F(z_0))}$, then $z_0 \notin \overline{W_{\mathcal{H}}(F)}$.

Proof. Assume that $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$. Then there exist sequences $(\lambda_k)_1^\infty \subset \mathbb{C}$ and $(x_k)_1^\infty \subset \mathcal{H}^\square$ such that $\det F_{x_k}(\lambda_k) = 0$, $k \in \mathbb{N}$, and $\lambda_k \to \lambda_0$, $k \to \infty$. Then

$$\|F_{x_k}(\lambda_k) - F_{x_k}(\lambda_0)\| = \|(F(\lambda_k) - F(\lambda_0))_{x_k}\| \le \|F(\lambda_k) - F(\lambda_0)\| \to 0, \quad k \to \infty,$$

as F is continuous in λ_0. From Lemma 1.33 (2) it follows immediately that also $\det F_{x_k}(\lambda_0) \to 0$, $k \to \infty$. But this means that $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$. □

To prove the converse in Proposition 2.9 under some additional assumptions is a major part of the following section.

Corollary 2.10. *Let H be finite dimensional, $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be a continuous operator function. Then $W_{\mathcal{H}}(F)$ is closed in Ω.*

Proof. Let $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$. Then, by Proposition 2.9 and because $\mathcal{H}^\square$ is compact, $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))} = D_{\mathcal{H}}(F(\lambda_0))$. This implies $\lambda_0 \in W_{\mathcal{H}}(F)$ by definition. □

To conclude this section, parts of the block numerical ranges of the matrix functions

$$F(z) = \begin{pmatrix} 1-z & \mathrm{i} & 1 \\ \mathrm{i} & z & -1 \\ 1 & -1 & z\sin z \end{pmatrix}, \quad G(z) = \begin{pmatrix} 2-z & \mathrm{i} & 1 & -\sin z \\ \mathrm{i} & 2-z & 3+\mathrm{i} & 1 \\ 1 & 3+\mathrm{i} & -2-z & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & -2-z \end{pmatrix}, \quad (2.4)$$

for $z \in \mathbb{C}$, are shown in Figure 2.2 and Figure 2.3.

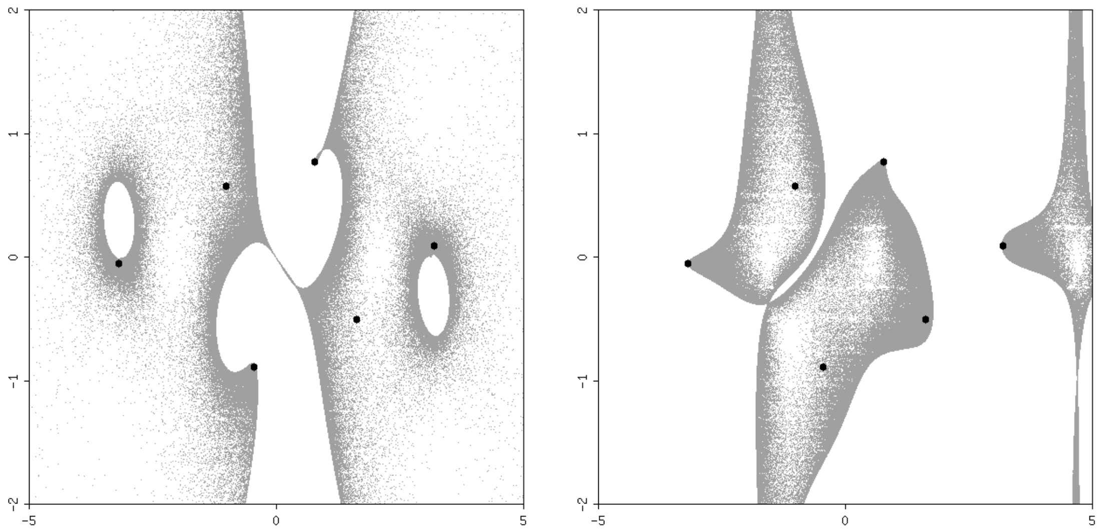


Figure 2.2: Parts of $W_{\mathbb{C}^2\times\mathbb{C}}(F)$ and $W_{\mathbb{C}\times\mathbb{C}^2}(F)$, where F is the matrix function from (2.4). Again, the circles denote the eigenvalues of F.

2.2 Spectral inclusion

We first give a very simple example of an operator function F for which the inclusion $\sigma(F) \subset \overline{W_{\mathcal{H}}(F)}$ does not hold. (Note that in this example the inclusion fails for every $\mathcal{H} \in \mathcal{Z}(H)$, in particular in the numerical range case $|\mathcal{H}| = 1$.)

Example 2.11. Let $f : \Omega \to \mathbb{C}$ be analytic, $f \not\equiv 0$, $A \in L(H)$, and

$$F : \Omega \to L(H), \quad F(z) := f(z)A.$$

Then, if $N_\Omega(f)$ is the set of zeros of f in Ω, we clearly have

$$\sigma(F) = \begin{cases} \Omega, & 0 \in \sigma(A), \\ N_\Omega(f), & 0 \notin \sigma(A), \end{cases} \qquad W_{\mathcal{H}}(F) = \begin{cases} \Omega, & 0 \in W_{\mathcal{H}}(A), \\ N_\Omega(f), & 0 \notin W_{\mathcal{H}}(A). \end{cases}$$

The latter follows immediately from

$$\det F_x(z) = \det(f(z)A_x) = f(z)^{|\mathcal{H}|} \det A_x, \quad z \in \Omega, \ x \in \mathcal{H}^\square.$$

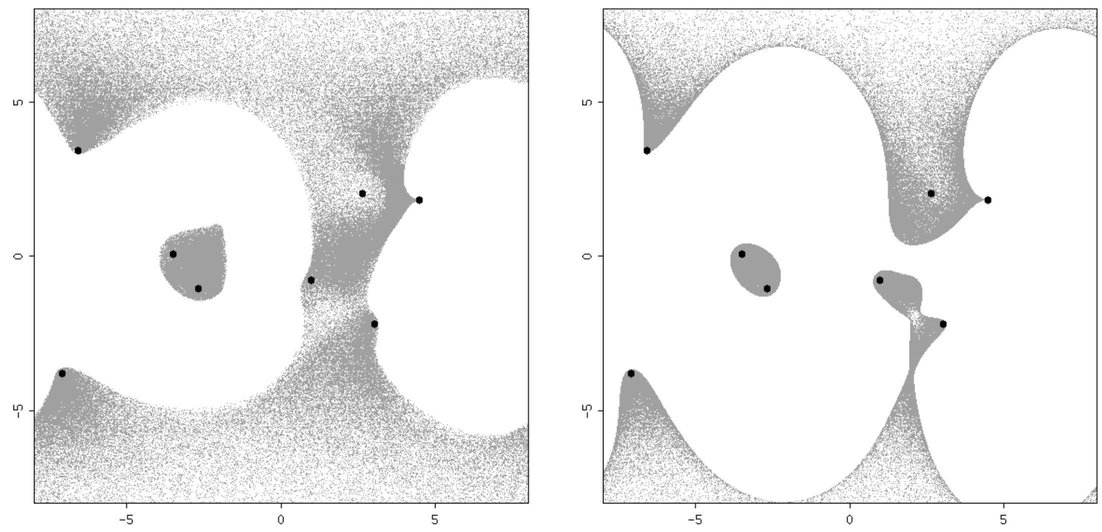


Figure 2.3: Parts of $W_{\mathbb{C}^2\times\mathbb{C}^2}(G)$ and $W_{\mathbb{C}\times\mathbb{C}\times\mathbb{C}^2}(G)$, where G is the matrix function from (2.4).

If we choose A such that $0 \in \sigma(A)$ and $0 \in \overline{W_{\mathcal{H}}(A)} \setminus W_{\mathcal{H}}(A)$ (which is possible only if $\dim H = \infty$, of course; take, e. g., $A := 1 - S$ with the left shift S from Example 1.8), then

$$\sigma(F) = \Omega \not\subset N_\Omega(f) = W_{\mathcal{H}}(F) = \overline{W_{\mathcal{H}}(F)} \cap \Omega.$$

Note that in this example the equivalence

$$z \in \overline{W_{\mathcal{H}}(F)} \iff 0 \in \overline{W_{\mathcal{H}}(F(z))}, \qquad z \in \Omega,$$

does not hold. In fact, as $\overline{W_{\mathcal{H}}(F(z))} = \overline{W_{\mathcal{H}}(f(z)A)} = f(z)\overline{W_{\mathcal{H}}(A)} \ni 0$, $z \in \Omega$, the right hand side is always true; in particular, the condition of Theorem 2.2 is not fulfilled for the numerical range case $|\mathcal{H}| = 1$.

Additionally, we already see from this simple example that refinements may dramatically shrink the block numerical range: If $\mathcal{H} \in \mathcal{Z}(H)$ is such that $0 \in W_{\mathcal{H}}(A)$ and $\mathcal{H}'$ is a refinement of $\mathcal{H}$ (see Definition 1.6) such that $0 \notin W_{\mathcal{H}'}(A)$, then, as we have seen above, $W_{\mathcal{H}}(F) = \Omega$, whereas $W_{\mathcal{H}'}(F) = N_\Omega(f)$ which may even be empty if f does not vanish at all.

The following proposition is a useful tool for deciding whether a point in Ω belongs to the closure of the block numerical range of an analytic operator function. It will be crucial later in the proof of the spectral inclusion.

Proposition 2.12. *Let $F : \Omega \to L(H)$ be an analytic operator function such that*

$$\exists\, s \in \mathbb{N}_0 \;\exists\, z_0 \in \Omega \quad 0 \notin \overline{\{(\det F_x)^{(s)}(z_0) \,:\, x \in \mathcal{H}^\square\}}, \tag{2.5}$$

where $^{(s)}$ denotes differentiation with respect to z. Then the equivalence

$$z \in \overline{W_{\mathcal{H}}(F)} \iff 0 \in \overline{W_{\mathcal{H}}(F(z))}, \qquad z \in \Omega, \tag{2.6}$$

holds.

Remark 2.13. Note that the condition

$$\exists\, z_0 \in \Omega \quad 0 \notin \overline{W_{\mathcal{H}}(F(z_0))}, \tag{2.7}$$

implies (2.5) and that, by Theorem 1.34, (2.6) is equivalent to

$$z \in \overline{W_{\mathcal{H}}(F)} \iff 0 \in \overline{D_{\mathcal{H}}(F(z))}, \qquad z \in \Omega. \tag{2.8}$$

Proof of Proposition 2.12. The implication "⇒" holds according to Proposition 2.9 as F is analytic. Vice versa, let $\lambda_0 \in \Omega$ be such that $0 \in \overline{W_{\mathcal{H}}(F(\lambda_0))}$, therefore $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$ by Theorem 1.34. Then there exists a sequence $(x_k)_1^\infty \subset \mathcal{H}^\square$ such that

$$\lim_{k\to\infty} \det F_{x_k}(\lambda_0) = 0.$$

The sequence $(\varphi_k)_1^\infty$ of analytic functions, defined by

$$\varphi_k : \Omega \to \mathbb{C}, \quad \varphi_k(z) := \det F_{x_k}(z),$$

is uniformly bounded on compact subsets of Ω. Hence by Montel's Theorem (see [Mar77, Theorem I.17.17 (p. 415_1)]) we may assume, that $(\varphi_k)_1^\infty$ converges uniformly to an analytic function $\varphi : \Omega \to \mathbb{C}$ on every compact subset of Ω. Now, by assumption,

$$\varphi(\lambda_0) = \lim_{k\to\infty} \det F_{x_k}(\lambda_0) = 0.$$

To show that $\varphi \not\equiv 0$, assume that $\varphi \equiv 0$. Let $s \in \mathbb{N}_0$ and $z \in \Omega$ be arbitrary. Then, using that $\varphi^{(s)}$ is the pointwise limit of the sequence $\big(\varphi_k^{(s)}\big)_1^\infty$ (even the uniform limit on compact subsets of Ω by Weierstraß' Theorem), we conclude that

$$0 = \varphi^{(s)}(z) = \lim_{k\to\infty} \varphi_k^{(s)}(z) = \lim_{k\to\infty} (\det F_{x_k})^{(s)}(z),$$

which yields that $0 \in \overline{\{(\det F_x)^{(s)}(z) \,:\, x \in \mathcal{H}^\square\}}$. As s and z were arbitrary, we have a contradiction to the assumption (2.5). Therefore, we have $\varphi \not\equiv 0$. As a consequence of Hurwitz' Theorem (see [Mar77, Theorem II.2.5 (p. 49_2)]), there exists a sequence $(\lambda_k)_1^\infty \subset \Omega$ such that $\lambda_k \to \lambda_0$, $k \to \infty$, and $\varphi_k(\lambda_k) = 0$ for all $k \in \mathbb{N}$. This implies that $0 \in W_{\mathcal{H}}(F(\lambda_k))$ for all $k \in \mathbb{N}$, thus $\lambda_k \in W_{\mathcal{H}}(F)$ and therefore $\lambda_0 \in \overline{W_{\mathcal{H}}(F)}$. □

Theorem 2.14. *Let $F : \Omega \to L(H)$ be a block operator function. Then*

$$\sigma_{\mathrm{p}}(F) \subset W_{\mathcal{H}}(F).$$

If, additionally, F is analytic and fulfills (2.5)*, then the spectral inclusion*

$$\sigma(F) \subset \overline{W_{\mathcal{H}}(F)}$$

holds.

Proof. Let $\lambda_0 \in \sigma_{\mathrm{p}}(F)$. Then $0 \in \sigma_{\mathrm{p}}(F(\lambda_0)) \subset W_{\mathcal{H}}(F(\lambda_0))$ by Theorem 1.4, thus, $\lambda_0 \in W_{\mathcal{H}}(F)$. If F is analytic such that (2.5) holds for F and $\lambda_0 \in \sigma(F)$, then $0 \in \sigma(F(\lambda_0)) \subset \overline{W_{\mathcal{H}}(F(\lambda_0))}$, and therefore $\lambda_0 \in \overline{W_{\mathcal{H}}(F)}$ by (2.6). □

Remark 2.15. In view of the equivalences (2.2) and (2.6) for analytic operator functions fulfilling (2.5), it would be natural to suspect that also the equivalence

$$z \in \partial W_{\mathcal{H}}(F) \iff 0 \in \partial W_{\mathcal{H}}(F(z)), \quad z \in \Omega,$$

is valid. But this is not true, not even for the numerical range of operator polynomials in the finite dimensional case. The following example is taken from [MP97]: Consider the operator polynomial

$$P(z) := z^2 - (1 + A)z + A, \quad z \in \mathbb{C},$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here, $P(1) = 0 \in M_2(\mathbb{C})$, thus $W(P(1)) = \{0\}$ and therefore $0 \in \partial W(P(1))$, but it is easy to see that $W(P) = \overline{B_{1/2}(1)}$, thus $1 \notin \partial W(P)$.

The following corollary corresponds to Lemma 1.31 in the operator case.

Corollary 2.16. *Let $F : \Omega \to L(H)$ be an analytic block operator function for which* (2.5) *holds. Then for each compact set $K \subset \Omega \setminus \overline{W_{\mathcal{H}}(F)}$*

$$\inf \{|\det F_x(z)| \,:\, z \in K,\ x \in \mathcal{H}^{\square}\} > 0.$$

Proof. Suppose there exist sequences $(x_k)_1^\infty \subset \mathcal{H}^\square$ and $(z_k)_1^\infty \subset K$ such that

$$\lim_{k\to\infty} \det F_{x_k}(z_k) = 0.$$

As K is compact, we may assume that $z_k \to z_0 \in K$, $k \to \infty$. Then

$$\|F_{x_k}(z_k) - F_{x_k}(z_0)\| \leq \|F(z_k) - F(z_0)\| \to 0, \quad k \to \infty,$$

and therefore

$$|\det F_{x_k}(z_0)| \leq |\det F_{x_k}(z_0) - \det F_{x_k}(z_k)| + |\det F_{x_k}(z_k)| \to 0, \quad k \to \infty,$$

thus $0 \in \overline{D_{\mathcal{H}}(F(z_0))}$ and, equivalently, $0 \in \overline{W_{\mathcal{H}}(F(z_0))}$. By (2.6) it follows that $z_0 \in \overline{W_{\mathcal{H}}(F)}$, which contradicts $z_0 \in K$ and $K \cap \overline{W_{\mathcal{H}}(F)} = \emptyset$. □

2.3 The norm of the resolvent

In this section we follow ideas from [MM01] where Theorem 2.18 and Corollary 2.20 are proved for the trivial decomposition $\mathcal{H} = \{H\}$. It turns out that the proofs given there apply to the general case without major changes.

First of all, we give an immediate corollary of Corollary 1.35.

Corollary 2.17. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $n := |\mathcal{H}|$ and $F : \Omega \to L(H)$ be an operator function. If $0 \notin \overline{D_{\mathcal{H}}(F(z))}$ for some $z \in \mathbb{C}$, then $F(z)$ is invertible and*

$$\|F^{-1}(z)\| \leq \frac{\|F(z)\|^{n-1}}{\inf |D_{\mathcal{H}}(F(z))|}. \tag{2.9}$$

Recall that for an analytic function $f : \Omega \to \mathbb{C}$ and $U \subset \Omega$ we defined $\nu_C(f)$ to be the number of zeros of f in U counting multiplicities.

Theorem 2.18. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $F : \Omega \to L(H)$ be an analytic operator function, and assume that* (2.5) *holds. If $C \subset W_{\mathcal{H}}(F)$ is a bounded connected component of $W_{\mathcal{H}}(F)$ such that*

$$\overline{C} \subset \Omega, \quad \overline{C} \cap \overline{W_{\mathcal{H}}(F) \setminus C} = \emptyset, \tag{2.10}$$

then the following assertions hold:

(1) *There is an integer $\nu \geq 1$ such that $\nu_C(\det F_x) = \nu$, $x \in \mathcal{H}^*$.*

(2) *If $U \subset \Omega$ is a bounded domain such that*

$$\overline{C} \subset U, \quad \overline{U} \cap \overline{W_{\mathcal{H}}(F) \setminus C} = \emptyset,$$

then there exists a constant $\gamma > 0$ such that

$$\|F^{-1}(z)\| \leq \frac{\gamma}{\operatorname{dist}(z, C)^{\nu}}, \quad z \in \overline{U} \setminus \overline{C}. \tag{2.11}$$

Proof. Let U be as in (2) and $V \subset \Omega$ be a domain such that $\Gamma := \partial V$ consists of finitely many piecewise smooth Jordan curves not intersecting each other and

$$\overline{U} \subset V, \quad \overline{V} \cap \overline{W_{\mathcal{H}}(F) \setminus C} = \emptyset.$$

(Note that such sets U and V exist by the assumption (2.10); see, e. g., [Mar88, Lemma 28.5].) Then (1) follows immediately from the connectedness of $\mathcal{H}^*$, the continuity of the index function $\operatorname{ind}_\Gamma$ and from $\nu_C(\det F_x) = \nu_{\overline{V}}(\det F_x) = \operatorname{ind}_\Gamma(\det F_x)$, $x \in \mathcal{H}^*$ (see [Mar88, §25.1]). To prove (2), let $\lambda_1(x), \dots, \lambda_\nu(x) \in C$ denote the zeros of $\det F_x$, $x \in \mathcal{H}^\square$, in C counted according to their multiplicities. Then the functions $g_x : \Omega \to \mathbb{C}$, $x \in \mathcal{H}^\square$, defined by

$$g_x(z) := \frac{\det F_x(z)}{(z - \lambda_1(x)) \cdots (z - \lambda_\nu(x))}, \quad z \in \Omega \setminus \{\lambda_1(x), \dots, \lambda_\nu(x)\},$$

are analytic and do not have zeros in $\overline{V}$. Moreover, we have

$$|\det F_x(z)| = |g_x(z)||z - \lambda_1(x)| \cdots |z - \lambda_\nu(x)| \geq |g_x(z)| \operatorname{dist}(z, C)^\nu, \quad x \in \mathcal{H}^\square,\ z \in \Omega. \tag{2.12}$$

Assume that we have shown that

$$\inf \{|g_x(z)| \,:\, x \in \mathcal{H}^\square,\ z \in \overline{U}\} =: d > 0. \tag{2.13}$$

Then $\inf |D_{\mathcal{H}}(F(z))| \geq d \operatorname{dist}(z, C)^\nu > 0$, $z \in \overline{U} \setminus \overline{C}$, by (2.12). Setting

$$\gamma := d^{-1} \max \{\|F(z)\|^{n-1} \,:\, z \in \overline{U}\},$$

we obtain from (2.9)

$$\|F^{-1}(z)\| \leq \frac{\|F(z)\|^{-1}}{\inf |D_{\mathcal{H}}(F(z))|} \leq \frac{\|F(z)\|^{-1}}{d \operatorname{dist}(z,C)^{\nu}} \leq \frac{\gamma}{\operatorname{dist}(z,C)^{\nu}}, \quad z \in \overline{U} \setminus \overline{C},$$

which had been claimed. To prove (2.13), assume that there are sequences $(z_k)_1^\infty \in \overline{U}$ and $(x_k)_1^\infty \subset \mathcal{H}^\square$ such that $g_{x_k}(z_k) \to 0$, $k \to \infty$. As $\overline{U}$ is compact, we may assume that $z_k \to z_0 \in \overline{U} \subset V$, $k \to \infty$. It follows from (2.12) that

$$|g_x(z)| \leq \frac{|\det F_x(z)|}{\operatorname{dist}(z,C)^{\nu}} \leq \frac{M}{\operatorname{dist}(\Gamma,C)^{\nu}} =: N, \quad z \in \Gamma,\ x \in \mathcal{H}^\square,$$

where $M := \sup\{|\det F_x(z)| \,:\, x \in \mathcal{H}^\square,\ z \in \Gamma\} < \infty$. The maximum modulus principle (see [Mar77, Theorem I.17.5]) yields $|g_x(z)| \leq N$, $x \in \mathcal{H}^\square$, $z \in V$, thus, by Montel's Theorem (see [Mar77, Theorem I.17.17]), we may assume that $(g_{x_k})_1^\infty$ converges uniformly on compact subsets of V to an analytic function $g : V \to \mathbb{C}$. From $g_{x_k}(z_k) \to 0$, $k \to \infty$, and the uniform convergence of the sequence $(g_{x_k})_1^\infty$ in a neighborhood of z_0 it follows easily that $g(z_0) = 0$. Moreover, for any $z \in V \setminus \overline{C}$, we have $\inf\{|g_{x_k}(z)| \,:\, k \in \mathbb{N}\} > 0$ by Corollary 2.16; in particular, $g \not\equiv 0$. According to Hurwitz' Theorem there is a sequence $(\mu_k)_1^\infty \subset U$ such that $\mu_k \to z_0$, $k \to \infty$, and $g_{x_k}(\mu_k) = 0$, $k \in \mathbb{N}$. This is a contradiction to $g_{x_k}(z) \neq 0$, $z \in U$, $k \in \mathbb{N}$. □

The following lemma ([MM01, Lemma 3]) allows to give an upper bound for the lengths of Jordan chains if an estimate of the resolvent norm $\|F^{-1}\|$ like (2.11) in a neighborhood of an eigenvalue of F is known.

Lemma 2.19. *Let $F : \Omega \to L(H)$ be an analytic operator function and $\lambda_0 \in \Omega$ be an eigenvalue of F. If for a natural number ν there exists a sequence $(\lambda_k)_1^\infty \subset \rho(F)$ such that $\lambda_k \to \lambda_0$, $k \to \infty$, and*

$$\|F^{-1}(\lambda_k)\| = \mathrm{O}(|\lambda_k - \lambda_0|^{-\nu}), \quad k \to \infty,$$

then the lengths of all Jordan chains of F in λ_0 do not exceed ν.

A boundary point λ_0 of some set $W \subset \mathbb{C}$ is said to have the *exterior cone property* (with respect to W) if there exists a closed cone K with vertex λ_0 and positive aperture, and an $r > 0$ such that

$$K \cap \overline{B_r(\lambda)} \cap \overline{W} = \{\lambda_0\}.$$

(See, e. g., [LMMT01, Section 3].) Using Theorem 2.18 and the preceding lemma, the proof of the following corollary is identical to that of [MM01, Theorem 2].

Corollary 2.20. *Let $F : \Omega \to L(H)$ be analytic such that* (2.5) *holds, $C \subset W_{\mathcal{H}}(F)$ be a bounded connected component of $W_{\mathcal{H}}(F)$ fulfilling* (2.10) *and ν as in Theorem* 2.18. *If an eigenvalue $\lambda_0 \in \partial C$ of F has the exterior cone property, then the lengths of all Jordan chains of F in λ_0 are at most ν.*

Chapter 3

The block numerical range of operator polynomials

An *operator polynomial on* H is an operator function of the form

$$P : \mathbb{C} \to L(H), \quad P(z) = A^{[d]}z^d + A^{[d-1]}z^{d-1} + \cdots + A^{[1]}z + A^{[0]},$$

where $d \in \mathbb{N}_0$, $A^{[l]} \in L(H)$, $l = 0, \ldots, d$, and either $d = 0$ or $d > 0$ and $A^{[d]} \neq 0$. The integer $d =: \deg P$ is then called the *degree of* P. The operator polynomial P is said to be *monic* if $A^{[d]} = 1_H$ and *linear* if $\deg P = 1$.

Spectral properties of operator polynomials frequently occur in many fields of mathematical physics, e. g., in differential equations, boundary value problems and hydrodynamics. As already in the operator case, the numerical range of operator polynomials plays an important role in the investigation of such problems. It has been studied intensively in the literature, both for the matrix polynomial case (see, e. g., [GLR82], [LR94], [MP97]) and in the general operator polynomial context (see, e. g., [Mar88] and [Rod89]).

Throughout this chapter we denote the operator coefficients of an operator polynomial P of degree d by $A^{[l]}$, $l = 0, \ldots, d$. First of all, let us consider some examples of block numerical ranges of operator polynomials. Recall that a complex number λ_0 belongs to $W_{\mathcal{H}}(P)$ if and only if there exists an $x \in \mathcal{H}^*$ such that $\det P_x(\lambda_0) = 0$.

Example 3.1. Let H be a Hilbert space and $\mathcal{H} \in \mathcal{Z}(H)$.

(1) That $W_{\mathcal{H}}(P_A) = W_{\mathcal{H}}(A)$ for the linear polynomial $P_A(z) = A - z$, $z \in \mathbb{C}$, where $A \in L(H)$, has already been mentioned in Remark 2.1.

(2) If $P(z) = p(z)A$, $z \in \mathbb{C}$, where $A \in L(H)$ and p is a complex polynomial, then (see Example 2.11)

$$W_{\mathcal{H}}(P) = \begin{cases} \mathbb{C}, & 0 \in W_{\mathcal{H}}(A), \\ N(p), & 0 \notin W_{\mathcal{H}}(A). \end{cases}$$

A non-constant operator polynomial of this form has, of course, always non-empty block numerical range. However, in contrast to the numerical range

$W(P)$, it may happen that $W_{\mathcal{H}}(P)$ is empty even if P is not a constant operator polynomial, as the following example shows.

(3) Let the operator polynomial P on $\mathcal{H} := H \times H$ be given by

$$P(z) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C}.$$

Then, as $P(z)$ is upper triangular, we have $W_{\mathcal{H}}(P(z)) = \{1\}$ for all $z \in \mathbb{C}$. In particular, $0 \notin W(P(z))$, $z \in \mathbb{C}$, thus $W_{\mathcal{H}}(P) = \emptyset$.

In the case of the numerical range of an operator polynomial P, the existence of a common non-zero isotropic vector $x \in H$ (i. e., $(A^{[d]}x, x) = \cdots = (A^{[0]}x, x) = 0$) implies that $W(P) = \mathbb{C}$. The condition of a block-analogue of this statement can not be as simple as the existence of some $x \in \mathcal{H}^{\square}$ with

$$\det A_x^{[d]} = \cdots = \det A_x^{[0]} = 0.$$

To see that this condition is not sufficient, consider the following two examples:

(4) With respect to the decomposition $\mathbb{C} \times \mathbb{C}$ of $\mathbb{C}^2$ for the matrix polynomial

$$P(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z^2 + z & 1 \\ z & 0 \end{pmatrix}, \quad z \in \mathbb{C},$$

we have $\det P_x(z) = -z = 0$ if and only if $z = 0$, that is, $W_{\mathbb{C}\times\mathbb{C}}(P) = \{0\}$ is bounded, although $\det A_x^{[l]} = 0$, $l = 0, 1, 2$, even for all $x \in \mathcal{H}^{\square}$.

(5) To have a less trivial example which also shows the complexity of the (block) numerical range of operator polynomials, consider

$$P(z) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) z^2 + \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) z + \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} z^2 & 0 & 0 \\ 0 & z & 1 \\ \hline 0 & z & z^2 \end{array}\right), \quad z \in \mathbb{C}, \tag{3.1}$$

with respect to the decomposition $\mathbb{C}^2 \times \mathbb{C} \in \mathcal{Z}(\mathbb{C}^3)$. First, we have

$$\det P(z) = z^3(z-1)(z+1),$$

so that $\sigma_{\mathrm{p}}(P) = \{-1, 0, 1\}$. Moreover, for the vector $x = ((0,1),1) \in (\mathbb{C}^2 \times \mathbb{C})^{\square}$,

$$\det A_x^{[0]} = \det A_x^{[1]} = \det A_x^{[2]} = 0.$$

To calculate $W_{\mathbb{C}^2\times\mathbb{C}}(P)$ it is sufficient to consider vectors $x = ((t, w), 1) \in \mathbb{C}^3$ such that $t \in [0, 1]$ and $|w|^2 = 1 - t^2$. For such x we have

$$\det P_x(z) = \det \begin{pmatrix} t^2 z^2 + |w|^2 z & \overline{w} \\ wz & z^2 \end{pmatrix} = z(t^2 z^3 + (1-t^2)(z^2-1)), \quad z \in \mathbb{C},$$

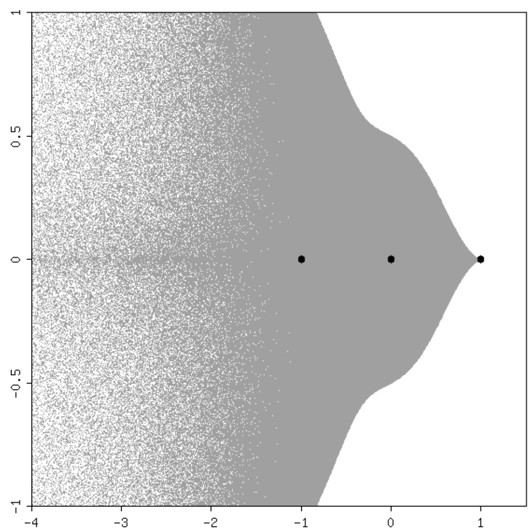
that is, $W_{\mathbb{C}^2\times\mathbb{C}}(P)$ consists of 0 and all zeros of the polynomials

$$p_t(z) = t^2 z^3 + (1-t^2)(z^2-1), \quad t \in [0, 1].$$

For $t = 0$ we get $\pm 1 \in W_{\mathbb{C}^2\times\mathbb{C}}(P)$, for $t = 1$ it follows again that $0 \in W_{\mathbb{C}^2\times\mathbb{C}}(P)$. If now $t \in (0,1)$ and $z \in \mathbb{C}$, $z \neq \pm 1$, we have

$$p_t(z) = 0 \iff \frac{z^3}{z^2-1} = \frac{t^2-1}{t^2} < 0,$$

thus, $z \in W_{\mathbb{C}^2\times\mathbb{C}}(P)$ if and only if $z^3/(z^2-1) < 0$ for $z \neq 0, \pm 1$. In particular, we have $W_{\mathbb{C}^2\times\mathbb{C}}(P) \cap \mathbb{R} = (-\infty, -1] \cup [0, 1]$, Thus, $W_{\mathbb{C}^2\times\mathbb{C}}(P)$ is unbounded and $\neq \mathbb{C}$. Parts of the (block) numerical range of P are shown in Figure 3.1.



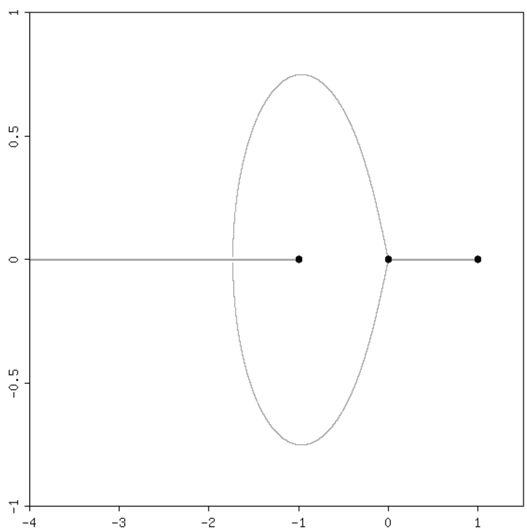


Figure 3.1: Parts of $W(P)$ and $W_{\mathbb{C}^2\times\mathbb{C}}(P)$ for the operator polynomial (3.1).

(6) The block numerical ranges of the operator polynomial in (3.1) were unbounded. An example of a matrix polynomial with bounded block numerical ranges is

$$P(z) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&0&1 \end{pmatrix} z^2 + \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} z + \begin{pmatrix} 1&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0\\ 0&0&0&1&0&1 \end{pmatrix} \tag{3.2}$$

for $z \in \mathbb{C}$. It is easy to see that $W(A^{[2]}) = \overline{B_{1/2}(1)}$, thus $0 \notin W(A^{[2]})$. It follows from [LR94, Theorem 2.3] that $W(P)$ is bounded. As $W_{\mathcal{H}}(P) \subset W(P)$, $\mathcal{H} \in \mathcal{Z}(H)$, by Proposition 2.6, $W_{\mathcal{H}}(P)$ is bounded for every $\mathcal{H} \in \mathcal{Z}(H)$. The block numerical ranges of P with respect to the decompositions $\mathcal{H}_1 = \mathbb{C}^3 \times \mathbb{C}^3$ and $\mathcal{H}_2 = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ are given in Figure 3.2.

Remark 3.2. (1) Let $P : \mathbb{C} \to M_n(\mathbb{C})$ be a matrix polynomial of degree d with

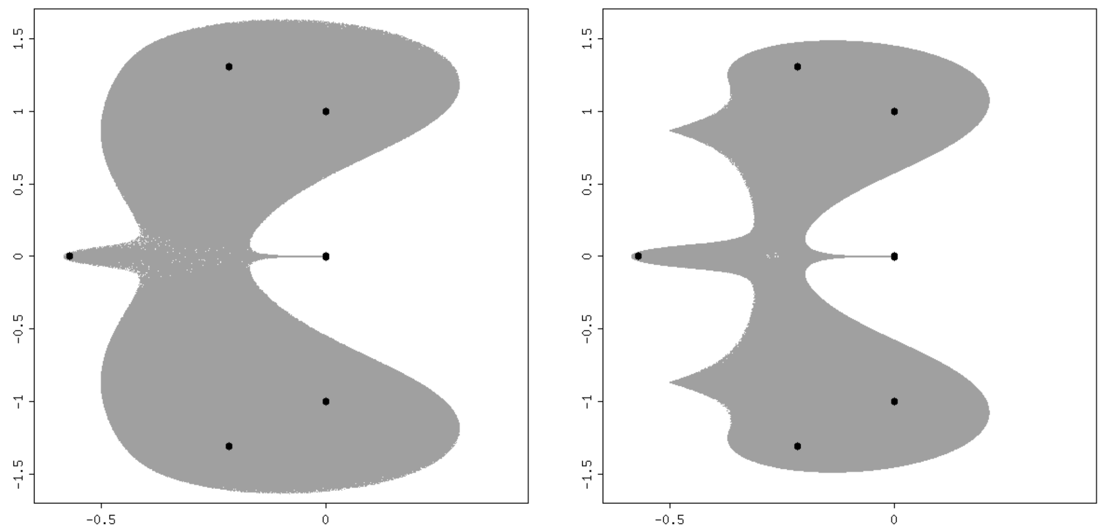


Figure 3.2: The block numerical ranges $W_{\mathbb{C}^3\times\mathbb{C}^3}(P)$ and $W_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(P)$ for the operator polynomial (3.2).

coefficients $A^{[l]} = \left(a_{ij}^{[l]}\right)_{i,j=1}^n$, $l = 0, \ldots, d$. Then

$$\det P(z) = \sum_{l=0}^{nd} \delta^{[l]} z^l = \det A^{[d]} z^{nd} + \sum_{l=1}^{nd-1} \delta^{[l]} z^l + \det A^{[0]},$$

where

$$\delta^{[l]} = \sum_{\substack{0\le l_1,\ldots,l_n\le d\\ l_1+\cdots+l_n=l}} \det\left(a_{ij}^{[l_i]}\right)_{i,j=1}^n, \quad l = 0, \ldots, nd.$$

(2) Let $P : \mathbb{C} \to L(H)$ be an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$. Then, writing $\mathcal{H} = \{M_1, \ldots, M_n\} \in \mathcal{Z}(H)$,

$$\det P_x(z) = \sum_{l=0}^{nd} \delta_x^{[l]} z^l = \det A_x^{[d]} z^{nd} + \sum_{l=1}^{nd-1} \delta_x^{[l]} z^l + \det A_x^{[0]}, \quad z \in \mathbb{C},\ x \in H, \tag{3.3}$$

where

$$\delta_x^{[l]} = \sum_{\substack{0\le l_1,\ldots,l_n\le d\\ l_1+\cdots+l_n=l}} \det\left(A_{ij}^{[l_i]} x_j, x_i\right)_{i,j=1}^n, \quad l = 0, \ldots, nd,\ x \in H. \tag{3.4}$$

(3) As an immediate consequence of (2), the block-analogue statement concerning the common non-zero isotropic vector mentioned on page 38 is: If there is a vector $x \in \mathcal{H}^*$ such that $\delta_x^{[l]} = 0$, $l = 0, \ldots, nd$, then $W_{\mathcal{H}}(P) = \mathbb{C}$. For $d = 1$ and $n = 2$ this condition reads

$$0 = \det\begin{pmatrix} (A_{11}^{[1]}x_1, x_1) & (A_{12}^{[1]}x_2, x_1) \\ (A_{21}^{[1]}x_1, x_2) & (A_{22}^{[1]}x_2, x_2) \end{pmatrix} = \det\begin{pmatrix} (A_{11}^{[0]}x_1, x_1) & (A_{12}^{[0]}x_2, x_1) \\ (A_{21}^{[0]}x_1, x_2) & (A_{22}^{[0]}x_2, x_2) \end{pmatrix}$$

$$= \det \begin{pmatrix} (A_{11}^{[0]}x_1, x_1) & (A_{12}^{[0]}x_2, x_1) \\ (A_{21}^{[1]}x_1, x_2) & (A_{22}^{[1]}x_2, x_2) \end{pmatrix} + \det \begin{pmatrix} (A_{11}^{[1]}x_1, x_1) & (A_{12}^{[1]}x_2, x_1) \\ (A_{21}^{[0]}x_1, x_2) & (A_{22}^{[0]}x_2, x_2) \end{pmatrix}$$

for some $x \in \mathcal{H}^*$.

(4) If $0 \in W_{\mathcal{H}}(A^{[0]})$, then $0 \in W_{\mathcal{H}}(P)$. In fact, if $0 = \det A_x^{[0]} = \delta_x^{[0]}$ for some $x \in \mathcal{H}^\square$, then clearly $\det P_x(0) = 0$.

Proof. (2) follows immediately from $P_x(z) = A_x^{[d]} z^d + \cdots + A_x^{[0]}$ and (1).

(1) For $z \in \mathbb{C}$ we have

$$P(z) = \begin{pmatrix} \sum_{l=0}^{d} a_{11}^{[l]} z^l & \ldots & \sum_{l=0}^{d} a_{1n}^{[l]} z^l \\ \vdots & & \vdots \\ \sum_{l=0}^{d} a_{n1}^{[l]} z^l & \ldots & \sum_{l=0}^{d} a_{nn}^{[l]} z^l \end{pmatrix}$$

and therefore, expanding the determinant of $P(z)$ and sorting by powers of z we obtain, denoting the set of permutations of $\{1, \ldots, n\}$ by Π_n,

$$\begin{aligned} \det P(z) &= \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) \Big(\sum_{l_1=0}^{d} a_{1\sigma(1)}^{[l_1]} z^{l_1} \Big) \cdots \Big(\sum_{l_n=0}^{d} a_{n\sigma(n)}^{[l_n]} z^{l_n} \Big) \\ &= \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) \sum_{m=0}^{nd} \Big(\sum_{\Sigma l_i = m} a_{1\sigma(1)}^{[l_1]} \cdots a_{\sigma(1)}^{[l_n]} \Big) z^m \\ &= \sum_{m=0}^{nd} \Big(\sum_{\Sigma l_i = m} \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)}^{[l_1]} \cdots a_{1\sigma(1)}^{[l_n]} \Big) z^m \\ &= \sum_{m=0}^{nd} \Big(\sum_{\Sigma l_i = m} \det \big(a_{ij}^{[l_i]} \big)_{i,j=1}^{n} \Big) z^m. \end{aligned}$$

□

Proposition 3.3. *If P is an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$ is such that $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$, then $0 \notin \overline{W_{\mathcal{H}}(P(z))}$ for large values of $|z|$. In particular, $W_{\mathcal{H}}(P)$ is bounded and P fulfills* (2.7). *Thus, the spectral inclusions*

$$\sigma_{\mathrm{p}}(P) \subset W_{\mathcal{H}}(P), \quad \sigma(P) \subset \overline{W_{\mathcal{H}}(P)},$$

and the equivalence

$$z \in \overline{W_{\mathcal{H}}(P)} \iff 0 \in \overline{W_{\mathcal{H}}(P(z))}, \quad z \in \mathbb{C},$$

hold.

Proof. Let $n := |\mathcal{H}|$ and $m := nd$. From the assumption $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$ and Theorem 1.34 it follows that $0 \notin \overline{D_{\mathcal{H}}(A^{[d]})}$. In particular, $\delta := \inf_{x \in \mathcal{H}^\square} |\det A_x^{[d]}| > 0$. Let

$C := \sup\{\delta_x^{[l]} \,:\, x \in \mathcal{H}^\square,\ l = 0, \dots, m-1\}$, where the $\delta_x^{[l]}$ are defined as in (3.4). Then for arbitrary $x \in \mathcal{H}^\square$,

$$|\det P_x(z)| = \left|\det A_x^{[d]} z^m + \sum_{l=0}^{m-1} \delta_x^{[l]} z^l\right| \geq \delta|z|^m - C\sum_{l=0}^{m-1} |z|^l \to \infty, \quad |z| \to \infty,$$

independently of $x \in \mathcal{H}^\square$. This shows that $\inf_{x\in\mathcal{H}^\square} |\det P_x(z)| > 0$, thus $0 \notin \overline{D_{\mathcal{H}}(P(z))}$ and therefore $0 \notin \overline{W_{\mathcal{H}}(P(z))}$ for large values of $|z|$ by Theorem 1.34. The remaining claims follow immediately from Proposition 2.12 and Theorem 2.14. □

Remark. Note that there is another way of seeing that (2.5) holds for P: From Remark 3.2 we know that

$$(\det P_x)^{(nd)}(z) = (nd)!\, \det A_x^{[d]}, \quad x \in \mathcal{H}^\square,\ z \in \mathbb{C}.$$

Now, by Theorem 1.34, $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$ implies $0 \notin \overline{D_{\mathcal{H}}(A^{[d]})}$. As a consequence,

$$0 \notin \overline{\{(nd)! \det A_x^{[d]} \,:\, x \in \mathcal{H}^\square\}} = \overline{\{(\det P_x)^{(nd)}(z) \,:\, x \in \mathcal{H}^\square\}}$$

even for arbitrary $z \in \mathbb{C}$.

3.1 On the boundedness of $W_{\mathcal{H}}(P)$

In [LR94] it is claimed (Theorem 2.3 and the note after example 3) that, if $A^{[d]} \neq 0$, the equivalence '$W(P)$ bounded $\iff$ $0 \notin \overline{W(A^{[d]})}$' holds. This is not true, as we have already seen in Example 3.1 (2): If H is not finite dimensional one may choose $A = A^{[d]} \in L(H)$ such that $0 \in \overline{W(A)} \setminus W(A)$. Then $W(P)$ is bounded although $0 \in \overline{W(A^{[d]})}$.

On the other hand, Proposition 3.3 shows that the implication

$$0 \notin \overline{W_{\mathcal{H}}(A^{[d]})} \Longrightarrow W_{\mathcal{H}}(P) \text{ bounded}$$

holds. In this section some criteria for the unboundedness of $W_{\mathcal{H}}(P)$ are given.

Definition 3.4. Let $\mathcal{H} \in \mathcal{Z}(H)$ be a decomposition of H. Then an operator polynomial P of degree d is called *regular with respect to* $\mathcal{H}$ or simply $\mathcal{H}$*-regular*, if there is an $x \in \mathcal{H}^\square$ such that $\det A_x^{[d]} \neq 0$.

Remark 3.5. (1) Every non-zero operator polynomial P is regular with respect to the trivial decomposition $\mathcal{H} = \{H\}$ because $0 = \det A_x^{[d]} = (A^{[d]}x, x)$ for all $x \in H$, $\|x\| = 1$, implies $A^{[d]} = 0$. In general, i. e., if $|\mathcal{H}| > 1$, the condition $A^{[d]} \neq 0$ is not sufficient for the $\mathcal{H}$-regularity of P, of course.

(2) By Remark 3.2, an operator polynomial P is $\mathcal{H}$-regular if and only if there exists an $x \in \mathcal{H}^\square$ such that the polynomial $\det P_x$ has maximal degree, that is, $\deg(\det P_x) = \deg P \cdot |\mathcal{H}|$.

(3) If P is an $\mathcal{H}$-regular non-constant operator polynomial, then $W_{\mathcal{H}}(P) \neq \emptyset$. This follows immediately from (2).

Example. (1) The polynomial P in Example 3.1 (4) is not $\mathbb{C} \times \mathbb{C}$-regular. Its block numerical range $W_{\mathcal{H}}(P) = \{0\}$ is bounded although $0 \in W_{\mathcal{H}}(A^{[2]})$.

(2) The polynomial P in Example 3.1 (5) is $\mathbb{C}^2 \times \mathbb{C}$-regular. In fact, $A_x^{[2]}$ is the identity in $M_2(\mathbb{C})$ for the vector $((1,0),1) \in (\mathbb{C}^2 \times \mathbb{C})^{\square}$. We have seen that $0 \in W_{\mathcal{H}}(A^{[2]})$, too, but the block numerical range $W_{\mathcal{H}}(P)$ is unbounded.

To be able to prove criteria for the unboundedness of the block numerical range of operator polynomials, we need the following lemma.

Lemma 3.6. *Let $p_k(z) = a_k^{[m]} z^m + \cdots + a_k^{[1]} z + a_k^{[0]}$, $z \in \mathbb{C}$, be a sequence of complex polynomials of degree m, i. e., $a_k^{[m]} \neq 0$, $k \in \mathbb{N}$, such that $a_k^{[l]} \to a^{[l]}$, $k \to \infty$, $l = 0, \ldots, m$. If $a^{[m]} = 0$ and $a^{[s]} \neq 0$ for some $s \in \{0, \ldots, m-1\}$, then there exists a sequence $(\lambda_k)_1^\infty \subset \mathbb{C}$ such that $|\lambda_k| \to \infty$, $k \to \infty$, and $p_k(\lambda_k) = 0$, $k \in \mathbb{N}$.*

Proof. Write $p_k(z) = a_k^{[m]}(z - \mu_{k1}) \cdots (z - \mu_{km})$ and $p(z) := a^{[m]} z^m + \cdots + a^{[1]} z + a^{[0]}$. Suppose the claim is false. Then there exists a constant $C > 0$ such that $|\mu_{kl}| \leq C$, $k \in \mathbb{N}$, $l = 0, \ldots, m$, thus

$$|p(z)| = \lim_{k\to\infty} |p_k(z)| \leq \lim_{k\to\infty} \left|a_k^{[m]}\right| (|z| + C)^m = \left|a^{[m]}\right| (|z| + C)^m = 0, \quad z \in \mathbb{C},$$

that is, p is the zero polynomial. But this contradicts the assumption $a^{[s]} \neq 0$ for some s. □

Proposition 3.7. *Let P be an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$. Then the following implications hold:*

(1) *If $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$, then $W_{\mathcal{H}}(P)$ is bounded.*

(2) *If P is $\mathcal{H}$-regular and $0 \in W_{\mathcal{H}}(A^{[d]})$, then $W_{\mathcal{H}}(P)$ is unbounded.*

(3) *If P is $\mathcal{H}$-regular and $0 \in \overline{W_{\mathcal{H}}(A^{[d]})}$ but $0 \notin \overline{W_{\mathcal{H}}(A^{[0]})}$, then $W_{\mathcal{H}}(P)$ is unbounded.*

Proof. Let $n := |\mathcal{H}|$ and $m := nd$. (1) has already been shown in Proposition 3.3. To prove (2), let

$$N := \{x \in \mathcal{H}^{\square} : \det A_x^{[d]} = 0\}.$$

Then $\emptyset \neq N \neq \mathcal{H}^{\square}$ by the assumptions. Let $x_0 \in \partial N \subset N$ and $(x_k)_1^\infty \subset \mathcal{H}^{\square} \setminus N$ be a sequence converging to x_0. For the sequences $\left(\delta_{x_k}^{[l]}\right)_{k=1}^\infty$ of the corresponding coefficients of the polynomials $\det P_{x_k}$ (see (3.3)) we have

$$\lim_{k\to\infty} \delta_{x_k}^{[l]} = \delta_{x_0}^{[l]}, \quad l = 0, \ldots, m.$$

Now consider two cases: If $\delta_{x_0}^{[l]} = 0$ for all $l = 0, \ldots, m$, then

$$\det P_{x_0}(z) = \lim_{k\to\infty} \det P_{x_k}(z) = \lim_{k\to\infty} \sum_{l=0}^{m} \delta_{x_k}^{[l]} z^l = 0, \quad z \in \mathbb{C}, \tag{3.5}$$

that is, $z \in W_{\mathcal{H}}(P)$ for every $z \in \mathbb{C}$, and thus $W_{\mathcal{H}}(P) = \mathbb{C}$ is unbounded. If, on the other hand, there is an $s \in \{0, \ldots, m-1\}$ such that $\delta_{x_0}^{[s]} \neq 0$, then applying Lemma 3.6 to the sequence of polynomials $\big(\det P_{x_k}\big)_1^\infty$ (note that $\delta_{x_0}^{[m]} = \det A_{x_0}^{[d]} = 0$ and $\delta_{x_k}^{[m]} \neq 0$, as $x_k \notin N$, $k \in \mathbb{N}$) yields an unbounded sequence $(\lambda_k)_1^\infty \subset \mathbb{C}$ such that $\det P_{x_k}(\lambda_k) = 0$, $k \in \mathbb{N}$. In particular, $\lambda_k \in W_{\mathcal{H}}(P)$, $k \in \mathbb{N}$. Again, $W_{\mathcal{H}}(P)$ is unbounded and the claim follows.

Considering assertion (3), we may assume that $0 \in \overline{W_{\mathcal{H}}(A^{[d]})} \setminus W_{\mathcal{H}}(A^{[d]})$; otherwise the claim already follows from (2). Then there exists a sequence $(x_k)_1^\infty \subset \mathcal{H}^\square$ such that $\det A_{x_k}^{[d]} \to 0$, $k \to \infty$, and $\det A_{x_k}^{[d]} \neq 0$, $k \in \mathbb{N}$. By passing to appropriate subsequences, we may assume that the sequences $\big(\delta_{x_k}^{[l]}\big)_{k=1}^\infty$, defined as in the proof of (2), converge to numbers $\delta^{[l]}$, $l = 0, \ldots, m$. Then $\delta^{[m]} = 0$, and $\delta^{[0]} \neq 0$ according to the assumption $0 \notin \overline{W_{\mathcal{H}}(A^{[0]})}$ (which is equivalent to $0 \notin \overline{D_{\mathcal{H}}(A^{[0]})}$ by Theorem 1.34). In particular, Lemma 3.6 is again applicable to the sequence of polynomials $\big(\det P_{x_k}\big)_1^\infty$; thus, the claim follows in the same way as in the second case in the proof of (2). □

Remark 3.8. (1) If P is not $\mathcal{H}$-regular, reasonable conditions on the (un)boundedness of $W_{\mathcal{H}}(P)$ seem to be hard to formulate due to the complexity of the middle coefficients of the polynomials $\det P_x$, $x \in \mathcal{H}^\square$ (see (3.4)).

(2) If P is $\mathcal{H}$-regular, the condition $0 \in \overline{W_{\mathcal{H}}(A^{[d]})}$ alone is not sufficient for the unboundedness of $W_{\mathcal{H}}(P)$. In fact, using the notation of the proof of Proposition 3.7 (3) we only get a sequence $(x_k)_1^\infty \subset \mathcal{H}^\square$ such that $\delta_{x_k}^{[l]} \to \delta^{[l]}$, $k \to \infty$, $l = 0, \ldots, m$, where $\delta^{[m]} = 0$. In this situation it may happen, of course, that $\delta^{[l]} = 0$, $l = 0, \ldots, m$. Then we obtain (compare (3.5))

$$\lim_{k\to\infty} \det P_{x_k}(z) = \lim_{k\to\infty} \sum_{l=0}^{m} \delta_{x_k}^{[l]} z^l = 0, \quad z \in \mathbb{C}.$$

Hence, $0 \in \overline{D_{\mathcal{H}}(P(z))}$, $z \in \mathbb{C}$. But now this does *not* imply that $z \in \overline{W_{\mathcal{H}}(P)}$, $z \in \mathbb{C}$, since the condition $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$ of Proposition 3.3 is not fulfilled.

3.2 Connected components and the norm of the resolvent

Proposition 3.9. *Let P be an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$. If $0 \notin W_{\mathcal{H}}(A^{[d]})$, then $W_{\mathcal{H}}(P)$ consists of at most $d \cdot |\mathcal{H}|$ connected components. For each connected component C of $W_{\mathcal{H}}(P)$, the number $\nu_C(\det P_x)$ of zeros of the polynomial $\det P_x$ in C counting multiplicities does not depend on $x \in \mathcal{H}^\square$.*

Proof. Let $n := |\mathcal{H}|$. From $0 \notin W_{\mathcal{H}}(A^{[d]})$, i.e., $\det A_x^{[d]} \neq 0$ for all $x \in \mathcal{H}^\square$, it follows that $\det P_x(z) = 0$ if and only

$$0 = z^{nd} + \sum_{l=0}^{nd-1} \frac{\delta_x^{[l]}}{\det A_x^{[d]}} z^l =: z^{nd} + \sum_{l=0}^{nd-1} \tilde{\delta}_x^{[l]} z^l =: p_x(z),$$

where the $\delta_x^{[l]}$ are defined as in (3.4). The mapping

$$B : \mathcal{H}^{\square} \to M_{nd}(\mathbb{C}), \quad B(x) := \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -\tilde{\delta}_x^{[0]} & -\tilde{\delta}_x^{[1]} & \cdots & -\tilde{\delta}_x^{[nd-1]} \end{pmatrix},$$

is continuous and for every $x \in \mathcal{H}^{\square}$ the zeros of the polynomial p_x coincide with the eigenvalues of $B(x)$ counting multiplicities. Therefore,

$$W_{\mathcal{H}}(P) = \bigcup_{x \in \mathcal{H}^{\square}} \sigma_{\mathrm{p}}(B(x)) = \sigma_{\mathrm{p}}(B(\mathcal{H}^{\square})).$$

As $B(\mathcal{H}^{\square}) \subset M_{nd}(\mathbb{C})$ is connected, it follows from Proposition 1.10 that $\sigma_{\mathrm{p}}(B(\mathcal{H}^{\square}))$ consists of at most nd connected components and $\nu_C(\det P_x) = \nu_C(p_x) = \nu_C(B(x))$ for a connected component C of $W_{\mathcal{H}}(P)$ does not depend on $x \in \mathcal{H}^{\square}$. $\square$

Let P be an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$ such that $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$. We have seen in Proposition 3.3 that $W_{\mathcal{H}}(P)$ is bounded and (2.7) holds; in particular, every connected component of $W_{\mathcal{H}}(P)$ is bounded. Thus, Theorem 2.18, which yields an estimate of the resolvent norm $\|P^{-1}\|$, applies to each connected component C of $W_{\mathcal{H}}(P)$ for which $\overline{C}$ is a connected component of $\overline{W_{\mathcal{H}}(P)}$. (It is not too hard to see that the condition (2.10) holds for such C; note that we have $\Omega = \mathbb{C}$.) However, the estimate occurring in (2.11) can be improved for operator polynomials. The proof of the following theorem is almost literally the same as the proof of Theorem 1.5 (see [TW03, Theorem 4.2]).

Theorem 3.10. *Let P be an operator polynomial of degree d, $\mathcal{H} \in \mathcal{Z}(H)$ and $n := |\mathcal{H}|$. If $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$, then the following estimate of the resolvent of P holds:*

$$\|P^{-1}(z)\| \leq \frac{\|P(z)\|^{n-1}}{\inf |D_{\mathcal{H}}(A^{[d]})| \cdot \operatorname{dist}(z, W_{\mathcal{H}}(P))^{nd}}, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(P)}.$$

More exactly, if $W_{\mathcal{H}}(P)$ consists of the connected components $C_1, \dots, C_s$ and $\nu_j := \nu_{C_j}(\det P_x)$, $x \in \mathcal{H}^{\square}$, $j = 1, \dots, s$, then

$$\|P^{-1}(z)\| \leq \frac{\|P(z)\|^{n-1}}{\inf |D_{\mathcal{H}}(A^{[d]})| \cdot \prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}}, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(P)}. \tag{3.6}$$

Proof. Note that from $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$ and Proposition 3.3 it follows that $\mathbb{C} \setminus \overline{W_{\mathcal{H}}(P)} \subset \rho(P)$ and $\inf |D_{\mathcal{H}}(A^{[d]})| > 0$ (the latter by Theorem 1.34). If $\lambda_1^{[j]}(x), \dots, \lambda_{\nu_j}^{[j]}(x)$ are the zeros of $\det P_x$ on C_j, $j = 1, \dots, s$, counted according to their multiplicities, we have

$$|\det P_x(z)| = |\det A_x^{[d]}| \cdot \prod_{j=1}^{s} \prod_{i=1}^{\nu_j} |z - \lambda_i^{[j]}| \geq \inf |D_{\mathcal{H}}(A^{[d]})| \cdot \prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}$$

for $x \in \mathcal{H}^{\square}$ and $z \in \mathbb{C}$. Hence,

$$\inf |D_{\mathcal{H}}(P(z))| \geq \inf |D_{\mathcal{H}}(A^{[d]})| \cdot \prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j} > 0, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(P)}.$$

The claim follows from Corollary 2.17. □

Corollary 3.11. *If, using the notations of Theorem* 3.10, *the operator polynomial P is monic, then*

$$\|P^{-1}(z)\| \leq \frac{\|P(z)\|^{n-1}}{\prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}} \leq \frac{\|P(z)\|^{n-1}}{\operatorname{dist}(z, W_{\mathcal{H}}(P))^{nd}}, \quad z \in \mathbb{C} \setminus \overline{W_{\mathcal{H}}(P)}.$$

Using the more precise estimate (3.6) instead of (2.11) we immediately get the following enhancement of Corollary 2.20. In contrast to Corollary 2.20, it allows to give a priory upper bounds on the lengths of Jordan chains of an operator polynomial in an eigenvalue having the exterior cone property with respect to $W_{\mathcal{H}}(P)$.

Corollary 3.12. *Let P be an operator polynomial of degree d and $\mathcal{H} \in \mathcal{Z}(H)$ such that $0 \notin \overline{W_{\mathcal{H}}(A^{[d]})}$. Moreover, let $C \subset W_{\mathcal{H}}(P)$ be a connected component of $W_{\mathcal{H}}(P)$ and $\nu := \nu_C(\det P_x)$, $x \in \mathcal{H}^{\square}$. If an eigenvalue $\lambda_0 \in \partial C$ of P has the exterior cone property, then the lengths of all Jordan chains of P in λ_0 are of length $\leq \nu \leq d \cdot |\mathcal{H}|$. In particular, if $W_{\mathcal{H}}(P)$ consists of the maximal number $d \cdot |\mathcal{H}|$ of connected components, then there are no associated vectors at λ_0.*

3.3 The companion polynomial

Recall that for an operator polynomial P with $d := \deg P > 0$ the *companion polynomial* $\mathcal{C}^P : \mathbb{C} \to L(H^d)$ is defined as (see, e. g, [Mar88, § 12.2])

$$\mathcal{C}^P(z) := \begin{pmatrix} 1_H & & & \\ & \ddots & & \\ & & 1_H & \\ & & & A^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -1_H & & \\ & & \ddots & \\ & & & -1_H \\ A^{[0]} & A^{[1]} & \dots & A^{[d-1]} \end{pmatrix}.$$

Note that $\mathcal{C}^P = P$ if $d = 1$ and, for a monic polynomial P, the companion polynomial has the form

$$\mathcal{C}^P(z) = z - \begin{pmatrix} 0 & 1_H & & \\ & & \ddots & \\ & & & 1_H \\ -A^{[0]} & -A^{[1]} & \dots & -A^{[d-1]} \end{pmatrix}.$$

In this case, the block operator matrix on the right hand side is called the *companion operator of P*.

Remark 3.13. The companion polynomial $\mathcal{C}^P$ of P is a linearization of P (see, e. g., [Mar88, § 12.2]). In particular, $\sigma(P) = \sigma(\mathcal{C}^P)$, and for finite dimensional H we have the equivalence

$$\det P(z) = 0 \iff \det \mathcal{C}^P(z) = 0, \qquad z \in \mathbb{C}. \tag{3.7}$$

To be able to relate the block numerical ranges of an operator polynomial and its companion polynomial, we need another definition concerning decompositions of products of Hilbert spaces.

Definition 3.14. Let H_i be Hilbert spaces, $\mathcal{H}_i \in \mathcal{Z}(H_i)$, and $\pi_i \in \Pi(\mathcal{H}_i)$, $i = 1, 2$. Then define $\mathcal{H}_1 \times \mathcal{H}_2 \in \mathcal{Z}(H_1 \times H_2)$ and $\pi_1 \times \pi_2 \in \Pi(\mathcal{H}_1 \times \mathcal{H}_2)$ by

$$\begin{aligned}
\mathcal{H}_1 \times \mathcal{H}_2 &:= \{M_1 \times \{0_{H_2}\} \,:\, M_1 \in \mathcal{H}_1\} \cup \{\{0_{H_1}\} \times M_2 \,:\, M_2 \in \mathcal{H}_2\}, \\
\pi_1 \times \pi_2 &: \{1, \dots, |\mathcal{H}_1| + |\mathcal{H}_2|\} \to \mathcal{H}_1 \times \mathcal{H}_2, \\
(\pi_1 \times \pi_2)(k) &:= \begin{cases} \pi_1(k) \times \{0_{H_2}\}, & k \le |\mathcal{H}_1|, \\ \{0_{H_2}\} \times \pi_2(k - |\mathcal{H}_1|), & |\mathcal{H}_1| < k. \end{cases}
\end{aligned}$$

This definition generalizes to the case of d Hilbert spaces $H_1, \dots, H_d$. If, in this case, $H_1 = \dots = H_d = H$, $\mathcal{H}_1 = \dots = \mathcal{H}_d = \mathcal{H} \in \mathcal{Z}(H)$ and $\pi_1 = \dots = \pi_d = \pi \in \Pi(\mathcal{H})$, we write

$$\mathcal{H}^d := \mathcal{H}_1 \times \dots \times \mathcal{H}_d, \quad \pi^d := \pi_1 \times \dots \times \pi_d.$$

The following proposition is a generalization of [TW03, Theorem 5.1], where the case of a monic polynomial P and the trivial decomposition $\mathcal{H} = \{H\}$ was considered.

Proposition 3.15. *For an operator polynomial P of degree d on H and $\mathcal{H} \in \mathcal{Z}(H)$ the inclusion*

$$W_{\mathcal{H}}(P) \subset W_{\mathcal{H}^d}(\mathcal{C}^P)$$

holds.

Proof. Let $n := |\mathcal{H}|$. Fix $\pi \in \Pi(\mathcal{H})$ and write $A_x^{[l]} = A_{x,\pi}^{[l]}$, $l = 1, \dots, d$, $x \in H$. For $x \in H$ and $z \in \mathbb{C}$ we have

$$P_x^\pi(z) = A_x^{[d]} z^d + \dots + A_x^{[0]} \in M_n(\mathbb{C}),$$

and

$$\mathcal{C}^{P_x^\pi}(z) = \begin{pmatrix} 1_{\mathbb{C}^n} & & & \\ & \ddots & & \\ & & 1_{\mathbb{C}^n} & \\ & & & A_x^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -1_{\mathbb{C}^n} & & \\ & & \ddots & \\ & & & -1_{\mathbb{C}^n} \\ A_x^{[0]} & A_x^{[1]} & \dots & A_x^{[d-1]} \end{pmatrix} \in M_{nd}(\mathbb{C}).$$

Moreover,

$$\mathcal{C}^{P}(z) = \begin{pmatrix} 1_H & & & \\ & \ddots & & \\ & & 1_H & \\ & & & A^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -1_H & & \\ & & \ddots & \\ & & & -1_H \\ A^{[0]} & A^{[1]} & \dots & A^{[d-1]} \end{pmatrix} \in L(H^d)$$

by definition. If, in addition, $x \in \mathcal{H}^{\square}$, then $x^d := (x, \ldots, x) \in \mathcal{H}^d$ and it follows that

$$\mathcal{C}^P_{x^d,\pi^d}(z) = \begin{pmatrix} 1_{\mathbb{C}^n} & & & \\ & \ddots & & \\ & & 1_{\mathbb{C}^n} & \\ & & & A_x^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -1_{\mathbb{C}^n} & & \\ & & \ddots & \\ & & & -1_{\mathbb{C}^n} \\ A_x^{[0]} & A_x^{[1]} & \ldots & A_x^{[d-1]} \end{pmatrix}.$$

Thus, we have the equality

$$\mathcal{C}^{P_x^\pi}(z) = \mathcal{C}^P_{x^d,\pi^d}(z), \quad z \in \mathbb{C}, \; x \in \mathcal{H}^{\square}.$$

In particular, if $\lambda \in W_{\mathcal{H}}(P)$, i. e., $\det P_x^\pi(\lambda) = 0$ for some $x \in \mathcal{H}^{\square}$, it follows from (3.7), applied to the matrix polynomial P_x^π, that $0 = \det \mathcal{C}^{P_x^\pi}(\lambda) = \det \mathcal{C}^P_{x^d,\pi^d}(\lambda)$ and therefore $\lambda \in W_{\mathcal{H}^d}(\mathcal{C}^P)$. $\square$

Example 3.16. The block numerical range of the matrix polynomial

$$P(z) = z^2 + \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) z + \left(\begin{array}{cc|cc|cc} -3 & 0 & 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right), \quad z \in \mathbb{C}, \tag{3.8}$$

with respect to the decomposition $\mathcal{H} = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ and the block numerical range of its companion polynomial with respect to the corresponding decomposition $\mathcal{H}^2 = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ are shown in Figure 3.3.

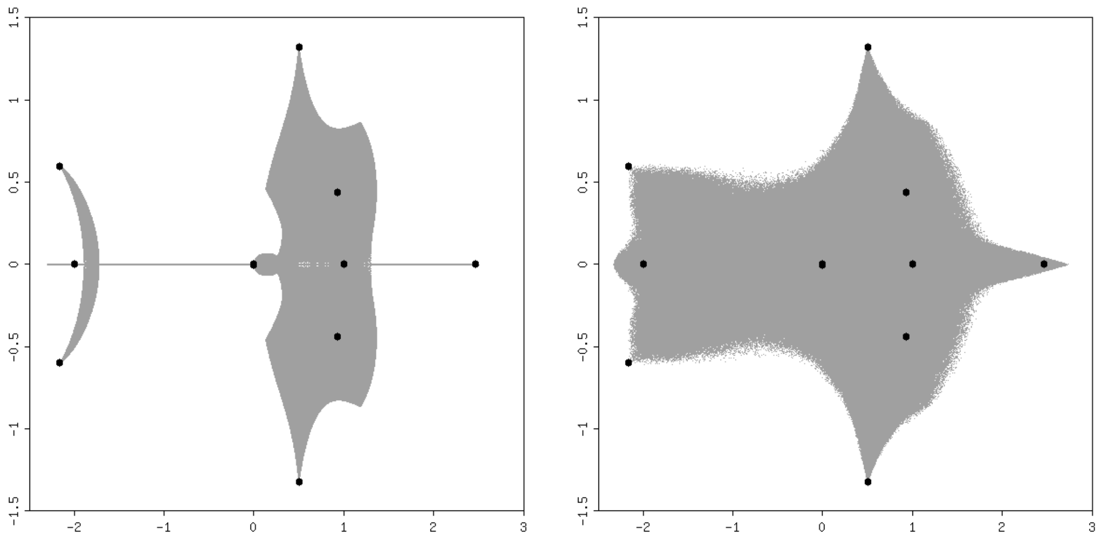


Figure 3.3: A block numerical range and the corresponding block numerical range of its companion operator (see Example 3.16).

Chapter 4

Block-diagonalization of operators

Let H_1, H_2 be Hilbert spaces, $\mathcal{H} := H_1 \times H_2$ and the block operator matrix $\mathcal{A} \in L(\mathcal{H})$ be given by

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It is known (see [ALT01, Lemma 2.1]) that an operator $K \in L(H_1, H_2)$ is a solution of the Riccati equation

$$KBK + KA - DK - C = 0 \tag{4.1}$$

if and only if its graph subspace

$$\mathcal{G}(K) := \left\{ \begin{pmatrix} x_1 \\ Kx_1 \end{pmatrix} : x_1 \in H_1 \right\} \subset H_1 \times H_2 \tag{4.2}$$

is $\mathcal{A}$-invariant. It is also known that the existence of solutions of the Riccati equation (4.1) is strongly related to a factorization of the Schur complement

$$S_A(z) = A - z - B(D - z)^{-1}C, \quad z \in \rho(D),$$

of the form

$$S_A(z) = T(z)(Z - z), \quad z \in \rho(D),$$

where $Z \in L(H_1)$, $\sigma(Z) \cap \sigma(D) = \emptyset$ and $T : \rho(D) \to L(H_1)$ is an analytic operator function (see [ALT01, Theorem 2.2]).

In this chapter we follow the lines of [LMMT01, Section 5], where it has been shown that the Riccati equation (4.1) has a solution K_1 if the closure $\overline{W^2(\mathcal{A})}$ of the quadratic numerical range of $\mathcal{A}$ consists of two connected components, and that the graph subspace of this solution coincides with the spectral subspace of $\mathcal{A}$ with respect to the spectrum contained in a connected component of $\overline{W^2(\mathcal{A})}$ (see [LMMT01, Theorem 5.1]). Moreover, in this case also the second Riccati equation corresponding to $\mathcal{A}$,

$$KCK + KD - AK - B = 0,$$

has a solution K_2, and $\mathcal{A}$ admits a block diagonalization with respect to $\mathcal{H} = H_1 \times H_2$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & K_2 \\ K_1 & 1 \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \begin{pmatrix} 1 & K_2 \\ K_1 & 1 \end{pmatrix}^{-1},$$

If $\overline{W^2(\mathcal{A})}$ is connected, this theorem is not applicable anymore. However, choosing a finer decomposition $\widetilde{\mathcal{H}} = \{H_1, H_{21}, \dots, H_{2m}\}$, where $H_2 = H_{21} \times \cdots \times H_{2m}$, and using the block numerical range of $\mathcal{A}$ with respect to this decomposition, there is, as $W_{\widetilde{\mathcal{H}}}(\mathcal{A}) \subset W^2(\mathcal{A})$, still a chance to $\widetilde{\mathcal{H}}$-separate a connected component of $W_{\widetilde{\mathcal{H}}}(\mathcal{A})$ and, as we will see, obtain a solution of (4.1).

Remark 4.1. Let, for example, H_2 be decomposed as $H_2 = H_{21} \times H_{22}$, and consider the refined decomposition $\widetilde{\mathcal{H}} = H_1 \times H_{21} \times H_{22}$ of $\mathcal{H} = H_1 \times H_2$. As a first guess one might suspect that, if $\overline{W_{\widetilde{\mathcal{H}}}(\mathcal{A})}$ consists of 3 connected components, then $\overline{W^2(\mathcal{A})}$ consists of 2 connected components – applying [LMMT01, Theorem 5.1] would then yield a solution of the Riccati equation (4.1). However, this guess is false. If, for example, $\mathcal{A}$ has the representation

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to $\widetilde{\mathcal{H}}$, then $W_{\mathcal{H}}(\mathcal{A}) = [-1, 1]$ is connected, although $W_{\widetilde{\mathcal{H}}}(\mathcal{A}) = \{-1, 0, 1\}$ has 3 connected components.

4.1 Schur complements

Let $\mathcal{H} \in \mathcal{Z}(H)$, $F : \Omega \to L(H)$ be an operator function and $M \subset H$ be a closed subspace of H. Then, with respect to the decomposition $H = M^\perp \times M$, F has the following block form:

$$F = \begin{pmatrix} F_{M^\perp} & F_{M^\perp M} \\ F_{MM^\perp} & F_M \end{pmatrix}.$$

Definition 4.2. The *Schur complement* $[F]_M$ *of* F *with respect to* M is defined to be the operator function

$$[F]_M : \rho(F_{M^\perp}) \to L(M), \quad [F]_M := F_M - F_{MM^\perp} F_{M^\perp}^{-1} F_{M^\perp M}.$$

For $A \in L(H)$ the Schur complement is then defined in the usual way by

$$[A]_M := [P_A]_M : \rho(A_{M^\perp}) \to L(M), \quad [A]_M(z) = A_M - z - A_{MM^\perp}(A_{M^\perp} - z)^{-1} A_{M^\perp M},$$

where $P_A(z) = A - z$, $z \in \mathbb{C}$. Note that with this definition we have

$$[F]_M(z) = [F(z)]_M(0), \quad z \in \mathbb{C}.$$

Remark 4.3. With respect to the decomposition $H = M^\perp \times M$ of H, the Schur–Frobenius factorization, that is, the identity

$$F = \begin{pmatrix} 1 & 0 \\ F_{MM^\perp}F_{M^\perp}^{-1} & 1 \end{pmatrix} \begin{pmatrix} F_{M^\perp} & 0 \\ 0 & [F]_M \end{pmatrix} \begin{pmatrix} 1 & F_{M^\perp}^{-1}F_{M^\perp M} \\ 0 & 1 \end{pmatrix}, \tag{4.3}$$

holds on $\rho(F_{M^\perp})$. In particular, we have $\rho(F_{M^\perp}) \cap \sigma(F) = \sigma([F]_M)$.

Theorem 4.4. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be an operator function. Then*

$$W([F]_M) \subset W_{\mathcal{H}}(F), \quad M \in \mathcal{H}.$$

Proof. Let $M \in \mathcal{H}$ and write $\mathcal{H} = \{M_1, \ldots, M_n\}$ such that $M = M_n$. Moreover, let $\lambda_0 \in W([F]_M)$ and $y \in M \setminus \{0\}$ such that $([F]_M(\lambda_0)y, y) = 0$, and define

$$x := y - F_{M^\perp}^{-1}(\lambda_0)F_{M^\perp M}(\lambda_0)y \in H, \tag{4.4}$$

and $J := \{j \in \{1, \ldots, n\} : x_j \neq 0\}$ (note that $n \notin J$). Abbreviating $A := F(\lambda_0)$, the factorization (4.3) yields that

$$\begin{aligned} Ax &= \begin{pmatrix} 1 & 0 \\ A_{MM^\perp}A_{M^\perp}^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_{M^\perp} & 0 \\ 0 & [A]_M(0) \end{pmatrix} \begin{pmatrix} 1 & A_{M^\perp}^{-1}A_{M^\perp M} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -A_{M^\perp}^{-1}A_{M^\perp M}y \\ y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ A_{MM^\perp}A_{M^\perp}^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_{M^\perp} & 0 \\ 0 & [A]_M(0) \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ [A]_M(0)y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ [F]_M(\lambda_0)y \end{pmatrix} \end{aligned}$$

with respect to the decomposition $H = M^\perp \times M$; that is, $P_nAx = [F]_M(\lambda_0)y$ and $P_iAx = 0$, $i = 1, \ldots, n$. Now define $\xi \in \mathcal{H}^*$ by

$$\xi_i := \begin{cases} \hat{x}_i, & x_i = 0 \text{ (i. e., } i \notin J), \\ x_i, & x_i \neq 0 \text{ (i. e., } i \in J), \end{cases} \qquad i = 1, \ldots, n,$$

where the vectors $\hat{x}_i \in M_i \setminus \{0\}$, $i = 1, \ldots, n$, are chosen arbitrarily. We have

$$\Big(\sum_{j\in J} A_{ij}\xi_j, \xi_i\Big) = \Big(\sum_{j=1}^{n} A_{ij}x_j, \xi_i\Big) = (P_iAx, \xi_i) = \begin{cases} 0, & i < n, \\ \big([F]_M(\lambda_0)y, y\big), & i = n. \end{cases}$$

It follows that

$$\det F_\xi(\lambda_0) = \left| \begin{pmatrix} (A_{11}\xi_1, \xi_1) & \cdots & (A_{1n}\xi_n, \xi_1) \\ \vdots & & \vdots \\ (A_{n1}\xi_1, \xi_n) & \cdots & (A_{nn}\xi_n, \xi_n) \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} (A_{11}\xi_1, \xi_1) & \cdots & (A_{1,n-1}\xi_{n-1}, \xi_1) & \Big(\sum_{j\in J} A_{1j}\xi_j, \xi_1\Big) \\ \vdots & & \vdots & \vdots \\ (A_{n-1,1}\xi_1, \xi_{n-1}) & \cdots & (A_{n-1,n-1}\xi_{n-1}, \xi_{n-1}) & \Big(\sum_{j\in J} A_{n-1,j}\xi_j, \xi_{n-1}\Big) \\ (A_{n1}\xi_1, x_n) & \cdots & (A_{n,n-1}\xi_{n-1}, \xi_n) & \big([F]_M(\lambda_0)y, y\big) \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} (A_{11}\xi_1, \xi_1) & \cdots & (A_{1,n-1}\xi_{n-1}, \xi_1) & 0 \\ \vdots & & \vdots & \vdots \\ (A_{n-1,1}\xi_1, \xi_{n-1}) & \cdots & (A_{n-1,n-1}\xi_{n-1}, \xi_{n-1}) & 0 \\ (A_{n1}\xi_1, x_n) & \cdots & (A_{n,n-1}\xi_{n-1}, \xi_n) & \big([F]_M(\lambda_0)y, y\big) \end{pmatrix} \right|$$
$$= \big([F]_M(\lambda_0)y, y\big) \det F_\xi^{\mathcal{H}\setminus\{M\}}(\lambda_0) = 0,$$

thus $\lambda_0 \in W_{\mathcal{H}}(F)$. □

Open Question. Let $\mathcal{H} \in \mathcal{Z}(H)$, $\mathcal{H}' \subset \mathcal{H}$ and $F : \Omega \to L(H)$ be an operator function. Does then the inclusion

$$W_{\mathcal{H}'}([F]_{\mathcal{H}'}) \subset W_{\mathcal{H}}(F)$$

hold?

Recall that for an analytic operator function $F : \Omega \to L(H)$ also the inverse operator function $F^{-1} : \rho(F) \to L(H)$ is analytic. In particular, for $\mathcal{H} \in \mathcal{Z}(H)$ and $M \in \mathcal{H}$, the Schur complement $[F]_M : \rho(F_{M^\perp}) \to L(M)$ is an analytic operator function.

Corollary 4.5. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$, $F : \Omega \to L(H)$ be analytic and $M \in \mathcal{H}$.*

(1) *The condition* (2.7) *holds for $[A]_M$. In particular, $[A]_M$ fulfills* (2.6) *and we have the spectral inclusion*

$$\sigma([A]_M) \subset \overline{W([A]_M)} \subset \overline{W_{\mathcal{H}}(A)}.$$

(2) *If $z_0 \in \Omega$ is such that $0 \notin \overline{W_{\mathcal{H}}(F(z_0))}$, then also $0 \notin \overline{W([F]_M(z_0))}$. In particular, if condition* (2.7) *holds for F, it also holds for $[F]_M$ and, in this case, we have*

$$\sigma([F]_M) \subset \overline{W([F]_M)} \subset \overline{W_{\mathcal{H}}(F)}.$$

Proof. (1) We have

$$\begin{aligned} \big\|[A]_M(z) + z\big\| &= \big\|A_M - A_{MM^\perp}(A_{M^\perp} - z)^{-1}A_{M^\perp M}\big\| \\ &\le \|A_M\| + \big\|A_{MM^\perp}\big\|\big\|(A_{M^\perp} - z)^{-1}\big\|\big\|A_{M^\perp M}\big\|, \end{aligned}$$

which tends to $\|A_M\|$ for $z \to \infty$ (since $\|(A_{M^\perp} - z)^{-1}\| \to 0$), hence, for $y \in M$, $\|y\| = 1$,

$$\big|([A]_M(z)y, y)\big| \ge |z| - \big|(([A]_M(z) + z)y, y)\big| \ge |z| - \big\|[A]_M(z) + z\big\| \ge C > 0,$$

for some $C > 0$ if $|z|$ is large enough. In particular, $0 \notin \overline{W([A]_M(z))}$ for such z.

(2) Let $A := F(z_0)$. From $W([A]_M) \subset W_{\mathcal{H}}(A)$ and $0 \notin \overline{W_{\mathcal{H}}(A)}$ it follows that $0 \notin \overline{W([A]_M)}$. In particular, by (2.6) for $z = 0$ (which holds for $[A]_M$ by (1)),

$$0 \notin \overline{W([A]_M(0))} = \overline{W([F(z_0)]_M(0))} = \overline{W([F]_M(z_0))},$$

as desired. □

4.2 Factorization of Schur complements

The aim of the current section is to prove, under some appropriate assumptions on A and $W_{\mathcal{H}}(A)$, respectively, a factorization of the Schur complement $[A]_M$ of the form

$$[A]_M(z) = T_M(z)(Z_M - z), \quad z \in \rho(A_{M^\perp}),$$

where $Z_M \in L(M)$, $\sigma(Z_M) = \sigma(A) \cap \overline{\kappa(M)}$ and $T_M : \rho(A_{M^\perp}) \to L(M)$ is an analytic operator function. (Recall that if the dimension condition (1.6) holds, then $\kappa(M)$ was defined to be the connected component of $W_{\mathcal{H}}(A)$ such that $W(A_M) \subset \kappa(M)$; see Lemma 1.16 (2).)

The following theorem from [MM75] on the factorization of analytic functions plays a crucial role in this factorization. It is cited here because of its importance and for the convenience of the reader.

Theorem 4.6. *Assume that the boundary of $U \subset \mathbb{C}$ is a piecewise smooth Jordan curve Γ and that $F : \overline{U} \to L(H)$ is continuous and analytic on U. If moreover*

$$\inf \{ |(F(z)x, x)| \,:\, z \in \Gamma,\ x \in H,\ \|x\| = 1 \} > 0$$

and $z \mapsto (F(z)x, x)$ has exactly one zero in U for some $x \in H \setminus \{0\}$ (thus, for any $x \in H \setminus \{0\}$ by [Mar88, Lemma 26.8]*), then there exist a continuous operator function $T : \overline{U} \to L(H)$ such that T is analytic on U and $\sigma(T) = \emptyset$, and an operator $Z \in L(H)$ with $\sigma(Z) \subset U$ such that*

$$F(z) = T(z)(Z - z), \quad z \in \overline{U}.$$

In addition, the operator

$$\int_\Gamma F^{-1}(z)\,\mathrm{d}z \in L(H)$$

is invertible.

Remark 4.7. Let $\mathcal{H} \in \mathcal{Z}(H)$, $M \in \mathcal{H}$ and $A \in L(H)$. Additionally assume that the dimension condition (1.6) holds and $\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ (see Definition 1.22). Let $\Gamma_M \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$ be a piecewise smooth Jordan curve such that $\operatorname{int}\Gamma_M \cap W_{\mathcal{H}}(A) = \kappa(M)$ and let $U_M := \operatorname{int}\Gamma_M$. Then (1.14) yields

$$\sigma(A_{M^\perp}) = \sigma(A_{\mathcal{H}\setminus\{M\}}) \subset \overline{W_{\mathcal{H}\setminus\{M\}}(A)} \subset \overline{W_{\mathcal{H}}(A) \setminus \kappa(M)},$$

thus, $[A]_M$ is defined on $\overline{U}_M$ (which implies that it is continuous on $\overline{U}_M$ and analytic on U_M). Moreover, we have seen in Corollary 4.5, that (2.7) holds for $[A]_M$; in particular, as $\Gamma_M \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)} \subset \mathbb{C} \setminus \overline{W([A]_M)}$ by Theorem 4.4, Corollary 2.16 yields

$$\inf \{ ([A]_M(z)y, y) \,:\, z \in \Gamma_M,\ y \in M,\ \|y\| = 1 \} > 0.$$

If we are able to show that the functions $z \mapsto ([A]_M(z)y, y)$, $y \in M \setminus \{0\}$, have exactly one zero in U_M, or, equivalently, $\nu_{\kappa(M)}([A]_M(\,\cdot\,)y, y) = 1$, $y \in M \setminus \{0\}$, because $W([A]_M) \cap U_M \subset W_{\mathcal{H}}(A) \cap U_M = \kappa(M)$, all conditions of Theorem 4.6 are fulfilled, and the desired factorization of the Schur complement $[A]_M$ is possible: There exist a continuous operator function $T : \overline{U}_M \to L(M)$ with $\sigma(T) = \emptyset$ which is analytic on U_M, and an operator $Z_M \in L(M)$ with $\sigma(Z_M) \subset U_M$ such that

$$[A]_M(z) = T(z)(Z_M - z), \quad z \in \overline{U}_M. \tag{4.5}$$

Clearly, $\sigma(Z_M) = \sigma([A]_M) \cap U_M = \sigma(A) \cap \rho(A_{M^\perp}) \cap U_M = \sigma(A) \cap \overline{\kappa(M)}$. Moreover, the definition

$$T_M : \rho(A_{M^\perp}) \to L(M), \quad T_M(z) := \begin{cases} T(z), & z \in \overline{U}_M, \\ [A]_M(z)(Z_M - z), & z \in \rho(A_{M^\perp}) \setminus \overline{U}_M, \end{cases}$$

makes T_M an analytic function such that $\sigma(T_M) \subset \overline{W_{\mathcal{H}}(A)} \setminus \overline{\kappa(M)}$ and (4.5) continues to hold with T replaced by T_M and $\overline{U}_M$ by $\rho(A_{M^\perp})$, respectively. Thus, we have shown the following theorem which, in combination with Theorem 4.23, will play the key role in solving Riccati equations connected to an operator A and a decomposition $\mathcal{H}$ in the following section.

Theorem 4.8. *Let $\mathcal{H} \in \mathcal{Z}(H)$ be such that the dimension condition* (1.6) *holds, $M \in \mathcal{H}$ and $A \in L(H)$. Assume that that $\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ and Γ_M is a piecewise smooth Jordan curve $\mathcal{H}$-separating $\kappa(M)$ in $W_{\mathcal{H}}(A)$. If, additionally,*

$$\nu_{\kappa(M)}([A]_M(\,\cdot\,)y, y) = 1, \quad y \in M \setminus \{0\}, \tag{4.6}$$

then the operator

$$\int_{\Gamma_M} [A]_M^{-1}(z)\,\mathrm{d}z \in L(M)$$

is invertible and there exist an analytic operator function $T_M : \rho(A_{M^\perp}) \to L(M)$ with $\sigma(T_M) \subset \overline{W_{\mathcal{H}}(A)} \setminus \overline{\kappa(M)}$ and an operator $Z_M \in L(M)$ with $\sigma(Z_M) = \sigma(A) \cap \overline{\kappa(M)}$ such that

$$[A]_M(z) = T_M(z)(Z_M - z), \quad z \in \rho(A_{M^\perp}).$$

The remaining part of this section is dedicated to find classes of block operator matrices for which (4.6) holds. We anticipate the most important results in the following theorem.

Theorem 4.9. *Let H be a separable Hilbert space, $\mathcal{H} \in \mathcal{Z}(H)$ fulfill the dimension condition* (1.6) *and $A \in L(H)$.*

(1) *If $M \in \mathcal{H}$ is such that $\kappa(M)$ is strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$, then* (4.6) *holds.* [Theorem 4.14.]

(2) *If $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, then* (4.6) *holds for all $M \in \mathcal{H}$ if one of the following conditions holds:*

   (a) $|\mathcal{H}| = 2$.

   (b) $A = A^*$. [Theorem 4.17.]

   (c) $\dim M = \infty$, $M \in \mathcal{H}$. [Theorem 4.19.]

That condition (2a) implies (4.6) was shown in [LMMT01] (see Remark 1.21). To prove the remaining assertions, we first restrict ourselves to the finite dimensional case and make use of the following theorem from [KMM93].

Theorem 4.10. *Let $\Omega \subset \mathbb{C}$ be a domain and $F : \Omega \to M_n(\mathbb{C})$ be analytic. If $\Gamma \subset \Omega \setminus W(F)$ is a Jordan curve such that $U := \operatorname{int}\Gamma \subset \Omega$, then*

$$\nu_U(\det F) = n \cdot \nu_U(F(\,\cdot\,)x, x), \quad x \in \mathbb{C}^n \setminus \{0\}.$$

Remark 4.11. Assume that in the situation of Remark 4.7 additionally H is finite dimensional and let $d_M := \dim_M$. Applying Theorem 4.10 to $\Omega = \rho(A_{M^\perp})$, $F = [A]_M$ and $\Gamma = \Gamma_M$, we get

$$\nu_{\kappa(M)}(\det[A]_M) = \nu_{U_M}(\det[A]_M) = d_M \cdot \nu_{U_M}([A]_M(\,\cdot\,)y, y) = d_M \cdot \nu_{\kappa(M)}([A]_M(\,\cdot\,)y, y) \tag{4.7}$$

for all $y \in M$. Then, using that $\kappa(M) \subset \rho(A_{M^\perp})$, the representation (4.3) yields

$$\det(A - z) = \det(A_{M^\perp} - z)\det[A]_M(z), \quad z \in \kappa(M).$$

Therefore, A and $[A]_M$ have the same eigenvalues including multiplicities in $\kappa(M)$, that is, $\nu_{\kappa(M)}(A) = \nu_{\kappa(M)}(\det[A]_M)$. By (4.7) we therefore conclude

$$\nu_{\kappa(M)}(A) = \dim M \cdot \nu_{\kappa(M)}([A]_M(\,\cdot\,)y, y), \quad y \in M \setminus \{0\}. \tag{4.8}$$

Proposition 4.12. *Let H be finite dimensional, $\mathcal{H} \in \mathcal{Z}(H)$ such that* (1.6) *holds and $A \in L(H)$.*

(1) *Let $M \in \mathcal{H}$. Then, each of the following conditions implies* (4.6)*:*
   - (a) *$\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ and A has $\dim M$ eigenvalues in $\kappa(M)$ counting multiplicities.*
   - (b) *$\kappa(M)$ is strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$.*

(2) *If $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, then* (4.6) *holds for all $M \in \mathcal{H}$ if one of the following conditions holds:*
   - (a) *$W([A]_M) \cap \kappa(M) \neq \emptyset$, $M \in \mathcal{H}$.*
   - (b) *$A = A^*$.*
   - (c) *The equation*

$$\sum_{M\in\mathcal{H}} \nu_M \dim M = \sum_{M\in\mathcal{H}} \dim M$$

   in the variables $\nu_M \in \mathbb{N}_0$, $M \in \mathcal{H}$, only has the trivial solution $\nu_M = 1$, $M \in \mathcal{H}$.

Proof. (1) Note first, that (1b) $\Rightarrow$ (1a) by Proposition 1.26 and (1a) $\Rightarrow$ (4.6) follows immediately from (4.8). Moreover, we have (2b) $\Rightarrow$ (2a) by Proposition 4.16.
(2) To prove the remaining implications, assume that $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, that is, all connected components of $W_{\mathcal{H}}(A)$ are $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$. Then (4.8) holds

for all $M \in \mathcal{H}$. As $\sigma_{\mathrm{p}}(A) \subset W_{\mathcal{H}}(A)$, every eigenvalue of A is located in some $\kappa(M)$ and, because A has $\dim H$ eigenvalues counting multiplicities, we get

$$\sum_{M\in\mathcal{H}} \dim M = \dim H = \sum_{M\in\mathcal{H}} \nu_{\kappa(M)}(A) = \sum_{M\in\mathcal{H}} \dim M \cdot \nu_{\kappa(M)}([A]_M(\,\cdot\,)y_M, y_M), \quad (4.9)$$

for arbitrary vectors $y_M \in M \setminus \{0\}$, $M \in \mathcal{H}$. Now, obviously (2c) implies (4.6). Finally, if (2a) holds, then for every $M \in \mathcal{H}$ there is a $y_M \in M \setminus \{0\}$ such that $\nu_{\kappa(M)}([A]_M(\,\cdot\,)y_M, y_M) \geq 1$, thus $\nu_{\kappa(M)}([A]_M(\,\cdot\,)y_M, y_M) \geq 1$ for every $y_M \in M \setminus \{0\}$ and $M \in \mathcal{H}$. Again, (4.6) follows from (4.9). □

Remark 4.13. Assume that H is a separable Hilbert space and let $\mathcal{H} \in \mathcal{Z}(H)$. Then there exists an increasing sequence $(H_k)_1^\infty$ of finite dimensional subspaces of H such that $\overline{\operatorname{span}\{H_k : k \in \mathbb{N}\}} = H$ and $H_1 \cap M \neq \{0\}$, $M \in \mathcal{H}$. Let $M_k := H_k \cap M$, $M \in \mathcal{H}$, $k \in \mathbb{N}$. Then $\mathcal{H}_k := \{M_k : M \in \mathcal{H}\} \in \mathcal{Z}(H_k)$, $k \in \mathbb{N}$. It is easy to see that, for $A \in L(H)$ and $A^{(k)} := A_{H_k} \in L(H_k)$, $k \in \mathbb{N}$, the inclusions $W_{\mathcal{H}_k}(A^{(k)}) \subset W_{\mathcal{H}}(A)$, $k \in \mathbb{N}$, hold. Moreover, if (1.6) holds for $\mathcal{H}_1$ (and thus also for $\mathcal{H}$ and $\mathcal{H}_k$, $k \in \mathbb{N}$), we even have $\kappa_k(M_k) \subset \kappa(M)$, where $\kappa_k(M_k)$ denotes the connected component of $W_{\mathcal{H}_k}(A^{(k)})$ which contains $W((A^{(k)})_{M_k})$.

Now let $M \in \mathcal{H}$ be such that $\kappa(M)$ is (strongly) $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ and Γ_M be a Jordan curve (strongly) $\mathcal{H}$-separating $\kappa(M)$ in $W_{\mathcal{H}}(A)$. Obviously, Γ_M is (strongly) $\mathcal{H}_k$-separating $\kappa_k(M_k)$ in $W_{\mathcal{H}_k}(A^{(k)})$ for every $k \in \mathbb{N}$. It has been shown in the proof of [LMMT01, Theorem 4.4] that

$$\sup_{z\in\Gamma} \left|([A^{(k)}]_{M_k}(z)y, y) - ([A]_M(z)y, y)\right| \to 0, \quad k \to \infty,$$

for every $y \in M_1 \subset M$. In fact, the only situation where properties of the quadratic numerical range of A are used in the proof of this theorem, is the uniform estimate of the resolvent norms, $\|(A_{M^\perp} - z)^{-1}\|$, $\|(A^{(k)}_{M_k^\perp} - z)^{-1}\|$ for $z \in \Gamma$, $k \in \mathbb{N}$. But this estimate is also valid in the general case considered here by Theorem 1.5: As $\Gamma_M \subset \mathbb{C} \setminus \overline{W_{\mathcal{H}}(A)}$ and $W_{\mathcal{H}_k}(A^{(k)}) \subset W_{\mathcal{H}}(A)$, $k \in \mathbb{N}$, we have

$$\|(A^{(k)}_{M_k^\perp} - z)^{-1}\|, \ \|(A_{M^\perp} - z)^{-1}\| \leq \frac{K}{\operatorname{dist}(\Gamma_M, W_{\mathcal{H}}(A))}, \quad z \in \Gamma_M, \ k \in \mathbb{N},$$

where $K := \max\{(\|A\| + |z|)^{|\mathcal{H}|-1} : z \in \Gamma_M\}$. The remainder of the proof may be adopted literally. From Hurwitz' Theorem it follows that the analytic functions $([A^{(k)}]_{M_k}(\,\cdot\,)x, x)$ and $([A]_M(\,\cdot\,)x, x)$ have the same number of zeros in $\operatorname{int}\Gamma$ for k large enough, that is,

$$\nu_{\kappa(M)}([A]_M(\,\cdot\,)x, x) = \lim_{k\to\infty} \nu_{\kappa_k(M_k)}([A^{(k)}]_{M_k}(\,\cdot\,)x, x). \quad (4.10)$$

Theorem 4.14. *Let H be separable, $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $M \in \mathcal{H}$ be such that $\kappa(M)$ is strongly $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$. Then* (4.6) *holds.*

Proof. Let $M \in \mathcal{H}$, $y \in M$ and $(H_k)_1^\infty$ be a sequence of finite dimensional subspaces of H as in Remark 4.13 such that $y \in H_1$. Then $\kappa_k(M_k)$ is strongly $\mathcal{H}_k$-separated in $W_{\mathcal{H}_k}(A^{(k)})$ for all $k \in \mathbb{N}$. Applying Proposition 4.12 (1b) and (4.10) yields

$$\nu_{\kappa(M)}([A]_M(\,\cdot\,)y, y) = \lim_{k\to\infty} \nu_{\kappa_k(M_k)}([A^{(k)}]_{M_k}(\,\cdot\,)y, y) = 1.$$ □

Thus, we have shown Theorem 4.9 (1). Considering condition (2b), $A = A^*$, we need a remark about the signs of the polynomials $\det(A-z)_x$ whose zeros determine $W_{\mathcal{H}}(A)$.

Remark 4.15. Let $\mathcal{H} \in \mathcal{Z}'(H)$, $M \in \mathcal{H}$ and $A \in L(H)$ be selfadjoint. It follows from $W_{\mathcal{H}}(A) \subset \mathbb{R}$ (see Proposition 1.3) that the connected components $C_1, \dots, C_s$ of $W_{\mathcal{H}}(A)$ are intervals. Assume that these intervals are ordered from left to right, that is, if we write $\overline{C}_i =: [a_i, b_i]$, $i = 1, \dots, s$, we have $b_i \le a_{i+1}$, $i = 1, \dots, s-1$. Moreover, let $\mathbb{R} \setminus W_{\mathcal{H}}(A) =: \omega_0 \,\dot\cup \dots \dot\cup\, \omega_{s+1}$ where the ω_i are intervals such that $\overline{\omega}_0 = (-\infty, a_1]$, $\overline{\omega}_i = [b_i, a_{i+1}]$, $i = 1, \dots, s-1$ and $\overline{\omega}_{s+1} = [b_s, \infty)$. For each $x \in \mathcal{H}^*$, the real valued polynomial

$$p_x : \mathbb{R} \to \mathbb{R}, \quad p_x(t) := \det(A-t)_x,$$

has degree $n := |\mathcal{H}|$ and its leading coefficient has the sign $(-1)^n$; in particular, we have $p_x(t) \to \infty$ for $t \to -\infty$. Its roots are located in $C_1 \cup \dots \cup C_s$, and for each $i \in \{1, \dots, s\}$ the number ν_i of roots of p_x in C_i does not depend on $x \in \mathcal{H}^*$. Therefore, it is clear that for $i \in \{0, \dots, s+1\}$ the sign of p_x on ω_i is given by

$$\operatorname{sgn} p_x(t) = (-1)^{\nu_1 + \dots + \nu_i}, \quad t \in \omega_i,$$

again independently of $x \in \mathcal{H}^*$.

Proposition 4.16. *Let $\mathcal{H} \in \mathcal{Z}(H)$ be such that* (1.6) *holds, $M \in \mathcal{H}$ and $A \in L(H)$ be selfadjoint. If $\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$, then*

$$W([A]_M) \cap \kappa(M) \ne \emptyset. \tag{4.11}$$

More precisely, for every $y \in M \setminus \{0\}$ the function $z \mapsto ([A]_M(\,\cdot\,)y, y)$ has at least one zero in $\kappa(M)$.

Proof. Let $\mathcal{H} = \{M_1, \dots, M_n\}$ be such that $M = M_n$ and $[a, b] := \overline{\kappa(M)}$. As $\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$, there is an interval $[\alpha, \beta] \subset \mathbb{R}$ such that

$$[\alpha, \beta] \cap \overline{W_{\mathcal{H}}(A)} = \overline{\kappa(M)} = [a, b] \subset (\alpha, \beta).$$

Now let $y \in M \setminus \{0\}$ be arbitrary. Similar to Equation (4.4) in the proof of Theorem 4.4, let $x : \rho(A_{M^\perp}) \to H$ be defined by

$$x(z) := y - (A_{M^\perp} - z)^{-1} A_{M^\perp M} y, \quad z \in \rho(A_{M^\perp}),$$

and $J := \{j \in \{1, \dots, n\} \,:\, x_j \not\equiv 0\}$ (note that $x_n \equiv y \ne 0$, thus $n \in J$). Moreover, define $\xi : \rho(A_{M^\perp}) \to H$ by

$$\xi_i(z) := \begin{cases} \hat{x}_i, & i \notin J, \\ x_i(z), & i \in J, \end{cases} \qquad z \in \rho(A_{M^\perp}),\ i = 1, \dots, n,$$

where $\hat{x}_i \in M_i \setminus \{0\}$ are arbitrary constant vectors. Then the ξ_i, $i = 1, \dots, n$, are analytic functions which do not vanish identically; in particular, they have only finitely many zeros in $[\alpha, a] \cup [b, \beta] \subset [\alpha, \beta] \subset \rho(A_{M^\perp})$. We conclude that there are $t_- \in (\alpha, a)$ and $t_+ \in (b, \beta)$ such that $\xi_i^\pm := \xi_i(t_\pm) \neq 0$, $i = 1, \dots, n$, that is, $\xi^\pm \in \mathcal{H}^*$. Calculations analogous to those in the proof of Theorem 4.4 (replace λ_0 by $t_\pm$) show that

$$([A]_M(t_\pm)y, y) = \frac{\det(A - t_\pm)_{\xi^\pm}}{\det(A - t_\pm)_{\xi^\pm}^{\mathcal{H}\setminus\{M\}}}.$$

As $[\alpha, \beta] \subset \mathbb{C} \setminus W_{\mathcal{H}\setminus\{M\}}(A)$ is an interval and by what has been said in Remark 4.15, the sign $s := \operatorname{sgn}\det(A - t)_\eta^{\mathcal{H}\setminus\{M\}} \neq 0$ does not depend on $\eta \in \mathcal{H}^*$ and $t \in [\alpha, \beta]$. As $\kappa(M)$ is $\mathcal{H}$-separated, the polynomial $t \mapsto \det(A_x - t)$ has exactly one zero in $\kappa(M)$. Hence, if we set $s_\pm := \operatorname{sgn}\det(A - t_\pm)_{\xi^\pm} \neq 0$, it follows that $s_- = -s_+$, thus

$$\operatorname{sgn}([A]_M(t_-)y, y) = \frac{s_-}{s} = -\frac{s_+}{s} = -\operatorname{sgn}([A]_M(t_+)y, y).$$

Hence, $t \mapsto ([A]_M(t)y, y)$ has at least one zero in (t_-, t_+); this zero lies in $\kappa(M)$ because $W([A]_M) \subset W_{\mathcal{H}}(A)$. □

Theorem 4.17. *Let H be a separable Hilbert space, $\mathcal{H} \in \mathcal{Z}(H)$ be such that the dimension condition* (1.6) *holds, $M \in \mathcal{H}$ and $A \in L(H)$ be selfadjoint. If $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, then* (4.6) *holds for every $M \in \mathcal{H}$.*

Proof. Let $M \in \mathcal{H}$, $y \in M$ and $(H_k)_1^\infty$ be a sequence of finite dimensional subspaces of H as in Remark 4.13 such that $y \in H_1$. Then $A^{(k)} = A_{H_k}$ is selfadjoint for every $k \in \mathbb{N}$. Therefore, (4.10) and Proposition 4.12 (2b) yield the claim as in the proof of Theorem 4.14. □

The proof of the subsequent Theorem 4.19 is based on the fact that the dimensions of the subspaces in Remark 4.7 may be chosen to fulfill condition (2c) of Proposition 4.12. To this end, we need the following lemma.

Lemma 4.18. *Let $m, \nu_0, \dots, \nu_m \in \mathbb{N}_0$.*

(1) *If*

$$\text{(a)} \quad \sum_{i=0}^m \nu_i \le m + 1, \qquad \text{(b)} \quad \sum_{i=0}^m \nu_i 2^i = 2^{m+1} - 1,$$

then $\nu_0 = \cdots = \nu_m = 1$.

(2) *Let $d \in \mathbb{N}$ be such that $d > (m + 1)(2^{m+1} - 1)$ and*

$$d_i := d + 2^i, \quad i = 0, \dots, m.$$

Then the equation

$$\sum_{i=0}^m \nu_i d_i = \sum_{i=0}^m d_i$$

only has the trivial solution $\nu_0 = \cdots = \nu_m = 1$.

Proof. (1) We proof the claim by induction on m. The case $m = 0$ is trivial. Now let $m \geq 1$. It follows from (b) that

$$\nu_0 + \sum_{i=1}^{m} \nu_i 2^i = 2^{m+1} - 1. \tag{4.12}$$

As the sum on the left is even and the number on the right hand side of (4.12) is odd, ν_0 has to be odd. We write $\nu_0 = 1 + 2k_0$ for some $k_0 \in \mathbb{N}_0$. Then, the last equation implies

$$k_0 + \sum_{i=0}^{m-1} \nu_{i+1} 2^i = 2^m - 1. \tag{4.13}$$

From (a) it follows that $\nu_0 \leq m + 1$, thus $k_0 = (\nu_0 - 1)/2 \leq m/2 < 2^m$. Therefore k_0 has the representation $k_0 = b_0 + 2b_1 + \cdots + 2^{m-1} b_{m-1}$ where $b_i \in \{0, 1\}$, $i = 0, \ldots, m-1$. Inserting this representation into (4.13) implies

$$2^m - 1 = \sum_{i=0}^{m-1} (\nu_{i+1} + b_i) 2^i = \sum_{i=0}^{m-1} c_i 2^i,$$

where $c_i := \nu_{i+1} + b_i$, $i = 0, \ldots, m-1$. Additionally, we get, by assumption (a),

$$\sum_{i=0}^{m-1} c_i = \sum_{i=1}^{m} \nu_i + \sum_{i=0}^{m-1} b_i \leq m + 1 - \nu_0 + \sum_{i=0}^{m-1} b_i 2^i = m + 1 - (1 + 2k_0) + k_0 \leq m.$$

Hence the induction hypothesis applies to $c_0, \ldots, c_{m-1}$ and we obtain $c_i = 1$, that is, $\nu_{i+1} = 1 - b_i$, $i = 0, \ldots, m-1$. We conclude that

$$m + 1 \geq \sum_{i=0}^{m} \nu_i = \nu_0 + \sum_{i=0}^{m-1} \nu_{i+1} = 1 + 2k_0 + m - \sum_{i=0}^{m-1} b_i,$$

thus

$$2k_0 \leq \sum_{i=0}^{m-1} b_i \leq k_0.$$

The last equation yields $k_0 = 0$ and $b_0 = \cdots = b_{m-1} = 0$, therefore $\nu_0 = 1$ and $\nu_1 = \cdots = \nu_m = 1$.

(2) By assumption we have

$$d \sum_{i=0}^{m} \nu_i + \sum_{i=0}^{m} \nu_i 2^i = \sum_{i=0}^{m} \nu_i d_i = \sum_{i=0}^{m} d_i = d(m+1) + 2^{m+1} - 1. \tag{4.14}$$

For $i = 0, \ldots, m$ we have the inequality $\nu_i d_i \leq d(m+1) + 2^{m+1} - 1$, thus

$$\nu_i \leq \frac{d(m+1) + 2^{m+1} - 1}{d + 2^i} \leq m + 1 + \frac{2^{m+1} - 1}{d} < m + 2,$$

hence $\nu_i \leq m + 1$. It follows that

$$\sum_{i=0}^{m} \nu_i 2^i \leq (m+1)(2^{m+1} - 1) < d.$$

Therefore, Equation (4.14) yields

$$\sum_{i=0}^{m} \nu_i = m+1, \quad \sum_{i=0}^{m} \nu_i 2^i = 2^{m+1}-1.$$

Applying (1) finishes the proof. □

Theorem 4.19. *Let H be a separable Hilbert space and $\mathcal{H} \in \mathcal{Z}(H)$ be such that $\dim M = \infty$, $M \in \mathcal{H}$. If $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated, then* (4.6) *holds for every $M \in \mathcal{H}$.*

Proof. Let $\mathcal{H} = \{M_1, \dots, M_n\}$, $M \in \mathcal{H}$ and $y \in M$. The sequence $(H_k)_1^\infty$ in Remark 4.13 may be chosen such that $y \in H_1$ and

$$d_i^{(k)} := \dim(H_k \cap M_i) = d^{(k)} + 2^{i-1}, \quad i = 1, \dots, n, \ k \in \mathbb{N},$$

where $(d^{(k)})_1^\infty \subset \mathbb{N}$ is such that $(n+2)(2^{n+2}-1) < d^{(k)}$, $k \in \mathbb{N}$, and $d^{(k)} \to \infty$, $k \to \infty$. By Lemma 4.18, for every $k \in \mathbb{N}$, the equation

$$\sum_{i=1}^{n} \nu_i d_i^{(k)} = \sum_{i=1}^{n} d_i^{(k)}$$

only has the trivial solution. Now, from Proposition 4.12 (2c) and (4.10) the claim follows. □

Remark 4.20. We close this section with a remark addressing the problems arising in the case $|\mathcal{H}| > 2$. Suppose we are – as for $|\mathcal{H}| = 2$ – able to show (4.11) without any further assumptions on H, $\mathcal{H}$ and the operator A, provided that (1.6) holds and $\kappa(M)$ is $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$. Then, the condition $A = A^*$ in Theorem 4.17 may be dropped, and Theorem 4.9 (2) holds without any of the additional conditions (2a), (2b) and (2c). Unfortunately, to be able to show (4.11) in the general non-selfadjoint case, we have to impose an additional condition on A – which is obviously always fulfilled if $|\mathcal{H}| = 2$. It might turn out that this condition always holds; nevertheless, a proof of this is not known by now.

Proposition 4.21. *Let $\mathcal{H} \in \mathcal{Z}(H)$ be such that* (1.6) *holds, $M \in \mathcal{H}$ and $A \in L(H)$. Moreover, let $\kappa(M)$ be $\mathcal{H}$-separated in $W_{\mathcal{H}}(A)$ and Γ_M be an $\mathcal{H}$-separating Jordan curve. If there exists a $y \in M \setminus \{0\}$ such that for all $N \in \mathcal{H} \setminus \{M\}$ the function*

$$x_N(z) := P_N(A_{M^\perp} - z)^{-1} A_{M^\perp M} y, \quad z \in \overline{\operatorname{int} \Gamma_M}, \tag{4.15}$$

is either identically zero or does not have any zero, then $W([A]_M) \cap \kappa(M) \neq \emptyset$. This is, in particular, the case if $A_{M^\perp M}$ has non-trivial kernel.

Proof. Let $\mathcal{H} = \{M_1, \dots, M_n\}$ such that $M = M_n$, $\Omega_M := \operatorname{int} \Gamma_M$ and, for $\eta \in M$, the function $x_\eta : \overline{\Omega}_M \to H$ be given by

$$x_\eta(z) := \eta - (A_{M^\perp} - z)^{-1} A_{M^\perp M} \eta, \quad z \in \overline{\Omega}_M.$$

By assumption, there is a $y \in M$ such that for $x := x_y$ we have either $x_i \equiv 0$ or $x_i(z) \neq 0$, $z \in \overline{\Omega}_M$, for every $i \in \{1, \dots, n-1\}$. Let $J := \{j \in \{1, \dots, n\} : x_j \not\equiv 0\}$

and define $\xi : \overline{\Omega}_M \to H$ just as in the proof of Proposition 4.16. Note that now we additionally know that $\xi_i(z) \neq 0$, $z \in \overline{\Omega}_M$. Therefore, we may define the continuous mapping $\hat{\xi} : \overline{\Omega}_M \to \mathcal{H}^\square$ by

$$\hat{\xi}_i(z) := \|\xi_i(z)\|^{-1}\xi_i(z), \quad z \in \overline{\Omega}_M,\ i = 1, \dots, n.$$

Now let $\lambda : \mathcal{H}^\square \to \kappa(M)$ be the continuous mapping such that $\lambda(x)$ is the eigenvalue of A_x in $\kappa(M)$ for every $x \in \mathcal{H}^\square$ (see Corollary 1.25). Then the composite function $\mu := \lambda \circ \hat{\xi} : \overline{\Omega}_M \to \kappa(M) \subset \overline{\Omega}_M$ is also continuous, and thus has a fixed point $\lambda_0 \in \kappa(M)$ according to Brouwer's Fixed Point Theorem (see, e. g., [Dol95, Corollary IV.2.6]). It follows that

$$0 = \det\left(A_{\hat{\xi}(\lambda_0)} - \lambda(\hat{\xi}(\lambda_0))\right) = \det(A_{\hat{\xi}(\lambda_0)} - \lambda_0) = ([A]_M(\lambda_0)y, y)\det\left(A^{\mathcal{H}\setminus\{M\}}_{\hat{\xi}(\lambda_0)} - \lambda_0\right),$$

according to what has been shown in the proof of Theorem 4.4. As $\lambda_0 \notin W_{\mathcal{H}\setminus\{M\}}(A)$, it follows that $\det\left(A^{\mathcal{H}\setminus\{M\}}_{\hat{\xi}(\lambda_0)} - \lambda_0\right) \neq 0$, hence $([A]_M(\lambda_0)y, y) = 0$, which finishes the proof. □

Remark 4.22. There is another situation where (4.11) may be shown in a similar way as in Proposition 4.21: If $A_{M^\perp M}$ is bijective, choose vectors $x_N \in N \setminus \{0\}$, $N \in \mathcal{H} \setminus \{M\}$, and let $x := \sum_{N \neq M} x_N$. Then, defining $y : \overline{\Omega}_M \to M \setminus \{0\}$ by

$$y(z) := A^{-1}_{M^\perp M}(A_{M^\perp} - z)x, \quad z \in \overline{\Omega}_M,$$

we have an equality like (4.15),

$$x_N = P_N(A_{M^\perp} - z)^{-1}A_{M^\perp M}y(z), \quad z \in \overline{\Omega}_M,$$

and the same arguments as in the proof of Proposition 4.21 show (4.11).

4.3 Invariant subspaces and Riccati equations

Before proceeding, we reformulate the correspondence between the solutions K of Riccati equations (4.1) and invariant graph subspaces (4.2) to fit the notations used so far: Let $M \subset H$ be a closed subspace of H and $A \in L(H)$. Then an operator $K_{M^\perp M} \in L(M, M^\perp)$ is a solution of the Riccati equation

$$KA_{MM^\perp}K + KA_M - A_{M^\perp}K - A_{M^\perp M} = 0 \tag{4.16}$$

if and only if its graph subspace $\mathcal{G}(K_{M^\perp M}) = \{y + K_{M^\perp M}y \,:\, y \in M\}$ is A-invariant.

The assertion of the following theorem is contained in the proof of [LMMT01, Theorem 5.1]. Indeed, only the factorization of the Schur complement of the form required in Theorem 4.23 is used in this proof – properties of the quadratic numerical range have only been needed to obtain such a factorization. (See also [ALT01, Theorem 2.2].)

Theorem 4.23. *Let $M \subset H$ be a closed subspace and $A \in L(H)$. Assume that $\Gamma_M \subset \rho(A_{M^\perp})$ is a Jordan curve such that $U_M := \operatorname{int}\Gamma_M \subset \rho(A_{M^\perp})$, and that there exist an analytic function $T_M : \rho(A_{M^\perp}) \to L(M)$ with $\sigma(T_M) \cap \overline{U}_M = \emptyset$, and an operator $Z_M \in L(M)$ with $\sigma(Z_M) \subset U_M$ such that $[A]_M$ admits the factorization*

$$[A]_M(z) = T_M(z)(Z_M - z), \quad z \in \rho(A_{M^\perp}).$$

Moreover, let the operator

$$\int_{\Gamma_M} [A]_M^{-1}(z)\,\mathrm{d}z \in L(M)$$

be invertible. Then, for the operator $K_{M^\perp M} \in L(M, M^\perp)$, given by

$$K_{M^\perp M} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_M} (A_{M^\perp} - z)^{-1} A_{M^\perp M} (Z_M - z)^{-1}\,\mathrm{d}z,$$

we have $R(P_{\sigma_M}) = \mathcal{G}(K_{M^\perp M})$, where $\sigma_M := \overline{\kappa(M)} \cap \sigma(A)$ and P_{σ_M} denotes the Riesz projection of A with respect to σ_M. In particular, $\mathcal{G}(K_{M^\perp M})$ is A-invariant, and therefore $K_{M^\perp M}$ satisfies the Riccati equation (4.16). *Moreover, Z_M and T_M have the representations*

$$\begin{aligned} Z_M &= A_M + A_{MM^\perp} K_{M^\perp M}, \\ T_M(z) &= 1 - A_{MM^\perp}(A_{M^\perp} - z)^{-1} K_{M^\perp M}, \quad z \in \rho(A_{M^\perp}). \end{aligned} \tag{4.17}$$

Using this theorem and the factorization results of the preceding section, we are able to solve Riccati equations and block diagonalize operators with respect to a decomposition $\mathcal{H}$.

As a first result we give the following theorem on strongly $\mathcal{H}$-separated connected components M of $W_\mathcal{H}(A)$. Here, in contrast to the other theorems in this section – where $W_\mathcal{H}(A)$ itself is always required to be $\mathcal{H}$-separated – we do not need additional information on the remaining part $W_\mathcal{H}(A) \setminus \kappa(M)$ of $W_\mathcal{H}(A)$.

Theorem 4.24. *Let H be separable, $\mathcal{H} \in L(H)$ fulfill the dimension condition* (1.6) *and let $A \in L(H)$. If $M \in \mathcal{H}$ is such that $\kappa(M)$ is strongly $\mathcal{H}$-separated in $W_\mathcal{H}(A)$, then there is an operator $K_{M^\perp M} \in L(M, M^\perp)$ such that $R(P_{\sigma_M}) = \mathcal{G}(K_{M^\perp M})$. In particular, the graph subspace $\mathcal{G}(K_{M^\perp M})$ is A-invariant and*

$$K_{M^\perp M}(A_M + A_{MM^\perp} K_{M^\perp M}) = A_{M^\perp M} + A_{M^\perp} K_{M^\perp M}. \tag{4.18}$$

Proof. The conditions of Theorem 4.23 are fulfilled by Theorem 4.9 (1) and Theorem 4.8. Equation (4.18) is a simple reformulation of (4.16). □

Assume that for every $M \in \mathcal{H}$ there exists an operator $K_{M^\perp M} \in L(M, M^\perp)$ as in Theorem 4.24. From

$$\sigma(A) = \overline{W_\mathcal{H}(A)} \cap \sigma(A) = \bigcup_{M \in \mathcal{H}} \big(\overline{\kappa(M)} \cap \sigma(A)\big) = \bigcup_{M \in \mathcal{H}} \sigma_M,$$

where $\sigma_M = \sigma(A) \cap \overline{\kappa(M)}$ as in Theorem 4.23, it follows that

$$H = \bigoplus_{M \in \mathcal{H}} R(P_{\sigma_M}) = \bigoplus_{M \in \mathcal{H}} \mathcal{G}(K_{M^\perp M}).$$

Define the operator $\mathcal{K} \in L(H)$ by $\mathcal{K}y := y + K_{M^\perp M}y$, $y \in M$, for $M \in \mathcal{H}$, that is,

$$\mathcal{K}x = \sum_{M \in \mathcal{H}} \mathcal{K}x_M = \sum_{M \in \mathcal{H}} (x_M + K_{M^\perp M}x_M) = x + \sum_{M \in \mathcal{H}} K_{M^\perp M}x_M, \quad x \in H.$$

Obviously, $\mathcal{K}|_M : M \to R(\mathcal{K}|_M) = \mathcal{G}(K_{M^\perp M})$ is an isomorphism for all $M \in \mathcal{H}$. Thus, also

$$\mathcal{K} : H = \bigoplus_{M \in \mathcal{H}} M \to \bigoplus_{M \in \mathcal{H}} \mathcal{G}(K_{M^\perp M}) = H$$

is an isomorphism. Define the $\mathcal{H}$-diagonal operator $\mathcal{D} \in L(H)$ by

$$\mathcal{D} := \sum_{M \in \mathcal{H}} Z_M P_M,$$

where the Z_M, $M \in \mathcal{H}$, are as in (4.17). We show that $A\mathcal{K} = \mathcal{K}\mathcal{D}$. To this end, let $M \in \mathcal{H}$ and $y \in M$. Then

$$\begin{aligned} A\mathcal{K}y &= A(y + K_{M^\perp M}y) = Ay + AK_{M^\perp M}y \\ &= (A_M y + A_{M^\perp M}y) + (A_{MM^\perp}K_{M^\perp M}y + A_{M^\perp}K_{M^\perp M}y) \\ &= (A_M + A_{MM^\perp}K_{M^\perp M})y + (A_{M^\perp M} + A_{M^\perp}K_{M^\perp M})y \\ &\overset{*}{=} Z_M y + K_{M^\perp M}Z_M y = \mathcal{K}Z_M y = \mathcal{K}\mathcal{D}y, \end{aligned}$$

where (4.17) and (4.18) have been used at $*$. As we have already seen that $\mathcal{K}$ is an isomorphism, and $\mathcal{D}$ is an $\mathcal{H}$-diagonal operator by definition, it follows that A is block diagonalizable with respect to $\mathcal{H}$:

$$\mathcal{K}^{-1}A\mathcal{K} = \mathcal{D}. \tag{4.19}$$

Writing $\mathcal{H} = \{M_1, \dots, M_n\}$, the last equation reads

$$\begin{pmatrix} 1 & \cdots & K_{1n} \\ \vdots & & \vdots \\ K_{n1} & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} 1 & \cdots & K_{1n} \\ \vdots & & \vdots \\ K_{n1} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} Z_1 & & \\ & \ddots & \\ & & Z_n \end{pmatrix}.$$

Therefore, we have proved the following theorem.

Theorem 4.25. *Let H be separable, $\mathcal{H} \in L(H)$ fulfill the dimension condition* (1.6) *and let $A \in L(H)$. If $W_{\mathcal{H}}(A)$ is strongly $\mathcal{H}$-separated, then A is block diagonalizable with respect to $\mathcal{H}$.*

Using the conditions of Theorem 4.9 (2) instead of (1), the proof of the following theorem is the same as above.

Theorem 4.26. *Let H be separable, $\mathcal{H} \in L(H)$ fulfill the dimension condition* (1.6) *and let $A \in L(H)$. Assume that $W_{\mathcal{H}}(A)$ is $\mathcal{H}$-separated and one of the following conditions holds:*

(1) $|\mathcal{H}| = 2$.

(2) $A = A^*$.

(3) $\dim M = \infty$, $M \in \mathcal{H}$.

Then, for every $M \in \mathcal{H}$, there exists an operator $K_{M^\perp M} \in L(M, M^\perp)$ such that $R(P_{\sigma_M}) = \mathcal{G}(K_{M^\perp M})$. In particular, the graph subspace $\mathcal{G}(K_{M^\perp M})$ is A-invariant and all Riccati equations (4.18) *hold. Moreover, A is block diagonalizable with respect to $\mathcal{H}$ as in* (4.19).

Chapter 5

Corners of block numerical ranges

It is well-known that for an operator $A \in L(H)$ every corner λ_0 of the numerical range $W(A)$ of A is contained in the spectrum of A and, if $\lambda_0 \in W(A)$, even is an eigenvalue of A (see, e. g., [GR97, Theorem 1.5-5]). It has been shown in [LMT01] that this theorem has generalizations to analytic operator functions and to the quadratic numerical range of operators:

(1) If $F : \Omega \to L(H)$ is an analytic operator function, $\lambda_0 \in W(F)$ is a corner of $W(F)$ of angle $\varphi < \pi/l$ for some $l \in \mathbb{N}$ and there exists an $x_0 \in H$, $\|x_0\| = 1$, such that $z \mapsto (F(z)x_0, x_0)$ has a zero of multiplicity $\leq l$ in λ_0, then λ_0 is an eigenvalue of F with eigenvector x_0, i. e., $F(\lambda_0)x_0 = 0$ ([LMT01, Theorem 2.7]).

(2) If the block operator matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L(H_1 \times H_2),$$

and λ_0 is a corner of $W^2(\mathcal{A})$, then $\lambda_0 \in \sigma(\mathcal{A}) \cup \sigma(A) \cup \sigma(D)$. If, additionally, $\lambda_0 \in W^2(\mathcal{A})$, then λ_0 is an eigenvalue of $\mathcal{A}$, A or D ([LMT01, Theorems 3.1 and 3.4]).

In the following it is our aim to prove corresponding theorems for the block numerical ranges of operators and operator functions.

First of all, we recall the definition of a corner of a set $W \subset \mathbb{C}$.

Definition 5.1. Let $W \subset \mathbb{C}$. Then $\lambda \in \mathbb{C}$ is called a *corner of* W if $\lambda \in \overline{W}$ and there exist $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\varepsilon > 0$, and $\varphi \in [0, \pi)$ such that

$$0 \leq \arg(\alpha(W - \lambda) \cap B_\varepsilon(0) \setminus \{0\}) \leq \varphi,$$

where arg takes values in $[-\pi, \pi)$.

The main property of a corner $\lambda \in W$ used subsequently is that $\gamma'(0) = 0$ for every differentiable function $\gamma : (-\varepsilon, \varepsilon) \to W$ for which $\gamma(0) = \lambda$.

5.1 Analytic perturbations of matrices

We need some facts about the behavior of eigenvalues of matrices under analytic perturbations. The following theorem contains an extract from [Bau72, pp. 86 ff.] including all information to be used later. (For a very concise treatment on this subject see also [HP05, Chapters 4.1 and 4.2].)

Theorem 5.2. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, $A : U \to M_n(\mathbb{C})$ be analytic, λ_0 be an eigenvalue of $A(0)$ of algebraic multiplicity m_0 and $\rho > 0$ be such that $\sigma_{\mathrm{p}}(A(0)) \cap B_\rho(\lambda_0) = \{\lambda_0\}$. Then there exist $r > 0$, $p \in \mathbb{N}$, $d_1, \dots, d_p \in \mathbb{N}$ and sequences $\big(a_k^{(i)}\big)_{k=1}^\infty \subset \mathbb{C}$, $i = 1, \dots, p$, such that for every $z \in B_r(0) \setminus \{0\}$ the matrix $A(z)$ has exactly $d_1 + \dots + d_p \le m_0$ distinct eigenvalues in $B_\rho(\lambda_0)$ which are represented by the Puiseux series*

$$s_i(z) = \lambda_0 + \sum_{k=1}^\infty a_k^{(i)} z^{k/d_i}, \quad i = 1, \dots, p, \tag{5.1}$$

To be more precise, fix $z \in B_r(0)$ and, for $i = 1, \dots, p$, an arbitrary d_i-th root w_i of z. Then, writing $\zeta_i := \exp(\frac{2\pi \mathrm{i}}{d_i})$, we have the following representation of $\sigma_{\mathrm{p}}(A(z))$ close to λ_0:

$$\sigma_{\mathrm{p}}(A(z)) \cap B_\rho(\lambda_0) = \Big\{\lambda_0 + \sum_{k=1}^\infty a_k^{(i)} (\zeta_i^j w_i)^k \,:\, i = 1, \dots, p,\ j = 1, \dots, d_i\Big\}. \tag{5.2}$$

Remark. Note that, if in the preceding theorem $d_i = 1$ for some $i \in \{1, \dots, p\}$, the corresponding Puiseux series s_i in (5.1) is an analytic function[1]. If we additionally assume that λ_0 is a corner of the set $\{\sigma_{\mathrm{p}}(A(t)) \,:\, t \in (-r, r)\}$, then, for the (differentiable) function $\mu := s_i|_{(-r,r)} : (-r, r) \to \mathbb{C}$, we have

$$\mu(0) = \lambda_0, \quad \mu'(0) = 0, \quad \mu(t) \in \sigma_{\mathrm{p}}(A(t)), \quad t \in (-r, r).$$

To prove the existence of such a function μ also in the case that $d_i > 1$, $i = 1, \dots, p$, is the aim of the current section.

Corollary 5.3. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, $A : U \to M_n(\mathbb{C})$ be an analytic matrix function and $\lambda_0 \in \sigma_{\mathrm{p}}(A(0))$. Then there exist constants $r > 0$, $d \in \mathbb{N}$, and a sequence $(a_k)_1^\infty$ of complex numbers, such that for the functions*

$$\mu_j : [0, r) \to \mathbb{C}, \quad \mu_j(t) := \lambda_0 + \sum_{k=1}^\infty a_k (\xi_d^j \sqrt[d]{t})^k, \quad j = 0, \dots, 2d-1,$$

where $\xi_d := \exp(\frac{\pi \mathrm{i}}{d})$, the inclusions

$$\mu_j(t) \in \sigma_{\mathrm{p}}\big(A((-1)^j t)\big), \quad t \in [0, r),\ j = 0, \dots, 2d-1,$$

hold.

[1] This is, for example, always the case if λ_0 is a simple eigenvalue of $A(0)$. Then it follows that $p = 1$, $d_1 = 1$ and $\sigma_{\mathrm{p}}(A(z)) \cap B_\rho(\lambda_0) = \{s_1(z)\}$ for all $z \in B_r(0)$.

Proof. We use the notations of Theorem 5.2. Let $d := d_1$ and $(a_k)_1^\infty := \big(a_k^{(1)}\big)_1^\infty$. Further, let $\zeta := \exp(\frac{2\pi \mathrm{i}}{d}) = \xi_d^2$. Then, for $t \in [0, r)$, and choosing $\sqrt[d]{-t} = \xi_d \sqrt[d]{t}$, we have

$$\xi_d^j \sqrt[d]{t} = \begin{cases} \xi_d^{2l} \sqrt[d]{t} = \zeta^l \sqrt[d]{t}, & j = 2l, \\ \xi_d^{2l+1} \sqrt[d]{t} = \zeta^l \xi_d \sqrt[d]{t} = \zeta^l \sqrt[d]{-t}, & j = 2l+1. \end{cases}$$

Therefore, (5.2) yields

$$\mu_j(t) = \lambda_0 + \sum_{k=1}^\infty a_k (\xi_d^j \sqrt[d]{t})^k = \begin{cases} \lambda_0 + \sum\limits_{k=1}^\infty a_k (\zeta^l \sqrt[d]{t})^k \in \sigma_{\mathrm{p}}(A(t)), & j = 2l, \\ \lambda_0 + \sum\limits_{k=1}^\infty a_k (\zeta^l \sqrt[d]{-t})^k \in \sigma_{\mathrm{p}}(A(-t)), & j = 2l+1. \end{cases}$$

In any case, $\mu_j(t) \in \sigma_{\mathrm{p}}(A((-1)^j t))$. □

Lemma 5.4. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, $A : U \to M_n(\mathbb{C})$ be an analytic matrix function and $\lambda_0 \in \sigma_{\mathrm{p}}(A(0))$. Assume that there exist $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $\varepsilon > 0$ such that*

$$\operatorname{Im}(\alpha(W - \lambda_0) \cap B_\varepsilon(0)) \geq 0, \tag{5.3}$$

where $W := \bigcup \{\sigma_{\mathrm{p}}(A(t)) : t \in U \cap \mathbb{R}\}$. Then there exists an $r > 0$ and a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that

$$\mu(0) = \lambda_0, \qquad \mu(t) \in \sigma_{\mathrm{p}}(A(t)), \quad t \in (-r, r).$$

Proof. Without loss of generality, we may assume that $\lambda_0 = 0$ and $\operatorname{Im} W \geq 0$. Let $(a_k)_1^\infty$ and $\mu_j : [0, r) \to \mathbb{C}$, $j = 0, \dots, 2d-1$, be as in Corollary 5.3. The claim is proved if we show that $a_1 = \dots = a_{d-1} = 0$: In this case, setting $\sqrt[d]{t} = \xi_d \sqrt[d]{-t}$ for $t \leq 0$, we define the function μ by

$$\mu(t) := \sum_{k=d}^\infty a_k \sqrt[d]{t}^k = \sum_{k=1}^\infty a_k \sqrt[d]{t}^k = \begin{cases} \mu_0(t) \in \sigma_{\mathrm{p}}(A(t)), & t \in [0, r), \\ \mu_1(-t) \in \sigma_{\mathrm{p}}(A(t)), & t \in (-r, 0], \end{cases}$$

by Corollary 5.3. Additionally, μ is differentiable and $\mu(0) = 0 = \lambda_0$ as required. If $d = 1$ there is nothing to show; thus, let $d \geq 2$. Set $a_0 := 0$ and assume by induction that $a_1 = \dots = a_{m-1} = 0$ for some $m \in \{1, \dots, d-1\}$ has already been shown. If $a_m \neq 0$, then $\operatorname{Im} a_m \xi_d^{\nu m} < 0$ for some $\nu \in \{0, \dots, 2d-1\}$. (In fact, from $1 \leq m \leq d-1$ it follows that $\operatorname{Im} \xi_d^m = \operatorname{Im} \exp(\frac{m}{d}\pi \mathrm{i}) > 0$, thus, as $2d - 1 \geq 3$, the set $\{a_m \xi_d^{\nu m} : \nu = 0, \dots, 2d-1\}$ consists of at least 3 points. As the rays through these points starting at 0 divide the plane in at least 3 sectors of the same angle, at least one such ray is pointing downwards.) Consider the mapping $f : [0, r) \to W$, $f(t) := \mu_\nu(t^{\frac{d}{m}})$. Then, using the induction hypothesis, we get

$$f(t) = \sum_{k=m}^\infty a_k (\xi_d^\nu \sqrt[d]{t^{\frac{d}{m}}})^k = \sum_{k=m}^\infty a_k \xi_d^{\nu k} t^{\frac{k}{m}} = a_m \xi_d^{\nu m} t + \sum_{k=m+1}^\infty a_k \xi_d^{\nu k} t^{\frac{k}{m}} =: a_m \xi_d^{\nu m} t + \beta(t).$$

Since $k > m$ in the sum defining β, the function β is differentiable and $\beta(0) = \beta'(0) = 0$. From this and $\operatorname{Im} a_m \xi_d^{\nu m} < 0$ it follows that $\operatorname{Im} f(t) < 0$ for small $t > 0$ which contradicts the fact that $\operatorname{Im} W \geq 0$. □

If, in the situation of Lemma 5.4, λ_0 is a corner of W, condition (5.3) is certainly fulfilled for appropriate α and ε. In this case, it follows additionally that $\mu'(0) = 0$, so that we have the following proposition.

Proposition 5.5. *Let $\rho > 0$, $A : B_\rho(0) \to M_n(\mathbb{C})$ be analytic and $\lambda_0 \in \sigma_{\mathrm{p}}(A(0))$ be a corner of the set $W := \bigcup\{\sigma_{\mathrm{p}}(A(t)) : t \in (-\rho, \rho)\}$. Then there exist $r > 0$ and a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that*

$$\mu(0) = \lambda_0, \quad \mu'(0) = 0, \quad \mu(t) \in \sigma_{\mathrm{p}}(A(t)), \quad t \in (-r, r).$$

Remark 5.6. It is worth noting that, if $A : (-r, r) \to M_n(\mathbb{C})$ is differentiable and $A(t)$ is diagonalizable for all $t \in (-r, r)$, then the preceding proposition can be proved with less effort: Due to [Kat95, Remark II.5.8] there are differentiable functions $\lambda_1, \dots, \lambda_n : (-r, r) \to \mathbb{C}$ such that $\sigma_{\mathrm{p}}(A(t)) = \{\lambda_1(t), \dots, \lambda_n(t)\}$ for every $t \in (-r, r)$. If, say, $\lambda_0 = \lambda_1(0)$, then we may choose $\mu = \lambda_1$.

5.2 Corners in $W_{\mathcal{H}}(A)$ belonging to $W_{\mathcal{H}}(A)$

In this section we consider the case that λ_0 is a corner of $\overline{W_{\mathcal{H}}(A)}$ and $\lambda_0 \in W_{\mathcal{H}}(A)$. The general situation, that is $\lambda_0 \in \overline{W_{\mathcal{H}}(A)}$, is then reduced to this case by extending both H and A to a larger Hilbert space with the help of a Banach limit – a method which has already been applied in [LMT01].

The following two lemmata serve to keep the proofs of the main theorems as short as possible. Before, we need a definition concerning cofactors of the matrices A_x.

Definition 5.7. Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $x \in H$. For $M, N \in \mathcal{H}$ we define complex numbers $r_{MN}(A, x)$ as follows: Writing $\mathcal{H} = \{M_1, \dots, M_n\}$, let $i, j \in \{1, \dots, n\}$ be such that $M = M_i$ and $N = M_j$. Then we set

$$r_{MN}(A, x) := (-1)^{i+j} \det(A_x)'_{ij}, \tag{5.4}$$

where $(A_x)'_{ij}$ denotes the matrix obtained from A_x by deleting the i-th row and the j-th column. For $\mathcal{H} = \{H\}$ the definition (5.4) should be read as $r_{HH}(A, x) = 1$. (As usual, this definition does not depend on the enumeration $\{M_1, \dots, M_n\}$ of $\mathcal{H}$.)

Lemma 5.8. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $A \in L(H)$ and $\xi \in \mathcal{H}^*$ be such that $\det A_\xi = 0$. Assume that, writing $r_{MN} := r_{MN}(A, \xi)$, $M, N \in \mathcal{H}$,*

$$\sum_{N \in \mathcal{H}} r_{MN} A_{MN} \xi_N = 0, \quad M \in \mathcal{H}, \tag{5.5}$$

and that $r_{MM} \neq 0$, $M \in \mathcal{H}$. Then 0 is an eigenvalue of A with eigenvector

$$\sum_{N \in \mathcal{H}} r_{KN} \xi_N \in \operatorname{span}\{\xi_N : N \in \mathcal{H}\}, \tag{5.6}$$

where $K \in \mathcal{H}$ is arbitrary[2].

[2] In fact, all the vectors in (5.6) for different $K \in \mathcal{H}$ are linearly dependent (see (5.8)).

Proof. From $\det A_\xi = 0$ and Jacobi's Theorem (see, e. g., [Pra94, Theorem I.2.5.2]) it follows that

$$r_{MN}r_{M'N'} - r_{MN'}r_{M'N} = \det\begin{pmatrix} r_{MN} & r_{MN'} \\ r_{M'N} & r_{M'N'} \end{pmatrix} = 0, \quad M, N, M', N' \in \mathcal{H};$$

in particular (with $M' = K$ and $N' = M$), using the assumption $r_{MM} \neq 0$, $M \in \mathcal{H}$, we get

$$r_{KN} = \frac{r_{KM}}{r_{MM}} r_{MN}, \quad M, N, K \in \mathcal{H}. \tag{5.7}$$

Fixing some $K \in \mathcal{H}$, it follows from $r_{KK}\xi_K \neq 0$ and (5.7) that

$$0 \neq \sum_{N\in\mathcal{H}} r_{KN}\xi_N = \sum_{N\in\mathcal{H}} \frac{r_{KM}}{r_{MM}} r_{MN}\xi_N = \frac{r_{KM}}{r_{MM}} \sum_{N\in\mathcal{H}} r_{MN}\xi_N, \tag{5.8}$$

thus,

$$A\Big(\sum_{N\in\mathcal{H}} r_{KN}\xi_N\Big) = \sum_{M\in\mathcal{H}} P_M A\Big(\frac{r_{KM}}{r_{MM}} \sum_{N\in\mathcal{H}} r_{MN}\xi_N\Big) = \sum_{M\in\mathcal{H}} \frac{r_{KM}}{r_{MM}} \sum_{N\in\mathcal{H}} r_{MN}A_{MN}\xi_N = 0$$

by (5.5), which shows the claim. □

Lemma 5.9. *Let $\mathcal{H} \in \mathcal{Z}(H)$, $F : \Omega \to L(H)$ be differentiable, $\lambda_0 \in W_{\mathcal{H}}(F)$, $\xi \in \mathcal{H}^*$ be such that $\det F_\xi(\lambda_0) = 0$, and $M \in \mathcal{H}$. Assume that, abbreviating $r_{KN} := r_{KN}(F(\lambda_0), \xi)$, $K, N \in \mathcal{H}$, for every*

$$x \in \Big\{ \beta \sum_{N\in\mathcal{H}} r_{MN}F_{MN}(\lambda_0)\xi_N \;:\; \beta = 0, \mathrm{i} \Big\} \subset M \tag{5.9}$$

there exist differentiable mappings $\mu_x : (-r_x, r_x) \to \mathbb{C}$ and $\eta_x : (-r_x, r_x) \to H$ such that

$$\begin{gathered} \mu_x(0) = \lambda_0, \quad \mu_x'(0) = 0, \quad \eta_x(0) = \xi, \quad \eta_x'(0) = x, \\ \mu_x(t) \in \sigma_{\mathrm{p}}(F_{\eta_x(t)}), \quad t \in (-r_x, r_x). \end{gathered} \tag{5.10}$$

Then $\displaystyle\sum_{N\in\mathcal{H}} r_{MN}F_{MN}(\lambda_0)\xi_N = 0.$

Proof. Write $\mathcal{H} = \{M_1, \dots, M_n\}$ such that $M = M_1$. Let $\beta \in \{1, \mathrm{i}\}$, $x \in M_1$ be the corresponding vector in (5.9) (note that $x_i = 0$, $i = 2, \dots, n$) and $\mu := \mu_x$ and $\eta := \eta_x$ be as in (5.10). In the following we use the notations $r_{ij} := r_{M_iM_j}$, $F_{ij}(z) = F(z)_{M_iM_j}$ and $\eta_j(z) = \eta(z)_{M_j}$ for $z \in \Omega$ and $i, j = 1, \dots, n$. For $i, j = 1, \dots, n$, we have

$$\big(F_{ij}(\mu(0))\eta_j(0), \eta_i(0)\big) = (F_{ij}(\lambda_0)\xi_j, \xi_i) =: b_{ij},$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big(F_{ij}(\mu(t))\eta_j(t), \eta_i(t)\big)\big|_{t=0} \\ &\quad= \big(F_{ij}'(\mu(0))\mu'(0)\eta_j(0), \eta_i(0)\big) + \big(F_{ij}(\mu(0))\eta_j'(0), \eta_i(0)\big) + \big(F_{ij}(\mu(0))\eta_j(0), \eta_i'(0)\big) \\ &\quad= (F_{ij}(\lambda_0)x_j, \xi_i) + (F_{ij}(\lambda_0)\xi_j, x_i) =: c_{ij}. \end{aligned}$$

Thus, $\mu(t) \in \sigma_{\mathrm{p}}(F_{\eta(t)})$, that is, $\det F_{\eta(t)}(\mu(t)) = 0$, $t \in (-r, r)$, implies

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t} \det F_{\eta(t)}(\mu(t))\Big|_{t=0} &= \begin{vmatrix} c_{11} & \dots & c_{1n} \\ b_{21} & \dots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{vmatrix} + \dots + \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n-1,1} & \dots & b_{n-1,n} \\ c_{n1} & \dots & c_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} r_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} (F_{ij}(\lambda_0) x_j, \xi_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} (F_{ij}(\lambda_0) \xi_j, x_i) \\
&= \sum_{i=1}^{n} r_{i1} (F_{i1}(\lambda_0) x_1, \xi_i) + \sum_{j=1}^{n} r_{1j} (F_{1j}(\lambda_0) \xi_j, x_1) \\
&= \Big(x_1, \sum_{i=1}^{n} \overline{r}_{i1} F_{i1}(\lambda_0)^* \xi_i\Big) + \Big(\sum_{j=1}^{n} r_{1j} F_{1j}(\lambda_0) \xi_j, x_1\Big) \\
&=: (x_1, \tau^*) + (\tau, x_1).
\end{aligned}$$

Note that

$$\tau = \sum_{j=1}^{n} r_{1j} F_{1j}(\lambda_0) \xi_j = \sum_{N \in \mathcal{H}} r_{MN} F_{MN}(\lambda_0) \xi_N,$$

that is, we have to show that $\tau = 0$. Recalling that $x_1 = x = \beta\tau$, where $\beta \in \{1, \mathrm{i}\}$ was arbitrary, the equation $0 = (x_1, \tau^*) + (\tau, x_1)$ yields

$$\begin{aligned}
0 &= (\tau, \tau^*) + (\tau, \tau) = (\tau, \tau^*) + \|\tau\|^2, \\
0 &= (\mathrm{i}\tau, \tau^*) + (\tau, \mathrm{i}\tau) = \mathrm{i}\big((\tau, \tau^*) - (\tau, \tau)\big) = \mathrm{i}\big((\tau, \tau^*) - \|\tau^2\|\big).
\end{aligned}$$

It follows that $2\|\tau\|^2 = 0$, hence $\tau = 0$. □

Theorem 5.10. *Assume that $\mathcal{H} \in \mathcal{Z}(H)$ fulfills the dimension condition* (1.6)[3] *and let $A \in L(H)$. Assume that $\lambda_0 \in W_{\mathcal{H}}(A)$ is a corner of $W_{\mathcal{H}}(A)$ and $\xi \in \mathcal{H}^{\square}$ is a vector for which $\lambda_0 \in \sigma_{\mathrm{p}}(A_\xi)$. Then*[4]

$$\lambda_0 \in \sigma_{\mathrm{p}}(A_{\mathcal{H}'})$$

for some $\mathcal{H}' \subset \mathcal{H}$ with an eigenvector in span $\{\xi_M : M \in \mathcal{H}'\}$.

Proof. Write $\mathcal{H} = \{M_1, \dots, M_n\}$. Let $x \in H$ be arbitrary and

$$\delta := \frac{1}{4 \max\{\|x_i\| : i = 1, \dots, n\}}.$$

It is easy to see that then the mapping

$$\eta : B_\delta(0) \to \mathcal{H}^*, \quad \eta(w) := \sum_{N \in \mathcal{H}} (\xi_N + w x_N),$$

[3] This condition is not necessary if $|\mathcal{H}| = 2$; see Remark 5.11.
[4] Recall that $A_{\mathcal{H}'} = P_{H'} A|_{H'} \in L(H')$ for $\mathcal{H}' \in \mathcal{Z}'(H)$ and $H' := \bigoplus \mathcal{H}'$.

is well-defined. Additionally, we have

$$\eta(0) = \xi, \quad \eta'(0) = x, \qquad (\eta_i(w), \eta_i(\overline{w})) \neq 0, \quad w \in B_\delta(0),\ i = 1, \ldots, n, \tag{5.11}$$

and

$$\frac{(A_{ij}\eta_i(w), \eta_j(\overline{w}))}{(\eta_i(w), \eta_i(\overline{w}))} = \frac{(A_{ij}\xi_j, \xi_i) + \big((A_{ij}\xi_j, x_i) + (A_{ij}x_j, \xi_i)\big)w + (A_{ij}x_j, x_i)w^2}{1 + 2\operatorname{Re}(\xi_i, x_i)w + \|x_i\|^2 w^2}$$

for $i, j \in \{1, \ldots, n\}$, $w \in B_\delta(0)$. Thus, the matrix function

$$B : B_\delta(0) \to M_n(\mathbb{C}), \quad B(w) := \begin{pmatrix} \frac{(A_{11}\eta_1(w),\eta_1(\overline{w}))}{(\eta_1(w),\eta_1(\overline{w}))} & \cdots & \frac{(A_{1n}\eta_n(w),\eta_1(\overline{w}))}{(\eta_1(w),\eta_1(\overline{w}))} \\ \vdots & & \vdots \\ \frac{(A_{n1}\eta_1(w),\eta_n(\overline{w}))}{(\eta_n(w),\eta_n(\overline{w}))} & \cdots & \frac{(A_{nn}\eta_n(w),\eta_n(\overline{w}))}{(\eta_n(w),\eta_n(\overline{w}))} \end{pmatrix}, \tag{5.12}$$

is an analytic perturbation of $B(0) = A_\xi$. For $w = t \in (-\delta, \delta)$, we have

$$B(t) = \begin{pmatrix} \frac{(A_{11}\eta_1(t),\eta_1(t))}{\|\eta_1(t)\|^2} & \cdots & \frac{(A_{1n}\eta_n(t),\eta_1(t))}{\|\eta_1(t)\|^2} \\ \vdots & & \vdots \\ \frac{(A_{n1}\eta_1(t),\eta_n(t))}{\|\eta_n(t)\|^2} & \cdots & \frac{(A_{nn}\eta_n(t),\eta_n(t))}{\|\eta_n(t)\|^2} \end{pmatrix}.$$

Therefore, introducing $F(z) := A - z$, $z \in \mathbb{C}$, we have

$$\det(B(t) - z) = \frac{\det(A - z)_{\eta(t)}}{\|\eta_1(t)\|^2 \cdots \|\eta_n(t)\|^2} = \frac{\det F_{\eta(t)}(z)}{\|\eta_1(t)\|^2 \cdots \|\eta_n(t)\|^2}, \quad t \in (-\delta, \delta),\ z \in \mathbb{C}.$$

In particular, $\sigma_{\mathrm{p}}(B(t)) = \sigma_{\mathrm{p}}(F_{\eta(t)})$ and therefore (see (2.3)) $\sigma_{\mathrm{p}}(B(t)) \subset W_{\mathcal{H}}(F)$, $t \in (-\delta, \delta)$, thus

$$W := \bigcup \{\sigma_{\mathrm{p}}(B(t)) \,:\, t \in (-\delta, \delta)\} \subset W_{\mathcal{H}}(F).$$

From $\lambda_0 \in \sigma_{\mathrm{p}}(F_\xi) = \sigma_{\mathrm{p}}(B(0)) \subset W \subset W_{\mathcal{H}}(F) = W_{\mathcal{H}}(A)$ and the assumption it follows that λ_0 is a corner of W, and according to Proposition 5.5 there is a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that

$$\mu(0) = \lambda_0, \quad \mu'(0) = 0, \qquad \mu(t) \in \sigma_{\mathrm{p}}(F_{\eta(t)}), \quad t \in (-r, r), \tag{5.13}$$

where again $\sigma_{\mathrm{p}}(B(t)) = \sigma_{\mathrm{p}}(F_{\eta(t)})$ was used. Thus, by (5.11) and (5.13), the conditions of Lemma 5.9 on F, ξ and λ_0 are fulfilled even for arbitrary $x \in H$. Therefore we conclude that

$$\sum_{N \in \mathcal{H}} r_{MN} F_{MN}(\lambda_0)\xi_N = 0, \quad M \in \mathcal{H}, \tag{5.14}$$

where $r_{MN} := r_{MN}(F(\lambda_0), \xi)$, $M, N \in \mathcal{H}$. Now we proceed by induction on $n = |\mathcal{H}|$. In the case $n = 1$, Equation (5.14) simply reads $F(\lambda_0)\xi = 0$. Thus, λ_0 is an eigenvalue of F with eigenvector ξ.

Assume that the theorem is true for $n - 1 \geq 1$. If $r_{MM} = 0$ for some $M \in \mathcal{H}$, then, letting $\mathcal{H}' := \mathcal{H} \setminus \{M\}$, we have

$$0 = r_{MM} = \det F_\xi^{\mathcal{H}'}(\lambda_0),$$

by the definition of r_{MM} (see (5.4)), that is, $\lambda_0 \in W_{\mathcal{H}'}(F)$. Moreover, as λ_0 is a corner of $W_{\mathcal{H}}(F)$, it is also a corner of $W_{\mathcal{H}'}(F) \subset W_{\mathcal{H}}(F)$. (The last inclusion holds by Proposition 2.6 (3) and the assumption (1.6).) Thus, the claim follows by the induction hypothesis. If, on the other hand, $r_{MM} \neq 0$ for all $M \in \mathcal{H}$, then the claim follows immediately from (5.14) and Lemma 5.8. □

Remark 5.11. For $|\mathcal{H}| = 2$ the additional assumption $\dim M \geq 2$, $M \in \mathcal{H}$, occurring in Theorem 5.10, is not necessary (see [LMT01, Theorem 3.1]). If in this case $\dim M = 1$ for some $M \in \mathcal{H}$, then A_M is simply the multiplication by a constant which allows a separate treatment: Writing $\mathcal{H} = \{M_1, M_2\}$, and, for simplicity, assuming that $\lambda_0 = 0$, Equation (5.14) reads

$$\begin{aligned} (A_{22}\xi_2, \xi_2)A_{11}\xi_1 - (A_{21}\xi_1, \xi_2)A_{12}\xi_2 &= 0, \\ -(A_{12}\xi_2, \xi_1)A_{21}\xi_1 + (A_{11}\xi_1, \xi_1)A_{22}\xi_2 &= 0. \end{aligned} \tag{5.15}$$

Only the case $A_{11}\xi_1 \neq 0$ and $A_{22}\xi_2 \neq 0$ is of interest here. If, say, $\dim H_2 = 1$, then A_{22} is the multiplication by a constant $\alpha \in \mathbb{C} \setminus \{0\}$, and (5.15) yields

$$A_{11}\xi_1 + A_{12}\Big(- \frac{(A_{21}\xi_1, \xi_2)}{\alpha}\xi_2\Big) = 0.$$

Additionally,

$$A_{21}\xi_1 + A_{22}\Big(- \frac{(A_{21}\xi_1, \xi_2)}{\alpha}\xi_2\Big) = A_{21}\xi_1 - (A_{21}\xi_1, \xi_2)\xi_2 = 0,$$

as $\{\xi_2\}$ is an orthonormal basis of M_2. Thus, 0 is an eigenvalue of A with eigenvector $\xi_1 - \alpha^{-1}(A_{21}\xi_1, \xi_2)\xi_2$. Unfortunately, it is not obvious how to handle this case separately if $|\mathcal{H}| \geq 3$ which seems much more complicated.

Example 5.12. For the matrix

$$A = \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & \mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 \\ \hline 1 & 0 & 0 & \mathrm{i} & 1+\mathrm{i} & 0 \\ 0 & 1 & \mathrm{i} & 0 & 0 & 1+\mathrm{i} \end{array}\right) \tag{5.16}$$

and the decomposition $\mathcal{H} = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 =: \{M_1, M_2, M_3\}$, the block numerical range $W_{\mathcal{H}}(A)$ has 10 corners. They all are eigenvalues of A (marked by •) or its sub-matrices

$$A_\circ = A_{M_1 \oplus M_2}, \quad A_\square = A_{M_1 \oplus M_3}, \quad A_\times = A_{M_2 \oplus M_3}, \quad A_\blacksquare = A_{M_3},$$

marked by the respective symbols.

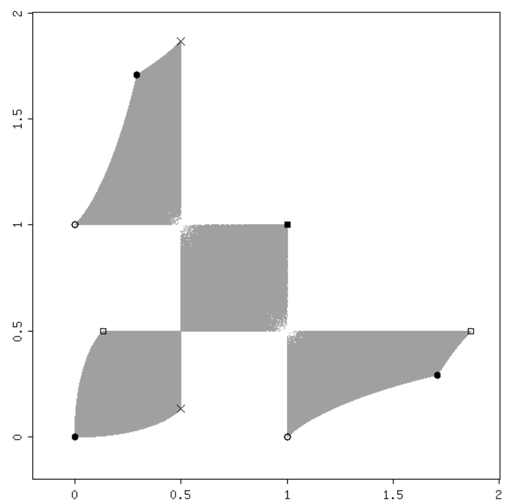


Figure 5.1: The block numerical range $W_{\mathcal{H}}(A)$ from Example 5.12 and its corners.

5.3 Corners in $W_{\mathcal{H}}(A)$ belonging to $\overline{W_{\mathcal{H}}(A)}$

In this section we prove the analogue of Theorem 5.10 without the additional condition $\lambda_0 \in W_{\mathcal{H}}(A)$ for the corner λ_0 of $W_{\mathcal{H}}(A)$. To this end, we fix a Banach limit b-lim, that is, b-lim is a linear functional on the space of bounded complex sequences l^∞ such that $\text{b-lim}_{k\to\infty} x_k := \text{b-lim}(x_k)_1^\infty = \lim_{k\to\infty} x_k$ whenever $(x_k)_1^\infty \in l^\infty$ converges and $\text{b-lim}(x_k)_1^\infty \geq 0$ for sequences $(x_k)_1^\infty$ with $x_k \geq 0$, $k \in \mathbb{N}$ (see e. g. [Ber62]).

With the use of b-lim we extend H to another Hilbert space $\widehat{H}$ as follows (see [Ber62]): Let $\widetilde{H}$ denote the space of all bounded sequences $(x_k)_1^\infty \subset H$. On $\widetilde{H}$ we define a scalar product $(\,\cdot\,,\cdot\,)_\sim$ by setting

$$((x_k)_1^\infty, (y_k)_1^\infty)_\sim := \operatorname*{b-lim}_{k\to\infty}(x_k, y_k), \quad (x_k)_1^\infty, (y_k)_1^\infty \in \widetilde{H}.$$

Now $\widehat{H}$ is defined as the completion of the quotient space $\widetilde{H}$ modulo the sequences $x \in \widetilde{H}$ for which $(x,x)_\sim = 0$. The resulting scalar product on $\widehat{H}$ induced by $(\,\cdot\,,\cdot\,)_\sim$ is again denoted by $(\,\cdot\,,\cdot\,)$.

If H' is another Hilbert space, every bounded operator $A \in L(H, H')$ admits an extension to an operator $\widehat{A} \in L(\widehat{H}, \widehat{H}')$ by setting

$$\widehat{A}(x_k)_1^\infty := (Ax_k)_1^\infty \in \widetilde{H}', \quad (x_k)_1^\infty \in \widetilde{H}.$$

It is known (see [Ber62, Theorem 1]) that in this case

$$\sigma_{\mathrm{p}}(\widehat{A}) = \sigma_{\mathrm{app}}(\widehat{A}) = \sigma_{\mathrm{app}}(A). \tag{5.17}$$

Furthermore, if $F : \Omega \to L(H)$ is an operator function, we define $\widehat{F} : \Omega \to L(\widehat{H})$, $\widehat{F}(z) := \widehat{F(z)}$.

Remark 5.13. Let $\mathcal{H} \in \mathcal{Z}(H)$ and $A \in L(H)$. Then

$$\widehat{\bigoplus \mathcal{H}} = \bigoplus \widehat{\mathcal{H}}, \qquad \widehat{A_{MN}} = \widehat{A}_{\widehat{M}\widehat{N}}, \quad M, N \in \mathcal{H},$$

where $\widehat{\mathcal{H}} := \{\widehat{M} \,:\, M \in \mathcal{H}\} \in \mathcal{Z}(\widehat{H})$.

Remark 5.14. Let $(z_k)_1^\infty \subset \mathbb{C}$ and $\hat{x} = (x_k)_1^\infty \subset H$. If $z_k \to z_0$, $k \to \infty$, and $(\|x_k\|)_1^\infty$ converges, then the equality $(z_k x_k)_1^\infty = z_0 \hat{x}$ holds in $\widehat{H}$. In particular, if $\hat{y} = (y_k)_1^\infty \subset H \setminus \{0\}$ is such that $(\|y_k\|)_1^\infty$ converges to some non-zero number, then $\|\hat{y}\| = \lim_{k\to\infty} \|y_k\|$ and $\|\hat{y}\|^{-1}\hat{y} = (\|y_k\|^{-1} y_k)_1^\infty$ in $\widehat{H}$.

Proof. From

$$\begin{aligned}\big((z_k x_k)_1^\infty - z_0\hat{x}, (z_k x_k)_1^\infty - z_0\hat{x}\big) &= (((z_k - z_0)x_k)_1^\infty, ((z_k - z_0)x_k)_1^\infty) \\ &= \operatorname*{b-lim}_{k\to\infty} |z_k - z_0|^2 (x_k, x_k) = \lim_{k\to\infty} |z_k - z_0|^2 \|x_k\|^2 = 0,\end{aligned}$$

the claim follows by definition of $\widehat{H}$. □

Lemma 5.15. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be a continuous operator function. Then*

$$\overline{W_{\mathcal{H}}(F)} \subset W_{\widehat{\mathcal{H}}}(\widehat{F}).$$

If, additionally, F is analytic and fulfills (2.5)*, then*

$$\sigma_{\rm p}(\widehat{F}_{\hat{\eta}}) \subset \overline{W_{\mathcal{H}}(F)}$$

for every $\hat{\eta} = (\eta^{(k)})_1^\infty \subset \mathcal{H}^*$ such that all the sequences $(\|\eta_M^{(k)}\|)_1^\infty$, $M \in \mathcal{H}$, converge to non-zero numbers and $(F_{\eta^{(k)}}(z))_1^\infty$ converges for all $z \in \Omega$.

Proof. Note first that if $\hat{\eta} := (\eta^{(k)})_1^\infty \subset H$ is bounded and $z \in \Omega$ is such that the sequence $(F_{\eta^{(k)}}(z))_1^\infty$ converges, then $F_{\eta^{(k)}}(z) \to \widehat{F}_{\hat{\eta}}(z)$, $k \to \infty$. In fact, writing $\mathcal{H} = \{M_1, \dots, M_n\}$ and $A := F(z)$ we have

$$(\widehat{A}_{ij}\hat{\eta}_j, \hat{\eta}_i) = ((A_{ij}\eta_j^{(k)})_1^\infty, (\eta_i^{(k)})_1^\infty) = \operatorname*{b-lim}_{k\to\infty}(A_{ij}\eta_j^{(k)}, \eta_i^{(k)}) = \lim_{k\to\infty}(A_{ij}\eta_j^{(k)}, \eta_i^{(k)}),$$

for $i, j = 1, \dots, n$.

Now let $\lambda_0 \in \overline{W_{\mathcal{H}}(F)}$. Then, as F is continuous, $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$ by Proposition 2.9, that is, $\det F_{\xi^{(k)}}(\lambda_0) \to 0$, $k \to \infty$, for some sequence $(\xi^{(k)})_1^\infty \subset \mathcal{H}^\square$. By passing to a subsequence, we may assume that $(F_{\xi^{(k)}}(\lambda_0))_1^\infty$ converges. Then $\hat{\xi} := (\xi^{(k)})_1^\infty \in \widehat{\mathcal{H}}^\square$ and, by what has been noted above, we have

$$0 = \lim_{k\to\infty} \det F_{\xi^{(k)}}(\lambda_0) = \det \widehat{F}_{\hat{\xi}}(\lambda_0),$$

that is, $\lambda_0 \in W_{\widehat{H}}(\widehat{F})$. Now let F and $\hat{\eta} = (\eta^{(k)})_1^\infty$ as in the second part of the lemma. As the sequences $\big(\|\eta_i^{(k)}\|\big)_1^\infty$, $i = 1, \dots, n$, converge to non-zero numbers, it follows from Remark 5.14 that $\|\hat\eta_i\| = \lim_{k\to\infty}\|\eta_i^{(k)}\| \neq 0$, $i = 1,\dots,n$, and, for $\hat\xi \in \widehat{\mathcal{H}}^\square$ and $\xi^{(k)} \in \mathcal{H}^\square$, $k \in \mathbb{N}$, defined by

$$\hat\xi_i := \frac{\hat\eta_i}{\|\hat\eta_i\|}, \quad (\xi_i^{(k)})_1^\infty := \Big(\frac{\eta_i^{(k)}}{\|\eta_i^{(k)}\|}\Big)_1^\infty, \qquad i = 1,\dots,n,$$

we have $\hat\xi = (\xi^{(k)})_1^\infty$ in $\widehat{H}$. Additionally,

$$\lim_{k\to\infty}\big(F_{ij}(z)\xi_j^{(k)}, \xi_i^{(k)}\big) = \lim_{k\to\infty}\frac{\big(F_{ij}(z)\eta_j^{(k)}, \eta_i^{(k)}\big)}{\|\eta_i^{(k)}\|\|\eta_j^{(k)}\|} = \frac{\big(\widehat{F}_{ij}(z)\hat\eta_j, \hat\eta_i\big)}{\|\hat\eta_i\|\|\hat\eta_j\|} = \big(\widehat{F}_{ij}(z)\hat\xi_j, \hat\xi_i\big)$$

for $z \in \Omega$, $i, j = 1,\dots,n$, that is, $F_{\xi^{(k)}}(z) \to \widehat{F}_{\hat\xi}(z)$, $k \to \infty$, $z \in \Omega$. Now, $\lambda_0 \in \sigma_{\mathrm{p}}(\widehat{F}_{\hat\eta})$ implies

$$0 = \|\hat\eta_1\|\cdots\|\hat\eta_n\| \det \widehat{F}_{\hat\eta}(\lambda_0) = \det \widehat{F}_{\hat\xi}(\lambda_0) = \lim_{k\to\infty} \det F_{\xi^{(k)}}(\lambda_0),$$

thus $\lambda_0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$ and (2.8), which applies because F is analytic and satisfies (2.5), yields $\lambda_0 \in \overline{W_{\mathcal{H}}(F)}$. □

Remark 5.16. It has been shown in [BO67] that $\overline{W(A)} = W(\widehat{A})$ for $A \in L(H)$. However, the proof of this fact given there relies on the convexity of the numerical range and it is not obvious how to generalize it to the block numerical range. (It is even not clear if $\overline{W_{\mathcal{H}}(A)} = W_{\widehat{\mathcal{H}}}(\widehat{A})$ holds at all for $|\mathcal{H}| \geq 2$.) Nevertheless, the assertions of Lemma 5.15 suffice for our purposes.

Theorem 5.17. *Assume that $\mathcal{H} \in \mathcal{Z}(H)$ fulfills the dimension condition* (1.6)[5] *and let $A \in L(H)$. If $\lambda_0 \in \overline{W_{\mathcal{H}}(A)}$ is a corner of $W_{\mathcal{H}}(A)$, then*

$$\lambda_0 \in \sigma_{\mathrm{app}}(A_{\mathcal{H}'}) \subset \sigma(A_{\mathcal{H}'})$$

for some $\mathcal{H}' \subset \mathcal{H}$.

Proof. We write $\mathcal{H} = \{M_1,\dots,M_n\}$ and define $F(z) := A - z$, $z \in \mathbb{C}$. By Theorem 1.34 there exists a sequence $\hat\xi = (\xi^{(k)})_1^\infty \subset \mathcal{H}^\square$ such that $\det(A_{\xi^{(k)}} - \lambda_0) = \det F_{\xi^{(k)}}(\lambda_0) \to 0$, $k \to \infty$. By passing to appropriate subsequences we may assume that the sequences

$$\begin{gathered}\big((A_{ij}\xi_j^{(k)}, \xi_i^{(k)})\big)_{k=1}^\infty, \quad \big((A_{il}A_{lj}\xi_j^{(k)}, \xi_i^{(k)})\big)_{k=1}^\infty, \\ \big((A_{ij}\xi_j^{(k)}, A_{il}\xi_l^{(k)})\big)_{k=1}^\infty, \quad \big((A_{il}A_{lj}\xi_j^{(k)}, A_{im}\xi_m^{(k)})\big)_{k=1}^\infty\end{gathered} \tag{5.18}$$

converge for all $i, j, l, m \in \{1,\dots,n\}$. Then each sequence converges to the corresponding expression in which A is replaced by $\widehat{A}$ and $\xi^{(k)}$ by $\hat\xi$, respectively. In particular, we have $F_{\xi^{(k)}}(z) \to \widehat{F}_{\hat\xi}(z)$, $k \to \infty$, $z \in \mathbb{C}$.

[5]Again, this condition is unnecessary if $|\mathcal{H}| = 2$ (see Remark 5.18).

Now, let $\beta \in \{1, \mathrm{i}\}$ and $\nu \in \{1, \dots, n\}$ be arbitrary, and (see (5.9) for the reason of this choice)

$$\begin{aligned} \hat{x} &:= \beta \sum_{\kappa=1}^{n} \hat{r}_{\nu\kappa} \widehat{F}_{\nu\kappa}(\lambda_0) \hat{\xi}_\kappa \in \widehat{M}_\nu, \\ x^{(k)} &:= \beta \sum_{\kappa=1}^{n} r_{\nu\kappa}^{(k)} F_{\nu\kappa}(\lambda_0) \xi_\kappa^{(k)} \in M_\nu, \quad k \in \mathbb{N}, \end{aligned} \tag{5.19}$$

where (recall the definition of the cofactors r_{MN} in (5.4))

$$\hat{r}_{ij} := r_{\widehat{M}_i \widehat{M}_j}(\widehat{F}(\lambda_0), \hat{\xi}),\ r_{ij}^{(k)} := r_{M_i M_j}(F(\lambda_0), \xi^{(k)}),\ k \in \mathbb{N}, \quad i, j = 1, \dots, n. \tag{5.20}$$

Then, by the convergence of the sequences in (5.18), it follows that $r_{ij}^{(k)} \to \hat{r}_{ij}$, $k \to \infty$, and Remark 5.14 yields $\hat{r}_{ij} \hat{\xi}_j = \big(r_{ij}^{(k)} \xi_{ij}^{(k)}\big)_1^\infty$ for $i, j = 1, \dots, n$. Thus,

$$\hat{x} = \beta \sum_{j=1}^{n} \widehat{F}_{\nu j}(\lambda_0) (r_{\nu j}^{(k)} \xi_j^{(k)})_1^\infty = \beta \Big(\sum_{j=1}^{n} F_{\nu j}(\lambda_0) r_{\nu j}^{(k)} \xi_j^{(k)} \Big)_1^\infty = (x^{(k)})_1^\infty.$$

Moreover, we have

$$(x_i^{(k)}, \xi_j^{(k)}) = \delta_{\nu i} \beta \sum_{\kappa=1}^{n} r_{\nu\kappa}^{(k)} (F_{\nu\kappa}(\lambda_0) \xi_\kappa^{(k)}, \xi_j^{(k)}) \to \delta_{\nu i} \beta \sum_{\kappa=1}^{n} \hat{r}_{\nu\kappa} (\widehat{F}_{\nu\kappa}(\lambda_0) \hat{\xi}_\kappa, \hat{\xi}_j) = (\hat{x}_i, \hat{\xi}_j)$$

for $k \to \infty$, where δ denotes the Kronecker symbol, and, likewise,

$$\begin{aligned} (x_i^{(k)}, x_j^{(k)}) \to (\hat{x}_i, \hat{x}_j),\ \ (A_{ij} \xi_j^{(k)}, x_i^{(k)}) \to (\widehat{A}_{ij} \hat{\xi}_j, \hat{x}_i), \\ (A_{ij} x_j^{(k)}, \xi_i^{(k)}) \to (\widehat{A}_{ij} \hat{x}_j, \hat{\xi}_i),\ \ (A_{ij} x_j^{(k)}, x_i^{(k)}) \to (\widehat{A}_{ij} \hat{x}_j, \hat{x}_i), \end{aligned}$$

for $k \to \infty$, and $i, j \in \{1, \dots, n\}$.

Further, let

$$\delta := \frac{1}{4 \sup \big\{ \|x_i^{(k)}\| \ :\ i = 1, \dots, n,\ k \in \mathbb{N} \big\}} > 0.$$

(See the definition of δ in the proof of Theorem 5.10.) Then, the functions

$$\hat{\eta} : B_\delta(0) \to \widehat{\mathcal{H}}^*,\ \hat{\eta}(w) := \hat{\xi} + w \hat{x}, \quad \eta^{(k)} : B_\delta(0) \to \mathcal{H}^*,\ \eta^{(k)}(w) := \xi^{(k)} + w x^{(k)} \tag{5.21}$$

for $k \in \mathbb{N}$ are well defined and we have

$$\begin{aligned} \lim_{k\to\infty} \|\eta_i^{(k)}(w)\|^2 &= \lim_{k\to\infty} \big(\|\xi_i^{(k)}\|^2 + ((\xi_i^{(k)}, x_i^{(k)}) + (x_i^{(k)}, \xi_i^{(k)}))w + \|x_i^{(k)}\|^2 w^2 \big) \\ &= \|\hat{\xi}_i\|^2 + ((\hat{\xi}_i, \hat{x}_i) + (\hat{x}_i, \hat{\xi}_i))w + \|\hat{x}_i\|^2 w^2 = \|\hat{\eta}_i(w)\|^2 \end{aligned}$$

for $w \in B_\delta(0)$. Moreover, for $k \to \infty$,

$$\begin{aligned} &(A_{ij} \eta_j^{(k)}(w), \eta_i^{(k)}(\overline{w})) \\ &\quad = (A_{ij} \xi_j^{(k)}, \xi_i^{(k)}) + ((A_{ij} x_j^{(k)}, \xi_i^{(k)}) + (A_{ij} \xi_j^{(k)}, x_i^{(k)}))w + (A_{ij} x_j^{(k)}, x_i^{(k)}) w^2 \\ &\quad \to (\widehat{A}_{ij} \hat{\xi}_j, \hat{\xi}_i) + ((\widehat{A}_{ij} \hat{x}_j, \hat{\xi}_i) + (\widehat{A}_{ij} \hat{\xi}_j, \hat{x}_i))w + (\widehat{A}_{ij} \hat{x}_j, \hat{x}_i) w^2 \\ &\quad = (\widehat{A}_{ij} \hat{\eta}_j(w), \hat{\eta}_i(\overline{w})), \end{aligned}$$

thus $A_{\eta^{(k)}(w)} \to \hat{A}_{\hat{\eta}(w)}$, $k \to \infty$, $w \in B_\delta(0)$. Clearly, then also $F_{\eta^{(k)}(w)}(z) \to \widehat{F}_{\hat{\eta}(w)}(z)$, $k \to \infty$, $w \in B_\delta(0)$, for all $z \in \mathbb{C}$. Applying Lemma 5.15, we see that

$$\sigma_{\mathrm{p}}(\widehat{F}_{\hat{\eta}(w)}) \subset \overline{W_{\mathcal{H}}(F)}, \quad w \in B_\delta(0). \tag{5.22}$$

Now define the analytic matrix function $\widehat{B}$ analogous to (5.12), that is,

$$\widehat{B} : B_\delta(0) \to M_n(\mathbb{C}), \quad \widehat{B}(w) := \begin{pmatrix} \frac{(\widehat{A}_{11}\hat{\eta}_1(w), \hat{\eta}_1(\overline{w}))}{(\hat{\eta}_1(w), \hat{\eta}_1(\overline{w}))} & \cdots & \frac{(\widehat{A}_{1n}\hat{\eta}_n(w), \hat{\eta}_1(\overline{w}))}{(\hat{\eta}_1(w), \hat{\eta}_1(\overline{w}))} \\ \vdots & & \vdots \\ \frac{(\widehat{A}_{n1}\hat{\eta}_1(w), \hat{\eta}_n(\overline{w}))}{(\hat{\eta}_n(w), \hat{\eta}_n(\overline{w}))} & \cdots & \frac{(\widehat{A}_{nn}\hat{\eta}_n(w), \hat{\eta}_n(\overline{w}))}{(\hat{\eta}_n(w), \hat{\eta}_n(\overline{w}))} \end{pmatrix}.$$

Then $\sigma_{\mathrm{p}}(\widehat{B}(t)) = \sigma_{\mathrm{p}}(\widehat{F}_{\hat{\eta}(t)})$, $t \in (-\delta, \delta)$, as we have already seen in the proof of Theorem 5.10. Hence, by (5.22),

$$\widehat{W} := \bigcup \{\sigma_{\mathrm{p}}(\widehat{B}(t)) \,:\, t \in (-\delta, \delta)\} \subset \overline{W_{\mathcal{H}}(F)}.$$

Again it follows that λ_0 is a corner of $\widehat{W}$ and, proceeding just as in the proof of Theorem 5.10, we get the equation corresponding to (5.14):

$$\sum_{\widehat{N} \in \widehat{\mathcal{H}}} r_{\widehat{M}\widehat{N}} \widehat{F}_{\widehat{M}\widehat{N}}(\lambda_0) \hat{\xi}_{\widehat{N}} = 0, \quad M \in \mathcal{H}. \tag{5.23}$$

Note that now, in contrast to the proof of Theorem 5.10, the conditions of Lemma 5.9 on $\widehat{F}$, $\hat{\xi}$ and λ_0 are not fulfilled for arbitrary $\hat{x} \in \widehat{H}$ anymore but still for those of the special form required in Lemma 5.9.

An induction as in the proof of Theorem 5.10 shows that $\lambda_0 \in \sigma_{\mathrm{p}}(\widehat{F}_{\widehat{\mathcal{H}'}})$ for some $\mathcal{H}' \subset \mathcal{H}$: If $r_{\widehat{M}\widehat{M}} \neq 0$ for all $M \in \mathcal{H}$, from Lemma 5.8 it follows again that λ_0 is an eigenvalue of $\widehat{F}$. If, on the other hand, $r_{\widehat{M}\widehat{M}} = 0$ for some $M \in \mathcal{H}$, we set $\mathcal{H}' := \mathcal{H} \setminus \{M\}$ and conclude that

$$0 = r_{\widehat{M}\widehat{M}} = \det \widehat{F}_{\hat{\xi}}^{\widehat{\mathcal{H}}'}(\lambda_0) = \lim_{k \to \infty} \det F_{\xi^{(k)}}^{\mathcal{H}'}(\lambda_0),$$

that is, by (2.8) which holds for $F = A - \cdot$, $\lambda_0 \in \overline{W_{\mathcal{H}'}(F)} \subset \overline{W_{\mathcal{H}}(F)}$ is a corner of $\overline{W_{\mathcal{H}'}(F)}$ and we may apply the induction hypothesis. The claim now follows immediately from (5.17). □

Remark 5.18. To see why the additional assumption $\dim M \geq 2$, $M \in \mathcal{H}$, may be dropped again if $|\mathcal{H}| = 2$, replace A by $\widehat{A}$ and ξ by $\hat{\xi}$ in Remark 5.11. It is clear that if $A_{22} = \alpha$, then also $\widehat{A}_{22} = \alpha$; the only situation where we need a new argument is in the proof of the equality $\widehat{A}_{21}\hat{\xi}_1 - (\widehat{A}_{21}\hat{\xi}_1, \hat{\xi}_2)\hat{\xi}_2 = 0$, as $\{\hat{\xi}_2\}$ is not an orthonormal basis of $\widehat{M}_2$ anymore. Using Remark 5.14 and that $\{\xi_2^{(k)}\}$ is an orthonormal basis of M_2 for every $k \in \mathbb{N}$, we get

$$\begin{aligned} (\widehat{A}_{21}\hat{\xi}_1, \hat{\xi}_2)\hat{\xi}_2 &= \lim_{k \to \infty} \big(A_{21}\xi_1^{(k)}, \xi_2^{(k)}\big) \cdot \big(\xi_2^{(k)}\big)_{k=1}^{\infty} = \big((A_{21}\xi_1^{(k)}, \xi_2^{(k)})\xi_2^{(k)}\big)_{k=1}^{\infty} \\ &= \big(A_{21}\xi_1^{(k)}\big)_{k=1}^{\infty} = \widehat{A}_{21}\hat{\xi}_1, \end{aligned}$$

which shows the desired equality.

5.4 Corners of $W_{\mathcal{H}}(F)$

Theorem 5.10 can be generalized to analytic operator functions with minor changes in the proof. In addition, we make use of Weierstraß' Preparation Theorem and the following polynomial version of Proposition 5.5:

Recall that, for an analytic function $f : \Omega \to \mathbb{C}$ and $U \subset \Omega$, the set of zeros of f in U is denoted by $N_U(f)$.

Corollary 5.19. *Let $U \subset \mathbb{C}$ be an open set, $\rho > 0$, $g_0, \dots, g_{n-1} : B_\rho(0) \to \mathbb{C}$ be analytic functions and*

$$G(z, w) := z^k + g_{n-1}(w)z^{k-1} + \cdots + g_1(w)z + g_0(w), \quad (z, w) \in U \times B_\rho(0).$$

If $\lambda_0 \in U$ is a corner of the set $N := \bigcup\{N_U(G(\,\cdot\,, t)) \,:\, t \in (-\rho, \rho)\}$ and $G(\lambda_0, 0) = 0$, then there exists an $r > 0$ and a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that

$$\mu(0) = \lambda_0, \quad \mu'(0) = 0, \quad G(\mu(t), t) = 0, \quad t \in (-r, r).$$

Proof. Define the analytic matrix function A by

$$A : B_\rho(0) \to M_k(\mathbb{C}), \quad A(w) := \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -g_0(w) & -g_1(w) & \cdots & -g_{k-1}(w) \end{pmatrix}.$$

and let $W := \{\sigma_{\mathrm{p}}(A(t)) \,:\, t \in (-\rho, \rho)\}$. Then $\lambda_0 \in \sigma_{\mathrm{p}}(A(0))$ and

$$\sigma_{\mathrm{p}}(A(w)) \cap U = N_U(G(\,\cdot\,, w)), \quad w \in B_\rho(0). \tag{5.24}$$

In particular, we have $W \cap U = N$, thus λ_0 is also a corner of W. Applying Corollary 5.19 yields a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that $\mu(0) = \lambda_0$, $\mu'(0) = 0$ and $\mu(t) \in \sigma_{\mathrm{p}}(A(t))$, $t \in (-r, r)$. By the continuity of μ we may additionally assume that $\mu(t) \in U$, $t \in (-r, r)$. But then we have $G(\mu(t), t) = 0$, $t \in (-r, r)$ by (5.24). □

Theorem 5.20. *Assume that $\mathcal{H} \in \mathcal{Z}(H)$ fulfills the dimension condition* (1.6) *or $|\mathcal{H}| = 2$, and let $F : \Omega \to L(H)$ be an analytic operator function. Assume that $\lambda_0 \in W_{\mathcal{H}}(F)$ is a corner of $W_{\mathcal{H}}(F)$ and $\xi \in \mathcal{H}^*$ is a vector for which* $\det F_\xi(\lambda_0) = 0$. *Then*

$$\lambda_0 \in \sigma_{\mathrm{p}}(F_{\mathcal{H}'})$$

for some $\mathcal{H}' \subset \mathcal{H}$ with an eigenvector in $\operatorname{span}\{\xi_M \,:\, M \in \mathcal{H}'\}$.

Proof. First of all note that the function $z \mapsto \det F_\xi(z)$ does not vanish identically. In fact, $\det F_\xi(z) = 0$ for all $z \in \Omega$ would imply $W_{\mathcal{H}}(F) = \Omega$, contradicting the fact that $\lambda_0 \in \Omega$ is a corner of $W_{\mathcal{H}}(F)$.

Write $\mathcal{H} = \{M_1, \dots, M_n\}$ and let $x \in H$ be arbitrary. Define $\delta > 0$ and the mapping $\eta : B_\delta(0) \to \mathcal{H}^*$ as in the beginning of the proof of Theorem 5.10, and let the function $b : \Omega \times B_\delta(0) \to \mathbb{C}$ be given by

$$b(z,w) := \det \begin{pmatrix} (F_{11}(z)\eta_1(w), \eta_1(\overline{w})) & \dots & (F_{1n}(z)\eta_n(w), \eta_1(\overline{w})) \\ \vdots & & \vdots \\ (F_{n1}(z)\eta_1(w), \eta_n(\overline{w})) & \dots & (F_{nn}(z)\eta_n(w), \eta_n(\overline{w})) \end{pmatrix}. \qquad (5.25)$$

Then b is analytic and $b(\,\cdot\,, 0) = \det F_\xi \not\equiv 0$ has a zero in $z = \lambda_0$, say, of multiplicity $k \geq 1$. By Weierstraß' Preparation Theorem (see [Mar77, Theorem II.3.9]) there are $\rho > 0$ and analytic functions $g_0, \dots, g_{k-1} : B_\rho(0) \to \mathbb{C}$, $Q : B_\rho(\lambda_0) \times B_\rho(0) \to \mathbb{C}$ such that

$$b(z,w) = (z^k + g_{k-1}(w)z^{k-1} + \dots + g_1(w)z + g_0(w))Q(z,w) =: G(z,w)Q(z,w)$$

and $Q(z,w) \neq 0$ for all $(z,w) \in B_\rho(\lambda_0) \times B_\rho(0)$. We have

$$G(z,t) = 0 \;\Rightarrow\; 0 = b(z,t) = \det F_{\eta(t)}(z) \;\Rightarrow\; z \in W_{\mathcal{H}}(F)$$

for $(z,t) \in B_\rho(\lambda_0) \times (-\rho, \rho)$. Hence, writing $U := B_\rho(\lambda_0)$,

$$N := \bigcup \{N_U(G(\,\cdot\,, t)) \,:\, t \in (-\rho, \rho)\} \subset W_{\mathcal{H}}(F) \qquad (5.26)$$

and, as $\lambda_0 \in N$ is a corner of $W_{\mathcal{H}}(F)$, it is also a corner of N. By Corollary 5.19, there exist $r > 0$ and a differentiable function $\mu : (-r, r) \to \mathbb{C}$ such that $\mu(0) = \lambda_0$, $\mu'(0) = 0$ and $G(\mu(t), t) = 0$, $t \in (-r, r)$. The latter implies that $\det F_{\eta(t)}(\mu(t)) = 0$, thus $\mu(t) \in \sigma_{\mathrm{p}}(F_{\eta(t)})$ for all $t \in (-r, r)$. That is, (5.13) in the proof of Theorem 5.10 holds again. The remaining part of the proof may now be literally taken from the proof of Theorem 5.10, additionally using Remark 5.11 for the case $|\mathcal{H}| = 2$. □

Example 5.21. As examples for Theorem 5.20 we again consider the functions F and G with respect to the decompositions $\mathbb{C}^2 \times \mathbb{C}$ and $\mathbb{C}^2 \times \mathbb{C}^2$, respectively, from (2.4). The • in the left picture denotes π which is clearly an eigenvalue of the matrix function $F_{22}(z) = (z \sin z)$. The • in the right picture is an eigenvalue of the matrix function G itself.

Theorem 5.22. *Assume that $\mathcal{H} \in \mathcal{Z}(H)$ fulfills the dimension condition* (1.6) *or $|\mathcal{H}| = 2$, and let $F : \Omega \to L(H)$ be an analytic operator function for which* (2.5) *holds. If $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$ is a corner of $W_{\mathcal{H}}(F)$, then*

$$\lambda_0 \in \sigma_{\mathrm{app}}(F_{\mathcal{H}'}) \subset \sigma(F_{\mathcal{H}'})$$

for some $\mathcal{H}' \subset \mathcal{H}$.

Proof. Write $\mathcal{H} = \{M_1, \dots, M_n\}$. It follows from $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$ and (2.8) that $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$. Therefore we may choose a sequence $\hat{\xi} = (\xi^{(k)})_1^\infty \subset \mathcal{H}^\square$ such that $\det F_{\xi^{(k)}}(\lambda_0) \to 0$, $k \to \infty$. For $k \in \mathbb{N}$ and $i, j, l, m \in \{1, \dots, n\}$, the functions

$$\begin{gathered} (F_{ij}(\,\cdot\,)\xi_j^{(k)}, \xi_i^{(k)}), \;\; (F_{il}(\,\cdot\,)F_{lj}(\lambda_0)\xi_j^{(k)}, \xi_i^{(k)}), \\ (F_{ij}(\,\cdot\,)\xi_j^{(k)}, F_{il}(\lambda_0)\xi_l^{(k)}), \;\; (F_{il}(\,\cdot\,)F_{lj}(\lambda_0)\xi_j^{(k)}, F_{im}(\lambda_0)\xi_m^{(k)}) \end{gathered} \qquad (5.27)$$

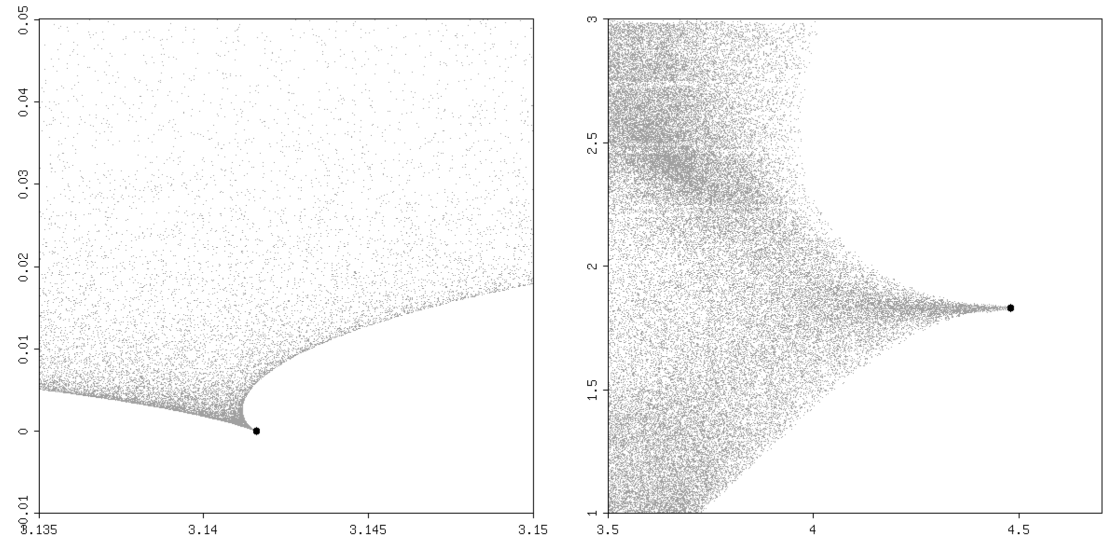


Figure 5.2: Parts of the block numerical ranges of the matrix functions F and G from (2.4) with respect to the decompositions $\mathbb{C}^2 \times \mathbb{C}$ and $\mathbb{C}^2 \times \mathbb{C}^2$, respectively.

are analytic. By Montel's Theorem, passing to appropriate subsequences, we may assume that, for $k \to \infty$, all these functions converge uniformly on compact subsets of Ω to the corresponding functions where F is replaced by $\widehat{F}$ and $\xi^{(k)}$ by $\hat{\xi}$, respectively.

As in the proof of Theorem 5.17, fix $\beta \in \{1, \mathrm{i}\}$ and $\nu \in \{1, \dots, n\}$. Define the vectors $\hat{x} \in \widehat{H}$, $x^{(k)} \in H$, the complex numbers $\hat{r}_{ij}$, $r_{ij}^{(k)}$ and the functions $\hat{\eta} : B_\delta(0) \to \widehat{\mathcal{H}}^*$, $\eta^{(k)} : B_\delta(0) \to \mathcal{H}^*$ as in (5.19), (5.20) and (5.21) in the proof of Theorem 5.17. Using the convergence of the functions in (5.27) (corresponding to (5.18)) for $k \to \infty$, we get $F_{\eta^{(k)}(w)}(z) \to \widehat{F}_{\hat{\eta}(w)}(z)$, $k \to \infty$, $w \in B_\delta(0)$, for all $z \in \Omega$ analogous to the proof of Theorem 5.17. Hence, as in (5.22),

$$\sigma_{\mathrm{p}}(\widehat{F}_{\hat{\eta}(w)}) \subset \overline{W_{\mathcal{H}}(F)}, \quad w \in B_\delta(0), \tag{5.28}$$

by Lemma 5.15. Now we proceed as in the proof of Theorem 5.20. Note that $\det \widehat{F}_{\hat{\xi}} \not\equiv 0$. In fact, $\det \widehat{F}_{\hat{\xi}} \equiv 0$ would imply that $0 \in \overline{D_{\mathcal{H}}(F(z))}$, thus, by (2.8), $z \in \overline{W_{\mathcal{H}}(F)}$ for every $z \in \Omega$. Therefore, $W_{\mathcal{H}}(F) = \Omega$, contradicting the fact that $\lambda_0 \in \Omega$ is a corner of $W_{\mathcal{H}}(F)$. Define the analytic function $\hat{b}$ as in (5.25), that is, $\hat{b} : \Omega \times B_\delta(0) \to \mathbb{C}$,

$$\hat{b}(z, w) := \det \begin{pmatrix} (\widehat{F}_{11}(z)\hat{\eta}_1(w), \hat{\eta}_1(\overline{w})) & \dots & (\widehat{F}_{1n}(z)\hat{\eta}_n(w), \hat{\eta}_1(\overline{w})) \\ \vdots & & \vdots \\ (\widehat{F}_{n1}(z)\hat{\eta}_1(w), \hat{\eta}_n(\overline{w})) & \dots & (\widehat{F}_{nn}(z)\hat{\eta}_n(w), \hat{\eta}_n(\overline{w})) \end{pmatrix}.$$

Again, $\hat{b}(\,\cdot\,, 0) = \det \widehat{F}_{\hat{\xi}} \not\equiv 0$ has a zero of finite multiplicity in $z = \lambda_0$. Thus, $\hat{b}$ allows a factorization $\hat{b}(z, w) = \widehat{G}(z, w)\widehat{Q}(z, w)$, $(z, w) \in B_\rho(\lambda_0) \times B_\rho(0)$, just as in the proof of Theorem 5.20. For $(z, t) \in B_\rho(\lambda_0) \times (-\rho, \rho)$ we have

$$\widehat{G}(z, t) = 0 \;\Rightarrow\; 0 = \hat{b}(z, t) = \det \widehat{F}_{\hat{\eta}(t)}(z) \;\Rightarrow\; z \in \sigma_{\mathrm{p}}(F_{\hat{\eta}(t)}) \subset \overline{W_{\mathcal{H}}(F)},$$

where the last inclusion is taken from (5.28). Writing $U := B_\rho(0)$, it follows that

$$\widehat{N} := \{N_U(\widehat{G}(\,\cdot\,,t)) \,:\, t \in (-\rho,\rho)\} \subset \overline{W_{\mathcal{H}}(F)}.$$

This is the equation corresponding to (5.26) in the proof of Theorem 5.20. Applying Corollary 5.19, we get a differentiable function $\hat{\mu} : (-r,r) \to \mathbb{C}$ such that $\hat{\mu}(0) = \lambda_0$, $\hat{\mu}'(0) = 0$ and $\hat{\mu}(t) \in \sigma_{\mathrm{p}}(\widehat{F}_{\hat{\eta}(t)})$, $t \in (-r,r)$. Hence, (5.13) in the proof of Theorem 5.10 holds again for $\widehat{F}$, $\hat{\xi}$ and λ_0 (and all $\hat{x}$ required in (5.9)). Proceeding as in the proof of Theorem 5.10 we get

$$\sum_{\widehat{N}\in\widehat{\mathcal{H}}} r_{\widehat{M}\widehat{N}} \widehat{F}_{\widehat{M}\widehat{N}}(\lambda_0)\hat{\xi}_{\widehat{N}} = 0, \quad M \in \mathcal{H},$$

which corresponds to (5.23). Now, the proof of the theorem is finished in the same way as the proof of Theorem 5.17 (note that (2.8) holds again by the assumption (2.5)), using Remark 5.18 for the case $|\mathcal{H}| = 2$. $\square$

Appendix A

Computing block numerical ranges

In the following, a description of the method used to generate the examples of block numerical ranges of matrices, matrix polynomials and matrix functions within this thesis is given. An algorithm is known to compute the numerical range of a matrix by determining an approximation of its boundary (see, e. g., [GR97, § 5.6]; note that this method heavily exploits the convexity of the numerical range). However, no such algorithm is known for the block numerical range by now. The method we present here is what would be called a 'brute force attack'.

To get an approximation of the block numerical range of a complex $m \times m$ **matrix** A with respect to a decomposition

$$\mathcal{H} = \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_n} \in \mathcal{Z}(\mathbb{C}^m),$$

we simply plot the eigenvalues of the matrices A_x for a reasonably large number of randomly chosen vectors $x \in \mathcal{H}^\square$.

(1) To generate a normed vector $x_i \in \mathbb{C}^{m_i}$, let

$$y_{is} = r_{is} \exp(2\pi\varphi_{is}), \quad s = 1, \dots, m_i,$$

where the r_{is}, $\varphi_{is} \in [0, 1)$ are randomly generated numbers using the algorithm from [MN98], and normalize the resulting vector $y_i = (y_{i1}, \dots, y_{im_i})$ to obtain x_i. Doing so for every $i = 1, \dots, n$, results in a random vector $x = (x_1, \dots, x_n) \in \mathcal{H}^\square$.

(2) To calculate, for some random $x \in \mathcal{H}^\square$ from (1), the eigenvalues of the matrix A_x, the `ZGGEV` routine[1] from LAPACK is used. This routine allows to determine the eigenvalues and eigenvectors of generalized eigenvalue problems of the form

$$Bv = \lambda Cv, \tag{A.1}$$

for arbitrary $k \times k$ matrices B and C.

To approximate the block numerical range of a **matrix polynomial** P of degree d, we have to determine the zeros of the polynomials $p_x(z) = \det P_x(z)$, $z \in \mathbb{C}$, again

[1] `http://www.netlib.org/lapack/complex16/zggev.f`

for a large number of randomly generated vectors $x \in \mathcal{H}^{\square}$. This is done by applying the `ZGGEV` routine mentioned above to the companion polynomial of the polynomial p_x which has the same zeros as p_x; that is, if $p_x(z) = a^{[nd]}z^{nd} + \cdots + a^{[1]}z + a^{[0]}$, the equation (A.1) is solved for the matrices

$$B = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a^{[0]} & -a^{[1]} & \cdots & -a^{[nd-1]} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a^{[nd]} \end{pmatrix}.$$

To calculate an approximation of the block numerical range of an analytic **matrix function** $F : \Omega \to M_m(\mathbb{C})$, a straightforward implementation of the argument principle is used. Recall that if $f : \Omega \to \mathbb{C}$ is an analytic function and $\Gamma : [0,1] \to \Omega$ is a piecewise smooth Jordan curve such that $U := \operatorname{int}\Gamma \subset \Omega$, then, provided that f has no zeros in Γ, the number $\nu_U(f)$ of zeros of f in U, counting multiplicities, is given by

$$\nu_U(f) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{[\arg f]_\Gamma}{2\pi},$$

where $[\arg f]_\Gamma$ denotes the increment of the argument of f along Γ. This increment may be approximated numerically: Let $0 = t_0 < t_1 < \cdots < t_K = 1$ be a subdivision of the interval $[0,1]$ and $z_k := \Gamma(t_k)$, $k = 0, \dots, K$. If $|z_k - z_{k-1}|$, $k = 1, \dots, K$, is small enough, then the number

$$\hat{\nu}_U(f) = \frac{1}{2\pi} \sum_{k=1}^{K} \big(\arg f(z_k) - \arg f(z_{k-1})\big) = \frac{1}{2\pi} \sum_{k=1}^{K} \arg\left(\frac{f(z_k)}{f(z_{k-1})}\right)$$

may be considered to be the number of zeros of f in U.

In practice, to determine the zeros of f in a given rectangle $R \subset \mathbb{C}$, apply this method to the boundary of R. If f is seen to have a zero in R, subdivide the rectangle into smaller rectangles and apply the method again. This procedure is iterated until the desired precision is reached, that is, the rectangles determined to contain zeros of f have a maximal diameter of some given ε.

To get an approximation of the block numerical range of F within a rectangle R, this method of determining zeros is applied to the analytic functions $f_x(z) = \det F_x(z)$, $z \in \Omega$, again for a sufficiently large number of randomly generated vectors $x \in \mathcal{H}^{\square}$.

Another approach. A different method to get an impression of the shape of the block numerical range of matrix functions is based on the following observation: Define, for $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$,

$$W^{\varepsilon}_{\mathcal{H}}(F) := \{z \in \Omega \,:\, \exists\, x \in \mathcal{H}^{\square} \;\; |\det F_x(z)| < \varepsilon\}, \quad \varepsilon > 0.$$

Lemma A.1. *Let $\mathcal{H} \in \mathcal{Z}(H)$ and $F : \Omega \to L(H)$ be a continuous operator function. Then*

$$\overline{W_{\mathcal{H}}(F)} \cap \Omega \subset \bigcap_{\varepsilon > 0} W^{\varepsilon}_{\mathcal{H}}(F).$$

If, additionally, F is analytic and (2.5) *holds for F, then*

$$\overline{W_{\mathcal{H}}(F)} \cap \Omega = \bigcap_{\varepsilon>0} W_{\mathcal{H}}^{\varepsilon}(F).$$

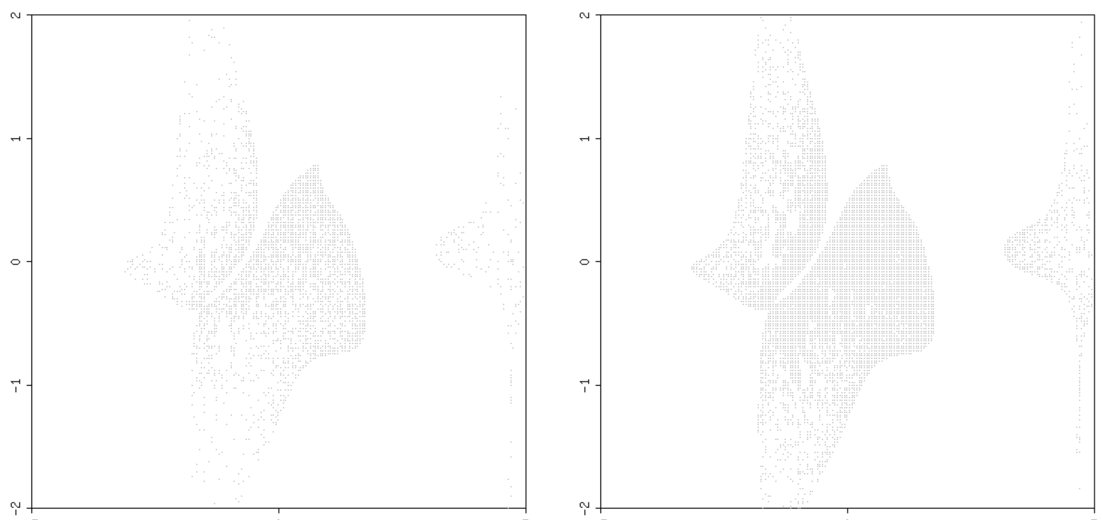
Proof. Let $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$. Then $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$ by Proposition 2.9. Hence, for every $\varepsilon > 0$ there is an $x \in \mathcal{H}^{\square}$ such that $|\det F_x(\lambda_0)| < \varepsilon$, i. e., $\lambda_0 \in W_{\mathcal{H}}^{\varepsilon}(F)$. If, on the other hand, $\lambda_0 \in W_{\mathcal{H}}^{\varepsilon}(F)$ for all $\varepsilon > 0$, then clearly $0 \in \overline{D_{\mathcal{H}}(F(\lambda_0))}$. Thus, if (2.5) holds, we obtain $\lambda_0 \in \overline{W_{\mathcal{H}}(F)} \cap \Omega$ by Proposition 2.12. □

Now let $F : \Omega \to M_m(\mathbb{C})$ be an analytic matrix function. Then, as $W_{\mathcal{H}}(F)$ is closed by Corollary 2.10, we have

$$W_{\mathcal{H}}(F) = \bigcap_{\varepsilon>0} W_{\mathcal{H}}^{\varepsilon}(F).$$

Therefore we may, for small $\varepsilon > 0$, consider the set $W_{\mathcal{H}}^{\varepsilon}(F)$ as an approximation of $W_{\mathcal{H}}(F)$ (from outside). Unfortunately, also $W_{\mathcal{H}}^{\varepsilon}(F)$ can not be calculated directly. Instead, we have to approximate $W_{\mathcal{H}}^{\varepsilon}(F)$ numerically (from within again). This may be done as follows: Let $\varepsilon > 0$, $z_0 \in \mathbb{C}$ and $K \in \mathbb{N}$. If the calculation of $|\det F_x(z_0)|$ for K randomly generated vectors $x \in \mathcal{H}^{\square}$ did not yield at least one value $\leq \varepsilon$, then we assume that $z_0 \notin W_{\mathcal{H}}^{\varepsilon}(F)$. Covering a region of the complex plane with a grid $G \subset \Omega$, and applying this method to every $z_0 \in G$, we may, for large values of K, expect to get some sort of approximation of $W_{\mathcal{H}}^{\varepsilon}(F)$. If, additionally, $\varepsilon > 0$ is small enough, this should be a reasonable approximation of $W_{\mathcal{H}}(F)$. In practice, of course, this approximation is far from good in most cases. However, the method presented here could be the starting point for more extensive considerations.

Example A.2. The pictures below show approximations of $W_{\mathcal{H}}^{\varepsilon}(F)$, where F is the matrix function from (2.4) and $\mathcal{H} = \mathbb{C} \times \mathbb{C}^2$. Here, $\varepsilon = 0.01$, $K = 20000$ (left) and $\varepsilon = 0.03$, $K = 10000$ (right).



Remark A.3. Care has to be taken in the choice of $\varepsilon > 0$. If, for example, F is $\mathcal{H}$-diagonal, then

$$|\det F_x(z)| = |(F_{11}(z)x_1, x_1)| \cdots |(F_{nn}(z)x_n, x_n)| \leq \|F_{11}\|_G \cdots \|F_{nn}\|_G,$$

where $\|\cdot\|_G$ denotes the maximum norm on the grid G. For every $\varepsilon > \prod_{i=1}^n \|F_{ii}\|_G$, we will get G itself as an approximation of $W_{\mathcal{H}}^{\varepsilon}(F)$. (To give an extreme example, let F be a constant operator function, $F(z)$ being the multiplication by a constant $\alpha \neq 0$, $z \in \Omega$. Then $W_{\mathcal{H}}(F) = \emptyset$ and $W_{\mathcal{H}}^{\varepsilon}(F) = \Omega$ for $\varepsilon \geq |\alpha|^n$.)

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