Representation Theoretical Construction of the Classical Limit and Spectral Statistics of Generic Hamiltonian Operators

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Contents

List of Figures

1 Introduction

The theory of spectral statistics is concerned with the spectral properties of ensembles of linear operators. Typically, these depend on a parameter N which is supposed to be very large or even approaching infinity. The origin of this field is quantum physics, where such ensembles arose as models for the energy spectra of large atoms.

Another branch of physics, namely semiclassical physics, is also concerned with such ensembles and their spectral statistics. In semiclassical physics large values of N should correspond to a quantum mechanical system which approaches classical mechanics. Details about these relations can be found in [Meh91] and [Haa99].

Finally, spectral statistics have been studied in the context of number theory, with the most famous example being the distribution of zeros of the Riemann ζ-function on the critical line. An introduction to this field is given in [Sna00].

Under the assumption of genericity one might hope that there exist natural sequences of operators taken from these ensembles such that the spectral properties of the individual operators reflect those of the ensembles.

We are concerned here with two examples, in which spectral statistics appear. The first being the theory of Random Matrices. In this theory natural sequences of symmetric spaces with invariant measures on them are given. These spaces have natural representations as matrices and one is interested in the limit of the spectral statistics as $N \to \infty$. An example is the sequence of unitary groups $\mathrm{U}(N)$ with the Haar measure. In [KS99] it is proven that a limit measure of a special kind of spectral statistics exists for this example.

The second example, in which spectral statistics appear, is given by the approach suggested in [GHK00]. In this article the authors consider two fixed operators in the universal enveloping algebra of $SL(3, \mathbb{C})$ in a sequence of irreducible representations of $SL(3, \mathbb{C})$ and study the spectral statistics by numerical methods. The motivation from the approach stems from a previous paper (cf. [GK98]) of two of the authors: Such a sequence of irreducible representations occurs in the construction of the classical mechanical system in the limit of a quantum mechanical system with $SL(3, \mathbb{C})$ symmetry. We will follow this approach in the following chapters.

Our main device in the study of spectral statistics is the nearest neighbor statistics, i.e. the normalized distribution of distances of neighboring eigenvalues (counted with multiplicity) of such linear operators. It is frequently drawn as a histogram (see Figure 1.1). A detailed explanation of this plot can be found in the Appendix.

The nearest neighbor statistics lead to Borel measures on the positive real line by putting a Dirac measure for every occurring distance of neighboring eigenvalues with proper normalization. Out of the wealth of notions of convergence for such

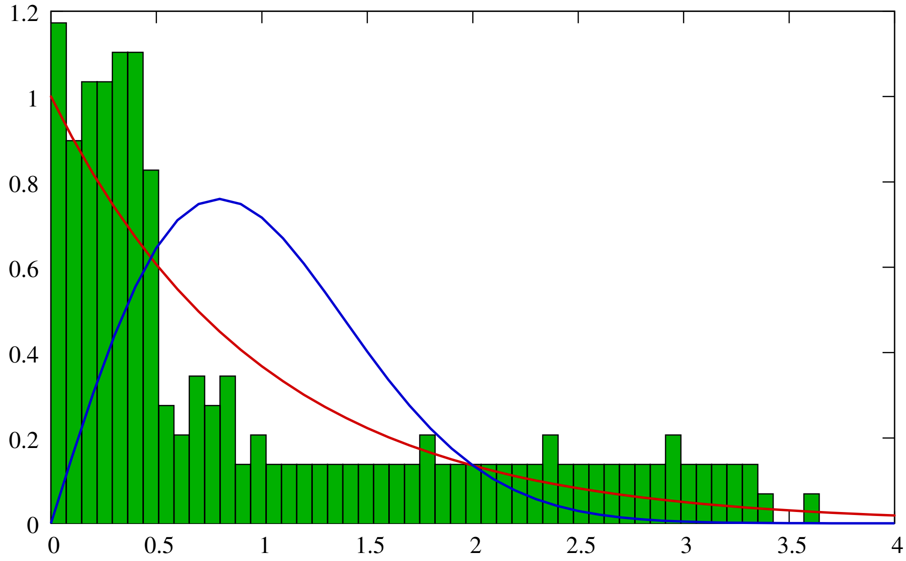

Figure 1.1: A sample histogram of the nearest neighbor statistics

measures we choose the weak convergence (in probability theory: convergence in distribution) and the Kolmogorov-Smirnov convergence. The Kolmogorov-Smirnov distance of two measures μ, ν is given by

$$d_{KS}(\mu, \nu) = \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{t} d\mu - \int_{-\infty}^{t} d\nu \right|, \tag{1.1}$$

i.e., Kolmogorov-Smirnov convergence is uniform convergence of the cumulative distribution functions. We will examine d_{KS} for sequences of individual operators relative to a fixed measure ν, but also average d_{KS} with respect to a fixed probability measure ν over the full ensemble. Here sequences of irreducible representations will arise.

This text is structured into six chapters. Following the approach in [GK98] we give a general construction of the classical limit for semi-simple compact Lie groups in Chapter 2. This can be done in a functorial way, but the objective of Chapter 2 is to give an interpretation as a mathematical limit as a parameter n converges to ∞.

Chapter 3 deals with the spectral statistics of operators in the Lie algebra along sequences of irreducible representations. It is necessary to discuss possible scalings of these operators in this context.

The goal of Chapter 4 is to study the spectral statistics of exponentiated operators, which satisfy certain conditions of genericity, in a certain completion of the universal enveloping algebra of a semi-simple complex Lie group. The main tools are Birkhoff's Ergodic Theorem and an estimation on d_{KS} for maximal tori of $U(N)$.

Chapter 5 is devoted to the proof of this estimation, where we follow the structure of [KS99] for the proof.

In the Appendix we collect the necessary background facts of representation theory and symplectic geometry for the readers' convenience. The Appendix closes with some general observations about nearest neighbor statistics.

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2 Representation Theoretical Construction of the Classical Limit

In this chapter we will give a construction of the classical limit of Hamiltonian mechanics by a representation theoretical approach. Our method is an abstract generalization of the method given in [GK98] and [Gnu00] and covers systems with compact semi-simple Lie groups as symmetry groups.

The following notation will be used without further notice (for details cf. Appendix): K is a compact semi-simple Lie group with complexification G and the corresponding Lie algebras are $\mathfrak{k}$ and $\mathfrak{g}$. Every representation of K will be assumed to be continuous, finite-dimensional and unitary, where the scalar product is denoted by $\langle \cdot, \cdot \rangle$. By convention the scalar product is complex linear in the second, and anti-linear in the first variable.

Furthermore we assume that we have fixed a Borel subgroup $B \subset G$ and obtain a notion of positivity of roots and weights. Recall that the choice of B also determines a maximal tours $T \subset K$.

2.1 The Classical Limit in the Simple Case

A guiding principle in quantum mechanics is that of correspondence. It states that quantum mechanical systems whose size is large compared to microscopical length scales can be described by classical physics. The classical system attached to the quantum mechanical system is called the classical limit (cf. [GK98]). So there should be some kind of functor from Hilbert spaces with Hamiltonian operators to symplectic manifolds with Hamiltonian functions. Actually, one might require that this functor is inverse to so-called geometric quantization. At least it should satisfy the Dirac correspondence, i.e., if ξ_{H_1} and ξ_{H_2} are two Hamiltonian operators with corresponding Hamiltonian functions h_1 and h_2, then the Lie bracket of ξ_{H_1} and ξ_{H_2} should correspond to the Poisson bracket of h_1 and h_2:

$$[\xi_{H_1}, \xi_{H_2}] \mapsto c\{h_1, h_2\}, \tag{2.1}$$

where c is a constant, usually $i\hbar$.

More often, one discusses the opposite direction, i.e., quantization (cf. [Woo97] Chapter 9.2). Therefore one may call the procedure presented here *dequantization*.

Let $\rho : K \to \mathrm{U}(V)$ be an irreducible representation. Let $\rho_* : \mathfrak{k} \to \mathrm{End}(V)$ be the induced representation of the Lie algebra. Both ρ and ρ_* extend to holomorphic resp. linear representations of the corresponding complexifications G and $\mathfrak{g}$. To keep notation as simple as possible we will also denote these by ρ and ρ_*

The map $\mu : \mathbb{P}(V) \to \mathfrak{k}^*$ given by

$$\mu^\xi([v]) = -2i\frac{\langle v, \rho_*(\xi).v\rangle}{\langle v, v\rangle} \forall \xi \in \mathfrak{k}, v \in \mathbb{P}(V) \tag{2.2}$$

is the momentum map with respect to the symplectic structure on $\mathbb{P}(V)$ induced by the Fubini-Study metric (cf. Appendix for details). Moreover, if $\lambda \in \mathfrak{t}^*$ is the highest weight of ρ, then

$$\mu([v_{max}]) = \lambda \tag{2.3}$$

for any vector v_{max} of highest weight.

Since μ is an K-equivariant map and the stabilizers of λ and v_{max} agree, this map is a symplectic diffeomorphism of the orbit $K.[v_{max}]$ onto the coadjoint orbit $K.\lambda$ with the Kostant-Kirillov form.

In the literature, this coadjoint orbit is called the **set of coherent states** (cf. [Per86], [Woo97]). To simplify notation we write $Z = K.\lambda$ for this set.

Equivariance implies that the map $\tilde{\mu} : \mathfrak{k} \to C^\infty(Z),\ \xi \mapsto \mu^\xi(\cdot)$, satisfies

$$\tilde{\mu}([\xi_1, \xi_2]) = \{\tilde{\mu}(\xi_1), \tilde{\mu}(\xi_2)\}. \tag{2.4}$$

If we compare this equation with the Dirac condition (2.1), then, up to constants, this is exactly what we are looking for. But the Lie algebra $\mathfrak{k}$ acts by *skew* self-adjoint operators on V. Thus we define $\mathrm{cl} : i\mathfrak{k} \to C^\infty(Z)$ for an element $\xi_H \in i\mathfrak{k}$ by

$$\mathrm{cl}(\xi_H)([x]) = \frac{1}{2}\tilde{\mu}(i\xi_H)(x) = \frac{\langle x, \rho_*(\xi_H).x\rangle}{\langle x, x\rangle}, \tag{2.5}$$

where the factor $\frac{1}{2}$ will become clear in the following. First note that while $i\xi_H$ is represented as a skew self-adjoint operator, ξ_H is self-adjoint. Now, we have the following version of the Dirac correspondence for the classical limit cl:

$$\begin{aligned} \mathrm{cl}(i[\xi_{H_1}, \xi_{H_2}]) &= \frac{1}{2}\tilde{\mu}(ii[\xi_{H_1}, \xi_{H_2}]) = \frac{1}{2}\tilde{\mu}([i\xi_{H_1}, i\xi_{H_2}]) \\ &= 2 \cdot \{\frac{1}{2}\tilde{\mu}(i\xi_{H_1}), \frac{1}{2}\tilde{\mu}(i\xi_{H_2})\} = 2 \cdot \{\mathrm{cl}(\xi_{H_1}), \mathrm{cl}(\xi_{H_2})\}. \end{aligned} \tag{2.6}$$

2.2 The Classical Limit in the General Case

So far our classical limit has been defined for those self-adjoint operators which can be expressed as the image of an element of $i\mathfrak{k}$ under ρ_*. But we want to define the classical limit for every self-adjoint linear operator on V. In fact, it will be defined for all linear operators on V, although in general we do not obtain real-valued functions on Z if we take the classical limit of an operator which is not self-adjoint.

Let $\mathcal{T}(\mathfrak{g})$ denote the full tensor algebra of $\mathfrak{g}$. The Lie algebra representation ρ_* extends uniquely to a representation $\rho_* : \mathcal{T}(\mathfrak{g}) \to \mathrm{End}(V)$. This map is surjective by the lemma of Burnside. Thus, in particular every self-adjoint operator is contained in the image of ρ_*.

We fix an $\mathbb{R}$-basis $\xi_1, \ldots, \xi_k$ of $i\mathfrak{k}$ for the rest of this chapter. Note that this is a $\mathbb{C}$-basis of $\mathfrak{g}$. Thus, an element ξ_H of $\mathcal{T}(\mathfrak{g})$ has a unique decomposition into homogeneous terms consisting of sums of "monomials" $\xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_p}$ for some indices $\alpha_j \in \{1, \ldots, k\}$. (These are not monomials in the usual sense because of the non-commutativity.)

Definition 2.1. *The classical limit of such a "monomial" is*

$$\mathrm{cl}(\xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_p}) := \mathrm{cl}(\xi_{\alpha_1}) \cdot \cdots \cdot \mathrm{cl}(\xi_{\alpha_p}). \tag{2.7}$$

The classical limit of

$$\xi_H = \sum \alpha_I \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_{p_I}} \quad \in \mathcal{T}(\mathfrak{g}) \tag{2.8}$$

is the sum of all classical limits of each "monomial" multiplied by the corresponding coefficient.

We call the resulting map $\mathrm{cl} : \mathcal{T}(\mathfrak{g}) \to C^\infty(Z, \mathbb{C})$ *the* ***classical limit map****.*

Let us discuss this definition. First note that if ξ_H is abstractly self-adjoint, then $\mathrm{cl}(\xi_H)$ is real-valued. To see this, we calculate

$$\mathrm{cl}(\xi_H) = \mathrm{cl}(\xi_H^\dagger) = \overline{\mathrm{cl}(\xi_H)}, \tag{2.9}$$

where the last step is due to (6.12) and (2.7). The converse is false since, in general, $\mathcal{T}(\mathfrak{g})$ contains nilpotent elements.

Remark 2.2. The map $\mathrm{cl} : \mathcal{T}(\mathfrak{g}) \to C^\infty(Z, \mathbb{C})$ has a natural factorization $\mathrm{cl}_S : \mathcal{S}(\mathfrak{g}) \to C^\infty(Z, \mathbb{C})$ to the full algebra of symmetric tensors $\mathcal{S}(\mathfrak{g})$.

In this way the classical limit map is a link between the non-commutative algebra $\mathcal{T}(\mathfrak{g})$ and a certain commutative subalgebra of $C^\infty(Z, \mathbb{C})$. But since $C^\infty(Z, \mathbb{C})$ is commutative, we have to work with the tensor algebra and cannot pass to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the definition of the classical limit, otherwise the quotient will not be well-defined. To see this, take any operators ξ_and ξ_b such that $[\xi_a, \xi_b] \neq 0$. Then it follows that $\mathrm{cl}(\xi_a\xi_b - \xi_b\xi_a - [\xi_a, \xi_b])$ is not equal to zero.

Let $x \in V$ be a vector of unit length. Reading cl as a map to $C^\infty(V \backslash \{0\}, \mathbb{C})$ we see that

$$\mathrm{cl}(\rho_*(\xi_a\xi_b))(x) = \mathrm{cl}(\xi_a)(x)\,\mathrm{cl}(\xi_b)(x) = \langle x, \rho_*(\xi_a)x\rangle \cdot \langle x, \rho_*(\xi_b)x\rangle, \tag{2.10}$$

which has a meaningful physical interpretation. Namely, if we think of ξ_a and ξ_b as observables, then in the classical limit the expectation value of the operator $\xi_a\xi_b$ is given by the product of the expectation values of ξ_a and ξ_b[1]. But this means that the operators ξ_a and ξ_b are stochastically independent in the classical limit.

[1]This remark has to be taken *cum grano salis*, because of the possible complex phases on the right-hand side. For probabilities one has to take the absolute value squared, which is an implicit convention in theoretical physics.

The main point of this chapter is to give an analytical realization of this purely algebraic construction, i.e., there will be a parameter and we will obtain the above classical limit as an analytical limit when this parameter goes to infinity. This will make the notion of $\hbar \to 0$ precise in our context. Here the theme of non-commutativity vs. commutativity will appear again.

2.3 Realizing the Classical Limit as an Analytical Limit

The Lie algebra $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{u}_+, \tag{2.11}$$

where $\mathfrak{t}^{\mathbb{C}}$ is the Lie algebra of the complexified maximal torus and $\mathfrak{u}_-$ and $\mathfrak{u}_+$ are unipotent Lie subalgebras corresponding to the positive and negative roots. We define the groups

$$U_+ = \exp(\mathfrak{u}_+),\ U_- = \exp(\mathfrak{u}_-), \text{ and } T^{\mathbb{C}} = \exp(\mathfrak{t}^{\mathbb{C}}). \tag{2.12}$$

Recall that the decomposition of the Lie algebra $\mathfrak{g}$ almost yields a decomposition of G. "Almost" in this context means that it is a decomposition of $G \backslash S$, where S is a Zariski-closed set,

$$G = \text{Zarsiki closure of } U_- T^{\mathbb{C}} U_+, \tag{2.13}$$

and even stronger

$$G \backslash S \simeq U_- \times T^{\mathbb{C}} \times U_+. \tag{2.14}$$

Let us again consider the representation $\rho_* : \mathcal{U}(\mathfrak{g}) \to \text{End(V)}$ and choose a vector of highest weight $v_{max} \in V$. By the definition of v_{max} we see that $U_+ \subset \text{Stab}_G(v_{max})$ and $\rho(T) \subset \mathbb{C}^* \cdot v_{max}$. Moreover, the K-orbit through $[v_{max}]$ agrees with the G-orbit through this point, i.e. $K.[v_{max}] = G.[v_{max}]$.

Thus, there exists a Zariski-closed set A in $K.[v_{max}]$ such that $K.[v_{max}] \backslash A$ is isomorphic to the orbit of U_- through v_{max} in V. Therefore, the U_--orbit is isomorphic to a dense, Zariski-open subset of Z if we identify $Z = K.\lambda$ with $K.[v_{max}]$ via the momentum map.

We will write cl as composition of two maps r and s:

$$r : i\mathfrak{k} \to \text{Vect}(V \backslash \{0\}), \xi \mapsto -\frac{1}{2} X_\xi, \text{ with } (X_\xi f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(-\xi t).x) \tag{2.15}$$

and

$$s : \text{Vect}(V \backslash \{0\}) \to C^\infty(V \backslash \{0\}, \mathbb{C}), X \mapsto \frac{1}{N}(XN), \tag{2.16}$$

where $N(x) = \|x\|^2$ is the norm function squared.

Slightly changing the definition of cl to a map to $C^\infty(V\backslash\{0\},\mathbb{C})$ the definition of the momentum map (2.2) yields the following commutative diagram:

$$\begin{array}{ccc} i\mathfrak{k} & \xrightarrow{\;r\;} & \mathrm{Vect}(V\backslash\{0\}) \\ & \searrow^{\mathrm{cl}} & \downarrow s \\ & & C^\infty(V\backslash\{0\},\mathbb{C}) \end{array} \tag{2.17}$$

Let us explicitly calculate the map s on the U_--orbit through v_{max}:

$$(X_\xi N)(x) = \left.\frac{d}{dt}\right|_{t=0} N(\exp(-t\xi).x). \tag{2.18}$$

Since x lies on the U_--orbit, there exists a $u \in U_-$ such that

$$x = u.v_{max}. \tag{2.19}$$

Now we can decompose $\exp(-t\xi)u$ uniquely as

$$\exp(-\xi t)u = u_-(t)l(t)u_+(t) \tag{2.20}$$

for t in a neighborhood of 0, where $u_-(t) \in U_-$, $l(t) \in T^{\mathbb{C}}$ and $u_+(t) \in U_+$. To see this note that we can decompose the identity and the set of decomposable elements is a Zariski open set by (2.13). Using the chain rule and self-adjointness of ξ, we obtain

$$(X_\xi N)(x) = 2\left\langle x, \left.\frac{d}{dt}\right|_{t=0} \exp(-\xi t).x \right\rangle = 2\left\langle x, \left.\frac{d}{dt}\right|_{t=0} u_-(t)l(t)u_+(t).v_{max} \right\rangle. \tag{2.21}$$

But since $u_+(t) \in U_+ \subset \mathrm{Stab}_G(v_{max})$ for all t we have

$$(X_\xi N)(x) = 2\left\langle x, \left.\frac{d}{dt}\right|_{t=0} u_-(t)l(t).v_{max} \right\rangle. \tag{2.22}$$

According to the product rule and using $l(0) = Id$, $u_-(0) = u$ we find

$$(X_\xi N)(x) = 2\langle x, u \left.\frac{d}{dt}\right|_{t=0} l(t).v_{max}\rangle + 2\left\langle x, \left.\frac{d}{dt}\right|_{t=0} u_-(t).v_{max} \right\rangle. \tag{2.23}$$

Due to the fact that $l(t) \in T$ acts as scalar on v_{max} this can be simplified as follows

$$(X_\xi N)(x) = 2\dot{l}(0)\langle x, x\rangle + 2\left\langle x, \left.\frac{d}{dt}\right|_{t=0} u_-(t).v_{max} \right\rangle. \tag{2.24}$$

Thus, we can read the right hand side as a differential operator applied to the norm function. This operator consists of a multiplication part with $2\dot{l}(0)$ and a vector field part which is tangential to the U_--orbit. Let $\mathcal{D}(U_-.v_{max})$ denote the algebra of linear differential operators on $U_-.v_{max}$. We claim that the above procedure affords a map

$$\tilde{r} : i\mathfrak{k} \to \mathcal{D}(U_-.v_{max}), \xi \mapsto m_\xi + \xi_{tan}, \tag{2.25}$$

where ξ_{tan} is the vector field tangent to the U_- orbit whose one parameter group at x is given by $2\left.\frac{d}{dt}\right|_{t=0} u_-(t)$ with respect to the above decomposition, and m_ξ is a smooth function on the U_--orbit with $m_\xi(x) = 2\dot{l}(0)$. The only thing we have to show is that the construction is independent of the choice of u in (2.19). But if we choose u' with

$$x = u.v_{max} = u'.v_{max} \tag{2.26}$$

then $u'u^{-1} \in \mathrm{Stab}_G(v_{max})$. So, $u' = ug$, where $g \in \mathrm{Stab}_G(v_{max})$. But as g acts trivially on v_{max} the calculation does not change.

The map $\tilde{r}$ will be the crucial point in the following. We will discuss it from an abstract point of view later on, but first we extend $\tilde{r}$ to $\mathcal{T}(\mathfrak{g})$ in the following manner

$$\tilde{r}(\xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_p}) = \tilde{r}(\xi_{\alpha_1}) \circ \cdots \circ \tilde{r}(\xi_{\alpha_p}). \tag{2.27}$$

This is well-defined because the $\tilde{\xi_j}$ are linear differential operators, so they respect scalar multiplication and addition.

Before we go into the details of the convergence, we need a fact about the norm.

Theorem 2.3. *Let λ be the highest weight of the representation ρ with decomposition into fundamental weights f_j as follows*

$$\lambda = \sum_{j=1}^{r} \lambda_j f_j. \tag{2.28}$$

Then the squared norm function N on the U_--orbit decomposes as

$$N(u.v_{max}) = c \cdot N_1(u.v_{max})^{\lambda_1} \cdot \dots \cdot N_r(u.v_{max})^{\lambda_r}, \tag{2.29}$$

where r is the rank of $\mathfrak{g}$ and $N_1, \dots, N_r$ are the squared norms of the fundamental unitary representations corresponding to the fundamental weights $f_1, \dots, f_r$.

Proof. For every fundamental representation $\rho_{(j)}$ we have a holomorphic line bundle $L_j \to G/B_-$ such that the representation of G on $\Gamma_{\mathrm{hol}}(G/B_-, L)$ is equivalent to $\rho_{(j)}$ (cf. Appendix Theorem 6.14).

By induction and Lemma 6.16, we find that the representation with highest weight $\lambda = \sum \lambda_j f_j$ is given by the action on the sections of

$$L = L_{(1)}^{\lambda_1} \otimes \ldots \otimes L_{(r)}^{\lambda_r}. \tag{2.30}$$

Let h_j denote the induced K-invariant, hermitian bundle metric on L_j, which is given in Lemma 6.15, and h the induced metric for L.

Choose a common open covering $\{W_k\}$ of G/B_-, such that L and all L_j are trivializable over each W_k. Without loss of generality we may assume that $W_1 =$

$U_- \cdot [v_{\max}]$. Each hermitian bundle metric h_j is given by a family $\{m_{k,j} : W_k \to \mathbb{R}_+\}$, h by the family $\{m_k : W_k \to \mathbb{R}_+\}$.

A direct calculation shows that the family $\{m'_k : W_k \to \mathbb{R}_+\}$ given by

$$m'_k := m_{k,1}^{\lambda_1} \cdot \ldots \cdot m_{k,r}^{\lambda_r} \tag{2.31}$$

represents a hermitian, K-invariant bundle metric h' on L. Thus, $h' = ch$ for some positive constant c. Using (6.23) we see that the norm on W_1 is defined by the bundle metric up to this scalar.

This completes the proof of Theorem 2.3. □

In the following we will consider a highest weight $\lambda = \sum_j \lambda_j f_j$. If we are given a function like

$$\frac{u + \bar{u}}{1 + \|u\|^2} \lambda_1 + 17\lambda_2 \tag{2.32}$$

then we can think of the function as a polynomial in λ_1, λ_2 where the coefficients are smooth functions. It is even a homogeneous polynomial of degree 1.

Notation 2.4. The ring of smooth functions on $U_-.v_{max}$ is denoted by the symbol R, i.e. $R := C^\infty(U_-.v_{max}, \mathbb{C})$, and the ring of polynomials in the λ_j with coefficients in R by $R[\lambda]$.

The key result of this chapter is the following:

Theorem 2.5. *Let $\lambda = \sum_j \lambda_j f_j$ be the highest weight of ρ and assume that at least one $\lambda_j > p$ for a fixed natural number p. Furthermore, let $\alpha = \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_p}$ be a "monomial" element of degree p in the generators ξ_j of $\mathfrak{g}$ as chosen above.*

Then $f(\lambda) := \frac{1}{N}\tilde{r}(\alpha)(N) \in R[\lambda]$ and $\deg f = p$. The homogeneous part of degree p of f is, up to a real, multiplicative constant, given by $\mathrm{cl}(\xi_{\alpha_1}) \cdot \cdots \cdot \mathrm{cl}(\xi_{\alpha_p})$, where we view the $\mathrm{cl}(\xi_{\alpha_j})$ as elements of $R[\lambda]$. Moreover, the constant does not depend on α.

Proof. By definition, every $\tilde{r}(\xi_{\alpha_j})$ is a first order partial differential operator. Hence the summands in the derivative of $N = N_1^{\lambda_1} \cdot \cdots \cdot N_r^{\lambda_r}$, after dividing by N, are polynomials in λ of degree at most p. On the other hand, at least one such summand must be a polynomial of degree at least p. If all were of lesser degree, one of the ξ_{α_j} would be multiplication by a constant, which is not the case, or the partial derivatives would lower every exponent λ_j to 0, which yields a contradiction because at least one λ_j is larger than p. This proves the first part of the theorem.

For the second part, we consider the case $p = 1$ first.Then there is no degree zero term in the polynomial $\frac{1}{N}\tilde{r}(\alpha)(N)$ since

$$\frac{1}{N}\tilde{r}(\alpha)(N) = \frac{1}{N} r(\alpha)(N) \tag{2.33}$$

in the above construction. But $r(\alpha)$ is a vector field and contains no multiplicative part, so we have only partial derivatives turning N into a homogeneous polynomial of degree 1 after dividing by N. This proves the second statement for $p = 1$.

Let $p \geq 2$ and $\xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_p}$ be given. We have $\tilde{r}(\xi_a) = c + \sum a_j \frac{\partial}{\partial z_j}$ for some a_j and c in some coordinate system $\{z_j\}$ on $U_{-}.v_{max}$. By the induction hypothesis

$$\mathrm{cl}(\xi_{\alpha_2} \otimes \cdots \otimes \xi_{\alpha_p}) = \mathrm{cl}(\xi_{\alpha_2}) \cdot \ldots \cdot \mathrm{cl}(\xi_{\alpha_p}) + q, \tag{2.34}$$

where q is a polynomial of degree less than $p-1$. Using the product rule of differentiation we calculate explicitly

$$\begin{aligned}
\mathrm{cl}(\xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_p}) &= \frac{1}{N}\tilde{r}(\xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_p})(N) = \frac{1}{N}\tilde{r}(\xi_{\alpha_1})(N\,\mathrm{cl}(\xi_{\alpha_2}) \ldots \mathrm{cl}(\xi_{\alpha_p}) + Nq) \\
&= \frac{1}{N}\left(c + \sum a_j \frac{\partial}{\partial z_j}\right)(N\,\mathrm{cl}(\xi_{\alpha_2}) \ldots \mathrm{cl}(\xi_{\alpha_p}) + Nq) \\
&= c \cdot \mathrm{cl}(\xi_{\alpha_2}) \ldots \mathrm{cl}(\xi_{\alpha_p}) + q + \frac{1}{N}\mathrm{cl}(\xi_{\alpha_2}) \ldots \mathrm{cl}(\xi_{\alpha_p}) \left(\sum a_j \frac{\partial}{\partial z_j}\right)(N) \\
&\quad + \frac{1}{N}\left(\sum a_j \frac{\partial}{\partial z_j}\right)(\mathrm{cl}(\xi_{\alpha_2}) \ldots \mathrm{cl}(\xi_{\alpha_p})) + \frac{1}{N}\left(\sum a_j \frac{\partial}{\partial z_j}\right)(Nq) \\
&= \frac{1}{N}(\mathrm{cl}(\xi_{\alpha_2}) \cdot \ldots \cdot \mathrm{cl}(\xi_{\alpha_p}))\left(\sum a_j \frac{\partial}{\partial z_j} + c\right)(N) + \text{ terms of degree less than } p. \\
&= \mathrm{cl}(\xi_{\alpha_1}) + (\mathrm{cl}(\xi_{\alpha_2}) \cdot \ldots \cdot \mathrm{cl}(\xi_{\alpha_p})) + \text{ terms of degree less than } p.
\end{aligned} \tag{2.35}$$

Here the first summand is a homogeneous polynomial of degree p, as claimed. The remaining summands are certainly of lower degree, because each $\mathrm{cl}(\xi_b)$ is of degree one and taking the partial derivatives can only lower the degree. □

After these preparations we define the classical limit along a ray in the following way.

Definition 2.6. *Let $\rho : K \to \mathrm{U}(V)$ be a non-trivial, irreducible, unitary representation of a semisimple, compact Lie group K on a finite-dimensional vector space V corresponding to the highest weight λ.*

*We call a sequence $(\rho_n : K \to \mathrm{U}(V_n))_{n \in N^*}$ of irreducible, unitary representations, each ρ_n corresponding to the highest weight $n \cdot \lambda$, the **ray through ρ**. For simplicity, we shall always assume that $\rho_1 = \rho$.*

Let $\xi_H \in \mathcal{T}(\mathfrak{g})$ be an abstractly hermitian operator and $\xi_1, \ldots, \xi_k$ be a basis of $i\mathfrak{k}$. We have a unique decomposition into "monomials" of $\xi_H = \sum_j a_j \xi_{j_1} \otimes \cdots \otimes \xi_{j_{d(j)}}$, where each a_j is a complex number. (Keep in mind that these are not monomials in the usual sense because of the non-commutativity.)

Definition 2.7. *The **n-th approximation of the classical limit** is*

$$\mathrm{cl}_n(\xi_H) = \sum_j a_j \frac{1}{n^{d(j)}} \frac{1}{N} \tilde{r}_n(\xi_{j_1} \otimes \cdots \otimes \xi_{j_{d(j)}})(N). \tag{2.36}$$

Here $\tilde{r}_n$ is defined as in (2.25) and (2.27) with respect to the representation ρ_n, i.e. we substitute every λ_j in the resulting polynomials by $n \cdot \lambda_j$.

Theorem 2.8. *Along a ray through the non-trivial, irreducible representation ρ the n-th approximations of the classical limit converge to the classical limit uniformly on compact subsets of $U_-.v_{max}$ for every fixed $\xi_H \in \mathcal{T}(\mathfrak{g})$, i.e.*

$$\mathrm{cl}_n(\xi_H) \to \mathrm{cl}(\xi_H) \text{ uniformly on compact subsets, as } n \to \infty. \tag{2.37}$$

Proof. Decompose ξ_H into its homogeneous parts:

$$\xi_H = \sum_j \xi_j \tag{2.38}$$

where each x_j is homogeneous of degree j. Since ρ is a non-trivial representation, at least one λ_j in the decomposition $\lambda = \sum_j \lambda_j f_j$ is not zero. Because ξ_H has only a finite degree, the conditions of Theorem 2.5 are satisfied for all n sufficiently big. Applying this theorem to each "monomial" in every ξ_j implies

$$\mathrm{cl}_n(\xi_j) = \mathrm{cl}(\xi_j) + \frac{1}{n}(\text{terms of lower degree}). \tag{2.39}$$

It follows that for any compact set M

$$\mathrm{cl}_n(\xi_H)(x) \to \mathrm{cl}(\xi_H)(x) \text{ as } n \to \infty \tag{2.40}$$

for all $x \in M$ uniformly. □

This completes the construction of the classical limit as a mathematical limit. The reader might wonder whether the convergence on a dense, open subset of Z suffices. Note that cl is defined on the whole of Z, but our U_- chart is not. Unfortunately, it is not clear that every approximation can be extended to Z, but nevertheless the limit does extend continuously.

Let us now discuss the procedure a more abstractly. The main step is the substitution of $\tilde{r}$ for r in the definition of the classical limit. After this, the other theorems follow from Theorem 2.3. But what are these deformed vector fields $\tilde{r}(\xi)$? In a way this is at least in a formal sense similar to a connection in a line bundle plus multiplicative function, like in geometric quantization. Indeed, we have a line bundle here. It is the tautological bundle $V\backslash\{0\} \to \mathbb{P}(V)$ restricted to $K.[v_{max}]$. Furthermore, the U_--orbit can be thought of as a section of this bundle over the dense open set $U_-.[v_{max}]$. Since U_- is biholomorphic to some $\mathbb{C}^p$, we get a chart for the bundle here. In this chart $\tilde{r}$ is in fact just a connection plus a multiplicative part.

A visualization of the situation is provided by Figure 2.1. Here we see the origin in V and v_{max}. Since K acts unitarily, the K-orbit preserves the metric and is drawn as a circular arc. The U_--orbit is non-compact and drawn as a very flat parabola. If we look at this in $\mathbb{P}(V)$, we see that the U_--orbit is not a global section of the tautological bundle because the horizontal axis has no intersection with the U_--orbit.

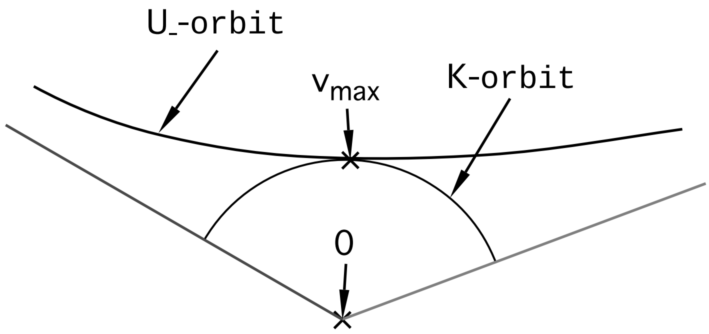

Figure 2.1: A picture of the U_--section.

3 Spectral Statistics of Simple Hamiltonian Operators

The spectral statistics of simple Hamiltonian operators, i.e., the nearest neighbor statistics for elements of some semi-simple Lie algebra, are discussed in this chapter. The main interest is in the behavior of the spectral statistics in irreducible representations as the dimension goes to infinity. Thereafter, the notion of rescaling is introduced and some consequences of the choice of rescaling are given.

3.1 A Convergence Theorem for Simple Operators

In this section we give an estimation on the number of weights of irreducible representations and in certain cases deduce from it the convergence of the spectral statistics for simple operators.

Here K always denotes a semi-simple, compact Lie group with a fixed maximal torus T and a fixed notion of positivity of roots. We write W for the Weyl group of K with respect to T. Further, let G be the complexification of K and denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$. For any hermitian matrix A we write μ_A for the nearest neighbor statistics of A, i.e.,

$$\mu_A := \mu(X(A)) \tag{3.1}$$

as defined in (6.30). If U is a unitary matrix we will write μ_U for the nearest neighbor statistics of unitary matrices (6.37), i.e.

$$\mu_U := \mu_c(X(A)). \tag{3.2}$$

It is clear by the subscript which kind of statistics is meant, so we use the same abbreviation.

We start with a lemma.

Lemma 3.1. *Let $\rho_\lambda : K \to U(V_\lambda)$ be an irreducible, unitary representation with highest weight λ. Let $\lambda = \sum \lambda_j f_j$ be the decomposition of λ into the basis of fundamental weights f_j. Then the number n_λ of possible weights of ρ_λ is bounded as follows*

$$n_\lambda \ \leq \ \operatorname{ord}(W) \cdot \prod_j (\lambda_j + 1). \tag{3.3}$$

Proof. Starting from λ we get all other weights by subtracting multiples of the roots. The lattice of roots is a sublattice of the lattice of weights, so we can reach every weight by subtracting multiples of the fundamental weights f_j.

There are at most $\prod_j(\lambda_j+1)$ of the such possible substractions that give positive weights and every weight is in the W-orbit of a positive weight, which has at most $|W|$ elements. □

Now we give a rough estimate for the dimension of an irreducible representation.

Lemma 3.2. *Under the assumptions of Lemma 3.1 we have the following inequality for the dimension of ρ_λ:*

$$\dim \rho_\lambda \geq \prod_{\alpha\in\Pi^+,\langle\lambda,\alpha\rangle>0} \frac{\langle\lambda,\alpha\rangle}{\langle\delta,\alpha\rangle}, \tag{3.4}$$

where Π^+ denotes the set of positive roots and $\delta = \frac{1}{2}\sum_{\alpha\in\Pi^+}\alpha$.

Proof. Weyl's dimension formula reads

$$\dim \rho_\lambda = \prod_{\alpha\in\Pi^+} \frac{\langle\delta+\lambda,\alpha\rangle}{\langle\delta,\alpha\rangle} = \prod_{\alpha\in\Pi^+}\left(1+\frac{\langle\lambda,\alpha\rangle}{\langle\delta,\alpha\rangle}\right). \tag{3.5}$$

Now, $\langle\lambda,\alpha\rangle \geq 0$ and $\langle\delta,\alpha\rangle > 0$ for all positive roots α. Thus, the inequality is clear. □

We write δ_{Dirac} for the Dirac measure with mass 1 at 0 and apply these lemmas to the situation of Chapter 2 where we looked at rays to infinity.

Theorem 3.3. *Let $\rho : K \to \mathrm{U}(V)$ be an irreducible representation with highest weight $\lambda = \sum \lambda_j f_j$ and the sequence $(\rho_n : K \to \mathrm{U}(V_n))_{n\in\mathbb{N}^*}$ be a ray through ρ.*

If $r := \operatorname{rank}(K) \geq 2$ and

$$r < {}^{\#}\{\alpha\in\Pi^+ : \langle\alpha,\lambda\rangle > 0\} \tag{3.6}$$

then for every $\xi \in i\mathfrak{k}\backslash\{0\}$

$$\mu_{\rho_{*,m\lambda}(\xi)} \to \delta_{\text{Dirac}} \text{ in } d_{KS}, \text{ as } m\to\infty. \tag{3.7}$$

Proof. Let $\xi \in i\mathfrak{k}$ be given. The element $i\xi\in\mathfrak{k}$ is conjugated to an element $\eta\in\mathfrak{t} = \mathrm{Lie}(T)$. We will show that

$$p_m := \frac{\text{number of (different) eigenvalues of } \rho_{*,m\lambda}(\eta)}{\dim \rho_{m\lambda}} \to 0 \tag{3.8}$$

as $m\to\infty$. This implies the convergence to δ_{Dirac} since the value of $\lim_{s\to 0}\int_0^s d\mu_A$ is

$$1 - \frac{\text{number of (different) eigenvalues}}{\text{number of rows of } A} \tag{3.9}$$

for any hermitian matrix A by the definition of the nearest neighbor statistics. Thus, $\mu_{\rho_{*,m\lambda}(\xi)}$ has mass $1 - p_m$ at zero, which proves the convergence.

It remains to show the claim about p_m. To do so, note that the eigenvalues of $\rho_{m\lambda}(\xi)$ are just the values of the weights of the representation evaluated at ξ. So, it is sufficient to prove that the ratio of the different weights and the dimension of $\rho_{m\lambda}$ converges to zero.

To show this we combine the inequalities of Lemma 3.1 and 3.2, but first we simplify the notation a bit. We denote by Q the set of $\alpha \in \Pi^+$, such that $\langle \alpha, \lambda \rangle > 0$ and by q the cardinality of Q. Finally, the number of different weights in $\rho_{m\lambda}$ is $n_{m\lambda}$.

We obtain

$$\frac{n_{m\lambda}}{dim \rho_{m\lambda}} \leq \frac{\left(\mathrm{ord}(W) \cdot \prod_{j=1}^{r} (\lambda_j + 1)\right) m^r}{\left(\prod_{\alpha \in Q} \frac{\langle \lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}\right) m^q} = c(\lambda)\, m^{r-q}. \tag{3.10}$$

Here $c(\lambda)$ is a constant, depending only on λ, and, since $r < q$ by the hypothesis, the ratio converges to zero as promised. This proves the theorem. □

Remark 3.4. The number q in the above proof is the complex dimension of the coadjoint orbit through λ, i.e., the complex dimension of the classical phase space in the classical limit of Chapter 2.

Corollary 3.5. *The conditions of the above theorem will be automatically satisfied if K is simple,* $\mathrm{rank}\, K \geq 2$, *and λ lies in the interior of the Weyl chamber.*

Proof. First, we remark that r equals the number of positive roots for any representation whose highest weight is in the interior of the Weyl chamber, since the interior is defined by the condition $\langle \lambda, \alpha \rangle > 0$ for every simple root α. But positive roots are positive integer combinations of simple roots $\langle \lambda, \alpha \rangle > 0$ for all positive roots α. This completes the proof. □

We now give another corollary.

Corollary 3.6. *Under the assumptions of the theorem let $t_1, \ldots, t_p \in \mathfrak{g}$ be given such that $\xi = t_1 \otimes \cdots \otimes t_p \in \mathcal{T}(\mathfrak{g})$ is abstractly hermitian in the sense of definition 6.10. Furthermore, let $\rho_{*,m\lambda}$ be the induced Lie algebra representation with highest weight $m\lambda$ extended to the full tensor algebra. Then*

$$\mu_{\rho_{*,m\lambda}(\xi)} \to \delta_{\mathrm{Dirac}} \textit{ weakly as } m \to \infty, \tag{3.11}$$

if $p \cdot r < {}^{\#}\{\alpha \in \Pi : \langle \alpha, \lambda \rangle > 0\}$.

Proof. We can assume without loss of generality that all t_j are always represented as diagonal matrices and we proceed by induction. From (3.10) it follows that for each t_j the number of its eigenvalues $n_{j,m\lambda}$ divided by the dimension is smaller than $c(\lambda)m^{r-q}$. But the maximal number of eigenvalues in a product of diagonal matrices is just the product of the number of eigenvalues of each matrix. Thus, we have a numerator m^{rp} here instead of m^r in (3.10). But by assumption $rp < q$, i.e. the number of eigenvalues of the product divided by the dimension is decreasing faster than $1/m$.

This proves the corollary. □

3.2 Rescaling

In this section we discuss the notion of rescaling. This concept appeared already in Chapter 2. There the classical limit along rays $(\rho_m : K \to \mathrm{U}(V_m))_{m \in \mathbb{N}^*}$ through a given representation ρ was considered and the scaling was given by substituting $\frac{1}{m}\xi_j$ for ξ_j. Since we are interested in the problem of scaling in general, we define the notion of a rescaling map abstractly.

Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$ and $\dagger$ the formal adjoint (cf. Appendix). We choose a fixed basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$ and write the elements of $\mathcal{U}(\mathfrak{g})$ as ordered polynomials in the ξ_j. Furthermore the multiindex notation Ξ^I will be used for $\xi_1^{i_1} \dots \xi_n^{i_n}$.

The basic problem can be seen if one considers the hermitian operators ξ and $\xi\eta$ in a sequence of irreducible representations. As the dimensions of the representations increase the maximal eigenvalues of $\xi\eta$ will in general grow faster than those of ξ. In principle, we would like the rate of growth to be the same, including the option of no growth at all. This motivates the following definition.

Definition 3.7. *A* ***rescaling map*** r_ρ *for the irreducible representation* $\rho : K \to \mathrm{U}(V)$ *is given by a map*

$$r_\rho : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \sum a_I \Xi^I \mapsto \sum \frac{1}{s^{|I|}} a_I \Xi^I, \tag{3.12}$$

where s *a positive integer number.*

Lemma 3.8. *Every rescaling map* r_ρ *is linear, injective and compatible with* $\dagger$.

Proof. This follows directly from the definition of r_ρ. □

Of all possible scalings the most natural one is the scaling by inverse dimension since we have no other natural quantity associated to arbitrary sequences of irreducible representations.

Definition 3.9. *Let* $\mathrm{Irr}(K)$ *denote the set of equivalence classes of irreducible, unitary representations of* K*. The* ***rescaling by inverse dimension*** *is the family of rescaling maps* $(i_\rho)_{\rho \in \mathrm{Irr}(K)}$ *given by*

$$i_\rho : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \ \sum a_I X^I \mapsto \sum \frac{1}{(\dim \rho)^{|I|}} a_I X^I \tag{3.13}$$

for each $\rho \in \mathrm{Irr}(K)$.

If we are considering rays through a fixed irreducible representation with highest weight λ, then we have another natural quantity: the parameter m for each $\rho_{m\lambda}$.

Definition 3.10. *Let $\rho : K \to \mathrm{U}(V)$ be an irreducible representation with highest weight $\lambda = \sum \lambda_j f_j$ and the sequence $(\rho_m : K \to \mathrm{U}(V_m))_{m\in\mathbb{N}^*}$ be a ray through ρ.*

The ***rescaling by inverse parameter*** *is the family of rescaling maps $(p_{\rho_{m\lambda}})$ given by*

$$p_{\rho_m} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}),\ \sum a_I X^I \mapsto \sum \frac{1}{m^{|I|}} a_I X^I. \tag{3.14}$$

3.2.1 Rescaling and Spectral Statistics

In the first section we considered simple operators only, i.e. Lie algebra elements. Rescaling has no effect in this case since for any self-adjoint matrix A and any $c > 0$

$$\mu_A = \mu_{c\cdot A}. \tag{3.15}$$

But rescaling has an effect if we consider operators whose monomial parts have different degrees, e.g.

$$\xi + \eta^2 \in \mathcal{U}(\mathfrak{g}). \tag{3.16}$$

Recall that for a highest weight λ the set Q is defined as $Q = \{\alpha \in \Pi_+ \,:\, \langle \alpha, \lambda\rangle\}$ and $q = {}^{\#}Q$. We state the following lemma:

Lemma 3.11. *Let $\xi_H = \sum_I a_I \Xi^I \in \mathcal{U}(\mathfrak{g})$ be given with $\xi_H^\dagger = \xi_H$ and consider the ray $(\rho_m : K \to \mathrm{U}(V_m))_{m\in\mathbb{N}^*}$ through an irreducible representation $\rho : K \to \mathrm{U}(V)$ of highest weight λ.*

Then

$$\|\rho_{*,m}(i_{\rho_m}(\xi_H))\|_{\mathrm{End}(V_m)} \le c_1(\lambda) \sum_I |a_I| c_2(\lambda)^{|I|} \cdot m^{|I|-q|I|} \tag{3.17}$$

where the $c_j(\lambda)$ are constants depending only on λ and $\|\cdot\|_{\mathrm{End}(V_m)}$ denotes the operator norm on $\mathrm{End}(V_m)$.

Proof. We use the explicit construction of irreducible representations by Borel-Weil. For this let

$$S_j = (s_1^{(j)}, \ldots, s_{d(j)}^{(j)}), j = 1, \ldots, r \tag{3.18}$$

denote a basis of the j-th fundamental representation. These are holomorphic sections in a holomorphic line bundle

$$L_j \to G/B \tag{3.19}$$

where B is a Borel subgroup of G and $L = G \times_{\chi_j} \mathbb{C}$, such that $\chi_j : B \to \mathbb{C}$ is the exponentiated character of the fundamental weight λ_j. The irreducible representation with highest weight λ is then given by the action on sections of the line bundle

$$L = L_1^{\otimes\lambda_1} \otimes \ldots \otimes L_j^{\otimes\lambda_j} \to G/B. \tag{3.20}$$

By the theorem of Borel-Weil the tensors of the form

$$S_1^{I_1} \otimes \ldots \otimes S_r^{I_r}, \tag{3.21}$$

with $I_1, \ldots, I_r$ multiindices of degree $|I_j| = \lambda_j$ constitute a generating system of the space of sections.

Without loss of generality we may take a basis $\xi_1, \ldots, \xi_n$ of $\mathfrak{g}$, such that ξ_1 is represented by a diagonal hermitian matrix of spectral norm 1 in every fundamental representation. Since the operator norm is equal to the spectral norm, we wish to give an estimate for the maximal absolute value of an eigenvalue of ξ_1 in $\rho_{*,\lambda}$.

But on the generating system of vectors given by (3.21) the action is on each factor separately, so we have

$$\|\rho_{*,\lambda}(\xi_1)\| \leqslant \lambda_1 + \ldots + \lambda_r =: |\lambda|. \tag{3.22}$$

Clearly, the same argument can be carried out for $\xi_2, \ldots, \xi_n$. So we have the following estimate

$$\|\rho_{*,m}(\xi_j)\| \leqslant m(\lambda_1 + \ldots + \lambda_r) = m|\lambda| \tag{3.23}$$

for all $j = 1, \ldots, n$.

Now, consider $\gamma = \sum_I a_I X^I$. Then

$$\|\tilde{\rho}_{*,m}(i_{\rho_m}(\gamma)t)\|_{\mathrm{End}(V_m)} \leqslant \sum_I \frac{1}{(\dim \rho_k)^{|I|}} \|\rho_{*,m}(\xi_1)\|^{i_1}_{\mathrm{End}(V_m)} \cdot \ldots \cdot \|\rho_{*,m}(\xi_n)\|^{i_n}_{\mathrm{End}(V_m)}. \tag{3.24}$$

Using the estimates given by (3.23) and Lemma 3.2, we see that

$$\|\tilde{\rho}_{*,m}(i_{\rho_m}(\gamma)t)\|_{\mathrm{End}(V_m)} \leq \sum_I \frac{|a_I|}{C \cdot m^{q|I|}} m^{|I|} \cdot |\lambda|^{|I|} = C' \sum_I |a_I||\lambda|^{|I|} m^{|I|-q|I|}, \tag{3.25}$$

where C and C' are constants depending only on λ, which completes the proof. □

We use this lemma to prove the following theorem.

Theorem 3.12. *Consider the ray $(\rho_m : K \to \mathrm{U}(V_m))_{m \in \mathbb{N}^*}$ through an irreducible representation $\rho : K \to \mathrm{U}(V)$ of highest weight λ and assume $q > 2$.*

Then for all $\xi_H = \eta + \sum_{|I| \geq 2} a_I \Xi^I \in \mathcal{U}(\mathfrak{g})$ with $\eta \in \mathfrak{g} \backslash \{0\}$ and $\xi_H^\dagger = \xi_H$

$$d_{KS}(\mu_{\rho_{*,m}(i_{\rho_m}(\xi_H))}, \mu_{\rho_{*,m}(\eta)}) \to 0 \text{ as } m \to \infty. \tag{3.26}$$

Proof. We claim, that

$$\lim_{m \to \infty} (\dim V_m) \cdot \left\| \rho_{*,m} \left(i_{\rho_m} \left(\sum_{|I| \geq 2} a_I \Xi^I \right) \right) \right\|_{\mathrm{End(V_m)}} = 0. \tag{3.27}$$

This implies the theorem, because the nearest neighbor statistics for hermitian matrices are scaling invariant, i.e.

$$\mu_{(\dim V_n)\rho_{*,m}(\xi_H)} = \mu_{\rho_{*,m}(\xi_H)} \tag{3.28}$$

and $(\dim V_m)\rho_{*,m}(i_{\rho_m}\eta) = \rho_{*,m}(\eta)$. Thus,

$$\lim_{m\to\infty} \|(\dim V_n)\rho_{*,m}(i_{\rho_m}(\xi_H)) - \rho_{*,m}(\eta)\|_{\mathrm{End}(V_m)} = 0. \tag{3.29}$$

It remains to proof (3.27). But by (3.25) we obtain

$$(\dim V_m)\sum_{|I|\geq 2} a_I \left\|\rho_{*,m}\left(i_{\rho_m}\left(\Xi^I\right)\right)\right\|_{\mathrm{End(V_m)}} \leq C\sum_I |a_I||\lambda|^{|I|}m^{|I|-(q-1)|I|}, \tag{3.30}$$

where C is a constant. Since λ is fixed and $q > 2$ the right hand side converges to zero. □

So, we only have to study the convergence of $\mu_{\rho_{*,m}(\eta)}$ to gain information about the convergence of the nearest neighbor distribution of the whole operator under rescaling by inverse dimension. For example, we may use Theorem 3.3.

3.2.2 Rescaling and exp

Rescaling can affect the limit measure of exponentiated operators as shown in the following lemma.

Lemma 3.13. *Let $\rho_k : K \to \mathrm{U}(V_k)$, $k \in \mathbb{N}$, be a sequence of irreducible, unitary representations and $\gamma \in \mathcal{U}(\mathfrak{g})$ with $\gamma^\dagger = \gamma$.*

Let us assume that

$$\lim_{n\to\infty} \|\tilde{\rho}_{*,k}(r_{\rho_k}(\gamma)t)\|_{\mathrm{End}(V_k)} = 0 \textit{ for all } t > 0, \tag{3.31}$$

where $\|\cdot\|_{\mathrm{End}(V_k)}$ denotes the operator norm on $\mathrm{End}(V_k)$.

Then $\mu_{\exp(\rho_{*,k}(r_{\rho_k}(\gamma))t)}$ does not converge to any Borel measure μ on the positive real line with

$$\int_0^{\frac{1}{2\pi}} d\mu < 1 \tag{3.32}$$

as n goes to infinity for any $t > 0$. In particular it does not converge to μ_{Poisson} or μ_{CUE}.

Proof. For simplicity set $\gamma_k = r_{\rho_k}(\gamma)$ and let $t > 0$ be fixed. Now by (3.31) we see that starting from a sufficiently large k_0 the spectrum of $\rho_{*,k}(\gamma_k)t$ is in the interval $[-\pi, -\pi]$.

Now we may consider a subsequence of ρ_{k_j} such that the spectrum of $\rho_{*,k}(\gamma_k)t$ is in the interval $]-\frac{1}{2j}, -\frac{1}{2j}[$. Analogously to the counterexample in Remark 6.28 in Chapter 6, one proves that a limit measure must necessarily have the whole mass between 0 and $1/2\pi$. □

The following theorem states that rescaling by inverse dimension will destroy convergence to μ_{Poisson} in many cases.

Theorem 3.14. *Choose a fixed irreducible, unitary representation* $\rho_\lambda : K \to \mathrm{U}(V_\lambda)$ *with highest weight* λ*, where* $\lambda = \lambda_1 f_1 + \ldots + \lambda_r f_r$ *is the decomposition into fundamental weights with every* $\lambda_j \geqslant 0$.

Let $\operatorname{rank}(\mathfrak{g}) \geqslant 2$ *and assume that at least two* f_j *are positive. Then for every* $\gamma \in \mathcal{U}(\mathfrak{g})$ *without constant term*

$$\lim_{k\to\infty} \|\tilde{\rho}_{*,k}(i_{\rho_k}(\gamma)t)\|_{\mathrm{End}(V_k)} = 0 \text{ for all } t > 0, \tag{3.33}$$

where $\rho_k : K \to \mathrm{U}(V_k)$ *is an irreducible representation with highest weight* $k \cdot \lambda$ *and* $\|\cdot\|_{\mathrm{End}(V_k)}$ *is the usual operator norm in* $\mathrm{End}(V_k)$.

Proof. Apply Lemma 3.11 and note that the right hand side of (3.17) converges to zero. □

Corollary 3.15. *Under the above assumptions* $\mu_{\exp(\rho_{*,k}(r_{\rho_k}(\gamma))t)}$ *does not converge to the measures* μ_{Poisson} *or* μ_{CUE}.

Proof. This follows from Lemma 3.13, since we proved that (3.31) is fulfilled. □

Remark 3.16. Note that there is an obvious counterexample to Theorem 3.14 if the rank of $\mathfrak{g}$ is 1. Namely, the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ on the homogeneous polynomials in two indeterminates.

Take $\xi = \operatorname{diag}(1,-1)$. Then $\|\rho_k(\xi)\|_{V_k} = k$ where V_k is the vector space of homogeneous polynomials of degree k. Therefore $\dim \rho_k(\xi) = k+1$. We see that

$$\|\rho_k(r_k(\xi))\|_{V_k} = \frac{k}{k+1} \to 1. \tag{3.34}$$

The reader may wonder what happens in the case of the rescaling by inverse parameter as in Chapter 2. There is no analogue of Theorem 3.14 in this case, because the denominator in (3.24) scales like the numerator, so there is no convergence to zero.

In fact, the statements of this chapter can be made more general by allowing rescaling maps which decrease operators faster than the rescaling by inverse parameter. The theorems will still be true in this case, although some corrections to the constants will be required.

4 Spectral Statistics of Generic Hamiltonian Operators

Having studied the spectral statistics of simple Hamiltonian operators, i.e., simple "polynomials" of Lie algebra elements in irreducible representations, we are now interested in more complicated operators.

In Chapter 2 "polynomials" in some basis of the Lie algebra were considered, which gave rise to Hamiltonians. But for a more analytic treatment of the matter, we investigate the spectral statistics in a completion of the polynomial algebra. Note that such a completion was already implicitly used in [GHK00], where the authors used the sine of a Lie algebra element.

Thereafter we will define the notion of a generic Hamiltonian operator and prove that the irreducible representations of the flows through the generic operators have spectral statistics converging to $\mu_{Poisson}$ under special assumptions on the dimensions of the representation spaces.

We will use the following notation throughout this chapter. Let K denote a compact semi-simple Lie group with complexification G. The corresponding Lie algebras are called $\mathfrak{k}$ and $\mathfrak{g}$. Every representation of K will be assumed to be continuous, finite-dimensional and unitary. The K-invariant inner product will be denoted by $\langle \cdot, \cdot \rangle$ without putting the representation space into the notation. It will be clear by the arguments or by the context which representation space is meant.

4.1 Topology and Completion of $\mathcal{U}(\mathfrak{g})$

In this section we introduce a topology on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and complete it to a Fréchet space. To do so, choose a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt Theorem we have a vector space isomorphism

$$\psi : \mathbb{C}[X_1, \dots, X_n] \to \mathcal{U}(\mathfrak{g}) \tag{4.1}$$

given by substituting ξ_i for X_i in every polynomial p in which we have ordered the indeterminates in each monomial lexicographically. Note that this ordering is necessary since ψ is only a vector space isomorphism, but not an algebra isomorphism.

We use ψ to give a topology to $\mathcal{U}(\mathfrak{g})$ by the natural embedding of $\mathbb{C}[X_1, \dots, X_n]$ into the algebra of holomorphic functions $\mathcal{O}(\mathbb{C}^n)$.

It is a well-known fact that $\mathcal{O}(\mathbb{C}^n)$ is a Fréchet space with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}^n$. If we change the basis of $\mathfrak{g}$ to $\eta_1, \dots, \eta_n$ we obtain *a priori* another completion of $\mathbb{C}[X_1, \dots X_n]$. But changing the basis is nothing more than a linear change of coordinates, yielding an induced linear homeomorphism of Fréchet spaces. So, a different choice of basis does not change the topology.

Remark 4.1. If a sequence of holomorphic functions on $\mathbb{C}^n$ converges to zero in the Fréchet topology, then the suprema of the coefficients in the Taylor expansion around the origin also converge to zero.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of holomorphic functions with Taylor expansion $f_j = \sum_I a_I^{(j)} X^I$, where I is a multiindex with the usual conventions.

By the general Cauchy integral formula in several variables we see that

$$a_I^{(j)} = \frac{1}{(2\pi i)^n} \oint_{\zeta\in\partial P} \frac{f_j(\zeta)}{\zeta^{I+(1,\ldots,1)}} d\zeta, \tag{4.2}$$

where P is the unit polycylinder in $\mathbb{C}^n$ and ∂P its distinguished boundary. From this we obtain

$$|a_I^{(j)}| \leqslant \sup_{\zeta\in\partial P} |f_j(\zeta)|. \tag{4.3}$$

The right hand side does not depend on I, so the inequality holds for the supremum of the $|a_I^{(j)}|$ for a fixed j, but the f_j converge uniformly on compact sets, especially on ∂P. □

Let $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible representation on a finite-dimensional complex vector space V. This map extends to an irreducible representation of $U(\mathfrak{g})$, which we will again call ρ_*.

Proposition 4.2. *The map $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ extends to a continuous, surjective, linear map*

$$\tilde{\rho}_* : \mathcal{O}(\mathbb{C}^n) \to \mathrm{End}(V) \tag{4.4}$$

with respect to the above completion of $\mathcal{U}(\mathfrak{g})$, where the topology on $\mathrm{End}(V)$ is given by the operator norm with respect to some norm on V.

Proof. Let $f = \sum a_I X^I \in \mathcal{O}(\mathbb{C}^n)$ be given. We define

$$\tilde{\rho}_*(f) = \sum a_I \rho_*(\xi_1)^{i_1} \ldots \rho_*(\xi_n)^{i_n}. \tag{4.5}$$

By the basic inequality for the operator norm

$$\|AB\| \leqslant \|A\| \cdot \|B\| \quad \forall A, B \in \mathrm{End}(V) \tag{4.6}$$

it follows that

$$\|a_I \rho_*(\xi_1)^{i_1} \ldots \rho_*(\xi_n)^{i_n}\| \leqslant |a_I| b_1^{i_1} \ldots b_n^{i_n} \tag{4.7}$$

for $b_i := \|\rho_*(\xi_i)\|$. This series is convergent since $f \in \mathcal{O}(\mathbb{C}^n)$. Moreover, $\tilde{\rho}_*$ is linear. To show the continuity, it suffices to show that $\tilde{\rho}_*$ is continuous at zero. So let

$(f_j)_{j\in\mathbb{N}}$ be a sequence of holomorphic functions on $\mathbb{C}^n$ converging to zero uniformly on compact subsets. We must show that

$$\lim_{j\to\infty} \tilde{\rho}_*(f_j) = 0, \tag{4.8}$$

but this is the claim that

$$\| \sum a_I^{(j)} \rho_*(\xi_1)^{i_1} \dots \rho_*(\xi_n)^{i_n} \| \to 0. \tag{4.9}$$

Note that

$$\| \sum a_I^{(j)} \rho_*(\xi_1)^{i_1} \dots \rho_*(\xi_n)^{i_n} \| \leqslant \sum |a_I^{(j)}| b_1^{i_1} \dots b_n^{i_n}. \tag{4.10}$$

Again the right-hand side converges to zero because the ξ_i can be chosen such that $|b_i| \leqslant \frac{1}{2}$ for all $i \in \{1, ., , n\}$, and the right-hand side is less or equal to

$$\sup |a_I^{(j)}| \sum \frac{1}{2^{|I|}}, \tag{4.11}$$

which converges to zero according to Remark 4.1. We can then scale back to the original ξ_i, which is just an isomorphism of Frechét spaces.

To see that $\tilde{\rho}_*$ is surjective we use the Lemma of Burnside which states that $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ is already surjective. □

4.2 A Notion of Hermitian Operators for $\mathcal{O}(\mathbb{C}^n)$

In the following a notion of self-adjointness or hermitian operators for $\mathcal{O}(\mathbb{C}^n)$ will be required. For this we will extend the definition of † on $\mathcal{U}(\mathfrak{g})$ by continuity.

Lemma 4.3. *The map † extends to a continuous involution of* $\mathcal{O}(\mathbb{C}^n)$.

Proof. We choose a basis of $\mathfrak{g}$ in the following way. First, fix a maximal torus $\mathfrak{t}$ in $\mathfrak{g}$. Let $\tau_1, \dots, \tau_r$ be a basis of the torus such that $\tau_i^\dagger = \tau_i$ for all i. Then choose a system Π of positive roots and a basis ξ_α of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Pi$ such that

$$\xi_\alpha^\dagger = \xi_{-\alpha}. \tag{4.12}$$

With this basis, † operates on the basis elements just by permutation.

Let $f = \sum_I a_I X^I$ be in $\mathcal{O}(\mathbb{C}^n)$. We define $f^\dagger := \sum_I \bar{a}_I (X^I)^\dagger$. Clearly, $f^\dagger$ is again everywhere convergent because we just changed the order of the summation and conjugated each coefficient.

Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of holomorphic functions on $\mathbb{C}^n$ converging to zero uniformly on compact subsets. To show that † is continuous, we must show that

$$\lim_{j\to\infty} (f_j^\dagger) = 0. \tag{4.13}$$

But since in each $f_j^\dagger$ we have only changed the order of the summands and conjugated to coefficients, this is also a series of holomorphic functions converging uniformly on compact subsets.

As stated before, the choice of basis has no effect on the topology. □

We define the notion of an abstractly hermitian operator as follows.

Definition 4.4. *$f \in \mathcal{O}(\mathbb{C}^n)$ is called an* abstractly hermitian operator *if $f^\dagger = f$. The set of all abstract hermitian operators is denoted by $\mathcal{H}$.*

Note that this definition is compatible with the one given for the tensor algebra in the Appendix.

Remark 4.5. $\mathcal{H}$ is a closed subspace of $\mathcal{O}(\mathbb{C}^n)$ and as such is a Fréchet space.

Proof. The linear map $\dagger - \mathrm{id}_{\mathcal{O}(\mathbb{C}^n)}$ is continuous and $\mathcal{H}$ is its kernel. □

Lemma 4.6. *Let $\rho : K \to \mathrm{U}(V)$ be an irreducible unitary representation and $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ the induced representation with extension $\tilde{\rho}_* : \mathcal{O}(\mathbb{C}^n) \to \mathrm{End}(V)$. Then the restriction of $\tilde{\rho}_*$ to $\mathcal{H}$ is surjective onto the subspace of self-adjoint linear operators of V.*

Proof. For $A \in \mathrm{End}(V)$ we denote by $A^\dagger$ the conjugate transpose of A. We remark that by the definition of † we have

$$\rho_*(\xi)^\dagger = \rho_*(\xi^\dagger)\ \forall \xi \in \mathfrak{g}. \tag{4.14}$$

Therefore

$$\rho_*(\mathcal{H} \cap \mathcal{U}(\mathfrak{g})) \subset \text{self-adjoint operators in End(V)}. \tag{4.15}$$

To show that the restriction is surjective, consider a self-adjoint operator $A \in \mathrm{End}(V)$. Since ρ_* is surjective, we find an $\alpha \in \mathcal{H} \cap U(\mathfrak{g})$, such that $\rho_*(\alpha) = A$. By (4.14) it follows that

$$\rho_*(\alpha^\dagger) = \rho_*(\alpha)^\dagger = A^\dagger = A. \tag{4.16}$$

Therefore we see that

$$\rho_*\left(\frac{1}{2}(\alpha + \alpha^\dagger)\right) = \frac{1}{2}\rho_*(\alpha) + \frac{1}{2}\rho_*(\alpha^\dagger) = \frac{1}{2}A + \frac{1}{2}A = A. \tag{4.17}$$

But

$$\frac{1}{2}(\alpha + \alpha^\dagger) \in \mathcal{H} \cap U(\mathfrak{g}), \tag{4.18}$$

so the restriction of $\tilde{\rho}_*$ to $\mathcal{H}$ is surjective. □

4.3 Examples of Convergence

In this section we will give a class of examples for the convergence of nearest neighbor statistics of abstractly hermitian operators in suitable sequences of irreducible representations.

Before these examples are considered we briefly discuss the effect of holomorphic maps on operators. Consider a holomorphic map $f : \mathbb{C} \to \mathbb{C}$. It induces a map

$$\tilde{f} : \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\mathbb{C}^n),\ g \mapsto f \circ g. \tag{4.19}$$

Let $\rho : K \to \mathrm{U}(V)$ be an irreducible representation and $\xi \in \mathcal{O}(\mathbb{C}^n)$ be a fixed operator. We are interested in the spectrum of $\tilde{\rho}_*(\tilde{f}(\xi))$.

Remark 4.7.

$$\operatorname{Spec}(\tilde\rho_*(\tilde f(\xi))) = f(\operatorname{Spec}(\tilde\rho_*(\xi))). \tag{4.20}$$

Proof. Let $\sum_j b_j z^j$ be the power series expansion for f at zero. Since $\tilde\rho_*$ is continuous, it follows that

$$\tilde\rho_*(\tilde f(\xi)) = \sum_j b_j \tilde\rho_*(\xi)^j. \tag{4.21}$$

Conjugating $\tilde\rho_*(\xi)$ to a diagonal matrix and inserting in the above equation gives then the desired result. □

Theorem 4.8. *Let $(\rho_m : K \to \mathrm{U}(V_m))_{m\in\mathbb{N}}$ be a sequence of irreducible representations with strictly increasing dimension. Assume that $\xi \in \mathcal{H}$ has the following properties:*

1. *Every eigenvalue of $\tilde\rho_{*,m}(\xi)$ has multiplicity one.*
2. *$S := \bigcup_{m\in\mathbb{N}} \operatorname{Spec}(\tilde\rho_{*,m}(\xi))$ is a discrete subset of $\mathbb{R}$.*

Then for every absolutely continuous measure μ on $\mathbb{R}^+$ with $\int_0^\infty x d\mu \in [0,1]$ there exists a function $f \in \mathrm{Hol}(\mathbb{C})$ and a subsequence $(\rho_{m_k} : K \to \mathrm{U}(V_{m_k}))_{k\in\mathbb{N}}$ such that $\eta := f(\xi)$ satisfies

$$d_{KS}(\mu_{\tilde\rho_{*,m_k}(\eta)}, \mu) \to 0 \text{ as } k \to \infty. \tag{4.22}$$

Proof. We begin by choosing a subsequence ρ_{m_k} in the following way. First, we set $r_{m_1} = \rho_1$ and proceed inductively by requiring that

$$N_{k+1} := \dim \rho_{m_k+1} \geq k(\dim \rho_{m_k} + 2). \tag{4.23}$$

Without loss of generality we assume that $N_1 \geq 3$ and find an N_1-tuple X_1 such that

$$d_{KS}(\mu(X_1), \mu) \leq \frac{2}{N_1}. \tag{4.24}$$

We now proceed inductively again, i.e. by Corollary 6.25 in the Appendix, there is an N_{k+1}-tuple X_{k+1} that contains the N_k-tuple X_k as subset such that

$$d_{KS}(\mu(X_{k+1}), \mu) \leq \frac{N_k + 2}{N_{k+1}} \leq \frac{1}{k}, \tag{4.25}$$

where the last inequality follows from (4.23).

Elementary complex analysis yields that there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$, such that

$$f(\operatorname{Spec}(\tilde\rho_{*,m_k}(\xi))) = X_k \ \forall k \in \mathbb{N}, \tag{4.26}$$

since S is a discrete subset in $\mathbb{R}$ and each $X_k \subset X_{k+1}$. By (4.20) it follows that

$$\operatorname{Spec}(\tilde\rho_{*,m_k}(\tilde f(\xi))) = X_k \ \forall k \in \mathbb{N}. \tag{4.27}$$

Thus, $\eta = \tilde f(\xi)$ has the property

$$d_{KS}(\mu_{\tilde\rho_{*,m_k}(\eta)}, \mu) \to 0 \text{ as } k \to \infty. \tag{4.28}$$

□

Operators ξ with the above properties will in general exist for every ray of irreducible representations. One strategy of producing them goes as follows:

Start with an operator ξ of degree 2 that fulfills condition 1. Such operators can be found for every simple group K and should exist in general. We now force condition 2 to hold by adding Casimir operators to ξ. Recall that Casimir operators act by scalar multiplication so they just add these scalars to the eigenvalues. If these scalars increase quickly enough, the spectra of ξ along the irreducible representations will lie in disjoint intervals and consequently condition 2 is satisfied.

The problem is that the operator ξ depends on the group K and we do not know if there is an abstract way of giving examples. So we will give here an example for $K = SU_n$ for the ray of irreducible representations through the standard representation.

Proposition 4.9. *Let $(\rho_m : SU_n \to \mathrm{U}(V_m))_{m\in\mathbb{N}}$ be the sequence of irreducible representations on the homogeneous polynomials of degree m in $\mathbb{C}[x_1, \ldots, x_n]$.*

Then there exists an operator $\xi \in \mathcal{U}(\mathfrak{g})$ that satisfies the conditions of Theorem 4.8.

Proof. Let α_j denote the $n \times n$-matrix with 1 in the j-th diagonal component and -1 in the $(j+1)$-th diagonal component. Every other component should be equal to zero. These matrices form a basis for the standard maximal torus in $SL_n(\mathbb{C}) = SU_n^{\mathbb{C}}$. They also define a system of simple roots (cf. the tables in Appendix C of [Kna02]).

The operation of α_j on the homogeneous polynomial $x_1^{a_1} \ldots x_n^{a_n}$ of degree m is given by

$$\rho_{*,n}(\alpha_j).x_1^{a_1} \ldots x_n^{a_n} = (a_j - a_{j+1})x_1^{a_1} \ldots x_n^{a_n}. \tag{4.29}$$

Therefore, the largest eigenvalue of $\rho_{*,n}\alpha_j$ is m and the smallest $-m$ and every other eigenvalue is an integer number in-between these extremes. Now, we consider the operator $\xi = \sum c_j\alpha_j$, where the c_j are real constants with $0 < c_j < \frac{1}{n}$ and which are linearly independent over $\mathbb{Q}$. Thus, $\rho_{*,m}(\xi)$ is represented as diagonal matrix and has eigenvalues with multiplicity greater than 1, since otherwise there would exist a linear relation between the c_j over $\mathbb{Q}$. Note that by the choice of the c_j the eigenvalues of ξ are still in the interval $[-m, m]$.

By now ξ satisfies condition 1 of Theorem 4.8 and we will now add the Laplace operator to ξ to guarantee that condition 2 holds. For this, let $\Omega \in \mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ be the Laplace operator associated to $\mathfrak{sl}_n(\mathbb{C})$. It acts on the homogeneous polynomials of degree m by $r_{\Omega,m} := \langle m\lambda, m\lambda + 2\delta\rangle_{\mathrm{Kil}}$, where λ is the highest weight of the standard representation of SU_k, δ denotes the half sum of positive weights and $\langle\cdot,\cdot\rangle_{\mathrm{Kil}}$ denotes the Killing form. It follows that

$$r_{\Omega,m+1} - r_{\Omega,m} = \langle\lambda, \lambda + 2\delta\rangle_{\mathrm{Kil}} + m\langle\lambda, \lambda\rangle_{\mathrm{Kil}} + m\langle\lambda, \lambda + 2\delta\rangle_{\mathrm{Kil}}. \tag{4.30}$$

Choosing a constant b such that

$$b(r_{\Omega,m+1} - r_{\Omega,m}) \geq 2m \ \forall \ m \in \mathbb{N}^* \tag{4.31}$$

yields that

$$\xi' := b\Omega + \sum_j c_j\alpha_j \tag{4.32}$$

fulfills conditions 1 and 2 of Theorem 4.8. □

4.4 Rational Independence of the Spectra in Representations

In this section we give a notion of generic operators in $\mathcal{H}$.

Definition 4.10. *An abstract hermitian operator $\alpha \in \mathcal{H}$ is called **generic** if for every irreducible representation ρ the eigenvalues of $\tilde{\rho}$ are linearly independent over $\mathbb{Q}$. We denote the set of generic operators in $\mathcal{H}$ by $\mathcal{H}_{\text{gen}}$.*

We start with the following theorem.

Theorem 4.11. *The set of generic operators $\mathcal{H}_{\text{gen}}$ is dense in $\mathcal{H}$.*

Before the prove is given, we need to fix the notation. The ordered tuple of eigenvalues with multiplicity of a hermitian matrix A will be denoted by $X(A)$ and the set of ordered n-tuples by $\mathbb{R}^n_{\text{ord}}$.

Lemma 4.12. *Let V be a unitary vector space of dimension n and $\operatorname{Herm}(V)$ be the real subspace of hermitian endomorphisms of V. For every $\lambda \in (\mathbb{Q}^n)^*$ the set*

$$S_\lambda := \{A \in \operatorname{Herm}(V) : \lambda(X(A)) = 0\} \tag{4.33}$$

is nowhere dense in $\operatorname{Herm}(V)$.

Proof. Let $\lambda \in (\mathbb{Q}^n)^*$ be a non-zero linear form. The set $\lambda^{-1}(0)$ is a hyperplane in $\mathbb{R}^n$, thus nowhere dense. In follows that the intersection of $\mathbb{R}^n_{\text{ord}} \cap \lambda^{-1}(0)$ is nowhere dense in $\mathbb{R}_{\text{ord}}$.

Now, let us fix a given point $x \in \mathbb{R}^n_{\text{ord}}$. From linear algebra we know that the set of hermitian operators with spectrum $\{x_1, \ldots, x_n\}$ is just the $U(n)$ orbit under matrix conjugation through the diagonal matrix

$$X = \operatorname{diag}(x_1, \ldots, x_n). \tag{4.34}$$

Therefore, the set $\mathbb{R}^n_{\text{ord}}$ can be identified with $\operatorname{Herm}(V)/U(n)$ and the projection map $p : \operatorname{Herm}(V) \to \operatorname{Herm}(V)/U(n) = \mathbb{R}^n_{\text{ord}}$ is an open map.

Because preimages of nowhere dense sets under open maps are nowhere dense, the lemma is proved. □

Proof. (Theorem 4.11) Since $\rho_* : \mathcal{U}(\mathfrak{g}) \to \operatorname{End}(V)$ is an irreducible, finite-dimensional representation, the induced mapping $\tilde{\rho}_* : \mathcal{H} \to \operatorname{Herm}(V)$ is a real linear, surjective mapping between Fréchet spaces. Therefore it is an open mapping by the open mapping theorem.

So for any given non-zero linear from $\lambda \in (\mathbb{Q}^{\dim V})^*$, the set

$$M_{\lambda,\rho} := \{\alpha \in \mathcal{H} \,:\, X(\tilde{\rho}_*(\alpha)) \in \lambda^{-1}(0)\} \tag{4.35}$$

is nowhere dense in $\mathcal{H}$. Otherwise, we could find an inner point in this set, but because $\tilde{\rho}$ is an open mapping this would contradict Lemma 4.12.

Thus, the set

$$M := \bigcup_{\rho\,\text{irrep.}, \lambda\in(\mathbb{Q}^{\dim V})^*} M_{\lambda,\rho} \tag{4.36}$$

contains no inner point by Baire's category theorem, i.e. its complement is dense. It follows that $\mathcal{H}_{\text{gen}}$ is dense. □

4.5 Ergodic Properties of $\mathcal{H}_{\text{gen}}$

Before we come to the main point of this section, we have to recall some terminology from ergodic theory. All details can be found in [Sin94] or [CFS82]. We follow the latter in terminology.

Let (X, μ) be a measure space, where μ denotes the measure on some σ-algebra in the power set of X. A measurable map $f : X \to X$ is called an automorphism of the measure space (X, μ), if f is bijective, f^{-1} is measurable again, and for all measurable sets $A \subset X$, we have

$$\mu(f(A)) = \mu(f^{-1}(A)) = \mu(A). \tag{4.37}$$

By a flow $(\varphi_t)_{t\in\mathbb{R}}$ of the measure space (X, μ), we mean a 1-parameter group of automorphisms of (X, μ), i.e., a group homomorphism of $\mathbb{R}$ into the group of all automorphisms of the measure space (X, μ) such that $\varphi : \mathbb{R}\times X \to X$ is measurable.

For us X will be an N-dimensional torus, i.e., $X = [0, 1]^N \bmod 1$ and the measure μ is the Haar measure on X, which is equal to the Lebesgue measure here. We consider some N-tuple $x = (x_1, .., x_n)$ such that $0 < x_i < 1$ for all $i \in \{1, \dots, N\}$ and the x_i's are linearly independent over the rational numbers. The map $\varphi_t : X \to X, z \mapsto z + t \cdot x \bmod 1$ defines a group homomorphism $\mathbb{R} \to \text{Diff}(X)$, $t \mapsto \varphi_t$, where $\text{Diff}(X)$ denotes the group of diffeomorphisms of X. It is a standard fact from ergodic theory that $(\varphi_t)_{t\in\mathbb{R}}$ is a flow of the measure space (X, μ) (cf. [CFS82]) Chapter 3, §1, Theorem 1).

A flow is called ergodic if for every $t \neq 0$, the only invariant sets of φ_t have measure either 0 or 1. We make use of the following

Theorem 4.13. (Birkhoff) *Let (X, μ) be a measure space with $\mu(X) = 1$ and $(\varphi_t)_{t\in\mathbb{R}}$ be a flow of the measure space (X, μ). Then for every integrable function $f : X \to \mathbb{R}$,*

$$\bar{f}(y) := \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(\varphi_\tau(y))d\tau = \int_X f(x)dx \tag{4.38}$$

for almost all $y \in T$ with respect to μ.

It is a standard result of ergodic theory that $(\varphi_t)_{t\in\mathbb{R}}$ is a uniquely ergodic flow, i.e., $\bar{f}$ is constant, (cf. [CFS82]) Chapter 3, §1, Theorem 2).

In this case, we obtain the formula for the characteristic function χ_A of a measurable set A:

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \chi_A(\varphi_\tau(y))d\tau = \mu(A) \ \forall y \in X. \tag{4.39}$$

Let us now consider an element $\alpha \in \mathcal{H}_{\text{gen}}$ and the induced irreducible, finite-dimensional representation $\rho_* : \mathcal{U}(\mathfrak{g}) \to \text{End}(V)$. Since $\tilde{\rho}_*(\alpha)$ is a self-adjoint operator, it follows that $(\exp(2\pi i \tilde{\rho}(\alpha)t))_{t\in\mathbb{R}}$ is a uniquely ergodic flow on the torus

$$T(V) = \text{closure}(\{\exp(2\pi i \tilde{\rho}_*(\alpha)t) | t \in \mathbb{R}\}). \tag{4.40}$$

This torus depends on the starting direction $\tilde{\rho}_*(\alpha)$, but we will in the following always assume that we have conjugated it into a diagonal matrix. There is no loss of generality because we are only interested in the eigenvalues and they do not change under conjugation. Thus, we will just write T_N for the N-dimensional torus, i.e.,

$$T_N = \{\text{diag}(e^{2\pi i \phi_1}, \dots, e^{2\pi i \phi_N}) : \phi_j \in [0, 1]\}. \tag{4.41}$$

4.6 The Sets B_N

In this section we will use the ergodic properties of $\mathcal{H}_{\text{gen}}$ in combination with a theorem of Chapter 5 to connect the spectral properties of an abstract hermitian operator with the Poisson-statistics. For this we first need to fix some notation.

For a unitary automorphism $A \in U(V)$ of a finite-dimensional unitary vector space V of dimension N we have the nearest neighbor statistics $\mu_c(X(A))$ as defined in Definition 6.26 of the Appendix. By μ_{Poisson} we denote the absolutely continuous probability measure on the positive real line with density function $\exp(-x)$ with respect to the Lebesgue measure. Finally, let us write $d_{\text{KS}}(\mu_1, \mu_2)$ for the Kolmogorov-Smirnoff distance (cf. (6.39) in the Appendix).

The following theorem is analogous to the second main theorem of [KS99] and is the main result of Chapter 5.

Theorem 4.14. *Let $\alpha > 0$ be given. Then there exists an natural number N_0 such that for every $N \geq N_0$*

$$\int_{T_N} d_{KS}(\mu_c(X(A)), \mu_{\text{Poisson}})dA < \frac{1}{e^{\alpha\sqrt{\log N}}}. \tag{4.42}$$

The rather technical proof is given in Chapter 5, cf. Theorem 5.20.

Corollary 4.15. *For all $\alpha \in \mathbb{R}$ with $\alpha > 0$ and any $N \geq N_0 = N_0(\alpha)$ we have*

$$d_{\text{KS}}(\mu_A, \mu_{\text{Poisson}}) \leqslant e^{-\frac{1}{2}\alpha\sqrt{\log(N)}} \tag{4.43}$$

for all A in a set in T_N of measure at least $1 - e^{-\frac{1}{2}\alpha\sqrt{\log(N)}}$.

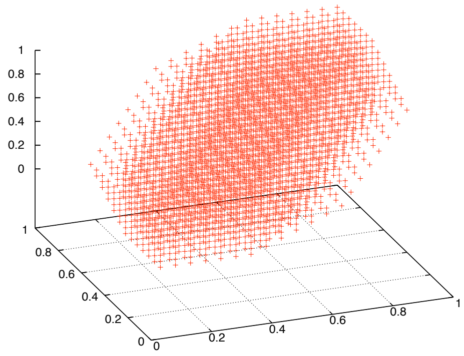

Figure 4.1: A picture of B_3.

Proof. Let us assume the contrary, i.e., we assume that

$$d_{\mathrm{KS}}(\mu_A, \mu_{\mathrm{Poisson}}) > e^{-\frac{1}{2}\alpha\sqrt{\log(N)}} \tag{4.44}$$

on a set M of measure at least $e^{-\frac{1}{2}\alpha\sqrt{\log(N)}}$. Then

$$\int_M d_{\mathrm{KS}}(\mu_A, \mu_{\mathrm{Poisson}}) d\,\mathrm{Haar}(A) > e^{-\frac{1}{2}\alpha\sqrt{\log(N)}} e^{-\frac{1}{2}\alpha\sqrt{\log(N)}} = e^{-\alpha\sqrt{\log(N)}}. \tag{4.45}$$

Since the integrand is always positive, this is a contradiction to Theorem 4.14. □

This motivates the following definition.

Definition 4.16. *Let $\alpha > 0$ be given. The set B_N is given by* [1]

$$B_N := \left\{ B \in T_N : d_{\mathrm{KS}}(\mu_B, \mu_{\mathrm{Poisson}}) \geq e^{-\frac{1}{2}\alpha\sqrt{\log(N)}} \right\}. \tag{4.46}$$

It is clear that B_N depends on the choice of α. However, for reasons of simplicity we suppress this fact in the notation. In the following we will always assume that the N are so large that Theorem 4.14 is valid, i.e. $N \geq N_0 \geq 2$.

Let us now collect some properties of B_N. First of all, B_N is not empty because the identity matrix E_N is in B_N. For this just recall that $\int_0^c \mu_{\mathrm{Poisson}}$ is close to zero for small c and that $\int_0^c d\mu_{E_N} = 1$ for every non-negative c, so $d_{\mathrm{KS}}(\mu_{E_n}, \mu_{\mathrm{Poisson}}) = 1$.

Due to the fact that the map $A \mapsto d_{\mathrm{KS}}(\mu_A, \mu_{\mathrm{Poisson}})$ is continuous (cf. Lemma 6.29), B_N is closed and the identity matrix is an inner point as a consequence of continuity.

Moreover, B_N is invariant under scalar multiplication with $z = e^{i\lambda}$, where $\lambda \in \mathbb{R}$, cf. Chapter 6.

The set B_3 for $\alpha = \frac{4}{3}$ is visualized by Figure 4.1. For the drawing, we have discretized the torus T_3 into a cubical lattice with $20 \times 20 \times 20$ points and calculated a

[1]The letter B in B_N is not an abbreviation for big. In fact these sets are small.

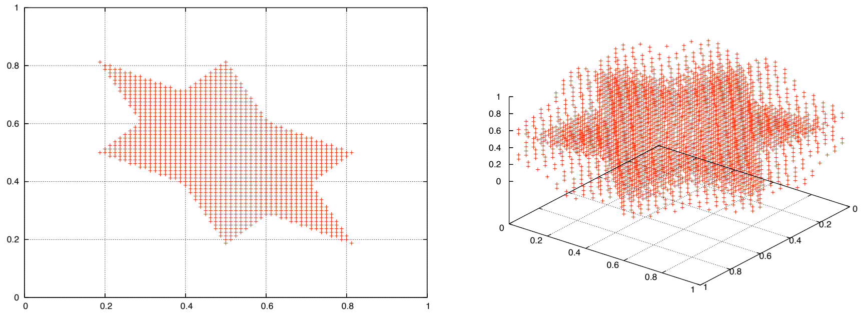

Figure 4.2: Pictures of B_3 and B_4 intersected with the hyperplane normal to the diagonal.

discretized version of d_{KS} for a grid size of 20 points. The axises show the coordinates ϕ_1, ϕ_2 and ϕ_3. The intersection of B_3 and the cubical grid is the drawn set of points. The definition of the discretized version is given as Definition 5.7. One can see the invariance under multiplication with $e^{i\phi}$ here as invariance under diagonal shifts.

Thus, it is enough to know the sets B_N only on that hyperplane which is normal to the diagonal and contains the point $\frac{1}{2}(1, \ldots, 1)$, i.e., the hyperplane given by

$$a_1\phi_1 + \cdots + a_N\phi_N = N/2. \tag{4.47}$$

Figure 4.2 shows these hyperplanes for $N = 3, 4$ parametrized by $\phi_1, \ldots, \phi_{N-1}$.

We now use the ergodic properties of $\mathcal{H}_{\text{gen}}$ to formulate our key lemma.

Lemma 4.17. *Let $\gamma \in \mathcal{H}_{\text{gen}}$ and $\rho : K \to \mathrm{U}(V)$ be an irreducible, finite-dimensional, unitary representation with $\dim V = N$ and denote the characteristic function of the set B_N by χ. Then*

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \chi(\exp(2\pi i\tilde{\rho}_*(\gamma)\tau))d\tau = \mathrm{vol}_{T_N}(B_N), \tag{4.48}$$

where $\mathrm{vol}_{T_N}(B_N)$ denotes the measure of B_N with respect to the Haar measure on T_N.

Proof. This is just the ergodic property of equation (4.39). □

We would like to emphasize the role of t in the above lemma. Consider the set $R(N)$ defined by

$$R(N) = \{t \in \mathbb{R} : \exp(2\pi i\tilde{\rho}_*(\gamma)t \in B_N\}. \tag{4.49}$$

Corollary 4.18. *Under the assumptions of the above lemma*

$$d_{\text{KS}}(\mu_{\exp(2\pi i\tilde{\rho}_*(\gamma)t}, \mu_{\text{Poisson}}) < e^{-\frac{1}{2}\alpha\sqrt{log(N)}} \tag{4.50}$$

for every $t \notin R(N)$.

Moreover $\mathbb{R}\backslash R(N)$ has infinite measure and he have the following estimation on the size of $R(N)$

$$0 < \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} \chi_{R(N)}(\tau)d\tau < e^{-\frac{1}{2}\alpha\sqrt{log(N)}}, \tag{4.51}$$

where $\chi_{R(N)}$ denotes the characteristic function of $R(N)$.

Proof. By virtue of equation (4.48) we obtain the corollary. □

4.7 Convergence to μ_{Poisson}

From now on, consider a sequence $(\rho_k)_{k\in\mathbb{N}}$ of irreducible, unitary representations $\rho_k : K \to \mathrm{U}(V_k)$ such that $d_k := \dim(V_k)$ is increasing. Before the main result can be stated, it is necessary to introduce two rather technical conditions.

Definition 4.19. *A sequence $(\rho_k)_{k\in\mathbb{N}}$ is said to be of **admissible growth**, if there exists an $\alpha > 0$ such that*

$$\sum_{k=0}^{\infty} e^{-\frac{1}{2}\alpha\sqrt{\log(d_k)}} < \infty. \tag{4.52}$$

Definition 4.20. *A generic hermitian operator $\gamma \in \mathcal{H}_{\text{gen}}$ is said to be **admissible of width ϵ** for the sequence $(\rho_k)_{k\in\mathbb{N}}$, where $0 < \epsilon < 1$ if there exists a k_0 and a t_0 such that for all $t \geq t_0$ and all $k \geq k_0$ the inequality*

$$\left| \mathrm{vol}(B_{d_k}) - \frac{1}{2t}\int_{-t}^{t} \chi_{B_{d_k}}(\exp(2\pi i \rho_{*,k}(\gamma)\tau)d\tau \right| < \epsilon \tag{4.53}$$

holds. Here $\chi_{B_{d_k}}$ denotes the characteristic function of the set B_{d_k} as defined above.

Let us briefly discuss these definitions. As will become clear in the following theorem the first describes a condition on the growth of the dimensions d_k. By a direct calculation we see that the condition requires d_k to grow faster than $e^{\left(\frac{2\log(k)}{\alpha}\right)^2}$. We will come back to this later.

The second definition guarantees that we are outside the sets B_N in each representation. For fixed k the condition can be fulfilled for every ϵ by Birkhoff's ergodic theorem. But we require here that t_0 as a function of k is bounded. So, condition (4.53) only fails, if

$$\left| \mathrm{vol}(B_{d_k}) - \frac{1}{2t}\int_{-t}^{t} \chi_{B_{d_k}}(\dots)d\tau \right| \to 1 \tag{4.54}$$

is true. This will happen if the leaving time, i.e., the supremum of all t, such that $\exp(2\pi i\rho_{*,k}(\gamma)\tau) \in B_{d_k}$, converges too rapidly to infinity as function of k. In Lemma 3.13 we saw this kind of behavior. The reader may wonder if operators of width ϵ do exist at all. But in Section 4.3 we saw examples of operators γ whose nearest neighbor statistics converge to a given measure μ. Although the situation is a little different here, because of the exponentiation, we could use the proof of Theorem 4.8 to construct operators γ such that $\exp(2\pi i\tilde{\rho}_{*,k})$ has nearest neighbor statistics which converge to μ_{Poisson}. These γ have a leaving time less than 1 by construction.

Now we state our key theorem in this chapter.

Theorem 4.21. *Let $\gamma \in \mathcal{H}_{\text{gen}}$ be admissible of width ϵ for a sequence $(\rho_k : K \to \mathrm{U}(V_k))_{k\in\mathbb{N}}$ of irreducible, unitary representations which is of admissible growth. Then for every $\epsilon' > 0$ there exists a set $R = R(\epsilon')$ in $\mathbb{R}$, such that*

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \chi_R(x)dx \leq \epsilon + \epsilon' \tag{4.55}$$

and

$$\mu_{\exp(2\pi i\tilde{\rho}_{*,k}(\gamma)t)} \to \mu_{\text{Poisson}} \text{ as } k \to \infty \tag{4.56}$$

for all $t \notin R$.

Before we prove the theorem, let us discuss the claim about the measure of R. Any bounded set R is of this type, or any set of measure 0. But from the point of view of percentage of real numbers, we prove that a fraction of $(1-\epsilon-\epsilon')$ of the real numbers yields convergence to μ_{Poisson} for the subsequence.

Proof. According to the condition of (4.53), we find a t_0 such that for all $t \geq t_0$ and all $k \geq k_0$

$$\left|\mathrm{vol}(B_{d_k}) - \frac{1}{2t}\int_{-t}^{t} \chi_{B_{d_k}}(\exp(2\pi i\rho_{*,k}(\gamma)\tau)d\tau\right| < \epsilon. \tag{4.57}$$

By the definition of admissible growth it follows that

$$\sum_{k=1}^{\infty} e^{-\frac{1}{2}\alpha\sqrt{log(d_k)}} < \infty. \tag{4.58}$$

Thus for every $\epsilon_1 > 0$ we find a natural number $N_0 = N_0(\epsilon_1)$ such that

$$\sum_{k=N_0}^{\infty} e^{-\frac{1}{2}\alpha\sqrt{log(d_k)}} < \epsilon_1. \tag{4.59}$$

Now set

$$R_{\epsilon_1} = \bigcup_{k=N_0}^{\infty} R(d_k), \tag{4.60}$$

where $R(d_k) = \{t \in \mathbb{R} : \exp(2\pi i\tilde{\rho}(\gamma)t \in B_{d_k}\}$. We set $Q_{\epsilon_1} = \mathbb{R}\backslash R_{\epsilon_1}$ and note that for all $t \in Q_{\epsilon_1}$

$$\mu_{\exp(2\pi i\tilde{\rho}_{*,k}(\gamma)t)} \to \mu_{\text{Poisson}} \text{ as } k \to \infty. \tag{4.61}$$

Now we have to show that $Q_{\epsilon_1} \neq \emptyset$.

By enlarging N_0 if necessary, we may also assume that $k_0 \leq N_0$. We fix an interval $[-t,t]$, where $t \geq t_0$, and obtain

$$2t(\mathrm{vol}(B_{d_k}) - \epsilon) \leq \mathrm{vol}(R(d_k) \cap [-t,t]) \leq 2t(\mathrm{vol}(B_{d_k}) + \epsilon). \tag{4.62}$$

Summing over all $k \geq N_0$ and applying (4.59) it follows that

$$\mathrm{vol}(R_{\epsilon_1} \cap [-t,t]) = \mathrm{vol}(\bigcup R(d_k) \cap [-t,t]) \leq 2t(\epsilon_1 + \epsilon). \tag{4.63}$$

We can choose ϵ_1 so small that $\epsilon_1 + \epsilon < 1$. This yields

$$Q_{\epsilon_1} \cap [-t,t] \neq \varnothing. \tag{4.64}$$

It remains to show (4.55). But since (4.63) holds for all $t \geq t_0$:

$$\mathrm{vol}(R_{\epsilon_1} \cap [-t,t]) = \int_{-t}^{t} \chi_{R_{\epsilon_1}}(s)ds \leq 2t(\epsilon_1 + \epsilon). \tag{4.65}$$

This completes the proof of the theorem. □

Let us briefly discuss this theorem. For every generic, admissible operator one has convergence of the nearest neighbor distributions for all $t \notin R$. But the reader may wonder how restrictive the condition of admissible width is. This will depend on the geometric structure of the sets B_N. If they are regular enough, the condition of admissible width should be automatically fulfilled for most generic operators. Unfortunately, we do not know enough about this structure yet, although in low dimensions the sets B_N are very regular (cf. Figures 4.1 and 4.2).

5 The Poisson Spectral Statistics for Tori

In this chapter we give a proof for the convergence of the nearest neighbor statistics of a real torus $T(N)$ to the Poisson spectral statistics in the sense of the Kolmogorov-Smirnov distance.

We follow the structure of the proof in [KS99] for the CUE case but will try to make this chapter as self-contained as possible, citing only some combinatorial lemmas and some facts about measures.

5.1 Some Combinatorics

We give here the basic definitions of Sep, Cor, Clump and so on from [KS99] again. To do this let $f : \mathbb{R} \to \mathbb{R}$ be a function, a be a non-negative integer called the *separation* and X be an N-tuple of real numbers in increasing order.

We define

$$\mathrm{Clump}(a, f, N, X) = \sum_{1 \le t_1 \le \ldots \le t_{a+2} \le N} f(x_{t_{a+2}} - x_{t_1}) \tag{5.1}$$

and

$$\mathrm{Sep}(a, f, N, X) = \sum_{1 \le t_1 \le \ldots \le t_{a+2} \le N, t_{j+1} - t_j = 1 \text{ for all } j} f(x_{t_{a+2}} - x_{t_1}). \tag{5.2}$$

Let us briefly discuss what these definitions signify, first taking a closer look at Clump. We sum over all $(a+2)$-tuples $(t_1, \ldots, t_{a+2})$ with increasing entries such that the last entry is smaller or equal than N, thereby evaluating the function f at the differences between $x_{t_{a+2}} - x_{t_1}$. If $a + 2 > N$ then there are no tuples to sum over, so Sep and Clump vanish identically.

Formally we can think of this as integrating the function f over a sum of Dirac measures at the points $x_{t_{a+2}} - x_{t_1}$. The same applies to the function Sep with the restriction that we sum only about the $(a+2)$-tuples of the form $(t_1, t_1 + 1, \ldots, t_1 + a + 1)$.

If we consider $a = 0$, then we evaluate f exactly at the nearest neighbor spacings. This may give a clear motivation why we are interested in Sep. The point in the definition of Clump will become clear later on. For the moment, let us just indicate that there will be a combinatorial identity expressing Sep as alternating sum over some versions of Clump.

By now, Sep and Clump are defined over increasing N-tuples X. We extend this definition to all N-tuples by first ordering the tuple X.

$$\mathrm{Clump}(a, f, N, \cdot) : \mathbb{R}^N \to \mathbb{R}, X \to \mathrm{Clump}(a, f, N, X \text{ ordered}) \tag{5.3}$$

and

$$\mathrm{Sep}(a, f, N, \cdot) : \mathbb{R}^N \to \mathbb{R}, X \mapsto \mathrm{Sep}(a, f, N, X \text{ ordered}). \tag{5.4}$$

Sep and Clump are special cases of a certain class of functions which we will deal with in the following. We define this class in the following way:

Definition 5.1. *Let $N \geq 2$ be an integer.*

*A function $f : \mathbb{R}^N \to \mathbb{R}$ is called a **function of class $\mathcal{T}(\mathbf{N})$** if f is Borel measurable, S_N-invariant and invariant under additive diagonal translations*

$$(x_1, \ldots, x_N) \mapsto (x_1 + t, \ldots, x_N + t), \tag{5.5}$$

with $t \in \mathbb{R}$.

*A function $f : \mathbb{R}^N \to \mathbb{R}$ is called a **function of class $\mathcal{T}_0(\mathbf{N})$** if f is a function of class $\mathcal{T}(N)$ and f vanishes outside the set $\{(x_1, \ldots, x_N) \in \mathbb{R}^N : \max_{i,j} |x_i - x_j| \leq \alpha\}$ for some $\alpha > 0$. We abbreviate this condition by*

$$\mathrm{supp}\, f \leq \alpha. \tag{5.6}$$

The following lemma lists some basic properties of Sep and Clump. This is Lemma 2.5.11 of [KS99].

Lemma 5.2. *For $a \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ Borel measurable and $N \geq 2$.*

Then $\mathrm{Sep}(a, f, N, \cdot)$ *and* $\mathrm{Clump}(a, f, N, \cdot)$ *are functions of class $\mathcal{T}(N)$. If f is continuous, then* $\mathrm{Sep}(a, f, N, \cdot)$ *and* $\mathrm{Clump}(a, f, N, \cdot)$ *are also continuous.*

If f vanishes outside the interval $[-\alpha, \alpha]$, $\mathrm{Sep}(a, f, N, \cdot)$ *and* $\mathrm{Clump}(a, f, N, \cdot)$ *are of class $\mathcal{T}_0(N)$ and*

$$\mathrm{supp}\,\mathrm{Sep}(a, f, N, \cdot) \leq \alpha \text{ and } \mathrm{supp}\,\mathrm{Clump}(a, f, N, \cdot) \leq \alpha. \tag{5.7}$$

Proof. See [KS99] p.52. □

Using Clump we define a third function for an integer k, $k \geq a$.

$$\mathrm{TClump}(k, a, f, N, \cdot) : \mathbb{R}^N \to \mathbb{R}, X \mapsto \binom{k}{a} \mathrm{Clump}(k, f, N, X). \tag{5.8}$$

Note that this definition may seem a bit superfluous, but it is added here to show the parallels to [KS99]. If we were working with multiple neighbor statistics, i.e. $r > 1$ in terms of [KS99], then TClump would be a more complicated sum.

We now relate this functions on $\mathbb{R}^N$ to functions on the torus $T(N)$. Again following [KS99], we name these functions Int for “integral”, Cor for “correlation” and TCor for “total correlation”.

These are defined as functions from $T(N)$ to $\mathbb{R}$ which map $A \in T(N)$ as follows

$$\begin{aligned}
\mathrm{Int}(a, f, T(N), A) &:= \frac{1}{N} \,\mathrm{Sep}\left(a, f, N, \frac{N}{2\pi} X(A)\right) \\
\mathrm{Cor}(a, f, T(N), A) &:= \frac{1}{N} \,\mathrm{Clump}\left(a, f, N, \frac{N}{2\pi} X(A)\right) \\
\mathrm{TCor}(k, a, f, T(N), A) &:= \frac{1}{N} \,\mathrm{TClump}\left(k, a, f, N, \frac{N}{2\pi} X(A)\right),
\end{aligned}$$

where $X(A)$ is $-i$ times the component-wise logarithm of A, i.e. for the matrix $A = \mathrm{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N})$ with $0 \leq \varphi_j < 2\pi$ for all j, we have $X(A) = (\varphi_1, \ldots, \varphi_N)$. It is now, obvious why we study these objects because

$$\mathrm{Int}(a, f, T(N), A) = \int_{\mathbb{R}} f d\mu(\mathrm{naive}, A, T(N), a). \tag{5.9}$$

It is exactly this $\mu(\mathrm{naive}, A, T(N), a)$ we want to study for a=0. For $a \geq 1$ we may take the above equation as definition of $\mu(\mathrm{naive}, A, T(N), a)$. In the notation of Chapter 1 this measure is given as

$$\mu(\mathrm{naive}, A, T(N), 0) = \frac{1}{N} \int_A \sum_{j=1}^{N-1} \delta\left(y - \frac{N}{2\pi} \cdot (\varphi_{j+1} - \varphi_j)\right) dy \tag{5.10}$$

if $a = 0$, which is almost identical to $\mu_c(X)(A)$ but the wrapped eigenangle between x_N and x_1 is missing. Therefore it is called "naive" in [KS99].

If we think of Int, Cor and TCor as random variables, we may calculate their expectation value. But instead of writing $E(\mathrm{Int}(a, f, T(N), A))$ we use capital letters:

$$\begin{aligned}
\mathrm{INT}(a, f, T(N), A) &:= \int_{T(N)} \mathrm{Int}(a, f, T(N), A) dA, \\
\mathrm{COR}(a, f, T(N), A) &:= \int_{T(N)} \mathrm{Cor}(a, f, T(N), A) dA, \\
\mathrm{TCOR}(k, a, f, T(N), A) &:= \int_{T(N)} \mathrm{TCor}(k, a, f, T(N), A) dA.
\end{aligned}$$

There are numerous relations between these functions, but we will stop the combinatorics here, coming back when we need it.

5.2 The Random Variable $Z[n, F, T(N)]$

Define the random variable $Z[n, F, T(N)]$ by

$$Z[n, F, T(N)](A) = \frac{1}{N} \sum_{\#T=n} F\left(\frac{N}{2\pi} \,\mathrm{pr}(T) X(A)\right), \tag{5.11}$$

where $\mathrm{pr}(T)$ is the projection from $T(N)$ to $T(n)$, $(x_1, \dots, x_N) \mapsto (x_{t_1}, \dots, x_{t_n})$ for a subset $T \subset \{1, \dots, N\}$ of cardinality n and $X(A)$ is the vector of angles for A.

We will later use this random variable with $F = \mathrm{TCor}$, but for the start we formulate our version of Theorem 4.2.2 of [KS99].

The following theorem should be thought of as a very special limit theorem for measures on the tori $T(N)$ as N goes to infinity. We fix a small torus of dimension n and sum over all projections of $T(N)$ to $T(n)$. In doing so we obtain induced measures on $T(n)$ and the statement of the following theorem can be interpreted as stating that these induced measures on $T(n)$ have a converging expectation value and decreasing variance.

Theorem 5.3. *Consider* $n \in \mathbb{N}, n \geq 2$ *and* $F \in \mathcal{T}_0(n)$ *with* $\operatorname{supp} F < \alpha$ *for* $\alpha > 0$. *Assume furthermore* $F \geq 0$.

1. *The sequence* $E(Z[n, F, T(N)])$ *converges for* $N \to \infty$ *to a limit* $E(n, F, \mathrm{univ})$ *and the estimation*
$$|E(Z[n, F, T(N)]) - E(n, F, \mathrm{univ})| \leq \|F\|_{\sup} \frac{1}{N} \frac{\alpha^{n-1}}{(n-2)!}. \tag{5.12}$$
is true for all $N \geq 2$.
2. *For all* $N \geq 2$ *the expectation is bounded as follows:*
$$|E(Z[n, F, T(N)])| \leq \|F\|_{\sup} \frac{\alpha^{n-1}}{(n-1)!}. \tag{5.13}$$
3. *For all* $N \geq 2$ *the variance is bounded as follows:*
$$\mathrm{Var}(Z[n, F, T(N)]) \leq \frac{\|F\|_{\sup}^2}{N} \max\{1, (2\alpha)^{2n-2}\} \frac{2n^2}{\left(\mathrm{floor}\left(\frac{n}{2}\right)!\right)^2}, \tag{5.14}$$
where floor *denotes the function rounding a real number down to the next integer.*

Proof. We start with the proof of statement 2.

By a direct calculation we see that

$$E(Z[n, F, T(N)]) = \frac{1}{N} \int_{[0,N]^n} \binom{N}{n} \frac{1}{N^n} F(x) dx_1 \dots dx_n. \tag{5.15}$$

Since $\operatorname{supp} F < \alpha$, we consider the set $\Delta(n, \alpha) = \{x \in \mathbb{R}^n : \sup_{i,j} |x_i - x_j| < \alpha\}$. By Lemma 5.8.3 of [KS99], we know

$$\frac{1}{N} \mathrm{Vol}(\Delta(n, \alpha) \cap [0, N]^n) \leq n\alpha^{n-1}. \tag{5.16}$$

Applying this to the above, it follows that

$$\begin{aligned} |E(Z[n, F, T(N)])| &\leq \frac{1}{N} \|F\|_{\sup} \frac{1}{N^n} \binom{N}{n} N \alpha^{n-1} n \\ &\leq \|F\|_{\sup} \frac{\alpha^{n-1}}{(n-1)!} \frac{N}{N} \cdot \frac{N-1}{N} \cdot \ldots \cdot \frac{N-n+1}{N}. \end{aligned} \tag{5.17}$$

The following inequality

$$\prod_{\nu=1}^{k} \left(1 - \frac{\nu}{N}\right) \leq 1 - \frac{k}{N}, \tag{5.18}$$

gives

$$|E(Z[n, F, T(N)])| \leq \|F\|_{\sup} \frac{\alpha^{n-1}}{(n-1)!} \left(1 - \frac{n-1}{N}\right) \leq \|F\|_{\sup} \frac{\alpha^{n-1}}{(n-1)!}. \tag{5.19}$$

Thus, statement 2 has been proven.

Now we wish to prove the first statement. For this, recall that $F \in \mathcal{T}_0(n)$ means, that F is S_n-invariant and invariant under diagonal addition. So

$$E(Z[n, F, T(N)]) = \frac{1}{N} \binom{N}{n} \frac{1}{N^n} n! \int_{[0,N]^n (\text{ordered})} F(x) dx_1 \ldots dx_n. \tag{5.20}$$

This is true since the tuples with two or more equal components are a zero set and can be neglected. Substituting

$$y_1 = x_1, y_2 = x_2 - x_1, \ldots, y_n = x_n - x_1 \tag{5.21}$$

yields

$$\begin{aligned} &E(Z[n, F, T(N)]) \\ &\quad = \frac{1}{N} \binom{N}{n} \frac{n!}{N^n} \int_0^N \left(\int_{[0,N-y_1]^{n-1} (\text{ordered})} F(0, y_2, \ldots, y_n) dy_2 \ldots dy_n \right) dy_1. \end{aligned} \tag{5.22}$$

We call the inner integral $g(y_1)$ and assume that $\alpha < N$. Note that

$$g(y_1) \leq \|F\|_{\sup} \int_{[0,\alpha]^{n-1} (\text{ordered})} dy_2 \ldots dy_n = \|F\|_{\sup} \frac{\alpha^{n-1}}{(n-1)!} \tag{5.23}$$

since $\operatorname{supp} F < \alpha$. Therefore the integral extends from 0 to $\min(\alpha, N - y_1)$. We now set

$$E(n, F, \text{univ}) := g_\alpha := \int_{[0,\alpha]^{n-1} (\text{orderhned})} F(0, y_2, \ldots, y_n) dy_2 \ldots dy_n \tag{5.24}$$

and consider the difference

$$D = |E(Z[n, F, T(N)]) - E(n, F, \text{univ})| = \left| \frac{1}{N} \binom{N}{n} \frac{n!}{N^n} \int_0^N g(y_1) dy_1 - g_\alpha \right|. \tag{5.25}$$

By splitting the integral into two parts, i.e., integrating from 0 to $N-\alpha$ and from $N-\alpha$ to N, it follows that in the first case $g(y_1) = g_\alpha$ because of the $\operatorname{supp} F < \alpha$ condition, and thus

$$D = \left| \frac{1}{N}\binom{N}{n}\frac{n!}{N^n}\int_0^{N-\alpha} g_\alpha dy_1 + \frac{1}{N}\binom{N}{n}\frac{n!}{N^n}\int_{N-\alpha}^{N} g(y_1)dy_1 - g_\alpha \right|. \tag{5.26}$$

Therefore we have

$$D \leq \left(\binom{N}{n}\frac{n!}{N^n}\frac{N-\alpha}{N} - 1\right) g_\alpha + \frac{\alpha}{N}\binom{N}{n}\frac{n!}{N^n} g_\alpha \tag{5.27}$$

which leads to

$$D \leq \left(\binom{N}{n}\frac{n!}{N^n} - 1\right) g_\alpha \leq \frac{n-1}{N} g_\alpha \leq \|F\|_{\text{sup}} \frac{\alpha^{n-1}}{(n-2)!}\frac{1}{N}. \tag{5.28}$$

This proves statement 1, if $N > \alpha$.

If $N \leq \alpha$, we define g_α as above, but immediately see that

$$E(n, F, \text{univ}) = g_\alpha = \int_{[0,N]^{n-1}(\text{ordered})} F(0, y_2, \ldots, y_n) dy_2 \ldots dy_n. \tag{5.29}$$

Inserting this into (5.25) it follows that statement 1 is fulfilled in this case as well.

For the proof of the last statement let us first look at

$$E(Z[\ldots]^2) = \frac{1}{N^2}\int_{T(N)} \sum_{\#T=n, \#S=n} F\left(\frac{N}{2\pi}\operatorname{pr}(T)X(A)\right) F\left(\frac{N}{2\pi}\operatorname{pr}(S)X(A)\right) dA, \tag{5.30}$$

where the sum extends over all subsets S and T of cardinality n of the set $\{1, \ldots, N\}$ and we write $E(Z[\ldots]^2)$ for $E(Z[n, F, T(N)]^2)$. This can be written as

$$\begin{aligned} E(Z[\ldots]^2) = {} & \frac{1}{N^2}\sum_{l=n}^{2n}\binom{N}{l}\frac{1}{N^l}\binom{l}{n}\binom{n}{l-n} \times \\ & \times \int_{[0,N]^l} F(x_1, \ldots, x_n) F(x_{l-n+1}, \ldots, x_l) dx_1 \ldots dx_l, \end{aligned} \tag{5.31}$$

as can seen by writing the double sum over T and S as a sum over the cardinality of $S \cup T$ and an inner sum. Using the S_n-invariance of F one obtains the above formula. Now we consider the summands with $l < 2n$. This means that

$$\operatorname{supp} F(x_1, \ldots, x_n) F(x_{l-n+1}, \ldots, x_l) \leq 2\alpha \tag{5.32}$$

because $|x_j - x_i| \leq |x_j - x_n| + |x_n - x_i|$ and $\operatorname{supp} F \leq \alpha$. Using again (5.16) and (5.18) we obtain

$$\begin{aligned} E(Z[\ldots]^2) \leq {} & \frac{1}{N}\sum_{l=n}^{2n-1}\left(1 - \frac{l-1}{N}\right)(2\alpha)^{l-1}\frac{l}{(l-n)!(l-n)!(2n-l)!}\|F\|_{\text{sup}}^2 \\ & + \frac{1}{N^2}\binom{N}{2n}\frac{1}{N^{2n}}\binom{2n}{n}\left(\int_{[0,N]^n} F(x_1, \ldots, x_n) dx_1 \ldots dx_n\right)^2. \end{aligned} \tag{5.33}$$

Now, compute the variance:

$$\begin{aligned}
&\operatorname{Var}(E(Z[n,F,T(N)])) = E(Z[n,F,T(N)]^2) - E(Z[n,F,T(N)])^2 \\
&\leq \frac{\|F\|_{\sup}^2}{N} \sum_{l=n}^{2n-1} (2\alpha)^{l-1} \frac{l}{((l-n)!)^2\,(2n-l)!} \\
&+ \frac{1}{N^2} \left(\int_{[0,N]^n} F(x_1,\dots,x_n) dx_1 \dots dx_n \right)^2 \left(\binom{2n}{n} \frac{1}{N^{2n}} \binom{N}{2n} - \left(\binom{N}{n} \frac{1}{N^n} \right)^2 \right) \\
&\leq \frac{\|F\|_{\sup}^2}{N} \sum_{l=n}^{2n-1} (2\alpha)^{l-1} \frac{l}{((l-n)!)^2\,(2n-l)!} \\
&+ \left(\alpha^{n-1}\right)^2 n^2 \left(\binom{2n}{n} \frac{1}{N^{2n}} \binom{N}{2n} - \left(\binom{N}{n} \frac{1}{N^n} \right)^2 \right) \|F\|_{\sup}^2 .
\end{aligned} \tag{5.34}$$

But the last summand is negative:

$$\begin{aligned}
&\binom{2n}{n} \frac{1}{N^{2n}} \binom{N}{2n} - \left(\binom{N}{n} \frac{1}{N^n} \right)^2 \\
&= \frac{1}{(n!)^2} \left(\prod_{\nu=0}^{2n-1} \left(1 - \frac{\nu}{N}\right) - \prod_{\nu=0}^{n-1} \left(1 - \frac{\nu}{N}\right)^2 \right) \\
&\leq 0
\end{aligned} \tag{5.35}$$

which gives the result

$$\operatorname{Var}(E(Z[n,F,T(N)])) \leq \frac{\|F\|_{\sup}^2}{N} \max\{(2\alpha)^{2n-2}, 1\} \sum_{p=0}^{n-1} \frac{n+p}{(p!)^2 (n-p)!} . \tag{5.36}$$

For simplicity we estimate further

$$\sum_{p=0}^{n-1} \frac{n+p}{(p!)^2 (n-p)!} \leq 2n \sum_{p=0}^{n-1} \frac{1}{n!} \frac{n!}{p!p!(n-p)!} \leq 2n \sum_{p=0}^{n-1} \frac{1}{n!p!} \binom{n}{p}. \tag{5.37}$$

For n even this yields

$$\binom{n}{p} \leq \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}. \tag{5.38}$$

and for n odd

$$\binom{n}{p} \leq \frac{n!}{\left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)!}. \tag{5.39}$$

So we have the following estimation for the variance:

$$\operatorname{Var}(Z[n,F,T(N)]) \leq \frac{\|F\|_{\sup}^2}{N} \max\{(2\alpha)^{2n-2}, 1\} \frac{2n^2}{\left(\text{floor}\left(\frac{n}{2}\right)!\right)^2}. \tag{5.40}$$

Combining everything finishes the proof of the last statement. □

5.3 Moving the Estimates to $\mathrm{TCor}(k, a, f, T(N))$

The reader may wonder how the above theorem is related to spectral statistics. The answer is given by the following theorem which transfers the above estimation on $Z[n, F, T(N)]$ to estimations about TCor.

Theorem 5.4. *Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a bounded, non-negative, Borel-measurable function with upper bound α and $a, k \in \mathbb{N}$ with $k \geq a$.*

1. *The sequence $\mathrm{TCOR}(k, a, f, T(N))$ converges for $N \to \infty$ to a limit which is denoted by $\mathrm{TCOR}(k, a, f, \mathrm{univ})$, and the following estimation*
$$|\,\mathrm{TCOR}(k, a, f, T(N)) - \mathrm{TCOR}(k, a, f, \mathrm{univ})| \leq \binom{k}{a} \|f\|_{\mathrm{sup}} \frac{1}{N} \frac{\alpha^{k+1}}{k!} \quad (5.41)$$
holds for all $N \geq 2$.
2. *For all $N \geq 2$ the expectation is bounded as follows:*
$$\mathrm{TCOR}(k, a, f, T(N)) \leq \binom{k}{a} \|f\|_{\mathrm{sup}} \frac{\alpha^{k+1}}{(k+1)!}. \quad (5.42)$$
3. *For all $N \geq 2$ the variance is bounded as follows:*
$$\begin{aligned} &\mathrm{Var}(A \mapsto \mathrm{TCOR}(k, a, f, T(N), A) \text{ on } T(N)) \\ &\qquad \leq \binom{k}{a}^2 \frac{\|f\|^2_{\mathrm{sup}}}{N} \max\{(2\alpha)^{2k+2}, 1\} \frac{2(k+2)^2}{\left(\mathrm{floor}\left(\frac{k}{2}+1\right)!\right)^2}. \end{aligned} \quad (5.43)$$

Proof. This is Proposition 4.2.3 of [KS99]. For self-containtedness we give the proof here again.

The idea is to use Theorem 5.3 for the function $F(X) = \mathrm{TClump}(k, a, f, k+2, X)$, where $k = n + 2$. We claim that
$$Z[k+2, F, T(N)](A) = \mathrm{TCor}(k, a, f, T(N), A) \quad (5.44)$$
then. This can be seen by unwinding the definitions and using a combinatorial identity for equation (5.45):
$$\begin{aligned} Z[k+2, F, T(N)](A) &= \frac{1}{N} \sum_{\#T = k+2} F\left(\frac{N}{2\pi} \mathrm{pr}(T) X(A)\right) \\ &= \frac{1}{N} \sum_{\#T = k+2} \mathrm{TClump}\left(k, a, f, k+2, \frac{N}{2\pi} \mathrm{pr}(T) X(A)\right) \\ &= \frac{1}{N} \mathrm{TClump}\left(k, a, f, N, \frac{N}{2\pi} X(A)\right) \quad (5.45) \\ &= \mathrm{TCor}(k, a, f, T(N), A). \end{aligned}$$
The theorem now follows from the fact that proved $F \in \mathcal{T}_0(n)$ and $\|f\|_{\mathrm{sup}}\binom{k}{a} \geq \|F\|_{\mathrm{sup}}$. But these are direct consequences of the definition of TClump as $\binom{k}{a}$ Clump and Lemma 5.2. □

5.4 The Weak Convergence of $\mu(\text{naive}, U(N), 1)$ to the Poisson Distribution

We only cite a part of Proposition 2.9.1 of [KS99] here without repeating the proof.

Theorem 5.5. (Katz, Sarnak) *Assume that $a \in \mathbb{N}$ is fixed. If for every $k \in \mathbb{N}$, $k \geq a$ and every $f : \mathbb{R} \to \mathbb{R}$ which is bounded, Borel measurable, non-negative and of compact support,*

$$\lim_{N\to\infty} \mathrm{TCOR}(k, a, f, T(N)) =: \mathrm{TCOR}(k, a, f, \text{univ}) \tag{5.46}$$

exists and moreover

$$\sum_{k\geq a} \mathrm{TCOR}(k, a, f, \text{univ}) < \infty, \tag{5.47}$$

then the limit measure $\mu(\text{naive}, a)$ exists and

$$\int_{\mathbb{R}} f d\mu(\text{naive}, a) = \sum_{k\geq a} (-1)^{k-a}\, \mathrm{TCOR}(k, a, f, \text{univ}). \tag{5.48}$$

Proof. [KS99], p. 58 and following. □

Since we are dealing with $T(N)$, it is possible to give an explicit formula for the Lebesgue density of $\mu(\text{naive}, 0)$.

Theorem 5.6. *The limit measure $\mu(\text{naive}, a)$ exists and for $a = 0$ it has the probability density e^{-x}.*

Proof. Here Theorem 5.5 will be applied to prove the convergence result. By statement 1 of Theorem 5.4, we know the existence of $\mathrm{TCOR}(k, a, f, \text{univ})$ and by statement 2 we see that

$$\mathrm{TCOR}(k, a, f, \text{univ}) \leq \binom{k}{a} \|f\|_{\sup} \frac{\alpha^{k+1}}{(k+1)!} \leq \|f\|_{\sup} \frac{(2\alpha)^{k+1}}{(k+1)!}. \tag{5.49}$$

Thus

$$\sum_{k\geq a} \mathrm{TCOR}(k, a, f, \text{univ}) \leq \|f\|_{\sup} \sum_{k=0}^{\infty} \frac{(2\alpha)^{k+1}}{(k+1)!} \leq \|f\|_{\sup} e^{2\alpha} < \infty. \tag{5.50}$$

It remains to prove the explicit form for $a = 0$. For this it suffices to calculate to

$$\int_{\mathbb{R}} f d\mu(\text{naive}, 0) \tag{5.51}$$

for the characteristic functions of intervals of the form $[0, p]$. But this integral can be calculated directly

$$\int_0^p d\mu(\text{naive}, 0) = \lim_{N\to\infty} \int_0^p d\mu(\text{naive}, T(N), 0), \tag{5.52}$$

where

$$\int_0^p d\mu(\text{naive}, T(N), 0) = \frac{1}{N}\sum_{j=1}^{N-1} N! \int_{T(N)(\text{ordered})} f\left(\frac{N}{2\pi}(x_{j+1} - x_j)\right) dA$$
$$= (N-1)! \sum_{j=1}^{N-1} \int_0^1 \int_0^{x_N} \dots \int_0^{x_{j+2}} \times \times \int_{x_{j+1}-p/N}^{x_{j+1}} \int_0^{x_j} \dots \int_0^{x_2} dx_1 \dots dx_N. \quad (5.53)$$

The desired result follows by evaluating the right-hand side. For this let us define the integrands I_j by

$$\int_0^p d\mu(\text{naive}, T(N), 0) = (N-1)! \sum_{j=1}^{N-1} I_J, \quad (5.54)$$

By a direct calculation we derive the recursion formula

$$I_{j+1} = I_j - (-1)^j \frac{1}{N!}\binom{N}{j+1}\left(\frac{p}{N}\right)^{j+1} \quad (5.55)$$

and thus the explicit formula for the I_j

$$I_j = \frac{1}{N!}\sum_{k=1}^{j}\binom{N}{k}\left(\frac{p}{N}\right)^k (-1)^{k+1}. \quad (5.56)$$

Now, we insert this into (5.54) and compare it to the power series for $1 - \exp(-p)$

$$\int_0^p d\mu(\text{naive}, T(N), 0) = (N-1)! \sum_{j=1}^{N-1} \frac{1}{N!} \sum_{k=1}^{j} \binom{N}{k}\left(\frac{p}{N}\right)^k (-1)^{k+1}$$
$$= \frac{1}{N}\sum_{j=1}^{N-1}\sum_{k=1}^{j}\binom{N}{k}\left(\frac{p}{N}\right)^k(-1)^{k+1} = -\frac{1}{N}\sum_{j=1}^{N-1}\sum_{k=1}^{j}\frac{(-p)^k}{k!}\prod_{\nu=1}^{k-1}\left(1-\frac{\nu}{N}\right)$$
$$= -\frac{1}{N}\sum_{k=1}^{N-1}\sum_{j=k}^{N-1}\frac{(-p)^k}{k!}\prod_{\nu=1}^{k-1}\left(1-\frac{\nu}{N}\right) = \sum_{k=1}^{N-1}\frac{(-p)^k}{k!}\frac{N-k}{N}\prod_{\nu=1}^{k-1}\left(1-\frac{\nu}{N}\right) \quad (5.57)$$
$$= \sum_{k=1}^{N-1} a_k(N)\frac{(-p)^k}{k!},$$

where the $a_k(N)$ are the coefficients defined above. For fixed k

$$a_k(N) = \prod_{\nu=1}^{k}\left(1-\frac{\nu}{N}\right) \to 1 \text{ as } N \to \infty, \quad (5.58)$$

which completes the proof. □

5.5 The M-grid

We would like to study the Kolmogorov-Smirnov distance d_{KS} for the nearest neighbor measures $\mu(\text{naive}, A, T(N), a)$. This is a quite complicated matter and therefore we discretize on the so-called M-grid.

For this let M be a (big) positive natural number. Divide the interval $[0,1]$ into pieces of length $\frac{1}{M}$. This defines a grid

$$-\infty = s(0) < s(1) < \ldots < s(M-1) < s(M) = +\infty, \tag{5.59}$$

where

$$\int_{-\infty}^{s(j)} \mu(\text{naive}, a) = \frac{j}{M} \text{ for all } 1 \leq j \leq M-1. \tag{5.60}$$

Definition 5.7. *Define the M-grid version of the Kolmogorov-Smirnov distance to be*

$$d_{M,\mathrm{KS}}(\mu, \nu) = \max_{i=1,\ldots,M-1} |\int_{s(1)}^{s(i)} d\mu - \int_{s(1)}^{s(i)} d\nu|. \tag{5.61}$$

Lemma 5.8. *For any Borel measure of total mass ≤ 1 we have the inequality*

$$d_{\mathrm{KS}}(\nu, \mu(\text{naive}, a)) \leq \frac{5}{M} + 2 \cdot d_{M,\mathrm{KS}}(\nu, \mu(\text{naive}, a)). \tag{5.62}$$

Proof. See [KS99] p.81. □

5.6 The Key Lemma

For simplicity we cite here Lemma 3.2.16 of [KS99].

Lemma 5.9. *Let $f \geq 0$ be a bounded, Borel measurable function with compact support and $L \geq a$ be an integer, then the following basic inequality*

$$\begin{aligned} &|\operatorname{INT}(a, f, \text{univ}) - \operatorname{Int}(a, f, T(N), A)| \\ &\qquad \leq \sum_{L \geq k \geq a} |\operatorname{TCOR}(k, a, f, T(N)) - \operatorname{TCor}(k, a, f, T(N), A)| \\ &\qquad + \sum_{L \geq k \geq a} |\operatorname{TCOR}(k, a, f, T(N)) - \operatorname{TCOR}(k, a, f, \text{univ})| \\ &\qquad + \operatorname{TCOR}(L, a, f, \text{univ}) + \operatorname{TCOR}(L+1, a, f, \text{univ}) \end{aligned} \tag{5.63}$$

holds.

Proof. See [KS99] p.83. □

Notation 5.10. Since it is too cumbersome to write the measure $\mu(\text{naive}, A, T(N), a)$, we abbreviate in the following

$$\mu = \mu(\text{naive}, a) \text{ and } \mu_A = \mu(\text{naive}, A, T(N), a) \tag{5.64}$$

for fixed a.

Corollary 5.11. *Let $R \subset [s(1), s(M-1)]$ be a Borel measurable. Then*

$$\begin{aligned}
|\mu(R) - \mu_A(R)| &\leq \sum_{L\geq k\geq a} |\operatorname{TCOR}(k, a, \chi, T(N)) - \operatorname{TCor}(k, a, \chi, T(N), A)| \\
&+ \binom{L}{a}\frac{\alpha^{L+1}}{(L+1)!} + \binom{L+1}{a}\frac{\alpha^{L+2}}{(L+2)!} + \frac{1}{N}\sum_{L\geq k\geq a}\binom{k}{a}\frac{\alpha^{k+1}}{k!} \\
&< \sum_{L\geq k\geq a} |\operatorname{TCOR}(k, a, \chi, T(N)) - \operatorname{TCor}(k, a, \chi, T(N), A)| \\
&+ \frac{(2\alpha)^{L+1}}{(L+1)!} + \frac{(2\alpha)^{L+2}}{(L+2)!} + \frac{1}{N}\sum_{L\geq k\geq a}\frac{(2\alpha)^{k+1}}{k!}
\end{aligned}$$

where χ is the characteristic function of R and $\alpha = \operatorname{diam}(R)$.

Proof. Apply the above Lemma and the TCOR estimations. □

Corollary 5.12. *Set $\beta = s(M-1) - s(1)$. The following estimation holds:*

$$\begin{aligned}
d_{M,\mathrm{KS}}(\mu, \mu_A) < \max_i\{ &\sum_{L\geq k\geq a} |\operatorname{TCOR}(k, a, \chi_{[s(1),s(i)]}, T(N)) \\
&- \operatorname{TCor}(k, a, \chi_{[s(1),s(i)]}, T(N), A)|\} \\
&+ \frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} + \frac{1}{N}\sum_{L\geq k\geq a}\frac{(2\beta)^{k+1}}{k!},
\end{aligned} \tag{5.65}$$

where χ_R denotes the characteristic function of the interval R.

Proof. This is clear from the definition of $d_{M,\mathrm{KS}}$. □

Lemma 5.13.

$$\begin{aligned}
\int_{T(N)} &|\operatorname{TCOR}(k, a, \chi_{[s(1),s(i)]}, T(N)) - \operatorname{TCor}(k, a, \chi_{[s(1),s(i)]}, T(N), A)| dA \\
&\leq \binom{k}{a}\sqrt{\frac{2}{N}}\max\{(2s(i) - 2s(1))^{k+1}, 1\}\frac{k+2}{\operatorname{floor}(\frac{k}{2}+1)!}.
\end{aligned} \tag{5.66}$$

Proof. This is just the Cauchy-Schwarz inequality

$$\int_{T(N)} |h(A)| dA \leq \sqrt{\int_{T(N)} |h(A)|^2 dA}, \tag{5.67}$$

where $h(A) = \operatorname{TCOR}(k, a, \chi_{[s(1),s(i)]}, T(N)) - \operatorname{TCor}(k, a, \chi_{[s(1),s(i)]}, T(N), A)$ combined with the statement about the variance of the TCOR estimations. □

Theorem 5.14. *For $N \to \infty$*

$$\int_{T(N)} d_{\mathrm{KS}}(\mu, \mu_A) dA \to 0. \tag{5.68}$$

Proof. Putting everything together, we obtain

$$\begin{aligned}\int_{T(N)} d_{\mathrm{KS}}(\mu, \mu_A) dA < \frac{5}{M} + 2 \cdot \left(\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} + \frac{1}{N} \sum_{L \geq k \geq 0} \frac{(2\beta)^{k+1}}{k!} \right) \\ + \sqrt{\frac{2}{N}} \max_i \left\{ \sum_{L \geq k \geq a} \binom{k}{a} \frac{k+2}{\mathrm{floor}(\frac{k}{2}+1)!} \max\{1, (2s(i) - 2s(1))^{k+1}\} \right\}\end{aligned} \tag{5.69}$$

where $\beta = s(M-1) - s(1)$ as above.

Now we combine the summands to make more explicit estimations

$$\sum_{L \geq k \geq 0} \frac{(2\beta)^{k+1}}{k!} < (2\beta) e^{2\beta} \tag{5.70}$$

and if $s(i) - s(1) \geq \frac{1}{2}$ we have the following estimation for the second sum

$$\begin{aligned}\sum_{L \geq k \geq 0} (4(s(i) - s(1)))^{k+1} \frac{k+2}{\mathrm{floor}(\frac{k}{2}+1)!} &\leq 8(s(i) - s(1)) \sum_{L \geq k \geq 0} \frac{(64(s(i) - s(1))^2)^{k/2}}{\mathrm{floor}(\frac{k}{2}+1)!} \\ &< 8(s(i) - s(1)) e^{64(s(i)-s(1))^2}.\end{aligned} \tag{5.71}$$

If $s(i) - s(1) < \frac{1}{2}$ we may estimate the sum as

$$\sum_{L \geq k \geq 0} 2^k \frac{k+2}{\mathrm{floor}(\frac{k}{2}+1)!} \leq \sum_{L \geq k \geq 0} 3 \cdot 2^k \leq 3L \cdot 2^L. \tag{5.72}$$

Applying this to the above it follows that

$$\begin{aligned}\int_{T(N)} d_{\mathrm{KS}}(\mu, \mu_A) dA <& \frac{5}{M} + 2 \left(\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} \right) \\ &+ \frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N}} (2\beta) e^{2\beta} + 8\beta\sqrt{2} e^{64\beta^2} + 6L \cdot 2^L \right).\end{aligned} \tag{5.73}$$

It is clear that β depends only on M. So given $\varepsilon > 0$, we first choose M so large, that

$$\frac{5}{M} < \frac{\varepsilon}{3}, \tag{5.74}$$

then we can choose L so large, that

$$\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} < \frac{\varepsilon}{6} \tag{5.75}$$

and finally N so large that

$$\frac{1}{\sqrt{N}} \left(\frac{1}{\sqrt{N}} (2\beta) e^{2\beta} + 8\beta\sqrt{2} e^{64\beta^2} + 6L \cdot 2^L \right) < \frac{\varepsilon}{3}. \tag{5.76}$$

□

5.7 The Final Estimation

In this last section we will give the final form of the estimation. But before we do so we state a series of lemmas which we will combine to give the main estimation.

We start by fixing two positive constants $\alpha, \gamma \in \mathbb{R}_{>0}$. Set the grid size M to be the largest integer smaller than $e^{\alpha\sqrt{\log N}}$ and the cut-off L to be the largest integer such that

$$(L-1)! \le N^{\gamma^2} \le L!\,. \tag{5.77}$$

Then

$$\log M \le \alpha\sqrt{\log N} \le \log(M+1) \tag{5.78}$$

and

$$\log(L-1)! \le \gamma^2 \log N \le \log L!\,. \tag{5.79}$$

Thus, we see that

$$\log M \le \frac{\alpha}{\gamma}\sqrt{\log L!}\,. \tag{5.80}$$

The following lemma is a useful corollary of Stirling's formula.

Lemma 5.15. *Given $\epsilon > 0$ and $c > 0$, there exists a k_0 such that for all $k \ge k_0$:*

1. $(\log k!)^{k+2} \le (k!)^{1+\epsilon}$.
2. $c^{k+2} \le (k!)^{\epsilon/2}$.

Proof. The proof can be found in [KS99] p.93. □

Next, note that $\beta = s(M-1) - s(0)) < \log M$ by construction. We will now give estimations for each summand in (5.89).

Lemma 5.16. *The following estimation holds:*

$$\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} \le \frac{1}{N^{\gamma^2-\epsilon\gamma^2}}. \tag{5.81}$$

Proof. By lemma 5.15 we see that

$$\begin{aligned}\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} &\le \frac{2(2\beta)^{L+2}}{(L+1)!} \le \frac{2}{L+1}\frac{(2\log M)^{L+2}}{L!} \\ &\le \frac{2}{L+1}\frac{\sqrt{L!}^{1+\epsilon}}{L!}\left(\frac{2\alpha}{\gamma}\right)^{L+2} \\ &\le (L!)^{\epsilon-1} \le \frac{1}{N^{\gamma^2-\epsilon\gamma^2}},\end{aligned} \tag{5.82}$$

which is the desired result. □

Lemma 5.17.

$$\frac{1}{N}\sum_{L\geq k\geq 0}\frac{(2\beta)^{k+1}}{k!}\leq\frac{1}{\sqrt{N}}\sum_{L\geq k\geq 0}\frac{(4\log M)^{k+1}(k+1)}{\text{floor}\left(\frac{k}{2}+1\right)!}. \tag{5.83}$$

Proof. This follows by direct calculation. □

Lemma 5.18.

$$\frac{1}{\sqrt{N}}\sum_{L\geq k\geq 0}\frac{(4\log M)^{k+1}(k+2)}{\text{floor}\left(\frac{k}{2}+1\right)!}\leq\frac{1}{\sqrt{N}}N^{\gamma^2+2\gamma^2\epsilon} \tag{5.84}$$

Proof.

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{L\geq k\geq 0}\frac{(4\log M)^{k+1}(k+2)}{\text{floor}\left(\frac{k}{2}+1\right)!} &\leq \frac{1}{\sqrt{N}}\sum_{L\geq k\geq 0}(8\log M)^{k+1}\\ &\leq \frac{1}{\sqrt{N}}L(8\log M)^{L+1}\leq\frac{1}{\sqrt{N}}(16\log M)^{L+1}\\ &\leq \frac{1}{\sqrt{N}}\left(\frac{16\alpha}{\beta}\right)^{L+1}\sqrt{\log L!}^{L+1}\leq\frac{1}{\sqrt{N}}(L!)^{\frac{1}{2}+\epsilon}\\ &\leq \frac{\left(N^{2\gamma^2}\right)^{\left(\frac{1}{2}+\epsilon\right)}}{\sqrt{N}},\end{aligned} \tag{5.85}$$

where in the last line we used that

$$L!\leq LN^{\gamma^2}\leq N^{2\gamma^2}. \tag{5.86}$$

□

Lemma 5.19.

$$\frac{1}{\sqrt{N}}\sum_{L\geq k\geq 0}\binom{k}{a}\frac{k+2}{\text{floor}(\frac{k}{2}+1)!}\leq 3N^{2\gamma^2-\frac{1}{2}}\ \textit{for sufficiently large } L. \tag{5.87}$$

Proof.

$$\sum_{L\geq k\geq 0}\binom{k}{a}\frac{k+2}{\text{floor}(\frac{k}{2}+1)!}\leq 3L\cdot 2^L\leq 3L!\text{ for sufficiently large }L. \tag{5.88}$$

□

Now, we want to combine these estimations. Starting with equation (5.89)

$$\int_{T(N)} d_{\mathrm{KS}}(\mu,\mu_A)dA < \frac{5}{M} + 2\cdot\left(\frac{(2\beta)^{L+1}}{(L+1)!} + \frac{(2\beta)^{L+2}}{(L+2)!} + \frac{1}{N}\sum_{L\geq k\geq 0}\frac{(2\beta)^{k+1}}{k!}\right)$$
$$+ \sqrt{\frac{2}{N}}\max_i\left\{\sum_{L\geq k\geq a}\binom{k}{a}\frac{k+2}{\mathrm{floor}(\frac{k}{2}+1)!}\max\{1,(2s(i)-2s(1))^{k+1}\}\right\} \quad (5.89)$$

the following intermediary result is a consequence of the above lemmas:

$$\int_{T(N)} d_{KS}(\mu,\mu_A)dA \leq \frac{5}{M} + \frac{2}{N^{\gamma^2-\epsilon\gamma^2}} + \frac{2\sqrt{2}}{N^{\frac{1}{2}-\gamma^2-2\gamma^2\epsilon}} + \frac{3\sqrt{2}}{N^{\frac{1}{2}-2\gamma^2}}. \quad (5.90)$$

The summand $\frac{5}{M}$ decreases like $\exp(-\alpha\sqrt{\log N})$ as N goes to infinity. The other summands decrease much faster. Therefore we may neglect them, i.e. for N sufficiently large, the left-hand side is smaller than $\frac{6}{M}$. If we substitute α from the beginning by $\alpha/2$ the constant 6 can also be neglected. Thus, we have proved the main theorem of this chapter.

Theorem 5.20. *Let α be a positive constant. Then the following estimation*

$$\int_{T(N)} d_{KS}(\mu,\mu_A)dA < \frac{1}{e^{\alpha\sqrt{\log N}}} \quad (5.91)$$

holds for N sufficiently large.

6 Appendix

In this Appendix the fundamental results from representation theory and momentum geometry which are used in the main body of the text are stated in detail. With few exceptions, for the proofs only references to the literature are given. We close this Appendix with elementary observations about nearest neighbor statistics.

6.1 Representation Theory

Throughout this text we are concerned with the representation theory of compact Lie groups. For the standard facts we refer the reader to [BtD85] and [Kna02].

We will always use the following conventions: K denotes a semi-simple, compact Lie group with Lie algebra $\mathfrak{k}$. Further, let G denote the complexification of K and $\mathfrak{g}$ be the Lie algebra of G. Furthermore for any unitary vector space V the symbol $\mathrm{U}(V)$ is used for the set of unitary automorphisms.

6.1.1 Representations of Compact Lie Groups

Fix a maximal torus T in K with Lie algebra $\mathfrak{t}$, i.e. T is a maximal, connected, commutative subgroup of K, and every irreducible representation can be decomposed into one dimensional representations of T. On each of these T acts by scalar multiplication, i.e., we are given a group homomorphism $f : T \to S^1 \subset \mathbb{C}^*$. We make the following definition.

Definition 6.1. *Let $\rho : K \to U(V)$ be an irreducible, unitary representation of K on some finite dimensional vector space V. Then a **weight** of ρ is an element $\lambda \in \mathfrak{t}^*$ such that there exists a non-trivial subspace V_λ of V with*

$$d_e\rho(t).x = 2\pi i\lambda(t)x \ \forall \ x \in V_\lambda, \ \ t \in \mathfrak{t}. \tag{6.1}$$

Note that these weights are sometimes called real infinitesimal weights.

Proposition 6.2. *The set of weights (with multiplicity) of an irreducible representation determines the representation uniquely.*

Proof. This is a very weak form of the Theorem 5.110 in [Kna02]. □

Moreover, one can order the set of all weights such that every irreducible representation has a unique highest weight. We will define such an ordering here, but we have to elaborate on the weights first.

Recall that the adjoint representation $Ad : K \to GL(\mathfrak{k})$ is given by $k \mapsto d_e \operatorname{int}(k)$, where $\operatorname{int} : K \to \operatorname{Aut}(K), k \mapsto (g \mapsto kgk^{-1})$. The complexified weights of the adjoint representation are called **roots**.

Of all Ad-invariant scalar products on $\mathfrak{g}$ the most important one is the so called **Killing form** $\langle \cdot, \cdot \rangle_{\mathrm{Kil}}$, which is defined by

$$\langle \xi, \eta \rangle_{\mathrm{Kil}} = \operatorname{trace}(ad(\xi) \circ ad(\eta)), \tag{6.2}$$

where $\xi, \eta \in \mathfrak{g}$ and $ad : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}), \xi \mapsto [\xi, \cdot]$.

Let us denote the set of roots by Δ. The following lemma summarizes some properties of the roots.

Lemma 6.3. *The set Δ has the following properties:*

1. *$\{\alpha \in \Delta\}$ generates $\mathfrak{t}^*$.*
2. *$\alpha \in \Delta$ if and only if $-\alpha \in \Delta$.*
3. *There exists a set of simple roots, i.e. a smallest subset Δ' of Δ, such that every $\alpha \in \Delta$ is an integer combination of simple roots.*
4. *In the integer combination either all coefficients are non-negative or all are non-positive.*
5. *The simple roots form a basis for $\mathfrak{t}^*$.*
6. *The non-negative linear combinations over $\mathbb{R}$ of the simple roots give a closed convex cone in $\mathfrak{t}^*$.*

Proof. Cf. [Kna02] Chapter II.5. □

The cone in the lemma above is usually called the **Weyl chamber** with respect to the system of simple roots. Identifying $\mathfrak{t}^*$ with $\mathfrak{t}$ via an Ad-invariant scalar product we can think of this cone as a subset of $\mathfrak{t}$.

Since every root is an integer combination of the simple roots, where all coefficients are either non-negative or non-positive, we divide the set Δ into the set of **positive roots**

$$\Pi_+ = \{\alpha \in \Delta \,:\, \alpha \text{ is non-negative combination of simple roots}\} \tag{6.3}$$

and **negative roots**

$$\Pi_- = \{\alpha \in \Delta \,:\, \alpha \text{ is non-positive combination of simple roots}\}. \tag{6.4}$$

The simultaneous eigenspace of a root α is denoted by $\mathfrak{g}_\alpha$, i.e.

$$\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} : \alpha(\tau)\xi = [\tau, \xi]\}. \tag{6.5}$$

This yields a direct sum decomposition of the Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{u}_-, \tag{6.6}$$

where

$$\mathfrak{u}_- := \bigoplus_{\alpha \in \Pi_-} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u}_+ := \bigoplus_{\alpha \in \Pi_+} \mathfrak{g}_\alpha. \tag{6.7}$$

Definition 6.4. *The group generated by the reflections on the faces of the Weyl chamber is called the **Weyl group**.*

We denote the Weyl group by W and remark that it is a finite group.

Definition 6.5. *The **ordering of weights** is given by*

$$\lambda \leq \mu :\Leftrightarrow \mathrm{Conv}(W.\lambda) \subset \mathrm{Conv}(W.\mu), \tag{6.8}$$

where λ, μ are weights.

Lemma 6.6. *Every weight is equivalent to a weight in the Weyl chamber under the action of the Weyl group.*

Proof. Cf. [Kna02] Corollary 2.68. □

The main statement about weights is called the Theorem of the Highest Weight.

Theorem 6.7. *Every irreducible representation has a unique highest weight in the Weyl chamber. Moreover, two irreducible representations are equivalent if and only if the highest weights are equal.*

Proof. Cf. [Kna02] Theorem 5.110. □

Connected to the above definitions are special complex subgroups of G, which are introduced subsequently.

Definition 6.8. *A **Borel subgroup** of G is a maximal, connected, solvable, complex subgroup of G. A **parabolic subgroup** is a complex subgroup which contains a Borel subgroup.*

Given a fixed torus and a notion of positivity of roots, we have two natural Borel subgroups, which are called B_+ and B_-. These can be obtained as follows:

$$B_- := \exp(\mathfrak{u}_- \oplus \mathfrak{t}^{\mathbb{C}}) \quad \text{and} \quad B_+ := \exp(\mathfrak{u}_+ \oplus \mathfrak{t}^{\mathbb{C}}). \tag{6.9}$$

6.1.2 The Universal Enveloping Algebra

Let $\mathcal{T}(\mathfrak{g})$ denote the full tensor algebra of $\mathfrak{g}$, .i.e. $\mathcal{T}(\mathfrak{g}) = \oplus_{j\in\mathbb{N}}(\otimes^j\mathfrak{g})$. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is given by the quotient algebra

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}, \tag{6.10}$$

where $\mathcal{I}$ is the ideal generated by all $\langle \xi\otimes\eta - \eta\otimes\xi - [\xi,\eta]\rangle$ for $\xi,\eta \in \mathfrak{g}$.

One directly checks that $\mathcal{U}(\mathfrak{g})$ is an associative algebra.

Theorem 6.9. *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has the following properties:*

1. *$\mathfrak{g}$ is embedded in $\mathcal{U}(\mathfrak{g})$ by $X \mapsto X + \mathcal{I}$.*
2. *Every Lie algebra representation $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ has a continuation as a homomorphism of associative algebras $\rho_* : \mathcal{T}(\mathfrak{g}) \to \mathrm{End}(V)$. The kernel of ρ_* contains $\mathcal{I}$ so this yields an induced homomorphism of associative algebras $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$.*
3. *(Lemma of Burnside) Let $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible Lie algebra representation on a finite dimensional vector space. Then $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$ is surjective.*
4. *(Theorem of Poincare-Birkhoff-Witt) Let $\xi_1,\ldots,\xi_n$ be a basis of $\mathfrak{g}$. Then the map*
$$\psi : \mathbb{C}[X_1,..,X_n] \to \mathcal{U}(\mathfrak{g}),\ \sum_I a_I X^I \to \sum_I a_I \xi^I \tag{6.11}$$
is an isomorphism of vector spaces, where it is assumed that every monomial in $\mathbb{C}[X_1,..,X_n]$ is ordered lexicographically.

Proof. The proof of the Lemma of Burnside can be found in [Far01] Chapter 3.3. The rest is proved in [Kna02] Chap. III. □

Note that ψ is not an isomorphism of algebras since $\mathbb{C}[X_1,..,X_n]$ is commutative and $\mathcal{U}(\mathfrak{g})$ is not.

In the text a notion of hermitian operators on the tensor algebra and on the universal enveloping algebra is needed.

Definition 6.10. *The $\mathbb{R}$-linear map $\dagger : \mathcal{T}(\mathfrak{g}) \to \mathcal{T}(\mathfrak{g})$ defined by*

1. *$(z\alpha_1 \otimes\cdots\otimes \alpha_n)^\dagger = \bar{z}\alpha_n^\dagger \ldots \alpha_1^\dagger \quad \forall \alpha_1,\ldots,\alpha_n \in \mathfrak{g}, z\in\mathbb{C}$*
2. *$\xi^\dagger = -\xi \quad \forall\xi \in \mathfrak{k}$*

*and, extended by $\mathbb{R}$-linearity to $\mathcal{T}(\mathfrak{g})$, is called the **formal adjoint**. An operator $\alpha \in \mathcal{T}(\mathfrak{g})$ is called **abstractly self-adjoint** or **abstractly hermitian**, if $\alpha^\dagger = \alpha$.*

Note that the formal adjoint is not complex linear because of the conjugation involved in condition 1.

Remark 6.11. The map † is compatible with ρ in the following sense:

$$\rho_*(\xi^\dagger) = \rho_*(\xi)^\dagger. \tag{6.12}$$

Lemma 6.12. *The map † induces a $\mathbb{R}$-linear map $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, which we also call †.*

Proof. We have to show that the ideal $\mathcal{I}$ in $\mathcal{T}(\mathfrak{g})$ is fixed by †. For this let $\xi = \xi_1 + i\xi_2$ and $\eta = \eta_1 + i\eta_2$ with ξ_1, ξ_2, η_1 and $\eta_2 \in \mathfrak{k}$ be given. We calculate

$$\begin{aligned}
&((\xi_1 + i\xi_2)(\eta_1 + i\eta_2) - (\eta_1 + i\eta_2)(\xi_1 + i\xi_2) - [\xi_1 + i\xi_2, \eta_1 + i\eta_2])^\dagger \\
&= (\xi_1\eta_1 - \eta_1\xi_1 - [\xi_1, \eta_1])^\dagger + (i(\xi_2\eta_1 - \eta_1\xi_2 - [\xi_2, \eta_1]))^\dagger \\
&+ (i(\xi_1\eta_2 - \eta_2\xi_1 - [\xi_1, \eta_2]))^\dagger - (\xi_2\eta_2 - \eta_2\xi_2 - [\xi_2, \eta_2])^\dagger \\
&= (\eta_1^\dagger\xi_1^\dagger - \xi_1^\dagger\eta_1^\dagger - [\xi_1, \eta_1]^\dagger) - i(\eta_1^\dagger\xi_2^\dagger - \xi_2^\dagger\eta_1^\dagger - [\xi_2, \eta_1]^\dagger) - i(\eta_2^\dagger\xi_1^\dagger - \xi_1^\dagger\eta_2^\dagger - [\xi_1, \eta_2]^\dagger) \\
&- (\eta_2^\dagger\xi_2^\dagger - \xi_2^\dagger\eta_2^\dagger - [\xi_2, \eta_2]^\dagger) \\
&= (\eta_1\xi_1 - \xi_1\eta_1 - [\eta_1, \xi_1]) - i(\eta_1\xi_2 - \xi_2\eta_1 - [\eta_1, \xi_2]) - i(\eta_2\xi_1 - \xi_1\eta_2 - [\eta_2, \xi_1]) \\
&- (\eta_2\xi_2 - \xi_2\eta_2 - [\xi_2, \eta_2]).
\end{aligned} \tag{6.13}$$

This proves the lemma. □

6.1.3 The Laplace Operator

A Casimir operator is by definition an element of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$. If we consider an irreducible representation $\rho_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V)$, then due to Schur's Lemma every Casimir operator has to act by scalar multiplication.

The most important example of a Casimir operator is the Laplace operator Ω. Sometimes it is even called *the* Casimir element, e.g. in [Kna02]. We do not give an explicit formula for the Laplace operator here, but just state that it is an operator of degree two in the basis elements of $\mathfrak{g}$.

Let δ denote half the sum of positive roots.

Lemma 6.13. *The Laplace operator Ω operates by the scalar $\langle\lambda, \lambda + 2\delta\rangle_{\mathrm{Kil}}$ in an irreducible representation of $\mathfrak{g}$ of highest weight λ.*

6.1.4 The Theorem of Borel-Weil and the Embedding Of Line Bundles

Let $H \subset G$ be a closed complex subgroup and $\rho : H \to \mathrm{End}(V)$ be a holomorphic representation. The fiber product $F := G \times_H V$ is the quotient space of $G \times V$ by the equivalence relation

$$(g_1, v_1) \sim (g_2, v_2), \text{ if } g_1 = g_2 h^{-1}, v_1 = \rho(h) v_2 \text{ for some } h \in H. \tag{6.14}$$

The projection $p : F \to G/H, [(g, v)] \mapsto gH$ is holomorphic and it can be shown by a direct calculation that $p : F \to G/H$ is a vector bundle with typical fiber V. We define a G-action on F by

$$x.[(g, v)] := [(xg, v)]. \tag{6.15}$$

This action induces a representation of G on the vector space of holomorphic sections[1] $\Gamma(G/H, F)$. For our purpose it is useful to give this representation in the context of H-invariant functions. Therefore, we identify the sections of $F \to G/H$ with the H-invariant functions $f : G \to V$, i.e.,

$$f(gh^{-1}) = \rho(h)f(g) \quad \forall\ h \in H, g \in G. \tag{6.16}$$

The G-action on these functions is given by

$$x.f(g) := f(x^{-1}g) \quad \forall\ g, x \in G. \tag{6.17}$$

In our context H will be a Borel subgroup of G.

After this preparation, we can formulate a weak version of the Borel-Weil Theorem. For a more complete version we refer to [Huc91] and [Akh91] for a treatment from the complex analytic point of view. An algebraic approach can be found in [WG99].

Theorem 6.14. (Borel-Weil) *Let $\rho : G \to \mathrm{End}(V)$ be an irreducible representation with highest weight λ and B_- the Borel subgroup of the negative roots. Then B_- acts by multiplication on V_λ with character $\chi : B_- \to \mathbb{C}^*$, where $d_e\chi|_{\mathfrak{t}} = 2\pi i\lambda$ and the representation on $\Gamma(G/B_-, G \times_{B_-} \mathbb{C})$ is isomorphic to ρ.*

Proof. Cf. [Akh91] Chap. 4.3. □

We now follow the classical construction of embedding a G-line bundle into the dual of the vector space of its sections. For this, set $L = G \times_{B_-} \mathbb{C}$ and fix a basis $s_0, \ldots, s_N$ of $\Gamma(G/B_-, L)$ and the corresponding dual basis $s_0^*, \ldots, s_N^*$.

Let $\mathcal{Z}$ be the zero section of L. In the view of $L = G \times \mathbb{C}/\sim$, the zero section is exactly given by the elements of the form $(g, 0)$ for $g \in G$. We claim that we obtain an equivariant, holomorphic map of $L\backslash\mathcal{Z}$ into $\Gamma(G/B_-, L)^*$ by the following construction. We think of the s_i's as B_--equivariant functions $G \to \mathbb{C}$ and define

$$\varphi : L\backslash\mathcal{Z} \to \Gamma(G/B_-, L)^*, [(g, z)] \mapsto \frac{1}{z}\sum_{j=0}^{N} s_j(g)s_j^*. \tag{6.18}$$

This is reasonable because z is not 0, otherwise we would have $[(g, 0)] \in \mathcal{Z}$. Moreover, φ is well-defined. Indeed, if we take another representative $(gb^{-1}, \chi(b)z)$, we get

$$\sum_{j=0}^{N} \frac{s_j(gb^{-1})}{\chi(b)z} s_j^* = \sum_{j=0}^{N} \frac{\chi(b)}{\chi(b)} \frac{s_j(g)}{z} s_j^* \tag{6.19}$$

because the s_j are equivariant under B_-, i.e.

$$s_j(gb) = \chi(b)^{-1}s_j(g). \tag{6.20}$$

[1]Since we only deal with holomorphic sections, we write $\Gamma(G/H, F)$ instead of $\Gamma_{hol}(G/H, F)$ for the rest of this chapter.

Next, we have to show the equivariance of φ with respect to the left action of G on L and the dual representation on $\Gamma(G/B_-, L)^*$. For this let $x^{-1}.s_j = \sum_{i=0}^N a_i s_i$ for a fixed $x \in G$ and we calculate

$$\begin{aligned}
x.\varphi([g,z])(s_j) &= \frac{1}{z}\left(x.\sum_{i=0}^N s_i(g)s_i^*\right)(s_j) \\
&= \frac{1}{z}\sum_{i=0}^N s_i(g)s_i^*(x^{-1}.s_j) \\
&= \frac{1}{z}\sum_{i=0}^N a_i s_i(g) \\
&= \frac{1}{z}(x^{-1}.s_j)(g) \\
&= \frac{1}{z}s_j(xg) \\
&= \frac{1}{z}\sum_{i=0}^N s_i(xg)s_i^*(s_j) \\
&= \varphi([xg,z])(s_j).
\end{aligned}$$

Now, we claim that a vector of maximal weight is in the image of φ. For this, consider the mapping

$$j : G/B_- \to \mathbb{P}(\Gamma(G/B_-, L)^*), x \mapsto [s_0(x) : \ldots : s_N(x)] \tag{6.21}$$

where the coordinates on the right hand side are the s_j^*. It is an equivariant, holomorphic map of G/B_- into the projective space of $\Gamma(G/B_-, L)^*$. Thus, the image is a closed orbit in $\mathbb{P}(\Gamma(G/B_-, L)^*)$. But the orbit of the projection of a maximal weight vector is the only such orbit (cf. [Huc91]). By comparison of (6.18) and (6.21) we obtain that a vector $v_{\max}$ of maximal weight is in the image of φ. Actually, every $c \cdot v_{\max}$, $c \neq 0$, is in the image then. By equivariance, we conclude that the whole U_--orbit through every vector of maximal weight is contained in the image of φ.

We state the following lemma.

Lemma 6.15. *Let $\varphi : L \backslash \mathcal{Z} \to \Gamma(G/B_-, L)^*$ be the equivariant embedding described above. Then any K-invariant unitary structure on $\Gamma(G/B_-, L)^*$ induces a K-invariant hermitian bundle metric which is unique up to multiplication by a constant.*

Proof. First, we recall that the K-action on G/B_- is transitive (cf. [Huc91]), so every K-invariant bundle metric is the same up to a constant factor and we have completed the proof once we find the induced bundle metric is indeed K-invariant.

For $[g, z_1], [g, z_2] \in L$ we define

$$h_g(z_1, z_2) = \begin{cases} \frac{1}{\langle \varphi([g,z_1]), \varphi([g,z_2]) \rangle} & \text{if } z_1, z_2 \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.22}$$

By the relation

$$\frac{1}{\langle \varphi([g,z_1]), \varphi([g,z_2])\rangle} = \overline{z_1} z_2 \frac{1}{\langle \sum_{i=0}^n f_i(g) f_i^*, \sum_{i=0}^n f_i(g) f_i^* \rangle} = \overline{z_1} z_2 \frac{1}{\|\varphi([g,1])\|^2} \tag{6.23}$$

we obtain a hermitian inner product at every point, since φ is well-defined and has only values different from zero. We claim that h_g is a smooth bundle metric. We see that h_g is continuous and smooth outside the zero section. Recall the standard fact that such a bundle metric is then smooth everywhere (cf. [Lan87] p.96). This metric is K-invariant because $\langle \cdot, \cdot \rangle$ is K-invariant and φ is equivariant. □

Lemma 6.16. *Let $L_1 \to G/B_-$ and $L_2 \to G/B_-$ be homogeneous complex line bundles that realize the representations corresponding to the highest weights λ_1 and λ_2.*

The representation of highest weight $\lambda_1 + \lambda_2$ is then realized by $\Gamma(G/B_-, L_1 \otimes L_2)$.

Proof. This is a corollary to the Theorem of the Highest Weight as written in [Huc91] Chap. 7.1. □

6.2 Symplectic geometry and momentum maps

In this section the basic definitions of symplectic manifolds and momentums maps are given.

By definition a symplectic manifold (M, ω) is a real manifold M with a non-degenerate two-form ω.

An action of a Lie group H on M is said to be symplectic if

$$h^* \omega = \omega \ \forall \ h \in H. \tag{6.24}$$

Before we define the notion of a momentum map, let us fix the notation.

The induced vector field of the flow $\exp(-\xi t)$ on M is denoted by X_ξ and the Lie derivative along X_ξ by $\mathcal{L}_{X_\xi}$. For a smooth map $\mu : M \to \mathrm{Lie}(H)^*$ we obtain an induced map $\mu^\xi : \mathrm{Lie}(H) \to C^\infty(M)$ by

$$\mu^\xi(x) := \mu(x)(\xi) \ \forall \ \xi \in \mathrm{Lie}(H). \tag{6.25}$$

Definition 6.17. *Let (M, ω) be a symplectic manifold on which H acts by symplectic transformations.*

*A **momentum map** is an equivariant, smooth map $\mu : M \to \mathrm{Lie}(H)^*$ such that*

$$d(\mu^\xi) = \omega(X_\xi, \cdot), \tag{6.26}$$

where the action on $\mathrm{Lie}(H)^*$ is the coadjoint action.

We will use the momentum map only in the context of representations of a compact Lie group. Let $\rho : K \to \mathrm{U}(V)$ be a unitary representation of the compact Lie group K on a finite-dimension vector space V. This representation induces an action of K on $\mathbb{P}(V)$ which is symplectic with respect to the Fubini-Study metric on $\mathbb{C}$. Recall that the Fubini-Study metric is given by the imaginary part of the form $\frac{i}{2}\partial\bar{\partial}\log||\cdot||^2$ pushed down from $V\backslash\{0\}$ to $\mathbb{P}(V)$.

Theorem 6.18. *Let $\rho : K \to \mathrm{U}(V)$ be an irreducible representation of highest weight λ.*

The map $\mu : \mathbb{P}(V) \to \mathfrak{k}^*$ given by

$$\mu^\xi([v]) = -2i\frac{\langle v, \rho_*(\xi).v\rangle}{\langle v, v\rangle}\forall\xi \in \mathfrak{k}, v \in \mathbb{P}(V) \tag{6.27}$$

is the unique momentum map and

$$\mu([v_{max}]) = \lambda \tag{6.28}$$

for any vector v_{max} of highest weight.

Proof. Cf. [Huc91] Chap. IV.7. □

6.3 Generalities on Level Spacings

In this section we summarize the foundational facts on level spacings.

6.3.1 The Nearest Neighbor Distribution

The material in this subsection applies to arbitrary N-tuples of real numbers, $N > 1$. Later on it will be used only for eigenvalues of hermitian matrices.

Definition 6.19. *Let $X = (x_1, \dots, x_N) \in \mathbb{R}^N$ be an N-tuple of real numbers, ordered by increasing value*

$$x_1 \leq x_2 \leq \cdots \leq x_N. \tag{6.29}$$

*The **nearest neighbor distribution** of X is the Borel measure on $\mathbb{R}$ given by*

$$\mu(X)(A) = \int_A \frac{1}{N}\sum_{i=1}^{N-1}\delta\left(y - \frac{N}{x_N - x_1}\cdot(x_{j+1} - x_j)\right)dy, \tag{6.30}$$

if $x_1 \neq x_N$, and

$$\mu(X)(A) = \frac{N-1}{N}\int_A \delta(y)dy, \tag{6.31}$$

if $x_1 = \ldots = x_N$, where A is a Borel set in $\mathbb{R}$ and $\delta(y-p)$ denotes the Dirac measure with mass one at the point p.

Thus, if $x_1 \neq x_N$, $\mu(X)$ is a measure of total mass[2] $1 - \frac{1}{N}$ with expectation value

$$\begin{aligned} E(\mu(X)) &= \int_{\mathbb{R}} y \, d\mu(X)(y) \\ &= \frac{1}{N} \sum_{j=1}^{N-1} \frac{N}{x_N - x_1} \cdot (x_{j+1} - x_j) \\ &= \frac{1}{x_N - x_1} \cdot (x_N - x_1) = 1. \end{aligned} \tag{6.32}$$

Remark 6.20. Note, that $\mu(X)$ does not change under scalar multiplication, i.e.,

$$\mu(aX) = \mu(X) \; \forall a \in \mathbb{R}, a \neq 0 \tag{6.33}$$

nor under diagonal addition

$$\mu((x_1 + a, \ldots, x_n + a)) = \mu((x_1, \ldots, x_n)). \tag{6.34}$$

If we know *a priori* that our N-tuple X is contained in $[a, b]^N \bmod 1$, it is customary to measure the wrapped around distance between x_N and x_1:

$$b - a - x_N + x_1 \tag{6.35}$$

and to replace $x_N - x_1$ in the denominator by $b - a$:

$$\begin{aligned} \mu_w(X)(A) = \frac{1}{N} \int_A \delta &\left(y - \frac{N}{b-a}(b - a - x_N + x_1) \right) \\ &+ \sum_{i=1}^{N-1} \delta \left(y - \frac{N}{b-a} \cdot (x_{j+1} - x_j) \right) dy. \end{aligned} \tag{6.36}$$

Note that the total mass of this measure is 1 and the expectation value is also 1.

Our main example for the above measure on the torus is given by the logarithms of eigenvalues of a unitary matrix U. Here $a = 0$ and $b = 2\pi$ and we obtain the following definition

$$\mu_c(X)(A) = \frac{1}{N} \int_A \delta \left(y - \frac{N}{2\pi}(2\pi - x_N + x_1) \right) + \sum_{i=1}^{N-1} \delta \left(y - \frac{N}{2\pi} \cdot (x_{j+1} - x_j) \right) dy \tag{6.37}$$

where $X = (x_1, \ldots, x_N)$ is the set of ordered logarithms of the eigenvalues with multiplicities, i.e. $\mathrm{spec}(U) = \{e^{ix_1}, \ldots, e^{ix_N}\}$. Here the differences $x_{j+1} - x_j$ are the angles between the eigenvalues and $2\pi - x_N + x_1$ is the angle between the first and the last eigenvalue.

In physical models such a wrapping occurs naturally because the only physical data is encoded in the difference of the arguments of the e^{ix_j}. Thus, the choice of the branch of the logarithm is artificial, i.e. the position of zero cannot be measured.

[2]Note that for this reason it is common to use the factor $\frac{1}{N-1}$ in front of the sum and $N - 1$ instead of N inside the δ measures, but we will see that this is of no importance for questions of convergence.

Note. The measures $\mu_c(X_N)$ and $\mu_w(X_N)$ are no longer invariant under scalar multiplication.

6.3.2 The Kolmogorov-Smirnov Distance

Since we want to discuss convergence of measures on the real line, we need a precise notion of the type of convergence we are dealing with. For us only two types of convergence are important: the weak convergence of distribution functions and the sup-norm convergence of distribution functions.

Recall that a sequence of measure μ_n is said to converge weakly to a measure μ if for every bounded, continuous function f the following holds:

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu. \tag{6.38}$$

Definition 6.21. *Let μ, ν be Borel measures on $\mathbb{R}$ of finite mass. The* ***Kolmogorov-Smirnov distance*** *d_{KS} of μ and ν is*

$$d_{KS}(\mu,\nu) = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} d\mu - \int_{-\infty}^{t} d\nu \right|, \tag{6.39}$$

which is the sup*-norm for the difference of the cumulative distribution functions.*

We say a sequence of Borel measures μ_N converges to μ if $d_{KS}(\mu_N, \mu)$ converges to zero.

Remark 6.22. Convergence with respect to the Kolmogorov-Smirnov distance implies weak convergence.

Proof. The convergence in the Kolmogorov-Smirnov distance implies the pointwise convergence of the cumulative distribution functions. But this implies weak convergence by a standard result of measure theory (cf. [Els04] chap. 8 Theorem 4.12). □

We now show that the scaling, with $N-1$ instead of N which is common in the literature (cf. [Meh91]), gives the same results.

Lemma 6.23. *Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of N-tuples such that $X_N \in \mathbb{R}^N$ and let ν be a Borel measure on $\mathbb{R}^+$ with continuous density function $p(x)$ with respect to the Lesbesgue measure. Then the following are equivalent:*

1. $\lim_{N\to\infty} \mu(X_N) = \nu$.
2. $\lim_{N\to\infty} \mu_1(X_N) = \nu$, *where*

$$\mu_1(X)(A) = \frac{1}{N-1} \int_A \sum_{i=1}^{N-1} \delta\left(y - \frac{N}{x_N - x_1} \cdot (x_{j+1} - x_j)\right) dy. \tag{6.40}$$

3. $\lim_{N\to\infty} \mu_2(X_N) = \nu$, *where*

$$\mu_2(X)(A) = \frac{1}{N-1} \int_A \sum_{i=1}^{N-1} \delta\left(y - \frac{N-1}{x_N - x_1} \cdot (x_{j+1} - x_j)\right) dy. \tag{6.41}$$

Proof. The equivalence of 1. and 2. is clear, since $d_{KS}(\mu_1(X)(A), \mu_2(X)(A)) = \frac{1}{N}$. For the proof of the equivalence of 2. and 3. we note that

$$\mu_1(X)([0,y]) = \frac{1}{N-1} \operatorname{card}\left\{j : \frac{x_{j+1} - x_j}{x_N - x_1} \cdot N \leq y\right\} \tag{6.42}$$

and

$$\mu_2(X)([0,y]) = \frac{1}{N-1} \operatorname{card}\left\{j : \frac{x_{j+1} - x_j}{x_N - x_1} \cdot (N-1) \leq y\right\}. \tag{6.43}$$

Therefore we see that

$$\mu_2(X)([0,y]) = \mu_1(X)\left(\left[0, \frac{N-1}{N} \cdot y\right]\right). \tag{6.44}$$

Now suppose 2. is true. Then

$$\begin{aligned} |\nu([0,y]) - \mu_2(X)([0,y])| &\leq \left|\mu_1(X)\left(\left[0, \frac{N-1}{N} \cdot y\right]\right) - \nu([0,y])\right| \\ &\leq \left|\mu_1(X)\left(\left[0, \frac{N-1}{N} \cdot y\right]\right) - \nu\left(\left[0, \frac{N-1}{N} \cdot y\right]\right)\right| \\ &+ \left|\nu\left(\left[0, \frac{N-1}{N} \cdot y\right]\right) - \nu([0,y])\right|. \end{aligned} \tag{6.45}$$

Since $p(x)$ is continuous, the cumulative density function of ν is uniformly continuous and the lemma follows from the estimation by a direct $\frac{\epsilon}{2}$ proof. Therefore, 2. implies 3. and, analogously, we see that the converse is true. □

In the literature one often comes across histograms with densities plotted into them for the nearest neighbor statistics. Compare Figure 1.1 in the introduction, where we see a histogram containing two curves.

Let us briefly discuss how the histogram in Figure 1.1 was built. We start with an N-tuple $X = (x_1, \dots, x_N)$ of non-decreasing real numbers as input and consider the $N-1$ rescaled nearest neighbor distances

$$\phi_j = \frac{N-1}{x_N - x_1} \cdot (x_{j+1} - x_j) \ \forall j = 1, \dots, N-1. \tag{6.46}$$

Now, we divide the real line into bins of some fixed width w and count the number of ϕ_j in each bin. At last we scale the height of the boxes with a common factor such that the total area of the histogram is one.

One usually has some measures with a continuous density function with which to compare the histogram. In Figure 1.1 two such densities are plotted.

This can be thought of as a visualization of the d_{KS}-convergence in the following sense: As the width w becomes smaller and the N-tuples become larger, the histogram should approach the density of the limit measure. This can be made precise in the following way. Fix $p \geq 0$ and think of the histogram restricted to $[0, p]$ as a Riemannian sum, which should converge to the integral of the density over $[0, p]$.

Unfortunately, this depends on the ratio of N and w. Being a bit sloppy we can say that at the locus ϕ_j we obtain a contribution of mass $1/(N-1)$ if the width is small enough. This is exactly the point of the definition of $\mu(X)$. Thus, a visualization of the convergence is obtained, although it is not without problems because of the new dependence on the parameter w.

6.3.3 Approximating N-tuples

The following lemma shows how to construct approximating N-tuples for any absolutely continuous measure.

Lemma 6.24. *Let μ be a measure on $\mathbb{R}_{\geq 0}$ with continuous density f such that*

$$\int_0^\infty x f(x) dx \in [0, 1]. \tag{6.47}$$

For every $N \geq 3$ there exists an N-tuple $X = (x_1, \ldots, x_N)$, $x_1 \leq \ldots \leq x_n$ such that

$$d_{KS}(\mu(X), \mu) \leq \frac{2}{N-1}. \tag{6.48}$$

Moreover, x_1 can be chosen to be 0.

Proof. First, define y_j by the requirement

$$\frac{j}{N} = \int_0^{y_j} d\mu \quad \forall j = 1, \ldots, N-1. \tag{6.49}$$

If we could choose X in such a way that $\mu(X)$ has mass $\frac{1}{N}$ exactly at the y_j, i.e.,

$$y_j \stackrel{!}{=} \frac{N}{x_N - x_1}(x_{j+1} - x_j) \quad \forall j = 1, \ldots, N-1, \tag{6.50}$$

then

$$\left|\int_0^y d\mu - \int_0^y d\mu(X)\right| \leq \frac{1}{N-1} \tag{6.51}$$

since the cumulative distribution functions agree at the y_j by construction and differ only by at most $\frac{1}{N-1}$ as indicated in the following picture for a certain measure.

Unfortunately, the system (6.50) might have no solution since

$$\sum_{j=1}^{N-1} y_j \neq N = \sum_{j=1}^{N-1} \frac{N}{x_N - x_1}(x_{j+1} - x_j). \tag{6.52}$$

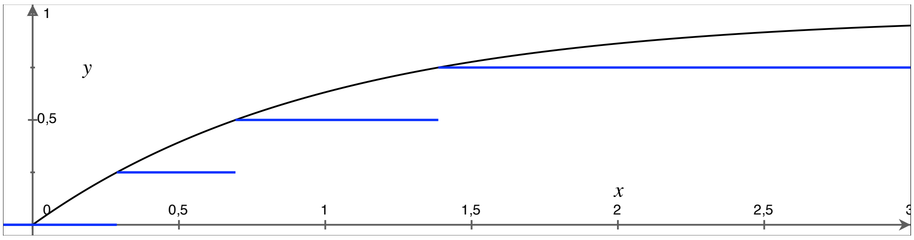

Figure 6.1: Approximation of μ_{Poisson}.

Thus, we redefine y_{N-1} in the following way

$$y_{N-1} = N - \sum_{j=1}^{N-2} y_j. \tag{6.53}$$

We claim that y_{N-1} is non-negative, i.e.,

$$N \geq \sum_{j=1}^{N-2} y_j. \tag{6.54}$$

This follows at once from the inequality

$$\begin{aligned}\int_0^\infty xf(x)dx &= \int_0^{y_1} xf(x)dx + \ldots + \int_{y_{N-2}}^\infty xf(x)dx \\ &\geq 0 \cdot \int_0^{y_1} f(x)dx + y_1 \cdot \int_{y_1}^{y_2} f(x)dx \ldots + y_{N-2} \int_{y_{N-2}}^\infty f(x)dx \\ &= y_1 \frac{1}{N} + \ldots y_{N-3} \frac{1}{N} + y_{N-2} \frac{2}{N} \geq \frac{1}{N} \sum_{j=1}^{N-2} y_j,\end{aligned} \tag{6.55}$$

because $\int_0^\infty xf(x)dx \in [0,1]$ by assumption.

Writing (6.50) as a linear system

$$(x_N - x_1)y_j - N(x_{j+1} - x_j) = 0 \tag{6.56}$$

and calculating the space of solutions, we see that the solutions depend on real parameters a and b:

$$x_j = a + \frac{b}{N} \sum_{1}^{j-1} y_j \quad \forall j = 1, \ldots, N. \tag{6.57}$$

Note that no solution with $b = 0$ solves the original problem (6.50).

For any solution X with $b \neq 0$ the estimate

$$d_{KS}(\mu(X), \mu) \leq \frac{2}{N-1} \tag{6.58}$$

holds since y_{N-1} is not in the optimal position any more. Hence, we have to adjust by the factor $\frac{2}{N-1}$. □

Corollary 6.25. *Let μ be an absolutely continuous measure on $\mathbb{R}_{\geq 0}$ with $\int_0^\infty x d\mu \in [0,1]$ and $p > 0$ be a fixed integer. Then for any p-tuple $(z_1, \dots, z_p)$ and any $N \geq p+2$ there is an N-tuple $X = (x_1, \dots, x_N)$ such that every z_j occurs as one of the x_k and*

$$d_{KS}(\mu(X), \mu) \leq \frac{2+p}{N-1}. \tag{6.59}$$

Proof. By Lemma 6.24 we find an N-tuple X such that

$$d_{KS}(\mu(X), \mu) \leq \frac{2}{N-1}. \tag{6.60}$$

Due to the invariance of $\mu(X)$ under scalar multiplication and diagonal addition, we may assume that

$$x_2 \leq z_j \leq x_N - 1 \;\forall\; j = 1, \dots, p. \tag{6.61}$$

Now, insert the z_j into the ordered sequence $x_1 \leq \ldots \leq x_N$ at the corresponding positions and remove the closest x_k for each z_j inserted, as long as x_k is neither x_1 nor x_N. In this case take the closest x_k in the middle. The resulting sequence is called $\tilde{X}$.

In the picture of Figure 6.1 we have changed p points of the jump loci of the approximating staircase function. Thus, we have to add an extra $\frac{p}{N-1}$ to the estimation. □

6.3.4 The Nearest Neighbor Statistics under exp

In this work the most important examples of sequences (X_N) of non-decreasing N-tuples are given by the spectra of sequences of Hamiltonian operators on finite-dimensional Hilbert spaces or, equally important, by the restrictions of Hamiltonian operators to finite dimensional subspaces of some infinite-dimensional Hilbert space such that the dimension of the finite-dimensional parts is approaching infinity.

In the setting of general finite-dimensional Hilbert spaces the Hamiltonians are just skew self-adjoint operators. The space of these operators is again a finite-dimensional vector space. If A is hermitian, the one-parameter group

$$\{\exp(iAt) : t \in \mathbb{R}\} \tag{6.62}$$

is a subgroup of the unitary group of this Hilbert space and the exponential mapping $A \mapsto \exp(iA)$ is surjective but not injective.

Now, we consider the spectrum of a unitary operator $\exp(iA)$ and take the nearest neighbor statistics μ_c of the eigenangles, i.e., the a_j in the eigenvalue e^{ia_j}, where $0 \leq a_j < 2\pi$.

Definition 6.26. *Let $U \in U(N)$ be a unitary matrix, whose eigenvalues are given as $e^{2i\pi\phi_1}, \dots, e^{2i\pi\phi_N}$, and $X(U) = (\phi_1, \dots, \phi_N)$. The nearest neighbor statistics of the unitary matrix U is $\mu_c(X(U))$.*

Frequently, we will write μ_A as an abbreviation for $\mu(X(A))$ and μ_U as abbreviation for $\mu_c(X(U))$.

The nearest neighbor statistics of $\exp(iA)$ will not agree with the nearest neighbor statistics of A for two reasons, the first being the wrapping discussed above and the second and more important is the problem of reordering.

The eigenvalues $x_1 \leq \cdots \leq x_N$ of A give the ϕ_j only modulo 2π, i.e.

$$a_j = \phi_j \bmod 2\pi \tag{6.63}$$

and it may happen that there are j_1 and j_2, such that $\phi_{j_1} < \phi_{j_2}$ but $a_{j_1} > a_{j_2}$.

If, however, all eigenvalues are sufficiently close to each other, meaning that they all lie in an interval of width 2π, one does not have to reorder, if choosing a different branch of the logarithm or just by adding a constant to all eigenvalues such that the smallest eigenvalue is 0.

Lemma 6.27. *Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of non-decreasing N-tuples such that $X_N \in [0, 2\pi[^N$ and let ν be a Borel measure on $\mathbb{R}^+$ with continuous density with respect to the Lebesgue measure. Assume that the difference between the largest and the smallest eigenvalue converges to 2π. Then the following are equivalent:*

1. $\lim_{N\to\infty} \mu(X_N) = \nu$.

2. $\lim_{N\to\infty} \mu_c(X_N) = \nu$.

Proof. Since the differences between the largest and the smallest eigenvalue converge to 2π, the $\mu(X_N)$ come arbitrarily close to the $\mu_c(X_N)$ as is evident by their definitions. □

Remark 6.28. The above lemma is false if we drop the assumption on the largest and smallest eigenvalues. Indeed, assume that every X_N is contained in the subinterval $[0, \frac{1}{N}[$ with smallest eigenvalue 0 and largest eigenvalue $\frac{1}{N}$, then $\mu_C(X_N)$ has the wrapping eigenangle given by

$$a_N = \frac{N}{2\pi}(2\pi - \frac{1}{N}) = N - \frac{1}{2\pi}, \tag{6.64}$$

all other eigenangles are less than or equal to $1/2\pi$. Therefore μ_c can only converge to a measure whose cumulative distribution function is 1 for all $t \in \mathbb{R}, t \geq 1/2\pi$.

To summarize, care has to be taken if considering the nearest neighbor statistics under exp. It is not enough to ensure that the eigenvalues of a hermitian operator are in an interval $[0, 2\pi]$ but one must also ensure that the difference between the smallest and the largest eigenvalue approaches 2π.

6.3.5 The Nearest Neighbor Statistics and the CUE Measure

As discussed above we are mainly interested in the nearest neighbor statistics associated to unitary matrices. We give some more details about these statistics here. In this section $\mu_c(X(A))$ will be abbreviated by μ_A.

The following lemma is necessary in certain of our applications.

Lemma 6.29. *If ν is an absolutely continuous probability measure on $\mathbb{R}$, then the map*

$$U(N) \to [0,1], \; A \mapsto d_{KS}(\nu, \mu_A) \tag{6.65}$$

is continuous.

Proof. Cf. [KS99] where the proof is given in lemma 1.0.11. and 1.0.12. □

Since $U(N)$ is a compact group, functions on $U(N)$ can be averaged. It is also possible to average the map $A \mapsto \mu_A$. This can be done in the following way. Let $\mu(U(N))$ denote the Borel measure given by

$$\mu(U(N))(X) := \int_{U(N)} (\mu_A(X)) \, d\,\mathrm{Haar}(A) \tag{6.66}$$

for any Borel-measurable set X.

We now state Lemma 1.2.1 of [KS99].

Lemma 6.30. *There exists an absolutely continuous probability measure ν on $\mathbb{R}$ with real analytic cumulative distribution function such that*

$$\mu(U(N)) \to \nu \textit{ weakly, as } n \to \infty. \tag{6.67}$$

We call this measure μ_{CUE}. In [KS99] the following theorem is given in a more general form as Lemma 1.2.6.

Theorem 6.31. *For every $\epsilon > 0$ there is a natural number N_0 such that*

$$\int_{U(N)} d_{KS}(\mu_{\mathrm{CUE}}, \mu_A) d\,\mathrm{Haar} \leq N^{\epsilon - 1/6} \tag{6.68}$$

for all $N \geq N_0$.

The complete proof of the lemma and the theorem is given in all detail in [KS99], where it takes the first half of the book, so it cannot be given here.

More details on μ_{CUE} can be found in [Meh91] and again in [KS99].

List of Symbols

$\dagger$ The formal adjoint on $\mathcal{T}(\mathfrak{g})$ or $\mathcal{U}(\mathfrak{g})$, page 62

Δ The set of roots of K, page 60

$\langle\cdot,\cdot\rangle_{\mathrm{Kil}}$ The Killing-Form on $\mathfrak{k}$ or $\mathfrak{g}$, page 60

$\mathbb{C}[X_1,..,X_n]$ The Ring of polynomials in n indeterminates with coefficients in $\mathbb{C}$, page 62

$\mathcal{O}(\mathbb{C}^n)$ The algebra of holomorphic functions on $\mathbb{C}^n$, page 29

$\mathcal{T}(\mathfrak{g})$ The full tensor algebra of $\mathfrak{g}$, page 62

$\mathcal{U}(\mathfrak{g})$ The universal enveloping algebra of $\mathfrak{g}$, page 62

$\mathfrak{g}$ The Lie algebra of G, page 59

$\mathfrak{k}$ The Lie algebra of K, page 59

$\mathfrak{t}$ The Lie algebra of T, page 59

$\mathfrak{u}_+$ The Lie algebra of positive roots, page 61

$\mathfrak{u}_-$ The Lie algebra of negative roots, page 61

$\mu(X)$ The nearest neighbor statistics of the tuple X, page 67

μ_A The nearest neighbor statistics of the eigenvalues of hermitian matrix A, page 74

$\mu_c(X)$ The nearest neighbor statistics of the angles X with wrapping at 2π, page 68

μ_U The nearest neighbor statistics of the eigenangles of unitary matrix U, page 74

μ_{CUE} The limit measure of the nearest neighborhood statistics of the unitary group, page 75

Ω The Laplace operator in $\mathcal{U}(\mathfrak{g})$, page 63

Π_+ The set of positive roots of K, page 60

Π_- The set of negative roots of K, page 60

ρ_*	The Lie algebra representation associated to a Lie group representation ρ, page 62
cl	The classical limit map, page 13
cl_n	n-approximation of the classical limit map, page 18
$\mathrm{U}(V)$	The set of unitary operators on the unitary vector space V, page 59
B_+	The Borel subgroup of positive roots of G, page 61
B_-	The Borel subgroup of negative roots of G, page 61
B_N	The set of unitary $N \times N$-matrices with nearest neighbor statistics not close to μ_{Poisson}, page 38
d_{KS}	The Kolmogorov-Smirnov distance, page 69
G	The complexification of K, page 59
K	A semi-simple, compact Lie group, page 59
T	A maximal torus of K, page 59
W	The Weyl group of K with respect to fixed Π_+, page 61

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