Quantum Frames and Uncertainty Principles arising from Symplectomorphisms

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Contents

$\mathbb{N}$omenclature

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Whole real numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	Positive real numbers
$\mathbb{T}$	Torus $\mathbb{R}/\mathbb{Z}$
$\mathbb{C}$	Complex numbers
$\widehat{\mathbb{R}}$	Dual group of $\mathbb{R}$, which is identifiable with $\mathbb{R}$ itself
i	Imaginary unit
$\mathfrak{R}$	Real part
$\mathfrak{I}$	Imaginary part
f, g, h, φ, ψ	Functions
$\widehat{f}, \widehat{g}, \widehat{h}, \widehat{\varphi}, \widehat{\psi}$	Spectrum resp. Fourier transform of functions
G, H	Locally compact (Lie) groups
$\mathfrak{g}, \mathfrak{h}$	Lie algebras
G/H	Group G modulo group H
π	Unitary irreducible representation of a locally compact group
ker	Kernel of a homomorphism
im	Image/Range of a mapping
dom	Domain of a mapping
supp	Support of a function
$\operatorname{ess\,sup}\lvert f\rvert$	Ess. supremum of f; $\inf\left\{ c \in \mathbb{R}_+ \;\middle|\; \mu\left(\lvert f\rvert^{-1}(c,\infty)\right) = 0 \right\}$
$\inf\lvert f\rvert$	Infimum of f; $\inf\limits_{x \in \operatorname{dom}(f)} \lvert f(x)\rvert$.
$\operatorname{tr}(T)$	Trace of T; $\sum_k T_k^k := \sum_k \langle Te_k, e_k \rangle$, with ONB $(e_k)_k$.

μ, ν, λ	Measures
$\mathrm{d}x, \mathrm{d}y, \mathrm{d}\lambda$	Lebesgue measure
$(X,\ \mathrm{d}\mu)$	Measure space with set X, Borel σ-algebra Σ_X and measure μ
$\langle \cdot, \cdot \rangle$	Inner Product, linear in the second variable
$\|\cdot\|$	Context-dependent Norm
$\mathcal{H}$	Hilbert space
$\mathcal{H}_\pi$	Hilbert representation space of π
$(\pi, \mathcal{H}_\pi)$	Unitary representation π, with representation space $\mathcal{H}_\pi$
$L^2(X,\ \mathrm{d}\mu)$	Hilbert space of square-integrable functions over (X, μ)
$L^2(\mathbb{R}^n)$	Hilbert space of n-dimensional square-integrable functions
$L^\infty(X,\ \mathrm{d}\mu)$	Essentially bounded functions over X; $\operatorname{ess\,sup} \lvert f \rvert < \infty$
$C(X)$	Continuous functions over topological space X
$C_0(X)$	Subspace of $C(X)$, vanishing at infinity resp. the boundary ∂X
$C^\infty(X)$	Space of smooth functions over X
$[A, B]$	Commutator of operators A and B; $[A, B] := AB - BA$
$\mathcal{F}$	Fourier transform; $\mathcal{F}f := \int_{\mathbb{R}^n} e^{-i2\pi\langle\,\cdot\,,\,x\,\rangle} f(x)\ \mathrm{d}x$
T_α	Translation operator; $T_\alpha : f \ \mapsto\ f(\bullet - \alpha)$
M_α	Modulation operator; $M_\alpha : f \ \mapsto\ \mathcal{F}^* T_\alpha \mathcal{F} f = e^{2\pi i\langle\,\alpha\,,\bullet\,\rangle} f$
σ	Diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$
J_σ	Jacobian matrix of σ; $(J_\sigma)^k_j := \frac{\partial \sigma_j}{\partial p_k}$
$\operatorname{div}(\sigma)$	Divergence of σ; $\operatorname{div}(\sigma) = \sum_k \frac{\partial \sigma_k}{\partial x_k} = \operatorname{tr}(J_\sigma)$
$\operatorname{div}(\sigma^{-1})$	Divergence of σ^{-1}; $\operatorname{div}(\sigma^{-1}) = \sum_k \frac{\partial \sigma_k^{-1}}{\partial x_k} = \operatorname{tr}(J_\sigma^{-1})$
$\widehat{\mathcal{W}}_\sigma$	"σ-warping" operator; $(\mathcal{W}_\sigma f)(x) := \lvert\det(J_{\sigma^{-1}}(x))\rvert^{1/2} f(\sigma^{-1}(x))$
$\mathcal{T}_\alpha$	Warped Translation operator; $\mathcal{T}_\alpha : f \ \mapsto\ \mathcal{F}^* \widehat{\mathcal{W}}_\sigma^* M_\alpha \mathcal{W}_\sigma \mathcal{F} f$
$\mathcal{D}^\sigma_\alpha$	Dilation operator; $\mathcal{D}^\sigma_\alpha : f \ \mapsto\ \mathcal{F}^* \widehat{\mathcal{W}}_\sigma^* T_\alpha \mathcal{W}_\sigma \mathcal{F} f$
$\mathcal{A}_\sigma$	$\left\{ f \in L^2(\mathbb{R}^n) \,\middle\vert\, n = \dim(\operatorname{dom}(\sigma)), \widehat{f} \in L^2(\operatorname{dom}(\sigma), \mathrm{d}\sigma) \right\}$
$\mathcal{S}_\sigma$	$\left\{ f \in L^2(\mathbb{R}^n) \,\middle\vert\, n = \dim(\operatorname{dom}(\sigma)), \widehat{f} \in L^2(\operatorname{dom}(\sigma),\ \mathrm{d}x) \right\}$
$\mathcal{H}_\sigma$	Reproducing kernel Hilbert space; $\mathcal{H}_\sigma := \pi^\sigma_\psi(\mathcal{S}_\sigma)$

$\mathcal{H}_\Sigma$	Reproducing kernel Hilbert space; $\mathcal{H}_\Sigma \coloneqq \pi^\Sigma_\psi(\mathcal{A}_\sigma)$
$\pi^\sigma\varphi$	$\pi^\sigma(x,y)\varphi \coloneqq T_x\mathcal{D}^\sigma_y\varphi$
$\pi^\sigma_\varphi f(x,y)$	σ-generalized wavelet transform; $\langle\, \pi^\sigma(x,y)\varphi\,,\, f\,\rangle$
δ_x	Dirac distribution; $\delta_x(\varphi) \coloneqq \int \delta_x\,\varphi\ \mathrm{d}t \coloneqq \varphi(x)$

x

Ich kann die ganze Prozedur nur als einen Akt der Verzweiflung charakterisieren, da ich von Natur aus friedlich bin und alle zweifelhaften Abenteuer ablehne.

— Max Planck [81]

1 Initiation

EPIPHANY HIT MAX PLANCK, in his endeavour to evade the consequences of the *Ultraviolettkatastrophe*, encouraging him to take a small, yet momentous step - the introduction of the elementary *Wirkungsquantum*, h. Ever since this monumental step and the establishment of the celebrated equation

$$E = h \cdot \nu, \tag{1.1}$$

linking energy and frequency of what is nowadays known as a photon, implicitly stated in [70], slowly but surely the understanding gained traction, that the emergence of a fundamental quanta of action necessarily demands a restructuring of physics. Although most of the contemporary physicists and chemists - among them Planck himself - did not immediately believe in a true quantization of the world, the revolution was inevitable.

It took an Einstein to realize in his work on the *photo effect* [22] in 1905, his *annus mirabilis*, that these quanta were not just a *Rechentrick*, as Planck called it, but implied a real quantization of the world - an idea which ultimately raised him among the Nobel laureates in 1921.

The implications of this realization, in particular the philosophical consequences, shook the world of physics to its very foundations and led to a paradigm shift in the years to come. Quanta began to pervade all of science and were eventually used to discretize phase spaces into cells, interpreted as that very subset of phase space a single particle occupies. As early as 1913, Otto Sackur [73] realized that for each particle and *each dimension* the size of these elementary phase space cells, had to be of the order of the elementary quanta of action, h, anticipating some of the revolutionary ideas which later led to the introduction of the mesmerizing term *Unbestimmtheit* by Werner Heisenberg in 1927.

In [42], Heisenberg introduced a relation to describe the mutual incompatibility of quantum measurements, although still on a more heuristic level. In fact, in the celebrated paper, Heisenberg gave the qualitative statement

$$q_1 \cdot p_1 \sim h, \tag{1.2}$$

where q_1 and p_1 are uncertainties in the variables of position and momenta, respectively. This relation is equivalent to the said decomposition of phase space into cells of size $\sim h$ and reveals that for each degree of freedom in the configuration space of a particle, its position and momentum cannot both be simultaneously determined to arbitrary precision, with the product of their respective "indeterminacies" being of the order of Planck's elementary quantum.

In the same year, Kennard [46] finally derived the world-renown uncertainty inequality

$$\Delta_q \cdot \Delta_p \gtrsim \hbar, \tag{1.3}$$

with Δ_q and Δ_p denoting the uncertainties in position and momentum for each degree of freedom of configuration space.

In 1923 [17], Louis de Broglie - inspired by the quanta of light, as introduced by Einstein - introduced waves of matter, where the associated equation

$$\lambda = \tfrac{h}{p}, \tag{1.4}$$

essentially states that particles with momentum p can show wave-like behavior of wavelength λ and vice versa. This idea was later generalized by Erwin Schrödinger in his wave mechanics, in which to each particle is assigned a wave-function, ψ. The famous Schrödinger equation

$$-\tfrac{i}{\hbar}\partial_t \psi(x,t) = \widehat{H}\psi(x,t), \tag{1.5}$$

then, determines the evolution of a particle's wave-function, ψ, over time in a deterministic way.

In the years to come, Heisenberg, Born, Schrödinger, von Neumann and others further developed non-relativistic quantum theory, which later was generalized by Paul Dirac to the relativistic case, out of which the more general quantum field theories arose.

Since a particle exhibits wave- as well as particle-like behavior - depending on the form of its associated phase space cell of size ~ h - and Schrödinger's equation determines its evolution, it is natural to ask whether it is possible to find a specific wave function, such that its associated phase space cell is as classical as possible, in the sense that it is as concentrated as possible around a classical point in phase space and such that measurements of observable quantities as well as its evolution over time is "optimal". The answer to this question is affirmative and establishes the so-called wave-packets, or *classical coherent states*. These classical coherent states are optimal in the sense that their indeterminacies in q and p attain the lower bound of the uncertainty principle and thus are as classical (point-like) as possible.

In 1944, Dennis Gabor [35] noted that the mathematics of quantum mechanics may also be utilized in information theory to decompose a given signal into atoms. These atoms extract the information from a certain cell in phase space to a (complex) number. More concretely, the atom is represented by a function, well localized in time and frequency simultaneously - a two-dimensional phase space - and maps, when integrated against a signal, the signal's content within this region to a number. This number, then, characterizes the said signal within that specific subset of phase space - a *quantum of information*. Consequently, these associated numbers define a function from the the set of phase space cells to the complex numbers, representing the signal on phase space. Although the idea was very influential, D. Gabor did not explicitly compute any decompositions and considered the phase space decomposition, respectively the set of associated functions, to resemble a basis in the sense of linear algebra.

Eight years later, Duffin and Schaeffer [21] enhanced the toolbox by loosening the concept of a basis to what is nowadays known as *frames*, by abdicating the necessity of linear independence of the basis vectors and thus introducing redundancy. Combining the atomic decompositions of Gabor in phase space with the redundancy of Duffin and Schaeffer's frames, puts the coherent states and wave packets of Schrödinger on a mathematical footing.

In the following decades, Glauber [37], Aslaksen and Klauder [5], Daubechies [11–14] and finally Ali et al. [1] developed the theory even further, culminating in the broad term of *wavelet theory*, encompassing a vast knowledge base about quantum mechanics and signal analysis.

In 2010, Maass et al.[59] posed the juicy question

Do uncertainty minimizers attain minimal uncertainty?

in which they showed that a certain uncertainty principle - related, but differing from the classical one above (*1.3*) - is not as meaningful as it seemed to be, since the lower bound could be made arbitrarily small.

In [78], Stark and Sochen systematically developed a program to categorize the dependency of the lower bound of the inequality for general pairs of generators of locally compact groups, proving that this conceptual inconsistency is not an isolated phenomenon for the “ax+b”-group but rather common among the Lie groups of practical interest, with the Heisenberg group embodying the exception.

This observation eventually yielded the research project *UNLocX* [83], funded by the *Future and Emerging Technologies (FET) programme*, within the *Seventh Framework Programme for Research of the European Commission*, of which this thesis is a late offshoot. Among other objectives, the research revolved around the establishment of optimal uncertainty principles and the identification of its equalizing waveforms, which is also one of the objectives of this monograph.

The *raison d'être* of this thesis is, *inter alia*, the installation of *(continuous) frames* in the sense of signal analysis, adapted to a coordinate system in phase space, specifically chosen for an application. The coordinate systems that are considered, arise via *cotangent lifts* of *diffeomorphisms* and the frames are defined as phase space translates of a probe, such that the chosen probe has a specific location in phase space - its associated *phase space cell*. The chosen coordinate system, respectively its associated Hamiltonian flows, then, shifts - and deforms - these phase space cells along the coordinate lines to tessellate the phase space in such a manner that eventually each point of phase space is contained within such a cell. A decomposition of a given signal with respect to this frame defines a coefficient mapping, such that each coefficient is a *quantum of information*, in the sense of Gabor, assigned to a given phase space cell. In accordance with quadratic phase space distributions like the *Wigner-Ville distribution* and the *Rihaczek distribution*, a new “warped distribution” will be introduced, having correct marginal distributions and giving a novel method at hand to define and calculate a spectrogram-like distribution for semi-direct product groups.

Further, two concurring types of uncertainty principles are established, specifically adapted to these coordinate systems in phase space. The “minimal uncertainty solutions” of these uncertainty principles are optimal in the sense that they declare a waveform to be optimal if its phase space picture is adapted to those coordinate lines in a precise manner.

The story of *research* is the story of *literature research*, since there already exists a vast knowledge base about most of the topics covered in this thesis on top of which this monograph's program is build.

As far as signal analysis, time-frequency analysis and wavelet theory is concerned, the books [1, 26, 27] give a comprehensive *state-of-the-art overview*, up to the year of printing, and include a plethora of topics, ranging from rigorous treatments of *linear signal transforms* through associated *uncertainty principles* to the concepts of *localization operators*, respectively multipliers, and the following years brought mainly deeper elaborations and generalizations of these concepts. Since this thesis is mainly concerned with the three concepts above, this is a good place to comment on the differences of this thesis to the one in these collections.

This monograph shall give an alternative approach to linear signal transforms like the *wavelet transform* and the *Short-Time Fourier Transform*, which are usually developed using (locally compact) group-theoretical arguments. This requirement is abandoned to gain more flexibility and define transforms solely via a loose collection of one-parameter flows of a certain diffeomorphism - the *spectral diffeomorphism*.

Moreover, the most general uncertainty principles are usually derived from a pair of two non-commuting self-adjoint operators - associated with a locally compact group -, defining a function to be optimal if an associated inequality is equalized. Although this concept is maintained in this program, the operators are very specific ones, namely *quantized Hamiltonians* corresponding to coordinate functions of a certain symplectic diffeomorphism - the *spectral cotangent lift*.

Further, the concept of localization operators, defined via multiplication operators weighting the coefficients of a signal transform, is carried over to that of a *spectral quantum frame multiplier*, again being independent of any locally compact group structure.

In [2], the authors introduced the notion of a *quantum frame*, meaning a quantized variant of a *classical frame of reference*. It is this term, which is seized in this monograph to describe a related but different term - a *quantum frame* shall denote a frame in the sense of Duffin and Schaeffer, such that the frame-vectors have a well-defined *phase space localization* associated with a *coordinate system in phase space*, chosen for a particular application to emphasize certain properties of signals.

1.1 Outline

In the second chapter, a very short trip from the foundations of classical mechanics through the quantum mechanical reinterpretation to frames in the sense of signal analysis will be taken. The reason is two-fold. Firstly, this is indicated as a lot of the arguments within the later chapters rely on the preliminaries of physics - classical and quantum - and its language. Secondly, this is a neat way to fix the notation without using a prevalent chapter of boring notational formalities. It is, however, symptomatic that not all topics can be covered, demanding nevertheless a short section about notations - following this one - and an appendix, in which various preliminaries as well as other miscellaneous fixations shall be given.

The third chapter is devoted to the development of a theory of *spectral diffeomorphisms* and its entourage. Essentially, a spectral diffeomorphism is a diffeomorphism in the usual sense, defined on the Fourier domain; that is, on the vector space dual of the reals, which indexes the exponential waves that make up the Fourier transform. This is often referred to as the spectrum - whence the diffeomorphism's prefix. A diffeomorphism deforms a differentiable manifold in a manner which is “manageable” in the sense that its local behavior is known and can be described using analytical methods. These methods will be used to design “spectral quantum frames” which are adapted to coordinate lines in phase space, arising from a spectral diffeomorphism and correspond to well-localized phase space cells. As a matter of fact, a diffeomorphism gives rise to differentiation and integration with respect to an enormous load of coordinate systems on the respective differentiable domain it is defined on. It is this fact, which is utilized to develop assertions which can be used to derive two differing generalized uncertainty principles in the subsequent chapter.

With each spectral quantum frame arises a possibility to localize and weight certain subsets of phase space, giving rise to spectral quantum frame multipliers. Finally, a quadratic distribution - the “warped distribution” - is defined, which characterizes a function's support in the warped phase space domain and leads via convolution to an analogue of a spectrogram.

Chapter four, then, gives the derivation of these two generalized uncertainty principles. The first principle is a generalization of the classical principle of uncertainty adapted to coordinate systems in phase space, associated with spectral diffeomorphisms. A function is said to be optimally adapted to these coordinate lines in the sense of this principle, if its phase space picture aligns optimally with respect to all coordinate lines as nicely as possible. This means that a waveform tries to "nestle" as widely as possible along all coordinate lines and refuses to stray far from these lines. Starting off from a generalized principle with respect to two-dimensional sub-planes of phase space, this principle is generalized to an n-dimensional version, incorporating all phase space coordinates and consequently leading to a system of differential equations whose simultaneous solutions exist, if a certain matrix - the spectral diffeomorphism's Jacobian - is diagonal. If they exist, these waveforms are optimally adapted to all coordinates in phase space simultaneously, in contrast to the usual approach leading to waveforms for individual sub-principles only.

The second generalized uncertainty principle is complementary to the first, as a waveform is said to be optimal in its sense if and only if it refuses to be aligned along coordinate lines and manages to minimize its spread along the coordinates. As before, a principle for two-dimensional sub-manifolds of phase space is given and thereafter generalized to a true n-dimensional version. The n-dimensional principle finally leads to a coupled system of ordinary differential equations, whose solutions - if existent - resemble genuine uncertainty equalizers for all coordinate lines simultaneously.

Finally, the fifth chapter is devoted to the utilization of the machinery assembled in the previous chapters and gives a surprising property of the warped distribution, namely that the convolution of certain warped distributions for spectral diffeomorphisms, arising from semi-direct product groups, resembles a spectrogram-like distribution. It is furthermore shown that the theory encompasses transforms and uncertainty principles, well-known in the community of signal analysis and physics such as

(i) the Short-Time Fourier Transform,

(ii) the one-dimensional Wavelet Transform,

(iii) the SIM(1,1)-Transform, or

(iv) the SIM(2)-Transform.

1.2 Notation

In this thesis, all physical constants, in particular Planck's constant, h, shall be of no interest and hence set to unity; of course, this can be interpreted as the utilization of natural units, $c = h = k_B = 1$, if one prefers this point of view.

To increase readability, we will indicate the end of theorems, definitions, corollaries, examples and so on with ◆ and proofs are closed with ■.

If not stated otherwise to be "up to sets of measure zero", equality of sets $A = B$ is defined to be element-wise, that is, A equals B if and only if each element of A is also in B and vice-versa. The set which contains each element of A, which is not in B, is denote as $A \backslash B := \{ x \in A \mid x \notin B \}$.

For a subset $E \subset X$, we will write $E^c := X \backslash E$ for the *complement* of E in X and a *neighborhood* of a point $x \in X$ is a subset $N \subset X$ containing it: $x \in N \subset X$.

The domain and image of a function are written $\mathrm{dom}(f)$ and $\mathrm{im}(f)$, respectively. If a function maps into a group G, we may speak of the set of points in its domain that are mapped to the neutral element, which is said to be its *kernel*

$$\ker(f) := \{ x \in \mathrm{dom}(f) \mid f(x) = e \in G \} ,$$

which is a subgroup of its domain, if the domain of f is a group, too. The complement of the kernel is its *support*, usually considered for mappings from a topological space into a normed space, which is defined as

$$\mathrm{supp}(f) := \{ x \in \mathrm{dom}(f) \mid f(x) \neq 0 \} .$$

By an inequality like $\epsilon > 0$ we imply that ϵ is a real (and positive) number, $\epsilon \in \mathbb{R}_+$. The supremum and infimum are defined as

$$\sup_{x \in \mathrm{dom}(f)} |f(x)| := \inf \left\{ c \in \mathbb{R}_+ \;\middle|\; \mathrm{dom}(|f|^{-1}) \cap (c, \infty) \text{ is empty} \right\}$$

and

$$\inf_{x \in \mathrm{dom}(f)} |f(x)| := \sup \left\{ c \in \mathbb{R}_+ \;\middle|\; \mathrm{dom}(|f|^{-1}) \cap [0, c) \text{ is empty} \right\}$$

and for *measurable functions*, defined on a measure space (X, Σ, μ), these extrema are generalized to be *essential extrema*. That is, defined up to sets of μ–measure zero, yielding

$$\operatorname*{ess\,sup}_{x \in \mathrm{dom}(f)} |f(x)| := \inf \left\{ c \in \mathbb{R}_+ \;\middle|\; \mu\big(\mathrm{dom}(|f|^{-1}) \cap (c, \infty)\big) = 0 \right\} .$$

To avoid clutter, we will occasionally drop the domain, if it is clear from the context.

A partial sign as ∂_x indicates a partial derivative as $\partial_x f := \frac{\partial}{\partial x} f$ and a dot indicates a time derivative, that is, $\dot{x} := \partial_t x$.

λ will denote (n-dimensional) Lebesgue measure, where its dimension should be clear from the context, and if $E \subseteq \mathbb{R}^n$ is measurable, $\lambda(E)$ is its measure.

When writing integrals involving measures, we will be free to drop the integration variable if no misunderstanding could arise, that is,

$$\int_X f \, \mathrm{d}\nu := \int_X f(x) \, \mathrm{d}\nu(x)$$

and sometimes even drop the integration domain if it is clear from the context. The same will be done for sums like

$$\sum_k f_k := \sum_{k=1}^n f_k \qquad \text{or} \qquad \oplus_k H_k := \oplus_{k=1}^n H_k,$$

whenever the limits are clear from the context. For reasons to become clear in the next lines, we will write inner products to be linear in the second factor, that is,

$$\langle f, g \rangle = \int \overline{f} \, g \, \mathrm{d}\nu,$$

which closely resembles the finite-dimensional case $\langle v, w \rangle := \overline{v}^T w$, where w is interpreted as a column vector and $\overline{v}^T$ is a complex conjugated row vector. The isometrical involution on a locally compact group will be denoted as

$$f \mapsto f^*.$$

We furthermore introduce a notation, which is very convenient when it comes to operator-function pairings - a pairing we are about to encounter in great abundance. Let T be a not necessarily bounded operator, $\psi \in D(T)$ and f a function in the pre-dual of the image of T. Then, we will use the shorthand

$$\langle T\psi, f \rangle =: T_\psi f,$$

that is, T_ψ is defined to be the mapping

$$T_\psi : f \longmapsto \langle T\psi, f \rangle = T_\psi f.$$

Given an operator-valued function $x \mapsto T(x)$, we may now express this conveniently as

$$(T_\psi f)(x) := \langle T(x)\psi, f \rangle.$$

The true power of this shorthand unfolds, when it comes to group representations and the generalization of such, which will be used in this thesis. Then, wavelet transforms, matrix coefficients and conjugation representations may all be represented in the same way. Let π be a unitary group representation, then

$$(\pi_\psi f)(x) := \langle \pi(x)\psi, f \rangle$$

is the associated transform of f with "window" ψ. Let P denote some appropriate operator and the group act on it by conjugation $P(x) := \pi(x)P\pi(x)^*$, then

$$(P_\psi f)(x) := \langle\, P(x)\psi\, ,\, f\,\rangle = \langle\, \pi(x)P\pi(x)^*\psi\, ,\, f\,\rangle\,.$$

The trace of an operator or a matrix, T, is denoted with $\mathrm{tr}(T)$. From time to time, Dirac's bra-ket notation will be adopted, as is custom in the community of quantum physicists. We'll write $|\,\Psi\,\rangle := \Psi \in \mathcal{H}$ for a state and $\langle\,\Psi\,| := \langle\,\psi\, ,\, \bullet\,\rangle$ for its "dual" in order to express rank-one projectors as

$$f \;\mapsto\; |\,\psi\,\rangle\langle\,\psi\,|f := \langle\,\psi\, ,\, f\,\rangle\,\psi.$$

Whenever we work in finite dimensions and explicit indices are in order, we will moreover use the notation of co- and contravariant tensors. In our notation, a contravariant vector has an upper index and will correspond to a column vector as

$$v^i := \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

whereas a covariant vector has a lower index and defines a co-vector

$$v_i := \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$$

A tensor of higher rank, then, necessarily has more than one index which can be upper, lower or both. A matrix, M, acting on some vector, v, is represented as

$$(Mv)^i =: \sum_j M^i_j v^j.$$

We will, however, not use metrics other than Kronecker delta to contract tensors and will refrain from using Einstein's summation convention and thus never drop the summation symbol.

The transposed inverse of a matrix J will be denoted with

$$(J^{-1})^T = (J^T)^{-1} =: J^{-T}.$$

For variables on phase space, we will be free to use (q, p), (x, ξ) or (x, y), where the first emphasizes its relation to mechanics, the second to time-frequency analysis and the third to a neutral set, bearing no special qualities.

Natura non facit saltus.

— Carolus Linnæus [84]

2
Quantization

CONTINUA AND INFINITIES are mathematical concepts, put on firm grounds by Georg Cantor, Richard Dedekind and others [53] and omnipresent in mathematics and mathematical physics.
Whether the emergence of these mathematical abstracta is imperative in the physical universe surrounding us, is still up to debate and shall not be of concern in this monograph. What is certain, however, is the occurrence of observable quantities, having purely discrete spectra - the quanta. It is these *quanta*, as well as the universality of *perspectives*, that this chapter is devoted to.

The world looks different from different perspectives - if not, there is a symmetry. This seemingly obvious fact is worth a few words, as it is a ubiquitous theme in this monograph. In classical physics, the frame of reference is a means to formalize this concept and the Galilean transformation in Newtonian mechanics as well as the Lorentz transformation in relativistic mechanics describe the respective changes of perspectives.

Mathematically, these transformations are coordinate transformations between families of grids, having certain characteristics assumed to be of relevance for the correct description of physical events. As a matter of fact, the physical phenomena - space-time events, as these are called in special relativity -, which constitute our world, should be independent of the observers perspective, and thus events and physical laws are presupposed to be covariant with such a coordinate change.

Albert Einstein's theory of general relativity [23] drives this point home, where the perspective is assumed to be dependent on the curvature of space-time and a frame of reference is distorted through the mere presence of mass and energy.

This curvature is derived from and manifests itself through a pseudo-metric tensor and a change of perspective inscribes the space-time manifold with a fresh set of coordinates which accordingly also alters said (pseudo)metric tensor. It is this change of perspective and the associated alteration of the metric which leads to the sensation of a constant gravitational field in a frame which is constantly accelerated and, thus, distorted. This is Einstein's equivalence principle.

The permissible coordinate transformations in general relativity are less restrictive than in the case of special relativity and, in fact, the only restriction is that these transforms are diffeomorphisms, which deform space-time in an invertible and differentiable manner.

In general, the structure preserving morphisms on *differentiable manifolds*, *Definition A.73 (Differentiable manifolds)*, are referred to as *diffeomorphisms*.

Definition 2.1 (Diffeomorphism)**.** Let X and Y be differentiable manifolds. A homeomorphism $\sigma : X \to Y$ which is continuously differentiable and has a continuously differentiable inverse is a *diffeomorphism*. ◆

In classical mechanics, the state of a system is given as a point in an abstract *phase space*, the permissible coordinates of which are called canonical and inscribe the phase space with pairs of conjugate coordinates. The associated coordinate transformations that map one set of canonical coordinates to another are accordingly called canonical transformations or symplectic diffeomorphisms. These symplectomorphisms, as these are often called, are diffeomorphisms preserving pairs of conjugate coordinates and are the ones that are of interest in the course of this chapter.

When speaking of signals, one usually means a “set of data”, containing interesting as well as irrelevant data and the challenging task is to separate these and extract what is of relevance. The explicit process of separating the wheat from the chaff, however, is dependent on how a signal is mathematically *represented*. Superficially, a signal is represented as an abstract function

$$f : X \to Y, \quad x \mapsto f(x),$$

associating to each $x \in X$ a value $f(x)$. This abstract concept can be made concrete in various ways, depending on the signal under consideration. For an exemplary audio signal, the domain is time, $X = \mathbb{R}$, and the value $f(x)$ is the loudness at time x, which is a real value, too. To be even more concrete, the signal's values represent the momentary deflection of, say, a speaker's membrane from its idle state.

Since the membrane is doomed to obey the laws of physics, common sense tells us that the membrane's deflection cannot change by an arbitrarily large amount in a short duration, so the membrane's deflection necessarily depicts a *continuous* path between two positions, since there are no true *quanta leaps* outside of the quantum world - *natura non facit saltus*.

This property - *continuity* - is a very desirable one in regard to any signal, since then no matter what the domain of or data contained in f is, a small change in x only leads to a small change in $f(x)$. Unfortunately, the quantum revolution made clear that the above is an indulgence in wishful thinking, since the vibrations of a clamped membrane is necessarily a clamped quantum oscillator, whose excitements are quantized.

Mathematically, this manifests itself in the fact that the continuity of the signal cannot always be guaranteed. Moreover, for continuous functions, there exists a *continuous* mapping

$$f \mapsto f(x), \qquad x \in \mathbb{R}^n,$$

which "measures the content" of f *at* the point $x \in \mathbb{R}^n$, i.e., it evaluates f at the point x. Whenever we are interested in further information on f around x - the role-model being the "frequency" of f at the instant of time x -, it is clear that we cannot extract any further information from $f(x)$, since it is just a single complex value. Thus, the information concerning the occurring frequencies must be encoded in a neighborhood of the point x, rather than the point x itself. This points to the general fact that in order to gain further information than just the apparent number-value of a function at some point, we have to take the neighborhood of that point into account, that is, the information is *coherent*.

Therefore, and because the continuity of the signal cannot always be guaranteed, we are forced to abstract the concept of a function to that of an equivalence class of such, that is, a *measurable* function, vanishing at infinity. The reason for the abstraction of functions to equivalence classes of these, is the fact that the topological dual of the measurable functions not necessarily contains the point-evaluation functionals. Rather, functionals

$$l : f \longmapsto l(f) \in \mathbb{C}$$

are defined as measures, which the measurable functions are *integrated* against.

This motivates the following definition.

Definition 2.2 (Signal)**.** A signal shall denote a complex-valued, measurable function - or an equivalence class of such - defined on some Lebesgue space of square-integrable equivalence class of functions, that is,

$$f \text{ is a signal} \iff f \in L^2(X,\ \mathrm{d}\nu; \mathbb{C}), \tag{2.1}$$

for some locally compact abelian group $(X,\ \mathrm{d}\nu)$, for which we may define a commutative Gelfand transform via Pontryagin duality, *Definition A.30 (Pontryagin duality)*.

In fact, we will solely use $\mathbb{R}^n$ - being its own dual - as the domain, for which said Gelfand transform boils down to the Fourier transform. ◆

This brings us to the next standing assumption of this chapter. Although there are more general domains that can be conceived, we henceforth claim that the relevant information of $n-$dimensional signals and functions that we shall consider may be indexed by (not necessarily countable) families of points, resp. subsets of the $2n-$dimensional phase space.

Definition 2.3 (Phase Space)**.** Let $\mathbb{R}^n$ be a configuration space and $\widehat{\mathbb{R}}^n$ its dual, then either its cotangent bundle

$$X := T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \widehat{\mathbb{R}}^n \tag{2.2}$$

or a subset thereof

$$X \subset \mathbb{R}^n \times \widehat{\mathbb{R}}^n \tag{2.3}$$

shall denote the *phase space*. ◆

Our definition of the phase space support of a function rests upon the possibility to distinguish between local and global behavior of a function for the purpose of which we will make heavy use of the Fourier transform.

Definition 2.4 (Fourier Transform)**.** The Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\ e^{-2\pi i \langle\, \xi\,,\, x\,\rangle}\ \mathrm{d}x\ , \qquad \xi \in \mathbb{R}^n, \tag{2.4}$$

maps functions on $L^1(\mathbb{R}^n)$, to functions on $C_0(\mathbb{R}^n)$ and can - using Plancherel's formula - be extended to a unitary map, $\mathcal{F}$, on $L^2(\mathbb{R}^n)$.

Then, $\mathcal{F}f$ is no longer defined in a pointwise sense but interpreted as

$$\langle\, f\,,\, g\,\rangle_{L^2(\mathbb{R}^n)} = \langle\, \mathcal{F}f\,,\, \mathcal{F}g\,\rangle_{L^2(\widehat{\mathbb{R}}^n)}\,, \tag{2.5}$$

that is, it is defined via Plancherel's theorem, *Theorem A.42 (Plancherel)*.

When restricting the Fourier transform to one parameter only, i.e.,

$$\mathcal{F}_k f(x_1,\ldots,\xi_k,\ldots,x_n) = \int_{\mathbb{R}} f(x_1,\ldots,x_k,\ldots,x_n)\ e^{-2\pi i \langle\, \xi_k\,,\, x_k\,\rangle}\ \mathrm{d}x_k, \quad \xi_k \in \mathbb{R}, \tag{2.6}$$

the properties, as stated above, still hold. ◆

In particular, this monograph is devoted to the decomposition of a vector space of signals into linear subspaces which have a well-known localization in phase space. That is, if we decompose the phase space into non-overlapping tiles, we wish to associate to each of these tiles a subspace of functions, whose phase space support is completely contained within this tile.

Heuristic 2.5 (Phase space decomposition)**.** *Let X denote phase space, $\mathcal{S}$ the vector space of signals and D_f the image of a function $f \in \mathcal{S}$ on phase space. Then,* ideally *we wish to find for all tiles $\tau \subset X$ an associated subspace $\mathcal{S}_\tau \subseteq \mathcal{S}$ such that all signals in $\mathcal{S}$ have phase space support in τ, i.e.,*

$$f \in \mathcal{S}_\tau \subseteq \mathcal{S} \;\Rightarrow\; \operatorname{supp}(D_f) \subseteq \tau.$$

◆

In fact, a classical phase space is an even-dimensional symplectic (differentiable) manifold, *Definition A.73 (Differentiable manifolds)*, whose structure arises from a nowhere vanishing two-form, called the symplectic form.

Definition 2.6 (Symplectic Form)**.** Let Ω be a differential two-form on phase space and Γ, Λ denote the set of vector fields and one forms on phase space, respectively. Then, if

(i) Ω is closed $\iff$ $\mathrm{d}\Omega = 0$,

(ii) alternating $\iff$ $\Omega(v,w) = -\Omega(w,v) \Rightarrow \Omega(v,v) = 0$, $\forall v,w \in \Gamma$,

(iii) and non-degenerate $\iff$ Ω identifies Λ with Γ ,

it is a *symplectic form.* ◆

In order to do explicit calculations on symplectic spaces, we need a local representation of these, for which the next theorem comes in handy.

Darboux's Theorem 2.7. *Assume that Ω is a symplectic form. Then, there exists a* local chart - *that is, a local coordinate system -, such that*

$$\Omega := \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q^i. \tag{2.7}$$

The corresponding local coordinates are the Darboux coordinates*.* ◆

Proof. Cf., e.g. [15]. ■

This moreover means that all symplectic manifolds of the same dimension may be identified, at least in local *Darboux coordinates*.

Let now (ϕ, U) be a local Darboux chart of phase space, with $\phi := (p_i, q^i)$. Then,

$$\phi(U) \subset \mathbb{R}^n \times \mathbb{R}^n$$

and, in this chart, the symplectic form may be expressed as

$$\Omega(w, v) := w^T J v \ ,$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

In a sense, *symplectic diffeomorphisms*, as defined in *Proposition 2.9 (Symplectomorphism)*, are a non-linear analogue to a symplectic matrix.

Proposition 2.8 (Symplectic Matrix)**.** *For a matrix S, the following are equivalent*

(i) S *is a* symplectic matrix

(ii) $\Omega(Sw, Sv) = \Omega(w, v)$,

(iii) *J is conjugation invariant under S,*

$$S^T J S = S J S^T = J,$$

(iv) *S is a block matrix of the form*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A^T D - C^T B = I_n$ *and* $A^T C$ *and* $B^T D$ *are symmetric matrices.* ◆

Proof. See, e.g. [15]. ■

Symplectomorphisms, mapping the set of canonical coordinates to itself, constitute a group, in fact a subgroup of the infinite-dimensional group of diffeomorphisms of a differentiable manifold. Whether a diffeomorphism is symplectic and thus in the symplectic subgroup, is characterized by its action on the manifolds cotangent bundle in the following sense.

Proposition 2.9 (Symplectomorphism)**.** *For a diffeomorphism σ, on a differentiable manifold M, the following are equivalent*

(i) σ *is a* symplectic diffeomorphism,

(ii) $\sigma^* \Omega = \Omega(\sigma(\bullet), \sigma(\bullet)) = \Omega$, *that is, the symplectic form is invariant under (the pullback of) σ,*

(iii) *the Jacobian matrix, J_σ, is a symplectic matrix, for all $(x, y) \in \mathcal{M}$.* ◆

Proof. Again, see, e.g. [15]. ■

2.1 From Classical States to Quantum States

2.1.1 Classical Mechanics

In classical mechanics, a system's state is represented as a point in phase space, $(q,p) \in \mathbb{R}^{2n}$, and its dynamics is determined by equations of evolution.

Classical States

Classical mechanics is deterministic in the sense that, assuming, for example, a system of n particles in a three-dimensional space, the system's state is determined completely by the $3 \cdot n$ positions and $3 \cdot n$ momenta of the particles in question.

That is, knowing the exact positions, masses and directions in which the particles are heading, as well as the existing forces at an instant of time, it is possible to determine the positions and momenta of all particles for all time. More concretely, the state of the system is given by $(q,p) \in \mathbb{R}^{3n} \times \widehat{\mathbb{R}}^{3n}$, where

$$q := (q_1, \dots, q_{3n}) \in \mathbb{R}^{3n} \qquad \text{and} \qquad p := (p_1, \dots, p_{3n}) \in \widehat{\mathbb{R}}^{3n}$$

are the positions and momenta, respectively. The set $X = \mathbb{R}^{3n} \times \widehat{\mathbb{R}}^{3n}$, the cotangent space of $\mathbb{R}^{3n}$, is the *phase space*, the space of all possible states the system can attain, where $\widehat{\mathbb{R}}^{3n}$ is the topological dual of the *configuration space* $\mathbb{R}^{3n}$, describing the momenta of the n particles. Here, $\widehat{\mathbb{R}}^{3n}$ can be identified with $\mathbb{R}^{3n} \simeq \widehat{\mathbb{R}}^{3n}$ itself.

Classical Observables and Evolution

The system's evolution over time may be expressed in various ways, usually derived from a Hamiltonian or Lagrangian. In Hamilton's mechanics, the Hamiltonian, $H : X \to \mathbb{R}$, is a real and smooth function on phase space, usually interpreted as the total energy of the system.

Definition 2.10 (Hamiltonian)**.** The Hamiltonian of a classical mechanical system is the observable for its energy and is, in the simplest case, given by the sum of kinetic, T, and potential energy, V, that is

$$H := T + V,$$

is the total energy of the system, called the *Hamiltonian.* ◆

When the system does not exchange energy with its surroundings, the energy necessarily stays constant and the Hamiltonians contour lines therefore coincide with the systems flow in time, for else it changed its energy, which contradicted the assumption of energy conservation. Thus, the flow is orthogonal to the Hamiltonian's gradient, determining Hamilton's differential equations.

Definition 2.11 (Hamilton's Equations)**.** Let $H \in C^\infty(\mathbb{R}^{2n})$ be a Hamiltonian, then

$$(\dot{q}^i, \dot{p}_i) = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}), \qquad i = 1, \ldots, n, \tag{2.8}$$

are Hamilton's equations which define the system's flow.

That is, the Hamiltonian vector field X_H, which determines the flow, is

$$X_H := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \nabla H \quad , \text{ with } \nabla H := \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix},$$

and I_n denoting the n-dimensional identity matrix $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$.

In differential geometry, vector fields are represented by differential operators, acting on smooth functions. Thus, the above may be rephrased as a differential operator of the form

$$X_H f := \frac{\partial H}{\partial p_i} \partial_q f - \frac{\partial H}{\partial q^i} \partial_p f, \quad f \in C^\infty(\mathbb{R}^{2n}),$$

and this is also how vector fields will be represented in this monograph. ◆

As a system without observable quantities other than its total energy is a boring one, we further introduce general *observables* of the system as real and smooth functions on the phase space, whose point evaluation

$$O \mapsto O(q,p), \qquad O \in C^\infty(X),$$

gives the observable quantity of the state. By virtue of these point evaluations, we may reinterpret the states of the system as Dirac delta functionals, which, when integrated against, evaluate the observables at the system's state in the sense of distributions. That is, writing $e_{(q,p)}$ for the evaluation functional, given as distributional integration against a Dirac delta functional, centered at (q,p), we have

$$\text{System is in state } (q,p) \in X \iff (q,p) \simeq e_{(q,p)} : O \mapsto O(q,p),$$

where the map

$$e : X \to C^\infty(X)^* \ , \quad (q,p) \mapsto e_{(q,p)}$$

associating to a point its evaluation functional is an injective map into the dual of the space of observables. Since the evaluation functionals are multiplicative, i.e.,

$$e_{(p,q)}(f \cdot g) = (f \cdot g)(p,q) = f(p,q) \cdot g(p,q) = e_{(p,q)}(f) \cdot e_{(p,q)}(g),$$

and C^∞ is invariant under pointwise multiplication, the observables constitute a *commutative algebra* and can thus be observed simultaneously, respectively shortly after each other, *without disturbing each other*.

By loosening the definition and smearing the differences between the system's Hamiltonian and the system's observables, we may define a plethora of other (Hamiltonian) flows this way. There are an infinite number of possible Hamiltonian flows, but in a two-dimensional phase space, the most common ones are given by

(i) the observable of momentum

$$P : (q,p) \mapsto p, \quad \text{with} \quad X_P := \partial_q,$$

inducing a flow of constant speed in q-direction,

(ii) the observable of position

$$Q : (q,p) \mapsto q, \quad \text{with} \quad X_Q := -\partial_p,$$

inducing a flow of constant speed in p-direction, as well as

(iii) the observable of energy of a harmonic oscillator

$$H : (q,p) \mapsto \tfrac{1}{2}\left(q^2 + p^2\right), \quad \text{with} \quad X_H := -q\partial_p + p\partial_q,$$

inducing a circular symmetric flow around the origin.

Apart from the ubiquitous ones above, there are other interesting ones, like

$$\tilde{H} : (q,p) \mapsto q \cdot p, \quad \text{with} \quad X_H := -p\partial_p + q\partial_q, \tag{2.9}$$

resulting in a flow along the contour lines, depicted in *Figure 2.1a*, or

$$\tilde{H} : (q,p) \mapsto \frac{q}{2+\cos(p)}, \quad \text{with} \quad X_H := -\frac{1}{2+\cos(p)}\partial_p + \frac{q \cdot \sin(p)}{(2+\cos(p))^2}\partial_q, \tag{2.10}$$

with a flow along the contour lines shown in *Figure 2.1b*.

Another approach to and in a sense a generalization of Hamilton's mechanics is via the Poisson structure of phase space.

Definition 2.12 (Poisson Bracket)**.** Let (q,p) be Darboux coordinates on phase space X. Then, the Poisson bracket

$$\{\ ,\ \} : (F,G) \mapsto \{F\,,\,G\}, \tag{2.11}$$

is representable as

$$\{ F , G \} = \sum_i \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}. \tag{2.12}$$

Poisson's bracket is a derivation and thus follows Leibniz's rule,

$$\{ ab , c \} = \{ a , c \} b + a \{ b , c \} ,$$

is bilinear,

$$\{ \alpha a + \beta b , c \} = \{ \alpha a , c \} + \{ \beta b , c \} \text{ and } \{ c , \alpha a + \beta b \} = \{ c , \alpha a \} + \{ c , \beta b \} ,$$

antisymmetric,

$$\{ a , b \} = - \{ b , a \} \;\Rightarrow\; \{ a , a \} = 0,$$

and, moreover, Jacobi's identity

$$\{ a , \{ b , c \} \} = \{ \{ a , b \} , c \} + \{ b , \{ a , c \} \}$$

holds. ◆

Note that the last identity above turns the Poisson bracket into a Lie bracket. Therefore, an associated Poisson algebra, which is a set of functions defined on phase space, closed under the Poisson bracket, is a Lie algebra.

The peculiarity of the Poisson bracket is that, for canonical coordinates, it holds that

$$\left\{ q^i , p_j \right\} = \delta^i_j,$$

and a canonical transformation - a symplectomorphism - is defined to be a transformation for which this relation is invariant, that is,

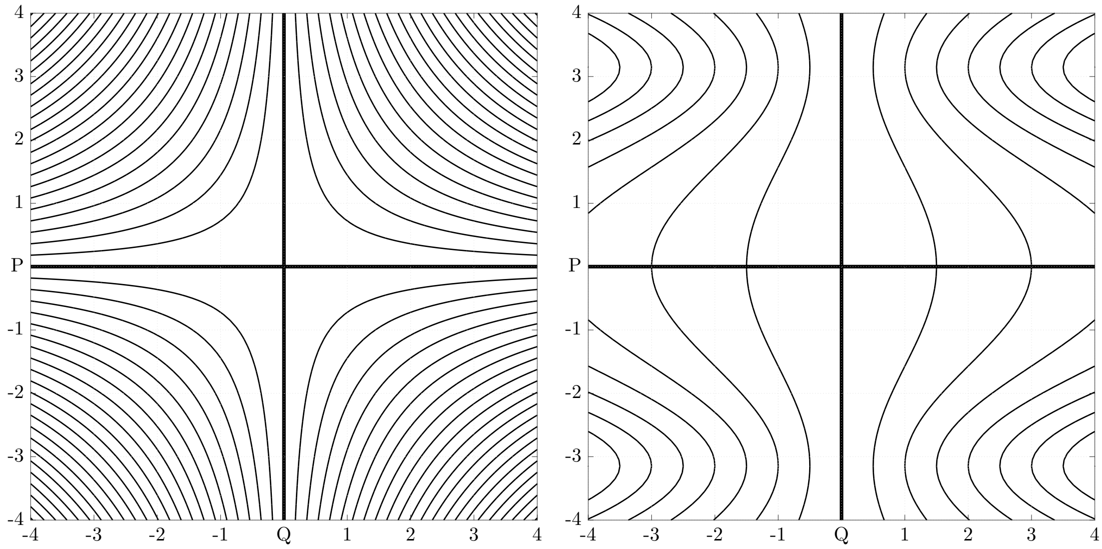
$$(q,p) \mapsto \mathcal{S}(q,p) \text{ is a symplectomorphism } \Leftrightarrow \mathcal{S}^* \{ F , G \} = \{ F , G \} ,$$

where $\mathcal{S}^* \{ F , G \} := \{ F \circ \mathcal{S} , G \circ \mathcal{S} \}$ denotes the pullback of the Poisson bracket under $\mathcal{S}$.

Moreover, Hamilton's equations may be elegantly expressed as

$$\dot{q}^i := \frac{\partial H}{\partial p_i} = \left\{ q^i , H \right\} \text{ and } \dot{p}_i := -\frac{\partial H}{\partial q^i} = \{ p_i , H \} , \quad i = 1, \dots, n, \tag{2.13}$$

which can be checked directly. This means that Poisson's bracket induces the (infinitesimal) symplectic flow, with respect to its second parameter, and gives another method of determining whether a quantity is invariant under a flow; often referred to as a *symmetry of the system.*



(a) Contour lines of $(q,p) \mapsto q \cdot p$ **(b)** Contour lines of $(q,p) \mapsto \frac{q}{2+cos(p)}$

Figure 2.1: Contour Lines

Definition 2.13 (Symmetry)**.** A *symmetry* shall denote a group of transformations, under which an equation is invariant. ◆

In the physics community - especially in quantum field theory -, the Hamiltonian approach to symmetries is uncommon, as symmetries of this kind are usually expressed via the Lagrangian (density) and the principle of stationary action, which plays no role in this program and will therefore not be discussed.

Nonetheless, from a physicist's point of view, [69, Ch. 2] or [68], this is intrinsically linked to

Noether's Theorem 2.14. *To each continuous symmetry, there corresponds a preserved quantity.* ◆

A quantity - in this case the value of an observable, evaluated for a specific state - which is preserved under a symmetry group, is necessarily conserved for the infinitesimal flow. Using Poisson's approach to mechanics, a flow along the contour lines of G is induced by $\{\bullet, G\}$ and an observable, F, which is conserved under the flow, induced by G, thus has a vanishing Poisson bracket with G

$$F \text{ is conserved under } G \iff \{F, G\} = 0.$$

From this, we conclude immediately the almost trivial fact that the flow induced by an observable preserves the same

$$\{G, G\} = 0.$$

That is, the symplectic flow along the contour lines preserves the value of the observable and thus the flow is given by a symmetry group, which manifests itself as

- conservation of momentum by translational symmetry,

$$((q,p) \mapsto p) \longmapsto -\partial_q,$$

- conservation of energy by time symmetry,

$$((q,p) \mapsto (p^2+q^2)/2) \longmapsto -p\partial_q + q\partial_p,$$

- etc.

From a signal analyst's point of view, however, there is no need to have physical pendants or even names for these kinds of symplectic maps, since - as we will see below - merely the possibility to flow along contour lines of these generalized Hamiltonians is what is wished for.

All these Hamiltonian flows share the all-important property of being symplectic diffeomorphisms and, as such, leave invariant the symplectic form as defined above. This means that the symplectic area of a subset of phase space under consideration is kept invariant and, moreover, the $2 \cdot n$-dimensional volume is kept constant. The former is the defining property of a diffeomorphism to be *symplectic*, while the latter is well known in classical mechanics and symplectic geometry under the name of

Liouville's Theorem 2.15. *Let (M,ω) be a symplectic manifold and σ a symplectic diffeomorphism on M, then σ preserves volume.* ◆

In fact, even more is true. In 1985, Gromov unveiled his *non-squeezing theorem* [40], stating that the volume-conservation of canonical transformations is merely one of the more apparent characteristics of symplectic diffeomorphisms, guaranteed by Liouville's theorem, and there's a more subtle and astonishing property.

Gromov's Theorem 2.16. *Gromov's* non-squeezing theorem *[40] highlights that canonical transformations cannot squeeze a ball of radius r into a cylinder of smaller radius, if the cylinder is perpendicular to a plane of conjugate variables, e.g.,*

$$C_i := \left\{ (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \,\middle|\, x_i^2 + y_i^2 \leq r \right\}.$$

That is, it is impossible for symplectic diffeomorphisms to squeeze a phase space subset such that its area in the planes of conjugate coordinates decreases below their initial value. ◆

This is surprising, as it is easy to imagine the ball being squeezed to a long needle until it fits inside the cylinder and has not been anticipated until 1985. In light of Gromov's non-squeezing theorem, before moving to the quantum realm, a short remark about the connection between classical and quantum is in order.

Remark 2.17 (Classical vs. Quantum)**.** According to De Gosson and Hiley [16], the non-squeezing theorem is - in a sense - a classical analogue of the uncertainty principle in quantum mechanics.

De Gosson realized that the connection between classical and quantum mechanics is deeper than it is perceived in the community of mathematical physics. He showed that there is a way to *derive* Schrödinger's equation from the classical equations of Hamilton, without taking recourse to any quantum arguments.

Although he introduced an arbitrary parameter, $\hbar$, without any physical significance, the path taken is legitimate and unearthed a more fundamental nexus between classical and quantum - at least from a mathematical point of view. ◆

2.1.2 Quantum Mechanics

Just like classical mechanics, quantum mechanics may be expressed in various ways, usually named *pictures*, expressing the quantum mechanical states, observables and its equations of evolution in seemingly different but mathematically equivalent ways.

Quantum States and Observables

In the *Schrödinger*, *Heisenberg* and *Dirac* pictures, the (pure) state of a system, which was formerly defined as a tuple of $2 \cdot n$ numbers, is now represented by a normalized vector in an abstract projective Hilbert space, $\mathcal{H}$. The observables are lifted to self-adjoint operators on this Hilbert space - which may be unbounded - and the Hamiltonian flow is replaced by strongly continuous one-parameter groups of unitary operators - the quantum symmetries of the system -, acting on the Hilbert state space.

In contrast to the classical case, the state of the system, respectively the observable quantities of the system, cannot be determined perfectly, since the state is modeled by a probability wave, which implies a certain uncertainty in the predictability of a measurement's outcome.

We will henceforth concentrate on the Schödinger picture, but note in passing that the same could developed within the framework of the other pictures.

In the spatial representation of the Schrödinger picture for a quantum system, consisting of a single particle in ordinary three-dimensional space,

$$|\psi(\vec{x},t)|^2 \ \mathrm{d}x \ \mathrm{d}y \ \mathrm{d}z$$

represents the probability of finding a particle in the spatial cube $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$, centered at $\vec{x} := (x, y, z) \in \mathbb{R}^3$, at time $t \in \mathbb{R}$, where

$$\|\psi\| = 1, \qquad \psi(\bullet, t) =: \psi_t \in L^2(\mathbb{R}^3,\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z), \quad \forall t \in \mathbb{R},$$

is the system's state represented in spatial or $|\, x\,\rangle$-representation. A particle's momentum is also given by a probability density and if

$$\widehat{\psi}(\vec{p}, t) := \iiint_{\mathbb{R}^3} \psi(\vec{x}, t)\ e^{-2\pi i \langle\, \vec{p}\,,\, \vec{x}\,\rangle}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{2.14}$$

is the Fourier transform of ψ - its *spectrum* - in its three spatial dimensions, then

$$\left|\widehat{\psi}(\vec{p}, t)\right|^2 \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$$

represents the probability of the particle to have a momentum in the "momentum cube" $\mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$, centered at $\vec{p} = (p_x, p_y, p_z) \in \widehat{\mathbb{R}}^3$, at the instant of time t.
A consequence of this is that the waveform ψ - as well as its Fourier transform $\widehat{\psi}$ - already encodes both the spatial and momentum representation, although they are only defined on either the configuration space, $\mathbb{R}^3$, or the space of momenta $\widehat{\mathbb{R}}^3$ and thus either ψ or $\widehat{\psi}$ each suffice for each and every measurement of observables. This is a general fact, whenever the state is represented with respect to the (generalized) Eigenbasis of an observable, having non-degenerate spectrum, but this shall not concern us here.

This is in strong contrast to the classical formalism, in which a particle's position and momentum is known exactly and given by the coordinate (q, p).

The observable properties of a given system in state ψ is given by a measurement process which, again, mathematically resembles an *evaluation functional.* These evaluation functionals, however, are themselves not defined as point evaluations of continuous functions, but as elements of the dual of the space of the observables in question, which are now given by self-adjoint operators.

Given an observable T, the outcome of a measurement of T, while the system is in the (pure) state ψ, is the evaluation mapping

$$e_\psi : T \longmapsto e_\psi(T) := \langle\, T\psi\,,\, \psi\,\rangle\,,$$

which is the *expectation value*, that is, the value we *expect* when measuring the observable T with respect to the system's probability wave function ψ. To each function corresponds a rank-one projector defined as follows.

Definition 2.18 (Rank-One Projector)**.** Let $\psi, f \in \mathcal{H}$ be a functions in the Hilbert space $\mathcal{H}$. Then,

$$P_\psi : \mathcal{H} \to \{ \lambda \cdot \psi \mid \lambda \in \mathbb{C} \}, \ f \mapsto |\tilde{\psi}\rangle\langle\tilde{\psi}| f := \tilde{\psi} \cdot \left\langle \tilde{\psi}, f \right\rangle, \tag{2.15}$$

with $\tilde{\psi} := \frac{1}{\|\psi\|}\psi$, defines the rank-one projection operator of ψ, projecting onto the one-dimensional subspace, spanned by ψ. ◆

By loosening the definition above, we end up with an operator which is no longer a projection, but still of rank one.

Definition 2.19 (Rank-One Operator)**.** Let $\psi, \varphi, f \in \mathcal{H}$ be a functions in the Hilbert space $\mathcal{H}$. Then,

$$P_{\psi,\varphi} : \mathcal{H} \to \{ \lambda \cdot \psi \mid \lambda \in \mathbb{C} \}, \ f \mapsto |\psi\rangle\langle\varphi| f := \psi \cdot \langle \varphi, f \rangle, \tag{2.16}$$

is rank-one operator. ◆

Rewriting the expectation value as

$$e_\psi(T) = \operatorname{tr}\left[T \ |\psi\rangle\langle\psi| \right],$$

we see that we may define the states to be rank-one projection operators

$$P_\psi := |\psi\rangle\langle\psi|, \quad \text{with} \quad \operatorname{tr}\left[P_\psi P_\varphi\right] = |\langle \psi, \varphi \rangle|^2 .$$

As in the classical case, we have an injection

$$\psi \overset{\sim}{\longmapsto} e_\psi,$$

identifying the states with the evaluation functions as a subset of the dual space of the algebra of observables.

Quantization of a classical System

The process of *quantization* of a classical system now associates to a classical system a quantum system, in which the rules of the game are necessarily changed although the behavior of the classical system shall be reflected as much as possible. If no particles are to be created, nor destroyed, and no Lorentzian invariance by incorporating special relativity is needed, the classical approach of quantum mechanics, which takes Hamiltonian mechanics, as above, as a starting point and is now often called the first quantization, suffices. This quantization is not unique, but the most common ways this is done is either via coherent state quantization - which we will postpone - or using *canonical quantization*, a.k.a. the Weyl correspondence [15].

Definition 2.20 (Weyl Correspondence)**.** Let $O \in C^\infty(X)$ denote a (real-valued) classical observable. Then, the *Weyl correspondence*

$$O \overset{\sim}{\longmapsto} \widehat{O},$$

associates to O a self-adjoint integral operator

$$O \longmapsto \int_{\mathbb{R}^n} e^{2\pi i\langle x-y\,,\,\xi\rangle}\, O\left(\tfrac{x+y}{2},\xi\right)\, \mathrm{d}\xi =: k(x,y), \tag{2.17}$$

to be interpreted as a map from a symbol - an observable on phase space - to an integral kernel, converging at least in the sense of distributions.

Conversely, let $T \in L(\mathcal{H})$ be a linear operator with kernel K, then

$$\mathcal{W}(T)(x,\xi) := \int_{\mathbb{R}^n} K(x-\tfrac{y}{2}, x+\tfrac{y}{2})\; e^{2\pi i\langle\, y\,,\,\xi\,\rangle}\; \mathrm{d}y \tag{2.18}$$

is its *Weyl symbol*, which is again real-valued if T is self-adjoint, a.k.a., the *Weyl map*.

The Weyl symbol of a rank-one operator is its *(cross) Wigner-Ville distribution*

$$\mathcal{W}(|\,f\,\rangle\langle\, g\,|)(x,\xi) := \int_{\mathbb{R}^n} f(x-\tfrac{y}{2})\; \overline{g(x+\tfrac{y}{2})}\; e^{2\pi i\langle\, y\,,\,\xi\,\rangle}\; \mathrm{d}y, \tag{2.19}$$

where the type of convergence of the last two integrals above depend on the function spaces of f, g and K. ◆

In this regard, we shall give an explicit definition of the Wigner-Ville Distribution for a function, which can be defined via its associated rank-one projector.

Definition 2.21 (Wigner-Ville Distribution)**.** The Wigner-Ville distribution of a function is the Weyl symbol of its rank-one projector

$$\mathcal{W}_f(x,\xi) := \mathcal{W}(|\,f\,\rangle\langle\, f\,|)(x,\xi) := \int_{\mathbb{R}^n} f(x-\tfrac{y}{2})\; \overline{f(x+\tfrac{y}{2})}\; e^{2\pi i\langle\, y\,,\,\xi\,\rangle}\; \mathrm{d}y. \tag{2.20}$$

◆

With this at hand, there is still missing some ingredient to explicitly calculate the quantized operators for often occurring observables like the canonical ones, p, q and polynomials in these. This is where the symmetric correspondence rule comes handy [63].

Definition 2.22 (Symmetric Correspondence Rule for Polynomials)**.** Let $O := p^n q^N$ be a polynomial observable, then the Weyl correspondence associates to it an operator which is the average of all possible orderings of the polynomials. Or, using one of the two of McCoy's formulas, this means

$$p^n q^N \mapsto \sum_{k=0}^{n} \binom{n}{k} P^k Q^N P^{n-k},$$

respectively

$$p^n q^N \mapsto \sum_{k=0}^{N} \binom{N}{k} Q^k P^n Q^{N-k}. \tag{2.21}$$

This will be referred to as the *symmetrization rule.* ◆

Example 2.1 (Hamiltonian)**.** By the Weyl correspondence, to the classical Hamiltonian,

$$H := T + V,$$

of a single-particle system of unit-mass $m = 1$, with T and V again denoting the kinetic and potential energy, is assigned the self-adjoint operator

$$f \;\mapsto\; \widehat{H} f := -\tfrac{1}{(2\pi)^2} \Delta f + V \cdot f, \quad f \in \operatorname{dom}(\widehat{H}),$$

with the Laplace operator Δ and V the potential, now reinterpreted as an operator of multiplication. ◆

In later chapters, we need a specific type of quantized observable and the following corollary summarizes all of the relevant features of the quantization scheme, which will be needed.

Corollary 2.23 (Quantization)**.** *Let $O, (O_k)_k$ be real-valued, analytic observables on $\mathbb{R}^{2n}$,* independent of the q coordinates*. Then, for all q^k, $k = 1, \dots, n$, (*k is a contravariant index and not a power*) we have*

(i) $\widehat{O \cdot q^k} = \frac{1}{2}(\widehat{O}\widehat{q}^k + \widehat{q}^k \widehat{O})$

(ii) *and* $\widehat{\sum_k O_k q^k} = \sum_k \frac{1}{2}(\widehat{O}_k \widehat{q}^k + \widehat{q}^k \widehat{O}_k)$.

Moreover,

(iii) the Weyl correspondence is linear,

(iv) the operator of a symbol, depending only on the q coordinates, is a multiplication operator in the spatial domain, and

(v) the operator of a symbol, depending only on the p coordinates, is a multiplication operator in the frequency domain. ◆

Proof. (iii), (iv) and (v) are obvious from the definition. To see (i), note that, since O is analytic and only depends on the p coordinates, there exist (c_n^j), such that

$$O \cdot q^k = \sum_{j=1}^{n} \sum_{i} c_i^j (p_j)^i q^k,$$

and from McCoy's second formula (*2.21*) and the linearity of quantization, we get

$$\begin{aligned}\widehat{O \cdot q^k} &= \sum_{j=1}^{n} \sum_{i} c_i^j \widehat{(p_j)^i q^k} \\ &= \frac{1}{2} \sum_{j=1}^{n} \sum_{i} \left(c_i^j (\widehat{p_j})^i \widehat{q^k} + \widehat{q^k} c_i^j (\widehat{p_j})^i \right) \\ &= \frac{1}{2} (\widehat{O} \widehat{q}^k + \widehat{q}^k \widehat{O}),\end{aligned}$$

which proves (i). (ii) follows from (i), again by linearity. ■

Another very common quantization principle, used in the theory of pseudo-differential operators, is the Kohn-Nirenberg correspondence, developed in great generality for specific classes of functions on X, dubbed *symbol classes*, but since there is no explicit dependence on any specific symbol classes, like, e.g., the important Hörmander Classes in the later chapters, no definition is given.

Definition 2.24 (Kohn-Nirenberg Correspondence)**.** Let $O \in C^\infty(\mathbb{R}^n)$, then

$$(\hat{O}(x, D) f)(x) := \int_{\mathbb{R}^n} O(x, \xi) e^{2\pi i \langle \xi , x \rangle} \hat{f}(\xi) \, \mathrm{d}\xi \tag{2.22}$$

is the pseudo-differential operator, assigned to the *symbol O*.

Conversely, let $T \in L(\mathcal{H})$ be a linear operator with kernel K, then

$$\begin{aligned}\mathcal{R}(T)(x, \xi) &:= \int_{\mathbb{R}^n} K(x, x + y) \; e^{2\pi i \langle y , \xi \rangle} \, \mathrm{d}y \\ &:= \int_{\mathbb{R}^n} K(x, y) \; e^{2\pi i \langle y , \xi \rangle} \, \mathrm{d}y \; e^{-2\pi i \langle x , \xi \rangle}\end{aligned} \tag{2.23}$$

is its *Kohn-Nirenberg symbol* on phase space.

The Kohn-Nirenberg symbol of a rank-one operator is the *(cross) Rihaczek distribution*

$$\begin{aligned}\mathcal{R}(| f \rangle \langle g |)(x, \xi) &:= \int_{\mathbb{R}^n} f(x) \; \overline{g(x + y)} \; e^{2\pi i \langle y , \xi \rangle} \, \mathrm{d}y \\ &:= f(x) \; \overline{g(\xi)} \; e^{-2\pi i \langle x , \xi \rangle},\end{aligned} \tag{2.24}$$

where the type of convergence of the last two integrals above are again dependent on the function spaces of f, g and K. ◆

Again, an explicit definition of the Rihaczek Distribution makes sense, hence the following

Definition 2.25 (Rihaczek Distribution)**.** The Rihaczek distribution of a function f is the Kohn-Nirenberg symbol of its rank-one projector

$$\mathcal{R}_f(x,\xi) := \mathcal{R}(|f\rangle\langle f|)(x,\xi) = f(x)\ \overline{\widehat{f}(\xi)}\ e^{-2\pi i\langle x,\xi\rangle}. \tag{2.25}$$

◆

And for the sake of completeness, the following definition gives the difference between the two quantization schemes above - the ordering.

Definition 2.26 (Ordered Correspondence Rule for Polynomials)**.** Let $O := p^n q^N$ be a polynomial observable, then the Kohn-Nirenberg correspondence associates to it an operator which is ordered as

$$p^n q^N \mapsto P^n Q^N.$$

◆

Unitary Flows

The canonical quantization scheme allows to assign a self-adjoint operator to each Hamiltonian in the same sense as we assign Hamiltonian vector fields to it. This map

$$\widehat{\ }\ :\ C^\infty(T^*M)\ \to\ L(\mathcal{H})\ ,\ H\ \mapsto\ \widehat{H}$$

now enables us to use *Schrödinger's epic equation*

$$-\frac{1}{2\pi i}\partial_t\psi(x,t) = \widehat{H}\psi(x,t) \tag{2.26}$$

and Stone's theorem, to further identify the Hamiltonian to a group of strongly continuous unitary operators, U_t, which solve Schrödinger's equation in the sense of operators, see, e.g. [72, Thm. 13.38] .

Stone's Theorem 2.27. *Let T be a self-adjoint operator on $\mathrm{dom}(T) \subseteq \mathcal{H}$ and $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. Then,*

$$\mathbb{R} \ni t \longmapsto e^{-2\pi i T t} \in U(\mathcal{H})$$

is a strongly continuous unitary group representation of $(\mathbb{R},+)$, that is, a one-parameter group of strongly continuous unitary operators, passing through the identity element at $t = 0$.

Conversely, if $(U_t)_{t\in\mathbb{R}}$ *is a one-parameter group of strongly continuous unitary operators, then there exists a unique operator* T*, defined and self-adjoint on* $\operatorname{dom}(T) \subseteq \mathcal{H}$*, such that*

$$e^{-i2\pi Tt}f =: U_t f, \quad f \in \operatorname{dom}(T),$$

in fact, by differentiating we get

$$\partial_t U_t f = -i2\pi T U_t f, \quad f \in \operatorname{dom}(T),$$

by which we may recover T *via passing* $t \to 0$*, guaranteed by its strong continuity .*

The operator T *will be referred to as the* (infinitesimal) generator *of the corresponding symmetry group, that is, of the unitary flow* U_t*.* ◆

Alternatively, a functional calculus, as in *Definition A.58 (Borel Functional Calculus)*, can be used to establish this correspondence.

Thus, the family of operators, defined by

$$\psi(x,t) := U_t \psi_0(x),$$

for some appropriate ground state ψ_0, solve (*2.26*), since we have

$$-\frac{1}{2\pi i}\partial_t U_t \psi_0 = \widehat{H} U_t \psi_0.$$

We may now assign two different but connected objects to each Hamiltonian - its Hamiltonian vector field and its Weyl operator. Let

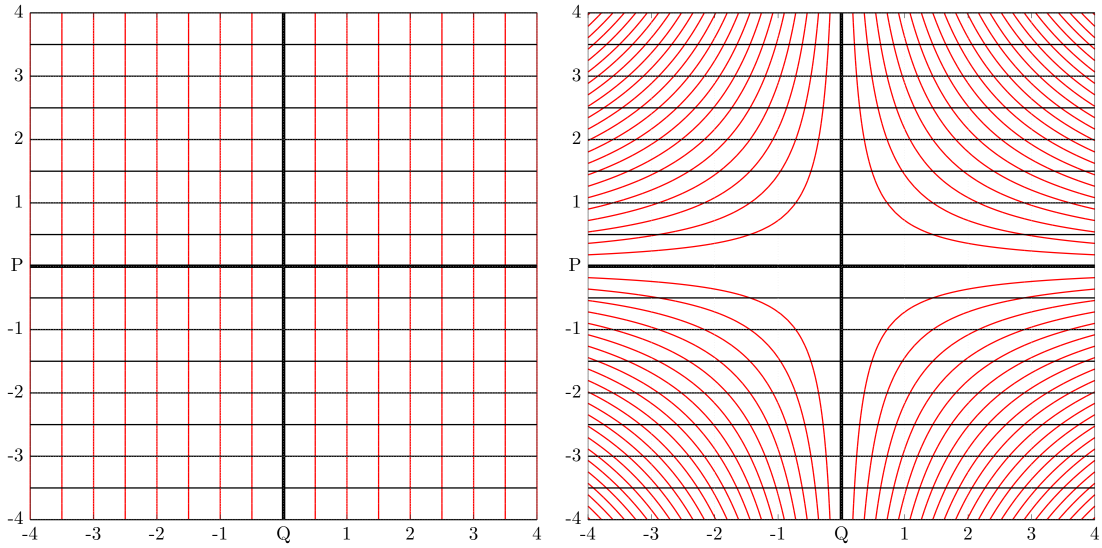
$$P : (q,p) \mapsto p \ , \quad Q : (q,p) \mapsto q$$

be the canonical coordinate functions on the *two-dimensional* phase space, then

$$X_P := -\partial_q, \quad X_Q := \partial_p$$

are the corresponding Hamiltonian vector fields, inducing the system's flow in phase space and

$$\widehat{P}f := \frac{1}{2\pi i}\frac{\partial f}{\partial x} \ , \quad \widehat{Q}f := x \cdot f$$



(a) Flow lines of X_P(black) and X_Q(red) **(b)** Flow lines of X_P(black) and X_D(red)

Figure 2.2: Flow Lines

are the associated self-adjoint generators, represented in the spatial domain. Using *Stone's Theorem 2.27*, to these self-adjoint operators we may now assign the well-known operator families of translation

$$U_P(t)f := e^{-i2\pi\widehat{P}t}f = e^{-\partial_x t}f = f(\cdot - t)$$

and modulation

$$U_Q(t)f := e^{-i2\pi\widehat{Q}t}f = e^{-i2\pi xt}\cdot f,$$

which represent the associated symmetry groups. *Figure 2.2a* shows the corresponding flow lines of the Hamiltonian vector fields X_P and X_Q.

When restricting to a half-plane of the two-dimensional phase space, e.g. when dealing with systems having a positive-valued configuration space only, a further pretty natural choice is the Hamiltonian

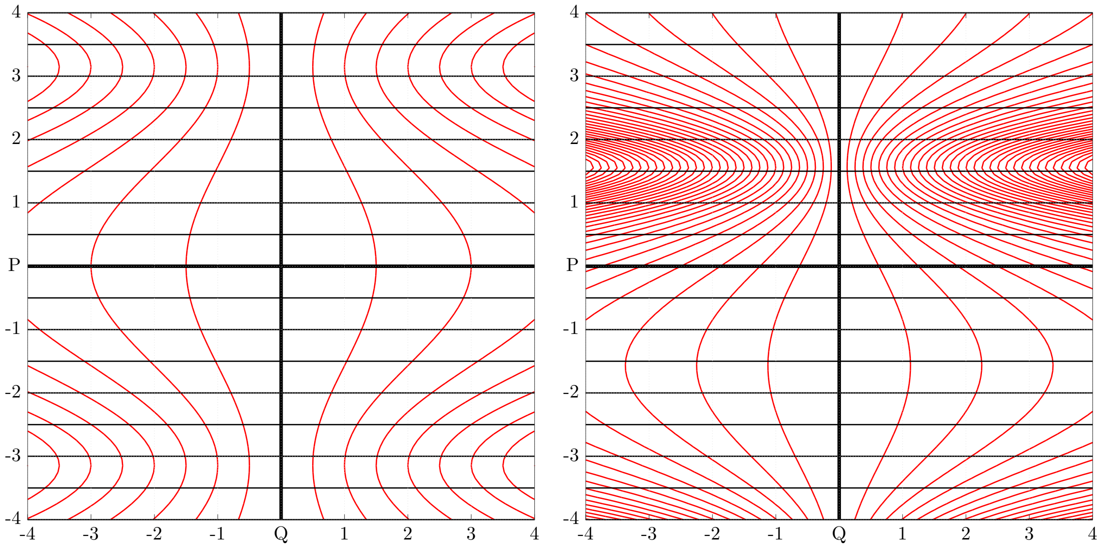
$$D : (q, p) \mapsto p \cdot q.$$

Applying the same steps as above to D, we get the Hamiltonian vector field

$$X_D := q\partial_q - p\partial_p$$

and the self-adjoint generator

$$\widehat{D} := \tfrac{1}{2}(\widehat{P}\widehat{Q} + \widehat{Q}\widehat{P}) = \tfrac{1}{2\pi i}(\tfrac{1}{2} + x\partial_x),$$



(a) Flow lines of X_P(black) and X_{S_1}(red) **(b)** Flow lines of X_P(black) and X_{S_2}(red)

Figure 2.3: Flow Lines

represented on the spatial domain, where we followed the symmetrization rule. Solving Schrödinger's equation for $\widehat{D}$, leads to the unitary group of dilation operators

$$(U_D(t)\Psi)(x) := e^{-t/2}\Psi(e^{-t}\cdot x).$$

Figure 2.2b shows the corresponding flow lines of the Hamiltonian vector fields X_P and X_D and *Figure 2.3* depicts the flow lines of X_P together with the Hamiltonian vector fields

$$X_{S_1}(q,p) := -\frac{1}{2+\cos(p)}\partial_p + \frac{q\cdot\sin(p)}{(2+\cos(p))^2}\partial_q$$

and

$$X_{S_2}(q,p) := -\frac{1}{1.25-\sin(p)}\partial_p + \frac{q\cdot\cos(p)}{(1.25-\sin(p))^2}\partial_q,$$

corresponding to the Hamiltonians

$$S_1 : (q,p) \mapsto \frac{q}{2+\cos(p)} \quad \text{and} \quad S_2 : (q,p) \mapsto \frac{q}{1.25-\sin(p)}.$$

Just like the Hamiltonian has its iron both in the fire of classical as well as quantum mechanics, the Poisson bracket plays a role in both descriptions, too. The quantization scheme(s) above have initially been introduced to associate with a given classical system a certain quantum system, which as closely as possible resembles the classical one. This is especially apparent, when considering the Poisson bracket.

Recall that in the classical description, a system of coordinates is canonical, iff

$$\left\{ q^i , p_j \right\} = \delta^i_j,$$

and a transformation is canonical, resp. a symplectomorphism, if

$$\mathcal{S}^* \{ F , G \} = \{ F \circ \mathcal{S} , G \circ \mathcal{S} \} = \{ F , G \},$$

holds for each pair of F, G from the Poisson algebra. In an adequate quantization, the Poisson bracket should be mapped to the commutator

$$\{ F , G \} \longmapsto 2\pi i \left[\widehat{F} , \widehat{G} \right], \tag{2.27}$$

where $\widehat{F}, \widehat{G}$ are the quantized observables and a pair of (quantized) observables are canonical if and only if an analogue of Poisson's relation above holds.

Definition 2.28 (Commutator)**.** Let S, T be operators, such that $\mathrm{im}(S) \subset \mathrm{dom}(T)$ and $\mathrm{im}(T) \subset \mathrm{dom}(S)$, then

$$[S,T]f := (ST - TS)f \ , \quad f \in \mathrm{dom}(S) \cap \mathrm{dom}(T) \cap \mathrm{dom}([S,T]), \tag{2.28}$$

defines its *commutator*. ◆

The quantized version of Poisson's canonical relation above now is the so-called canonical commutation relation.

Definition 2.29 (Canonical Commutation Relation)**.** Let $(P_i)_i$ and $(Q^j)_j$ be self-adjoint. Then

$$[P_i, Q^j]f := \tfrac{1}{2\pi i} \delta^j_i f, \quad f \in \mathrm{dom}(P_i) \cap \mathrm{dom}(Q^j), \tag{2.29}$$

is the *canonical commutation relation*; the *CCR*. ◆

Apart from the CCR, the commutator now gives another criterion, whether an observable quantity, S, is invariant under the unitary flow, induced by some generator T, namely

$$S \text{ is invariant under the unitary flow of } T \iff [\, S , T \,] = 0.$$

Since to each Hamiltonian corresponds a group of symplectomorphisms of classical mechanics, as the flow of its Hamiltonian vector field - the classical symmetry -, as well as a group of unitary evolution operators - the quantum symmetry -, via the Weyl correspondence, Schrödinger's equation and Stone's theorem, it is in fact not far-fetched to seek an inverse connection between the *unitary flow* of a quantum state in $\mathcal{H}$ and the *symplectic flow* of some image - which is yet to be determined - of this state on phase space. That is,

Heuristic 2.30 (Phase space flow)**.** *Let D_ψ denote the image of the quantum state ψ on phase space and H a Hamiltonian on phase space. Let moreover $\phi_H(x,t)$ denote its symplectic flow on phase space, generated by the Hamiltonian vector field X_H, that is,*

$$\partial_t \phi_H(x,t) := X_H(\phi_H(x,t)).$$

Let furthermore $U_H(t)$ be its unitary evolution group.

Then there should be some nexus such that

$$D_{U_H(t)\psi}(x) \sim D_\psi(\phi(x,t)).$$

◆

It turns out that this is indeed the case and that the theory of coherent states, along with its associated (de-)quantization method(s), is the right tool to show that the heuristic rule above is in fact a strict one.

2.2 From Classical Frames to Quantum Frames

The theory of coherent states [1], as developed by Schrödinger [74], Sudarshan [80], Glauber [37] and Klauder [49–51], associates to each (classical) point of phase space a finite-rank projection operator - usually of rank one - in such a way that the operator-valued integral of these projections, with respect to some measure on phase space, converges to a bounded, positive and invertible operator on the Hilbert space of states.

The whole theory is independent of phase space and can be defined as follows [1].

Definition 2.31 (Abstract Coherent States)**.** Let (X,μ) be a measure space and

$$\mathbb{F} := \{\, \varphi_x \,|\, x \in (X,\mu) \,\} \subseteq \mathcal{H}$$

a family of functions in the Hilbert space $\mathcal{H}$, then $\mathbb{F}$ is a *coherent state system* if and only if there exists a bounded, positive operator $\mathcal{A}$, possessing a densely defined inverse, such that

$$\mathcal{A} := \int_X |\varphi_x\rangle\langle\varphi_x| \, \mathrm{d}\mu(x), \tag{2.30}$$

converges in the weak sense. ◆

Note that the operator $\mathcal{A}$ in the definition above need not have a bounded inverse $\mathcal{A}^{-1}$, if so, the coherent state system is a frame, by which we shall mean *a redundant generalization of a Hilbert space basis.*

Definition 2.32 (Continuous Frame)**.** Let (X, μ) be a measure space and

$$\mathbb{F} := \{\, \varphi_x \,|\, x \in (X, \mu) \,\} \subseteq \mathcal{H}$$

a family of functions in the Hilbert space $\mathcal{H}$, then $\mathbb{F}$ is a *continuous frame* if and only if there exist $0 < A \leq B < \infty$, such that

$$A \, \| f \|^2 \leq \int_X |\langle\, \varphi_x \,,\, f \,\rangle|^2 \;\mathrm{d}\mu(x) \leq B \, \| f \|^2 \;\;, \quad f \in \mathcal{H}. \tag{2.31}$$

The frame is *tight* if and only if $A = B$ and may then be characterized by its resolution of the identity

$$\mathcal{A}^{-1} \int_X |\,\varphi_x\,\rangle\langle\,\varphi_x\,| \;\mathrm{d}\mu(x) = 1_{\mathcal{H}}, \tag{2.32}$$

where the integral converges in the weak sense.

With $\operatorname{spec}(\mathcal{A})$ denoting the spectrum of $\mathcal{A}$, by choosing

$$A := \inf \operatorname{spec}(\mathcal{A}) \quad \text{and} \quad B := \sup \operatorname{spec}(\mathcal{A}),$$

this definition coincides with *Definition 2.31 (Abstract Coherent States)* for a boundedly invertible resolution operator $\mathcal{A}$. ◆

Before moving to the definition of a quantum frame, there is one final generalization of a frame, which will play some (minor) role during the course of the later chapters.

Definition 2.33 (Frames of Rank N)**.** Let $\{\, \varphi_x^i \,|\, x \in (X,\ \mathrm{d}\nu), i = 1, \ldots, N \,\}$ constitute a frame in the sense that

$$\mathcal{A} := \int_X \sum_i |\,\varphi_x^i\,\rangle\langle\,\varphi_x^i\,| \;\mathrm{d}\nu(x) \tag{2.33}$$

is a positive, bounded operator with *bounded inverse.* Then it is called a *frame of rank N*, if, for each $x \in X$, the set

$$\left\{\, \varphi_x^i \,|\, i = 1, \ldots, N \,\right\}$$

is linearly independent. As before, it is a *tight frame* of rank N, if $\mathcal{A} = \lambda 1_{\mathcal{H}}$, $\lambda \in \mathbb{C}$, and *normalized tight* if $\lambda = 1$. ◆

Whenever only the decomposition with respect to $x \in X$ is considered - ignoring the individual dependencies on the i -, the rank-N definition above lacks a certain uniqueness, as then the map

$$F : x \mapsto \sum_i |\varphi_x^i\rangle\langle\varphi_x^i|,$$

which defines a function taking its values in the positive rank-n operators over $\mathcal{H}$, becomes the main matter of interest. Then, whenever $N > 1$, these are not uniquely defined, since there is a certain freedom in choosing, for each $x \in X$, the vectors φ_x^i giving rise to $F(x)$.

As a matter of fact, these degrees of freedom lead to the equivalence of frames, as a given frame of rank N is simply a specific choice of basis vectors for each point $x \in X$.

Definition 2.34 (Frame Equivalence)**.** Assume that $\{\varphi_x^i \,|\, x \in X, i = 1, \ldots, N\}$ is a frame of rank N and denote with F the corresponding rank-N operator-valued function

$$x \mapsto F(x) := \sum_i |\varphi_x^i\rangle\langle\varphi_x^i|.$$

Then, the set $\{\tilde{\varphi}_x^i \,|\, x \in X, i = 1, \ldots, N\}$ is *Gauge equivalent*, if for each $x \in X$ there exists a unitary operator $U(x) \in U(N)$ giving rise to

$$\tilde{\varphi}_x^i = \sum_k U_k^i(x)\varphi_x^k.$$

◆

Proposition 2.35 (Equivalence)**.** *Gauge equivalent rank-N frames share the same projectors.* ◆

Proof. Let $\{\varphi_x^i \,|\, x \in X, i = 1, \ldots, N\}$ and $\{\tilde{\varphi}_x^i \,|\, x \in X, i = 1, \ldots, N\}$ be gauge equivalent frames of rank N, with $F(x)$ and $\tilde{F}(x)$ denoting their projectors. Denote with $U(x) \in U(N)$ the relating field of operators, we have

$$\begin{aligned}
\tilde{F}(x) := \sum_i |\tilde{\varphi}_x^i\rangle\langle\tilde{\varphi}_x^i| &= \sum_i |\sum_k U_k^i(x)\varphi_x^k\rangle\langle\sum_{k'} U_{k'}^i(x)\tilde{\varphi}_x^{k'}| \\
&= \sum_{i,k,k'} U_k^i(x)|\varphi_x^k\rangle\langle\tilde{\varphi}_x^{k'}|\overline{U_i^{k'}}(x) \\
&= \sum_{k,k'}\sum_i U_k^i(x)\overline{U_i^{k'}}(x)|\varphi_x^k\rangle\langle\tilde{\varphi}_x^{k'}| \\
&= \sum_{k,k'} \delta_k^{k'}|\varphi_x^k\rangle\langle\tilde{\varphi}_x^{k'}| = \sum_k |\varphi_x^k\rangle\langle\tilde{\varphi}_x^k| = F(x)
\end{aligned}$$

■

Since a frame - of whatever rank - is a generalization of a basis in the sense that it is possible to decompose the whole space of states, $\mathcal{H}$, with respect to subspaces associated with each of the projectors $|\varphi_i\rangle\langle\varphi_i|$, that is,

$$\bigoplus_{x\in X} \operatorname{im}\left(\sum_i |\varphi_x^i\rangle\langle\varphi_x^i|\right) \text{ is dense in } \mathcal{H} \iff \bigoplus_{x\in X}\bigoplus_i \varphi_x^i \text{ is dense in } \mathcal{H},$$

it is the perfect tool to decompose our space of states into subspaces, assigned to subsets of phase space.

This motivates the following

Terminology 2.36 (Quantum Frame)**.** By a *quantum frame* we shall mean a frame in the sense of *Definition 2.32 (Continuous Frame)*, for which

(i) (X, μ) denotes phase space with some yet undetermined measure,

(ii) which is *adapted* to some application-dependent *coordinate system* in phase space, which need not be canonical, and

(iii) such that each of the projectors localizes along the coordinate lines of the chosen frame of reference in a well-concentrated manner.

This, then, enables us to speak of the quantization of a classical frame of reference. ◆

Of course, this terminology is very vague and its clarification is what the rest of this monograph is all about. The first two of the three points in the last definition are what the next chapter is devoted to, whereas chapter 4 is dedicated to the third point above.

2.3 From Quantum Frames to Signal Analysis Frames

In the 1st chapter, it was noted that D. Gabor introduced the mathematics of quantum mechanics into signal analysis in the sense that with each phase space cell is associated a waveform which, by integration against a signal, extracts a specific complex number from this signal, which characterizes the signal within this region of phase space. By anticipating the uncertainty principle, which will be introduced in 4th chapter, this region of phase space - and thus the information contained within it - is determined and its minimal size controlled by the uncertainty principle.

Terminology 2.37 (Quantum of Information)**.** A *quantum of information* shall denote an indivisible piece of information, contained within a region of phase space. ◆

Coherent state map

Let now $\mathbb{F}$ be a normalized tight frame - and thus a system of coherent states -, whose characteristic localizations in phase space is known and let $\mathcal{S}$ denote the space of signals. We may then decompose a function, in fact the whole reservoir of relevant functions, with respect to certain subsets of phase space. This decomposition can be understood in two closely related ways.

- Mapping a function to another function on phase space

  $$f \mapsto \langle \varphi_x , f \rangle, \quad x \in X, f \in \mathcal{S},$$

  such that each x contains a quantum of information (which may be redundant), contained within the neighborhood of x, where the specific neighborhood and how certain parts contribute is dependent on φ_x

- Mapping the function to another function in $\mathcal{S}$,

  $$P_E : \mathcal{S} \to \mathcal{S}, \ f \mapsto f_E$$

  which contains only that information of f which was essentially contained within the region $E \subseteq X$

The former is more general in the sense that, whenever $\mathbb{F}$ is a frame, the latter is a reconstruction of the former after setting all values in the complement of E to zero, that is,

$$P_E f := \int_E \langle \varphi_x , f \rangle \varphi_x \, \mathrm{d}\mu(x),$$

converging with respect to some relevant topology - usually the weak one.

Thus, since the relevant information about f for a particular application is encoded in the function

$$x \mapsto \langle \varphi_x , f \rangle, \quad x \in X,$$

it makes sense to shift the focus from the resolution property of frames to the linear map

$$f \mapsto \pi_{\{\varphi\}} f := \{\langle \varphi_x , f \rangle\}_{x \in X},$$

which maps a function to another function, defined on phase space.

Proposition 2.38 (Coherent state map)**.** *Let $\mathbb{F} \subseteq \mathcal{H}$ be a normalized tight quantum frame. Then, the linear map*

$$\pi_{\{\varphi\}} : \mathcal{H} \to L^2(X,\ \mathrm{d}\mu), f \mapsto \pi_{\{\varphi\}} f := \{\langle \varphi_x , f \rangle\}_x, \tag{2.34}$$

called the coherent state map*, or* coefficient mapping*, is unitary.*

Using the weakly converging integral

$$f := \int_X \langle \varphi_x , f \rangle \varphi_x \ \mathrm{d}\mu(x), \qquad f \in \mathcal{H}, \tag{2.35}$$

it is possible to reconstruct f from its coefficients $\langle \varphi_x , f \rangle$. ◆

Proof. Since the frame is normalized tight, it holds that

$$\int_X |\varphi_x\rangle\langle\varphi_x| \ \mathrm{d}\mu(x) = 1_{\mathcal{H}},$$

and thus

$$\int_X |\langle \varphi_x , f \rangle|^2 \ \mathrm{d}\mu(x) = \langle f , f \rangle, \quad f \in \mathcal{H},$$

proving unitarity of the coherent state map. By polarizing the last identity we have

$$\begin{aligned} &\int_X \langle \varphi_x , f \rangle \langle h , \varphi_x \rangle \ \mathrm{d}\mu(x) = \langle h , f \rangle, \quad f, h \in \mathcal{H}, \\ \Leftrightarrow \ &\left\langle h , \int_X \langle \varphi_x , f \rangle \varphi_x \ \mathrm{d}\mu(x) \right\rangle = \langle h , f \rangle, \quad f, h \in \mathcal{H}, \end{aligned}$$

which shows the weak convergence of the reconstruction integral. ∎

Corollary 2.39 (Rank-N Coefficient function)**.** *Let $\mathbb{F} \subseteq \mathcal{H}$ be a normalized tight frame of rank N. Then, the coefficient function*

$$\mathcal{H} \to \mathbb{C}^N \otimes L^2(X,\ \mathrm{d}\nu) \simeq L^2(X,\ \mathrm{d}\nu; \mathbb{C}^N) \ , \quad f \mapsto \left\{\left\langle \varphi_x^i , f \right\rangle\right\}_{x,i}$$

associates to each f a vector-valued function on X.

The reconstruction of f is given by

$$\int_X \sum_i \varphi_x^i \left\langle \varphi_x^i , f \right\rangle \ \mathrm{d}\nu(x) = f, \qquad f \in \mathcal{H},$$

to be interpreted in the weak sense. ◆

Proof. The coefficient function is a definition and the reconstruction formula is immediate from

$$\begin{aligned} \int_X \sum_i \varphi_x^i \left\langle \varphi_x^i , f \right\rangle \ \mathrm{d}\nu(x) &= \int_X \sum_i |\varphi_x^i\rangle\langle\varphi_x^i| f \ \mathrm{d}\nu(x) \\ &= \int_X \sum_i |\varphi_x^i\rangle\langle\varphi_x^i| \ \mathrm{d}\nu(x) f \\ &= f, \end{aligned}$$

where we used the resolution of the identity of the normalized tight frame and convergence at least with respect to the weak operator topology. ∎

Corollary 2.40 (Reproducing Kernel Hilbert Space). *Let $\mathbb{F} \subseteq \mathcal{H}$ be a frame with resolution operator $\mathcal{A}$. Then, to it corresponds the reproducing kernel*

$$k(y,x) := \left\langle \varphi_y \,, \mathcal{A}^{-1}\varphi_x \right\rangle,$$

giving rise to a reproducing kernel Hilbert space, $\mathcal{H}_{\{\varphi\}}$, on $(X,\ \mathrm{d}\mu)$.

Moreover, this reproducing kernel Hilbert space is the image of the coherent state map, that is,

$$\pi_{\{\varphi\}} : \mathcal{H} \to \mathcal{H}_{\{\varphi\}},\ f \mapsto \pi_{\{\varphi\}} f,$$

with $\mathcal{H}_{\{\varphi\}} := \pi_{\{\varphi\}}\mathcal{H}$ being the r.k.H.s. ◆

Proof. Using the frame property, we have the weak identity

$$\mathcal{A}^{-1} \int_X \langle \varphi_x \,, f \rangle \varphi_x \,\mathrm{d}\mu(x) = f, \qquad f \in \mathcal{H},$$

and thus, after applying the coherent state map again, it follows that

$$\begin{aligned}
\left\langle \varphi_y \,, \mathcal{A}^{-1} \int_X \langle \varphi_x \,, f \rangle \varphi_x \,\mathrm{d}\mu(x) \right\rangle &= \int_X \langle \varphi_x \,, f \rangle \ \left\langle \varphi_y \,, \mathcal{A}^{-1}\varphi_x \right\rangle \ \mathrm{d}\mu(x) \\
&= \int_X k(y,x) \ \langle \varphi_x \,, f \rangle \ \mathrm{d}\mu(x) \\
&= \left\langle k(y,\bullet) \,, \pi_{\{\varphi\}} f \right\rangle \\
&= \langle \varphi_y \,, f \rangle
\end{aligned}$$

which proves the claim. ■

In analogy to the spectrogram of the Short-Time Fourier Transform and the scaleogram, for the Wavelet Transform, to each (normalized tight) frame is assigned a quadratic representation, which - in lack of a better name - is herein referred to as *frameogram.*

Definition 2.41 (Frameogram). Let $\mathbb{F}$ be a normalized tight frame. Then, the squared modulus of its coefficients

$$f \mapsto |\langle \varphi_x \,, f \rangle|^2, \tag{2.36}$$

is its *frameogram.* ◆

Localization

In the 4th chapter, the localization properties of frames, respectively their prototype functions, φ will be examined. It is the coherent state map,

$$x \mapsto |\varphi_x\rangle\langle\varphi_x|, \quad x \in X, \tag{2.37}$$

as defined above, associating to each point in phase space a projection operator of rank one, on which a whole theory of quantization may be built. Composing (*2.37*) with the Weyl or Kohn-Nirenberg map

$$x \mapsto \mathcal{W}(|\varphi_x\rangle\langle\varphi_x|) = \mathcal{W}_f(\bullet - x), \qquad x \in X,$$

respectively

$$x \mapsto \mathcal{R}(|\varphi_x\rangle\langle\varphi_x|) = \mathcal{R}_f(\bullet - x), \qquad x \in X,$$

we find that the (coherent states) quantization scheme maps a single point in phase space to a distribution, centered at x. By abuse of geometrical language, quantization "smears" a classical state around its classical position - and thus introduces an uncertainty in its phase space. However, since we are interested in the process of decomposing phase space with respect to functions having no a priori kinship to quantum states, we will not dwell any longer on the quantum nature of any of these and take a first step towards a phase space decomposition.

Clearly, this "smearing uncertainty" depends on the associated projector, which suggests the following definition.

Definition 2.42 (Phase Space Localization)**.** Let X denote phase space and

$$F : x \mapsto \sum_i |\varphi_x^i\rangle\langle\varphi_x^i|, \quad x \in X,$$

a positive operator valued function on X. Let moreover $\mathcal{D}$ denote a map, associating to an operator of finite rank, a function on phase space. Then

$$x \mapsto \mathcal{D}(F(x))$$

shall denote the (F-dependent) quantized phase space cell, assigned to x. ◆

It is this kind of localization, which will be elaborated on in chapter 4; specifically with respect to the theory of spectral diffeomorphisms, as developed in the next chapter.

Remark 2.43 (A note on Quantization)**.** In signal processing, the word quantization has a different but related meaning, namely quantizing a continuous value via a sample-and-hold circuit in an analog-to-digital converter.

When a time-varying signal is digitized, it needs to be sampled which, ultimately, takes more than single instant of time to accomplish. Thus, the circuit does not measure the quantity, in this case a voltage, instantaneously but some weighted average over a certain time-interval, where the weight - the “window” - depends on the device itself. In the simplest idealized case of a switch followed by an (ideal) RC low-pass and an A/D-Converter, it is an increasing exponential waveform followed by a decreasing one. On the other hand, if in parallel a second device is attached, which changes its polarization while the sampling step is in progress, the A/D-device may sample two streams simultaneously where the former samples the lower and the latter samples the higher frequency part, effectively doubling the sampling frequency as both streams may be superimposed to half the sampling time-constant.

Therefore, for each sample-step, the circuit digitizes a certain, circuit-dependent *quantum of information* in the sense of *Terminology 2.37 (Quantum of Information)* and the phase space of the analog signal, which is to be sampled, undergoes a quantization in the sense of the “phase space smearing” above and each sampling step is a coherent state map with a time-shifted window, depending on the device.

A quantization in signal processing is therefore, in an obvious sense, related to quantization in quantum mechanics. ◆

Panta rhei.

— (attributed to) Heraklit

3 Symplectomorphization

UBIQUITOUS IN MATHEMATICS, and hence in this monograph, is the concept of morphisms. In this chapter, a very special morphism, the *symplectic diffeomorphism*, will play the lead.

In order to decompose phase space in a manner which is adapted to certain applications, we are in need of a machinery, to make this process transparent and easily generalizable. Since the signals we shall decompose are initially not defined on phase space but on a Lagrangian subspace of phase space - like the time or frequency domain in case of a two-dimensional phase space -, it is desirable to have a method at hand which assigns a symplectomorphism in phase space to each diffeomorphism on some Lagrangian subspaces [61, Ch. 6].

Meet the Cotangent Lift:

Theorem 3.1 (Cotangent Lift). *Let $\sigma : M \to M$ be a diffeomorphism on the manifold M, and let T^*M denote the cotangent bundle of M. Then*

$$\Sigma \; : \; T^*M \; \to \; T^*M \; , \; (q,p) \; \mapsto \; (\sigma(q), J_\sigma^{-T}(q)p) \; , \quad q \in M, \; p \in T_x^*M, \tag{3.1}$$

is a symplectomorphism and will be referred to as cotangent lift *or* symplectomorphization *of the diffeomorphism σ.* ◆

Proof. The Jacobian of Σ is

$$\begin{pmatrix} J_\sigma & \\ C & J_\sigma^{-T} \end{pmatrix},$$

with $C := \frac{\partial}{\partial q}(\sum_k p^k \cdot (J_{\sigma^{-1}}^T)_k^\bullet(\sigma(\bullet))) = \sum_k p^k \cdot H_{\sigma_k^{-1}}(\sigma(\bullet))J_\sigma$, where $H_\bullet$ denotes the (symmetric) Hessian matrix. Consequently, $J_\sigma^T C = \sum_k p^k \cdot J_\sigma^T H_{\sigma_k^{-1}}(\sigma(\bullet))J_\sigma$, being the sum of symmetric matrices, is symmetric. Thus, by *Proposition 2.8 (Symplectic Matrix)*(iv), the Jacobian of Σ is a symplectic matrix and, therefore, by *Proposition 2.9 (Symplectomorphism)*(iii), it is a symplectomorphism. ∎

Since in the later chapters we will often use the index notation for Jacobian matrices, we can use this proof as a warm-up and show another proof.

Second proof of Theorem 3.1. From *Darboux's Theorem 2.7* it follows that one can always find local coordinates such that the symplectic form can be represented as

$$\Omega := \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$$

and thus, since the components of the Jacobian $(J_\sigma(q))^i_j$ are scalars, we have

$$\begin{aligned}
\Sigma^*\Omega &= \sum_i (\mathrm{d}\sigma(q))^i \wedge (\mathrm{d}(J_\sigma^{-T}(q)p))_i \\
&= \sum_i \Big(\sum_j (J_\sigma(q))^i_j \mathrm{d}q^j\Big) \wedge \Big(\sum_k (J_\sigma^{-1}(q))^k_i \mathrm{d}p_k\Big) \\
&= \sum_{i,j,k} \big((J_\sigma(q))^i_j (J_\sigma^{-1}(q))^k_i\big) \mathrm{d}q^j \wedge \mathrm{d}p_k \\
&= \sum_{j,k} \delta^k_j \mathrm{d}q^j \wedge \mathrm{d}p_k \\
&= \sum_k \mathrm{d}q^k \wedge \mathrm{d}p_k = \Omega,
\end{aligned}$$

where in the second line it was used that $(J_\sigma^{-T}(q))^i_k = (J_\sigma^{-1}(q))^k_i$. ∎

Using the cotangent lift above, we assign to each diffeomorphism on a manifold a specific symplectomorphism, preserving canonical coordinates. In fact, we will bring a specific type of diffeomorphism into focus.

Definition 3.2 (Spectral Diffeomorphism)**.** Let $T^*\mathbb{R}^n := \mathbb{R}^n \times \widehat{\mathbb{R}}^n$ denote the cotangent bundle of the Euclidean manifold $\mathbb{R}^n$, with $\widehat{\mathbb{R}}^n$ denoting its dual. Then, $\widehat{\mathbb{R}}^n$ is a differentiable manifold and a diffeomorphism

$$\sigma : \mathrm{dom}(\sigma) \subseteq \widehat{\mathbb{R}}^n \to \mathrm{im}(\sigma) \subseteq \widehat{\mathbb{R}}^n$$

will be referred to as a *spectral diffeomorphism* or *spectral warp.* ◆

Since these kinds of diffeomorphisms necessarily deform the domains of functions and measures, it is inevitable that with each spectral diffeomorphism comes along a

(i) a *spectral cotangent lift* to the subset $\mathbb{R}^n \times \mathrm{dom}(\sigma)$ of phase space,

(ii) two kinds of *symplectic flows*, induced by the *spectral cotangent lift*,

(iii) three *spectral measure spaces*, defined on $\mathrm{dom}(\sigma), \mathrm{im}(\sigma)$ and $\mathbb{R}^n \times \mathrm{im}(\sigma)$,

(iv) two spectral *unitary dilation operators*,

(v) two *spectral reservoirs*, interpreted as signals and "admissible windows" and represented as abstract Hilbert spaces of equivalence classes of functions, square-integrable with respect to the former two measures above,

(vi) a reproducing kernel Hilbert spaces of genuine functions on $\mathbb{R}^n \times \mathrm{im}(\sigma)$, square-integrable with respect to the latter of the three measures above,

(vii) two quadratic phase space distributions,

(viii) and various mappings interconnecting these.

This chapter is devoted to the definition of these companions of a spectral diffeomorphism and the last but one chapter of this text to the affiliation of this program with the various transforms in harmonic analysis and signal processing, often arising from group theory.

3.1 Symplectomorphisms induced by Spectral Diffeomorphisms

Lemma 3.3 (Spectral Cotangent Lift)**.** *Assume that σ is a spectral diffeomorphism. Then*

$$\Sigma_\sigma(q,p) := (J_\sigma^{-T}(p)q, \sigma(p)), \quad (q,p) \in \mathbb{R}^n \times \mathrm{dom}(\sigma) \tag{3.2}$$

is a cotangent lift, called the spectral cotangent lift. ◆

Proof. Since the classical phase space, $\mathbb{R}^n \times \widehat{\mathbb{R}}^n$, is self-dual, we may reinterpret $\mathbb{R}^n$ as the dual of $\widehat{\mathbb{R}}^n$ and thus phase space may be reinterpreted as the cotangent bundle of $\widehat{\mathbb{R}}^n$, i.e., $\mathbb{R}^n \times \widehat{\mathbb{R}}^n \simeq T^*\widehat{\mathbb{R}}^n$. Thus, the cotangent lift also applies to diffeomorphisms on $\widehat{\mathbb{R}}^n$. ■

Definition 3.4 (Spectral Measure Spaces)**.** If σ is a spectral diffeomorphism and Σ is the standard Borel-Sigma algebra of $\widehat{\mathbb{R}}^n$, denote with Σ_{dom} and Σ_{im} the induced Borel Sigma-Algebra of $\mathrm{dom}(\sigma) \subseteq \widehat{\mathbb{R}}^n$ and $\mathrm{im}(\sigma) \subseteq \widehat{\mathbb{R}}^n$ respectively. Consider $\mathrm{dom}(\sigma)$ as a subspace of $\widehat{\mathbb{R}}^n$ and equip it with the standard Lebesgue measure $\mathrm{d}x$ in n dimensions. Let $\mathrm{im}(\sigma)$ be equipped with the measure $\mathrm{d}\nu$, which depends on the structure of $\mathrm{im}(\sigma)$, but is absolutely continuous with respect to the Lebesgue measure on $\mathrm{im}(\sigma) \subseteq \widehat{\mathbb{R}}^n$, to wit,

$$\mathrm{d}\nu := \tfrac{\mathrm{d}\nu}{\mathrm{d}y}\,\mathrm{d}y,$$

where $\frac{\mathrm{d}\nu}{\mathrm{d}y}$ is the Radon-Nikodym derivative, *Definition A.16 (Radon-Nikodym)*, of ν with respect to $\mathrm{d}y$.

Then,

$$(\mathrm{dom}(\sigma), \Sigma_{\mathrm{dom}}, \mathrm{d}x) \quad\text{and}\quad (\mathrm{im}(\sigma), \Sigma_{\mathrm{im}}, \mathrm{d}\nu),$$

are measure spaces and the latter is Borel isomorphic to

$$(\mathrm{dom}(\sigma), \sigma^{-1}(\Sigma_{\mathrm{im}}), \mathrm{d}\sigma),$$

where $\mathrm{d}\sigma := \mathrm{d}\nu \circ \sigma$ is the pullback measure, *Definition A.18 (Pullback of a measure)*, of $\mathrm{d}\nu$ under σ.

Moreover, we have the measure space

$$(\mathbb{R}^n \times \mathrm{im}(\sigma), \Sigma \times \Sigma_{\mathrm{im}}, \mathrm{d}\mu)$$

with $\mathrm{d}\mu(x,y) := \mathrm{d}x\mathrm{d}\nu(y)$ and $\mathrm{d}\nu$ as above. ◆

Definition 3.5 (Spectral Reservoirs)**.** Let σ denote a spectral diffeomorphism, defined on $\mathrm{dom}(\sigma) \subseteq \widehat{\mathbb{R}}^n$. Then, we associate with it the following two Hilbert spaces.

(i) A Hilbert space of (equivalence classes of) functions, *interpreted as signals*, whose spectra are supported on $\mathrm{dom}(\sigma)$ and are square-integrable with respect to the standard Lebesgue measure on $\mathrm{dom}(\sigma)$

$$\mathcal{S}_\sigma := \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, \widehat{f} \in L^2(\mathrm{dom}(\sigma), \mathrm{d}x) \right\} \tag{3.3}$$

(ii) A Hilbert space of (equivalence classes of) functions, *interpreted as admissible windows* for an associated transform, whose spectra are supported on $\mathrm{dom}(\sigma)$ and are square-integrable with respect to $\mathrm{d}\sigma(\xi) := \mathrm{d}\nu \circ \sigma$

$$\mathcal{A}_\sigma := \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, \widehat{f} \in L^2(\mathrm{dom}(\sigma), \mathrm{d}\sigma) \right\}. \tag{3.4}$$

◆

Proposition 3.6 (Measure Mappings). *Let σ be a spectral diffeomorphism and $\mathcal{S}_\sigma$, $\mathcal{A}_\sigma$ as above. Then*

$$\iota_{\mathcal{A}\to\mathcal{S}} : \mathcal{A}_\sigma \to \mathcal{S}_\sigma, \psi \mapsto \mathcal{F}^* \sqrt{\tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi}} \mathcal{F}\psi \tag{3.5}$$

and its inverse

$$\iota_{\mathcal{A}\to\mathcal{S}}^{-1} = \iota_{\mathcal{S}\to\mathcal{A}} : \mathcal{S}_\sigma \to \mathcal{A}_\sigma, f \mapsto \mathcal{F}^* \left(\sqrt{\tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi}}\right)^{-1} \mathcal{F}f. \tag{3.6}$$

are unitary mappings from the space of signals *to* the space of admissible windows *and vice versa.* ◆

Proof. For $\iota_{\mathcal{A}\to\mathcal{S}}$, see

$$\begin{aligned}
\|\iota_{\mathcal{A}\to\mathcal{S}}\psi\|_{\mathcal{S}_\sigma}^2 &:= \int_{\mathrm{dom}(\sigma)} \left|\sqrt{\tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi}}\widehat{\psi}\right|^2 \,\mathrm{d}\xi \\
&= \int_{\mathrm{dom}(\sigma)} \left|\widehat{\psi}\right|^2 \tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi} \,\mathrm{d}\xi \\
&= \int_{\mathrm{dom}(\sigma)} \left|\widehat{\psi}\right|^2 \mathrm{d}\sigma = \|\psi\|_{\mathcal{A}_\sigma}^2
\end{aligned}$$

and

$$\begin{aligned}
\|\iota_{\mathcal{S}\to\mathcal{A}}f\|_{\mathcal{A}_\sigma}^2 := \int_{\mathrm{dom}(\sigma)} \left|(\sqrt{\tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi}})^{-1}\widehat{f}\right|^2 \mathrm{d}\sigma &= \int_{\mathrm{dom}(\sigma)} |f|^2 \,(\tfrac{\mathrm{d}\sigma}{\mathrm{d}\xi})^{-1}\mathrm{d}\sigma \\
&= \int_{\mathrm{dom}(\sigma)} |f|^2 \,\mathrm{d}\xi = \|f\|_{\mathcal{S}_\sigma}^2 ,
\end{aligned}$$

for its inverse. Note that this is a special case of *Lemma A.69* ■

As is always the case with coordinate transforms or "deformations" of a function's domain, an action on the domain gives rise to an action on the function space over the domain, which will be baptized as *warping.*

Proposition 3.7 (Spectral Warping Transform). *Assume that σ is a spectral diffeomorphism. Then, with*

$$j_\sigma := |\det J_\sigma|^{1/2},$$

(i) *the spectral warping transforms*

$$\widehat{\mathcal{W}}_\sigma : L^2(\mathrm{dom}(\sigma),\ \mathrm{d}\xi) \to L^2(\mathrm{im}(\sigma),\ \mathrm{d}\xi),\ \widehat{f} \mapsto j_{\sigma^{-1}} \cdot \widehat{f} \circ \sigma^{-1} \tag{3.7}$$

and

$$\widehat{\widehat{\mathcal{W}}}_\sigma : L^2(\mathrm{dom}(\sigma),\ \mathrm{d}\sigma) \to L^2(\mathrm{im}(\sigma),\ \mathrm{d}\xi),\ \widehat{f} \mapsto \widehat{f} \circ \sigma^{-1}. \tag{3.8}$$

are unitary.

(ii) *Furthermore, by extending functions, defined on the domain or the image of* σ *to all of* $\mathbb{R}^n$ *by defining them to be zero on the complements, we may conjugate the spectral warping transforms above with the Fourier transform, to get*

$$\mathcal{W}_\sigma : \mathcal{S}_\sigma \to L^2(\mathbb{R}^n,\ \mathrm{d}x), f \mapsto \mathcal{F}^* \widehat{\mathcal{W}}_\sigma \mathcal{F} f \tag{3.9}$$

and

$$\widetilde{\mathcal{W}}_\sigma : \mathcal{A}_\sigma \to L^2(\mathbb{R}^n,\ \mathrm{d}x), f \mapsto \mathcal{F}^* \widehat{\widetilde{\mathcal{W}}}_\sigma \mathcal{F} f\ . \tag{3.10}$$

◆

Proof. For (i), the calculations

$$\begin{aligned}
\left\|\widehat{\mathcal{W}}_\sigma f\right\|^2_{L^2(\mathrm{im}(\sigma),\ \mathrm{d}x)} &= \int_{\mathrm{im}(\sigma)} \left|j_{\sigma^{-1}} \cdot f \circ \sigma^{-1}\right|^2\ \mathrm{d}x \\
&= \int_{\mathrm{dom}(\sigma)} |f|^2\, j^2_{\sigma^{-1}} \mathrm{d}\sigma \\
&= \int_{\mathrm{dom}(\sigma)} |f|^2\ \mathrm{d}x = \|f\|_{L^2(\mathrm{dom}(\sigma),\ \mathrm{d}x)}
\end{aligned}$$

and

$$\begin{aligned}
\left\|\widehat{\mathcal{W}}_\sigma f\right\|^2_{L^2(\mathrm{im}(\sigma),\ \mathrm{d}x)} &= \int_{\mathrm{im}(\sigma)} \left|f \circ \sigma^{-1}\right|^2\ \mathrm{d}x \\
&= \int_{\mathrm{dom}(\sigma)} |f|^2\, \mathrm{d}\sigma = \|f\|_{L^2(\mathrm{dom}(\sigma),\mathrm{d}\sigma)}
\end{aligned}$$

show that these maps are unitary. Finally, (ii) follows, since the unitary operators constitute a group, the Fourier transform is unitary and composition of unitary maps is a unitary map again. ■

Remark 3.8. The operators $\mathcal{W}$ above are actually unitary representations of the group of diffeomorphisms sharing a common domain, i.e.,

$$\mathcal{W}_{\sigma_1 \circ \sigma_2} = \mathcal{W}_{\sigma_1} \circ \mathcal{W}_{\sigma_2}, \quad \mathrm{dom}(\sigma_1) \subseteq \mathrm{im}(\sigma_2)$$

as well as

$$\mathcal{W}^*_\sigma = \mathcal{W}^{-1}_\sigma = \mathcal{W}_{\sigma^{-1}} \quad \text{and} \quad \mathcal{W}_\sigma \circ \mathcal{W}_{\sigma^{-1}} = \mathcal{W}_{\sigma \circ \sigma^{-1}} := \mathcal{W}_{\mathbf{1}} := \mathbf{1}\,.$$

The same holds for $\widetilde{\mathcal{W}}$. ◆

Whenever a translation is defined on the image of σ, another interesting operator arises as the conjugation of the translation operator with the spectral warping operators above.

Lemma 3.9 (Spectral Dilation Operators)**.** *Let σ be a spectral warp, with* $\mathrm{im}(\sigma)$ *having the structure of a locally compact abelian group, with multiplication being written additively and translational invariant (Haar) measure* $\mathrm{d}x$*. Let moreover* $f \in \mathcal{S}_\sigma, \psi \in \mathcal{A}_\sigma$ *and* $T_\alpha : f \mapsto f(\bullet + \alpha)$ *denote the translation operator on the locally compact abelian group* $\mathrm{im}(\sigma)$*. Then, using*

$$j_\sigma := |\det J_\sigma|^{1/2},$$

we have that

$$\begin{aligned}(\widehat{\mathcal{D}}^\sigma_\alpha \widehat{f})(x) &:= (\mathcal{W}^*_\sigma T_\alpha \mathcal{W}_\sigma \widehat{f})(x) \\ &= j_\sigma(x) j_{\sigma^{-1}}(\sigma(x) + \alpha) \widehat{f}(\sigma^{-1}(\sigma(x) + \alpha))\end{aligned}$$

and

$$\begin{aligned}(\widehat{\widetilde{\mathcal{D}}}{}^\sigma_\alpha \widehat{\psi})(x) &:= (\widehat{\mathcal{W}}^* T_\alpha \widehat{\mathcal{W}} \widehat{\psi})(x) \\ &= \widehat{\psi}(\sigma^{-1}(\sigma(x) + \alpha))\end{aligned}$$

are unitary operators.

Conjugating these with the Fourier transform, we get the operators

$$\mathcal{D}^\sigma_\alpha := \mathcal{F}^* \widehat{\mathcal{D}}^\sigma_\alpha \mathcal{F} \quad \text{and} \quad \widetilde{\mathcal{D}}^\sigma_\alpha := \mathcal{F}^* \widehat{\widetilde{\mathcal{D}}}{}^\sigma_\alpha \mathcal{F},$$

which are again unitary. ◆

Proof. Since the translation action on a locally compact abelian group is well-defined and the associated Haar measure is invariant, translation is a unitary operation. Finally, since all other operators are unitary, their composition is unitary again, which concludes the proof. ■

Remark 3.10 (Dilation Operators)**.** Note that in the case of a "traditional", one-dimensional dilation of the $x \mapsto e^a \cdot x$, as e.g., it is used in the wavelet transform, the operator $\widetilde{\mathcal{D}}^\sigma_\alpha$ above reads

$$(\widetilde{\mathcal{D}}_\alpha \psi)(t) = e^{-\alpha} \psi(e^{-\alpha} t) \text{ and } (\widehat{\widetilde{\mathcal{D}}_\alpha} \widehat{\psi})(\xi) = \widehat{\psi}(e^\alpha t),$$

as opposed to the more usual version,

$$(\mathcal{D}_\alpha f)(t) = e^{-\alpha/2} f(e^{-\alpha} t) \text{ and } (\widehat{\mathcal{D}_\alpha} \widehat{f})(\xi) = e^{\alpha/2} \widehat{f}(e^\alpha t),$$

having symmetric normalization factors. The former is the way to go for admissible wavelets, since these are defined on $\mathcal{A}_\sigma$, for which the $\widetilde{\mathcal{D}}^\sigma_\alpha$ are unitary. ◆

3.2 Hamiltonians induced by Spectral Diffeomorphisms

Having defined the cotangent lift of the spectral diffeomorphism - its "symplectomorphization" - and its spectral dilation operators, it is worthwhile, before heading to the first main theorem of this chapter, to discuss the implications, as these are not obvious.

Start with the standard phase space, $\mathbb{R}^{2n}$, and equip it with the canonical coordinates, given by (q,p). Then, as laid out in the previous chapter, the coordinate functions induce flows along their respective contour lines - symmetries of the system -, meaning that $(q,p) \mapsto q$ induces flows in p direction for constant values of q and $(q,p) \mapsto p$ induces flows in q direction for constant values of p - that is, flows along their respective coordinate lines. A symplectic map like the one above now maps these standard canonical coordinates to another set of canonical coordinates

$$(q,p) \mapsto (J_\sigma^{-T}(p)q, \sigma(p)) \;, \quad (q,p) \in \mathbb{R}^n \times \mathrm{dom}(\sigma),$$

and accordingly, the induced flows are along the contour lines of the Hamiltonians

$$(q,p) \mapsto (J_\sigma^{-T}(p)q)^i \quad \text{and} \quad (q,p) \mapsto \sigma_j(p), \quad i,j = 1,\dots,n.$$

As discussed, via the Weyl correspondence, Schrödinger's equation and Stone's theorem, to each of these symplectic flows corresponds a family of unitary operators, playing the role of a quantized variant of these, acting on states in a Hilbert space. The dilation operators in *Lemma 3.9 (Spectral Dilation Operators)* above, now are exactly these quantized flows of the former kind, i.e., of the Hamiltonians

$$(q,p) \mapsto (J_\sigma^{-T}(p)q)^i, \quad i = 1,\dots,n,$$

acting on the Hilbert spaces $\mathcal{S}_\sigma$, in the case of $\mathcal{D}^\sigma_\alpha$, and on $\mathcal{A}_\sigma$, for $\widetilde{\mathcal{D}}^\sigma_\alpha$.

Thus, strictly speaking,

(i) to each spectral diffeomorphism on a $2n$-dimensional phase space, correspond n spectral dilation operator, one for each q coordinate, that is,

$$(q,p) \mapsto (J_\sigma^{-T}(p)q)^i =: H^i \;\Rightarrow\; \mathcal{D}^{\sigma^i}_{\alpha^i} \text{ or } \widetilde{\mathcal{D}}^{\sigma^i}_{\alpha^i},$$

where $H^i := (J_\sigma^{-1}(p')q')^i$ is the i-th Hamiltonian, inducing the unitary flow via

$$\psi \mapsto e^{-2\pi i \widehat{H^i}\alpha^i}\psi =: \mathcal{D}^{\sigma^i}_{\alpha^i}\psi, \tag{3.11}$$

(ii) and another family of unitary flows, one for each p coordinate, defined as

$$\psi \mapsto e^{-2\pi i \widehat{H_j}\beta_j}\psi =: \mathcal{T}^{\sigma_j}_{\beta_j}\psi, \tag{3.12}$$

where $H_j := \sigma(p')_j$, the j-th component of σ, is the j-th Hamiltonian.

Lemma 3.11 (Spectral-warped Translation Operator)**.** *Assume again that σ is a spectral warp and $\psi \in \mathcal{S}_\sigma$ or $\psi \in \mathcal{A}_\sigma$.*

Then

$$\begin{aligned}\widehat{(\mathcal{T}^\sigma_\beta \psi)}(\xi) &:= \widetilde{\mathcal{W}}^*_\sigma M_\beta \widetilde{\mathcal{W}}_\sigma \widehat{\psi}(\xi) \\ &= \widehat{\psi}(\xi) e^{-2\pi i \langle\, \sigma(\xi)\,,\, \beta\,\rangle}, \quad \beta \in \mathbb{R}^n, \xi \in \mathrm{dom}(\sigma),\end{aligned}$$

is the unitary flow, associated with

$$(x,\xi) \mapsto \sigma(\xi), \tag{3.13}$$

interpreted as a vector-valued Hamiltonian. ◆

Proof. Unitarity is immediate from

$$\int \left|\widehat{\psi}(\xi) e^{-2\pi i \langle\, \sigma(\xi)\,,\, \beta\,\rangle}\right|^2 \,\mathrm{d}\nu = \int \left|\widehat{\psi}(\xi)\right|^2 \,\mathrm{d}\nu,$$

where $\mathrm{d}\nu$ is either $\mathrm{d}x$ or $\mathrm{d}\sigma$. ■

Theorem 3.12 (Hamiltonian)**.** *Let σ be a spectral diffeomorphism. Then*

$$(q,p) \mapsto (J_\sigma^{-T}(p)q)^i =: H^i \quad \textit{and} \quad (q,p) \mapsto \sigma_j(p) =: H_j, \quad i,j = 1,\dots,n,$$

are Hamiltonians which induce the canonical unitary flows

$$\psi \mapsto e^{-2\pi i \widehat{H^i}\alpha^i} f =: \mathcal{D}^{\sigma^i}_{\alpha^i} f, \quad f \in \mathcal{S}_\sigma, \tag{3.14}$$

and

$$\psi \mapsto e^{-2\pi i \widehat{H_j}\beta_j} f =: \mathcal{T}^{\sigma_j}_{\beta_j} f, \quad f \in \mathcal{S}_\sigma. \tag{3.15}$$

Both flows can also be defined on $\mathcal{A}_\sigma$. ◆

Proof. Combine *Lemma 3.9 (Spectral Dilation Operators)* and *Lemma 3.11 (Spectral-warped Translation Operator)* to find the expressions for the unitary flows. The final claim is due to the fact, that the map $\iota_{\mathcal{S}\to\mathcal{A}}$ maps $\mathcal{S}_\sigma$ unitarily into $\mathcal{A}_\sigma$. ■

With this in mind, it is almost trivial to show that we may utilize the composition of the unitary families of operators above to define an action on some prototype function, such that its associated phase space distribution is shifted along the contour lines such that each point on phase space is reached.

Remark 3.13 (Panta rhei)**.** Indeed, that is what a set of coordinates can actually be defined to be. A grid, such that starting from an arbitrary point on the coordinate lines, the chosen origin, we may reach every point on the grid, for which it is defined - in our case, this is the set $\mathbb{R}^n \times \mathrm{dom}(\sigma)$. We define the coordinates with respect to the chosen origin, by "flowing" along the $2 \cdot n$ coordinate lines for specific amounts of time, determined by the unique tuple of $2 \cdot n$ numbers.

This, of course, continues to hold for the quantized version of phase space, since the quantized phase space cell, as defined in *Definition 2.42 (Phase Space Localization)*, associates to each classical point an "ensemble" of other points, each of which now flows along the coordinate lines it is located on. Thus, still, each and every point in the local coordinate chart is reached - once, for each point in the quantized phase space cell. This introduces a redundancy in the description of the system, which is ultimately a manifestation of the quantum mechanical uncertainty principle. ◆

Before defining the all-important unitary action, by which we will build frames, associated with a spectral warp, another observation is needed, as the next chapter relies on it.

Lemma 3.14 (Spectral Hamiltonians)**.** *Let σ be an* analytic *spectral diffeomorphism and*

$$A^i := (J_\sigma^{-T}(p)q)^i, \quad B_i' := \sigma_i(p), \quad B_i := p_i \tag{3.16}$$

the (canonical and non-canonical) Hamiltonians, then the quantized Hamiltonians, represented in the Fourier domain, are

$$\widehat{A^i}\hat{f} = \tfrac{-1}{2\pi i}\sum_k \left(\tfrac{1}{2}\partial_{p_k}(j_k^i(p)) + j_k^i(p)\partial_{p_k}\right)\hat{f} \tag{3.17}$$

$$\widehat{B}_i\hat{f} = p_i\hat{f} \tag{3.18}$$

$$\widehat{B}_i'\hat{f} = \sigma_i(p)\hat{f}, \tag{3.19}$$

where $j_k^i := (J^{-T})_k^i$, are the components of the transposed inverse of the Jacobian, J_σ, the k in x^k is an upper index and not a power and f is assumed to be in the appropriate domains of these operators. ◆

Proof. Since σ is analytic, the components of the Jacobian are analytic functions, for which *Corollary 2.23 (Quantization)*(ii) applies and since the components of the Jacobian are are depended on p only, these quantize to multiplication operators on the Fourier domain. Thus, setting $j^i_k(p) := (J^{-T}(p))^i_k$, which is a multiplication operator on the Fourier domain, and noting that q^k quantizes to $\widehat{q^k} := \frac{-1}{2\pi i}\partial_{p_k}$ on the Fourier domain, we get

$$\begin{aligned}
\widehat{A^i}\hat{f} &:= (\widehat{J_\sigma^{-T}(p)q})^i\hat{f} \\
&:= \sum_k \widehat{j^i_k(p)q^k}\hat{f} \\
&= \tfrac{1}{2}\sum_k(\widehat{j^i_k(p)}\widehat{q^k} + \widehat{q^k}\widehat{j^i_k(p)}) \\
&= \tfrac{-1}{2\pi i}\tfrac{1}{2}\sum_k(j^i_k(p)\partial_{p_k} + \partial_{p_k}j^i_k(p)) \\
&= \tfrac{-1}{2\pi i}\tfrac{1}{2}\sum_k(2j^i_k(p)\partial_{p_k} + \partial_{p_k}(j^i_k(p))) \\
&= \tfrac{-1}{2\pi i}\sum_k(\tfrac{1}{2}\partial_{p_k}(j^i_k(p)) + j^i_k(p)\partial_{p_k}).
\end{aligned}$$

B_i and B'_i are immediate, since these only depend on p and thus, they are only reinterpreted as operators of multiplication on the Fourier domain. ∎

Corollary 3.15 (Commutators of Spectral Hamiltonians)**.** *Let σ be an* analytic *spectral diffeomorphism, $J := J_\sigma$ its Jacobian and $\widehat{A}^i, \widehat{B}_i, \widehat{B}'_i$ its quantized Hamiltonians. Then,*

(i) $\left[\, \widehat{A}^i \,,\, \widehat{B}'_k \,\right] = \frac{-1}{2\pi i}\delta^i_k, \quad i, k = 1, \ldots, n,$ *and*

(ii) $\left[\, \widehat{A}^i \,,\, \widehat{B}_k \,\right] = \frac{-1}{2\pi i}(J^{-T})^i_k, \quad i, k = 1, \ldots, n.$ ◆

Proof. To see (i) and (ii), we represent the A^i in the Fourier domain and take some function, ρ, on the same domain to calculate

$$\begin{aligned}
\left[\, \widehat{A}^i \,,\, \rho \,\right] &= \tfrac{-1}{2\pi i}\sum_m \left[\, \tfrac{1}{2}\partial_{p_m}(j^i_m(p)) + j^i_m(p)\partial_{p_m} \,,\, \rho \,\right] \\
&= \tfrac{-1}{2\pi i}\sum_m \left[\, j^i_m(p)\partial_{p_m} \,,\, \rho \,\right] \\
&= \tfrac{-1}{2\pi i}\sum_m j^i_m(p)\left[\, \partial_{p_m} \,,\, \rho \,\right] \\
&= \tfrac{-1}{2\pi i}\sum_m j^i_m(p)\partial_{p_m}(\rho).
\end{aligned}$$

Now, for (i), set $\rho := B'_k := \sigma_k(p)$ and resubstitute $j^i_m := (J^{-T})^i_m$ to find

$$\begin{aligned}
\tfrac{-1}{2\pi i}\sum_m j^i_m(p)\partial_{p_m}(\sigma_k(p)) &= \tfrac{-1}{2\pi i}\sum_m j^i_m(p)(J^T)^m_k(p) \\
&= \tfrac{-1}{2\pi i}\sum_m (J^{-T})^i_m(p)(J^T)^m_k(p) \\
&= \tfrac{-1}{2\pi i}\sum_m (J^{-T})^i_m(p)(J^T)^m_k(p) \\
&= \tfrac{-1}{2\pi i}\delta^i_k.
\end{aligned}$$

(ii) follows by setting $\rho \coloneqq B_k \coloneqq p_k$ and

$$\begin{aligned}\tfrac{-1}{2\pi i}\sum_m j^i_m(p)\partial_{p_m}(p_k) &= \tfrac{-1}{2\pi i}\sum_m j^i_m(p)\delta^m_k \\ &= \tfrac{-1}{2\pi i} j^i_k(p) \\ &= \tfrac{-1}{2\pi i}(J^{-T})^i_k(p).\end{aligned}$$

∎

Before moving on, two examples shall demonstrate the above for the n-dimensional Weyl-Heisenberg group and the simple n-dimensional affine group.

Example 3.1 (Weyl-Heisenberg group)**.** For the Weyl-Heisenberg group, the diffeomorphism is the identical one, that is, $\sigma : (p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n)$, with

$$J_\sigma = J^{-T} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Thus, the Hamiltonians boil down to the canonical ones

$$\widehat{A}^i \hat{f} = \tfrac{-1}{2\pi i}\partial_{p_i}\hat{f} \tag{3.20}$$
$$\widehat{B}_i \hat{f} = p_i \hat{f} \tag{3.21}$$
$$\widehat{B}'_i \hat{f} = p_i \hat{f} \tag{3.22}$$

along with the canonical commutation relations

(i) $\left[\widehat{A}^i, \widehat{B}'_k\right] = \tfrac{-1}{2\pi i}\delta^i_k, \quad i, k = 1, \ldots, n$, and

(ii) $\left[\widehat{A}^i, \widehat{B}_k\right] = \tfrac{-1}{2\pi i}\delta^i_k, \quad i, k = 1, \ldots, n$. ◆

Example 3.2 (Affine group)**.** For the n-dimensional "ax+b" group, we have that $\sigma : (p_1, \ldots, p_n) \mapsto (\log p_1, \ldots, \log p_n)$ and thus

$$J_\sigma = \begin{pmatrix} \frac{1}{p_1} & & \\ & \ddots & \\ & & \frac{1}{p_n} \end{pmatrix}$$

and

$$J_\sigma^{-T} = \begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{pmatrix}.$$

From this, it follows that the spectral Hamiltonians are

$$\widehat{A}^i \hat{f} = \tfrac{-1}{2\pi i}\tfrac{1}{2}(p_i\partial_{p_i} + \partial_{p_i}p_i)\hat{f} = \tfrac{-1}{2\pi i}(\tfrac{1}{2} + p_i\partial_{p_i})\hat{f} \tag{3.23}$$
$$\widehat{B}_i \hat{f} = p_i \hat{f} \tag{3.24}$$
$$\widehat{B}'_i \hat{f} = \log(p_i)\hat{f} \tag{3.25}$$

with the commutation relations

(i) $\left[\widehat{A}^i \,,\, \widehat{B}'_k \right] = \frac{-1}{2\pi i}\delta^i_k, \quad i,k = 1,\dots,n$, and

(ii) $\left[\widehat{A}^i \,,\, \widehat{B}_k \right] = \frac{-1}{2\pi i}\delta^i_k \widehat{B}_k, \quad i,k = 1,\dots,n$,

where the commutation relations in (i) resemble the canonical commutation relations, since the operators $\widehat{A}^i$ and $\widehat{B}'_k$ are unitarily equivalent to the canonical ones. ◆

Now, we may introduce a unitary action, which translates a prototype function in phase space, along (canonical) coordinates.

Proposition 3.16 (Canonical Unitary Action)**.** *As usual, assume that σ is a spectral warp and that $\mathcal{T},\mathcal{D}$ are the associated spectral translation and dilation operator. If furthermore $\psi \in \mathcal{A}_\sigma$ and $f \in \mathcal{S}_\sigma$, then*

$$\pi \circ \Sigma(\beta,\alpha)\psi = \mathcal{T}^\sigma_\beta \mathcal{D}^\sigma_\alpha f, \quad \alpha,\beta \in \mathbb{R}^n, \tag{3.26}$$

and

$$\widetilde{\pi \circ \Sigma}(\beta,\alpha)\psi = \mathcal{T}^\sigma_\beta \widetilde{\mathcal{D}}^\sigma_\alpha \psi, \quad \alpha,\beta \in \mathbb{R}^n, \tag{3.27}$$

are unitary. ◆

Proof. The unitarity of both is immediate from the group property of unitary operators. ■

This action is in fact a unitary and projective group representation, unitarily equivalent to a representation of the Weyl-Heisenberg group. This is a simple consequence of the

Stone-von Neumann Theorem 3.17. *[71] Let T^1 and T^2 be self-adjoint and fulfill the* canonical commutation relation*, then T^1 and T^2 are unitarily equivalent to $\frac{1}{i2\pi}\partial_x$ and x.*

Put another way, let $e^{-2\pi i T^1\alpha}$ and $e^{-2\pi i T^2\beta}$ be strongly continuous unitary one-parameter groups, then if

$$e^{-2\pi i T^1\alpha} e^{-2\pi i T^2\beta} = e^{-2\pi i\beta\alpha} e^{-2\pi i T^2\beta} e^{-2\pi i T^1\alpha},$$

holds, then these operators are unitarily equivalent to the (projective) Schrödinger representation of the Weyl-Heisenberg group and thus there exists a unitary intertwining operator $\mathcal{W}$, such that

$$\mathcal{W} e^{-2\pi i T^1\alpha} \mathcal{W}^* =: e^{-2\pi i \frac{1}{i2\pi}\partial_x\alpha} \quad \textit{and} \quad \mathcal{W} e^{-2\pi i T^2\beta} \mathcal{W}^* =: e^{-2\pi i x\beta} \ .$$

◆

Corollary 3.18 (Unitary Equivalence). *The unitary action $\pi \circ \Sigma$ is a projective unitary group representation of the Weyl-Heisenberg group.* ◆

Proof. Since the spectral cotangent lift of σ is a symplectomorphism, we have

$$\left\{ (J_\sigma^{-1}(p)q)^i \,,\, \sigma(p)_j \right\} = \left\{ q^i \,,\, p_j \right\} = \delta^i_j,$$

which, along with heuristic equation (*2.27*), suggests that the quantized operators $\widehat{(\sigma(q))}_i$ and $(\widehat{J_\sigma(p)q})^j$ also fulfill

$$2\pi i \Big[(\widehat{J_\sigma^{-1}(p)q})^i \,,\, \widehat{\sigma(p)}_j \Big] = 2\pi i \left[\widehat{q}^i \,,\, \widehat{p}_j \right] = \pm \delta^i_j.$$

Indeed, this follows from *Corollary 3.15 (Commutators of Spectral Hamiltonians)*.

The *Stone-von Neumann Theorem 3.17*, then, tells us that there exists an intertwining operator, $\mathcal{W}$, under which this representation is unitary equivalent to a (projective) representation of the Weyl-Heisenberg group, which was to be proven. ■

Although this was used in the proof of *Corollary 3.15 (Commutators of Spectral Hamiltonians)*, it is nonetheless worthwhile to note that the arising intertwining operator in the last proof above is in fact the spectral warping transform, $\mathcal{W}_\sigma$, and therefore, for each pair of conjugate and, thus, non-commuting operators, we have

$$\mathcal{W}_\sigma e^{-2\pi i (\widehat{J_\sigma(p)q})^j \beta} \mathcal{W}_\sigma^* =: e^{-2\pi i \frac{1}{i2\pi}\partial_x \alpha} \quad \text{and} \quad \mathcal{W}_\sigma e^{-2\pi i \widehat{(\sigma(q))}_i \beta} \mathcal{W}_\sigma^* =: e^{-2\pi i x \beta} \;.$$

Although, as used above, the canonically conjugate coordinates to $J_\sigma^{-1}(p)q$ are given by $\sigma(p)$, reflected by invariance under Poisson's bracket

$$\left\{ (J_\sigma^{-1}(p)q)^i \,,\, \sigma(p)_j \right\} = \left\{ q^i \,,\, p_j \right\} = \delta^i_j,$$

we shall not only use the spectral translation operator, $\mathcal{T}^\sigma_\beta$, but also the "standard" translation operator

$$\mathcal{T}^{\mathbf{1}}_\beta := T_\beta = e^{-2\pi i \langle\, \hat{p} \,,\, \beta \,\rangle},$$

given as the quantized unitary flow of the standard coordinate p. The reason for this is that although the canonical ones are in a sense the optimal choice, some applications demand that signals may be decomposed into "spectral channels", meaning that the "form" of the projector's phase space distribution should not change through translation along the spatial coordinate.

Thus, another proposition is in order.

Proposition 3.19 (Non-canonical Unitary Action). *Let σ be a spectral warp, $\mathcal{D}_\alpha^\sigma$ be the associated spectral dilation, $\psi \in \mathcal{A}_\sigma$, $f \in \mathcal{S}_\sigma$ and let furthermore T_β denote the standard translation operator. Then we define*

$$\pi^\sigma(\beta,\alpha)f = T_\beta \mathcal{D}_\alpha^\sigma f, \quad \alpha,\beta \in \mathbb{R}^n, \tag{3.28}$$

and

$$\widetilde{\pi^\sigma}(\beta,\alpha)\psi = T_\beta \widetilde{\mathcal{D}}_\alpha^\sigma \psi, \quad \alpha,\beta \in \mathbb{R}^n, \tag{3.29}$$

which are, again, unitary operators. ◆

Proof. Again, since unitary operators constitute a group, the claim follows. ■

Although we try to minimize the algebraic preconditions, since we want to translate in the image of the spectral diffeomorphism, everything works as nicely as possible, if we restrict to cases for which there exists a locally compact abelian group structure on $\mathrm{im}(\sigma)$.

Now, here comes the first main theorem for the chapter. In the following theorem, the actions, unitary on $\mathcal{A}_\sigma$, are used to build frames. Almost all of what follows stems from the fact that to each spectral warp is assigned a pair of *spectral quantum frames*.

Theorem 3.20 (Spectral Quantum Frames). *Let σ be a spectral diffeomorphism, with $\mathrm{im}(\sigma)$ constituting a locally compact abelian group and denote its Haar measure with $\mathrm{d}\nu$. Let moreover $\varphi \in \mathcal{A}_\sigma$, with*

$$c_\varphi := \|\varphi\|_{\mathcal{A}_\sigma}^2 = \int_{\mathrm{dom}(\sigma)} |\varphi|^2 \, \mathrm{d}\sigma = \int_{\mathrm{im}(\sigma)} \left|\varphi \circ \sigma^{-1}\right|^2 \, \mathrm{d}\nu. \tag{3.30}$$

Then, the following holds.

(i) *With the measure $\mathrm{d}\mu(x,y) := \mathrm{d}x \, \mathrm{d}\nu(y)$ on $\mathbb{R}^n \times \mathrm{im}(\sigma)$, the family*

$$\mathbb{F}_\sigma := \left\{ \widetilde{\pi^\sigma}(x,y)\varphi \mid (x,y) \in (\mathbb{R}^n \times \mathrm{im}(\sigma), \ \mathrm{d}\mu) \right\} \tag{3.31}$$

is a continuous tight frame for $\mathcal{S}_\sigma$, with frame bounds $A = B = c_\varphi$, called the spectral quantum frame*.*

(ii) *With the measure $\mathrm{d}x \, \mathrm{d}y$ on $\mathbb{R}^n \times \mathrm{im}(\sigma)$,*

$$\mathbb{G}_\sigma := \left\{ \widetilde{\pi \circ \Sigma}(x,y)\varphi \mid (x,y) \in (\mathbb{R}^n \times \mathrm{im}(\sigma), \ \mathrm{d}x \, \mathrm{d}y) \right\} \tag{3.32}$$

is a continuous tight frame for $\mathcal{A}_\sigma$, with frame bounds $A = B = c_\varphi$, called the canonical spectral quantum frame*.* ◆

Proof. Suppressing the integration domains, a straight-forward calculation gives

$$\begin{aligned}
&\iint \overline{\langle \widetilde{\pi}^\sigma(b,a)\varphi\,,\, f\,\rangle}\ \langle \widetilde{\pi}^\sigma(b,a)\varphi\,,\, h\,\rangle\ \mathrm{d}\mu \\
&= \iiiint \widehat{\varphi}(\sigma^{-1}(\sigma(\xi)+a))\overline{\widehat{\varphi}(\sigma^{-1}(\sigma(\xi')+a))}\overline{\widehat{f}(\xi)}\widehat{h}(\xi')e^{-2\pi i\langle\, \xi-\xi'\,,\, b\,\rangle}\ \mathrm{d}\xi\ \mathrm{d}\xi'\ \mathrm{d}b\ \mathrm{d}\nu \\
&= \iint \left|\widehat{\varphi}(\sigma^{-1}(\sigma(\xi)+a))\right|^2 \overline{\widehat{f}(\xi)}\widehat{h}(\xi)\ \mathrm{d}\xi\ \mathrm{d}\nu \\
&= \iint \left|\widehat{\varphi}(\sigma^{-1}(a))\right|^2 \overline{\widehat{f}(\xi)}\widehat{h}(\xi)\ \mathrm{d}\xi\ \mathrm{d}\nu \\
&= \int |\widehat{\varphi}(\zeta)|^2\ \mathrm{d}\sigma(\zeta) \int \overline{\widehat{f}(\xi)}\widehat{h}(\xi)\ \mathrm{d}\xi \\
&= \|\varphi\|^2_{\mathcal{A}_\sigma} \langle\, f\,,\, h\,\rangle_{\mathcal{S}_\sigma} = c_\varphi \langle\, f\,,\, h\,\rangle_{\mathcal{S}_\sigma}\,,
\end{aligned}$$

which shows that

$$c_\varphi^{-1} \iint |\widetilde{\pi}^\sigma(b,a)\varphi\rangle\langle\widetilde{\pi}^\sigma(b,a)\varphi|\ \mathrm{d}\mu(b,a) = 1_{\mathcal{S}_\sigma}$$

is a resolution of the identity, which - by *Definition 2.32 (Continuous Frame)* - identifies it as a continuous frame having a resolution operator which is a multiple of the identity.

To see (ii), note that

$$\begin{aligned}
\langle\, \pi\circ\Sigma(x,y)\varphi\,,\, f\,\rangle &:= \left\langle\, \widetilde{\mathcal{W}}_\sigma^* T_x M_y \widetilde{\mathcal{W}}_\sigma\varphi\,,\, f\,\right\rangle_{\mathcal{A}_\sigma} \\
&:= \left\langle\, T_x M_y \widetilde{\mathcal{W}}_\sigma\varphi\,,\, \widetilde{\mathcal{W}}_\sigma f\,\right\rangle_{L^2(\mathbb{R}^n)} \\
&:= \left\langle\, T_x M_y \widetilde{\varphi}\,,\, \widetilde{f}\,\right\rangle_{L^2(\mathbb{R}^n)}\,,
\end{aligned}$$

for $f \in A_\sigma$, $\widetilde{\varphi} := \widetilde{\mathcal{W}}_\sigma\varphi \in L^2(\mathbb{R}^n)$ and $\widetilde{f} := \widetilde{\mathcal{W}}_\sigma f \in L^2(\mathbb{R}^n)$. Then, using a shortened calculation like the one above

$$\begin{aligned}
&\iint \overline{\pi_\varphi f}\ \pi_\varphi h\ \mathrm{d}b\ \mathrm{d}a \\
&= \iiiint \widehat{\varphi}(\sigma^{-1}(\sigma(\xi)+a))\overline{\widehat{\varphi}(\sigma^{-1}(\sigma(\xi')+a))} \\
&\times \overline{\widehat{f}(\xi)}\widehat{h}(\xi')e^{-2\pi i\langle\, \sigma(\xi)-\sigma(\xi')\,,\, b\,\rangle}\mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\xi')\ \mathrm{d}b\ \mathrm{d}a \\
&= \iiiint \widehat{\widetilde{\varphi}}(y+a)\overline{\widehat{\widetilde{\varphi}}(y'+a)}\overline{\widehat{\widetilde{f}}(y)}\widehat{\widetilde{h}}(y')e^{-2\pi i\langle\, y-y'\,,\, b\,\rangle}\ \mathrm{d}y\ \mathrm{d}y'\ \mathrm{d}b\ \mathrm{d}a \\
&= \iint \left|\widehat{\widetilde{\varphi}}(y+a)\right|^2 \overline{\widehat{\widetilde{f}}(y)}\widehat{\widetilde{h}}(y)\ \mathrm{d}y\ \mathrm{d}a \\
&= \|\varphi\|^2_{\mathcal{A}_\sigma} \left\langle\, \widetilde{f}\,,\, \widetilde{h}\,\right\rangle_{L^2(\mathbb{R}^n)} = c_\varphi \langle\, f\,,\, h\,\rangle_{\mathcal{A}_\sigma}\,,
\end{aligned}$$

which again shows that

$$c_\varphi^{-1} \iint |\widetilde{\pi}\circ\Sigma(b,a)\varphi\rangle\langle\widetilde{\pi}\circ\sigma(b,a)\varphi|\ \mathrm{d}b\ \mathrm{d}a = 1_{\mathcal{A}_\sigma}$$

is a resolution of the identity. ∎

Corollary 3.21 (Spectral Quantum Frames of Rank N). *Let again σ be a spectral diffeomorphism, with* $\mathrm{im}(\sigma)$ *constituting a locally compact abelian group and denote its Haar measure with* $\mathrm{d}\nu$. *Let moreover $\varphi^i \in \mathcal{A}_\sigma$, $i = 1, \ldots, n$, with*

$$c_{\varphi^i} := \left\|\varphi^i\right\|^2_{\mathcal{A}_\sigma} = \int_{\mathrm{dom}(\sigma)} \left|\varphi^i\right|^2 \, \mathrm{d}\sigma = \int_{\mathrm{im}(\sigma)} \left|\varphi^i \circ \sigma^{-1}\right|^2 \, \mathrm{d}\nu. \tag{3.33}$$

Then, the following holds.

(i) *With the measure* $\mathrm{d}\mu(x,y) := \mathrm{d}x \, \mathrm{d}\nu(y)$ *on* $\mathbb{R}^n \times \mathrm{im}(\sigma)$*, the family*

$$\mathbb{F}_\sigma := \left\{ \widetilde{\pi^\sigma}(x,y)\varphi^i \mid (x,y) \in (\mathbb{R}^n \times \mathrm{im}(\sigma), \, \mathrm{d}\mu), i = 1, \ldots, n \right\} \tag{3.34}$$

is a continuous tight frame of rank N for $\mathcal{S}_\sigma$, with frame bounds $A = B = \sum_i c_{\varphi^i}$, called the spectral quantum frame of rank N.

(ii) *With the measure* $\mathrm{d}x \, \mathrm{d}y$ *on* $\mathbb{R}^n \times \mathrm{im}(\sigma)$*, the family*

$$\mathbb{G}_\sigma := \left\{ \widetilde{\pi \circ \Sigma}(x,y)\varphi^i \mid (x,y) \in (\mathbb{R}^n \times \mathrm{im}(\sigma), \, \mathrm{d}x \, \mathrm{d}y), i = 1, \ldots, n \right\} \tag{3.35}$$

is a continuous tight frame of rank N for $\mathcal{A}_\sigma$,with frame bounds $A = B = \sum_i c_{\varphi^i}$, called the canonical spectral quantum frame of rank N. ◆

Proof. For (i), note that

$$\begin{aligned}
&\iint \sum_i \overline{\langle \widetilde{\pi}^\sigma(b,a)\varphi^i \, , \, f \rangle} \, \left\langle \widetilde{\pi}^\sigma(b,a)\varphi^i \, , \, h \right\rangle \, \mathrm{d}\mu \\
&= \sum_i \iint \overline{\langle \widetilde{\pi}^\sigma(b,a)\varphi^i \, , \, f \rangle} \, \left\langle \widetilde{\pi}^\sigma(b,a)\varphi^i \, , \, h \right\rangle \, \mathrm{d}\mu \\
&= \sum_i \left\langle f \, , \, \iint |\widetilde{\pi}^\sigma(b,a)\varphi^i \rangle \langle \widetilde{\pi}^\sigma(b,a)\varphi^i | \, \mathrm{d}\mu \; h \right\rangle_{\mathcal{S}_\sigma} \\
&= \sum_i \langle f \, , \, h \rangle_{\mathcal{S}_\sigma} \left\|\varphi^i\right\|^2_{\mathcal{A}_\sigma} = \langle f \, , \, h \rangle_{\mathcal{S}_\sigma} \left(\sum_i c_{\varphi^i} \right),
\end{aligned}$$

where the last line follows from *Theorem 3.20 (Spectral Quantum Frames)*.

A similar argument

$$\begin{aligned}
&\iint \sum_i \overline{\left\langle \widetilde{\pi \circ \Sigma}(b,a)\varphi^i \, , \, f \right\rangle} \, \left\langle \widetilde{\pi \circ \Sigma}(b,a)\varphi^i \, , \, h \right\rangle \, \mathrm{d}\mu \\
&= \sum_i \iint \overline{\left\langle \widetilde{\pi \circ \Sigma}(b,a)\varphi^i \, , \, f \right\rangle} \, \left\langle \widetilde{\pi \circ \Sigma}(b,a)\varphi^i \, , \, h \right\rangle \, \mathrm{d}\mu \\
&= \sum_i \left\langle f \, , \, \iint |\widetilde{\pi \circ \Sigma}(b,a)\varphi^i \rangle \langle \widetilde{\pi \circ \Sigma}(b,a)\varphi^i | \, \mathrm{d}\mu \; h \right\rangle_{\mathcal{A}_\sigma} \\
&= \sum_i \langle f \, , \, h \rangle_{\mathcal{A}_\sigma} \left\|\varphi^i\right\|^2_{\mathcal{A}_\sigma} = \langle f \, , \, h \rangle_{\mathcal{A}_\sigma} \left(\sum_i c_{\varphi^i} \right),
\end{aligned}$$

shows (ii). ∎

Corollary 3.22 (Spectral Reproducing Kernel Hilbert Spaces)**.** *Let σ be a spectral warp and $\mathbb{F}_\sigma$ and $\mathbb{G}_\sigma$ the associated spectral quantum frames, with π_ψ^σ and π_ψ^Σ denoting the respective coherent state maps. Then*

$$\mathcal{H}_\sigma := \pi_\psi^\sigma(\mathcal{S}_\sigma) \quad \textit{and} \quad \mathcal{H}_\Sigma := \pi_\psi^\Sigma(\mathcal{A}_\sigma), \tag{3.36}$$

are the associated spectral reproducing kernel Hilbert spaces. ◆

Proof. This follows from *Theorem 3.20 (Spectral Quantum Frames)* and *Corollary 2.40 (Reproducing Kernel Hilbert Space).* ■

Before moving on, a few final observations concerning the canonical one of the spectral quantum frames are in order, since it resembles essentially a warped Short-time Fourier transform.

Corollary 3.23 (Spectral-warped STFT)**.** *Assume that σ is a spectral warp, $f, \psi \in \mathcal{A}_\sigma$ and denote with $\mathcal{W}_\sigma$ the associated unitary spectral warping transform. Then, the coherent state map*

$$f \;\mapsto\; \pi_\psi^\Sigma f(b,a), \tag{3.37}$$

associated with the canonical spectral quantum frame $\mathbb{G}_\sigma$ is a spectral warped Short-time Fourier transform*, such that*

$$\pi_\psi^\Sigma f(b,a) := STFT_{\widetilde{\mathcal{W}}_\sigma \psi}(\widetilde{\mathcal{W}}_\sigma f) \tag{3.38}$$

holds. ◆

Proof. This follows from

$$\begin{aligned} f \;\mapsto\; \pi_\psi^\Sigma f(b,a) &= \left\langle\, \tilde{\mathcal{W}}_\sigma^* T_b T_a \tilde{\mathcal{W}}_\sigma \psi \,,\, f \,\right\rangle \\ &= \int_{\mathrm{dom}(\sigma)} e^{2\pi i \langle\, b \,,\, \sigma(\xi) \,\rangle} \overline{\widehat{\psi}\,(\sigma^{-1}(\sigma(\xi)+a))}\; \widehat{f}(\xi)\; \mathrm{d}\sigma(\xi) \\ &= \int_{\mathrm{im}(\sigma)} e^{2\pi i \langle\, b \,,\, \xi' \,\rangle} \overline{(\widehat{\psi} \circ \sigma^{-1})(\xi'+a)}\; (\widehat{f} \circ \sigma^{-1})(\xi')\; \mathrm{d}\xi' \\ &= \int_{\mathbb{R}^n} \overline{T_b M_a \tilde{\psi}(x)}\; \tilde{f}(x)\; \mathrm{d}x, \end{aligned}$$

with $\tilde{f} := \mathcal{W} f$ and $\tilde{\psi} := \mathcal{W}\psi$. ■

The following definition gives the generalization of the STFT for the case of a possibly sampled - to wit, discrete - phase space [26, 27].

Definition 3.24 (Gabor Frame)**.** Let $g \in L^2(\mathbb{R}^n,\ \mathrm{d}x)$ and $a, b \in \mathbb{R}^n$. Then, whenever

$$G(g,a,b) := \{\, T_{b\cdot n} M_{a\cdot m} g \mid n, m \in \mathbb{Z}^n \,\} \tag{3.39}$$

constitutes a frame, it will be referred to as a *Gabor frame*. For vanishing a, b, the convention

$$G(g,0,0) := \{\, T_\beta M_\alpha g \mid \alpha, \beta \in \mathbb{R}^n \,\} \tag{3.40}$$

defines a *continuous Gabor frame*, whose coherent state map is the well-known *Short-Time Fourier Transform.* ◆

This brings us to the final corollary of this section. It essentially says that in order to define a phase space tessellation along the canonical coordinates, given by the spectral cotangent lift, all that is needed is a Gabor frame for the standard coordinates. Then, the symplectomorphization of the spectral diffeomorphism, Σ, respectively its quantized variant, the spectral warping transform $\mathcal{W}_\sigma$, maps this standard Gabor frame to a unitarily equivalent one, but with localization properties adapted to the canonical reference frame, defined by $\Sigma(q,p)$.

Corollary 3.25 (Warped Gabor Frame)**.** *Assume that σ is a spectral diffeomorphism and $G(g,a,b)$ a Gabor frame for $L^2(\mathbb{R}^n)$, in the sense of* Definition 3.24 (Gabor Frame)*. Then*

$$G_\sigma(g,a,b) := \left\{\, \widetilde{\mathcal{W}}_\sigma^*\ T_{b\cdot n} M_{a\cdot m} g \mid n, m \in \mathbb{Z}^n \,\right\} \tag{3.41}$$

is a frame for $\mathcal{A}_\sigma$.

Moreover, if $a = b = 0$, this frame coincides with the canonical spectral quantum frame $\mathbb{G}_\sigma$ *for the window $\psi := \widetilde{\mathcal{W}}_\sigma^* g$.* ◆

Proof. Since, by *Proposition 3.7 (Spectral Warping Transform)*, $\widetilde{\mathcal{W}}_\sigma$ maps $\mathcal{A}_\sigma$ unitarily to $L^2(\mathbb{R}^n)$, we have by implication that $\widetilde{\mathcal{W}}^* T_{b\cdot n} M_{a\cdot m} g$ constitutes a frame for $\mathcal{A}_\sigma$.

This continues to hold for the case $a \to 0$ and $b \to 0$. Then,

$$\begin{aligned}
G_\sigma(g,0,0) &:= \left\{\, \widetilde{\mathcal{W}}_\sigma^* T_\beta M_\alpha \widetilde{\mathcal{W}}_\sigma (\widetilde{\mathcal{W}}_\sigma^* g) \mid \alpha, \beta \in \mathbb{R}^n \,\right\} \\
&= \left\{\, \widetilde{\mathcal{W}}_\sigma^* T_\beta T_\alpha \widetilde{\mathcal{W}}_\sigma \psi \mid \alpha, \beta \in \mathbb{R}^n \,\right\}, \\
&= \left\{\, \mathcal{T}_\beta^\sigma \mathcal{D}_\alpha^\sigma \psi \mid \alpha, \beta \in \mathbb{R}^n \,\right\},
\end{aligned}$$

with $\psi := \widetilde{\mathcal{W}}_\sigma^* g$, which is exactly the definition of $\mathbb{G}_\sigma$. ∎

3.3 Multiplier arising from Spectral Diffeomorphism

Since to each spectral diffeomorphism corresponds a spectral quantum frame which is adapted to the associated coordinate system in phase space, there is a straightforward definition of a *multiplier* - id est, a *phase space localization operator* - for these frames, cf. [11, 26, 27] for details on these kinds of operators. These phase space localization operators weight the coefficients that are associated with a phase space cell and can thus be used to alter the phase space content of a signal.

Ipso facto, these operators do have especially fetching localization properties, if the associated window function is also adapted to the chosen coordinate system, as developed in the next chapter.

Proposition 3.26 (Spectral Quantum Frame Multiplier). *Let σ be a spectral diffeomorphism and $\mathbb{F}_\sigma$ and $\mathbb{G}_\sigma$ the associated spectral quantum frames, defined on $\mathbb{R}^n \times \operatorname{im}(\sigma)$. Let furthermore m be a function on $\mathbb{R}^n \times \operatorname{im}(\sigma)$. Then, the at least weakly convergent operator-valued integrals*

$$M^m_{\mathbb{F}_\sigma} := \int_{\mathbb{R}^n \times \operatorname{im}(\sigma)} m(x,y) |\varphi_{x,y}\rangle\langle\varphi_{x,y}| \; \mathrm{d}x \; \mathrm{d}\nu, \tag{3.42}$$

and

$$M^m_{\mathbb{G}_\sigma} := \int_{\mathbb{R}^n \times \operatorname{im}(\sigma)} m(x,y) |\varphi_{x,y}\rangle\langle\varphi_{x,y}| \; \mathrm{d}x \; \mathrm{d}y, \tag{3.43}$$

define bounded operators on $\mathcal{S}_\sigma$, respectively $\mathcal{A}_\sigma$, whenever $m \in L^\infty(\mathbb{R}^n \times \operatorname{im}(\sigma), \; \mathrm{d}x \; \mathrm{d}\nu)$, respectively $m \in L^\infty(\mathbb{R}^n \times \operatorname{im}(\sigma), \; \mathrm{d}x \; \mathrm{d}y)$. ◆

Proof. The coherent state maps of both continuous frames above are multiples of an isometry and thus bounded operators $\mathcal{S}_\sigma \to \mathcal{H}_\sigma$ with bounded inverses (defined on their images) $\mathcal{H}_\sigma \to \mathcal{S}_\sigma$, respectively $\mathcal{A}_\sigma \to \mathcal{H}_\sigma$ and $\mathcal{H}_\sigma \to \mathcal{A}_\sigma$. Since a multiplication with an essentially bounded $m \in L^\infty(\mathbb{R}^n \times \operatorname{im}(\sigma), \; \mathrm{d}x \; \mathrm{d}y)$, respectively $m \in L^\infty(\mathbb{R}^n \times \operatorname{im}(\sigma), \; \mathrm{d}x \; \mathrm{d}\nu)$, defines a bounded operator on $\mathcal{H}_\sigma$, and the composition of bounded operators is again a bounded operator, the claim follows. ■

There is no good reason to restrict the discussion to frames, such that to each point of phase space is assigned only a single window and thus - when acting on a signal - only a single quantum of information. It is a natural step to extend this discussion to sets of functions, respectively projectors, which assign more degrees of freedom to each point in phase space. This means

(i) For each point in phase space, the projectors become operators of higher rank and thus may extract more than a single quantum of information

(ii) The coefficient functions become vector-valued, $\pi_\varphi f : X \to \mathbb{C}^N$,

(iii) The phase space localization operators, respectively the frame multipliers, no longer act by pointwise multiplication but are lifted to fields of matrices which may rotate the vector-valued coefficient functions and reduce to the localization operators in the usual sense only when they are diagonal matrices at each point.

The rest of this section is devoted to this slight extension.

Lemma 3.27 (Spectral Quantum Frame Multiplier of Rank N)**.** *Let $\mathbb{F}_\sigma$ and $\mathbb{G}_\sigma$ be the quantum frames of rank N, assigned to the spectral diffeomorphism σ. Let $X := \mathbb{R}^n \times \mathrm{im}(\sigma)$ and*

$$M : X \to Mat(N) \ , \quad x \mapsto M(x)$$

be a matrix-valued function on X. Then,

$$f \mapsto \int_X \sum_i \varphi_x^i \left[\sum_j (M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}\nu(y) \tag{3.44}$$

and

$$f \mapsto \int_X \sum_i \varphi_x^i \left[\sum_j (M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}y \tag{3.45}$$

define frame multiplier of rank N*, which are bounded operators on $\mathcal{S}_\sigma$ and $\mathcal{A}_\sigma$, respectively.* ◆

Proof. Since we may rewrite the above as

$$\begin{aligned} &\int_X \sum_i \varphi_x^i \left[\sum_j (M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}\nu(y) \\ =& \sum_{i,j} \int_X \varphi_x^i \ \left[(M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}\nu(y) \end{aligned}$$

and

$$\begin{aligned} &\int_X \sum_i \varphi_x^i \left[\sum_j (M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}y \\ =& \sum_{i,j} \int_X \varphi_x^i \ \left[(M(x))^i_j \left\langle \varphi_x^j , f \right\rangle \right] \mathrm{d}x \ \mathrm{d}y, \end{aligned}$$

we immediately see that these are finite sums of operators, which by *Proposition 3.26 (Spectral Quantum Frame Multiplier)* are bounded operators; but finite sums of bounded operators are bounded operators, which finishes the proof. ■

For specific choices of the matrix field, we have interesting special cases, the most direct is characterized by the following

Corollary 3.28 (Multiple Frame Multiplier)**.** *Let everything as in* Lemma 3.27 (Spectral Quantum Frame Multiplier of Rank N)*. In the special case that the matrices are diagonal, for each $x \in X$, the multiplier is representable via a sum of standard multipliers as*

$$f \mapsto \int_X \sum_i \varphi_x^i \; M^i(x) \left\langle \varphi_x^i \,, f \right\rangle \, \mathrm{d}\nu(x). \tag{3.46}$$

◆

Proof. This follows from *Lemma 3.27 (Spectral Quantum Frame Multiplier of Rank N)* as it is a special case. ■

Note that since the rank-N projectors

$$f \mapsto \sum_i |\varphi_x^i\rangle\langle\varphi_x^i|f$$

span N-dimensional subspaces for each $x \in X$, the matrix field M, used in the construction of the multiplier above, may rotate these in a manner to make apparent certain properties of f not easily accessible in the rank-one setting and the following gives three interesting examples of these types of operators.

Example 3.3 (Rank-two Multiplier)**.** Let $\mathbb{F}$ be a frame of rank two and $M : X \to Mat(2)$ a matrix-valued function as used below.

- $M := \begin{pmatrix} m_1 & 0 \\ 0 & 0 \end{pmatrix}$ ⇒ The multiplier projects on the first components at each point and boils down to a standard multiplier,

$$f \mapsto \int_X m_1(x) |\varphi_x^1\rangle\langle\varphi_x^1| + 0 \, \mathrm{d}\nu(x) \; f \;.$$

- $M := \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ ⇒ The multiplier weights each component, at each point $x \in X$, individually,

$$f \mapsto \int_X m_1(x) |\varphi_x^1\rangle\langle\varphi_x^1| + m_2(x) |\varphi_x^2\rangle\langle\varphi_x^2| \, \mathrm{d}\nu(x) \; f \;.$$

- $M := \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}$ ⇒ The multiplier "rotates" the fiber over each point,

$$f \mapsto \int_X |\varphi_x^1\rangle\langle\varphi_x^2| \pm |\varphi_x^2\rangle\langle\varphi_x^1| \, \mathrm{d}\nu(x) \; f \;.$$

◆

3.4 A Quadratic Phase Space Representation

One of the main goals of this program is to ensure that signals may be decomposed with respect to (spectral) quantum frames, whose projectors have a well-controlled phase space localization.

The quasi-standard in this case is the Wigner-Ville distribution

$$f \longmapsto \mathcal{W}_f(t,\xi) := \int_{\mathbb{R}^n} f(x+t/2)\overline{f(x-t/2)}e^{2\pi i\langle\, x\,,\,xi\,\rangle}\,\mathrm{d}x, \quad (t,\xi)\in X, \tag{3.47}$$

which will play a significant role in the next chapter. There is, however, a whole family of other (quadratic) phase space distributions, all related to the Wigner-Ville distribution in the sense that these may be represented as a Wigner-Ville distribution, convolved (in the standard, abelian manner) with a convolution kernel,

$$f \longmapsto \mathcal{D}^{\kappa}_f(t,\xi) := \int \; W_f(x,y)\;\kappa(t-x,\xi-y)\;\mathrm{d}x\;\mathrm{d}y, \quad (t,\xi)\in X, \tag{3.48}$$

where the distribution is completely characterized by this convolution kernel, see, e.g. [39] for a review. Another well-known but less used distribution is the Rihaczek map

$$f \longmapsto \mathcal{R}_f(t,\xi) := f(t)\overline{\widehat{f}(\xi)}e^{-2\pi i\langle\, \xi\,,\,t\,\rangle}, \quad (t,\xi)\in X, \tag{3.49}$$

which is the one that is generalized below.

With each spectral warp, we may associate two warped variants of the Rihaczek distribution.

Definition 3.29 (Spectral Warped Distributions)**.** Let σ be a spectral warp, $j_\sigma :=$ $\det J_\sigma^{-1}$ and $f\in\mathcal{S}_\sigma$, $\psi\in\mathcal{A}_\sigma$. Then

$$D^\sigma_f(x,y) := j_\sigma(y)\cdot f(x)\overline{\widehat{f}(\sigma^{-1}(y))}e^{-2\pi i\langle\, \sigma^{-1}(y)\,,\,x\,\rangle}, \quad (x,y)\in\mathbb{R}^n\times \mathrm{im}(\sigma), \tag{3.50}$$

and

$$\widetilde{D}^\sigma_\psi(x,y) := \psi(x)\overline{\widehat{\psi}(\sigma^{-1}(y))}e^{-2\pi i\langle\, \sigma^{-1}(y)\,,\,x\,\rangle}, \quad (x,y)\in\mathbb{R}^n\times \mathrm{im}(\sigma), \tag{3.51}$$

will be referred to as the *warped distribution*, on $\mathcal{S}_\sigma$ and $\mathcal{A}_\sigma$, respectively.

◆

For each of these distributions, the following holds.

Proposition 3.30 (Warped Distributions)**.** *Let σ be a spectral warp and $\mathcal{D}^\sigma_f$, $\widetilde{\mathcal{D}}^\sigma_\psi$ as above. Let all equalities hold up to measure zero, that is, almost everywhere with respect to the respective measure spaces .Then,*

(i) D_f^σ *has the marginal densities*

$$\int_{\mathrm{im}(\sigma)} D_f^\sigma \, \mathrm{d}\nu = |f(x)|^2 \tag{3.52}$$

$$\int_{\mathbb{R}^n} D_f^\sigma \, \mathrm{d}x = j_\sigma(y) \left|\widehat{f}(\sigma^{-1}(y))\right|^2 \tag{3.53}$$

and

$$\int_{\mathrm{im}(\sigma)} \int_{\mathbb{R}^n} D_f^\sigma \, \mathrm{d}x \, \mathrm{d}\nu = \|f\|_{\mathcal{S}_\sigma}^2 . \tag{3.54}$$

(ii) $\widetilde{D}_\psi^\sigma$ *has the marginal densities*

$$\int_{\mathrm{im}(\sigma)} \widetilde{D}_\psi^\sigma \, \mathrm{d}\nu = \left|\mathcal{F}^*(j_\sigma^{-1/2} \cdot \widehat{\psi})(x)\right|^2 , \tag{3.55}$$

$$\int_{\mathbb{R}^n} \widetilde{D}_\psi^\sigma \, \mathrm{d}x = \left|\widehat{\psi}(\sigma^{-1}(y))\right|^2 \tag{3.56}$$

and

$$\int_{\mathrm{im}(\sigma)} \int_{\mathbb{R}^n} \widetilde{D}_\psi^\sigma \, \mathrm{d}x \, \mathrm{d}\nu = \|\psi\|_{\mathcal{A}_\sigma}^2 . \tag{3.57}$$

◆

Proof. The proofs are trivial and follow immediately from integrating the distributions with respect to the relevant measures. ■

In fact, these distributions can be used to define a "spectral frameogram", that is, an analogon to the classical spectrogram from time-frequency analysis, but adapted to the coordinates, defined by the spectral diffeomorphism.

Corollary 3.31 (Spectral Frameogram)**.** *Let σ be a spectral warp. With $f \in \mathcal{S}_\sigma$ and $\psi \in \mathcal{A}_\sigma$, we have*

$$\left\langle D_{\pi^\sigma(\beta,\alpha)\psi}^\sigma , D_f^\sigma \right\rangle = \left|\pi_\psi^\sigma f(\beta,\alpha)\right|^2 , \tag{3.58}$$

which is a form of a frameogram *for the associated spectral quantum frame.* ◆

Proof. Using $\psi_{(\beta,\alpha)} \coloneqq \pi^\sigma(\beta,\alpha)\psi$, the calculation

$$
\begin{aligned}
&\left\langle D^\sigma_{\pi^\sigma(\beta,\alpha)\psi}\, ,\, D^\sigma_f \right\rangle\\
&= \iint \overline{\psi_{(\beta,\alpha)}(x)\overline{\widehat{\psi}_{(\beta,\alpha)}(\sigma^{-1}(y))}e^{-2\pi i\langle\, \sigma^{-1}(y)\, ,\, x\,\rangle}} j_\sigma(y)\cdot f(x)\overline{\widehat{f}(\sigma^{-1}(y))}e^{-2\pi i\langle\, \sigma^{-1}(y)\, ,\, x\,\rangle}\ \mathrm{d}x\ \mathrm{d}\nu\\
&= \int \overline{\psi_{(\beta,\alpha)}(x)} f(x)\ \mathrm{d}x \int j_\sigma(y)\cdot\widehat{\psi}_{(\beta,\alpha)}(\sigma^{-1}(y))\overline{\widehat{f}(\sigma^{-1}(y))}\ \mathrm{d}\nu(y)\\
&= \int \overline{\psi_{(\beta,\alpha)}(x)} f(x)\ \mathrm{d}x \int \widehat{\psi}_{(\beta,\alpha)}(\xi)\overline{\widehat{f}(\xi)}\mathrm{d}\xi\\
&= \int \overline{\psi_{(\beta,\alpha)}(x)} f(x)\ \mathrm{d}x \overline{\int \overline{\widehat{\psi}_{(\beta,\alpha)}(\xi)}\widehat{f}(\xi)\mathrm{d}\xi}\\
&= \int \overline{\psi_{(\beta,\alpha)}(x)} f(x)\ \mathrm{d}x \overline{\int \overline{\psi_{(\beta,\alpha)}(x)} f(x)\ \mathrm{d}x}\\
&= \left|\pi^\sigma_\psi f(\beta,\alpha)\right|^2,
\end{aligned}
$$

proves the claim. ∎

Location, Location, Location.

— Real estate mantra

4 Localization

UNCERTAINTIES ARE OMNIPRESENT, yet rarely a thing to desire, in particular when it comes to measurements of fundamental qualities such as positions or momenta.

In the previous chapters, spectral quantum frames, associated with a coordinate system in phase space, were defined. This chapter is devoted to the localization properties of these frames, id est, to the determination of its localization within phase space.

As is ubiquitous in this monograph, the motivation for the consideration of these sets of frames is the decomposition and alteration of signals and quantum states with respect to "arbitrary" phase space cells. As was already stated, the uncertainty principle sets a lower bound on the phase space concentration of functions and its associated projectors.

In this chapter, the true localization properties of frames, defined on phase space, respectively their associated template will be examined. In the previous chapter, transforms were considered, which decompose a given signal with respect to subsets of phase space.

Terminology 4.1 (Phase Space Cell)**.** A *phase space cell* shall denote a subset of the phase space of size ~ 1. ◆

To each rank-one projector corresponds a certain elementary phase space cell and hence to each phase space cell, is assigned a subspace of the universe of signals which are to be analyzed. It is in this manner that we shall refer to a window as having a certain phase space localization. Utilizing the theory of discrete frames, this may be rephrased as follows. The phase space is tessellated, such that the phase space cells of the rank-one projectors of a frame cover the whole phase space.

Heuristic 4.2 (Phase Space Location). *A function ψ is well localized around a phase space point $(p,q) \in X$, if there exists a* phase space cell $B \subset X$ *such that* $\operatorname{supp}(D_\psi) \sim B$, *where D_ψ is some phase space distribution of the function ψ, which is yet to be determined.* ◆

It is the purpose of this chapter, to make the heuristic idea above a concrete one.

There have been numerous attempts to quantitatively define the true localization of a given function, respectively its associated projector. We have already met the most prominent ones, as the most well-known ones are the Wigner distribution,

$$\mathcal{W}_\psi(x,\xi) := \int_{\mathbb{R}^n} \psi(x - \tfrac{y}{2})\ \overline{\psi(x + \tfrac{y}{2})}\ e^{2\pi i \langle\, y\,,\,\xi\,\rangle}\ \mathrm{d}y. \tag{4.1}$$

the Rihaczek distribution

$$\mathcal{R}_\psi(x,y) := \psi(x)\overline{\widehat{\psi}(y)}e^{-2\pi i \langle\, x\,,\,y\,\rangle} \tag{4.2}$$

and the spectrogram

$$SPEC_\varphi\psi(x,y) := |\pi_\varphi\psi(x,y)|^2\ , \tag{4.3}$$

where the variables $(x,y) \in X$ are interpreted as points in phase space and φ should be well localized in phase space, for this argument to make sense.

The first two above are quadratic and actually equivalent in the sense that there exists an isomorphism mapping one into the other.

Lemma 4.3 (Wigner vs Rihaczek). *Let W_ψ denote the Wigner distribution of the function $\psi \in L^2(\mathbb{R}^n)$, then*

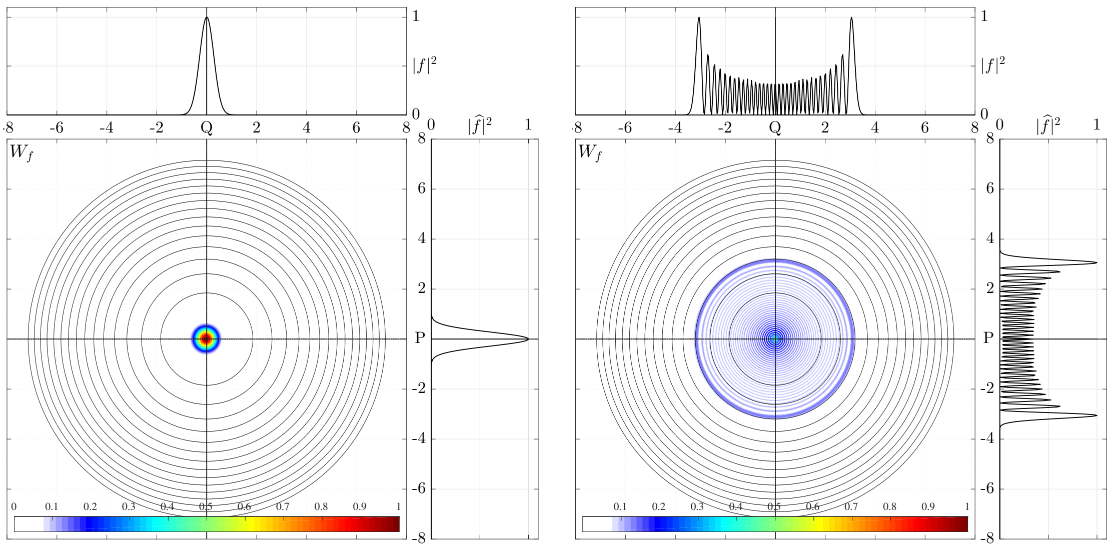
$$R_\psi(x,y) = \int_{\mathbb{R}^{2n}} e^{-2\pi i \langle\,(x,y)\,,\,(x',y')\,\rangle}\ e^{-\pi i \langle\,(x,y)\,,\,(x',y')\,\rangle}\widehat{W_\psi}(x',y')\ \mathrm{d}x'\ \mathrm{d}y' \tag{4.4}$$

is the Rihaczek distribution of ψ and the mapping is invertible as

$$W_\psi(x,y) = \int_{\mathbb{R}^{2n}} e^{-2\pi i \langle\,(x,y)\,,\,(x',y')\,\rangle}\ e^{+\pi i \langle\,(x,y)\,,\,(x',y')\,\rangle}\widehat{R_\psi}(x',y')\ \mathrm{d}x'\ \mathrm{d}y' \tag{4.5}$$

◆

Proof. See, e.g. [39] ■



(a) Wigner distribution of Gaussian.

(b) Wigner distribution of 32nd Hermite function.

Figure 4.1: Wigner distributions of Hermite functions.

The spectrogram is special in the sense that it is linear and depends on a window. It is actually a "smeared" version of the signal's Wigner distribution - and, thus, in light of *Lemma 4.3 (Wigner vs Rihaczek)* also of the Rihaczek distribution -, where the smearing is given as a convolution with the Wigner (resp. Rihaczek) distribution of the window. That is,

$$\psi \longmapsto \int W_\psi(x,y)\,\overline{W_\varphi(x'-x,y'-y)}\,\mathrm{d}x\,\mathrm{d}y \coloneqq |\pi_\varphi\psi(x',y')|^2 \tag{4.6}$$

respectively

$$\psi \longmapsto \int R_\psi(x,y)\,\overline{R_\varphi(x'-x,y'-y)}\,\mathrm{d}x\,\mathrm{d}y \coloneqq |\pi_\varphi\psi(x',y')|^2 \tag{4.7}$$

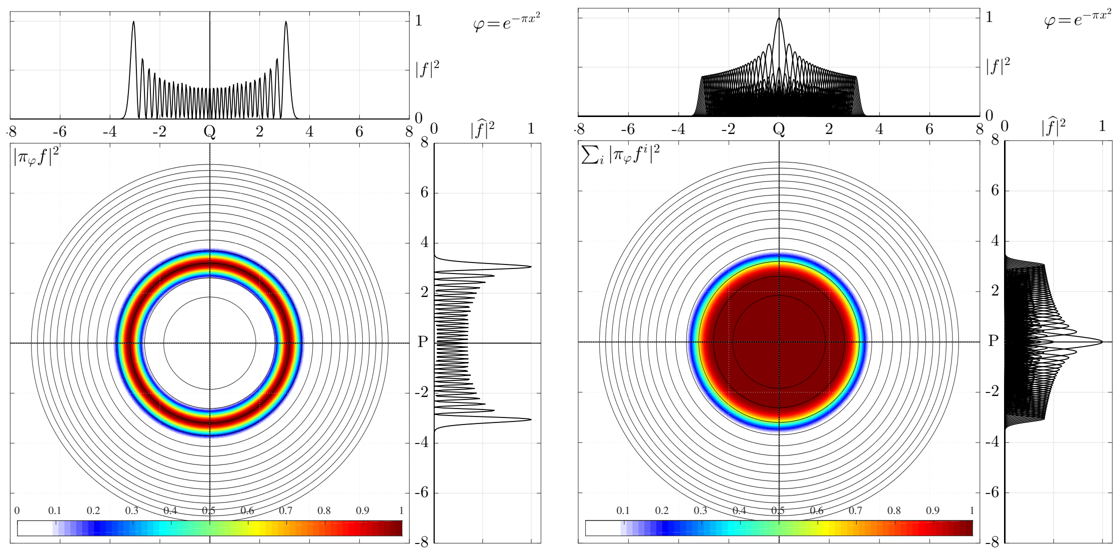
defines the spectrogram.

The spectrogram is the instrument of choice, whenever the function is not well localized around a single point, as then its quadratic behavior overlaps parts of the signal of different regions of phase space. E.g., if a function decomposes into two separate parts, $\psi = f + g$, then

$$W_\psi = W_{f+g} = W_f + W_g + W(f,g),$$

where $W(f,g)$ is a cross-term, encoding the nonlinear behavior which may overlap with the supports of W_f and W_g.

In the course of this chapter and the next, the arguments shall be strengthened by various plots of phase space localizations. Whenever possible - that is, when the



(a) Spectrogram of 32$^{\text{nd}}$ Hermite function.

(b) Spectrogram of the sum of the projection operators of the first 32 Hermite functions.

Figure 4.2: Spectrogram of Hermite functions.

distributions are localized and connected -, these plots will be given in terms of the Wigner distribution. In all other cases, the spectrogram will be given in order to make the support in phase space clear. To illustrate this, we use the well-known Hermite functions. The standard example of a well localized function in phase space is the Gaussian waveform, plotted in *Figure 4.1a*, which is the “0$^{\text{th}}$ Hermite function”.

Its support is concentrated to a small phase space cell and thus, the Wigner distribution gives a clear picture of what is happening. The Hermite functions are the eigenfunctions of the Harmonic oscillator with each Hermite function of higher degree occupying an annulus of phase space, further away from the origin. In *Figure 4.1b*, the Wigner distribution of the 32$^{\text{nd}}$ Hermite function is depicted, which seems to show that the whole region around the origin is allocated by a single function. *Figure 4.2a*, however, makes clear that this is an artifact of the “quadraticity” of the Wigner distribution. That is, the *sum* of the Wigner distributions of the first 32 Hermite functions - as well as the sum of their spectrograms - sum up to a disc in phase space, as plotted in *Figure 4.2b*.

This is a consequence of *Lemma 4.36 (Phase Space Distributions of Rank-N Operators)* below.

4.1 Uncertainty Principles

In quantum mechanics, the principle of uncertainty arose as a means to quantitatively describe the incompatibility of simultaneous measurements of position and momentum. This is expressed via the classical uncertainty principle mentioned in the first chapter, which, by a little abuse of notation, may be expressed as

$$\Delta(f)\,\Delta(\widehat{f}) \gtrsim \hbar\ , \tag{4.8}$$

where $\Delta(f)$ and $\Delta(\widehat{f})$ are measures of the concentration of the function f in the spatial as well as the spectral domain. We will find a more explicit expression below, but let's elaborate on this a little more.

For a motivating warmup, one may consider an audio signal. Then, it is clear that measuring the frequencies around an instant in time demands the incorporation of a certain time-interval, since frequencies are spread in time by nature and the proper identification of an occurring frequency takes an amount of time, related to the frequency's period. Another phenomenon, which is easily understood to be reasonable, is the following. The determination of the speed, resp. the momentum, of an object demands that an object has to travel an amount of space in a certain time to calculate its speed. So, decreasing the space traveled more and more inevitably leads to inaccuracies in the calculation of its speed and increasing the space traveled means that the object's position is smeared along its path, since during the measurement it has been at each and every point along the path, limiting the determination of its actual position.

From a mathematical point of view, whenever f and its Fourier transform, $\widehat{f}$, are both absolutely integrable, the calculation

$$\begin{aligned}\mathcal{F}\{f(a\,\bullet)\}(y) &= \int f(ax)e^{-i\langle\, y\,,\,x\,\rangle}\mathrm{d}x\\ &= \int f(x)e^{-i\langle\, y\,,\,x/a\,\rangle}\mathrm{d}(x/a)\\ &= \frac{1}{a}\int f(x)e^{-i\langle\, y/a\,,\,x\,\rangle}\mathrm{d}x\\ &= \frac{1}{a}\widehat{f}(y/a)\end{aligned}$$

shows that *compressing* a function in time or space by factor a is - up to a multiplicative factor a^{-1}, which preserves the integral's value - equivalent to *spreading* its Fourier transform by the very same factor. Thus, the better localized the function becomes in the time resp. space domain, the more "unlocalized" its Fourier transform, $\widehat{f}$, becomes. Due to the properties of the Fourier transform, the same holds in the opposite direction. Since dilating a function does not lead to a better combined localization, we cannot take an arbitrary function and squeeze it, but rather need to look for a function, having good resp. the best combined localization in the first place.

The quantification of this uncertainty between these "conjugate" measurements first brings us to a more general phenomenon. A comparable trade-off between two measurements occurs in different situations in physics and mathematics. In general, the product of the variances of two *self-adjoint operators*, which do not commute, calculated for the same function, is greater than or equal to a certain bound. We will first state this general uncertainty principle and afterwards specialize to the time-frequency case discussed above, see, e.g. [30].

Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)**.** *Let A, B be self-adjoint operators, $\alpha, \beta \in \mathbb{R}$ and*

$$\psi \in D(A) \cap D(B) \cap D(AB - BA),$$

then

$$\|(A-\alpha)\psi\|_2 \, \|(B-\beta)\psi\|_2 \geq \frac{1}{2} |\langle\, (AB - BA)\psi \,,\, \psi \,\rangle| \tag{4.9}$$

with the inequality becoming an equality if

$$(A-\alpha)\psi = -i\mu(B-\beta)\psi, \qquad \mu \in \mathbb{R}, \tag{4.10}$$

making ψ a minimizing waveform. ◆

This theorem shows that in order to find an optimal function for a pair of self-adjoint operators, which has "minimal uncertainty", we only need to solve a single equation. But, although this theorem is well-known and very prominent in the literature, we shall state the well-known proof of this theorem, as this will be used often in the course of this chapter.

Proof of Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators). The Cauchy-Schwarz *Theorem A.41 (Cauchy-Schwarz)*, provides us with

$$\begin{aligned}\|(A-\alpha)\psi\|_2^2 \, \|(B-\beta)\psi\|_2^2 &\geq |\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle|^2 \\ &= \Re\left(\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle\right)^2 \\ &\quad + \Im\left(\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle\right)^2 ,\end{aligned}$$

where equality holds if and only if $(A-\alpha)\psi = \lambda(B-\beta)\psi$, $\lambda \in \mathbb{C}$, i.e., if $(A-\alpha)\psi$ and $(B-\beta)\psi$ are linearly dependent. Furthermore

$$\begin{aligned}|\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle|^2 &\geq |\Im\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle|^2 \\ &= \left| \frac{\langle\, (A-\alpha)\psi \,,\, (B-\beta)\psi \,\rangle - \langle\, (B-\beta)\psi \,,\, (A-\alpha)\psi \,\rangle}{2i} \right|^2 ,\end{aligned}$$

with equality if and only if $\langle\, (A-\alpha)\psi\,,\, (B-\beta)\psi\,\rangle$ is purely imaginary. Combining both, we find

$$\begin{aligned}\|(A-\alpha)\psi\|^2\,\|(B-\beta)\psi\|^2 &\geq \Im\langle\,(A-\alpha)\psi\,,\,(B-\beta)\psi\,\rangle^2\\ &= \frac{1}{4}\,|\langle\, BA\psi - AB\psi\,,\,\psi\,\rangle|^2\,,\end{aligned}$$

with equality if and only if

$$(A-\alpha)\psi = -i\mu(B-\beta)\psi, \quad \mu\in\mathbb{R},$$

that is, iff $(A-\alpha)\psi$ and $(B-\beta)\psi$ are *purely imaginary multiples* of each other. ■

In the famous case of time and frequency, respectively position and momentum, mostly known for its philosophical implications and referred to as the "Heisenberg uncertainty principle", the operators above are given by $A := x$, which "measures" the spatial positions or instants of time, and $B := \frac{1}{2\pi i}\frac{\partial}{\partial x}$, measuring momenta or frequencies.

Theorem 4.5 (Classical Uncertainty Principle). *Let $f\in L^2(\mathbb{R}^n)$, then*

$$\left\|(x^j-\beta)f\right\|_2\left\|(x_k-\alpha)\widehat{f}\right\|_2 \geq \frac{1}{4\pi}\,\|f\|^2\,\delta^j_k, \tag{4.11}$$

with $\|(x^j-\alpha)f\|_2\left\|(x_k-\beta)\widehat{f}\right\|_2 = \frac{1}{4\pi}$ iff $j=k$ and for each k, f is a "Gaussian" of the form

$$f_{\alpha,\beta,\mu}(x) := e^{2\pi x(\mu\alpha - i\beta)}e^{-\pi\mu x^2}, \quad \mu\in\mathbb{R}_+,\ \alpha,\beta\in\mathbb{R}. \tag{4.12}$$

◆

Proof. Let $A = x_j-\beta$ and $B = \frac{1}{2\pi i}\frac{\partial}{\partial x_k}-\alpha$, then

$$(AB-BA)f = i\frac{1}{2\pi}\left(\frac{\partial}{\partial x_k}x_j - x_j\frac{\partial}{\partial x_k}\right)f = \begin{cases}\frac{i}{2\pi}f, & j=k\\ 0\,, & else\end{cases},$$

and applying *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)* for $j=k$ leads to (4.11) with equality for

$$-i\mu(x^k-\beta)f = \left(\frac{1}{2\pi i}\frac{\partial}{\partial x_k}-\alpha\right)f.$$

Solving this first order partial differential equation for f gives

$$f_{\alpha,\beta,\mu}(x) := e^{2\pi x(\mu\alpha - i\beta)}e^{-\pi\mu x^2}, \quad \mu\in\mathbb{R}_+,\ \alpha,\beta\in\mathbb{R}.$$

which finishes the proof. ■

The principle above comprises only pairs of self-adjoint operators, whose viability is limited. In fact, there are two ways to generalize the principle above to an n-dimensional pendant, the first of which utilizes tensor products, the second uses the language of vector fields.

Higher Dimensions

The principles hold for each two-dimensional symplectic subspace, associated with $L^2(\mathbb{R})$, and thus the tensor construction enables us to write the whole space as

$$L^2(\mathbb{R}^n) \simeq \bigotimes_k L^2(\mathbb{R}) \tag{4.13}$$

where the linear combinations of the pure tensors, $\otimes_k f_k$, are dense in $L^2(\mathbb{R}^n)$ and one further needs to take the Hilbert space completion with respect to the norm on $L^2(\mathbb{R}^n)$. The inner product and the induced norm of the tensor-product is

$$\left\langle \bigotimes_k f_k \, , \bigotimes_k g_k \right\rangle_{L^2(\mathbb{R}^n)} := \prod_k \langle\, f_k \, , \, g_k \,\rangle_{L^2(\mathbb{R})} \quad \text{and} \quad \|\otimes_k f_k\| := \prod_k \|f_k\| \, , \tag{4.14}$$

respectively linear combinations of the above for non-pure tensors.

It is easy to check, that this inner product yields the norm of $L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \otimes_k f_k \, , \otimes_k g_k \rangle_{L^2(\mathbb{R}^n)} &:= \prod_k \langle\, f_k \, , \, g_k \,\rangle_{L^2(\mathbb{R})} \\
&= \prod_k \int_{\mathbb{R}} \overline{f_k(x_k)} g_k(x) \; \mathrm{d}x_k \\
&= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \overline{(\prod_k f_k(x_k))} (\prod_k g_k(x_k)) \; \mathrm{d}x_1 \dots \; \mathrm{d}x_n \\
&= \int_{\mathbb{R}^n} \overline{f} g \; \mathrm{d}x,
\end{aligned}$$

again, with the obvious extension by linearity. There are in fact only very few cases, in which an element of such a tensor product is pure and even if one starts out with a pure tensor, $\otimes_k f_k$, this form is not necessarily preserved by the action of linear operators, but those operators that actually do, are of the very intuitive form

$$\mathcal{T} := \bigotimes_k T_k : L^2(\mathbb{R}^n) \;\to\; L^2(\mathbb{R}^n), \otimes_k f_k \;\to\; \otimes_k T_k \otimes_k f_k = \otimes_k (T_k f_k), \tag{4.15}$$

with each T_k acting on and preserving the structure of a specific factor $L^2(\mathbb{R})$, for each k.

It turns out that the operators $\widehat{Q}_k$ and $\widehat{P}_k$ can be utilized to build

$$\widehat{Q} := \bigotimes_k Q^k := \bigotimes_k x^k \quad \text{and} \quad \widehat{P} := \bigotimes_k P_k = \bigotimes_k \tfrac{1}{2\pi i} \frac{\partial}{\partial x^k} \tag{4.16}$$

and it is these tensor products of operators that are the patron of the next theorem.

Theorem 4.6 (Classical Uncertainty Principle for Tensor Products)**.** *Let everything as in* Theorem 4.5 (Classical Uncertainty Principle)*. Let moreover*

$$\widehat{Q} := \bigotimes_k Q^k := \bigotimes_k x^k \quad \textit{and} \quad \widehat{P} := \bigotimes_k P_k = \bigotimes_k \tfrac{1}{2\pi i}\frac{\partial}{\partial x^k} \tag{4.17}$$

and $\alpha, \beta \in \mathbb{R}^n$, lifted to a tensor product of multiplication operators, such that

$$\widehat{Q} - \alpha := \otimes_k (Q^k - \alpha^k)$$

and the same for P and β. Then

$$\left\|(\widehat{Q}-\alpha)f\right\|_{L^2(\mathbb{R}^n)} \left\|(\widehat{P}-\beta)f\right\|_{L^2(\mathbb{R}^n)} \geq \tfrac{1}{\cdot(4\pi)^n} \|f\|^2_{L^2(\mathbb{R}^n)} \tag{4.18}$$

Furthermore, if f_k denote the equalizing waveforms for each of the uncertainty principles, interpreted as elements of $L^2(\mathbb{R})$, then their tensor product,

$$L^2(\mathbb{R}^n) \ni f_0 := \otimes_k f_k \in \otimes_k L^2(\mathbb{R}),$$

*is the equalizer of (*4.18*) and reads*

$$f_0(x) := Ce^{-\pi\mu|x|^2} e^{-2\pi\langle\, \vec{x}\,,\, \mu\vec{\alpha} - i\vec{\beta}\,\rangle}. \tag{4.19}$$

◆

Proof. Let $f := \otimes_k f_k$ be a linear superposition of elementary tensors, then

$$\begin{aligned}
&\left\|(\widehat{Q}-\alpha)f\right\|_{L^2(\mathbb{R}^n)} \left\|(\widehat{P}-\beta)f\right\|_{L^2(\mathbb{R}^n)} \\
=& \left\|\otimes_k(\widehat{Q^k}-\alpha^k)f_k\right\|_{L^2(\mathbb{R}^n)} \left\|\otimes_k(\widehat{P_k}-\beta_k)f_k\right\|_{L^2(\mathbb{R}^n)} \\
=& \prod_k \left\|(\widehat{Q^k}-\alpha^k)f_k\right\|_{L^2(\mathbb{R})} \left\|(\widehat{P_k}-\beta_k)f_k\right\|_{L^2(\mathbb{R})} \\
\geq& \prod_k \left|\left\langle\, Q^k f_k\,,\, P_k f_k\,\right\rangle_{L^2(\mathbb{R})}\right| \\
\geq& \tfrac{1}{2^n} \prod_k \left|\left\langle\, [\,P_k\,,\, Q^k\,]\, f_k\,,\, f_k\,\right\rangle_{L^2(\mathbb{R})}\right| \\
=& \tfrac{1}{2^n} \prod_k \left|\left\langle\, \tfrac{1}{2\pi i} f_k\,,\, f_k\,\right\rangle_{L^2(\mathbb{R})}\right| \\
=& \tfrac{1}{2^n} \tfrac{1}{(2\pi)^n} \left|\prod_k \langle\, f_k\,,\, f_k\,\rangle_{L^2(\mathbb{R})}\right| \\
=& \tfrac{1}{\cdot(4\pi)^n} \|f\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}$$

and since each of the factors in the inequalities above are equalized for a Gaussian, a tensor product of Gaussians equalizes the generalized tensor principle

$$\begin{aligned}
f_0(x) &:= (\otimes_k f_k)(x) \\
&:= Ce^{-\pi\mu|x|^2} e^{-2\pi\langle\, \vec{x}\,,\, \mu\vec{\alpha} - i\vec{\beta}\,\rangle}.
\end{aligned}$$

■

Although the generalization is straight-forward, the outcome is not satisfactory, since

(i) it is not clear how this principle can be further generalized for cases in which more than individual pairs of operators have non-vanishing commutators and, even more strikingly,

(ii) the lower bound contains the factor $\frac{1}{(4\pi)^n}$, which, reinstalling Planck's constant for a moment, reads $(\frac{h}{4\pi})^n := (\frac{\hbar}{2})^n$. It is, however, known since 1913 - as stated in the first chapter -, that for each degree of freedom of the configuration space, the elementary phase space cell is of size of the *elementary Wirkungsquantum* independent of the other degrees of freedom. Thus, the lower bound should be $\sim n \cdot \frac{\hbar}{2}$, since there is no constraint on concentration on non-symplectic subspaces.

It turns out, that the generalization of the classical principle above, using the language of vector fields - which is well-known in the literature -, is the way to go. Without further preliminary skirmishing, it is easy to see that

$$L^2(\mathbb{R}^n) \;\to\; \mathbb{C}^n \otimes L^2(\mathbb{R}^n), f \;\mapsto\; \vec{P}f := \tfrac{1}{2\pi i}\vec{\nabla} f \tag{4.20}$$

and

$$L^2(\mathbb{R}^n) \;\to\; \mathbb{C}^n \otimes L^2(\mathbb{R}^n), f \;\mapsto\; \vec{Q}f := \vec{x}f \tag{4.21}$$

on $L^2(\mathbb{R}^n)$, turn f into vector fields and that the Hilbert space $\mathbb{C}^n \otimes L^2(\mathbb{R}^n)$ has the inner product

$$\left\langle \vec{h}\,,\, \vec{g}\,\right\rangle_{\mathbb{C}^n\otimes L^2(\mathbb{R}^n)} = \sum_k \langle\, h_k\,,\, g_k\,\rangle_{L^2(\mathbb{R}^n)}\,,$$

which brings us the following theorem.

Theorem 4.7 (Classical Uncertainty Principle in n Dimensions)**.** *Let everything as before, $\alpha, \beta \in \mathbb{R}^n$ and*

$$\vec{P}f := \tfrac{1}{2\pi i}\vec{\nabla} f \quad \textit{and} \quad \vec{Q}f := \vec{x}f. \tag{4.22}$$

Then

$$\left\|\vec{P}f - \vec{\alpha}f\right\| \left\|\vec{Q}f - \vec{\beta}f\right\| \geq \tfrac{n}{4\pi}\, \|f\|^2\,, \tag{4.23}$$

with equality if and only if

$$f(x) := Ce^{-\pi\mu|x|^2} e^{-2\pi\langle\, \vec{x}\,,\, \mu\vec{\alpha} - i\vec{\beta}\,\rangle}. \tag{4.24}$$

◆

Proof. Using Cauchy-Schwarz, we get

$$
\begin{aligned}
\left\|\vec{P}f - \vec{\alpha}f\right\| \left\|\vec{Q}f - \vec{\beta}f\right\| &\geq \left|\left\langle \vec{P}f - \vec{\alpha}f \,,\, \vec{Qf} - \vec{\beta}f \right\rangle\right| \\
&= \left| \sum_k \tfrac{1}{2\pi i} \left\langle (\partial_{x_k} - \alpha^k) f \,,\, (x_k - \beta_k) f \right\rangle_{L^2(\mathbb{R}^n)} \right| \\
&\geq \left| \sum_k \tfrac{1}{2} \tfrac{1}{2\pi i} \left\langle [\, \partial_{x_k} \,,\, x_k \,] \, f \,,\, f \right\rangle_{L^2(\mathbb{R}^n)} \right| \\
&= \left| \sum_k \tfrac{1}{2} \tfrac{1}{2\pi i} \left\langle 1 f \,,\, f \right\rangle_{L^2(\mathbb{R}^n)} \right| \\
&= \tfrac{n}{4\pi} \left\| f \right\|^2
\end{aligned}
$$

and equality holds, if and only if

$$
\left(\tfrac{1}{2\pi i}\vec{\nabla} - \alpha\right) f = i\mu(\vec{x - \beta}) f,
$$

with $\mu \in \mathbb{R}^n$. Setting $\widetilde{x} := \mu(x - \beta) - i\alpha$, we have that

$$
\vec{\nabla} f = -2\pi \vec{\widetilde{x}} f,
$$

which, using the Ansatz $f(x) := e^{g(x)}$ reads

$$
\nabla g = -2\pi \vec{\widetilde{x}}.
$$

Thus, $-2\pi\vec{\widetilde{x}}$ is a conservative vector field, with potential g. The solution, then, is given by a line integral of the form

$$
g(x) = -2\pi \int_c \left\langle \vec{\widetilde{x}} \,,\ \vec{\mathrm{d}x} \right\rangle + g(x_0),
$$

along a path c, starting at some point x_0 and ending at x. Re-substitution of f then gives

$$
f(x) := e^{g(x)} = e^{g(x_0)} e^{-2\pi \int_c \left\langle \vec{\widetilde{x}} \,,\ \mathrm{d}x \right\rangle}.
$$

Computing the integrals and re-substitution of $\widetilde{x}$ then finishes the proof. ∎

4.2 Uncertainty Principles for Spectral Diffeomorphisms

In order for the coherent state map of a spectral quantum frame to localize information of a signal as concentrated as possible around a phase space point and to be adapted to the chosen coordinates, it is necessary that the window, φ, is as localized as possible in phase space. Since the spectral quantum frames are adapted to a specific coordinate system in phase space, chosen to fit the application, the localization of the prototypical probe, used to decompose phase space, should be adapted to the chosen coordinate system, as well.

Hereafter, two opposing themes shall be examined. The first is the “classical” idea that a function is optimally adapted to a coordinate system around a given point, if its phase space picture is centered at that very point and it aligns with the coordinate lines, running through that point, as nicely as possible. That is, a function tries to stretch along all coordinate lines simultaneously and the optimal function is the one which optimizes this balancing act, without diverging to much from any of the coordinate lines. The second is a new form of localization and is - although still adapted to a frame of reference in phase space -, a completely opposing state of affairs. By a little abuse of language, a function is thought to be optimal, if it is as “unaligned” as possible with respect to all coordinates. This means, that the optimal function can be interpreted as “aligned along the canonical conjugate coordinate” of each of the individual coordinates of the chosen frame of reference.

Although these notions seem to be mutually exclusive, there is a specific case, in which both of these notions coincide!

In the previous section, the generalized version of the uncertainty principle for two non-commuting observables was presented and the special case for canonical coordinates in quantum mechanics was calculated. The optimal waveform, the Gaussian, is optimally adapted to the canonical coordinates in the sense that it is as concentrated as possible around a classical point on phase space and at the same time it aligns as good as possible along the classical rectangular coordinate lines. But the coordinate system is a canonical one, so the canonically conjugate coordinates are simply a relabeling of the operators involved and thus, the Gaussian also holds the scepter for the new conjugated principle.

That these two seemingly opposing facts - the optimal concentration around a point and optimal alignment along the coordinate lines - coincide, turns out to be a specialty of canonical coordinates - underlining once again the peculiarity of canonical coordinates and the Gaussian waveform.

Concerning the second and new principle, another approach shall be mentioned. As stated in the initial chapter, in [59], it was noted that the classical uncertainty principle - which in this monograph is the principle of optimal alignment - does not necessarily lead to fixed lower bounds. For the special case of the affine “ax+b” group, it was shown that the lower bound can be made arbitrarily small, which ultimately led to the research project *UNLocX* [83].

In [58], the authors introduced the notion of "adjoint observables" - essentially a canonically conjugate operator to an observable -, defined via the commutator, which is an alternative approach to the problem, posed in [59]. These adjoint observables lead to a principle, closely related to the one developed in this monograph.

The key was that to each (self-adjoint) observable, T, which generates the one-parameter subgroup $(e^{-2\pi i T t})_{t\in\mathbb{R}}$ of G, corresponds an adjoint observable, $\breve{T}$, (non-uniquely) defined by the canonical commutation relation

$$\left[T, \breve{T}\right] = \tfrac{1}{2\pi i}. \tag{4.25}$$

The descriptive name was chosen in order to emphasize the fact that these do represent a measurement process, measuring the "position" of the state in the direction which the one-parameter subgroup translates the state. Note that the name was given, using a bit of notational abuse, since these operators are neither necessarily observables in the mathematical sense of self-adjoint operators nor in the sense of quantum mechanics of measuring a quantum property of the state. As a matter of fact, when no quantum mechanical probability distributions on measurable quantities - given by diagonalization of quantum observables - is needed, it may be enough to speak of symmetric operators, which admittedly lack the essential fact that their adjoints have the same domain, but can be defined more easily be imposing restrictions on their respective domains, e.g., such that boundary terms in partial integration vanish.

4.2.1 Optimal Alignment

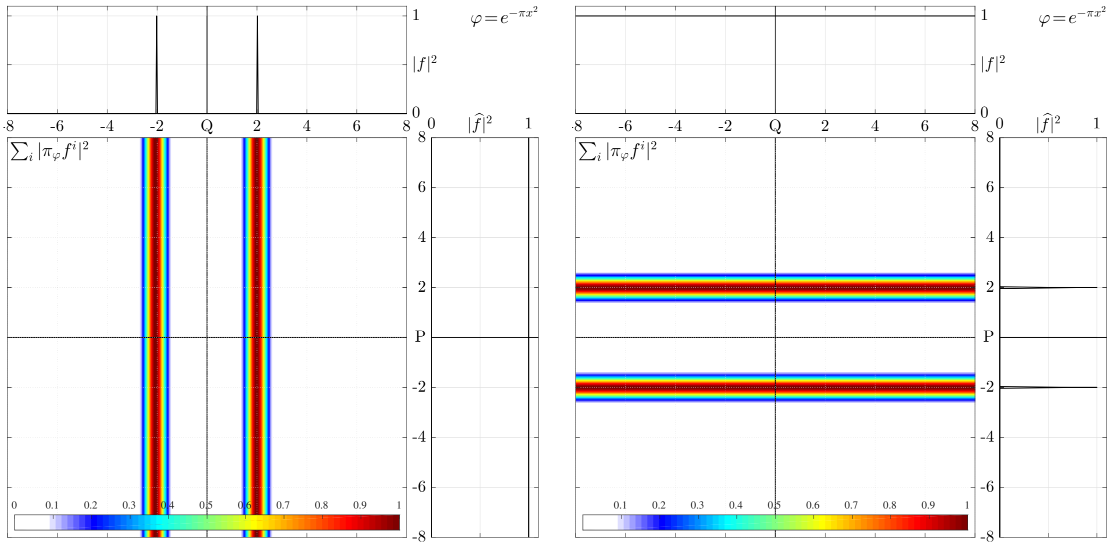
Before defining a generalized version of the uncertainty principle for a generalized coordinate system in phase space, a quick look on eigenfunctions of the quantized versions of coordinate functions is appropriate and since this is a motivation, no proofs are given.

Let P be one of the coordinate functions in the two-dimensional phase space. Then, its quantized operator, represented in the time domain, is given by

$$\hat{P} = \tfrac{1}{2\pi i}\partial_x$$

and its - generalized - eigenfunctions are the exponential waves

$$\hat{P}e^{2\pi i \xi x} = \xi e^{2\pi i \xi x}.$$



(a) Eigenfunctions of $Q : (q, p) \mapsto q$
- Dirac deltas -

(b) Eigenfunctions of $P : (q, p) \mapsto p$
- exponential waves -

Figure 4.3: (Distributional) Spectrograms of Eigenfunctions.

In phase space, these exponential waves align perfectly along the contour lines of the coordinate function P, where each of the eigenfunctions corresponds to exactly one of coordinates values, that is,

$$P := \xi \;\Rightarrow\; e^{2\pi i \xi x} \text{ aligns perfectly along } \xi \text{ and vanishes elsewhere.}$$

Let now Q be the other coordinate function in the two-dimensional phase space. Then, its quantized operator, represented in the time domain, is given by

$$\hat{Q} = x$$

and its - generalized - eigenfunctions are the Dirac deltas

$$\hat{Q}\delta_x = x\delta_x.$$

In phase space, these Dirac deltas align perfectly along the contour lines of the coordinate function Q, where each of the eigenfunctions corresponds, again, to exactly one of coordinates values, that is,

$$Q := x \;\Rightarrow\; \delta_x \text{ aligns perfectly along } x \text{ and vanishes elsewhere.}$$

Figure 4.3 depicts spectrograms of eigenfunctions of P and Q, as referenced above, and *Figure 4.4* shows the same for two non-canonical coordinate systems in phase space, defined in (*2.9*) and (*2.10*).

There seems to be a pattern, which indeed continues to hold for more general coordinate functions. Let A and B be coordinates on the two-dimensional phase space, such that the coordinate lines are *not closed*, since closed coordinate lines lead to periodicity, which in turn leads to compact operators after quantization, having discrete spectrum [85]. This is the ultimate mathematical reason for the quantized energy spectrum of the harmonic oscillator - its contour lines are closed.

Then, each of these coordinate functions partitions the phase space X, that is,

$$X \coloneqq \cup_{a\in \mathrm{im}(A)} A^{-1}(a) \quad \text{and} \quad X \coloneqq \cup_{b\in \mathrm{im}(B)} B^{-1}(b),$$

where the images $\mathrm{im}(A)$ and $\mathrm{im}(B)$ are exactly the *spectrum* of the quantized operators and to each value of the spectrum corresponds a generalized eigenfunction, aligning with the associated coordinate line in phase space.

This means, ignoring issues of convergence and other subtleties, we have the *purely formal* decomposition of the coordinate functions A and B into integrals

$$A \coloneqq \int_{\mathrm{im}(A)} a \cdot \chi_{A^{-1}(a)} \; \mathrm{d}a \quad \text{and} \quad B \coloneqq \int_{\mathrm{im}(B)} b \cdot \chi_{B^{-1}(b)} \; \mathrm{d}b,$$

where $\chi_{A^{-1}(a)}$ is the characteristic function of the contour-line of A of height a,

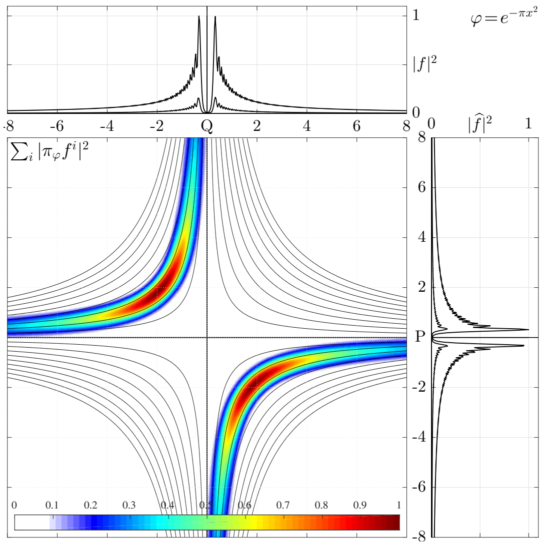
$$\chi_{A^{-1}(a)}(q,p) \coloneqq \begin{cases} 1 & , \; A(q,p) = a \\ 0 & , \text{else} \end{cases}$$

and analogously for $\chi_{B^{-1}(b)}$. Now, reinterpreting $\mathrm{im}(A)$ as $\mathrm{spec}(\widehat{A})$, we have the correspondence

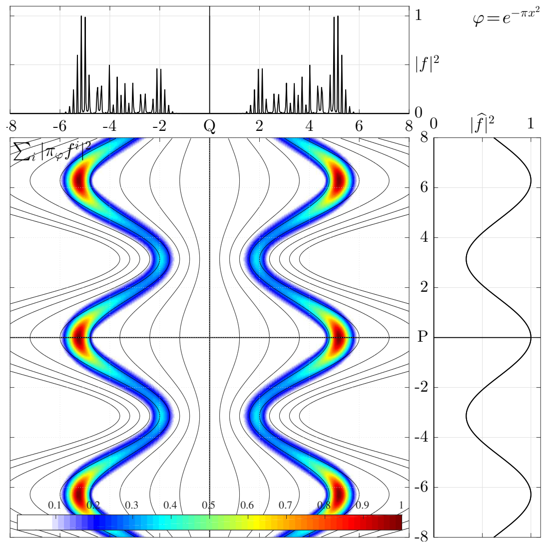
$$A \coloneqq \int_{\mathrm{im}(A)} a \cdot \chi_{A^{-1}(a)} \; \mathrm{d}a \;\overset{\sim}{\longmapsto}\; \widehat{A} \coloneqq \int_{\mathrm{spec}(A)} a \; \mathrm{d}P_a \sim \int_{\mathrm{spec}(A)} a \; |a\rangle\langle a| \; \mathrm{d}a$$

where now $\mathrm{d}P_a$ is the projection-valued measure, projecting onto subsets of the spectrum and $|a\rangle\langle a|$ is a purely formal expression for a generalized projector onto the "eigenspace" of a.

A second reason for the demand for non-closedness of the coordinate lines is the uncertainty principle, which states that each (generalized) function occupies at least a region of $\hbar$ in phase space. If the lines are non-compact, and thus have infinite extend, the uncertainty principle is defied, leading to "infinitesimally" thin lines in phase space, like for the Dirac deltas or the exponential waves. But, since the ultimate aim is to quantize phase space, such that to each classical point corresponds a phase space cell, which aligns with the coordinate lines, the generalized functions above, that are "infinitesimally thin" in phase space, are to be abandoned. We wish to decomposes phase space in a manner, which is optimal for the application,



(a) Eigenfunctions of $D:(q,p) \mapsto q \cdot p$.



(b) Eigenfunctions of $S:(q,p) \mapsto \frac{q}{2+\cos(p)}$.

Figure 4.4: (Distributional) Spectrograms of Eigenfunctions.

where the application determines the classical frame (of reference) in phase space, and via the process of quantization also the optimal quantum frame, consisting of atoms, aligned optimally along concurring coordinate lines.

This, then, motivates the following notion.

Terminology 4.8 (Optimal Alignment)**.** A function's representation on phase space is said to be *optimally aligned* in the sense of a chosen frame of reference, if it is as aligned to both coordinate lines of each two-dimensional subspace of conjugate variables as a specific inequality, an *uncertainty principle*, admits.

The rest of this section is devoted to the specification of this terminology. ◆

As a function cannot be an eigenfunction of two Hamiltonians simultaneously, unless these have the same projection-valued measure, we need a possibility to determine those functions, which minimize the trade-off between the deviations of optimal alignment, that is, a function which is a compromise between the eigenfunctions of both operators.

A generic approach in quantum mechanics to this, but usually without reference to any coordinate functions or any phase space localization, is the Ladder approach, cf. e.g. [64], associating to the self-adjoint operators A and B - in the case above the quantized coordinate functions -, the operator

$$\widehat{L} := \widehat{A} + i\widehat{B}$$

and its adjoint

$$\widehat{L}^* := \widehat{A} - i\widehat{B}.$$

Definition 4.9 (Generalized Ladder and Number Operators)**.** Let $\widehat{A}$ and $\widehat{B}$ be two self-adjoint operators, then

$$\widehat{L} := \widehat{A} + i\widehat{B} \tag{4.26}$$

and its adjoint

$$\widehat{L}^* := \widehat{A} - i\widehat{B}. \tag{4.27}$$

will be referred to as the *generalized Ladder operators.*

The composed operator

$$\widehat{N} := \widehat{L}^*\widehat{L} := \widehat{A}^2 + \widehat{B}^2 - i\left[\,\widehat{B}\,,\,\widehat{A}\,\right], \tag{4.28}$$

is called the (generalized) *Number operator.* ◆

Remark 4.10 (Ladder)**.** For the canonical operators of the CCR in their spatial representation - corresponding to the *canonical phase space coordinates* p and q -, reading

$$\widehat{P} := \tfrac{1}{2\pi i}\partial_x \quad \text{and} \quad \widehat{Q} := x,$$

the operators $\widehat{L}$ and $\widehat{L}^*$ are the actual Ladder operators for the quantum harmonic oscillator, called the lowering operator ($\widehat{L}$) and the raising operator ($\widehat{L}^*$), respectively. In quantum field theory - lifting fields of numbers to fields of operators [60, 64, 69] - these are the annihilation and creation operators, that create and annihilate particles in the various decoupled Fourier modes of the quantized fields. ◆

The operator $\widehat{L}$ now has complex spectrum and the associated *generalized coherent states* are defined by

$$(\widehat{A} + i\widehat{B})|\,\alpha + i\beta\,\rangle = (\alpha + i\beta)|\,\alpha + i\beta\,\rangle.$$

This definition is, up to some constant, μ, equivalent to the general uncertainty principle for two non-commuting self-adjoint operators, since we may rewrite this as

$$(\widehat{A} - \beta)|\,\alpha + i\beta\,\rangle = -i(\widehat{B} - \beta)|\,\alpha + i\beta\,\rangle.$$

The generalized eigenfunctions of $\widehat{L}$ therefore minimize, for some specific constant $\mu = 1$, the general uncertainty principle of $\widehat{A}$ and $\widehat{B}$, as stated in *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)*, and the ground state - respectively vacuum state in quantum field theory - is determined by setting $\alpha = \beta = 0$.

We'll also need a shifted version of these.

Lemma 4.11 (Shifted Generalized Ladder Operators)**.** *Let $\alpha, \beta \in \mathbb{R}$, then, the generalized Ladder operators of*

$$\widehat{A} - \alpha \quad \text{and } \widehat{B} - \beta$$

are

$$\widehat{L}_{\alpha\beta} := \widehat{L} - (\alpha + i\beta) \quad \text{and} \quad \widehat{L}^*_{\alpha\beta} := \widehat{L}^* - (\alpha - i\beta).$$

Moreover, the associated Number operator becomes

$$\begin{aligned}\widehat{N}_{\alpha\beta} &:= \widehat{N} - 2(\alpha\widehat{A} + \beta\widehat{B}) + \alpha^2 + \beta^2 \\ &:= \widehat{A}^2 + \widehat{B}^2 - i[\widehat{B}, \widehat{A}] - 2(\alpha\widehat{A} + \beta\widehat{B}) + \alpha^2 + \beta^2.\end{aligned}$$

◆

Proof. The proof is direct

$$\widehat{L}_{\alpha\beta} := (\widehat{A} - \alpha) + i(\widehat{B} - \beta) = (\widehat{A} + i\widehat{B}) - (\alpha + i\beta)$$

and analogous for $\widehat{L}^*$. For $\widehat{N}$, as in (*4.28*), find

$$\begin{aligned}\widehat{N}_{\alpha\beta} &= ((\widehat{A} - i\widehat{B}) - (\alpha - i\beta))(\widehat{A} + i\widehat{B}) - (\alpha + i\beta) \\ &= (\widehat{A} - i\widehat{B})(\widehat{A} + i\widehat{B}) - (\alpha - i\beta)(\widehat{A} + i\widehat{B}) - (\alpha + i\beta)(\widehat{A} - i\widehat{B}) + (\alpha - i\beta)(\alpha + i\beta) \\ &= \widehat{N} - 2\alpha\widehat{A} - 2\beta\widehat{B} + \alpha^2 + \beta^2,\end{aligned}$$

which was the claim. ■

Spectral Diffeomorphisms

Let now σ be a spectral diffeomorphism and

$$A^i := (J_\sigma^{-1}(p)q)^i, \quad B'_i := \sigma_i(p), \; B_i := p_i$$

the associated spectral Hamiltonians. Then, by *Theorem 3.14 (Spectral Hamiltonians)*, to these correspond the self-adjoint operators

$$\begin{aligned}\widehat{A}_i &= \tfrac{-1}{2\pi i} \sum_k \left(\tfrac{1}{2}\partial_{p_k}(j^i_k(p)) + j^i_k(p)\partial_{p_k}\right) \\ \widehat{B}_i &= p_i \\ \widehat{B}'_i &= \sigma_i(p).\end{aligned}$$

From these, we now build the spectral warped Ladder operators $\widehat{L}_j$ and $\widehat{L}'_j$.

Definition 4.12 (Spectral-warped Ladder Operators)**.** Let σ be an analytic spectral diffeomorphism and

$$\widehat{A}_i = \tfrac{-1}{2\pi i} \sum_k \big(\tfrac{1}{2}\partial_{p_k}(j^i_k(p)) + j^i_k(p)\partial_{p_k}\big) f \tag{4.29}$$

$$\widehat{B}_i = p_i \tag{4.30}$$

$$\widehat{B}'_i = \sigma_i(p), \tag{4.31}$$

the associated quantized spectral Hamiltonians. Then

$$\widehat{L}_j := \widehat{A}_j + i\widehat{B}_j = \tfrac{-1}{2\pi i} \sum_k \big(\tfrac{1}{2}\partial_{p_k}(j^i_k(p)) + j^i_k(p)\partial_{p_k}\big) f + ip_j, \tag{4.32}$$

will be called the *warped Ladder operator*, and

$$\widehat{L}'_j := \widehat{A}_j + i\widehat{B}'_j = \tfrac{-1}{2\pi i} \sum_k \big(\tfrac{1}{2}\partial_{p_k}(j^i_k(p)) + j^i_k(p)\partial_{p_k}\big) f + i\sigma_j(p), \tag{4.33}$$

referred to as the *canonical warped Ladder operator*, both represented on the Fourier domain. ◆

Returning to our discussion of coordinates on phase space, we turn the observations above into a proposition.

Definition 4.13 (Optimal Aligned Waveform For Pairs Of Conjugate Variables)**.** Let $(A^k)_k$ and $(B_k)_k$ be (not necessarily canonical) coordinate functions on the $2n$-dimensional phase space. Denote with

$$(\widehat{A}^k)_k \quad \text{and} \quad (\widehat{B}_k)_k$$

their self-adjoint quantized pendants, represented on the Fourier domain. Then, a waveform, $\widehat{\varphi}$, is sait to be optimally aligned to the phase space coordinates above in the sense of *Terminology 4.8 (Optimal Alignment)*, and around a classical point, (α_k, β_k), in each of the n symplectic subspaces of phase space, if it fulfills

$$(\widehat{A}^k + i\mu\widehat{B}_k)\widehat{\varphi} := (\alpha + i\mu_k\beta)\widehat{\varphi}, \quad k = 1, \dots, n. \tag{4.34}$$

where the parameter $\mu_k \in \mathbb{R}$ determines whether the function aligns better along the B_k or the A^k coordinate. ◆

Proposition 4.14 (Uncertainty Minimization)**.** *The waveform $\widehat{\varphi}$ is optimally aligned in the sense of 4.13 if it equalizes the uncertainty inequality*

$$\big\|(\widehat{A}^k - \alpha)\widehat{\varphi}\big\| \, \big\|(\widehat{B}_k - \beta)\widehat{\varphi}\big\| \geq \tfrac{1}{4\pi} \big|\big\langle [\widehat{A}^k, \widehat{B}_k] f \, , \, f \big\rangle\big|,$$

as given in Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators) *and the parameter $\mu_k \in \mathbb{R}$ determines whether the function aligns better along the B_k or the A^k coordinate.* ◆

Proof. From *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)*, we know that $\widehat{\varphi}$ equalizes

$$\left\|(\widehat{A}^k - \alpha)\widehat{\varphi}\right\| \left\|(\widehat{B}_k - \beta)\widehat{\varphi}\right\| \geq \tfrac{1}{4\pi} \left|\left\langle [\widehat{A}^k, \widehat{B}_k] f \,,\, f \right\rangle\right|$$

if

$$(\widehat{A}^k - \alpha)\widehat{\varphi} = -i\mu_k(\widehat{B}_k - \beta)\widehat{\varphi},$$

which is (*4.34*).

Finally, to see the claim about μ, the equation above essentially equates the deviations from being eigenfunctions, that is, if $(\widehat{A}^k - \alpha)\widehat{\varphi} = 0$, than φ is a generalized eigenfunction of $\widehat{A}$, and if $(\widehat{B}_k - \beta)\widehat{\varphi} = 0$, than φ is a generalized eigenfunction of $\widehat{B}$. Thus, for $\mu = 1$, this deviations are of the same order. Thus, μ is the constant of proportionality between both deviations and the bigger μ, the more $\widehat{\varphi}$ deviates from being an eigenfunction of $\widehat{A}^k$ and the more closely it resembles an eigenfunction of $\widehat{B}_k$. ■

The proposition above does not make clear what happens for arbitrary pairs of generators, nor does it give explicit inequalities. Of course, one could argue that not only pairs of the same indices are relevant but all pairs, (A^i, B_k), for which the quantized Hamiltonians do not commute. Indeed, this makes sense, which brings us to our generalized uncertainty principle.

Theorem 4.15 (Uncertainty Principle of Optimal Alignment)**.** *Let σ be a spectral diffeomorphism and A^i, B_k the spectral Hamiltonians.*

Then

$$\left\|(\widehat{A}^i - \alpha)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B}_k - \beta)f\right\|_{\mathcal{S}_\sigma} \geq \tfrac{1}{4\pi} \left|\left\langle (J_\sigma^{-T})^i_k \widehat{f} \,,\, \widehat{f} \right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|, \quad i, k = 1, \ldots, n. \tag{4.35}$$

Moreover, the inequality turns into an equality, if and only if

$$\widehat{f} = Ce^{-2\pi\mu \int p_k \mathrm{d}\sigma_i} e^{2\pi(\mu\beta - i\alpha)\sigma_i} e^{-\frac{1}{2}\sum_n \int \partial_{p_n}\left(\frac{\sigma_n^{-1}}{\partial\sigma_i}\right)\mathrm{d}\sigma_i}, \tag{4.36}$$

for some $\mu \in \mathbb{R}_+$. ◆

Proof. Use *Corollary 3.15 (Commutators of Spectral Hamiltonians)* to find

$$\left[\widehat{A}^i \,,\, \widehat{B}_k\right] = -\tfrac{1}{2\pi i}(J_\sigma^{-T})^i_k$$

and then apply *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators).*

To see the form of the equalizing function, use *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)* again, and find

$$
\begin{aligned}
(\widehat{A}^i - \alpha)\widehat{f} &= -i\mu(\widehat{B}_k - \beta)\widehat{f} \\
\Leftrightarrow\ \widehat{A}^i\widehat{f} &= -i(\mu\widehat{B}_k - \mu\beta + i\alpha)\widehat{f} \\
\Leftrightarrow\ \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\widehat{f} + \sum_n(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\partial_{p_n}\widehat{f} &= -2\pi(\mu\widehat{B}_k - \mu\beta + i\alpha)\widehat{f} \\
\Leftrightarrow\ \sum_n(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\partial_{p_n}\widehat{f} &= -2\pi(\mu\widehat{B}_k - \mu\beta + i\alpha)\widehat{f} - \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\widehat{f} \\
\Leftrightarrow\ \partial_{\sigma_i}\widehat{f} &= -2\pi(\mu p_k - \mu\beta + i\alpha)\widehat{f} - \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\widehat{f}
\end{aligned}
$$

As usual, the form of this differential equation suggests that an exponential Ansatz is appropriate. Assuming $\widehat{f}$ is positive on all of $\mathrm{dom}(\sigma)$, we substitute $\widehat{f} = e^g$ and get

$$
\begin{aligned}
\partial_{\sigma_i}e^g &= -2\pi(\mu p_k - \mu\beta + i\alpha)e^g - \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})e^g \\
\Leftrightarrow\ \partial_{\sigma_i}g &= -2\pi(\mu p_k - \mu\beta + i\alpha) - \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i}) \\
\Leftrightarrow\ g &= \int \left(-2\pi(\mu p_k - \mu\beta + i\alpha) - \tfrac{1}{2}\sum_n \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\right)\mathrm{d}\sigma_i + \log(C) \\
&= -2\pi\mu\int p_k\mathrm{d}\sigma_i + 2\pi(\mu\beta - i\alpha)\sigma_i - \tfrac{1}{2}\sum_n\int \partial_{p_n}(\tfrac{\partial\sigma_n^{-1}}{\partial\sigma_i})\mathrm{d}\sigma_i + \log(C).
\end{aligned}
$$

Thus, after re-substitution, we arrive at

$$
\widehat{f} = Ce^{-2\pi\mu\int p_k\mathrm{d}\sigma_i}e^{2\pi(\mu\beta - i\alpha)\sigma_i}e^{-\frac{1}{2}\sum_n\int\partial_{p_n}\left(\frac{\partial\sigma_n^{-1}}{\partial\sigma_i}\right)\mathrm{d}\sigma_i},
$$

which was the claim. ∎

Even though the proof contained a lengthy calculation, it is not as rewarding as one might hope for, mainly for the following two reasons.

(i) the expression for the equalizing waveform is *an ugly one*, in the sense that it is in-transparent, whether the various sub-principles are compatible or mutually exclusive. And (ii), in order to define the "perfect" function, adapted to all coordinate lines simultaneously as good as possible, it is highly questionable, whether the description via pairs of non-commuting observables, like in the theorem above, is the right way to do it.

In fact, one can show, that there are cases in which no waveform exists, which equalizes all of the inequalities simultaneously, see, e.g., [9], for a counter-example in the case of the SIM(2) group (which is a special case of the uncertainty principle above, as we will see in the next chapter, which will be dedicated to applications). This points to the fact that for more general cases one should search for differential equations involving all generators at the same time instead of concentrating on sub-cases individually, while keeping fingers crossed to hope that these individual solutions miraculously conspire to a general solution.

In the end, this can only be expected, if the cases do really decompose into independent parts - completely analogous to the case of irreducible sub-representations or other instances of the general divide-and-conquer approach in mathematics.

In fact, one of these special cases, for which this is possible, is when the principles and their solutions are restricted to each of the factors of a tensor representation. Which is the one, the next corollary is all about - the case of a diagonal Jacobian.

Corollary 4.16 (Diagonal Jacobians). *Let everything as in* Theorem 4.15 (Uncertainty Principle of Optimal Alignment)*. Let moreover the Jacobian (and hence its transposed inverse), be a diagonal matrix. Then*

$$\sum_k \left\|(\widehat{A}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} \geq \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma^{-1})\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|. \tag{4.37}$$

Moreover, if f_k denote the equalizing waveforms for each of the uncertainty principles associated with the diagonal entries of the Jacobian, we have that its tensor product,

$$\mathcal{S}_\sigma \ni f_0 := \otimes_k f_k,$$

*is the equalizer of (*4.37*) and is the optimally aligned waveform for pairs of conjugate variables in* Proposition 4.14 (Uncertainty Minimization). ◆

Proof. Since the Jacobian is diagonal, only alike indices contribute. For each $k = 1, \ldots, n$, we have

$$\begin{aligned}\left\|(\widehat{A}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} &\geq \tfrac{1}{2}\left|\left\langle -\tfrac{1}{2\pi i}(J_\sigma^{-1})^k_k\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right| \\ &= \tfrac{1}{4\pi}\left|\left\langle (J_\sigma^{-1})^k_k\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|\end{aligned}$$

Adding all inequalities - and using the same f -, we get

$$\begin{aligned}\sum_k \left\|(\widehat{A}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} &\geq \sum_k \tfrac{1}{4\pi}\left|\left\langle (J_\sigma^{-1})^k_k\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right| \\ &\geq \tfrac{1}{4\pi}\left|\left\langle \sum_k (J_{\sigma^{-1}})^k_k\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right| \\ &= \tfrac{1}{4\pi}\left|\left\langle \operatorname{tr}(J_{\sigma^{-1}})\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right| \\ &= \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma^{-1})\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|,\end{aligned}$$

where we used the convexity of the modulus and the last line is due to $\operatorname{tr} J_{\sigma^{-1}} =: \operatorname{div}(\sigma^{-1})$.

Finally, since the Jacobian is diagonal, each σ_k^{-1} is dependent on ξ_k only, that is, $\sigma_k^{-1}(\vec{\xi}) = \sigma_k^{-1}(\xi_k)$, which means that all $\hat{f}_k$ are only dependent on ξ_k also, since

$$(\widehat{B}^k - \beta^k)\hat{f}_k = -i\mu(\widehat{A}_k - \alpha_k)\hat{f}_k$$

involves only the ξ_k coordinate. So, by defining the function

$$\widehat{f}_0(\vec{\xi}) \coloneqq (\oplus_k \widehat{f}_k)(\xi_k) = \prod_k \widehat{f}_k(\xi_k),$$

we have that

$$\begin{aligned}((\widehat{B}^k - \beta^k) + i\mu(\widehat{A}_k - \alpha_k))\widehat{f}_0 =& ((\widehat{B}^k - \beta^k) + i\mu(\widehat{A}_k - \alpha_k))\prod_j \widehat{f}_j \\ =& (\prod_{j\neq k} \widehat{f}_j)\left((\widehat{B}^k - \beta^k) + i\mu(\widehat{A}_k - \alpha_k)\right)\widehat{f}_k \\ =& (\prod_{j\neq k} \widehat{f}_j)\cdot 0 = 0,\end{aligned}$$

for each $k = 1, \ldots, n$. Hence, f_0 still equalizes all k uncertainty principles, and thus

$$\sum_k \left\|(\widehat{A}_k - \alpha_k)f_0\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B}^k - \beta^k)f_0\right\|_{\mathcal{S}_\sigma} = \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma^{-1})\widehat{f}_0\,,\, \widehat{f}_0\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|,$$

which finishes the proof. ∎

Remark 4.17 (Diagonal Jacobians)**.** Note that the case of diagonal Jacobians includes those for which the Jacobians contains one non-zero entry for each row and columns, as by a reordering of the components of σ, these can be brought into diagonal form.

This essentially amounts to a permutation of the coordinates. ◆

In *Corollary 4.16 (Diagonal Jacobians)*, the explicit form of the equalizing function has not been stated, which is due to the fact that it needs a further ingredient and will be given in the following corollary.

Corollary 4.18 (Equalizing Waveform for Diagonal Jacobians)**.** *Let σ be a spectral diffeomorphism, having a diagonal Jacobian. Then, the individual equalizers have the form*

$$\widehat{f}_k = C\cdot\left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-2\pi\mu_k\int p_k \mathrm{d}\sigma_k} e^{2\pi(\mu_k\beta_k - i\alpha^k)\sigma_k}, \tag{4.38}$$

and thus the equalizing function attains the rather pleasant form

$$\begin{aligned}\widehat{f_0} &= C\left|\det(J_\sigma)\right|^{1/2} e^{-2\pi\sum_k \mu_k \int p_k \mathrm{d}\sigma_k} e^{2\pi\sum_k(\mu_k\beta_k - i\alpha^k)\sigma_k} \\ &= C\left|\det(J_\sigma)\right|^{1/2} e^{-2\pi\int\langle\, \vec{\mu p}\,,\, \vec{\mathrm{d}\sigma}\,\rangle} e^{2\pi\langle\, \mu\beta - i\alpha\,,\, \sigma\,\rangle},\end{aligned} \tag{4.39}$$

where $\int_c \left\langle \vec{\mu p}\,,\, \vec{\mathrm{d}\sigma} \right\rangle$ *denotes a line integral along some path, c, starting from a fixed point and ending at* $\vec{\xi}$*.*

Its pendant on $\mathcal{A}_\sigma$ *is*

$$\widehat{\psi_0} := Ce^{-2\pi\int\langle\, \vec{\mu p}\,,\, \vec{\mathrm{d}\sigma}\,\rangle} e^{2\pi\langle\, \mu\beta - i\alpha\,,\, \sigma\,\rangle}. \tag{4.40}$$

◆

Proof. Since only alike indices contribute, due to the diagonality of the Jacobian, we have that each entry of the Jacobian is independent of the others and thus, the inverse function theorem applies to each of the diagonal entries individually. Thus $\frac{\partial\sigma_k^{-1}}{\partial\sigma_k}\circ\sigma = \left(\frac{\partial\sigma_k}{\partial p_k}\right)^{-1}$ is the inverse function of $\frac{\partial\sigma_k}{\partial p_k}$ for each k. Therefore,

$$\sum_n \partial_{p_n}\left(\frac{\partial\sigma_n^{-1}}{\partial\sigma_k}\circ\sigma\right) = \partial_{p_k}\left(\frac{\partial\sigma_k^{-1}}{\partial\sigma_k}\circ\sigma\right) = \partial_{p_k}\left(\frac{\partial\sigma_k}{\partial p_k}\right)^{-1} = -\frac{\partial_{p_k}\left(\frac{\partial\sigma_k}{\partial p_k}\right)}{\left(\frac{\partial\sigma_k}{\partial p_k}\right)^2}.$$

Integrating with respect to $\mathrm{d}\sigma_k$, we get

$$\begin{aligned}-\int \frac{\partial_{p_k}\left(\frac{\partial\sigma_k}{\partial p_k}\right)}{\left(\frac{\partial\sigma_k}{\partial p_k}\right)^2}\mathrm{d}\sigma_k = -\int \frac{\partial_{p_k}\left(\frac{\partial\sigma_k}{\partial p_k}\right)}{\left(\frac{\partial\sigma_k}{\partial p_k}\right)^2}\frac{\partial\sigma_k}{\partial p_k}\,\mathrm{d}p_k &= -\int \frac{\partial_{p_k}\left(\frac{\partial\sigma_k}{\partial p_k}\right)}{\left(\frac{\partial\sigma_k}{\partial p_k}\right)}\,\mathrm{d}p_k \\ &= -\log\left|\frac{\partial\sigma_k}{\partial p_k}\right|.\end{aligned}$$

Inserting into the expression for the equalizer, we get

$$\begin{aligned}\widehat{f_k}(\xi_k) &= Ce^{-2\pi\mu_k\int p_k\mathrm{d}\sigma_k} e^{2\pi(\mu_k\beta_k - i\alpha^k)\sigma_k(\xi_k)} e^{-\frac{1}{2}\left(-\log\left|\frac{\partial\sigma_k}{\partial p_k}\right|\right)} \\ &= C\cdot\left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-2\pi\mu_k\int p_k\mathrm{d}\sigma_k} e^{2\pi(\mu_k\beta_k - i\alpha^k)\sigma_k(\xi_k)}.\end{aligned}$$

Finally, taking the tensor product of all individual equalizers, we arrive at

$$\begin{aligned}\widehat{f} := (\otimes_k\widehat{f_k}) &= \prod_k \widehat{f_k} \\ &= C\prod_k \left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-2\pi\mu_k\int p_k\mathrm{d}\sigma_k} e^{2\pi(\mu_k\beta_k - i\alpha^k)\sigma_k} \\ &= C\prod_k \left(\left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2}\right) e^{-2\pi\sum_k\mu_k\int p_k\mathrm{d}\sigma_k} e^{2\pi\sum_k(\mu_k\beta_k - i\alpha^k)\sigma_k} \\ &= C\left|\det(J_\sigma)\right|^{1/2} e^{-2\pi\int_c\langle\, \vec{\mu p}\,,\, \vec{\mathrm{d}\sigma}\,\rangle} e^{2\pi\langle\, \mu\beta - i\alpha\,,\, \sigma\,\rangle},\end{aligned}$$

where $\int_c \langle \vec{\mu p}, \mathrm{d}\vec{\sigma} \rangle$ is interpreted as a path integral along some path, starting at a fixed point and ending at $\vec{\xi}$.

Finally,

$$\widehat{\psi}_0 := \widehat{\iota_{\mathcal{S} \to \mathcal{A}} f} = Ce^{-2\pi \int_c \langle \vec{\mu p}, \mathrm{d}\vec{\sigma} \rangle} e^{2\pi \langle \mu\beta - i\alpha, \sigma \rangle},$$

which was to be demonstrated. ■

Although the principle above is nice in the sense that it incorporates all of the non-commuting observables and a proof that there exists a simultaneous equalizer for all of the sub-principles, it is a somewhat "botchy" inequality as no explicit treatment of a true n-dimensional uncertainty principle is given, which we try to catch up on in the next theorem. In the proof of *Corollary 4.32 (Diagonal Jacobians)*, the fact was used, that the tensor product of all individual equalizers is an equalizer of the extended principle. The principle shows what to expect of a general principle, if it existed. Since the Hamiltonians $\widehat{A}^k$ induce flows along the rectangular grid in the symplectically "warped" phase space, it makes sense to define a vector field for these

$$\widehat{\widehat{\mathcal{W}}}_\sigma^* \nabla \widehat{\widehat{\mathcal{W}}}_\sigma \widehat{\psi} \in \mathbb{C}^n \otimes \mathcal{F}\mathcal{A}_\sigma,$$

which leads to

$$\left(\nabla(\widehat{\psi} \circ \sigma^{-1})\right) \circ \sigma = J_\sigma^{-T} \nabla \widehat{\psi}, \tag{4.41}$$

since vector fields transform tensorially.

Pictured in phase space, these derivatives are along the coordinate lines, defined by the spectral diffeomorphism respectively its cotangent lift. Thus, these curvilinear derivatives induce the Hamiltonian flows along these coordinate lines and equating with $-2\pi\vec{\mu}\xi\widehat{\psi}$ gives

$$J_\sigma^{-T} \nabla \widehat{\psi} = -2\pi\vec{\mu}\xi\widehat{\psi},$$

respectively

$$J_\sigma^{-T} \nabla \widehat{\psi} = -2\pi(\vec{\mu}\xi - \vec{\beta} + i\vec{\alpha})\widehat{\psi}, \tag{4.42}$$

when we include the constants. Above, we have been rather sloppy, since the fundamental differential object, assigned to a function is not its gradient but its total differential, which is definable for every differentiable function on a differentiable manifold. The definition of a gradient, on the other hand, demands the existence of a non-degenerate two-form by means of which one may identify a differential form with a vector field. Although $\mathrm{dom}(\sigma)$ as a subspace of $\widehat{\mathbb{R}}^n$ can be equipped with the standard two-form from Euclidean space, it makes sense to restate the above as

$$\nabla\widehat{\psi} = -2\pi J_\sigma^T(\vec{\mu\xi} - \vec{\mu\beta} + i\vec{\alpha})\widehat{\psi},$$

by the invertibility of a diffeomorphism's Jacobian and get

$$\mathrm{d}\widehat{\psi} = -2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial \xi_k}(\mu_j\xi_j - \mu_j\beta_j + i\alpha_j)\widehat{\psi}\ \mathrm{d}\xi_k, \tag{4.43}$$

which is a one-form.

Definition 4.19 (Weighted Diffeomorphism)**.** Let σ be a spectral diffeomorphism and $\mu \in \mathbb{R}^n$, then

$$(\sigma_\mu)_k := (\mu_k\sigma_k)_k \tag{4.44}$$

shall denote the *weighted diffeomorphism.* ◆

Lemma 4.20 (Symmetry Condition)**.** *Let J_σ be the Jacobian of the diffeomorphism σ and define $\sigma_\mu := (\mu_k\sigma_k)_k$. Then, there exists a solution ψ, such that*

$$\mathrm{d}\widehat{\psi} = -2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial \xi_k}(\mu_j\xi_j - \mu_j\beta_j + i\alpha_j)\widehat{\psi}\ \mathrm{d}\xi_k. \tag{4.45}$$

if the Jacobian matrix of the weighted diffeomorphism σ_μ is a symmetric matrix,

$$J_{\sigma_\mu} = J_{\sigma_\mu}^T, \tag{4.46}$$

and if $\mathrm{dom}(\sigma)$ *is simply-connected.* ◆

Proof. Using the Ansatz $\widehat{\psi} = e^g$, with $\vec{\xi} = (x_1, \dots, x_n)$, leads to

$$\mathrm{d}g = -2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial x_k}(\mu_j\xi_j - \mu_j\beta_j + i\alpha_j)\mathrm{d}x_k,$$

which is an equation for an exact form - interpretable as a conservative vector field - and its potential. There exists a potential, g, for $-2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial x_k}(\mu_j\xi_j - \mu_j\beta_j + i\alpha_j)\mathrm{d}x_k$ if it is a closed form,

$$\begin{aligned}
&\mathrm{d}\left(-2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial x_k}(\mu_j\xi_j - \mu_j\beta_j + i\alpha_j)\mathrm{d}x_k\right)\\
&= \mathrm{d}\left(-2\pi \sum_k \sum_j \frac{\partial \sigma_j}{\partial x_k}(\mu_j\xi_j)\mathrm{d}x_k\right)\\
&=0,
\end{aligned}$$

and the domain is simply-connected. Since we assume the space to be simply-connected, only its closed-ness is left to be proven. Since $\mathrm{d}x_j \wedge \mathrm{d}x_k = -\mathrm{d}x_k \wedge \mathrm{d}x_j$, we have

$$\begin{aligned}\mathrm{d}\left(-2\pi \sum_k \sum_i \frac{\partial \sigma_i}{\partial x_k}\mu_i x_i \ \mathrm{d}x_k\right) &= -2\pi \sum_i \sum_k \sum_j \partial_j \left(\frac{\partial \sigma_i}{\partial x_k}\mu_i x_i\right) \ \mathrm{d}x_j \wedge \mathrm{d}x_k \\ &= 2\pi \sum_i \sum_k \sum_{j>k} \left(\partial_j \left(\frac{\partial \sigma_i}{\partial x_k}\mu_i x_i\right) - \partial_k \left(\frac{\partial \sigma_i}{\partial x_j}\mu_i x_i\right)\right) \ \mathrm{d}x_k \wedge \mathrm{d}x_j,\end{aligned}$$

which vanishes if and only if

$$\sum_i \partial_j \left(\frac{\partial \sigma_i}{\partial x_k}\mu_i x_i\right) - \sum_i \partial_k \left(\frac{\partial \sigma_i}{\partial x_j}\mu_i x_i\right) = 0,$$

that is

$$\sum_i \partial_j \left(\frac{\partial (\mu\sigma)_i}{\partial x_k} x_i\right) = \sum_i \partial_k \left(\frac{\partial (\mu\sigma)_i}{\partial x_j} x_i\right)$$

for all $j, k = 1, \ldots, n$. Explicitly, this means that

$$\begin{aligned}\sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_j \partial x_k} x_i + \tfrac{\partial(\sigma_\mu)_i}{\partial x_k}\tfrac{\partial x_i}{\partial x_j}\right) &= \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_k \partial x_j} x_i + \tfrac{\partial(\sigma_\mu)_i}{\partial x_j}\tfrac{\partial x_i}{\partial x_k}\right) \\ \Leftrightarrow \quad \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_j \partial x_k} x_i\right) + \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_k}\delta_i^j\right) &= \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_k \partial x_j} x_i\right) + \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_j}\delta_i^k\right) \\ \Leftrightarrow \quad \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_k}\delta_i^j\right) &= \sum_i \left(\tfrac{\partial(\sigma_\mu)_i}{\partial x_j}\delta_i^k\right) \\ \Leftrightarrow \quad \tfrac{\partial(\sigma_\mu)_j}{\partial x_k} &= \tfrac{\partial(\sigma_\mu)_k}{\partial x_j}\end{aligned}$$

by the commutativity of partial derivatives. But this means that

$$J_{\sigma_\mu} = J_{\sigma_\mu}^T,$$

which was the assertion. ∎

Note that the closed-ness of the one-form in the proof above means, reinterpreted as a vector field, that it fulfills the *integrability condition*

$$\partial_n(-2\pi J_\sigma^T \vec{\mu}\xi)_j = \partial_j(-2\pi J_\sigma^T \vec{\mu}\xi)_n. \tag{4.47}$$

Using this lemma, we may now generalize the uncertainty principle a step further, although the discussion following it shows that its usability is narrow and will thus not be pursued any further hereafter.

Theorem 4.21 (Optimal Alignment for Weighted Jacobians)**.** *Let σ be a spectral diffeomorphism with simply-connected domain and symmetric weighted Jacobian, as demanded by* Lemma 4.20 (Symmetry Condition)*, and let further $\psi \in \mathcal{A}_\sigma$. Then,*

$$\left\| \left(\tfrac{-1}{2\pi i} J_\sigma^{-T} \nabla - \vec{\alpha}\right)\widehat{\psi} \right\| \left\| (\vec{\xi} - \vec{\alpha})\widehat{\psi} \right\| \geq \tfrac{1}{4\pi} \left| \left\langle \operatorname{div}(\sigma^{-1})\widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right|, \tag{4.48}$$

with $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$, is the generalized uncertainty principle for symmetric Jacobians and

$$\widehat{\psi}(\xi) := C(\xi_0) e^{-2\pi \int_c \langle \vec{\xi}\,,\, \mathrm{d}\sigma_\mu \rangle} e^{2\pi \langle \mu\beta - i\alpha\,,\, \sigma \rangle}, \tag{4.49}$$

for some path, c, starting at ξ_0 and ending at ξ, is the equalizing waveform.

As usual, a final map

$$\psi \;\mapsto\; \iota_{\mathcal{A} \to \mathcal{S}} \psi =: f,$$

gives the solution

$$f(\xi) := \left|\det(J_\sigma)\right|^{1/2} e^{g(\xi_0)} e^{-2\pi \int_c \langle \vec{\xi}\,,\, \mathrm{d}\sigma_\mu \rangle} e^{2\pi \langle \mu\beta - i\alpha\,,\, \sigma \rangle}, \tag{4.50}$$

on $\mathcal{S}_\sigma$. ◆

Proof. Cauchy-Schwarz gives

$$\begin{aligned}
\left\| \left(\tfrac{-1}{2\pi i} J_\sigma^{-T} \nabla - \vec{\alpha}\right)\widehat{\psi} \right\| \left\| (\vec{\xi} - \vec{\alpha})\widehat{\psi} \right\| &\geq \left| \left\langle \tfrac{-1}{2\pi i} J_\sigma^{-T} \nabla \widehat{\psi}\,,\, \vec{\xi}\widehat{\psi} \right\rangle \right| \\
&= \left| \tfrac{-1}{2\pi i} \sum_k \left\langle \Big(\sum_j \big(\tfrac{\partial \sigma_j^{-1} \circ \sigma}{\partial \xi_k} \partial_{\xi_j}\big) - \beta_k\Big)\widehat{\psi}\,,\, (\xi_k - \alpha_k)\widehat{\psi} \right\rangle \right|
\end{aligned}$$

and using the same arguments as has been used in the derivation of the uncertainty principles above, we have

$$\begin{aligned}
\left| \tfrac{-1}{2\pi i} \sum_k \left\langle \Big(\sum_j \big(\tfrac{\partial \sigma_j^{-1} \circ \sigma}{\partial \xi_k} \partial_{\xi_j}\big) - \beta_k\Big)\widehat{\psi}\,,\, (\xi_k - \alpha_k)\widehat{\psi} \right\rangle \right| &\geq \left| \tfrac{-1}{2\pi i} \sum_{k,j} \tfrac{1}{2} \left\langle \left[\tfrac{\partial \sigma_j^{-1} \circ \sigma}{\partial \xi_k} \partial_{\xi_j}\,,\, \xi_k \right] \widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right| \\
&= \tfrac{1}{4\pi} \left| \left\langle \sum_{k,j} \tfrac{\partial \sigma_j^{-1} \circ \sigma}{\partial \xi_k} \left[\partial_{\xi_j}\,,\, \xi_k \right] \widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right| \\
&= \tfrac{1}{4\pi} \left| \left\langle \sum_{k,j} \tfrac{\partial \sigma_j^{-1} \circ \sigma}{\partial \xi_k}\; \delta_k^j\; \widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right| \\
&= \tfrac{1}{4\pi} \left| \left\langle \sum_k \tfrac{\partial \sigma_k^{-1} \circ \sigma}{\partial \xi_k} \widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right| \\
&= \tfrac{1}{4\pi} \left| \left\langle \operatorname{tr} J_\sigma^{-1} \widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right| \\
&= \tfrac{1}{4\pi} \left| \left\langle \operatorname{div}(\sigma^{-1})\widehat{\psi}\,,\, \widehat{\psi} \right\rangle \right|,
\end{aligned}$$

and thus

$$\left\| \left(\tfrac{-1}{2\pi i} J_\sigma^{-T} \nabla - \vec{\alpha} \right) \widehat{\psi} \right\| \left\| (\vec{\xi} - \vec{\alpha}) \widehat{\psi} \right\| \geq \tfrac{1}{4\pi} \left| \left\langle \operatorname{div}(\sigma^{-1}) \widehat{\psi} \, , \, \widehat{\psi} \right\rangle \right| .$$

To find the equalizing waveform, we note that the steps in the derivation of the inequality are the same as above and thus, equality is attained for some $\psi \in \mathcal{A}_\sigma$, fulfilling

$$\tfrac{-1}{2\pi i} J_\sigma^{-T} \nabla \widehat{\psi} = -i(\vec{\mu\xi} - \vec{\beta} + i\vec{\alpha}) \widehat{\psi},$$

which leads to

$$\nabla \widehat{\psi} = -2\pi J_\sigma^T (\vec{\mu\xi} - \vec{\beta} + i\vec{\alpha}) \widehat{\psi},$$

and using the Ansatz $\widehat{\psi} = e^g$, again, we end up with

$$\nabla g = -2\pi J_\sigma^T (\vec{\mu\xi} - \vec{\beta} + i\vec{\alpha}),$$

which is an equation for a conservative vector field and its potential. But, by *Lemma 4.20 (Symmetry Condition)*, there exists a solution, g, if and only if the weighted Jacobian is symmetric and the domain is simply-connected; which is assumed to hold, so there's a solution and the potential may be found by a line integral of the form

$$g(\xi) = -2\pi \int_c \left\langle J_\sigma^T (\vec{\mu\xi} - \vec{\beta} + i\vec{\alpha}) \, , \, \mathrm{d}\vec{\xi} \right\rangle + g(\xi_0),$$

again for some path, c, from ξ_0 to ξ. Noting that

$$\begin{aligned} \left\langle J_\sigma^T (\vec{\mu\xi} - \vec{\beta} + i\vec{\alpha}) \, , \, \mathrm{d}\vec{\xi} \right\rangle &= \left\langle \vec{\mu\xi} - \vec{\beta} + i\vec{\alpha} \, , \, J_\sigma \mathrm{d}\vec{\xi} \right\rangle \\ &= \left\langle \vec{\mu\xi} - \vec{\beta} + i\vec{\alpha} \, , \, \mathrm{d}\vec{\sigma} \right\rangle, \end{aligned}$$

and re-substituting for $\widehat{\psi}$, gives

$$\widehat{\psi}(\xi) = e^{g(\xi)} = e^{g(\xi_0)} e^{-2\pi \int_c \langle \vec{\xi} \, , \, \mathrm{d}\sigma_\mu \rangle} e^{2\pi \langle \mu\beta - i\alpha \, , \, \sigma \rangle},$$

which was the main point to be demonstrated. Since the principle above was formulated on $\mathcal{A}_\sigma$, a final $\widehat{\psi} \mapsto \iota_{\mathcal{A} \to \mathcal{S}} \widehat{\psi} =: \widehat{f}$

gives the solution

$$\widehat{f}(\xi) := \left| \det(J_\sigma) \right|^{1/2} e^{g(\xi_0)} e^{-2\pi \int_c \langle \vec{\xi} \, , \, \mathrm{d}\sigma_\mu \rangle} e^{2\pi \langle \mu\beta - i\alpha \, , \, \sigma \rangle}, \tag{4.51}$$

on $\mathcal{S}_\sigma$, which finishes the proof. ■

Unfortunately, there is no fixed ordering in the components of a spectral diffeomorphism, and, therefore, all re-orderings of its Jacobian's are possible, which somewhat limits the usability of the theorem above, as different orderings lead to completely different solutions, partially nonsensical. The following example shall make this clear.

Let σ be the identical diffeomorphism in two-dimensions,

$$\sigma : (x, y) \;\mapsto\; (x, y),$$

then its Jacobian is the identity and solving

$$\mathrm{d}g(\xi) = -2\pi\left((1x + 0y)\mathrm{d}x + (0x + 1y)\mathrm{d}y\right) = -2\pi(x \;\mathrm{d}x + y \;\mathrm{d}y),$$

gives

$$g(\xi) = -2\pi(x^2/2 + y^2/2) + \log(C).$$

A re-substitution $\widehat{f} := e^g$ gives

$$\widehat{f(\xi)} = e^{g(\xi)} = Ce^{-\pi(x^2+y^2)},$$

which is a reasonable solution.

Swapping the components,

$$\sigma : (x, y) \;\mapsto\; (y, x),$$

however, gives

$$J_\sigma(x, y) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and solving

$$\mathrm{d}g(\xi) = -2\pi\left((0x + 1y)\mathrm{d}x + (1x + 0y)\mathrm{d}y\right) = -2\pi(y \;\mathrm{d}x + x \;\mathrm{d}y),$$

gives

$$g(\xi) = -2\pi(xy + yx) + \log(C),$$

After a re-substitution $\widehat{f} := e^g$, we have

$$\widehat{f(\xi)} = e^{g(\xi)} = Ce^{-4\pi(x\cdot y)},$$

which is nonsensical.

Remark 4.22 (A Note on Implication). (i) It is worthwhile to stop here for a moment and discuss the meaning of the minimizing waveforms above. To begin with, the solutions above, if they exist, do not necessarily equalize the individual principles - they do, if the Jacobian is diagonal -, since these are no longer independent of each other. As a matter of fact, a derivative along the curvilinear coordinate given by, say, σ_k, is not necessarily constraint to a single, two-dimensional symplectic subspace of phase space and thus, an uncertainty principle only makes sense, if it incorporates all of the domains affected by this derivation, which are exactly the ones, for which the Jacobian does not vanish. The equation (*4.51*), written as a linear system of coupled ordinary differential equations, reads

$$\begin{array}{lll} \frac{\partial f}{\partial_{\xi_1}} & = -2\pi & \left(\frac{\partial \sigma_1}{\partial_{\xi_1}}(\mu_1\xi_1 - \mu_1\beta_1 + i\alpha_1) + \cdots + \frac{\partial \sigma_n}{\partial_{\xi_1}}(\mu_n\xi_n - \mu_n\beta_n + i\alpha_n)\right) f \\ \vdots & \quad\vdots & \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \frac{\partial f}{\partial_{\xi_n}} & = -2\pi & \left(\frac{\partial \sigma_1}{\partial_{\xi_n}}(\mu_1\xi_1 - \mu_1\beta_1 + i\alpha_1) + \cdots + \frac{\partial \sigma_n}{\partial_{\xi_n}}(\mu_n\xi_n - \mu_n\beta_n + i\alpha_n)\right) f \end{array}, \qquad (4.52)$$

which is the system, the solution function $f \in \mathcal{S}_\sigma$ (or $\psi \in \mathcal{A}_\sigma$) solves individually.

(ii) Moreover, when doing signal analysis, one is interested in a waveform, ψ, which not only equalizes an uncertainty principle but is also admissible, which in the language of this monograph reads $\psi \in \mathcal{A}_\sigma$.

The construction above, however, reveals that this is not necessarily compatible, as the differential equations, which a minimizing waveform has to satisfy, are derived from the generators, which are - by construction - defined on $\mathcal{S}_\sigma$ and not on $\mathcal{A}_\sigma$. That is, the Hamiltonians $\widehat{A}^i$ generate flows, which are unitary on $\mathcal{S}_\sigma$, but not on $\mathcal{A}_\sigma$; $\widehat{B}'_k$ and $\widehat{B}_k$ are actually the same operators on both spaces and thus do not contribute to a better understanding here. Taking the loss of anticipating some of the later results, an example for the one-dimensional wavelet transform shall make this clear. Let $\widehat{A} := \frac{-1}{2\pi i}\left(\frac{1}{2} + \xi\partial_\xi\right)$ be the infinitesimal generator of dilation, defined on the Fourier domain, inducing the flow

$$\left(e^{-2\pi i\left(\frac{-1}{2\pi i}\left(\frac{1}{2}+\xi\partial_\xi\right)\right)t}\widehat{f}\right)(\xi) = e^{t/2}\left(e^{\xi\partial_\xi t}\widehat{f}\right)(\xi) := e^{t/2}\widehat{f}(e^t\xi), \quad f \in \mathcal{S}_{\log}, \qquad (4.53)$$

which is unitary on $\mathcal{S}_{\log}$,

$$\left\| e^{-2\pi i\left(\frac{-1}{2\pi i}\left(\frac{1}{2}+\xi\partial_\xi\right)\right)t}\widehat{f}\right\|_{\mathcal{S}_{\log}} = \|\widehat{f}\|_{\mathcal{S}_{\log}}, \quad f \in \mathcal{S}_{\log},$$

but not on $\mathcal{A}_{\log}$. Let further $\widehat{B} := \xi$ be the generator of spatial or temporal translation, again defined on the Fourier domain. Then, for $\xi > 0$, some adequate α, β and $\widehat{f}$, we have

$$\begin{aligned}(\widehat{A} - \alpha)\widehat{f} &= -i\mu(\widehat{B} - \beta)\widehat{f} \\ \Leftrightarrow \ \left(\tfrac{-1}{2\pi i}\left(\tfrac{1}{2} + \xi\partial_\xi\right) - \alpha\right)\widehat{f} &= -i\mu(\xi - \beta)\widehat{f} \\ \Leftrightarrow \ \xi\partial_\xi\widehat{f} &= \left(\tfrac{1}{2} - 2\pi i\alpha + 2\pi\mu\beta - 2\pi\mu x\right)\widehat{f},\end{aligned}$$

which is solved by

$$\widehat{f}(\xi) := \xi^{-1/2}e^{-2\pi\mu\xi}\xi^{2\pi\mu\beta}e^{-2\pi i\alpha}. \tag{4.54}$$

This function is, a priori, for well-chosen constants, thought to be defined on

$$\mathcal{S}_{\log} \ni f := \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, \widehat{f} \in L^2(\mathbb{R}_+,\ \mathrm{d}\xi) \right\}$$

by construction - since the operators act on this space in the sense that their induced unitary flows act on $\mathcal{S}_\sigma$.

Although it is possible to tweak the constant, such that f is also an admissible wavelet, this is not the way to go for more general relations. It is more natural - and correct - to use $\iota_{\mathcal{S}\to\mathcal{A}}$, from *Proposition 3.6 (Measure Mappings)* to map this waveform to the space of the admissible wavelets, which in the case of the wavelet transform means

$$\begin{aligned}\widehat{f} \ \mapsto \ \widehat{\iota_{\mathcal{S}\to\mathcal{A}}f} &= \left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{-1/2}\widehat{f} \\ &= \xi^{+1/2}\widehat{f} \\ &= e^{-2\pi\mu\xi}\xi^{2\pi\mu\beta}e^{-2\pi i\alpha},\end{aligned} \tag{4.55}$$

which cancels the factor $\xi^{-1/2}$, which is exactly $\left(\frac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{1/2}$. Even more is true. The waveform (*4.55*),

$$\widehat{\psi}(\xi) = e^{-2\pi\mu\xi}\xi^{2\pi\mu\beta}e^{-2\pi i\alpha} \qquad \in \mathcal{A}_{\log} \tag{4.56}$$

is the solution to

$$\begin{aligned}\iota_{\mathcal{S}\to\mathcal{A}}(\widehat{A} - \alpha)\iota_{\mathcal{A}\to\mathcal{S}}\widehat{\psi} &= -i\mu\iota_{\mathcal{S}\to\mathcal{A}}(\widehat{B} - \beta)\iota_{\mathcal{A}\to\mathcal{S}}\widehat{\psi} \\ \left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{-1/2}(\widehat{A} - \alpha)\left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{1/2}\widehat{\psi} &= -i\mu\left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{-1/2}(\widehat{B} - \beta)\left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{1/2}\widehat{\psi} \\ \Leftrightarrow \ \left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{-1/2}\left(\tfrac{-1}{2\pi i}\left(\tfrac{1}{2} + \xi\partial_\xi\right) - \alpha\right)\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}^{1/2}\widehat{\psi} &= -i\left(\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}\right)^{-1/2}\mu(\xi - \beta)\tfrac{\mathrm{d}\log}{\mathrm{d}\xi}^{1/2}\widehat{\psi} \\ \Leftrightarrow \ \left(\tfrac{-1}{2\pi i}\xi\partial_\xi - \alpha\right)\widehat{\psi} &= -i\mu(\xi - \beta)\widehat{\psi},\end{aligned}$$

which now equates the appropriate - unitarily equivalent, but nonetheless different - generators defined on $\mathcal{A}_\sigma$, on which they induce the appropriate unitary flows.

Summarizing, solving

$$\left(\tfrac{-1}{2\pi i}\xi\partial_\xi - \alpha\right)\widehat{\psi} = -i\mu(\xi - \beta)\widehat{\psi}, \tag{4.57}$$

leads to an admissible wavelet on $\mathcal{A}_{\log}$, ready to be used for the wavelet transform, whereas a solution to the equation

$$\left(\tfrac{-1}{2\pi i}\left(\tfrac{1}{2} + \xi\partial_\xi\right) - \alpha\right)\widehat{f} = -i\mu(\xi - \beta)\widehat{f}, \tag{4.58}$$

is not a priori in the space of admissible wavelets and therefore a final map

$$\mathcal{S}_{\log} \ni f \;\mapsto\; \iota_{\mathcal{S}\to\mathcal{A}} f \in \mathcal{A}_{\log}$$

ought to be done. ◆

With the experience above at hand, we may now prove the following special case.

Corollary 4.23 (The Canonical Case). *Let σ be an analytic spectral diffeomorphism,* $\dim(\mathrm{dom}(\sigma)) = n$ *and* $(L'_k)_k := \widehat{A}^k + i\mu_k\widehat{B}'_k$ *the* canonical warped Ladder operators. *Then,*

$$\sum_k \left\|(\widehat{A}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B'}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} \geq \tfrac{n}{4\pi}\left\|\widehat{f}\right\|^2_{\mathcal{S}_\sigma} \tag{4.59}$$

and the simultaneous equalizer is a warped Gaussian of the form

$$\widehat{f_0}(\xi) := C\left|\det(J_\sigma)\right|^{1/2} e^{-\pi|\sigma_\mu(\xi)|^2} e^{2\pi\langle\, \mu^2\beta - i\alpha\,,\,\sigma\,\rangle}, \tag{4.60}$$

with $|\sigma_\mu(\xi)|^2 := \sum_k \mu_k^2\sigma_k^2(\xi)$.

Setting $C = 2^{\frac{n}{4}}, \mu \equiv 1$ *and* $\alpha = \beta = 0$ *we get the normalized, simultaneous ground state of all Ladder operators, which reads*

$$\widehat{f_0}(\xi) := 2^{n/4}\left|\det(J_\sigma)\right|^{1/2} \cdot e^{-\pi|\sigma(\xi)|^2}, \quad \|f\|_{\mathcal{S}_\sigma} = 1, \tag{4.61}$$

with its admissible pendant reading

$$\widehat{\psi_0}(\xi) := 2^{n/4} e^{-\pi|\sigma(\xi)|^2}, \quad \|\psi\|_{\mathcal{A}_\sigma} = 1. \tag{4.62}$$

◆

Proof. The proof is surprisingly simple, but nonetheless somewhat lengthy. Since the warped canonical Ladder operator

$$(L'_k)_k := \widehat{A}^k + i\widehat{B}'_k,$$

consists of pairs of canonically conjugate operators, we have by *Theorem 3.15 (Commutators of Spectral Hamiltonians)* that the associated matrix of commutation relations - which replaces the Jacobian in this context - is diagonal and in fact the $-\frac{1}{2\pi i}$-multiple of the identity. Hence, the same arguments as in *Corollary 4.16 (Diagonal Jacobians)*, only with $B_k := p_k$ replaced by the canonical $B'_k := \sigma_k$, lead to the expression

$$\begin{aligned}\sum_k \left\|(\widehat{A}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{B'}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} &\geq \tfrac{1}{2}\left|\left\langle \sum_k \left[\, \widehat{A}_k \,,\, \widehat{B'}^k \,\right] \widehat{f} \,,\, \widehat{f} \right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right| \\ &= \tfrac{n}{4\pi} \left\langle \widehat{f} \,,\, \widehat{f} \right\rangle_{\mathcal{F}\mathcal{S}_\sigma}.\end{aligned}$$

Moreover, using μ_k^2 to get a nicer formula, we get

$$\begin{aligned}\widehat{f_k}(\xi_k) &= Ce^{-2\pi\mu_k^2 \int \sigma_k \mathrm{d}\sigma_k} e^{2\pi(\mu_k^2\beta_k - i\alpha^k)\sigma_k} e^{\frac{1}{2}\log\left|\frac{\partial\sigma_k}{\partial p_k}\right|} \\ &= C \cdot \left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-2\pi \int \mu_k^2 \sigma_k \mathrm{d}\sigma_k} e^{2\pi(\mu_k^2\beta_k - i\alpha^k)\sigma_k}\end{aligned}$$

for the individual equalizers. Since $\int \sigma_k \mathrm{d}\sigma_k = \frac{1}{2}\sigma_k^2$, this becomes

$$\widehat{f_k}(\xi_k) = C \cdot \left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-\pi\mu_k^2\sigma_k^2(\xi_k)} e^{2\pi(\mu_k^2\beta_k - i\alpha^k)\sigma_k(\xi_k)}$$

and taking the tensor product of all individual equalizers, again, we arrive at

$$\begin{aligned}\widehat{f}(\vec{\xi}) := (\otimes_k \widehat{f_k})(\xi_k) &= \prod_k \widehat{f_k}(\xi_k) \\ &= C \prod_k \left|\frac{\partial\sigma_k}{\partial p_k}\right|^{1/2} e^{-\pi\mu_k^2\sigma_k^2(\xi_k)} e^{2\pi(\mu_k^2\beta_k - i\alpha^k)\sigma_k(\xi_k)} \\ &= C \left|\det(J_\sigma)\right|^{1/2} e^{-\pi \sum_k \mu_k^2\sigma_k^2(\xi_k)} e^{2\pi \sum_k (\mu_k^2\beta_k - i\alpha^k)\sigma_k(\xi_k)} \\ &= C \left|\det(J_\sigma)\right|^{1/2} e^{-\pi|\sigma_\mu(\xi)|^2} e^{2\pi\langle\, \mu^2\beta - i\alpha \,,\, \sigma \,\rangle}\end{aligned}$$

Setting the constants to $C = 2^{n/4}$,$\mu = 1$ and $\alpha = \beta = 0$, we arrive at

$$\widehat{f_0} := 2^{n/4} \left|\det\left(J_\sigma\right)\right|^{1/2}\ e^{-\pi|\sigma|^2},$$

and

$$\begin{aligned}\widehat{\psi}_0 (= \iota_{\mathcal{S} \to \mathcal{A}} \widehat{f_0} (&= \left|\det\left(J_\sigma\right)\right|^{-1/2} \widehat{f_0} \\ &= 2^{n/4} e^{-\pi|\sigma|^2}.\end{aligned}$$

Normalization can be checked easily by

$$\begin{aligned}\|f_0\|^2_{\mathcal{S}_\sigma} &:= 2^{n/2}\int_{\mathrm{dom}(\sigma)} |\det(J_\sigma)|\, e^{-2\pi|\sigma(\xi)|^2} \mathrm{d}\xi \\ &= 2^{n/2}\int_{\mathrm{dom}(\sigma)} e^{-2\pi|\sigma(\xi)|^2} \mathrm{d}\sigma := \|\psi_0\|_{\mathcal{A}_\sigma} \\ &= 2^{n/2}\int_{\mathrm{im}(\sigma)} e^{-\pi|\sqrt{2}y|^2} 2^{n/2}\ \mathrm{d}y \\ &= 2^{n/2}\int_{\mathbb{R}^n} e^{-\pi|y|^2} 2^{-n/2}\ \mathrm{d}y = 1,\end{aligned}$$

where we used the well-known Gaussian integral

$$\int_{\mathbb{R}} e^{-\pi x^2}\ \mathrm{d}x = 1,$$

exactly n times. ∎

By abuse of language, one could say that the optimally aligned waveform resembles the unique phase space cell - corresponding to a square-integrable function -, which aligns along both coordinate lines. In the next section, it shall be made clear that this does not necessarily lead to optimally concentrated phase space cells around some chosen point of phase space but quite the opposite. There's even another way of looking at those waveforms, torn between two coordinate directions.

Let X be a two-dimensional symplectic subspace of phase space and the Hamiltonians A and B be two smooth coordinate functions, defined on X. Then, as these are coordinates, through each point in this symplectic plane goes exactly one coordinate line - that is, a contour line - of each of the coordinate functions. Pick an abstract point $x \in X$ and let $(\alpha, \beta) := (A(x), B(x))$ be its local coordinates with respect to A and B. Then, that same abstract point becomes the origin if we readjust the coordinates to

$$\tilde{A} := A - \alpha \quad \text{and} \quad \tilde{B} := B - \beta,$$

that is,

$$\tilde{A}(x) = A(x) - \alpha = A(x) - A(x) = 0 \quad \text{and} \quad \tilde{B}(x) = B(x) - \beta = B(x) - B(x) = 0.$$

Squaring those Hamiltonians now gives us a means to describe the *distance from the origin* - $x \in X$ - in the sense of the coordinates defined by A and B, that is,

$$\mathrm{dist}^2_0(q,p) := \tilde{A}^2(q,p) + \tilde{B}^2(q,p).$$

The interpretation is the following. If a point lies exactly along the coordinate line, defined by $\tilde{A}(0)$, its (squared) distance from it is 0 and if it is not, the amount by which it goes astray is penalized by the squared distance, in the sense of the coordinate A. The same holds for B and thus a “cuddly” waveform, which tries to nestle along both coordinate lines, is also the one which concentrates the best within this generalized parabola, which is made up from “sub-parabolas” - one for each probably curved coordinate. The orientation of each of this sub-parabola is “canonically orthogonal” to the coordinate line in the sense that each “gains height” when straying from the contour line. Then, owing to the uncertainty principle, it is impossible for a function to be concentrated to a single point and thus it is smeared over phase space, where this phase space smearing is optimally adapted to the (A,B)-parabola above.

The following trivial lemma and the corollary following it, show that this is actually an equivalent way of expressing this.

Lemma 4.24 (Binomials). *Let $a,b \in \mathbb{R}$, then*

$$a^2 + b^2 \geq 2a \cdot b, \tag{4.63}$$

with equality if and only if $a = b$. ◆

Proof. $(a-b)^2 \geq 0 \iff a^2+b^2 \geq 2ab$ and $(a-a)^2 = (b-b)^2 = 0$. ∎

Corollary 4.25 (Phase Space Parabola). *Let X be a two-dimensional symplectic subspace of phase space and A and B (not necessarily canonical) coordinates on X. Let (α,β) be the (A,B) coordinates of some point on X and $\widehat{A}$, $\widehat{B}$ such that $\left\|(\widehat{A}-\alpha)f\right\| = \left\|\mu(\widehat{B}-\beta)f\right\|$.*

Then, the following are equivalent

(i) *f is the uncertainty equalizer for $\hat{A}$ and $\hat{B}$,*

(ii) *f is an eigenfunction of $N := L^*L$, corresponding to the infimum of its spectrum (which is zero).*

◆

Proof. Since $L := (\widehat{A}-\alpha) + i\mu(\widehat{B}-\beta)$, we get

$$\begin{aligned} \|Lf\|^2 = \langle L^*Lf\,,\,f\rangle &= \left\langle ((\widehat{A}-\alpha) - i\mu(\widehat{B}-\beta))((\widehat{A}-\alpha) + i\mu(\widehat{B}-\beta))f\,,\,f\right\rangle \quad (4.64)\\ &= \left\langle (\widehat{A}-\alpha)^2 f\,,\,f\right\rangle + \left\langle \mu^2(\widehat{B}-\beta)^2 f\,,\,f\right\rangle + i\mu\left\langle [\widehat{B},\widehat{A}]f\,,\,f\right\rangle \\ &\geq 0, \end{aligned}$$

by the non-negativity of the norm. Rearranging, we have

$$\left\|(\widehat{A}-\alpha)f\right\|^2 + \left\|\mu(\widehat{B}-\beta)f\right\|^2 \geq i\mu\left\langle [\widehat{A},\widehat{B}]f\,,\,f\right\rangle$$

and *Lemma 4.24 (Binomials)* gives that

$$\left\|(\widehat{A}-\alpha)f\right\|^2+\left\|\mu(\widehat{B}-\beta)f\right\|^2\geq 2\left\|(\widehat{A}-\alpha)f\right\|\left\|\mu(\widehat{B}-\beta)f\right\|$$

with equality if and only if $\left\|(\widehat{A}-\alpha)f\right\|=\left\|\mu(\widehat{B}-\beta)f\right\|$, which was a prerequisite. Thus,

$$\begin{aligned}2\left\|(\widehat{A}-\alpha)f\right\|\left\|\mu(\widehat{B}-\beta)f\right\|&=\left\|(\widehat{A}-\alpha)f\right\|^2+\left\|\mu(\widehat{B}-\beta)f\right\|^2\\&\geq i\mu\left\langle[\widehat{A},\widehat{B}]f\,,\,f\right\rangle,\end{aligned}$$

and taking the absolute value, we get

$$\left\|(\widehat{A}-\alpha)f\right\||\mu|\left\|(\widehat{B}-\beta)f\right\|\geq\tfrac{|\mu|}{2}\left|\left\langle[\widehat{A},\widehat{B}]f\,,\,f\right\rangle\right|,\tag{4.65}$$

which is exactly the uncertainty inequality for $(\widehat{A}-\alpha)$ and $(\widehat{B}-\beta)$, after canceling the factor $|\mu|$. Thus, those functions for which (*4.65*) holds an equality coincide with those, for which (*4.64*) holds an equality.

Moreover, since the lower bound is attained, we have that it coincides with $\inf(\operatorname{spec}(L^*L))$, which equals 0. ∎

In the next section, we will make the same approach as in this section, but with all coordinates replaced by their canonical conjugate coordinates, which will lead to another uncertainty principle.

4.2.2 Optimal Concentration

Above, the argument has been used, that eigenfunctions of the Ladder operators try as hard to be aligned along the induced flow lines in phase space as possible and that a minimizing waveform strays the least from these paths. Although these waveforms are, in a sense, the best one could hope for, these do not lead to a decomposition of phase space into cells which are as concentrated around a single point as possible. One could even argue that they are trying to do the opposite - stretching as thinly along the flow lines as possible.

In this section, another uncertainty principle will be used to find waveforms - again being initially defined on $\mathcal{S}_\sigma$ by construction and thus not resembling admissible wavelets a priori - which have the opposite property. While still being associated with the flow lines, these waveforms minimize their spread along each of the contour lines and thus try to linger around a point as packaged as Lord uncertainty admits.

In the aftermath of the previous section, we noted that an optimal aligning waveform also lies within a phase space parabola, defined via the chosen coordinates and that the orientation of this parabola is "canonically orthogonal" to the coordinate lines. From this, we may now argue that, instead of defining parabola which penalize going astray from a contour line, we now make the phase space point itself the epicenter of our arguments and give a penalty for the distance along each of the coordinate lines. Since the coordinate lines themselves measure "distances" by "numbering" the coordinate lines which partition the space, there is no direct way of expressing a distance along a coordinate. The closest one could come to doing such a thing is in defining a canonically conjugate coordinate, with respect to a given coordinate, and "count" the number of canonically conjugate coordinate lines crossed, while traveling along a contour line. Indeed, this is what is done in this section and will ultimately lead to new uncertainty principles.

Let A^k and B_j be coordinates on phase space, then, by

$$\left\{ A^k \,,\, \widetilde{A}_{k'} \right\} = \delta^k_{k'} \quad \text{and} \quad \left\{ B_j \,,\, \widetilde{B}^{j'} \right\} = \delta^{j'}_j \,, \quad j, j', k, k' = 1, \ldots, n \tag{4.66}$$

we define sets of conjugate coordinates, where $\{\ ,\ \}$ is Poisson's bracket.

Terminology 4.26 (Optimal Concentration)**.** A function's representation on phase space is said to be *optimally concentrated* in the sense of a chosen frame of reference, if it is as aligned to both *canonically conjugate* coordinate lines of each two-dimensional subspace of conjugate variables as a specific inequality, an uncertainty principle, admits.

As before, the rest of this section shall make this terminology more precise. ◆

Completely analogue to the modus operandi in the previous section, one may now define the generalized parabola - by means of which distances in phase space could be measured -, as well as the associated quantized operators, from which the generalized Ladder operators and its eigenfunctions can be found.

So let's stop doing the talk and do the walk.

Definition 4.27 (Conjugate Ladder)**.** Assume that A^k and B_j are coordinates on phase space and define their canonically conjugate coordinates $\widetilde{A}_{k'}$ and $\widetilde{A}_{j'}$ by

$$\left\{ A^k \,,\, \widetilde{A}_{k'} \right\} = \delta^k_{k'} \quad \text{and} \quad \left\{ B_j \,,\, \widetilde{B}^{j'} \right\} = \delta^{j'}_j \,, \quad j, j', k, k' = 1, \ldots, n. \tag{4.67}$$

Then, $\widetilde{L}$ will be called a *conjugate spectral Ladder operator*, if it results in the replacement of its constituting quantized operators by the quantized operators of their conjugate counterparts and

$$\widetilde{N} := \widetilde{L}^* \widetilde{L} \tag{4.68}$$

is the conjugate generalized Number operator. ◆

Spectral Diffeomorphisms

We have already met all actors in this subsection, since in order to define the uncertainty principle for spectral diffeomorphisms, all we have to do is to find the conjugate coordinates, associated with the coordinates defined by the spectral cotangent lift. This could not be easier, since the cotangent lift itself, being a symplectomorphism, pairs conjugate variables.

Lemma 4.28 (Conjugate Variables for Spectral Warps)**.** *Let Σ be the spectral cotangent lift*

$$\Sigma(q,p) := \big(J^{-T}(p)q, \sigma(p)\big) \tag{4.69}$$

and set, as usual,

$$A^k(q,p) := (J^{-T}(p)q)^k,\ k = 1,\dots,n, \quad \textit{and} \quad B_j(q,p) := p_j,\ j = 1,\dots,n. \tag{4.70}$$

Then, these coordinates have the conjugate canonical counterparts

$$\widetilde{A}_k(q,p) := \sigma_k(p),\ k = 1,\dots,n, \quad \textit{and} \quad \widetilde{B}^j := q^j,\ j = 1,\dots,n. \tag{4.71}$$

◆

Proof. Since the cotangent lift is a symplectomorphism, it is clear that

$$\widetilde{A}_k(q,p) := \sigma_k(p)$$

is the canonically conjugate coordinate to A^k. $\widetilde{B}^j := q^j$ follows since (q^j, p_k) are the standard canonically coordinates. ■

These conjugate variables can themselves be regarded as Hamiltonians on phase space and, in complete analogy to what has been done before, quantized. This, then, necessarily leads to commutator relations, controlling the incompatibility of simultaneous observation.

Lemma 4.29 (Canonically Conjugate spectral Hamiltonians)**.** *Let σ, Σ and*

$$A^k(q,p) := (J^{-T}(p)q)^k,\ k = 1,\dots,n, \tag{4.72}$$

and

$$B_j(q,p) := p_j,\ j = 1,\dots,n, \tag{4.73}$$

as well as

$$\widetilde{A}_k(q,p) := \sigma_k(p),\ k = 1,\dots,n, \tag{4.74}$$

and

$$\widetilde{B}^j := q^j,\ j = 1,\dots,n \tag{4.75}$$

as before. Then

$$\widehat{\widetilde{A}}_k(q,p) := \sigma_k(p),\ k = 1,\dots,n, \tag{4.76}$$

and

$$\widehat{\widetilde{B}}^j := -\tfrac{1}{2\pi i}\partial_{p^j},\ j = 1,\dots,n, \tag{4.77}$$

are the quantized conjugate Hamiltonians and

(i) $\left[\widehat{\widetilde{A}}_k, \widehat{\widetilde{A}}_j\right] = 0, \quad j,k = 1,\dots,n,$

(ii) $\left[\widehat{\widetilde{B}}^k, \widehat{\widetilde{B}}^j\right] = 0, \quad j,k = 1,\dots,n,$

(iii) $\left[\widehat{\widetilde{A}}_k, \widehat{\widetilde{B}}^j\right] = -\tfrac{1}{2\pi i}(J_\sigma)^j_k, \quad j,k = 1,\dots,n,$

are their commutation relations. ◆

Proof. Since the σ_k are only dependent on p, from *Corollary 2.23 (Quantization)* it follows that these quantize to multiplication operator on the Fourier domain and thus their commutators vanish, which proves (i). The correspondence $q^j \mapsto -\frac{1}{2\pi i}\partial_{p^j}$ has already turned up multiple times and since partial derivatives commute, their commutators vanish also and thus (ii) holds true.

To see (iii), note that

$$\begin{aligned}\left[-\tfrac{1}{2\pi i}\partial_{p^j}, \sigma_k\right] f &= -\tfrac{1}{2\pi i}\left(\partial_{p^j}(\sigma_k f) - \sigma_k \partial_{p^j} f\right)\\ &= -\tfrac{1}{2\pi i}\left(\frac{\partial \sigma_k}{\partial p^j} f\right) = -\tfrac{1}{2\pi i}(J_\sigma)^j_k f,\end{aligned}$$

holds, whenever $f \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(\partial_{p^j})$. ■

With this at hand, we are now able to define a new uncertainty principle for spectral diffeomorphisms, which will be the main theorem for this section.

Theorem 4.30 (Uncertainty Principle of Optimal Concentration)**.** *Let σ be a spectral diffeomorphism and denote with $\widehat{\widehat{A}}_j$ and $\widehat{\widehat{B}}^k$ its quantized conjugate Hamiltonians. Let furthermore $f \in \mathcal{S}_\sigma$ and $\alpha_k, \beta^j \in \mathbb{R}$, for all $j, k = 1, \ldots, n$. Then*

$$\left\|(\widehat{\widehat{A}}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{\widehat{B}}^j - \beta^j)f\right\|_{\mathcal{S}_\sigma} \geq \tfrac{1}{4\pi}\left|\left\langle (J_\sigma)^j_k \widehat{f}, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|, \quad j, k = 1, \ldots, n. \tag{4.78}$$

Moreover, the waveform $f_0 \in \mathcal{S}_\sigma$, given by

$$\widehat{f}_0(\xi) = Ce^{-2\pi\mu \int \sigma_k \, \mathrm{d}x^j} e^{2\pi x(\mu\alpha_k - i\beta^j)}, \tag{4.79}$$

equalizes (4.78)*, for some $\mu \in \mathbb{R}_+$.* ◆

Proof. Use *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)* and *Lemma 4.29 (Canonically Conjugate spectral Hamiltonians)* to get the expression for the uncertainty principle. For the equalizing waveform, find that *Theorem 4.4 (Uncertainty Principle for Self-Adjoint Operators)* gives us that this holds true if and only if there exists a constant $\mu \in \mathbb{R}$, such that

$$(\widehat{\widehat{B}}^j - \beta^j)f = -i\mu(\widehat{\widehat{A}}_k - \alpha_k)f.$$

Using *Lemma 4.29 (Canonically Conjugate spectral Hamiltonians)* again, we get

$$(\tfrac{-1}{2\pi i}\partial_{p^j} - \beta^j)f = -i\mu(\sigma_k - \alpha_k)f,$$

which is solved by

$$\widehat{f}_0 := Ce^{-2\pi\mu \int \sigma_k \, \mathrm{d}x^j} e^{2\pi x(\mu\alpha_k - i\beta^j)}, \tag{4.80}$$

for some C. ■

From the last theorem it trivially follows that the same arguments as in *Corollary 4.25 (Phase Space Parabola)* apply, which gives the following short corollary.

Corollary 4.31 (Conjugate Phase Space Parabola)**.** *Let everything as in* Corollary 4.25 (Phase Space Parabola)*, with the coordinates replaced by their canonically conjugate ones, $\widehat{\widehat{A}}$ and $\widehat{\widehat{B}}$. Then, the following are equivalent*

(i) *f is the uncertainty equalizer for $\widehat{\widehat{A}}$ and $\widehat{\widehat{B}}$,*

(ii) *f is the eigenfunction of the conjugate number operator $\widetilde{N} := \widetilde{L}^* \widetilde{L}$, with the lowest spectral value* 0. ◆

Proof. The proof goes exactly as in *Corollary 4.25 (Phase Space Parabola)*, with each operator replaced by its canonically conjugate sidekick. ■

As in the case of the "classical" uncertainty principle, one cannot expect that for all possible spectral diffeomorphisms there exists a function which equalizes all of the above inequalities simultaneously.

There is, however, a special case for which all works well - namely, again, the lovely case of a diagonal Jacobian.

Corollary 4.32 (Diagonal Jacobians)**.** *Let everything as in* Theorem 4.30 (Uncertainty Principle of Optimal Concentration)*. Let moreover the Jacobian be a diagonal matrix. Then*

$$\sum_k \left\|(\widehat{\widehat{A}}_k - \alpha_k)f\right\|_{\mathcal{S}_\sigma} \left\|(\widehat{\widehat{B}}^k - \beta^k)f\right\|_{\mathcal{S}_\sigma} \geq \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma)\widehat{f}\,,\, \widehat{f}\right\rangle_{\mathcal{F}\mathcal{S}_\sigma}\right|. \tag{4.81}$$

Moreover, if f_k denote the equalizing waveforms for each of the uncertainty principles associated with the diagonal entries of the Jacobian, we have that

$$\mathcal{S}_\sigma \ni f_0 := \prod_k f_k$$

*is the equalizer of (*4.81*) and has, expressed on $\mathcal{F}\mathcal{S}_\sigma$, the beautiful form*

$$\begin{aligned}\widehat{f_0}(\xi) &:= Ce^{-2\pi\sum_k \mu_k \int \sigma_k(\omega_k)\ \mathrm{d}\omega_k} e^{2\pi\langle\, \xi\,,\, \mu\alpha - i\beta\,\rangle} \\ &= Ce^{-2\pi\int_c \langle\, \vec{\sigma_\mu}\,,\ \vec{\mathrm{d}\omega}\,\rangle} e^{2\pi\langle\, \xi\,,\, \mu\alpha - i\beta\,\rangle},\end{aligned} \tag{4.82}$$

where $\int_c \left\langle\, \vec{\sigma}_\sigma\,,\ \vec{\mathrm{d}\omega}\,\right\rangle$ denotes a line integral along some path, c, starting from a fixed point and ending at ξ.
As before, to f_0 corresponds the admissible window

$$\widehat{\psi}_0(\xi) := C\left|\det(J_\sigma)\right|^{-1/2} \cdot e^{-2\pi\int_c\langle\, \vec{\sigma_\mu}\,,\ \vec{\mathrm{d}\omega}\,\rangle} e^{2\pi\langle\, \xi\,,\, \mu\alpha - i\beta\,\rangle}, \tag{4.83}$$

ready to be used in a coherent state map. ◆

Proof. Everything, except for the form of the equalizer, goes exactly as in *Corollary 4.16 (Diagonal Jacobians)*, with σ replaced by σ^{-1} and all $\widehat{A}$ and $\widehat{B}$ replaced by their canonically conjugate counterparts $\widehat{\widehat{A}}$ and $\widehat{\widehat{B}}$, respectively.

Thus, *quod esset demonstrandum* is (*4.82*) and (*4.83*). To see these, note that

$$\begin{aligned}\widehat{f_0}(\xi) &:= Ce^{-2\pi\sum_k \int (\mu_k\sigma(\omega_k)_k - \mu_k\alpha_k + i\beta_k)\ \mathrm{d}\omega_k} \\ &= Ce^{-2\pi\int_c\langle\, \vec{\sigma_\mu}\,,\ \vec{\mathrm{d}\omega}\,\rangle} e^{2\pi\langle\, \xi\,,\, \mu\alpha - i\beta\,\rangle}\end{aligned}$$

follows from taking the product of all the $\widehat{f_k}$ and writing it in a form, involving a path integral, ending at ξ. It is defined on $\mathcal{S}_\sigma$ since the operators involved are

"conjugate canonical quantizations" of coordinates which correspond to operators, inducing unitary flows on $\mathcal{S}_\sigma$.
And, finally, by applying the map $\iota_{\mathcal{S}\to\mathcal{A}}$, we arrive at

$$\widehat{\psi}_0(\xi) := \iota_{\mathcal{S}\to\mathcal{A}}\widehat{f}_0(\xi) = C\left|\det(J_\sigma)\right|^{-1/2} \cdot e^{-2\pi\int_c\langle\,\vec{\sigma_\mu}\,,\;\vec{\mathrm{d}\omega}\,\rangle} e^{2\pi\langle\,\xi\,,\,\mu\alpha - i\beta\,\rangle}, \tag{4.84}$$

which finishes the claim. ∎

Having defined the special case of diagonal Jacobians, one might think - analogous to the principle for waveforms optimally aligned along various curvilinear coordinates simultaneously - that this special case can be further generalized to non-diagonal cases, given that the weighted Jacobian is symmetric.

This time, the derivations are along the rectangular grid on $\mathcal{S}_\sigma$, that is, on the standard Fourier domain and a pullback of the standard vector field $\vec{\mu}\vec{y}$ from the warped domain to the rectangular grid, leads to an equation of the form

$$\tfrac{-1}{2\pi i}\vec{\nabla}\widehat{f} = -i(\vec{\mu}\vec{\sigma} - \vec{\mu}\vec{\alpha} + i\vec{\beta})\widehat{f}, \tag{4.85}$$

suggested to be the equation for an n-dimensional equalizing waveform on $\mathcal{S}_\sigma$ for symmetric weighted Jacobians, having a solution if the associated one-form is exact, respectively closed and its domain simply-connected.

Theorem 4.33 (Optimal Concentration for Weighted Jacobians)**.** *Let everything as in* Corollary 4.32 (Diagonal Jacobians)*, except the restriction to a diagonal Jacobian. Let* $\operatorname{dom}(\sigma)$ *be simply-connected, the weighted Jacobian,* J_{σ_μ}*, be symmetric and let furthermore* $f \in \mathcal{S}_\sigma$*. Then,*

$$\left\|\left(\tfrac{-1}{2\pi i}\nabla - \vec{\beta}\right)\widehat{f}\right\| \left\|(\vec{\sigma} - \vec{\alpha})\widehat{f}\right\| \geq \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma)\widehat{f}\,,\,\widehat{f}\right\rangle\right| \tag{4.86}$$

is the generalized uncertainty principle for symmetric Jacobians and

$$\widehat{f}(\xi) := C(\xi_0)e^{-2\pi\int_c\langle\,\vec{\sigma_\mu}\,,\,\vec{\mathrm{d}\xi}\,\rangle} e^{2\pi\langle\,\xi\,,\,\mu\alpha - i\beta\,\rangle}, \tag{4.87}$$

for some path, c starting at ξ_0 *and ending at* ξ*, is the equalizing waveform on* $\mathcal{S}_\sigma$ *and*

$$\widehat{\psi}(\xi) := (\widehat{\iota_{\mathcal{S}\to\mathcal{A}}f})(\xi) = C(\xi_0)\left|\det(J_\sigma)\right|^{-1/2} e^{-2\pi\int_c\langle\,\vec{\sigma_\mu}\,,\,\vec{\mathrm{d}\xi}\,\rangle} e^{2\pi\langle\,\xi\,,\,\mu\alpha - i\beta\,\rangle}, \tag{4.88}$$

is its admissible counterpart. ◆

Proof. The proof goes along the same lines as above and therefore, a shortcut shall suffice. We have

$$\begin{aligned}\left\|\left(\tfrac{-1}{2\pi i}\nabla - \vec{\beta}\right)\widehat{f}\right\| \left\|(\vec{\sigma} - \vec{\alpha})\widehat{f}\right\| &\geq \tfrac{1}{2\pi}\left|\sum_k \langle\, (\partial_{\xi_k} - \beta_k) f\,,\, (\sigma_k - \alpha_k) f\,\rangle\right| \\ &\geq \tfrac{1}{4\pi}\left|\sum_k \left\langle\, \frac{\partial \sigma_k}{\partial \xi_k} f\,,\, f\,\right\rangle\right| \\ &\geq \tfrac{1}{4\pi}\left|\langle\, \operatorname{div}(\sigma) f\,,\, f\,\rangle\right|\end{aligned}$$

and this inequality is equalized if and only if

$$\tfrac{-1}{2\pi i}\vec{\nabla}\widehat{f} = -i(\vec{\mu\sigma} - \vec{\mu\alpha} + i\vec{\beta})\widehat{f},$$

which leads, with $e^g =: \widehat{f}$, to

$$\nabla g = -2\pi(\vec{\mu\sigma} - \vec{\mu\alpha} + i\vec{\beta}).$$

This equation is solvable if and only if the domain is simply-connected and if it fulfills the integrability condition

$$-2\pi\partial_{\xi_n}(\mu_j\sigma_j) = -2\pi\partial_{\xi_j}(\mu_n\sigma_n),$$

that is, if the Jacobian of the weighted diffeomorphism is symmetric. Since the space is assumed to be simply-connected and J_{σ_μ} to be symmetric, it is, therefore, solved by

$$\widehat{f}(\xi) := C(\xi_0) e^{-2\pi\int_c \langle\, \vec{\sigma_\mu}\,,\, \mathrm{d}\vec{\xi}\,\rangle} e^{2\pi\langle\, \xi\,,\, \mu\alpha - i\beta\,\rangle}, \tag{4.89}$$

with c a path from ξ_0 to ξ.

The statement about its admissible counterpart is trivial. ∎

Before moving on, it makes sense to cherish the resemblance of the two complementing uncertainty principles and their equalizers above. With $\alpha = \beta = 0$, in the case of waveforms optimally aligned along coordinate lines, the uncertainty inequality reads

$$\left\|\tfrac{-1}{2\pi i} J_\sigma^{-T}\nabla\widehat{\psi}\right\| \left\|\vec{\xi}\widehat{\psi}\right\| \geq \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma^{-1})\widehat{\psi}, \widehat{\psi}\right\rangle\right| \tag{4.90}$$

whereas

$$\left\|\tfrac{-1}{2\pi i}\nabla\widehat{f}\right\| \left\|\vec{\sigma}\widehat{f}\right\| \geq \tfrac{1}{4\pi}\left|\left\langle \operatorname{div}(\sigma)\widehat{f}, \widehat{f}\right\rangle\right| \tag{4.91}$$

for the principle of optimal concentration. For the canonical case, both inequalities coincide, since $\sigma = \sigma^{-1} = \boldsymbol{1}$ and $J_\sigma = J_\sigma^{-1} = I_n$ and $\operatorname{div}(\sigma) = \operatorname{div}(\sigma^{-1}) = n$.

Assuming that μ can be chosen such that the weighted diffeomorphism has a symmetric Jacobian, the equalizing waveforms in the space of signals, respectively in the space of admissible waveforms, of the first principle read

$$\widehat{f}(\xi) := |\det J_\sigma|^{1/2} C(\xi_0) e^{-2\pi \int_c \langle \vec{\xi} \,,\, \mathrm{d}\sigma_\mu \rangle} \quad \text{and} \quad \widehat{\psi}(\xi) := C(\xi_0) e^{-2\pi \int_c \langle \vec{\xi} \,,\, \mathrm{d}\sigma_\mu \rangle}, \tag{4.92}$$

whereas

$$\widehat{f}(\xi) := C(\xi_0) e^{-2\pi \int_c \langle \vec{\sigma_\mu} \,,\, \mathrm{d}\vec{\xi} \rangle} \quad \text{and} \quad \widehat{\psi}(\xi) := |\det J_\sigma|^{-1/2} C(\xi_0) e^{-2\pi \int_c \langle \vec{\sigma_\mu} \,,\, \mathrm{d}\vec{\xi} \rangle} \tag{4.93}$$

for the second. In the beautiful case of canonically conjugate variables, both principles coincide and the minimizing waveform is the σ-warped Gaussian. Captivating.

4.3 A Principle for the Affine Group

Another concept of localization arises from a Lie group structure [6, 65] as follows. To comply with standard terminology, the notation is slightly altered and greek letters are used as indices.

The Lie algebra of a Lie group gives a coordinate system on the connected component of the identity via left- or right-invariant vector fields. Thus, each point can be parallel transported along the resulting flow lines - corresponding to the one-parameter subgroups of a Lie group -, by virtue of which the differentiable manifold of a Lie group has a natural coordinate system. Then, one may pass to the tangent space of the identity, T_eG - equivalent to its Lie algebra -, to describe the manifold, at least in some open neighborhood of the identity, on which the exponential map is a homeomorphism. Since the tangent space at the identity component is a vector space, one may equip it with an inner product and by means of left (or right) action of the group, translate this inner product to all points reachable from the identity, to define an inner product for each point of the whole connected component of the identity. That is,

$$\begin{aligned} g_p(v,w) &:= \left\langle \mathrm{d}L_p^{-1} v \,,\, \mathrm{d}L_p^{-1} w \right\rangle \\ &= \left\langle (\mathrm{d}L_p^{-1})^* \mathrm{d}L_p^{-1} v \,,\, w \right\rangle \\ &= \langle M(p) v \,,\, w \rangle\,, \quad p \in G, \end{aligned}$$

where v and w are vector fields on G, $\mathrm{d}L_p$ is the differential of translation and $M(p) := \mathrm{d}L_{p^{-1}}^* \mathrm{d}L_{p^{-1}}$ is a matrix-valued function (a $(1,1)$-tensor field), which for some fixed inner product, $\langle\ ,\ \rangle$, on T_eG completely characterizes the metric. This metric now defines a right- (or left)-invariant Riemannian structure on the manifold, often written in a form of the line element, akin to

$$(\mathrm{d}s)^2 := \sum_{\mu,\nu} g_{\mu,\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu, \tag{4.94}$$

by which one may measure distances of points along the paths of the associated flows. That is,

$$d(b;a) := \int_c \sqrt{\sum_{\mu,\nu} g_{\mu,\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu} = \int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))} \, \mathrm{d}t, \tag{4.95}$$

is the distance from a to b, where c is the "shortest path" between $a =: c(0)$ and $b =: c(1)$ in the sense of this integral. These shortest paths are known as geodesics and are a means to generalize the concept of a straight line to curved spaces such as Lie group manifolds. Solving the associated Euler-Lagrange equations, *Definition A.79 (Euler-Lagrange Equations)*, for the integral

$$E := \int_0^1 g_{c(t)}(\dot{c}(t), \dot{c}(t)) \, \mathrm{d}t, \tag{4.96}$$

which is the action, related to the distance integral above, gives the *geodesic equation(s)*

$$\frac{\mathrm{d}^2 c^\lambda}{\mathrm{d}t^2} + \sum_{\mu,\nu} \Gamma^\lambda_{\mu\nu} \frac{\mathrm{d}c^\mu}{\mathrm{d}t} \frac{\mathrm{d}c^\nu}{\mathrm{d}t} = 0 \;, \qquad \lambda = 1, \ldots, n, \tag{4.97}$$

for a curve $(c^\lambda)_\lambda$, with the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, *Definition A.80 (Christoffel symbols)*.

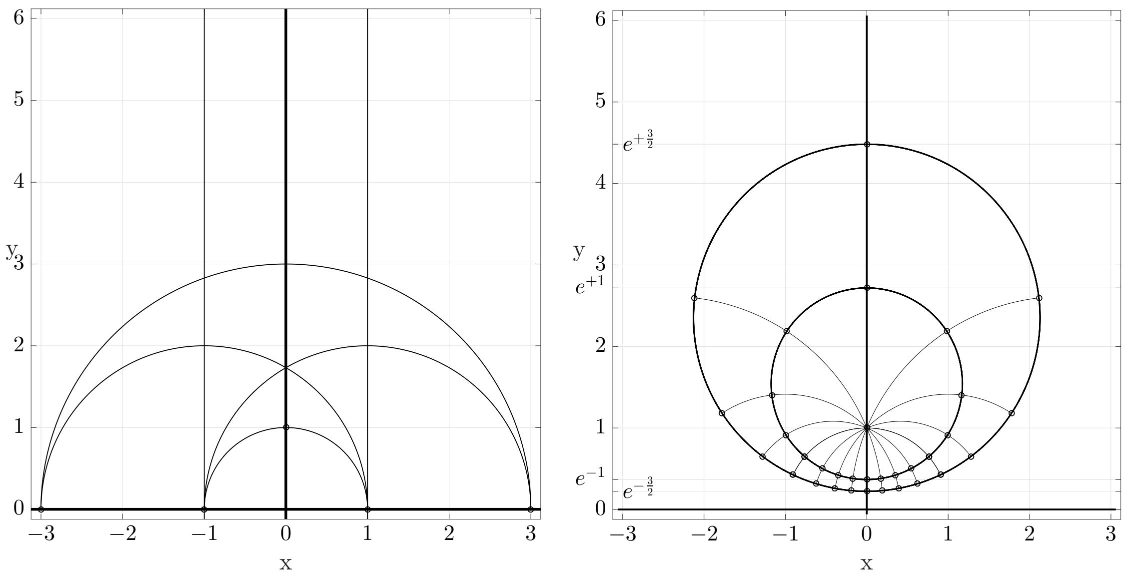
Returning to the Affine group example, we find that the matrix M above, by which we define the metric, is

$$\begin{aligned} M(x,y) &:= \mathrm{d}L^*_{(x,y)^{-1}} \mathrm{d}L_{(x,y)^{-1}} \\ &= \begin{pmatrix} \frac{1}{y} & \\ & \frac{1}{y} \end{pmatrix}^* \begin{pmatrix} \frac{1}{y} & \\ & \frac{1}{y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{y^2} & \\ & \frac{1}{y^2} \end{pmatrix}, \end{aligned}$$

which induces the metric

$$(\mathrm{d}s)^2 := \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{4.98}$$

But this is the hyperbolic metric of the Poincaré half-plane, $\mathbb{H}$, which gives us the whole power of hyperbolic geometry at hand to proceed even further, cf. e.g. [4, 77].



(a) Hyperbolic geodesics of infinite length.

(b) Isomorphic hyperbolic geodesics through $(0,1) \in \mathbb{H}$ and hyperbolic circles.

Figure 4.5: The Hyperbolic Plane.

The arising distance function of the Poincaré half-plane, induced from this metric, is

$$d(x,y;x',y') := \cosh^{-1}\left(1 + \frac{(x-x')^2 + (y-y')^2}{2yy'}\right), \quad (x,y),(x',y') \in \mathbb{H}, \tag{4.99}$$

and gives a method by which we may decide whether two points have the same distance from another point and thus describe generalized circles on the manifold to define "geodesically circular" symmetric functions on the manifold.

It is a well-known fact that the geodesics of the Poincaré half-plane are given by half-circles, orthogonal to the x-axis, as well as straight lines, parallel to the y-axis, that is, either

$$x = \text{const} \quad \text{or} \quad (x-x_0)^2 + y^2 = r^2\ , \quad x, x_0 \in \mathbb{R}, y, r > 0. \tag{4.100}$$

Starting from a fixed point and flowing along each geodesic, we arrive at the set of points equidistant from the starting point, which constitutes a generalized circle for this kind of geometry.

Interestingly, although the hyperbolic metric is different from the Euclidean one, the set of points equidistant to another point on the half-plane is again a circle, just like in ordinary Euclidean spaces. Its center, however, is different, as the infinitesimal line segments get stretched, respectively squeezed, depending on the y-coordinate. Indeed, flowing from the point $(0,1) \in \mathbb{H}$ towards the point $(0,\epsilon) \in \mathbb{H}$ takes the same "amount of time" as flowing from $(0,1) \in \mathbb{H}$ to $(0,1/\epsilon) \in \mathbb{H}$, and in fact an "infinite amount of time" to reach a point like $(0,0) \in \mathbb{H}$ on the x-axis - it is actually not even part of the set.

Figure 4.5a plots some hyperbolic geodesics and *Figure 4.5b* shows two hyperbolic circles around the point $(0,1) \in \mathbb{H}$, along with isomorphic geodesics (of equal length) through the point $(0,1) \in \mathbb{H}$.

Optimal localization in the sense of this - in fact of any - geometry suggests that a well localized function should be geodesically symmetric and therefore constant along these generalized circles. That is, for a function which is centered at $(0,1) \in \mathbb{H}$, its values on points, equidistant from this center, should be the same

$$f_{opt}(x,y) := \widetilde{f}(d(x,y;0,1)), \quad (x,y) \in \mathbb{H},$$

and thus it should be a function of geodesic distance to the point $(0,1) \in \mathbb{H}$ only. That is, the function should only depend on

$$\begin{aligned} d(x,y;0,1) &:= \cosh^{-1}\left(1 + \frac{x^2 + (y-1)^2}{2y}\right) \\ &= \cosh^{-1}\left(1 + \tfrac{x^2}{2y} + \tfrac{1}{2}y - 1 + \tfrac{1}{2y})\right) \\ &= \cosh^{-1}\left(\tfrac{1}{2}(\tfrac{x^2}{y} + y + \tfrac{1}{y})\right), \end{aligned} \tag{4.101}$$

for $(x,y) \in \mathbb{H}$. Thus, an optimally localized window function - in the sense of the wavelet transform - should have a (hyperbolical) circular symmetric localization on the group manifold. Although this does not fix the window uniquely, it is a hint as to what one should expect.

A first step might be the analogy to Euclidean spaces, in which the geodesic distance from the origin, or, to be more precise, the geodesic flow, is utilized in the definition of the heat kernel

$$\vec{x} \in \mathbb{R}^n \mapsto e^{-\frac{|\vec{x}|^2}{2t}}.$$

That is, starting at time $t = 0$ from a single point, $c \in \mathbb{H}$, interpreted as a heat source, the heat spreads symmetrically along the geodesics and the heat distribution, at time $t > 0$, is thus akin to the Euclidean heat kernel.

In fact, the identification of the heat kernel for a given geometry is an interesting topic on its own and whole books have been written on it [6].

Adopting Claude Shannon's Ansatz [75], that a nice (centered) function should link entropy with variance, that is,

$$H(|f|^2) := -\int \log|f(x)|^2 \; |f(x)|^2 \; \mathrm{d}x \overset{!}{=} \lambda \int |x|^2 |f(x)|^2 \; \mathrm{d}x, \quad \lambda \in \mathbb{R}_+,$$

with the well-known family of Gaussian solutions, $e^{-c|x|^2}, c \in \mathbb{R}_+$, for which

$$\begin{aligned} H(e^{-c|x|^2}) &:= -\int \log e^{-c|x|^2}\ e^{-c|x|^2}\ \mathrm{d}x \\ &= c\int |x|^2\ e^{-c|x|^2}\ \mathrm{d}x. \end{aligned}$$

holds, suggests that a first shot at a minimizing waveform may be

$$(x,y)\ \mapsto e^{-c\cdot d(x,y;0,1)^2}, \quad (x,y)\in\mathbb{H},$$

as it resembles the Euclidean nexus for the hyperbolic case

$$\begin{aligned} H(e^{-c\cdot d(x,y;0,1)^2}) &:= -\int \log e^{-c\cdot d(x,y;0,1)^2}\ e^{-c\cdot d(x,y;0,1)^2}\ \mathrm{d}x \\ &= c\int d(x,y;0,1)^2\ e^{-c\cdot d(x,y;0,1)^2}\ \mathrm{d}x, \end{aligned}$$

with $\int d(x,y;0,1)^2\ e^{-c\cdot d(x,y;0,1)^2}\ \mathrm{d}x$ being an analogue to the Euclidean variance. In fact, the heat kernel for the hyperbolic geometries has been identified, see, e.g., [38], with

$$k(r,t) := \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_r^\infty \frac{s e^{-s^2/(4t)}}{\sqrt{\cosh s - \cosh r}} \mathrm{d}s, \tag{4.102}$$

being the one for $n = 2$. Although the resemblance to the Euclidean case is tempting, it is not clear (i) how to show that this function is an optimally localized reproducing kernel for the group and (ii) how to find a window function whose image on the group is (akin to) the kernel, as suggested above.

We can, however, take recourse to the fact that an optimally localized function should be geodesically symmetric and as compressed around its center as possible. Thus, there is a chance that the optimal window is an eigenfunction of the quantized operator, associated with the phase space symbol of the distance function, pulled back to phase space under the map

$$(b,a)\ \mapsto\ (b,1/a)\ , \qquad \mathrm{d}b\ \mathrm{d}a\ \mapsto\ \mathrm{d}b\ \mathrm{d}(1/a) =: \frac{\mathrm{d}b\ \mathrm{d}a}{a^2} \tag{4.103}$$

which identifies phase space with the affine group manifold [1, Sec. 12.4].

Recalling that a Hamiltonian function, as well as its quantized Hamiltonian operator, induces a flow along its contour lines, the Hamiltonian

$$s(x,y) := 2\cosh d(x,1/y) := x^2y + y + 1/y, \quad (x,y)\in\mathbb{H}, \tag{4.104}$$

should induce a flow along the pullback of the hyperbolic circles, equidistantly from the point $(0,1)\in\mathbb{H}$ and the eigenfunctions of the quantized Hamiltonian are constant along these curves - varying only by a phase-factor along the curves.

Remark 4.34 (Symmetric Functions)**.** Above, we rather used $2\cosh d$ to get a nice symbol. In the end, a factor of 2 should not do any harm and cosh is strictly increasing on the positive half-line, which is the image of the distance function, since distances are necessarily non-negative. The only thing which changes in our construction, are the eigenvalues which are not relevant for our construction so long as we make sure, that the symmetric function we choose - our distance function -, has smaller values near the center and increases with distance from the origin. This restriction is needed in order to make sure that the eigenfunction, corresponding to the lowest eigenvalue, is localized around the central point $(0,1) \in \mathbb{H}$. ◆

Using the symmetric quantization scheme from *Definition 2.22 (Symmetric Correspondence Rule for Polynomials)*, leads to

$$
\begin{aligned}
x^2y + y + 1/y \longmapsto & \tfrac{1}{3}\big(\big(\tfrac{-1}{2\pi i}\partial_y\big)^2 y + \big(\tfrac{-1}{2\pi i}\partial_y\big)y\big(\tfrac{-1}{2\pi i}\partial_y\big) + y\big(\tfrac{-1}{2\pi i}\partial_y\big)^2\big) + y + \tfrac{1}{y} \\
&= \tfrac{1}{3\cdot(4\pi^2)}\big(-3\partial_y - 3y\partial_y^2\big) + y + \tfrac{1}{y} \\
&= -\tfrac{1}{(4\pi^2)}\partial_y - \tfrac{1}{\cdot(4\pi^2)}y\partial_y^2 + y + \tfrac{1}{y} =: \widehat{T},
\end{aligned}
\tag{4.105}
$$

which is a self-adjoint operator $L^2(\widehat{\mathbb{R}},\ \mathrm{d}x) =: \mathcal{FS}_{\log}$, diagonalizable by a family of functions, geodesically symmetric in the group manifold.

The operator $\widehat{T}$ is an interesting one since this is not the first time this operator turns up. As a matter of fact, the path of diagonalization has already been taken by Daubechies and Paul in [14], although by another construction.

In that paper, the construction relied on the induced flow of $e^{2\pi i T t}$ not on phase space itself, but on the homeomorphic group manifold, that is, the Poincaré half-plane.

We will take our own approach to this and not rely on the results in [14], so let's move on.

On first sight, it is not obvious how to diagonalize the operator $\widehat{T}$, but its construction suggests that there is a chance that it commutes with the affine Number operator, $N := L^*L$ - and thus their eigenfunctions coincide -, since both rely on specific coordinates on the manifold. In the case of the abstract Number operator, those coordinates are given by the contour lines of Hamiltonians which induce the flow along exactly those paths, which the one-parameter subgroups of the affine group induce. This is also a good chance to explicitly construct a spectral warped Number operator, as defined above.

Example 4.1 (Affine Number Operator)**.** Let $A := q \cdot p$ and $B := p$ be the affine coordinates and $\alpha, \beta \in \mathbb{C}$ some yet undetermined numbers. Then

$$\widehat{A} := \widehat{qp} = (\widehat{q}\widehat{p} + \widehat{p}\widehat{q})/2 = -\tfrac{1}{2\pi i}\left(\tfrac{1}{2} + p\partial_p\right), \tag{4.106}$$

$$\widehat{B} := \widehat{p} := p \tag{4.107}$$

and the shifted spectral Ladder operator becomes

$$L_{\alpha\beta} := (\widehat{A} + i\widehat{B}) - (\alpha + i\beta) = \left(-\tfrac{1}{2\pi i}\left(\tfrac{1}{2} + p\partial_p\right) + ip\right) - (\alpha + i\beta),$$

where all operators are represented on the Fourier domain.

The associated number operator is

$$\begin{aligned} N_{\alpha\beta} &:= -\tfrac{1}{4\pi^2}p^2\partial_p^2 - \left(\tfrac{1}{2\pi^2} - \tfrac{\alpha}{i\pi}\right)p\partial_p - \left(\tfrac{1}{2\pi} + 2\beta\right)p + p^2 + \left(\alpha^2 + \beta^2 - \tfrac{1}{16\pi^2} + \tfrac{\alpha}{2\pi i}\right) \\ &:= p\left(-\tfrac{1}{4\pi^2}p\partial_p^2 - \left(\tfrac{1}{2\pi^2} - \tfrac{\alpha}{i\pi}\right)\partial_p - \left(\tfrac{1}{2\pi} + 2\beta\right) + p + \frac{\alpha^2+\beta^2-\frac{1}{16\pi^2}+\frac{\alpha}{2\pi i}}{p}\right), \end{aligned} \tag{4.108}$$

which looks like a mess. We can, however, compare it with (*4.105*) to find that if we set the parameters to $\alpha = \frac{i}{4\pi}$ - note the imaginary unit i - and $\beta = \pm 1$, and substitute $x = q$ and $y = p$, we have

$$\begin{aligned} N_{\alpha\beta} &:= y\left(-\tfrac{1}{\cdot(4\pi^2)}\partial_y - \tfrac{1}{\cdot(4\pi^2)}y\partial_y^2 + y + \tfrac{1}{y} + \left(\tfrac{1}{2\pi} \pm 2\right)\right) \\ &:= y\left(\widehat{T} + \left(\tfrac{1}{2\pi} \pm 2\right)\right). \end{aligned} \tag{4.109}$$

Since the Number operator is a positive operator, we have that

$$N_{\alpha\beta}\widehat{f} = y\left(\widehat{T} + \left(\tfrac{1}{2\pi} \pm 2\right)\right)\widehat{f} \geq 0. \tag{4.110}$$

After getting rid of the factor y - which is legitimate since $y > 0$ and actually exactly $i[\widehat{A}, \widehat{B}]$ - and rearranging, we arrive at

$$\widehat{T}\widehat{f} \geq \left(\mp 2 - \tfrac{1}{2\pi}\right)\widehat{f}, \tag{4.111}$$

which becomes an eigenequation for $\widehat{T}$, if we enforce equality and makes sense if we choose +2 on the right-hand side, for else the spectrum were negative.

Before moving on, the mysterious imaginary unit in α should be discussed. Inserting α, we get

$$\begin{aligned} \widehat{\widehat{A}} = \widehat{A} - \alpha &= -\tfrac{1}{2\pi i}\left(\tfrac{1}{2} + p\partial_p\right) - \tfrac{i}{4\pi} \\ &= -\tfrac{1}{2\pi i}\left(\tfrac{1}{2} + p\partial_p\right) - \tfrac{-1/2}{2\pi i} \\ &= -\tfrac{1}{2\pi i}(p\partial_p), \end{aligned} \tag{4.112}$$

which is the generator of dilation on $\mathcal{A}_{\log}$!. Therefore, the function f, if it exists, will actually be a wavelet, defined on $\mathcal{A}_{\log}$. But we know that for a function for which

$$L_{\alpha\beta}\psi = ((A-\alpha) + i(B-\beta))\psi = 0, \quad \psi \in \mathcal{A}_{\log},$$

it necessarily also holds that

$$N_{\alpha\beta}\psi = L^*_{\alpha\beta}L_{\alpha\beta}\psi = 0, \quad \psi \in \mathcal{A}_{\log}.$$

We have therefore recovered the situation of *Theorem 4.21 (Optimal Alignment for Weighted Jacobians)* and have

$$\begin{aligned} (A-\alpha)f &= -i(B-\beta)f \\ \Leftrightarrow\ -\tfrac{1}{2\pi i}(p\partial_p)\widehat{\psi} &= -i(p-1)\widehat{\psi} \\ \Leftrightarrow\ p\partial_p\widehat{\psi} &= -2\pi(p-1)\widehat{\psi} \\ \Leftrightarrow\ \partial_p\widehat{\psi} &= (\tfrac{2\pi}{p} - 2\pi)\widehat{\psi}, \end{aligned} \tag{4.113}$$

with solution

$$\widehat{\psi}(p) := C\cdot p^{2\pi}e^{-2\pi p}. \tag{4.114}$$

This is a specific choice for the (admissible!) general solution of the uncertainty principle for the affine group, as found in (*4.56*), by setting the parameters (of (*4.56*)) to $\mu\beta = 1$ and $\alpha = 1$.

Returning to (*4.111*), we find that the function $\widehat{\psi}$ is an eigenfunction of $\widehat{T}$ in the sense that it equalizes (*4.111*),

$$\widehat{T}\widehat{\psi} = \lambda\widehat{\psi},$$

and has eigenvalue $\lambda := 2 - \frac{1}{2\pi}$ ◆

This points to a bigger picture. Namely, that the geodesically circular symmetric functions on a Lie group and especially the "ground" or "vacuum" state are specific choices of the solutions to the standard uncertainty principle of the infinitesimal generators of the Lie group and thus are always optimally concentrated in the sense of geodesic distances of the given geometry of the Lie group manifold, as defined by the associated Riemannian metric. This connection seems even more likely for Lie groups, for which Kirillov's orbit method works [48].

The author, however, has not yet been able to locate a proof for it, nor prove it himself - be it due to a lack of time or a lack of competence -, so this is a conjecture.

Conjecture 4.35 (Geodesically Circular Symmetric Functions)**.** *Let $G := \mathbb{R}^n \rtimes H$ be a well-behaved semi-direct product Lie group, with $\dim H = n$ and $(\pi, \mathcal{H}_\pi)$ denoting some unitary and irreducible representation, for which A^k and B_k are the corresponding infinitesimal generators on $\mathcal{H}$ of H and $\mathbb{R}^n$, respectively.*

Then, for the minimizing waveforms for which

$$(A^k - \alpha^k) f_{\alpha\beta} = -i(B_k - \beta_k) f_{\alpha\beta}, \quad f \in \operatorname{dom}(A^k) \cap \operatorname{dom}(B_k),$$

holds, there exists some choices $\alpha^k, \beta_k \in \mathbb{C}$, such that the $f_{\alpha\beta}$ also correspond to the optimal geodesically circular symmetric functions on the Lie group manifold, in the sense of geodesic circles equidistant from the neutral element of G. ◆

So far, the emphasize has been on the localization properties of functions in a way which reflects only whether a function has some sort of information or "energy" contained within a neighborhood of a phase space point, where the details of that neighborhood are determined by the "window", respectively its associated rank-one projector. This naturally led to the localization properties of that window and its projector, respectively the uncertainty principles above, and ultimately gave rise to phase space localization operators, a.k.a. frame multiplier.
The next section gives a slight generalization to rank-N operators.

4.4 On Phase Space Localization of Rank-N Frames

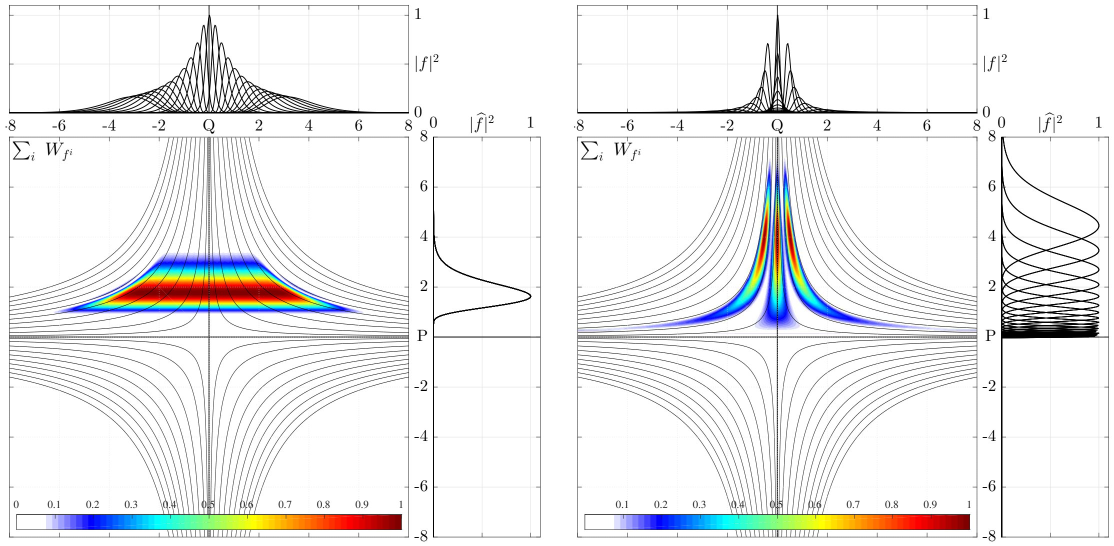
As a matter of principle, no matter the size of the associated phase space subset, the amount of information gained per rank-one projector is necessarily a single quantum of information. Thus, it is not far to seek a decomposition of functions with respect to arbitrary phase space tilings, with each tile being associated with one or more (complex) numbers, completely characterizing a function within this tile.

A somewhat idealized situation would be the following. The phase space X is tessellated into countable number of arbitrarily small tiles $\tau_i \subset X$, such that $\bigcup_i \tau_i = X$. To each τ_i we assign a "projection operator", P_i, defined on a reservoir of interesting functions $\mathcal{R}$, such that their sum resolute the identity on $\mathcal{R}$, that is,

$$\sum_i P_i = 1_{\mathcal{R}}.$$

Although this seems like a good start, it is of no use without the possibility to characterize a function's content within these individual tiles of the phase space, which is not apparent from the map

$$f \;\mapsto\; P_i f,$$



(a) **(b)**

Figure 4.6: Multiple Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma(x) = \log_2(x)$.

as the image is again a function, not a number. Thus, either a composition with a (not necessarily linear) functional

$$P_i f \mapsto F(P_i f) \in \mathbb{C}$$

is needed, which is clearly not unique, or a further decomposition of the operator P_i is needed. The latter boils down to the decomposition of the projectors into a sum of finite-rank projectors - ideally of rank one -, by writing

$$P_i := \sum_{j \in J_i} |\phi_i^j\rangle\langle\phi_i^j|,$$

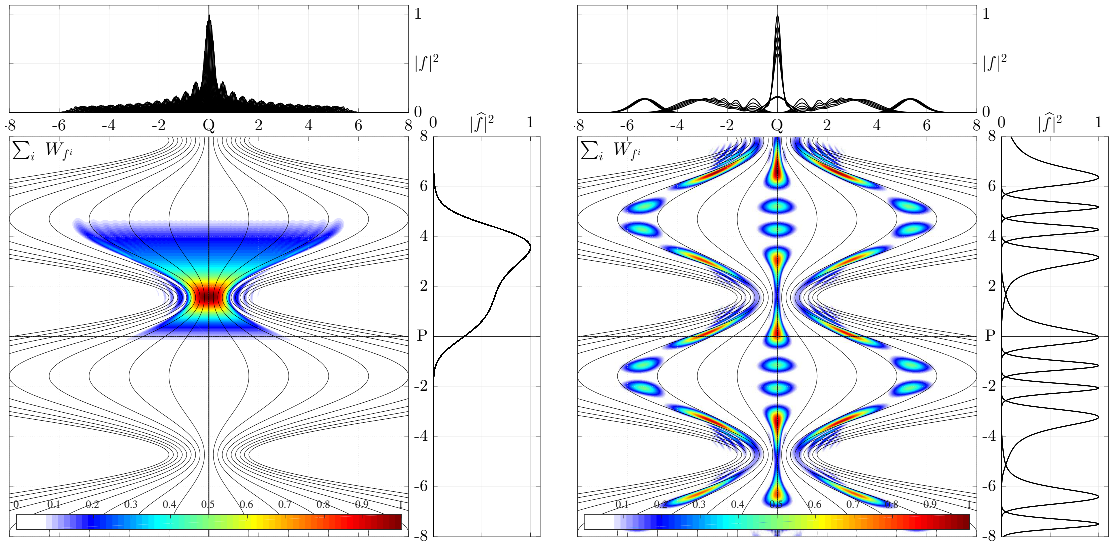
where convergence is, again, assumed with respect to the weak operator topology, as the ultimate interest is in numbers. Now, the action of P_i decomposes into an analysis and a synthesis step, that is,

$$f \mapsto \sum_{j \in J_i} \phi_i^j \left\langle \phi_i^j , f \right\rangle,$$

which suggests that, for fixed i - which ultimately labels a phase space tile - the vector

$$j \mapsto \left\langle \phi_i^j , f \right\rangle,$$

characterizes f within the tile $\tau_i \subset X$. In a perfect world, this vector would moreover have finite rank and thus *arbitrary* phase space decompositions were possible and computable in finite time. The uncertainty principle, once again being



(a) **(b)**

Figure 4.7: Multiple Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma(x) = 1.25 \cdot x + \cos(x)$.

the spoilsport, however, makes things more complicated - and more interesting. For the sum of the projectors above to be ideally localized in phase space up to arbitrary precision, an infinite number of terms is needed. Since an infinite number of computations for each of the phase space tiles is absurd, the restriction to projection operators of finite rank is the way to go, which naturally lead to the definitions of rank-N quantum frames above.

The following lemma characterizes the Wigner distribution of rank-N operators.

Lemma 4.36 (Phase Space Distributions of Rank-N Operators)**.** *Let M be a rank-N operator of the form*

$$M := \sum_i \lambda_i |\varphi^i\rangle\langle\varphi^i|. \tag{4.115}$$

Then, the Weyl symbol distribution of M is given by

$$W(M) := \sum_i \lambda_i W_{\varphi^i}. \tag{4.116}$$

In particular, for a rank-N projection operator its Wigner distribution is the sum of the Wigner distributions of its rank-one projectors. ◆

Proof. As the Wigner distribution of an operator is its Weyl symbol, which is representable via a trace, the claim follows from the linearity of the trace

$$\operatorname{tr}\left(T_{(x,\xi)} \sum_i \lambda_i |\varphi^i\rangle\langle\varphi^i|\right) = \sum_i \lambda_i \operatorname{tr}\left(T_{(x,\xi)}|\varphi^i\rangle\langle\varphi^i|\right) = \sum_i \lambda_i W_{\varphi^i},$$

where the interchange of trace and sum is permitted whenever the sum converges, which is trivially the case for finite sums. ■

The addressed fact that there is a gauge freedom in each fiber of X - the N-dimensional vector space over each point of the base manifold -, suggests that the combined localization properties of the individual rank-ones, which make up the rank-N projectors, should be gauge invariant, too. Indeed.

Lemma 4.37 (Gauge Equivalence of Localization)**.** *Let $\mathbb{F}$ be a frame of rank N and $F(x)$ the rank-N projectors for each $x \in X$. Then, the localization properties of $F(x)$ are well-defined, that is, the phase space cell associated with $F(x)$ is gauge invariant.* ◆

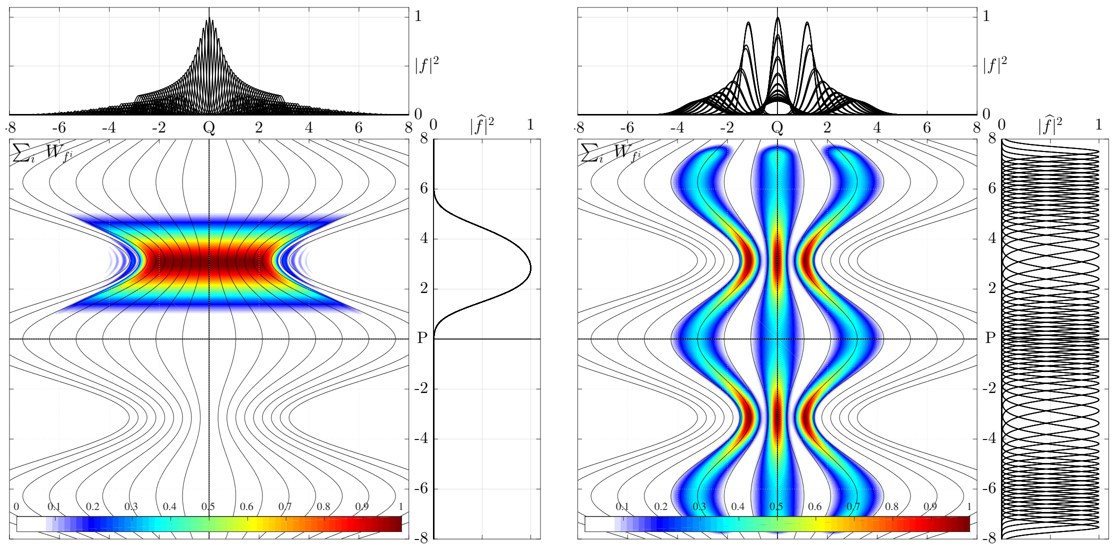
Proof. Since our concept of localization hinges on quadratic phase space representations and, by *Lemma 4.36 (Phase Space Distributions of Rank-N Operators)*, the phase space distribution of a rank-N operator is the sum of its N rank-one distributions, it suffices to show that for each $U(x) \in U(N)$, the sum of the distributions is invariant. But, since, for each $x \in X$, the operator $U(x)$ commutes with $T_{(x',\xi')}$, we necessarily have

$$\begin{aligned}
\operatorname{tr}\left(T_{(x',\xi')} U(x) \sum_i |\varphi_x^i\rangle\langle\varphi_x^i| U(x)^* \right) &= \operatorname{tr}\left(U(x) T_{(x',\xi')} \sum_i |\varphi_x^i\rangle\langle\varphi_x^i| U(x)^* \right) \\
&= \operatorname{tr}\left(T_{(x',\xi')} \sum_i |\varphi_x^i\rangle\langle\varphi_x^i| \right) \\
&= \sum_i W_{\varphi^i},
\end{aligned}$$

by the invariance of the trace under unitary conjugation. ■

In the previous chapter, it was noted that with each spectral diffeomorphism comes along a pair of spectral quantum frames, by means of which multipliers may be introduced, which localize or weight different coefficients. Since to these coefficients correspond a certain phase space cell, a multiplier can be seen as a means to change the importance - to weight - each quantum of information of phase space of a given signal. We now know that if with each point in phase space is associated an operator of higher rank, its phase space picture is given by the sum of its constituting rank-ones. Thus, using optimally localized functions associated with a coordinate system in phase space leads to localized phase space cells, containing various quanta of information.

In *Figure 4.6*, *Figure 4.7* and *Figure 4.8*, multiple Wigner distributions of "warped Gaussians" - for two spectral diffeomorphisms, defined in the next chapter



(a)

(b)

Figure 4.8: Multiple Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma(x) = 2 \cdot x + \sin(x)$.

-, along with coordinate lines are depicted. These show that the associated phase space localization can be made to be aligned along the coordinates. In this regard, also the sum of the projectors of the first 32 Hermite functions - including the "0^{th} Hermite function", a.k.a. the Gaussian -, can be used as a "rank-33 quantum probe" associated with the rectangular coordinate grid in phase space. These assign to each point in phase space a circular phase space cell, as depicted in *Figure 4.2b*, containing 33 quanta of information.

Hoper Edei Deixai - *Quod Erat Demonstrandum* ■

— Euclid [41]

5

Application

HITHERTO, THE *EMPHASIS* has been on the development of the abstract theory of spectral diffeomorphisms, its associated frames and connected localization principles in phase space.
Since theory without practice is meaningless, this scriptum is yet missing a significant ingredient - it is time for concretization.

The theory of spectral diffeomorphisms and its associated symplectomorphisms encompasses a great number of linear signal transforms, omnipresent in the literature of signal analysis. In this chapter, various examples of spectral diffeomorphisms, along with their cotangent lifts, will be elaborated and the associated transforms will be identified.

But, in order to do so, a quick refresh on dual orbits, arising from locally compact semi-direct product groups is needed, if only to fix some notation. Confer, e.g., [1, 31, 34] for virtuous treatments of the theory of locally compact groups and further material.

5.1 Dual Orbits of Semi-Direct Product Groups

We start with a proposition, summarizing most of the obvious facts.

Proposition 5.1 (Semi-Direct Product Group). *Let* $G := \mathbb{R}^n \rtimes H$ *be a semi-direct product, with* $\mathbb{R}^n$ *denoting the abelian Euclidean group and* $(H, \mathrm{d}\nu)$ *some connected abelian Lie group, with* $\dim(H) = n$ *and Haar measure* $\mathrm{d}\nu$*. Then*

(i) *the multiplication and inversion laws are given by*

$$(b,a)(x,h) = (a.x+b, ah), \qquad (x,h),(b,a) \in \mathbb{R}^n \times H, \tag{5.1}$$

and

$$(x,h)^{-1} = (-h^{-1}x, h^{-1}), \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.2}$$

(ii) *the left and right Haar measures are given by*

$$\mathrm{d}\mu_L(x,h) = \det(h)^{-1}\ \mathrm{d}x\ \mathrm{d}\nu, \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.3}$$

and

$$\mathrm{d}\mu_R(x,h) = \ \mathrm{d}x\ \mathrm{d}\nu, \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.4}$$

with

$$\mathrm{d}\mu_R(x,h) = \ \mathrm{d}\mu_L((x,h)^{-1}), \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.5}$$

(iii) *the modular function, which is a Radon-Nikodym derivative,* Definition A.16 (Radon-Nikodym)*, reads*

$$\Delta(x,h) := \frac{\mathrm{d}\mu_L(x,h)}{\mathrm{d}\mu_R(x,h)} = \det(h)^{-1}, \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.6}$$

(iv) *for some $F \in L^1(G,\ \mathrm{d}\mu_L)$, the isometric involution,* Proposition A.67 (Isometric Involution)*, on G is given by*

$$\begin{aligned} F^*(x,h) =& \Delta^{-1}(x,h)\overline{F(-h^{-1}x,h^{-1})}, \qquad (x,h) \in \mathbb{R}^n \times H, \\ =& \det(h)\overline{F(-h^{-1}x,h^{-1})}, \qquad (x,h) \in \mathbb{R}^n \times H. \end{aligned} \tag{5.7}$$

◆

Proof. (i) is part of the definition of a semi-direct product group. (ii) can be checked directly by

$$\begin{aligned} \mathrm{d}\mu_L((b,a)(x,h)) =& \ \mathrm{d}\mu_L(a.x+b, ah) \\ &= \det(ah)^{-1}\mathrm{d}(ax+b)\ \mathrm{d}\nu(ah) \\ &= \det(h)^{-1}\det(a)^{-1}\det(a)\ \mathrm{d}x\ \mathrm{d}\nu(h) \\ &= \det(h)^{-1}\ \mathrm{d}x\ \mathrm{d}\nu(h) \end{aligned}$$

and

$$\begin{aligned}
d\mu_R((x,h)(b,a)) &= \ d\mu_R(h.b+x, ha) \\
&= d(h.b+x)\ d\nu(ah) \\
&= \ dx\ d\nu(h)
\end{aligned}$$

and

$$\begin{aligned}
d\mu_L((x,h)^{-1}) &= \ d\mu_L(-h^{-1}x, h^{-1}) \\
&= \det(h^{-1})^{-1} d(-h^{-1}x)\ d\nu(h^{-1}) \\
&= \det(h)\det(h^{-1})\ dx\ d\nu(h) \\
&= \ dx\ d\nu(h).
\end{aligned}$$

(iii) follows immediately from (ii) via

$$\begin{aligned}
\Delta(x,h) &:= \frac{d\mu_L(x,h)}{d\mu_R(x,h)}, \\
&= \frac{\det(h)^{-1}\ dx\ d\nu(h)}{dx\ d\nu(h)}, \\
&= \det(h)^{-1}.
\end{aligned}$$

To see (iv), note that

$$\begin{aligned}
\int_G |F^*(x,h)|\ d\mu_L &= \int_G \left|\det(h)\overline{F(-h^{-1}x, h^{-1})}\right|\ d\mu_L(x,h) \\
&= \int_G \left|\det(h)\overline{F(x,h)}\right|\ d\mu_L((x,h)^{-1}) \\
&= \int_G \det(h)\left|\overline{F(x,h)}\right|\ d\mu_R(x,h) \\
&= \int_G |F(x,h)|\det(h)\ d\mu_R(x,h) \\
&= \int_G |F(x,h)|\ d\mu_L(x,h),
\end{aligned}$$

where we used (i), (ii) and (iii). ■

Let now $G \coloneqq \mathbb{R}^n \rtimes H$ be such a semi-direct product, with $\mathbb{R}^n$ denoting the abelian Euclidean group and $(H,\ \mathrm{d}\nu)$ some connected abelian Lie group, with $\dim(H) = n$ and Haar measure $\mathrm{d}\nu$. Let furthermore H act *freely* via matrices on $\mathbb{R}^n$ as

$$x \mapsto h.x, \quad x \in \mathbb{R}^n, h \in H, \tag{5.8}$$

where an action is said to be free if the stabilizer of each point is trivial, that is, if there are no fixed points of the action.

Then, via dual pairing with $\widehat{\mathbb{R}}^n$,

$$\langle \xi\,,\, h.x \rangle = \left\langle h^T.\xi\,,\, x \right\rangle, \quad x \in \mathbb{R}^n, \xi \in \widehat{\mathbb{R}}^n, \tag{5.9}$$

H also acts on $\widehat{\mathbb{R}}^n$ with the induced dual action

$$\xi \mapsto h^{-T}.\xi, \quad \xi \in \widehat{\mathbb{R}}^n, h \in H, \tag{5.10}$$

where the inversion of h is needed in order to assure that it is a group action

$$(h' \circ h)^{-T}.\xi_0 = h'^{-T}.(h^{-T}.\xi_0). \tag{5.11}$$

Since the groups we shall consider are abelian, h' and h necessarily commute and thus this fact is superfluous and can be relaxed, if the calculations benefit from it.

Then, from these two actions, there arise two unitarily equivalent *group representations* of H, *Definition A.60 (Unitary representations)*, acting on $L^2(\mathbb{R}^n,\ \mathrm{d}x)$, ρ, as well as on $L^2(\widehat{\mathbb{R}}^n,\ \mathrm{d}\xi)$, $\hat{\rho}$, defined as

$$(\rho(h)f)(x) \coloneqq \det h^{-1/2} f(h^{-1}.x) \text{ and } (\hat{\rho}(h)\hat{f})(\xi) \coloneqq \det h^{1/2} \hat{f}(h^T.\xi) \tag{5.12}$$

and the quasi-regular representations of G, π and $\hat{\pi}$, defined by

$$(\pi(b,h)f)(x) \coloneqq \det h^{-1/2} f(h^{-1}.(x-b)), \quad f \in L^2(\mathbb{R}^n,\ \mathrm{d}x) \tag{5.13}$$

and

$$(\hat{\pi}(b,h)\hat{f})(\xi) \coloneqq \det h^{1/2} \hat{f}(h^T.\xi) e^{-2\pi i \langle \xi\,,\, b \rangle}, \quad \hat{f} \in L^2(\widehat{\mathbb{R}}^n,\ \mathrm{d}\xi). \tag{5.14}$$

The representation ρ - and thus by equivalence $\hat{\rho}$ as well - of H and the quasi-regular representation π (and $\hat{\pi}$) are *irreducible*, *Definition A.71 (Irreducibility)*, if and only if the dual action of H is transitive on $\widehat{\mathbb{R}}^n$, meaning that it only has a single *orbit*, *Definition A.23 (Orbits of group actions)*. In general, $\widehat{\mathbb{R}}^n$ foliates into orbits of H - each of which has associated an irreducible unitary representation -, such that the *orbit space*

$$\widehat{\mathbb{R}}^n / G \coloneqq \left\{ \mathcal{O} \subseteq \widehat{\mathbb{R}}^n \mid \exists \xi_0 \in \mathcal{O} : G.\xi_0 \coloneqq \mathcal{O} \right\}, \tag{5.15}$$

partitions $\widehat{\mathbb{R}}^n$ and the orbit space may possess some measure μ with which one may do non-commutative harmonic analysis, speak of direct integrals of representations, decompose a regular representation of a locally compact group into the discrete and continuous series and much more; for short - from this point, a lot of non-trivial representation theory departs.

Since the program of spectral diffeomorphisms tries to circumvent most of the restricting group structure to gain more generality and uses only diffeomorphisms and families of one-parameter groups - which do not necessarily constitute a group as a whole -, it suffices to concentrate on a single dual orbit of the group H and the associated irreducible representation of G, $(\pi, \mathcal{H}_\pi)$. The same steps may then be applied to each of the orbits and its associated irreducible representations, provided the orbits have the same dimension as the group. *Schur's Lemma A.72*, then, tells us that the only operators which commute with the whole image of an irreducible representation, within the unitary operators over the representation space $\mathcal{H}_\pi$, are multiples of the identity and given a cyclic and admissible vector, φ, *Definition A.71 (Irreducibility)*, we may decompose the representation space via the set

$$\{\, \pi(x)\varphi \mid x \in G \,\}, \tag{5.16}$$

giving a first step towards continuous frames.

Then, since the action of H on this orbit, say $\mathcal{O}$, is both free and transitive, the orbit is a *principal homogeneous space* of H, *Definition A.31 (Homogeneous Spaces)*, and the map

$$H \ni h \;\mapsto\; h^{-T}.\xi_0 \in \mathcal{O}, \quad \xi_0 \in \mathcal{O}, h \in H, \tag{5.17}$$

is a homeomorphism from H to the orbit, if and only if H is σ-compact [31]. Note that the element, ξ_0, is associated with the neutral element of H and thus may serve as an origin on $\mathcal{O}$. Since the group is assumed to be abelian and connected, its Lie algebra, $\mathfrak{h}$, is abelian, too, and the exponential map is a homeomorphism.

Then, we may represent any element of H by the local coordinates, $h(\alpha_1, \ldots, \alpha_n)$, defined by the Lie algebra. And since it is abelian, we have

$$\exp(-2\pi i \sum_k \alpha^k X_k) = \prod_k e^{-2\pi i \alpha^k X_k}, \tag{5.18}$$

where $\alpha \in \mathbb{R}^n$ and the $\{\, X_k \mid k = 1, \ldots, n \,\}$ are a basis of generators on the Lie algebra.

Now, here is the main takeaway of the above said.

Theorem 5.2 (Spectral Diffeomorphisms for Dual Orbits)**.** *Let $G := \mathbb{R}^n \rtimes H$ be a locally compact semi-direct product group, $(H,\ \mathrm{d}\nu)$ a connected abelian Lie group, with $\dim(H) = n$ and Haar measure $\mathrm{d}\nu$, assumed to be σ-compact. Let H act freely and transitive on the dual orbit*

$$\mathcal{O}_0 := \left\{ h^{-T}.\xi_0 \,|\, h \in H \right\} \subseteq \widehat{\mathbb{R}}^n, \tag{5.19}$$

for some fixed $\xi_0 \in \widehat{\mathbb{R}}^n$. Denote

$$\sigma^{-1}(h) := h^{-T}.\xi_0 \quad \textit{and} \quad \sigma(\xi) := h_\xi, \;\; \textit{s.t.}\; \xi = h_\xi^{-T}.\xi_0 \tag{5.20}$$

and $\mathrm{d}\sigma(\xi) := (\mathrm{d}\nu \circ \sigma)(\xi)$.

Then, σ is a spectral diffeomorphism *and the machinery of this thesis applies.* ◆

Proof. See, e.g., [31, Prop. 2.44], for a proof that the map

$$h \;\mapsto\; h^{-T}.\xi_0 =: \sigma^{-1}(h)$$

is a homeomorphism. Supplying both spaces - the group and the orbit - with the Borel σ-algebra, we get *measurable spaces*, where the group can be equipped with its natural translation invariant Haar measure, $\mathrm{d}\mu$, to get a true *measure space*. The orbit, then, may be equipped with the pullback measure $\mathrm{d}\sigma(\xi) := (\mathrm{d}\nu \circ \sigma)(\xi)$ and the homeomorphism becomes a measurable mapping.

The mapping is furthermore a diffeomorphism since the group is a Lie group and thus a differentiable manifold. Therefore, it acts on differentiable manifolds in a differentiable way, which makes way for diffeomorphisms. And, finally, it is a *spectral diffeomorphism*, since it acts on the dual space of $\mathbb{R}^n$ and thus we may apply a *spectral cotangent lift*, to set the *spectral warping machine* in motion. ∎

Since the image of σ is H, we have that

$$\sigma^{-1}(a^{-1}h) = (a^{-1}h)^{-T}.\xi_0 = a^T h^{-T}.\xi_0 = a^T \sigma^{-1}(h), \qquad a, h \in H, \tag{5.21}$$

and

$$j_\sigma(h) = \det(J_\sigma^{-1}) = \det(h^{-T}) = \det(h)^{-1}, \tag{5.22}$$

and therefore

$$\det(J_\sigma) \circ \sigma = \det(h) \circ \sigma. \tag{5.23}$$

The associated spectral warped distribution, as introduced in *Definition 3.29 (Spectral Warped Distributions)*, can be seen as being defined on $G := \mathbb{R}^n \times H$, that is, we have that

$$D_f^\sigma(x,h) =: D_f(x,h), \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.24}$$

and

$$\widetilde{D}_\varphi^\sigma(x,h) =: \widetilde{D}_\varphi(x,h), \qquad (x,h) \in \mathbb{R}^n \times H, \tag{5.25}$$

are functions on G and we may introduce a convolution, *Definition A.64 (Convolution)*, on G. Further, for the latter distribution, we may define the involution

$$\begin{aligned} \widetilde{D}_\varphi^*(x,h) &:= \Delta^{-1}(x,h)\overline{\widetilde{D}_\varphi((x,h)^{-1})} \\ &= \Delta^{-1}(x,h)\overline{\widetilde{D}_\varphi(-h^{-1}x, h^{-1})} \\ &= (\det(h))^{+1}\overline{\widetilde{D}_\varphi(-h^{-1}x, h^{-1})} \end{aligned} \tag{5.26}$$

which gives the following.

Corollary 5.3 (Frameogram for Semi-Direct Product Groups)**.** *Let $G := \mathbb{R}^n \rtimes H$ be a locally compact semi-direct product group, $(H,\ \mathrm{d}\nu)$ a connected abelian Lie group, with $\dim(H) = n$ and Haar measure $\mathrm{d}\nu$, assumed to be σ-compact. Let H act freely and transitive on the dual orbit*

$$\mathcal{O}_0 := \left\{\, h^{-T}.\xi_0 \mid h \in H \,\right\} \subseteq \widehat{\mathbb{R}}^n, \tag{5.27}$$

for some fixed $\xi_0 \in \widehat{\mathbb{R}}^n$. Denote

$$\sigma^{-1}(h) := h^{-T}.\xi_0 \quad \textit{and} \quad \sigma(\xi) := h_\xi, \quad \textit{s.t.} \quad \xi = h_\xi^{-T}.\xi_0 \tag{5.28}$$

and $\mathrm{d}\sigma(\xi) := (\mathrm{d}\nu \circ \sigma)(\xi)$. Let furthermore $f \in \mathcal{S}_\sigma$ and $\varphi \in \mathcal{A}_\sigma$, with D_f and $\widetilde{D}_\varphi$ denoting their respective warped distributions. Then

$$D_f * \widetilde{D}_\varphi^* = \left|\pi_\varphi^\sigma f\right|^2. \tag{5.29}$$

◆

Proof. In order to prove (*5.29*), we do a lengthy calculation.

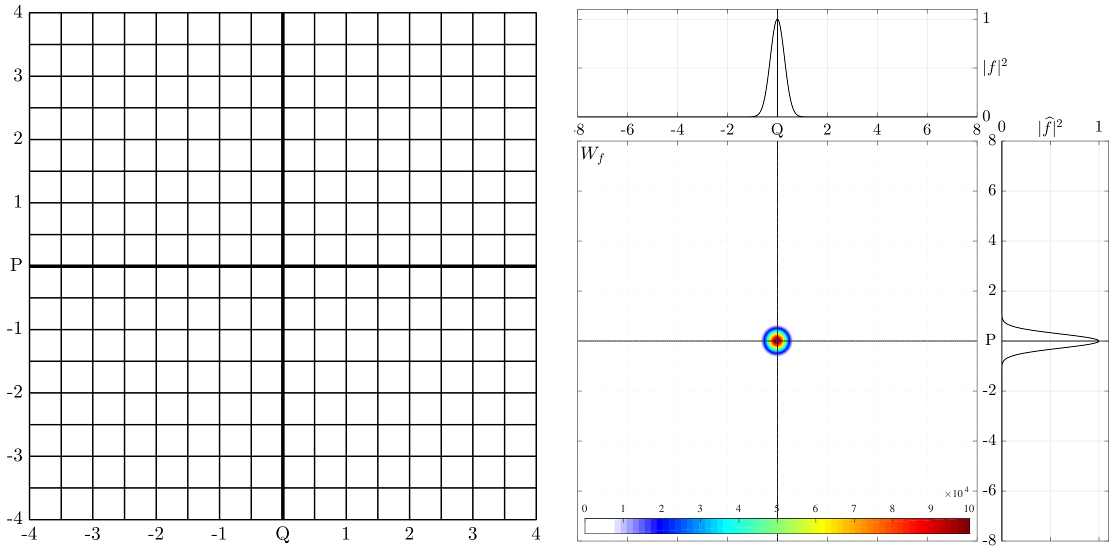
$$
\begin{aligned}
&(D_f * \widetilde{D}_\varphi^*)(b,a) \\
=&\int_G \widetilde{D}_\varphi^*((x,h)^{-1}(b,a))\ D_f(x,h)\ \mathrm{d}\mu_L(x,h) \\
=&\int_G \Delta^{-1}\overline{\widetilde{D}_\varphi(((x,h)^{-1}(b,a))^{-1})}D_f(x,h)\ \mathrm{d}\mu_L(x,h) \\
=&\int_G \Delta^{-1}\overline{\widetilde{D}_\varphi((b,a)^{-1}(x,h))}D_f(x,h)\ \mathrm{d}\mu_L(x,h) \\
=&\int_G \det(a^{-1}h)\overline{\widetilde{D}_\varphi(a^{-1}(x-b),a^{-1}h)}D_f(x,h)\ \mathrm{d}\mu_L(x,h) \\
=&\int_G \det(a)^{-1}\det(h)\overline{\varphi(a^{-1}(x-b))\overline{\widehat{\varphi}(\sigma^{-1}(a^{-1}h))}e^{-2\pi i\langle\,\sigma^{-1}(a^{-1}h)\,,\,a^{-1}(x-b)\,\rangle}} \\
&\times\ \det(h)^{-1}\cdot f(x)\overline{\widehat{f}(\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,x\,\rangle}\ \mathrm{d}\mu_L(x,h) \\
=&\int_G \overline{\det(a)^{-1}\varphi(a^{-1}(x-b))\overline{\widehat{\varphi}(a^T\sigma^{-1}(h))}e^{-2\pi i\langle\,a^T\sigma^{-1}(h)\,,\,a^{-1}(x-b)\,\rangle}} \\
&\times\ f(x)\overline{\widehat{f}(\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,x\,\rangle}\ \mathrm{d}\mu_L(x,h) \\
=&\int_{\mathbb{R}^n\times H} \overline{\det(a)^{-1}\varphi(a^{-1}(x-b))\overline{\widehat{\varphi}(a^T\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,x-b\,\rangle}} \\
&\times\ f(x)\overline{\widehat{f}(\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,x\,\rangle}\det(h)^{-1}\ \mathrm{d}x\ \mathrm{d}\nu(h) \\
=&\int_{\mathbb{R}^n\times H} \overline{\det(a)^{-1}\varphi(a^{-1}(x-b))\overline{\widehat{\varphi}(a^T\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,b\,\rangle}} \\
&\times\ f(x)\overline{\widehat{f}(\sigma^{-1}(h))}\det(h)^{-1}\ \mathrm{d}x\ \mathrm{d}\nu(h) \\
=&\int_{\mathbb{R}^n} \overline{\det(a)^{-1}\varphi(a^{-1}(x-b))}\ f(x)\ \mathrm{d}x \\
&\times\ \int_{\mathrm{im}(\sigma)} \widehat{\varphi}(a^T\sigma^{-1}(h))\overline{\widehat{f}(\sigma^{-1}(h))}e^{-2\pi i\langle\,\sigma^{-1}(h)\,,\,b\,\rangle}\det(h)^{-1}\ \mathrm{d}\nu(h) \\
=&(\pi_\varphi^\sigma f)(b,a)\int_{\mathrm{dom}(\sigma)} \widehat{\varphi}(a^T\xi)\overline{\widehat{f}(\xi)}e^{-2\pi i\langle\,\xi\,,\,b\,\rangle}\det(J_\sigma^{-1})\ \mathrm{d}\sigma(\xi) \\
=&(\pi_\varphi^\sigma f)(b,a)\int_{\mathrm{dom}(\sigma)} \widehat{\varphi}(a^T\xi)\overline{\widehat{f}(\xi)}e^{-2\pi i\langle\,\xi\,,\,b\,\rangle}\det(J_\sigma^{-1})\det(J_\sigma)\mathrm{d}\xi \\
=&(\pi_\varphi^\sigma f)(b,a)\overline{\int_{\mathrm{dom}(\sigma)} \overline{e^{2\pi i\langle\,\xi\,,\,b\,\rangle}\ \widehat{\varphi}(a^T\xi)}\ \widehat{f}(\xi)\mathrm{d}\xi} \\
=&(\pi_\varphi^\sigma f)(b,a)\overline{(\pi_\varphi^\sigma f)(b,a)} \\
=&\left|(\pi_\varphi^\sigma f)(b,a)\right|^2
\end{aligned}
\tag{5.30}
$$

This calculation was indeed lengthy! ∎

5.2 Explicit Phase Space Decompositions

5.2.1 The Trivial Case

Before looking at some special cases, there is one peculiar case which is the most beautiful.



(a) Phase space tessellation. **(b)** Wigner distribution of a Gaussian.

Figure 5.1: Phase space, associated with $\sigma(x) = x$.

Example 5.1 (Identity)**.** Let $f \in L^2(\mathbb{R}^n)$. When we are willing to accept the fact that each point in the spatial domain, as well as each point in the Fourier domain, is equally important, there is no need to apply any spectral diffeomorphism and thus, by setting $\sigma = \sigma^{-1} = \boldsymbol{1}$, we have that

$$J_\sigma = J_\sigma^{-T} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}. \tag{5.31}$$

Figure 5.1a plots the associated phase space tessellation and *Figure 5.1b* shows the Wigner distribution of a Gaussian, which is the optimal waveform - for both uncertainty principles -, adapted to the rectangular grid.

The arising quantum frames are the continuous Gabor frame and the associated coherent state map is the well known

Definition 5.4 (Short-Time Fourier Transform)**.** Let $f, \psi \in \mathcal{S}_{\boldsymbol{1}} = \mathcal{A}_{\boldsymbol{1}}$, then

$$f \mapsto \left\langle \mathcal{T}_\beta^{\boldsymbol{1}} \widetilde{\mathcal{D}}_\alpha^{\boldsymbol{1}} \psi \, , \, f \right\rangle = \int_{\mathbb{R}^n} \overline{e^{2\pi i \alpha (x-\beta)} \psi\left(x - \beta\right)} \; f(x) \; \mathrm{d}x, \tag{5.32}$$

is the *Short-Time Fourier Transform.* ◆

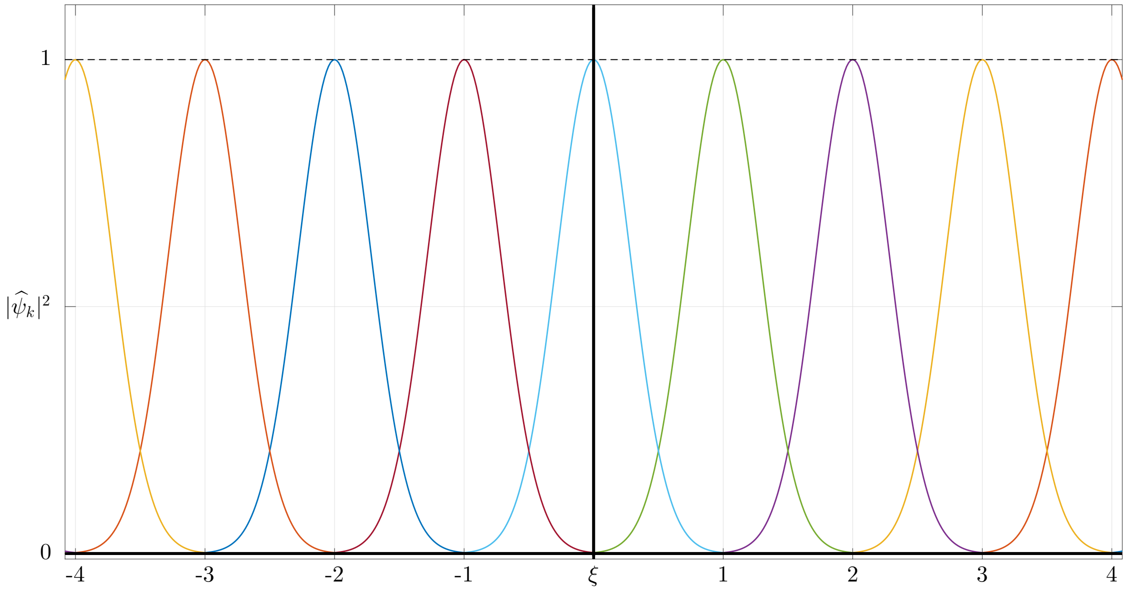


Figure 5.2: Superimposed spectra of "warped" Gaussians, $\psi_k := \sqrt{|\frac{\mathrm{d}\sigma}{\mathrm{d}\xi}|} e^{-\pi(\sigma(\xi)-k)^2}$, spectral dilated with $\sigma(\xi) := \mathbf{1}(\xi) = \xi$. The dotted line shows $\frac{\mathrm{d}\sigma}{\mathrm{d}\xi} = \frac{\mathrm{d}\xi}{\mathrm{d}\xi} \equiv 1$.

Localization

Both notions of localization coincide and thus the equalizing waveform for optimal alignment as well as optimal concentration is given by

$$\begin{aligned}\widehat{f}(x) &= e^{-2\pi \sum_k \int x_k \ \mathrm{d}\mathbf{1}_k(x)} \\ &= e^{-2\pi \sum_k \int \mathbf{1}_k(x) \ \mathrm{d}x_k} \\ &= e^{-\pi |x|^2},\end{aligned} \tag{5.33}$$

where the constants are chosen trivially. ◆

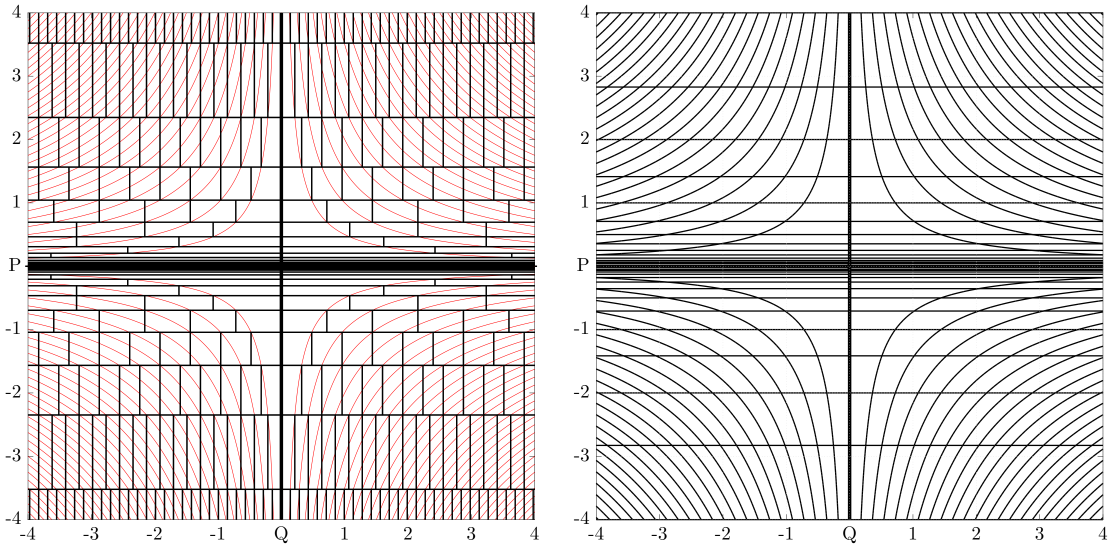
5.2.2 The Half-Line

To get a taste for a more non-trivial use case, we will take "a closer look" at what will ultimately lead to what is known as the *wavelet transform*. A transform, which can be understood as a mathematical microscope and is thus - by its very definition - the paragon of "looking more closely".

The Logarithm

Let f be a measured audio signal.

Example 5.2 (Logarithm)**.** Speaking of audio signals, their values are real, the spatial domain is time and the relevant information about the occurring frequencies



(a) Rectangular phase space tessellation **(b)** Curved phase space tessellation

Figure 5.3: Phase space tessellation, associated with $\sigma(x) = \log_2(x)$.

is supported on the *positive frequency axis only*, since there is no physical equivalent of a negative vibration.

Moreover, since audio signals are assumed to have finite energy, by a little abuse of notation, it holds

$$f \text{ is an audio signal} \quad \Rightarrow \widehat{f} \in L^2(\mathbb{R}_+, \mathrm{d}x).$$

It therefore makes sense to map the positive half-axis $\widehat{\mathbb{R}}_+$ onto the whole reals $\widehat{\mathbb{R}}$ by utilizing the logarithm - and, of course, its inverse, the exponential -, which is a diffeomorphism,

$$\log : \mathbb{R}_+ \to \mathbb{R}, \qquad \log^{-1} = \exp : \mathbb{R} \to \mathbb{R}_+,$$

for it is known to model this correspondence. Of course, the actual base of the logarithm is of no relevance and may thus be chosen at will. For numerical reasons and its connection to octaves, a popular choice is the $\log_2$, but in general, one is free to use

$$\log_b : \mathbb{R}_+ \to \mathbb{R}, \qquad \log_b^{-1} = b^{\bullet} : \mathbb{R} \to \mathbb{R}_+, \quad \log_b^{-1}(x) = b^x,$$

for any real $b > 0$.

What makes the logarithm, respectively its inverse, the exponential function, as the underlying morphism so special, is that it maps a translation, that is, an addition in the warped domain to a dilation, i.e., a multiplication, in the "unwarped"

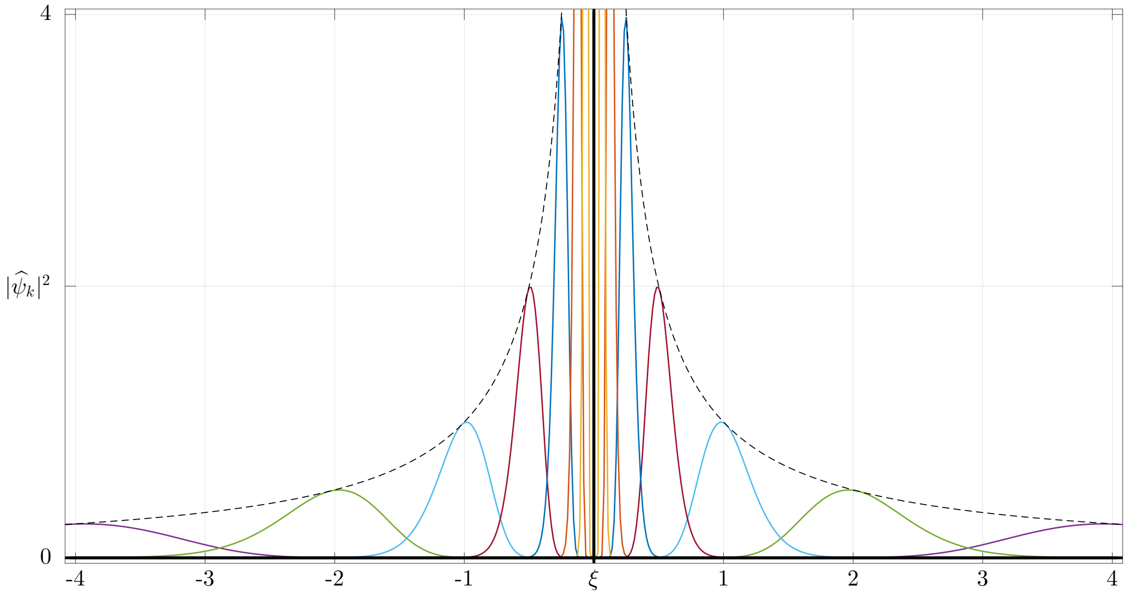


Figure 5.4: Superimposed spectra of warped Gaussians, $\psi_k := \sqrt{\left|\frac{\mathrm{d}\log_2}{\mathrm{d}\xi}\right|} e^{-\pi(\log_2|\xi|-k)^2}$, spectral dilated with $\sigma(\xi) = \log_2|\xi|$. The dotted line shows $\left|\frac{\mathrm{d}\log_2}{\mathrm{d}\xi}\right|$.

original domain and vice versa. That is, the induced spectral dilation operator is a dilation operator in the standard sense of a dilation - in fact, this is where its name was derived from. *Figure 5.4* illustrates this fact, for some well-localized g - a Gaussian waveform - in the warped domain and where the base of the logarithm was chosen to be 2, in order to increase the readability.

This is the first of four cases we shall consider, to which *Theorem 5.2 (Spectral Diffeomorphisms for Dual Orbits)* applies. Let $H := \mathbb{R}$, acting via

$$\xi \ \mapsto \ e^{\alpha}\xi, \qquad \xi \in \mathbb{R}_+, \alpha \in \mathbb{R}, \tag{5.34}$$

on $\mathbb{R}_+$. Then, with $\xi_0 = 1$, we have that

$$\alpha \ \mapsto \ e^{\alpha}\xi_0 = e^{\alpha} =: \sigma^{-1}(\alpha) \tag{5.35}$$

is a *spectral* diffeomorphism from $H := \mathbb{R} = \mathrm{im}(\sigma)$ to $\mathrm{dom}(\sigma) =: \mathbb{R}_+$. It is straightforward to check that the left-invariant measure on $\mathbb{R} \rtimes H$ is $\mathrm{d}\mu_L(x,y) := \frac{\mathrm{d}x\ \mathrm{d}y}{e^y}$, whereas the right invariant measure and the modular function are $\mathrm{d}\mu_R(x,y) := \mathrm{d}x\ \mathrm{d}y$ and $\Delta(y) := e^{-y}$.

As for the *frameogram*, associated with the logarithm - the *scaleogram* -, we need to be more pedantic when it comes to the spectral diffeomorphism above and re-instantiate the inversion from (*5.10*) and redefine the above as

$$\alpha \ \mapsto \ e^{-\alpha}\xi_0 = e^{-\alpha} =: \sigma^{-1}(\alpha), \tag{5.36}$$

in order to derive the frameogram. Since *Corollary 5.3 (Frameogram for Semi-Direct Product Groups)* is very general, this shall be the only explicit derivation of a spectral frameogram via warped distributions in this chapter.

The warped distributions are defined as

$$\mathcal{D}_f^{\log}(b,a) := e^{-a} f(b)\overline{\widehat{f}(e^{-a})} e^{-2\pi i e^{-a} b} \tag{5.37}$$

and

$$\begin{aligned}
(\widetilde{\mathcal{D}}_\psi^{\log})^*(b,a) &:= \Delta^{-1}(a)\widetilde{\mathcal{D}}_\psi^{\log}((b,a)^{-1}) \\
&= \Delta^{-1}(a)\widetilde{\mathcal{D}}_\psi^{\log}(-\tfrac{b}{e^a}, -a) \\
&= \overline{e^{+a}\psi(-\tfrac{b}{e^a})\overline{\widehat{\psi}(e^{+a})} e^{-2\pi i e^{+a}(-\frac{b}{e^a})}} \\
&= \overline{e^{a}\psi(-\tfrac{b}{e^a})\overline{\widehat{\psi}(e^{a})} e^{2\pi i b}}
\end{aligned} \tag{5.38}$$

and convolution gives

$$\begin{aligned}
&(\mathcal{D}_f^{\log} * (\widetilde{\mathcal{D}}_\psi^{\log})^*)(b,a) \\
&= \iint e^{-y} f(x)\overline{\widehat{f}(e^{-y})} e^{-2\pi i e^{-y} x}\, \overline{e^{y-a}\psi(-\tfrac{\frac{b-x}{e^y}}{e^{(a-y)}})\overline{\widehat{\psi}(e^{(a-y)})} e^{2\pi i \frac{b-x}{e^y}}}\, \mathrm{d}\mu_L \\
&= \iint e^{-a} f(x)\overline{\widehat{f}(e^{-y})} e^{-2\pi i e^{-y} x}\, \overline{\psi(\tfrac{x-b}{e^y e^{(a-y)}})\overline{\widehat{\psi}(e^a e^{-y})} e^{2\pi i e^{-y}(b-x)}}\, \mathrm{d}\mu_L \\
&= \iint e^{-a} f(x)\overline{\psi(\tfrac{x-b}{e^a})}\,\overline{\widehat{f}(e^{-y})} e^{-2\pi i e^{-y} x}\widehat{\psi}(e^a e^{-y}) e^{2\pi i e^{-y}(x-b)} \frac{\mathrm{d}x\ \mathrm{d}y}{e^y} \\
&= \int e^{-a} f(x)\overline{\psi(\tfrac{x-b}{e^a})}\ \mathrm{d}x \int \overline{\widehat{f}(e^{-y})}\widehat{\psi}(e^a e^{-y}) e^{-2\pi i e^{-y} b} \frac{\mathrm{d}y}{e^y} \\
&= \int e^{-a} f(x)\overline{\psi(\tfrac{x-b}{e^a})}\ \mathrm{d}x\, \overline{\int \widehat{f}(\xi)\overline{\widehat{\psi}(e^a \xi)} e^{-2\pi i \xi b}\ \mathrm{d}\xi} \\
&= \left|\pi_\psi^{\log} f(b,a)\right|^2,
\end{aligned} \tag{5.39}$$

which is the *scaleogram*.

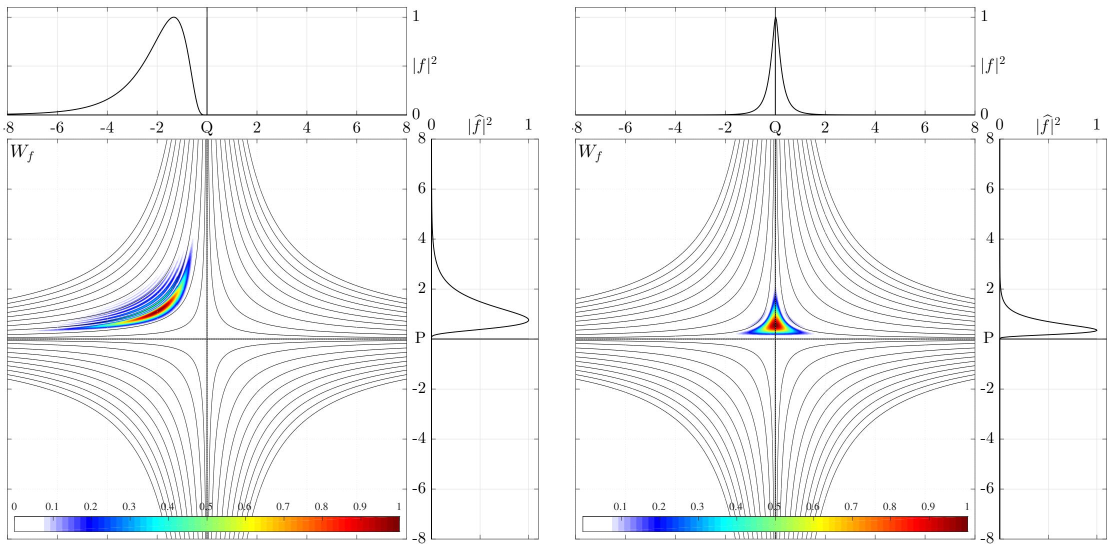
The spectral cotangent lift is defined by

$$\Sigma_{\log}(x,\xi) := \left(\tfrac{d\log(\xi)}{\mathrm{d}\xi}^{-1} \cdot x, \log(\xi)\right) = (\xi \cdot x, \log(\xi)), \tag{5.40}$$

and the associated Hamiltonians are

$$A(x,\xi) := \xi \cdot x \quad \text{and} \quad B'(x,\xi) := \log(\xi),\ B(x,\xi) := \xi, \tag{5.41}$$

where A induces dilation and B', respectively B, denote the Hamiltonians inducing the warped and standard translations, respectively. In *Figure 5.3a*, a phase space decomposition by *rectangles* of size $\sim \frac{1}{4}$, induced by the logarithmic spectral diffeomorphism is depicted.



(a) Translated Warped Gaussian **(b)** Centered Warped Gaussian

Figure 5.5: Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma(x) = \log_2(x)$.

To each of the rectangles could in principle correspond an elementary phase space cell of a single "quantum state". *Figure 5.3b* by contrast, depicts another phase space tessellation, this time by curved phase space cells, again of size $\sim \frac{1}{4}$, optimally adapted to the cotangent lift of the logarithm and thus constitutes a warped Gabor frame decomposition. In that regard, *Figure 5.5a* and *Figure 5.5b* show the Wigner distribution of two associated logarithmically-warped Gaussians in phase space.

By using the identity $\mathrm{d}\log := \frac{\mathrm{d}x}{x}$ we find that the associated reservoir of admissible windows is

$$\mathcal{A}_{\log} := \left\{ f \in L^2(\mathbb{R}) \,\middle|\, \widehat{f} \in L^2(\mathrm{dom}(\log), \tfrac{\mathrm{d}x}{x}) \right\}, \tag{5.42}$$

from which we recover the well-known admissibility condition

$$\psi \text{ is admissible} \iff \int_{\mathrm{dom}(\log)} \left|\widehat{\psi}(\xi)\right|^2 \frac{\mathrm{d}\xi}{\xi} \iff \psi \in \mathcal{A}_{\log}. \tag{5.43}$$

The induced spectral dilation operators are defined as

$$(\widehat{\mathcal{D}}_\alpha^{\log} \widehat{f})(\xi) := \sqrt{\frac{\mathrm{d}\log(\xi)}{\xi}}\, \widehat{f}(\exp(\log(\xi) + \alpha)) = e^{\alpha/2} \widehat{f}(e^\alpha \xi), \tag{5.44}$$

for $f \in \mathcal{S}_{\log}$ and

$$(\widehat{\widehat{\mathcal{D}}}_\alpha^{\log} \widehat{\psi})(\xi) := \widehat{\psi}(\exp(\log(\xi) + \alpha)) = \widehat{\psi}(e^\alpha \xi), \tag{5.45}$$

for $\psi \in \mathcal{A}_{\log}$. Writing $\mathcal{T}_b^{\log}$ and T_b for the warped and the standard translation operators, respectively, we end up with the spectral quantum frames

$$\mathbb{F}_{\log} := \left\{ T_\beta \widetilde{\mathcal{D}}_\alpha^{\log} \psi \mid \beta, \alpha \in \mathbb{R} \right\}, \quad \psi \in \mathcal{A}_{\log}, \tag{5.46}$$

and

$$\mathbb{G}_{\log} := \left\{ \mathcal{T}_\beta^{\log} \widetilde{\mathcal{D}}_\alpha^{\log} \psi \mid \beta, \alpha \in \mathbb{R} \right\}, \quad \psi \in \mathcal{A}_{\log}, \tag{5.47}$$

where the latter is canonical and essentially a logarithmically warped Gabor frame.

Definition 5.5 (Wavelet transform)**.** Let $f \in \mathcal{S}_{\log}$, $\psi \in \mathcal{A}_{\log}$ and $\widetilde{\pi}^{\log}(\beta,\alpha) = T_\beta \widetilde{\mathcal{D}}_\alpha^{\log}$, then

$$\begin{aligned} f \ \mapsto \ (\pi_\psi^{\log} f)(\beta,\alpha) := \left\langle T_\beta \widetilde{\mathcal{D}}_\alpha^{\log} \psi \,,\, f \right\rangle &= \int_{\mathrm{dom}\,(\log)} \overline{e^{-2\pi i \xi \beta} \hat{\psi}\left(e^\alpha \xi\right)} \ \hat{f}(\xi) \ \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \overline{e^{-\alpha} \psi\left(e^{-\alpha}(x-\beta)\right)} \ f(x) \ \mathrm{d}x \end{aligned} \tag{5.48}$$

is the *wavelet transform.* ◆

If the form of the wavelet need not necessarily be invariant under spatial translations, the following generalization of the wavelet transform to a logarithmically warped Short-Time Fourier transform, makes sense. Of course, giving up the "covariance" of the wavelet with respect to spatial translations means giving up on the interpretation of the wavelet transform as a *mathematical zoom* for analyzing, e.g., fractal structures.

Definition 5.6 (Logarithmic Short-Time Fourier Transform)**.** Let $f \in \mathcal{A}_{\log}$ and $\psi \in \mathcal{A}_{\log}$, then

$$\begin{aligned} f \ \mapsto \ \left\langle \mathcal{T}_\beta^{\log} \widetilde{\mathcal{D}}_\alpha^{\log} \psi \,,\, f \right\rangle &= \int_{\mathrm{dom}(\log)} \overline{e^{-2\pi i \log \xi \beta} \hat{\psi}\left(e^\alpha \xi\right)} \ \hat{f}(\xi) \mathrm{d}\log(\xi) \\ &= \int_{\mathbb{R}} \overline{e^{2\pi i a(x-\beta)} \widetilde{\psi}\left(x-\beta\right)} \ \widetilde{f}(x) \ \mathrm{d}x, \end{aligned} \tag{5.49}$$

with $\widetilde{\psi} := \mathcal{W}_{\log}\psi$ and $\widetilde{f} := \mathcal{W}_{\log} f$, is the *logarithmically warped Short-Time Fourier Transform.* ◆

Localization

For the logarithm, respectively the wavelet transform, the notions of localization do indeed differ.

The equalizing waveform, on $\mathcal{S}_{\log}$, for the principle of optimal alignment, is

$$\mathcal{S}_{\log} \ni \widehat{f}(\xi) = \left(\tfrac{\mathrm{d}\log}{\mathrm{d}x}\right)^{1/2} e^{-2\pi \int \xi \ \mathrm{d}\log(\xi)} e^{2\pi(\mu\beta - i\alpha)\log(\xi)} \tag{5.50}$$

And, for $\alpha = 0$, $\beta \cdot \mu = C = 1$,

$$\widehat{f}(\xi) = e^{-2\pi \int 1 \ \mathrm{d}x} |\xi|^{-1/2} e^{2\pi(1)\log(\xi)} \tag{5.51}$$
$$= e^{-2\pi\xi}\xi^{2\pi} |\xi|^{-1/2}, \tag{5.52}$$

we recover the equalizing waveform, already encountered twice in this thesis. Its admissible counterpart is

$$\mathcal{A}_{\log} \ni \widehat{\psi}(\xi) = e^{-2\pi\xi}\xi^{2\pi}, \tag{5.53}$$

with the same constants.

The equalizing waveform, on $\mathcal{S}_{\log}$, for the principle of optimal concentration, is

$$\begin{aligned} \mathcal{S}_{\log} \ni \widehat{f}(\xi) &= Ce^{-2\pi \int \log(\xi) \ \mathrm{d}\xi} e^{2\pi \langle \, \xi \, , \, \mu\alpha - i\beta \, \rangle} \\ &= Ce^{-2\pi(\xi\log(\xi)-\xi)} e^{2\pi \langle \, \xi \, , \, \mu\alpha - i\beta \, \rangle} \end{aligned} \tag{5.54}$$

and its admissible counterpart reads

$$\mathcal{A}_{\log} \ni \widehat{\psi}(\xi) = C\xi^{+1/2} e^{-2\pi(\xi\log(\xi)-\xi)} e^{2\pi \langle \, \xi \, , \, \mu\alpha - i\beta \, \rangle}. \tag{5.55}$$

◆

Other Diffeomorphisms on the Half-Line

Speaking of audio signals, which are usually intended to be perceived by the ears of human beings or other species in the *biological kingdom Animalia*, the emphasis on the (positive) frequency spectrum with all frequency bands being considered *equally important* might not be the right thing to do. This idea is, for example, taken for the design of the wavelet transform, which we already discussed above, where the frequencies are consolidated into sub-bands with logarithmically increasing size. More application-oriented transformations are taken by the ideas of the constant-Q scale, the Bark scale and the ERB scale, which will be discussed next.

Example 5.3 (Constant-Q scale)**.** The CQ scale associates the origin in the CQ domain to some predefined frequency ξ_0 and integer steps in the CQ domain correspond to doubling the frequency every B-th step. That is, if σ_{CQ} is the diffeomorphism, the point $k \in \mathbb{Z} \subset \mathbb{R} = \mathrm{im}(\sigma_{CQ})$ corresponds to $2^{k/B}\xi_0 \in \mathbb{R}_+ = \mathrm{dom}(\sigma_{CQ})$. The explicit formula for the CQ diffeomorphisms are

$$\sigma_{CQ}(\xi) := \log_2(\xi/\xi_0) \cdot B \quad \text{and} \quad \sigma_{CQ}^{-1}(z) := 2^{z/B} \cdot \xi_0 \tag{5.56}$$

and the only real difference between the CQ scale and the classical log scale is that the corresponding axes are stretched by factors B and ξ_0, respectively.

To define the associated transform, only the spectral dilation operator on $\mathcal{A}_{\sigma_{CQ}}$ shall be given, the rest should be clear by now. Letting the dependence on B and ξ_0 implicit, we have

$$(\widetilde{\mathcal{D}}_{\alpha}^{\sigma_{CQ}}\psi)(x) \coloneqq 2^{-\alpha/B}\psi(2^{-\alpha/B}x), \tag{5.57}$$

respectively

$$(\widehat{\widetilde{\mathcal{D}}}_{\alpha}^{\sigma_{CQ}}\widehat{\psi})(\xi) \coloneqq \widehat{\psi}(2^{\alpha/B}\xi), \tag{5.58}$$

which gives us the following definition

Definition 5.7 (Constant-Q transform)**.** Let $f \in \mathcal{S}_{\sigma_{CQ}}$, $\psi \in \mathcal{A}_{\sigma_{CQ}}$ and $\pi^{\sigma_{CQ}}(\beta,\alpha) = T_\beta\widetilde{\mathcal{D}}_{\alpha}^{\sigma_{CQ}}$, then

$$\begin{aligned} f \;\mapsto\; (\pi_{\psi}^{\sigma_{CQ}} f)(\beta,\alpha) \coloneqq \left\langle T_\beta\widetilde{\mathcal{D}}_{\alpha}^{\sigma_{CQ}}\psi\,,\, f\right\rangle &= \int_{\operatorname{dom}\sigma_{CQ}} \overline{e^{-2\pi i\xi\beta}\hat{\psi}\left(2^{\alpha/B}\xi\right)}\;\hat{f}(\xi)\;\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \overline{2^{-\alpha/B}\psi\left(2^{-\alpha/B}(x-\beta)\right)}\;f(x)\;\mathrm{d}x \end{aligned} \tag{5.59}$$

is the *Constant-Q transform.* ◆

Definition 5.8 (CQ-warped Short-Time Fourier Transform)**.** Let $f \in \mathcal{A}_{\sigma_{CQ}}$ and $\psi \in \mathcal{A}_{\sigma_{CQ}}$, then

$$\begin{aligned} f \;\mapsto\; \left\langle \mathcal{T}_{\beta}^{\sigma_{CQ}}\widetilde{\mathcal{D}}_{\alpha}^{\sigma_{CQ}}\psi\,,\, f\right\rangle &= \int_{\operatorname{dom}\sigma_{CQ}} \overline{e^{-2\pi i\log_2(\xi/\xi_0)B\beta}\hat{\psi}\left(2^{\alpha/B}\xi\right)}\;\hat{f}(\xi)\mathrm{d}\sigma_{CQ}(\xi) \\ &= \int_{\mathbb{R}} \overline{e^{2\pi i a(x-\beta)}\widetilde{\psi}\left(x-\beta\right)}\;\widetilde{f}(x)\;\mathrm{d}x, \end{aligned} \tag{5.60}$$

with $\widetilde{\psi} \coloneqq \mathcal{W}_{\sigma_{CQ}}\psi$ and $\widetilde{f} \coloneqq \mathcal{W}_{\sigma_{CQ}}f$, is the *CQ-warped Short-Time Fourier Transform.* ◆

◆

Example 5.4 (BARK scale)**.** The Bark scale [82] is an *application-oriented* frequency scale, on which equal distances correspond to equal perceptual distances in the human ear, without stretching the positive axis to the whole reals. That is, there exists a mapping

$$\sigma_{bark} : \mathbb{R}_+ \to \mathbb{R}_+, \quad \xi \;\mapsto\; \sigma_{bark}(\xi) \tag{5.61}$$

which deforms the frequency axis in such a way that equidistant steps on the deformed axis correspond to equidistant frequency steps to the human ear. Of course, this "scale" was defined empirically, so one cannot expect that an absolutely

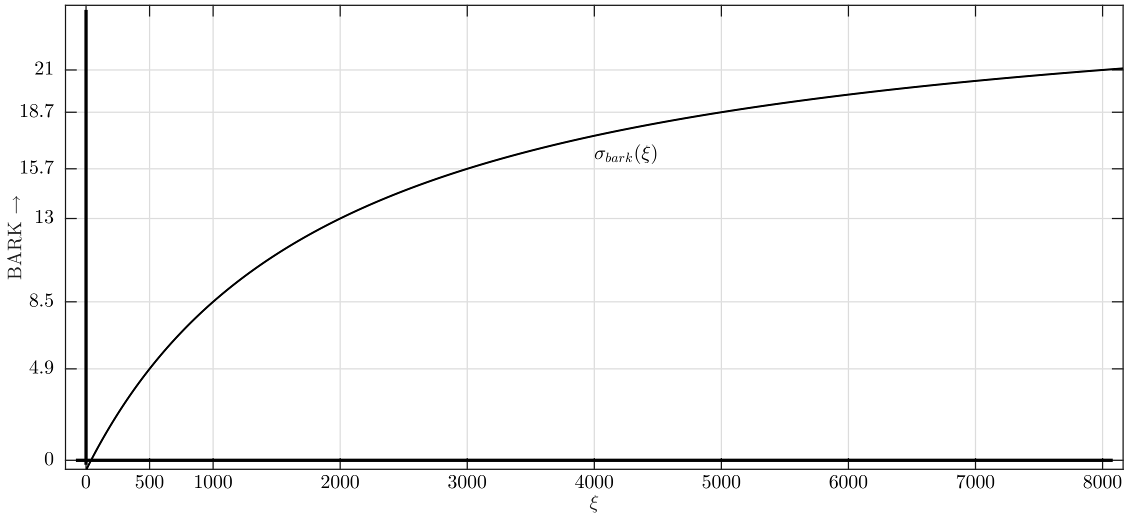


Figure 5.6: BARKscale $\sigma_{bark}(\xi)$ vs. frequency ξ

valid mathematical formula exists, which models this connection, even less when we restrict ourselves to diffeomorphic mappings. For the definition of this morphism, there are more proposals than one. Two of them, proposed in [82], are

$$\tilde{\sigma}_{bark}(\xi) = 13\,\mathrm{atan}\left(7.6 \cdot 10^{-4} \cdot \xi\right) + 3.5\,\mathrm{atan}\left((\xi/7500)^2\right), \tag{5.62}$$

and

$$\sigma_{bark}(\xi) = \frac{26.81\xi}{1960+\xi} - 0.53, \tag{5.63}$$

where the latter one has a simple inverse, given by

$$\sigma_{bark}^{-1}(z) = 1960\frac{0.53+z}{26.28-z} \tag{5.64}$$

and is plotted in *Figure 5.6*.

This is the first time we have a diffeomorphism for which the associated dilation operator does not look very tidy, since the diffeomorphism is no group-homomorphism. This, however, does no harm, except for a more filthy looking formula

$$(\widehat{\widetilde{\mathcal{D}}}_{\alpha}^{\sigma_{CQ}}\widehat{\psi})(\xi) := \widehat{\psi}\left(1960\frac{0.53 + ((\frac{26.81\xi}{1960+\xi} - 0.53) + \alpha)}{26.28 - ((\frac{26.81\xi}{1960+\xi} - 0.53) + \alpha)}\right),$$

which gives us the following definition.

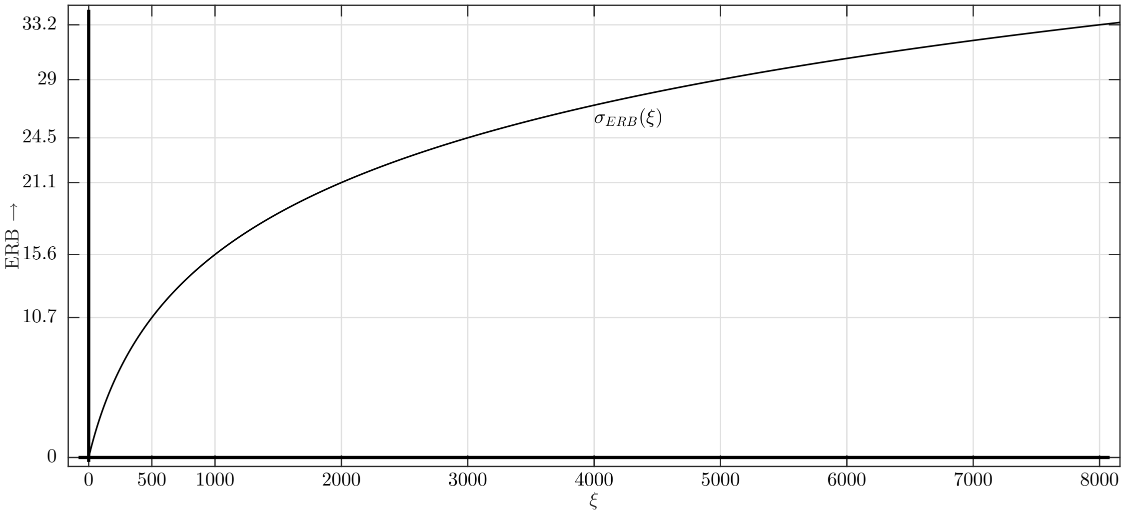


Figure 5.7: ERBscale $\sigma_{ERB}(\xi)$ vs. frequency ξ

Definition 5.9 (Bark transform)**.** Let $f \in \mathcal{S}_{\sigma_{bark}}$, $\psi \in \mathcal{A}_{\sigma_{bark}}$ and $\pi^{\sigma_{bark}}(\beta, \alpha) = T_\beta \widetilde{\mathcal{D}}_\alpha^{\sigma_{bark}}$, then

$$\begin{aligned} f \mapsto & (\pi_\psi^{\sigma_{bark}} f)(\beta, \alpha) \\ & := \left\langle T_\beta \widetilde{\mathcal{D}}_\alpha^{\sigma_{bark}} \psi \, , \, f \right\rangle \\ & = \int_{\operatorname{dom} \sigma_{bark}} \overline{e^{-2\pi i \xi \beta} \widehat{\psi} \left(1960 \frac{0.53 + ((\frac{26.81\xi}{1960+\xi} - 0.53) + \alpha)}{26.28 - ((\frac{26.81\xi}{1960+\xi} - 0.53) + \alpha)} \right)} \hat{f}(\xi) \, \mathrm{d}\xi \end{aligned} \tag{5.65}$$

is the *Bark transform.* ◆

Definition 5.10 (Bark-warped Short-Time Fourier Transform)**.** Let $f \in \mathcal{A}_{\sigma_{bark}}$ and $\psi \in \mathcal{A}_{\sigma_{bark}}$, then

$$\begin{aligned} f \mapsto & \left\langle \mathcal{T}_\beta^{\sigma_{bark}} \widetilde{\mathcal{D}}_\alpha^{\sigma_{bark}} \psi \, , \, f \right\rangle \\ & = \int_{\operatorname{dom} \sigma_{bark}} \overline{e^{-2\pi i \sigma_{bark}(\xi) \beta} \hat{\psi} \left(\sigma_{bark}^{-1} (\sigma_{bark}(\xi) + \alpha) \right)} \hat{f}(\xi) \, \mathrm{d}\sigma_{bark}(\xi) \\ & = \int_{\mathbb{R}} \overline{e^{2\pi i a (x - \beta)} \widetilde{\psi} \left(x - \beta \right)} \widetilde{f}(x) \, \mathrm{d}x, \end{aligned} \tag{5.66}$$

with $\widetilde{\psi} := \mathcal{W}_{\sigma_{bark}} \psi$ and $\widetilde{f} := \mathcal{W}_{\sigma_{bark}} f$, is the *bark-warped Short-Time Fourier Transform.* ◆

◆

Example 5.5 (ERB scale)**.** The ERB scale [36, 66], for *Equivalent Rectangular Bandwidth*, with diffeomorphism

$$\sigma_{ERB}(\xi) = \frac{1000}{24.7 \cdot 4.37} \ln \left(1 + \frac{4.37 \cdot \xi}{1000} \right) \tag{5.67}$$

$$\approx 9.265 \cdot \ln \left(1 + 4.37 \cdot 10^{-3} \cdot \xi \right), \tag{5.68}$$

and inverse

$$\sigma_{ERB}^{-1}(z) = \frac{e^{z/9.265} - 1}{4.37 \cdot 10^{-3}} \tag{5.69}$$

is another such scale, emphasizing again those parts of the frequency spectrum that are most relevant for the human ear. Or, more specifically, the human ear has a frequency filter bank, consisting of different band-passes, which are - although not necessarily physical correct - modeled to be rectangular band-passes.

In [67] the authors introduced the so-called “ERBlet”-transform, which is a discrete transform, applying these ideas in linear audio processing. Although it is a discrete transform, by using the diffeomorphism, σ_{ERB}, which deforms the frequency axis such that the interesting features of the spectrum of f are distributed in equally spaced distances, it seems possible to design a continuous analogue of this transform, such that sampling this transform in equidistant steps leads to the classical ERB transform.

Again, no nice formula can be given, so we will, if only to avoid clutter, refrain from inserting the explicit expressions of the diffeomorphism into the transform below and shall be content with the following expression

$$(\widehat{\widetilde{\mathcal{D}}_{\alpha}^{\sigma_{ERB}}\psi})(\xi) \tag{5.70}$$

$$:= \widehat{\psi}\left(4.37^{-1} \cdot 10^{3}\left(\exp\left(9.265^{-1}\left(\tfrac{1000}{24.7 \cdot 4.37}\ln\left(1 + \tfrac{4.37 \cdot \xi}{1000}\right) + \alpha\right)\right) - 1\right)\right). \tag{5.71}$$

Definition 5.11 (ERB transform)**.** Let $f \in \mathcal{S}_{\sigma_{ERB}}$, $\psi \in \mathcal{A}_{\sigma_{ERB}}$ and $\pi^{\sigma_{ERB}}(\beta,\alpha) = T_\beta \widetilde{\mathcal{D}}_{\alpha}^{\sigma_{ERB}}$, then

$$\begin{aligned} f \mapsto (\pi_{\psi}^{\sigma_{ERB}} f)(\beta,\alpha) &:= \left\langle T_\beta \widetilde{\mathcal{D}}_{\alpha}^{\sigma_{ERB}}\psi \,,\, f \right\rangle \\ &= \int_{\mathrm{dom}\,(\sigma_{ERB})} \overline{e^{-2\pi i \xi\beta}\widehat{\psi}\left(\sigma_{ERB}^{-1}(\sigma_{ERB}(\xi) + \alpha)\right)}\, \hat{f}(\xi)\, \mathrm{d}\xi \end{aligned} \tag{5.72}$$

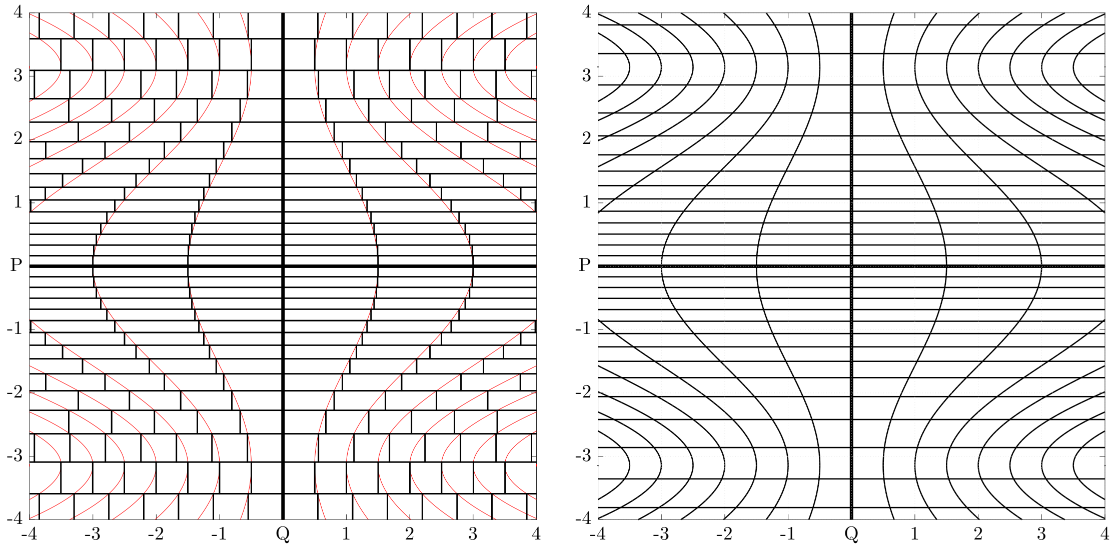
is the *continuous ERB transform.* ◆

Definition 5.12 (ERB-warped Short-Time Fourier Transform)**.** Let $f \in \mathcal{A}_{\sigma_{ERB}}$ and $\psi \in \mathcal{A}_{\sigma_{ERB}}$, then

$$\begin{aligned} f \mapsto &\left\langle \mathcal{T}_{\beta}^{\sigma_{ERB}} \widetilde{\mathcal{D}}_{\alpha}^{\sigma_{ERB}}\psi \,,\, f \right\rangle \\ &= \int_{\mathrm{dom}\,(\sigma_{ERB})} \overline{e^{-2\pi i \sigma_{ERB}(\xi)\beta}\hat{\psi}\left(\sigma_{ERB}^{-1}(\sigma_{ERB}(\xi) + \alpha)\right)}\, \hat{f}(\xi)\, \mathrm{d}\sigma_{ERB}(\xi) \\ &= \int_{\mathbb{R}} \overline{e^{2\pi i a(x-\beta)}\widetilde{\psi}\,(x - \beta)}\, \widetilde{f}(x)\, \mathrm{d}x, \end{aligned} \tag{5.73}$$

with $\widetilde{\psi} := \mathcal{W}_{\sigma_{ERB}}\psi$ and $\widetilde{f} := \mathcal{W}_{\sigma_{ERB}} f$, is the *ERB-warped Short-Time Fourier Transform.* ◆

◆



(a) Rectangular phase space tessellation **(b)** Curved phase space tessellation

Figure 5.8: Phase space tessellation, associated with $\sigma_1(x) = 2 \cdot x + \sin(x)$.

Localization for *CQ*, *ERB* and *BARK*

Again, for all of the three cases above, the notions of localization do differ. We will assemble all three cases and write σ.

The equalizing waveforms, on $\mathcal{S}_\sigma$, for the principle of optimal alignment, are

$$\mathcal{S}_\sigma \ni \widehat{f}(\xi) = \left|\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right|^{1/2} e^{-2\pi \int \xi \ \mathrm{d}\sigma(\xi)} e^{2\pi(\mu\beta - i\alpha)\sigma(\xi)} \tag{5.74}$$

and their admissible counterparts are

$$\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) = e^{-2\pi \int \xi \ \mathrm{d}\sigma(\xi)} e^{2\pi(\mu\beta - i\alpha)\sigma(\xi)} \tag{5.75}$$

with the same constants.

The equalizing waveform, on $\mathcal{S}_\sigma$, for the principle of optimal concentration, is

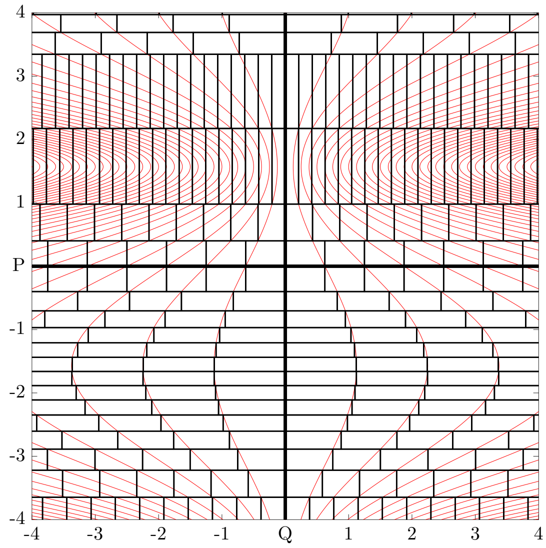
$$\mathcal{S}_\sigma \ni \widehat{f}(\xi) = C e^{-2\pi \int \sigma(\xi) \ \mathrm{d}\xi} e^{2\pi\xi(\mu\alpha - i\beta)} \tag{5.76}$$

and its admissible counterpart is

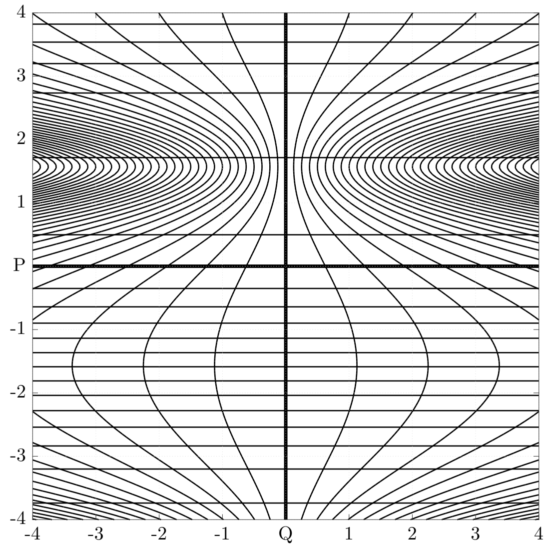
$$\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) = C \left|\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right|^{-1/2} e^{-2\pi \int \sigma(\xi) \ \mathrm{d}\xi} e^{2\pi\xi(\mu\alpha - i\beta)} \tag{5.77}$$

5.2.3 A Miscellaneous Example on the full line

Example 5.6 (A Miscellaneous Example)**.** Assume we are interested in all the harmonic frequency-bands which are multiples of a given ground frequency and



(a) Rectangular phase space tessellation



(b) Curved phase space tessellation

Figure 5.9: Phase space tessellation, associated with $\sigma_2(x) = 1.25 \cdot x + \cos(x)$.

thus we wish to give more credit to those bands than for others. Then, a spectral diffeomorphism, spreading those frequencies we are interested in and squeezing the uninteresting ones, similar to

$$\sigma_1(\xi) := 2\xi + \sin(\xi) \quad \text{or} \quad \sigma_2(\xi) := 1.25\xi + \cos(\xi) \tag{5.78}$$

does the trick. The one-dimensional Jacobians, that is, the derivatives, and its inverses are

$$\tfrac{\mathrm{d}\sigma_1}{\mathrm{d}\xi} := 2 + \cos(\xi) \quad \text{and} \quad \left(\tfrac{\mathrm{d}\sigma_1}{\mathrm{d}\xi}\right)^{-1} := \tfrac{1}{2+\cos(\xi)} \tag{5.79}$$

and

$$\tfrac{\mathrm{d}\sigma_2}{\mathrm{d}\xi} := 1.25 - \sin(\xi) \quad \text{and} \quad \left(\tfrac{\mathrm{d}\sigma_2}{\mathrm{d}\xi}\right)^{-1} := \tfrac{1}{1.25-\sin(\xi)} \tag{5.80}$$

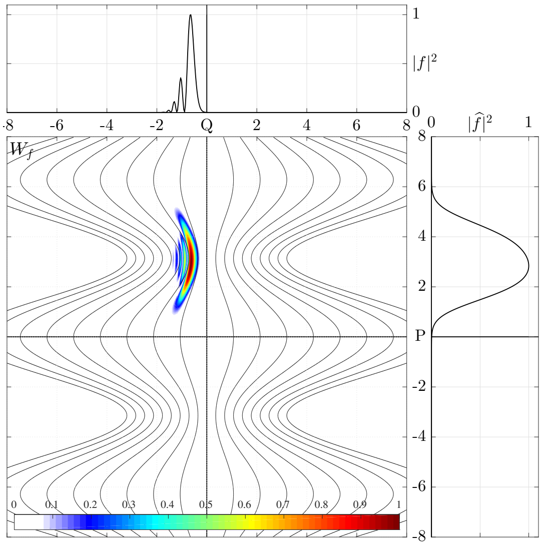
and the corresponding Hamiltonian vector fields are

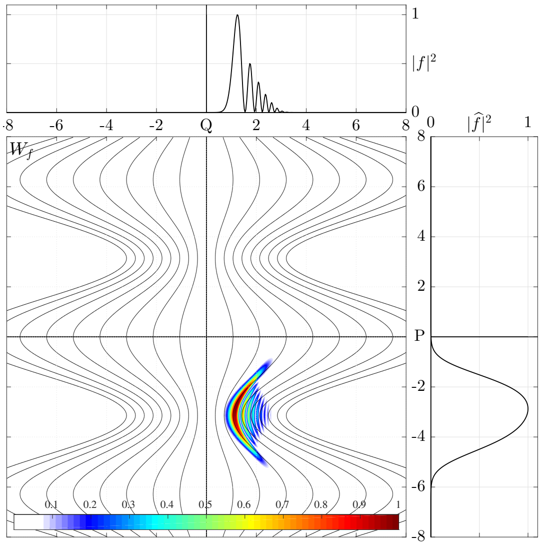
$$X_1(q,p) := -\frac{1}{2+\cos(p)}\partial_p + \frac{q \cdot \sin(p)}{(2+\cos(p))^2}\partial_q \tag{5.81}$$

and

$$X_2(q,p) := -\frac{1}{1.25-\sin(p)}\partial_p + \frac{q \cdot \cos(p)}{(1.25-\sin(p))^2}\partial_q. \tag{5.82}$$

Figure 5.8a depicts the associated phase space tessellation of σ_1 into rectangular boxes and *Figure 5.8b* shows the same with curved phase space cells, as associated with the warped STFT. *Figure 5.9a* depicts the associated phase space tessellation





(a) Translated Warped Gaussian (b) Translated Warped Gaussian

Figure 5.10: Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma_1(x) = 2 \cdot x + \sin(x)$.

of σ_2 into rectangular boxes and *Figure 5.9b* shows the same with curved phase space cells. All tessellations are into boxes of size $\sim \frac{1}{4}$.

Figure 5.10a and *Figure 5.10b* show the Wigner distribution of two associated σ_1-warped Gaussians in phase space and *Figure 5.11a* and *Figure 5.11b* show the same for σ_2-warped Gaussians in phase space.

Although this should have become clear by now, for the sake of thoroughness, in the following we will give the associated transforms and localization properties of $\sigma_1 =: \sigma$.

Definition 5.13 (σ Transform)**.** Let $f \in \mathcal{S}_\sigma$, $\psi \in \mathcal{A}_\sigma$ and $\pi^\sigma(\beta,\alpha) = T_\beta \widetilde{\mathcal{D}}^\sigma_\alpha$, then

$$\begin{aligned} f \mapsto (\pi^\sigma_\psi f)(\beta,\alpha) &:= \left\langle T_\beta \widetilde{\mathcal{D}}^\sigma_\alpha \psi \,,\, f \right\rangle \\ &= \int_{\operatorname{dom}\sigma} \overline{e^{-2\pi i \xi \beta} \widehat{\psi}\left(\sigma^{-1}(\sigma(\xi)+\alpha)\right)}\ \hat{f}(\xi)\ \mathrm{d}\xi \end{aligned} \tag{5.83}$$

is the σ *transform.* ◆

Definition 5.14 (σ-warped Short-Time Fourier Transform)**.** Let $f \in \mathcal{A}_\sigma$ and $\psi \in \mathcal{A}_\sigma$, then

$$\begin{aligned} f \mapsto & \left\langle \mathcal{T}^\sigma_\beta \widetilde{\mathcal{D}}^\sigma_\alpha \psi \,,\, f \right\rangle \\ &= \int_{\operatorname{dom}\sigma} \overline{e^{-2\pi i \sigma(\xi)\beta} \hat{\psi}\left(\sigma^{-1}(\sigma(\xi)+\alpha)\right)}\ \hat{f}(\xi)\ \mathrm{d}\sigma(\xi) \\ &= \int_{\mathbb{R}} \overline{e^{2\pi i a(x-\beta)} \widetilde{\psi}\,(x-\beta)}\ \widetilde{f}(x)\ \mathrm{d}x, \end{aligned} \tag{5.84}$$

with $\widetilde{\psi} := \mathcal{W}_\sigma \psi$ and $\widetilde{f} := \mathcal{W}_\sigma f$, is the *$\sigma$-warped Short-Time Fourier Transform.* ◆

Localization

The equalizing waveforms on $\mathcal{S}_\sigma$, for the principle of optimal alignment, are

$$\begin{aligned}\mathcal{S}_\sigma \ni \widehat{f}(\xi) &= C\left|\tfrac{\mathrm{d}\sigma}{\mathrm{d}x}\right|^{1/2} e^{-2\pi\mu\int\xi\;\mathrm{d}\sigma(\xi)}e^{2\pi\sigma(\xi)(\mu\alpha-i\beta)} \\ &= \sqrt{2+\cos(\xi)}Ce^{-2\pi\mu(\xi^2+\xi\sin(\xi)+\cos(\xi)-1)}e^{2\pi(2x+\sin(x))(\mu\alpha-i\beta)}\end{aligned} \tag{5.85}$$

and its admissible counterpart is

$$\begin{aligned}\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) &= Ce^{-2\pi\mu\int\xi\;\mathrm{d}\sigma(\xi)}e^{2\pi\sigma(\xi)(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu(\xi^2+\xi\sin(\xi)+\cos(\xi)-1)}e^{2\pi(2x+\sin(x))(\mu\alpha-i\beta)}.\end{aligned} \tag{5.86}$$

The equalizing waveform, on $\mathcal{S}_\sigma$, for the principle of optimal concentration, is

$$\begin{aligned}\mathcal{S}_\sigma \ni \widehat{f}(\xi) &= Ce^{-2\pi\mu\int\sigma(\xi)\;\mathrm{d}\xi}e^{2\pi\xi(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu(\xi^2+1-\cos(\xi))}e^{2\pi\xi(\mu\alpha-i\beta)}\end{aligned} \tag{5.87}$$

and its admissible counterpart is

$$\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) = \frac{1}{\sqrt{2+\cos(\xi)}}Ce^{-2\pi\mu(\xi^2+1-\cos(\xi))}e^{2\pi\xi(\mu\alpha-i\beta)}. \tag{5.88}$$

◆

5.2.4 The Plane

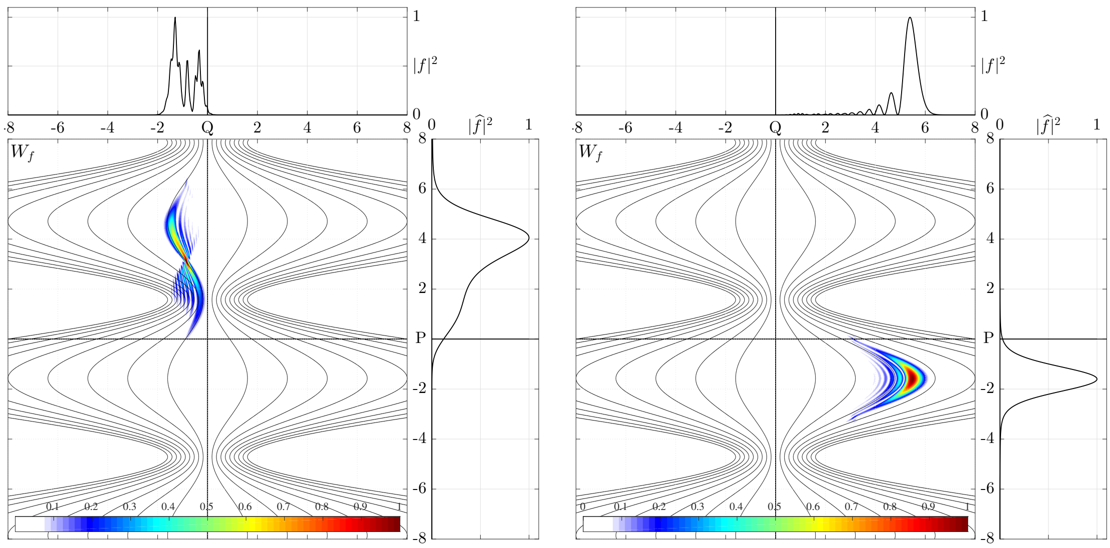
When dealing with more picturesque two-dimensional data, such as images, this idea may be applied as well.

Example 5.7 (Independent Logarithms)**.** The most direct application is the simple, two-dimensional wavelet transform, which - stubbornly following the exact same arguments as in the one-dimensional case - may not be readily found to be as suitable as above, but a slight adjustment will take care of this.

Although not very common, let nonetheless $f \in L^2(\mathbb{R}^2)$ be a signal with interesting features contained in the positive frequency quadrant

$$Q_{++} := \left\{\,(x,y) \in \widehat{\mathbb{R}}^2 \mid x > 0, y > 0\,\right\} \tag{5.89}$$

only. Then, two-dimensional log-warping, that is, warping in each direction independently, $\sigma := (\log,\log)$, takes the Q_{++} quadrant to $\mathbb{R}^2$. Completely analogue to the one-dimensional case, equidistant steps in the warping domain, $\mathrm{im}(\sigma) = \mathbb{R}^2$, correspond to exponentially scaled ones on $\mathrm{dom}(\sigma) = Q_{++}$.



(a) Translated Warped Gaussian **(b)** Translated Warped Gaussian

Figure 5.11: Wigner distributions of warped Gaussians in phase space, adapted to the coordinate system associated with $\sigma_2(x) = 1.25 \cdot x + \cos(x)$.

The Jacobian and its transposed inverse are given by

$$J_\sigma(\xi_1, \xi_2) = \begin{pmatrix} \frac{1}{\xi_1} & \\ & \frac{1}{\xi_2} \end{pmatrix} \implies \frac{-1}{2\pi i} J_\sigma(\xi_1, \xi_2) = \begin{pmatrix} \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_2\right] \\ \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_2\right] \end{pmatrix} \tag{5.90}$$

and

$$J_\sigma^{-T}(\xi_1, \xi_2) = \begin{pmatrix} \xi_1 & \\ & \xi_2 \end{pmatrix} \implies \frac{-1}{2\pi i} J_\sigma^{-T}(\xi_1, \xi_2) = \begin{pmatrix} \left[\widehat{A}^1, \widehat{B}_1\right] & \left[\widehat{A}^1, \widehat{B}_2\right] \\ \left[\widehat{A}^2, \widehat{B}_1\right] & \left[\widehat{A}^2, \widehat{B}_2\right] \end{pmatrix} \tag{5.91}$$

Thus, the spectral cotangent lift is

$$\Sigma(\xi; x) = (\sigma(\xi); J_\sigma^{-T}(\xi)x) \tag{5.92}$$

$$= (\log(\xi_1), \log(\xi_2)); \xi_1 x_1, \xi_2 x_2). \tag{5.93}$$

From this, we find that the Hamiltonians

$$A^i(\xi; x) := (J_\sigma^{-T}(\xi)x)^i \text{ and } B_j(\xi; x) := \sigma_j(\xi), \tag{5.94}$$

are

$$A^1 := \xi_1 x_1 \quad \text{and} \quad A^2 := \xi_2 x_2 \tag{5.95}$$

and

$$B_1 := \log(\xi_1) \quad \text{and} \quad B_2 := \log(\xi_2), \tag{5.96}$$

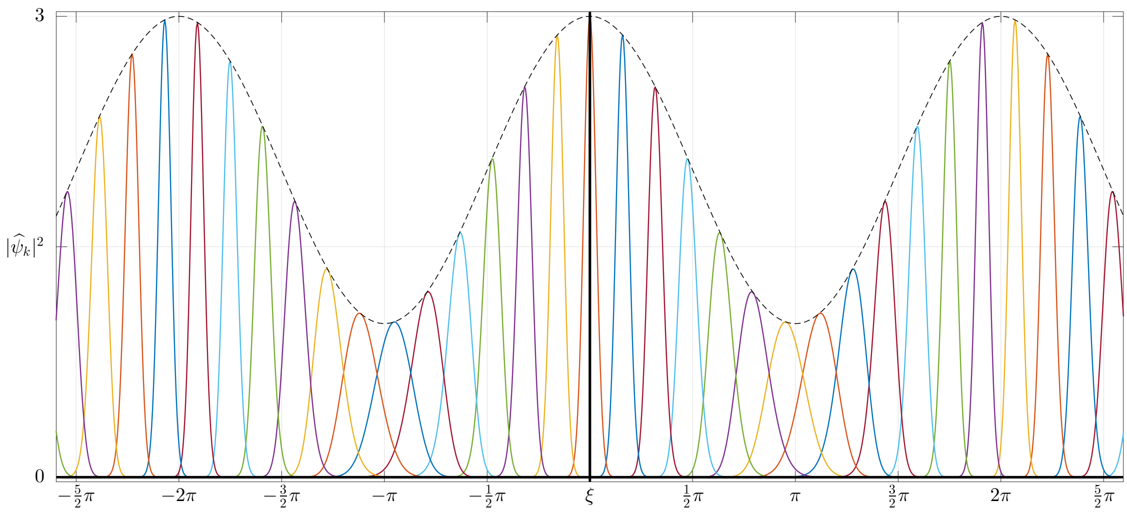


Figure 5.12: Superimposed spectra of warped Gaussian, $\psi_k := \sqrt{\frac{\mathrm{d}\sigma_1}{\mathrm{d}\xi}} e^{-\pi(\sigma_1(\xi)-\frac{k}{2})^2}$, spectral dilated with $\sigma_1(\xi) := 2\xi + \sin(\xi)$. The dotted line shows $\frac{\mathrm{d}\sigma_1}{\mathrm{d}\xi}$.

where the latter induce the warped translations and are thus only interesting for the warped Gabor case. The former two Hamiltonians lead to the Hamiltonian vector fields

$$X_{A^1} = -[\xi_1 \partial_{\xi_1}] + [x_1 \partial_{x_1}] \tag{5.97}$$

and

$$X_{A^2} = -[\xi_2 \partial_{\xi_2}] + [x_2 \partial_{x_2}]\,. \tag{5.98}$$

Since the extension from the one-dimensional case is immediate, we will jump to the definition.

Definition 5.15 (The two-dimensional Wavelet Transform)**.** Let $f \in \mathcal{S}_\sigma$ and $\psi \in \mathcal{A}_\sigma$ be in the appropriate domains. Then

$$\begin{aligned} f \;\mapsto\; & \left\langle T_{\vec{b}} \widetilde{\mathcal{D}}^{\log}_{\alpha_1} \widetilde{\mathcal{D}}^{\log}_{\alpha_2} \psi \,,\, f \right\rangle \\ := & \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \overline{\widehat{\psi}\left(e^{\alpha_1}(\xi_1 - b_1), e^{\alpha_2}(\xi_2 - b_2)\right)}\; \widehat{f}(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2, \\ = & \iint_{\mathbb{R}^2} \overline{e^{-(\alpha_1+\alpha_2)} \psi\left(e^{-\alpha_1}(x_1 - b_1), e^{-\alpha_2}(x_2 - b_2)\right)}\; f(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \end{aligned} \tag{5.99}$$

is the *two-dimensional wavelet transform* . ◆

Definition 5.16 (Logarithmic STFT in 2D)**.** Let $f \in \mathcal{A}_{\log}$ and $\psi \in \mathcal{A}_{\log}$, then

$$f \;\mapsto\; \left\langle \widetilde{T}^{\log}_{\vec{b}} \widetilde{\mathcal{D}}^{\log}_{\alpha_1} \widetilde{\mathcal{D}}^{\log}_{\alpha_2} \psi \,,\, f \right\rangle = \iint_{\mathbb{R}^2} \overline{e^{2\pi i \langle \vec{\alpha}\,,\, \vec{x}-\vec{b} \rangle} \widetilde{\psi}\left(\vec{x} - \vec{b}\right)}\; \widetilde{f}(x)\, \mathrm{d}x, \tag{5.100}$$

with $\widetilde{\psi} := \mathcal{W}_{\log}\psi$ and $\widetilde{f} := \mathcal{W}_{\log} f$, is the *logarithmic Short-Time Fourier Transform in 2D.* ◆

The extension to n dimensions is straight-forward without any rude awakenings and the same holds for its localization measures, which is the next topic.

Localization

Again, the notions of localization differ, but the Jacobian is diagonal, so we have perfect equalizers, given as two-dimensional tensor-products of the one-dimensional logarithmic case, namely the following.

The equalizing waveform, on $\mathcal{S}_{\log}$, for the principle of optimal alignment, is

$$\mathcal{S}_\sigma \ni \widehat{f}(\xi) = |\xi_1\xi_2|^{-1/2} (\xi_1\xi_2)^{2\pi} e^{-2\pi\xi_1\xi_2} e^{2\pi\langle \mu\beta - i\alpha \, , \, \sigma(\xi) \rangle} \tag{5.101}$$

and its admissible counterpart is

$$\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) = (\xi_1\xi_2)^{2\pi} e^{-2\pi\xi_1\xi_2} e^{2\pi\langle \mu\beta - i\alpha \, , \, \sigma(\xi) \rangle} \tag{5.102}$$

with the same constants.

The equalizing waveform, on $\mathcal{S}_\sigma$, for the principle of optimal concentration, is

$$\mathcal{S}_\sigma \ni \widehat{f}(\xi) = Ce^{-2\pi \sum_k (\xi_k \log(\xi_k) - \xi_k)} e^{2\pi\langle \xi \, , \, \mu\alpha - i\beta \rangle} \tag{5.103}$$

and its admissible counterpart is

$$\mathcal{A}_\sigma \ni \widehat{\psi}(\xi) = C \cdot |\xi_1\xi_2|^{+1/2} e^{-2\pi \sum_k (\xi_k \log(\xi_k) - \xi_k)} e^{2\pi\langle \xi \, , \, \mu\alpha - i\beta \rangle} \tag{5.104}$$

◆

When speaking of images and other two-dimensional data, there is no restriction on the positive quadrant for any relevant data, since the frequency data now encode the directions of, say, edges in an image which are definitely not restricted to certain directions. Decomposing an arbitrary $f \in L^2(\mathbb{R}^2)$ into four components, each supported on one of the four frequency quadrants only, that is,

$$f \mapsto (f_1, f_2, f_3, f_4), \quad \text{with} \ \ \widehat{f_k} \in L^2(Q_k), \tag{5.105}$$

we can still apply the same steps as above, which we will leave alone, as it results in four identical copies of the one above, with each having an extra index for its quadrant.

Unfortunately, the viability with respect to anisotropic features like sharp edges in varying directions is highly questionable and thus the utility of a more "anisotropic" frequency domain is desirable. Without a doubt, one of the most relevant coordinates for the incorporation of directions are the *polar coordinates*, which is our next concern.

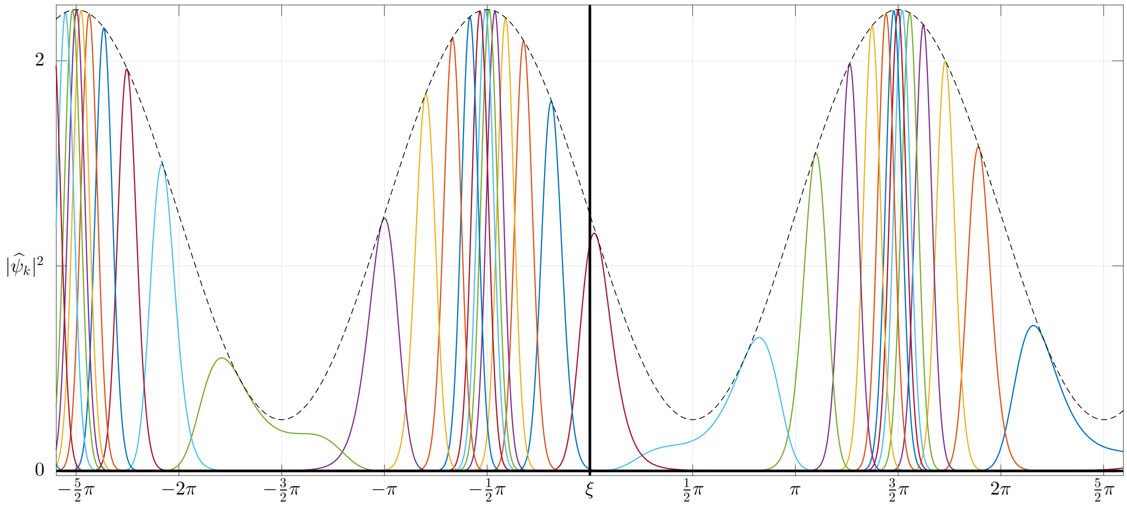


Figure 5.13: Superimposed spectra of warped Gaussians, $\psi_k := \sqrt{\frac{\mathrm{d}\sigma_2}{\mathrm{d}\xi}} e^{-\pi(\sigma_2(\xi)-\frac{k}{2})^2}$, spectral dilated with $\sigma_2(\xi) := 1.25\xi + \cos(\xi)$. The dotted line shows $\frac{\mathrm{d}\sigma_2}{\mathrm{d}\xi}$.

The Log-Polar Diffeomorphism

Let $f \in L^2(\mathbb{R}^2)$, having real values only. It then suffices to describe the signal via half of the Fourier domain, since the values are point-symmetric to the origin.

Example 5.8 (Polar Coordinates)**.** Let f have significant anisotropic features, like "edges" in a picture, that need to be characterized along with their occurring spatial positions. Say, we want to find the position and "directions" of edges. To this end, we deform the Fourier plane via the diffeomorphism

$$\sigma_{pol} : \begin{cases} (\mathbb{R}\backslash\{0\}) \times \mathbb{R} & \to \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ (x, y) & \mapsto \left(\log(\sqrt{x^2 + y^2}), \operatorname{atan}(y/x)\right) \end{cases} \tag{5.106}$$

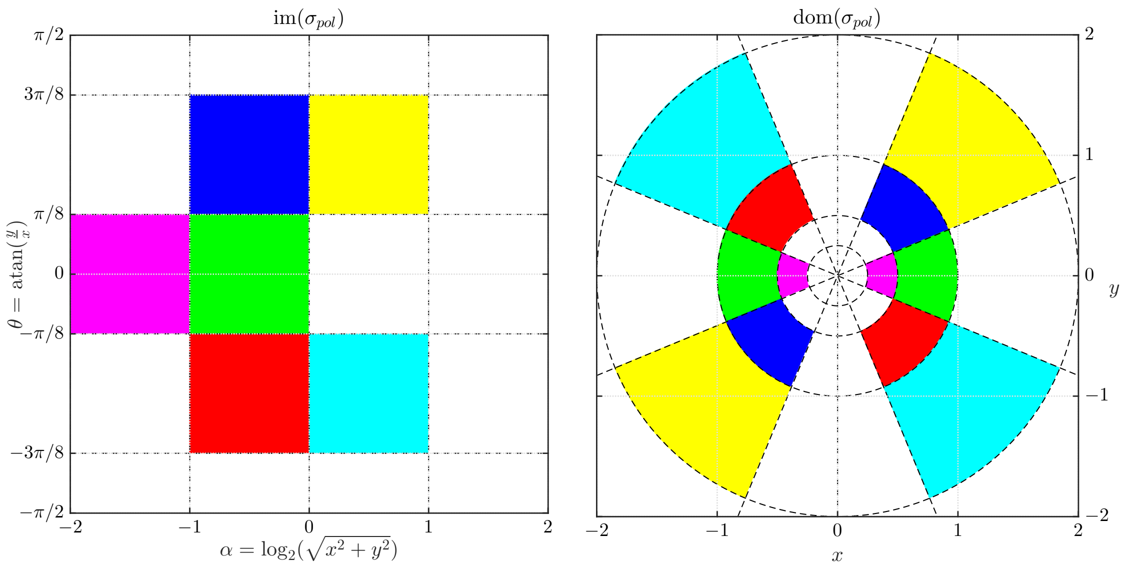
which is the well-known polar mapping, where we furthermore applied a log-warping to the positive radial component in order to map it to the whole real axis for the same reasons as in the one-dimensional wavelet transform above. And for the same reasons as pointed out in the wavelet example, the base of the logarithm is a degree of freedom here.

Note that the diffeomorphism itself takes care of the symmetry

$$\sigma_{pol}(-x, -y) = \left(\log(\sqrt{(-x)^2 + (-y)^2}),\ \operatorname{atan}((-y)/(-x))\right) \tag{5.107}$$

$$= \left(\log(\sqrt{x^2 + y^2}),\ \operatorname{atan}(y/x)\right) = \sigma_{pol}(x, y), \tag{5.108}$$

so if we not restrict our domain to either side of $\mathbb{R}^2$, σ_{pol} is not injective and thus a point $\left(\log(\sqrt{x^2 + y^2}),\ \operatorname{atan}(y/x)\right) \in \operatorname{im}(\sigma_{pol})$ corresponds to $\{(x, y), (-x, -y)\} \subset \operatorname{dom}(\sigma_{pol})$.



(a) Image of σ_{pol}.

(b) Domain of σ_{pol}.

Figure 5.14: Translation vs. Polar-Scaling

To illustrate this, we define the "rectangular window", i.e., an indicator function,

$$\Psi(x, y) := \varphi\left(\log_2(\sqrt{x^2 + y^2})\right) \cdot \eta\left(\mathrm{atan}(y/x)\right)$$

with

$$\varphi(r) := \begin{cases} 1 & , \ -1 < r \leq 0 \\ 0 & , \ \text{else} \end{cases} \quad \text{and} \quad \eta(\theta) := \begin{cases} 1 & , \ -\frac{\pi}{8} < \theta \leq \frac{\pi}{8} \\ 0 & , \ \text{else} \end{cases}.$$

Translating it in the image-domain, $\mathrm{im}(\sigma_{pol})$, and pulling it back to the Cartesian coordinates in the Fourier domain leads to a radially-circular scaling. This is shown in *Figure 5.14* for six exemplary shifting values, plotted in various colors for identification of corresponding pairs.

Remark 5.17 (atan 2)**.** It is worth noting that, since $\tan^{-1}$ is injective only on the half circle, it is possible to use the atan 2 function, if it is important to explicitly distinguish between the values on the left and right half-circle, which definitely is the case if the signal under investigation is complex valued.
This atan 2 function is an extension of the classical atan, such that its image is $\mathrm{im}(\mathrm{atan}\,2) = (-\pi, \pi]$, instead of $\mathrm{im}(\mathrm{atan}) = (-\pi/2, \pi/2)$ for the classical one. Its explicit definition is given by

$$\mathrm{atan}\,2(x) = \begin{cases} +\mathrm{atan}\left(\frac{y}{x}\right) & , \quad x > 0, \\ +\mathrm{atan}\left(\frac{y}{x}\right) \pm \pi & , \quad x < 0, \\ +\frac{\pi}{2} & , \quad y > 0 \wedge x = 0, \\ -\frac{\pi}{2} & , \quad y < 0 \wedge x = 0, \\ \text{undefined} & , \quad x = y = 0. \end{cases}$$

◆

The inverse of the polar diffeomorphism is easily recognized to be

$$\sigma_{pol}^{-1} : \mathbb{R} \times (-\pi/2, \pi/2) \to (\mathbb{R}\backslash\{0\}) \times \mathbb{R}, \quad (\alpha, \theta) \mapsto (e^{\alpha}\cos(\theta), e^{\alpha}\sin(\theta)) \tag{5.109}$$

and, in fact, this the second of the four cases, to which *Theorem 5.2 (Spectral Diffeomorphisms for Dual Orbits)* applies.

Let $H := \mathbb{R} \times SO(2)$, acting as

$$(\widehat{\mathbb{R}}^2, H) \ni (\xi, (\alpha, \theta)) \mapsto e^{\alpha} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^T .\xi \tag{5.110}$$

on $\widehat{\mathbb{R}}^2$ and set $\xi_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\begin{aligned} \sigma^{-1}(\alpha, \theta) &:= e^{\alpha} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^T \xi_0 \\ &= \begin{pmatrix} e^{\alpha}\cos(\theta) & -e^{\alpha}\sin(\theta) \\ e^{\alpha}\sin(\theta) & e^{\alpha}\cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\alpha}\cos(\theta) \\ e^{\alpha}\sin(\theta) \end{pmatrix} \end{aligned} \tag{5.111}$$

and we have recovered the diffeomorphism.

The Jacobian and its transposed inverse are given by

$$J_{\sigma}(\xi_1, \xi_2) = \begin{pmatrix} \frac{\xi_1}{\xi_1^2+\xi_2^2} & \frac{\xi_2}{\xi_1^2+\xi_2^2} \\ -\frac{\xi_2}{\xi_1^2+\xi_2^2} & \frac{\xi_1}{\xi_1^2+\xi_2^2} \end{pmatrix} \Longrightarrow \frac{-1}{2\pi i} J_{\sigma}(\xi_1, \xi_2) = \begin{pmatrix} \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_2\right] \\ \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_2\right] \end{pmatrix} \tag{5.112}$$

and

$$J_{\sigma}^{-T}(\xi_1, \xi_2) = \begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix} \Longrightarrow \frac{-1}{2\pi i} J_{\sigma}^{-T}(\xi_1, \xi_2) = \begin{pmatrix} \left[\widehat{A}^1, \widehat{B}_1\right] & \left[\widehat{A}^1, \widehat{B}_2\right] \\ \left[\widehat{A}^2, \widehat{B}_1\right] & \left[\widehat{A}^2, \widehat{B}_2\right] \end{pmatrix} \tag{5.113}$$

Thus, the spectral cotangent lift is

$$\Sigma(\xi; x) = (\sigma(\xi); J_{\sigma}^{-T}(\xi)x) \tag{5.114}$$

$$= (\log(\sqrt{\xi_1^2 + \xi_2^2}), \operatorname{atan}(\tfrac{\xi_2}{\xi_1}); \xi_1 x_1 + \xi_2 x_2, \xi_2 x_1 - \xi_1 x_2). \tag{5.115}$$

From this, we find that the Hamiltonians

$$A^i(\xi; x) := (J_{\sigma}^{-T}(\xi)x)^i \quad \text{and} \quad B_j(\xi; x) := \sigma_j(\xi), \tag{5.116}$$

are

$$A^1 := \xi_1 x_1 + \xi_2 x_2 \quad \text{and} \quad A^2 := \xi_2 x_1 - \xi_1 x_2 \tag{5.117}$$

and

$$B_1 := \log(\sqrt{\xi_1^2 + \xi_2^2}) \quad \text{and} \quad B_2 := \operatorname{atan}(\tfrac{\xi_2}{\xi_1}), \tag{5.118}$$

where the latter induce the warped translations and are thus only interesting for the warped Gabor case. The former two Hamiltonians induce the Hamiltonian vector fields

$$X_{A^1} = [\xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2}] - [x_1 \partial_{x_1} + x_2 \partial_{x_2}] \tag{5.119}$$

and

$$X_{A^2} = [\xi_1 \partial_{\xi_2} - \xi_2 \partial_{\xi_1}] + [x_1 \partial_{x_2} - x_2 \partial_{x_1}], \tag{5.120}$$

where the first is an infinitesimal symplectic flow, circular symmetric "away from the origin" for the ξ coordinates and circular symmetric towards the origin in x coordinates. The second induces a rotation around the origin in both the x and ξ coordinates. This is in accordance with the expectations, since this is essentially why this spectral diffeomorphism is interesting and leads to the SIM(2)-transform, which will be defined below.

But before doing so, by quantizing the Hamiltonians, using the symmetric quantization rule, we get

$$\begin{aligned} \xi_1 x_1 + \xi_2 x_2 = A^1 \longmapsto \widehat{A}^1 &= \tfrac{1}{2}\left[\left(\xi_1 \tfrac{-1}{2\pi i}\partial_{\xi_1} + \tfrac{-1}{2\pi i}\partial_{\xi_1}\xi_1\right) + \left(\xi_2 \tfrac{-1}{2\pi i}\partial_{\xi_2} + \tfrac{-1}{2\pi i}\partial_{\xi_2}\xi_2\right)\right] \\ &= -\tfrac{1}{2\pi i}\left[\left(\tfrac{1}{2} + \xi_1\partial_{\xi_1}\right) + \left(\tfrac{1}{2} + \xi_2\partial_{\xi_2}\right)\right] \end{aligned} \tag{5.121}$$

and

$$\begin{aligned} \xi_2 x_1 - \xi_1 x_2 = A^2 \longmapsto \widehat{A}^2 &= \tfrac{1}{2}\left[\left(\xi_2 \tfrac{-1}{2\pi i}\partial_{\xi_1} + \tfrac{-1}{2\pi i}\partial_{\xi_1}\xi_2\right) - \left(\xi_1 \tfrac{-1}{2\pi i}\partial_{\xi_2} + \tfrac{-1}{2\pi i}\partial_{\xi_2}\xi_1\right)\right] \\ &= -\tfrac{1}{2\pi i}\left[\xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2}\right], \end{aligned} \tag{5.122}$$

where $\widehat{A}^1$, represented on the Fourier domain, is the infinitesimal generator of unitary dilation in two dimensions,

$$e^{-2\pi i \widehat{A}^1 \alpha} = e^{\left((\frac{1}{2} + \xi_1\partial_{\xi_1}) + (\frac{1}{2} + \xi_2\partial_{\xi_2})\right)\alpha}, \tag{5.123}$$

which induces a two-dimensional unitary dilation on $\mathcal{S}_\sigma$. The Hamiltonian $\widehat{A}^2$ generates rotation around the origin, that is,

$$e^{-2\pi i \widehat{A}^2 \theta} = e^{(\xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2})\theta}, \tag{5.124}$$

is unitary on $\mathcal{F}\mathcal{S}_\sigma$ and rotates the Fourier domain around the origin.

Again, this is what we expected and proceed towards the definition of the associated spectral quantum frames.

The associated corresponding operator,

$$\mathcal{W}_{\sigma_{pol}} : L^2(\mathrm{dom}(\sigma_{pol})) \to L^2\left(\mathbb{R} \times (-\pi/2, \pi/2),\ \mathrm{d}\alpha\ \mathrm{d}\theta\right) := L^2(\mathrm{im}(\sigma_{pol})), \tag{5.125}$$

on the Fourier domain, acts as

$$\begin{aligned} (\mathcal{W}_{\sigma_{pol}} \hat{f})(\alpha, \theta) &= \sqrt{\frac{\mathrm{d}\lambda(\sigma_{pol}^{-1}(\alpha,\theta))}{\mathrm{d}\alpha\ \mathrm{d}\theta}}\, \hat{f}(\sigma_{pol}^{-1}(\alpha,\theta)) \\ &= \sqrt{\left|\det(J_{\sigma_{pol}^{-1}})(\alpha,\theta)\right|} \cdot \hat{f}(\sigma_{pol}^{-1}(\alpha,\theta)) \\ &= e^{\alpha} \hat{f}(e^{\alpha}\cos\theta, e^{\alpha}\sin\theta), \end{aligned} \tag{5.126}$$

and its pendant on the space of admissible windows is

$$\widetilde{\mathcal{W}}_{\sigma_{pol}} : \begin{cases} \mathcal{A}_{\sigma_{pol}} & \to\ L^2(\mathrm{im}(\sigma_{pol})) \\ \psi & \mapsto\ \hat{\psi}(e^{\alpha}\cos\theta, e^{\alpha}\sin\theta), \end{cases} \tag{5.127}$$

where the equalities hold almost everywhere. Conjugating a translation with this warping operator leads to the associated spectral dilation operator

$$(\widehat{\widetilde{\mathcal{D}}}{}^{\sigma_{pol}}_{(\alpha,\theta)} \hat{f})(x,y) = \hat{f}\left(e^{\alpha} R_{\theta}(x,y)\right), \tag{5.128}$$

where R_θ is the two-dimensional rotation in the Fourier plane. We already know that to this spectral warp is assigned a spectral frame, so the remaining mystery is its coherent state map. Composing with one of the translation operators and conjugating with the Fourier transform leads to

$$\widetilde{\pi}^{\sigma_{pol}}(\beta_1, \beta_2; \alpha, \theta) := T_{\vec{\beta}} \widetilde{\mathcal{D}}^{\sigma_{pol}}_{(\alpha,\theta)}, \tag{5.129}$$

for standard translation and

$$\widetilde{\pi} \circ \Sigma_{pol}(\beta_1, \beta_2; \alpha, \theta) := \mathcal{T}^{\sigma_{pol}}_{\vec{\beta}} \widetilde{\mathcal{D}}^{\sigma_{pol}}_{(\alpha,\theta)}, \tag{5.130}$$

for the warped translation, which gives the following.

Definition 5.18 (The SIM(2) transform)**.** Let $f \in \mathcal{S}_{\sigma_{pol}}$ be real-valued and $\psi \in \mathcal{A}_{\sigma_{pol}}$, then

$$\begin{aligned} f \ \mapsto\ & (\pi_{\psi}^{\sigma_{pol}} f)(\vec{\beta}, \alpha, \theta) \\ = & \iint_{\mathbb{R}^2} \overline{e^{-2\pi i \langle \vec{\xi}, \vec{\beta} \rangle} \widehat{\psi}\left(e^{\alpha}(\xi_1\cos\theta - \xi_2\sin\theta), e^{\alpha}(\xi_2\cos\theta + \xi_1\sin\theta)\right)} \\ & \quad \times \widehat{f}(\xi_1, \xi_2)\ \mathrm{d}\xi_1\ \mathrm{d}\xi_2 \\ = & \iint_{\mathbb{R}^2} \overline{e^{-2\alpha}\psi\left(e^{-\alpha}((x-\beta_1)\cos\theta - (y-\beta_2)\sin\theta), e^{\alpha}((y-\beta_2)\cos\theta + (x-\beta_1)\sin\theta)\right)} \\ & \quad \times f(x,y)\ \mathrm{d}x\ \mathrm{d}y \end{aligned} \tag{5.131}$$

is *the SIM(2) transform.* ◆

Definition 5.19 (Polar-warped STFT in 2D)**.** Let $f \in \mathcal{A}_{\sigma_{pol}}$ and $\psi \in \mathcal{A}_{\sigma_{pol}}$, then

$$f \mapsto \left\langle \widetilde{\mathcal{T}}_{\vec{b}}^{\sigma_{pol}} \widetilde{\mathcal{D}}_{\alpha}^{\sigma_{pol}} \psi \,,\, f \right\rangle = \iint_{\mathbb{R}^2} \overline{e^{2\pi i \langle \vec{\alpha}\,,\, \vec{x} - \vec{b} \rangle} \widetilde{\psi}\left(\vec{x} - \vec{b}\right)}\, \widetilde{f}(x)\, \mathrm{d}x, \tag{5.132}$$

with $\widetilde{\psi} := \mathcal{W}_{\sigma_{pol}} \psi$ and $\widetilde{f} := \mathcal{W}_{\sigma_{pol}} f$, is the *polar-warped Short-Time Fourier transform in 2D.* ◆

Localization

Again, the notions of localization differ, the Jacobian is non-diagonal and thus we cannot expect to have perfect equalizers, given as tensor-products of any lower-dimensional sub-solutions. Therefore, the general principle applies and the optimal waveform, for the principle of optimal alignment, has to fulfill the following system of differential equations, with $\xi = (x, y)$, reading

$$\begin{pmatrix} \frac{\partial \widehat{\psi}}{\partial_x} \\ \frac{\partial \widehat{\psi}}{\partial_y} \end{pmatrix} = -2\pi \begin{pmatrix} \frac{x}{x^2+y^2} & -\frac{y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} (\mu_x x - \mu_x \beta_x + i\alpha_x)\widehat{\psi} \\ (\mu_y y - \mu_y \beta_y + i\alpha_y)\widehat{\psi} \end{pmatrix}. \tag{5.133}$$

Unfortunately, the domain is not simply-connected, as it is the punctured plane $\mathbb{R}^2 \setminus \{0\}$, so there is no guarantee for a solution.

Hoping for the best, however, it is easy to see that choosing $\mu := \mu_x = -\mu_y$, we get

$$\begin{pmatrix} \frac{\partial \widehat{\psi}}{\partial_x} \\ \frac{\partial \widehat{\psi}}{\partial_y} \end{pmatrix} = -2\pi\mu \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} (x - \beta_x)\widehat{\psi} \\ (y - \beta_y)\widehat{\psi} \end{pmatrix} - 2\pi i \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} \alpha_x \widehat{\psi} \\ \alpha_y \widehat{\psi} \end{pmatrix} \tag{5.134}$$

that is, the weighted Jacobian becomes symmetric, and thus the general principle applies. Setting $\mu_x = -\mu_y$, $\alpha_x = \alpha_y = 0$, and $\beta_x = 1/2, \beta_y = 0$, leading to a solution, centered at the point $(0, 1/2)$, this system is solved by

$$\begin{aligned} \widehat{\psi}(\xi) &= e^{-2\pi\left(\int \mu \frac{x^2+y^2}{x^2+y^2}\, \mathrm{d}x + \int \mu \frac{xy-yx}{x^2+y^2}\, \mathrm{d}y\right)} e^{+2\pi\mu\frac{1}{2}\left(\int \frac{x}{x^2+y^2}\, \mathrm{d}x + \int \frac{y}{x^2+y^2}\, \mathrm{d}y\right)} \\ &= e^{-2\pi\left(\int \mu \frac{x^2+y^2}{x^2+y^2}\, \mathrm{d}x + \int \mu \frac{xy-yx}{x^2+y^2}\, \mathrm{d}y\right)} e^{2\pi\mu \log\left(\sqrt{x^2+y^2}\right)} \\ &= C e^{-2\pi\mu x} \sqrt{x^2+y^2}^{2\pi\mu}, \end{aligned} \tag{5.135}$$

respectively its counterpart

$$\widehat{f}(\xi) = \frac{1}{\sqrt{x^2+y^2}} C e^{-2\pi\mu x} \sqrt{x^2+y^2}^{2\pi\mu}, \tag{5.136}$$

on $\mathcal{S}_{\sigma_{pol}}$, which cannot be taken seriously as a reasonable waveform for doing signal analysis in two-dimensions, so one should stick to partial equalizing waveforms.

After setting the constants to zero - which is legitimate since the spectra of the operators are the full line -, the system of differential equations for the principle of optimal concentration is

$$\begin{pmatrix} \frac{\partial \widehat{f}}{\partial_x} \\ \frac{\partial \widehat{f}}{\partial_y} \end{pmatrix} = \begin{pmatrix} -2\pi\mu_x \log(\sqrt{x^2+y^2})\widehat{f} \\ -2\pi\mu_y \operatorname{atan}\left(\frac{y}{x}\right)\widehat{f} \end{pmatrix}, \tag{5.137}$$

which, for $\mu_x = -\mu_y$, is solved by the waveform

$$\begin{aligned} \widehat{f}(\xi) &:= e^{-2\pi(\mu_x \int \log(\sqrt{x^2+y^2})\ \mathrm{d}x + \mu_y \int \operatorname{atan}(\frac{y}{x})\ \mathrm{d}y)} \\ &= Ce^{-2\pi\mu_x(\frac{x}{2}\log(x^2+y^2)+y\operatorname{atan}(\frac{x}{y})-x)} e^{-2\pi\mu_y(-\frac{x}{2}\log(x^2+y^2)+y\operatorname{atan}(\frac{y}{x}))} \\ &= Ce^{-2\pi(\mu_x-\mu_y)(\frac{x}{2}\log(x^2+y^2))} e^{2\pi\mu_x x} e^{-2\pi y(\mu_x\operatorname{atan}(\frac{x}{y})+\mu_y\operatorname{atan}(\frac{y}{x}))} \\ &= Ce^{-2\pi\mu(x\log(x^2+y^2)-x)} e^{-2\pi y\mu(\operatorname{atan}(\frac{x}{y})-\operatorname{atan}(\frac{y}{x}))} \end{aligned} \tag{5.138}$$

respectively its admissible counterpart

$$\widehat{\psi}(\xi) = \sqrt{x^2+y^2}\cdot Ce^{-2\pi\mu(x\log(x^2+y^2)-x)} e^{-2\pi y\mu(\operatorname{atan}(\frac{x}{y})-\operatorname{atan}(\frac{y}{x}))} \tag{5.139}$$

on $\mathcal{A}_{\sigma_{pol}}$, which again does not make sense, this time because it does not solve the associated system of differential equations, due to the space being not simply-connected.

Regarding the individual equalizers, we note that

$$\sum_n \partial_{p_n}\left(\frac{\sigma_n^{-1}}{\partial\sigma_i}\right) = \sum_n \partial_{p_n}\left((J^{-T})_n^i\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \tag{5.140}$$

Thus, we get the following equalizing waveforms for the principle of optimal alignment:

$$\begin{aligned} \widehat{\psi}_{1,1}(\xi) &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_1} e^{2\pi(\mu\beta-i\alpha)\sigma_1} e^{-\frac{1}{2}\int 2\mathrm{d}\sigma_1} \\ &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_1} e^{2\pi(\mu\beta-i\alpha)\sigma_1} e^{-\sigma_1} \\ &= C\sqrt{x^2+y^2}e^{-2\pi\mu(x\log(\sqrt{x^2+y^2})-y\operatorname{atan}(\frac{y}{x})+x)} e^{2\pi(\mu\beta-i\alpha)\log(\sqrt{x^2+y^2})} \end{aligned} \tag{5.141}$$

$$\widehat{\psi}_{1,2}(\xi) = C\sqrt{x^2+y^2}e^{-2\pi\mu(y\log(\sqrt{x^2+y^2})-x\operatorname{atan}(\frac{x}{y})+y)} e^{2\pi(\mu\beta-i\alpha)\log(\sqrt{x^2+y^2})} \tag{5.142}$$

$$\begin{aligned} \widehat{\psi}_{2,1}(\xi) &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_2} e^{2\pi(\mu\beta-i\alpha)(\operatorname{atan}(y/x))} \\ &= Ce^{-2\pi\mu(x\operatorname{atan}(\frac{y}{x})-y\log(\sqrt{x^2+y^2}))} e^{2\pi(\mu\beta-i\alpha)(\operatorname{atan}(y/x))} \end{aligned} \tag{5.143}$$

$$\begin{aligned} \widehat{\psi}_{2,2}(\xi) &= Ce^{-2\pi\mu\int y\mathrm{d}\sigma_2} e^{2\pi(\mu\beta-i\alpha)(\operatorname{atan}(y/x))} \\ &= Ce^{-2\pi\mu(-y\operatorname{atan}(\frac{x}{y})+x\log(\sqrt{x^2+y^2}))} e^{2\pi(\mu\beta-i\alpha)(\operatorname{atan}(y/x))}, \end{aligned} \tag{5.144}$$

all on $\mathcal{S}_\sigma$ and the usual alterations apply to find the associated functions on $\mathcal{S}_\sigma$.

For the principle of optimal concentration, we jump again straight to the formulas:

$$\begin{aligned}\widehat{f_{1,1}}(x,y) &= Ce^{-2\pi\mu\int \log\sqrt{y^2+x^2}\ \mathrm{d}x} e^{2\pi x(\mu\alpha - i\beta)} \\ &= Ce^{-2\pi\mu(x(\log\sqrt{y^2+x^2}-1)+y\,\mathrm{atan}(x/y))} e^{2\pi x(\mu\alpha - i\beta)}\end{aligned} \tag{5.145}$$

$$\begin{aligned}\widehat{f_{1,2}}(x,y) &= Ce^{-2\pi\mu\int \mathrm{atan}(y/x)\ \mathrm{d}x} e^{2\pi x(\mu\alpha - i\beta)} \\ &= Ce^{-2\pi\mu(y\log\sqrt{y^2+x^2}+x\,\mathrm{atan}(y/x))} e^{2\pi x(\mu\alpha - i\beta)}\end{aligned} \tag{5.146}$$

$$\begin{aligned}\widehat{f_{2,1}}(x,y) &= Ce^{-2\pi\mu\int \log\sqrt{y^2+x^2}\ \mathrm{d}y} e^{2\pi y(\mu\alpha - i\beta)} \\ &= Ce^{-2\pi\mu(y(\log\sqrt{y^2+x^2}-1)+y\,\mathrm{atan}(x/y))} e^{2\pi y(\mu\alpha - i\beta)}\end{aligned} \tag{5.147}$$

$$\begin{aligned}\widehat{f_{2,2}}(x,y) &= Ce^{-2\pi\mu\int \mathrm{atan}(y/x)\ \mathrm{d}y} e^{2\pi y(\mu\alpha - i\beta)} \\ &= Ce^{-2\pi\mu(-x\log\sqrt{y^2+x^2}+y\,\mathrm{atan}(y/x))} e^{2\pi x(\mu\alpha - i\beta)}\ .\end{aligned} \tag{5.148}$$

◆

The Log-Shear Diffeomorphism

Instead of the very natural polar coordinates, we may introduce scale-shear coordinates to describe the Fourier domain; of course, again only for half-space.

Example 5.9 (Shear Coordinates)**.** As before, we wish to be able to mask out certain features with specific "directions" and size resp. scale. Note that the term *direction* should now be taken with a grain of salt, as its "directed-ness" is a little biased and not as directly transferable to, say, some angle θ as above. This should become more clear below.

Again, we resort to exponential scaling and define

$$\sigma_{sh}(x,y) = \left(\log(x), \tfrac{y}{x}\right) \quad \text{and} \quad \sigma_{sh}^{-1}(\alpha,s) = (e^\alpha, e^\alpha s)\,. \tag{5.149}$$

This is the third case to which *Theorem 5.2 (Spectral Diffeomorphisms for Dual Orbits)* applies. Let $H := \mathbb{R}\times\mathbb{R}$, acting as

$$(\widehat{\mathbb{R}}^2, H) \ni (\xi,(\alpha,s)) \;\mapsto\; e^\alpha \begin{pmatrix} 1 & s \\ & 1\end{pmatrix}^T .\xi \tag{5.150}$$

on $\widehat{\mathbb{R}}^2$ and set again to $\xi_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\begin{aligned} \sigma^{-1}(\alpha,\theta) &:= e^{\alpha}\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}^T \xi_0 \\ &= \begin{pmatrix} e^{\alpha} & \\ e^{\alpha}s & e^{\alpha} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\alpha} \\ e^{\alpha}s \end{pmatrix} \end{aligned} \tag{5.151}$$

and we have recovered the diffeomorphism.

Apart from the logarithmic scaling, this diffeomorphism associates lines, parallel to the α-axis in $\mathrm{im}(\sigma_{sh})$, to lines through the origin in the original Fourier domain. To illustrate this, we define, as before, a rectangular window

$$\Psi(x,y) := \varphi\left(\log_2(x)\right)\cdot\eta\left(y/x\right)$$

with

$$\varphi(\alpha) := \begin{cases} 1 & , -1 < \alpha \leq 0 \\ 0 & , \text{else} \end{cases} \quad \text{and} \quad \eta(s) := \begin{cases} 1 & , -\frac{1}{2} < s \leq \frac{1}{2} \\ 0 & , \text{else} \end{cases}.$$

Shifting this function in the image-domain, $\mathrm{im}(\sigma_{sh})$, corresponds to scaled shearing in Cartesian coordinates. An illustration of this is given by *Figure 5.15*, again exemplary for six shifting values, plotted in various colors for identification of corresponding pairs. For comparison with *Figure 5.14*, the same shifting values as well as colors have been chosen.

The Jacobian and its transposed inverse are given by

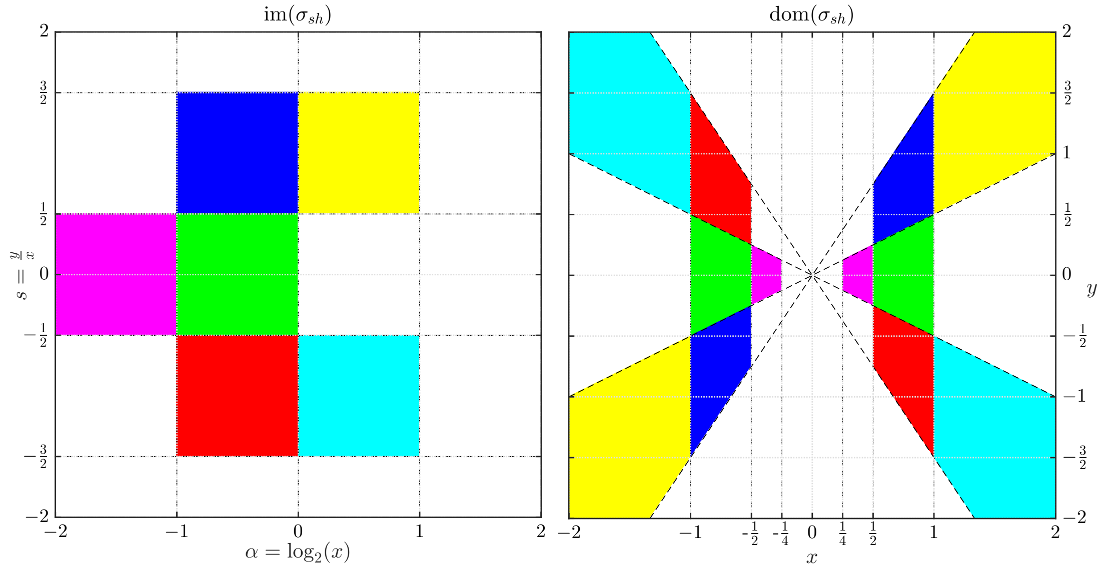
$$J_{\sigma}(\xi_1,\xi_2) = \begin{pmatrix} \frac{1}{\xi_1} & 0 \\ -\frac{\xi_2}{\xi_1^2} & \frac{1}{\xi_1} \end{pmatrix} \Longrightarrow \frac{-1}{2\pi i}J_{\sigma}(\xi_1,\xi_2) = \begin{pmatrix} \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_2\right] \\ \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_2\right] \end{pmatrix} \tag{5.152}$$

and

$$J_{\sigma}^{-T}(\xi_1,\xi_2) = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & \xi_1 \end{pmatrix} \Longrightarrow \frac{-1}{2\pi i}J_{\sigma}^{-T}(\xi_1,\xi_2) = \begin{pmatrix} \left[\widehat{A}^1, \widehat{B}_1\right] & \left[\widehat{A}^1, \widehat{B}_2\right] \\ \left[\widehat{A}^2, \widehat{B}_1\right] & \left[\widehat{A}^2, \widehat{B}_2\right] \end{pmatrix}. \tag{5.153}$$

Thus, the spectral cotangent lift is

$$\begin{aligned} \Sigma(x;\xi) =& (\sigma_{sh}(\xi); J_{\sigma_{sh}}^{-T}(\xi)x) \\ =& (\log(\xi_1), \tfrac{\xi_2}{\xi_1}; \xi_1 x_1 + \xi_2 x_2, \xi_1 x_2). \end{aligned} \tag{5.154}$$



(a) Image of σ_{sh}. **(b)** Domain of σ_{sh}.

Figure 5.15: Translation vs. Non-Parabolic Shear-Scaling

From this, we find that the Hamiltonians

$$A^i(x;\xi) := (J_\sigma^{-T}(\xi)x)^i \text{ and } B_j(x;\xi) := \sigma_j(\xi), \tag{5.155}$$

are

$$A^1(x;\xi) := \xi_1 x_1 + \xi_2 x_2 \quad \text{and} \quad A^2(x;\xi) := \xi_1 x_2 \tag{5.156}$$

and

$$B_1(x;\xi) := \log(\xi_1) \quad \text{and} \quad B_2(x;\xi) := \tfrac{\xi_2}{\xi_1}, \tag{5.157}$$

where, as in the polar case, the latter induce the warped translations and are thus only interesting for the warped Gabor case. The former two Hamiltonians induce the Hamiltonian vector fields

$$X_{A^1} = [\xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2}] - [x_1 \partial_{x_1} + x_2 \partial_{x_2}] \tag{5.158}$$

and

$$X_{A^2} = \xi_1 \partial_{\xi_2} - x_2 \partial_{x_1}, \tag{5.159}$$

where the first is again a infinitesimal symplectic flow, circular symmetric “away from the origin” for the ξ coordinates and circular symmetric “towards the origin” in x coordinates. The second induces a shear in negative x_1 direction for x and positive ξ_1 direction for the ξ coordinates. Quantizing the Hamiltonians, using the symmetric quantization rule, we get

$$\begin{aligned}\xi_1 x_1 + \xi_2 x_2 = A^1 \longmapsto \widehat{A}^1 &= \tfrac{1}{2}\left[(\xi_1 \tfrac{-1}{2\pi i}\partial_{\xi_1} + \tfrac{-1}{2\pi i}\partial_{\xi_1}\xi_1) + (\xi_2 \tfrac{-1}{2\pi i}\partial_{\xi_2} + \tfrac{-1}{2\pi i}\partial_{\xi_2}\xi_2)\right] \\ &= -\tfrac{1}{2\pi i}\left[(\tfrac{1}{2} + \xi_1\partial_{\xi_1}) + (\tfrac{1}{2} + \xi_2\partial_{\xi_2})\right]\end{aligned} \tag{5.160}$$

and

$$\xi_1 x_2 = A^2 \longmapsto \widehat{A}^2 = -\tfrac{1}{2\pi i}\xi_1\partial_{\xi_2}, \tag{5.161}$$

where $\widehat{A}^1$, represented on the Fourier domain, is as before the infinitesimal generator of unitary dilation in two dimensions

$$e^{-2\pi i \widehat{A}^1 \alpha} = e^{\left((\frac{1}{2}+\xi_1\partial_{\xi_1})+(\frac{1}{2}+\xi_2\partial_{\xi_2})\right)\alpha}, \tag{5.162}$$

inducing a two-dimensional unitary dilation on $\mathcal{S}_{\sigma_{sh}}$, and the Hamiltonian $\widehat{A}^2$ generates a shear, that is,

$$e^{-2\pi i \widehat{A}^2 s} = e^{\xi_1 \partial_{\xi_2} s}, \tag{5.163}$$

in positive ξ_1 direction and is unitary on $\mathcal{F}\mathcal{S}_{\sigma_{sh}}$.

Again, the map $\sigma_{sh} \mapsto \widetilde{\mathcal{W}}_{\sigma_{sh}}$ lifts the diffeomorphism to the level of functions, with the associated operator

$$(\widetilde{\mathcal{W}}_{\sigma_{sh}} f)(\alpha, s) := f(\sigma_{sh}^{-1}(\alpha, s)) = f(e^{\alpha}, e^{\alpha} s), \tag{5.164}$$

to be read in the sense of almost everywhere equivalence. To define the action of the spectral dilation operator - induced by the Hamiltonians above -, we write $\sigma := \sigma_{sh}$ and calculate

$$\begin{aligned}\sigma^{-1}(\sigma(x,y) + (a,s)) &:= \sigma^{-1}\left(\log(x) + a, \frac{y}{x} + s\right) \\ &= \sigma^{-1}\left(\log(xe^{a}), \frac{y + xs}{x}\right) \\ &= \left(e^{a}x, e^{a}x\frac{y+xs}{x}\right) \\ &= \left(e^{a}x, e^{a}(y + xs)\right),\end{aligned} \tag{5.165}$$

which gives us the associated spectral dilation operator

$$(\widehat{\widehat{\mathcal{D}}}{}^{\sigma_{sh}}_{(\alpha,s)}\widehat{\psi})(x,y) = \widehat{\psi}(e^{\alpha}x, e^{\alpha}(y + sx)). \tag{5.166}$$

As usual, we define the operator $\pi^{\sigma_{sh}}(\vec{\beta}, \alpha, s) := T_{\vec{\beta}}\mathcal{F}^{*}\widehat{\widehat{\mathcal{D}}}{}^{\sigma_{sh}}_{(\alpha,s)}\mathcal{F}$ and for some $\psi \in \mathcal{A}_{\sigma_{sh}}$, we get a transform

$$\pi_{\psi}^{\sigma_{sh}} : \mathcal{S}_{\sigma_{sh}} \to L^2(\mathbb{R}^2 \times \operatorname{im}(\sigma_{sh})), \; f \mapsto \left\langle \pi^{\sigma_{sh}}(\vec{\beta}, \alpha, s)\psi \,,\, f \right\rangle, \tag{5.167}$$

which we will baptize as the *non-parabolic shearlet transform*, as there does not seem to be a definition in the literature. Confer, e.g, [10] for a treatment of the parabolic version.

Definition 5.20 (Non-Parabolic Shearlet Transform)**.** Let $f \in \mathcal{S}_{\sigma_{sh}}$ be real-valued and $\psi \in \mathcal{A}_{\sigma_{sh}}$. Let moreover

$$\sigma_{sh}(x,y) := (\log(x), y/x)$$

be the shear-diffeomorphism. Then,

$$\begin{aligned} f \mapsto & \left(\pi_{\psi}^{\sigma_{sh}} f\right)(\vec{\beta}, \alpha, s) \\ := & \left\langle T_{\vec{\beta}} \widetilde{\mathcal{D}}_{\alpha,s}^{\sigma_{sh}} \psi \,,\, f \right\rangle \\ = & \iint_{\mathrm{dom}(\sigma_{sh})} \overline{e^{-2\pi i \langle\, \vec{\xi} \,,\, \vec{\beta} \,\rangle} \widehat{\psi}\left(e^{\alpha}\xi_1, e^{\alpha}(\xi_2 + s\xi_1)\right)} \widehat{f}(\xi_1, \xi_2) \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2 \end{aligned} \tag{5.168}$$

is the *non-parabolic shearlet transform.* ◆

Definition 5.21 (Non-Parabolic Shear-warped STFT in 2D)**.** Let $f \in \mathcal{A}_{\sigma_{sh}}$ and $\psi \in \mathcal{A}_{\sigma_{sh}}$, then

$$\begin{aligned} f \mapsto & \left\langle \widetilde{\mathcal{T}}_{\vec{b}}^{\sigma_{sh}} \widetilde{\mathcal{D}}_{\alpha,s}^{\sigma_{sh}} \psi \,,\, f \right\rangle \\ = & \iint_{\mathrm{dom}(\sigma_{sh})} \overline{e^{-2\pi i \langle\, \sigma_{sh}(\vec{\xi}) \,,\, \vec{\beta} \,\rangle} \widehat{\psi}\left(e^{\alpha}\xi_1, e^{\alpha}(\xi_2 + s\xi_1)\right)} \widehat{f}(\xi_1, \xi_2) \,\mathrm{d}\sigma_{sh}(\xi_1, \xi_2) \\ = & \iint_{\mathbb{R}^2} \overline{e^{2\pi i \langle\, \vec{\alpha} \,,\, \vec{x} - \vec{b} \,\rangle} \widetilde{\psi}\left(\vec{x} - \vec{b}\right)} \,\widetilde{f}(x) \,\mathrm{d}x, \end{aligned} \tag{5.169}$$

with $\widetilde{\psi} := \mathcal{W}_{\sigma_{sh}}\psi$ and $\widetilde{f} := \mathcal{W}_{\sigma_{sh}} f$, is the *non-parabolic shear-warped Short-Time Fourier transform in 2D.* ◆

What differentiates this transform from the standard (parabolic) shearlet transform, as found in the literature, is the commutativity of the shearing and scaling translations, along with some other minor adjustments, arising from this difference. In the next section, these minor adjustments will be made precise and finally lead to the parabolic shearlet transform.

Localization

Again, the notions of localization differ, the Jacobian is non-diagonal and thus we cannot expect to have perfect equalizers, given as tensor-products of any lower-dimensional sub-solutions. Therefore, the general principle applies and the optimal waveform, for the principle of optimal alignment, has to fulfill the following system of differential equations, with $\xi = (x, y)$, reading

$$\begin{pmatrix} \frac{\partial \widehat{\psi}}{\partial_x} \\ \frac{\partial \widehat{\psi}}{\partial_y} \end{pmatrix} = -2\pi \begin{pmatrix} \frac{1}{x} & -\frac{y}{x^2} \\ 0 & \frac{1}{x} \end{pmatrix} \begin{pmatrix} (\mu_x x - \mu_x \beta_x + i\alpha_x)\widehat{\psi} \\ (\mu_y y - \mu_y \beta_y + i\alpha_y)\widehat{\psi} \end{pmatrix}. \tag{5.170}$$

Unfortunately, the Jacobian cannot be made symmetric and thus the admissible waveform

$$\begin{aligned}\widehat{\psi}(\xi) &:= e^{-2\pi(\int \mu_x - \mu_y (\frac{y}{x})^2 \ \mathrm{d}x + \mu_y \int \frac{y}{x} \ \mathrm{d}y)} \\ &= Ce^{-2\pi(\mu_x x + \mu_y \frac{y^2}{x} + \mu_y \frac{y^2}{2x})} \\ &= Ce^{-2\pi\mu_x x} e^{-2\pi \frac{3}{2}\mu_y \frac{y^2}{x}},\end{aligned} \tag{5.171}$$

respectively its counterpart

$$\widehat{f}(\xi) = \tfrac{1}{x} C e^{-2\pi\mu_x x} e^{-2\pi\frac{3}{2}\mu_y \frac{y^2}{x}} \tag{5.172}$$

on $\mathcal{S}_{\sigma_{sh}}$ are no explicit equalizing waveform for the uncertainty principle and the partial equalizers are the way to go.

Again, the spectra of the operators are the full line and after setting the constants to zero, the system of differential equations, for the principle of optimal concentration, reads

$$\begin{pmatrix} \frac{\partial \widehat{f}}{\partial_x} \\ \frac{\partial \widehat{f}}{\partial_y} \end{pmatrix} = \begin{pmatrix} -2\pi\mu_x \log(x) \widehat{f} \\ -2\pi\mu_y \frac{y}{x} \widehat{f} \end{pmatrix}, \tag{5.173}$$

which cannot be symmetrized also and thus the waveform

$$\begin{aligned}\widehat{f}(\xi) &:= e^{-2\pi(\mu_x \int \log(x) \ \mathrm{d}x + \mu_y \int \frac{y}{x} \ \mathrm{d}y)} \\ &= Ce^{-2\pi\mu_x (x\log(x)+x)} e^{-2\pi\mu_y \frac{y^2}{2x}}\end{aligned} \tag{5.174}$$

respectively its admissible counterpart

$$\widehat{\psi}(\xi) = xC \cdot e^{-2\pi\mu_x(x\log(x)+x)} e^{-\pi\mu_y \frac{y^2}{x}} \tag{5.175}$$

on $\mathcal{A}_{\sigma_{sh}}$ are again no solutions and partial equalizers are the next best thing.

For the individual uncertainty principles, we have, as in the polar case, that

$$\sum_n \partial_{p_n} \left(\frac{\sigma_n^{-1}}{\partial \sigma_i} \right) = \sum_n \partial_{p_n} \left((J^{-T})^i_n \right). = \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \tag{5.176}$$

Thus, we get the following equalizing waveforms for the principle of optimal alignment:

$$\begin{aligned}\widehat{\psi}_{1,1}(x,y) &= Ce^{-2\pi\mu\int x\mathrm{d}\log}e^{2\pi(\mu\beta-i\alpha)\log}e^{-\frac{1}{2}\int 2\mathrm{d}\log} \\ &= Ce^{-2\pi\mu x}e^{2\pi(\mu\beta-i\alpha)\log(x)}e^{-\log|x|} \\ &= C\left|x\right|^{-1}e^{-2\pi\mu x}x^{2\pi(\mu\beta-i\alpha)} \end{aligned} \tag{5.177}$$

$$\begin{aligned}\widehat{\psi}_{1,2}(x,y) &= Ce^{-2\pi\mu\int y\mathrm{d}\log}e^{2\pi(\mu\beta-i\alpha)\log}e^{-\frac{1}{2}\int 2\mathrm{d}\log} \\ &= Ce^{-2\pi\mu y\log(x)}e^{2\pi(\mu\beta-i\alpha)\log(x)}e^{-\log|x|} \\ &= C\left|x\right|^{-1}e^{-2\pi\mu y\log(x)}x^{2\pi(\mu\beta-i\alpha)} \end{aligned} \tag{5.178}$$

$$\begin{aligned}\widehat{\psi}_{2,1}(x,y) &= Ce^{-2\pi\mu\int x\mathrm{d}(y/x)}e^{2\pi(\mu\beta-i\alpha)(y/x)} \\ &= Ce^{-2\pi\mu(\int x\frac{-y}{x^2}\ \mathrm{d}x+\int x\frac{1}{x}\ \mathrm{d}y)}e^{2\pi(\mu\beta-i\alpha)(y/x)} \\ &= Ce^{-2\pi\mu(-y\log|x|+y)}e^{2\pi(\mu\beta-i\alpha)(y/x)} \end{aligned} \tag{5.179}$$

$$\begin{aligned}\widehat{\psi}_{2,2}(x,y) &= Ce^{-2\pi\mu\int y\mathrm{d}(y/x)}e^{2\pi(\mu\beta-i\alpha)(y/x)} \\ &= Ce^{-2\pi\mu(\int y\frac{-y}{x^2}\ \mathrm{d}x+\int y\frac{1}{x}\ \mathrm{d}y)}e^{2\pi(\mu\beta-i\alpha)(y/x)} \\ &= Ce^{-2\pi\mu(\frac{3y^2}{2x})}e^{2\pi(\mu\beta-i\alpha)(y/x)}, \end{aligned} \tag{5.180}$$

all on $\mathcal{S}_\sigma$ and the map ι takes these to their admissible pendants.

For the principle of optimal concentration, we again shall be content with presenting the formulas for the equalizing waveforms on $\mathcal{S}_\sigma$:

$$\begin{aligned}\widehat{f}_{1,1}(x,y) &= Ce^{-2\pi\mu\int\log x\ \mathrm{d}x}e^{2\pi x(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu(x\log x-x)}e^{2\pi x(\mu\alpha-i\beta)} \end{aligned} \tag{5.181}$$

$$\begin{aligned}\widehat{f}_{1,2}(x,y) &= Ce^{-2\pi\mu\int\frac{y}{x}\ \mathrm{d}x}e^{2\pi x(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu y\log(x)}e^{2\pi x(\mu\alpha-i\beta)} \end{aligned} \tag{5.182}$$

$$\begin{aligned}\widehat{f}_{2,1}(x,y) &= Ce^{-2\pi\mu\int\log x\ \mathrm{d}y}e^{2\pi y(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu y\log x}e^{2\pi y(\mu\alpha-i\beta)} \end{aligned} \tag{5.183}$$

$$\begin{aligned}\widehat{f}_{2,2}(x,y) &= Ce^{-2\pi\mu\int\frac{y}{x}\ \mathrm{d}y}e^{2\pi y(\mu\alpha-i\beta)} \\ &= Ce^{-2\pi\mu\frac{y^2}{x}}e^{2\pi y(\mu\alpha-i\beta)}. \end{aligned} \tag{5.184}$$

◆

The Log-Hyperbolic Diffeomorphism

Before heading to a final non-abelian generalization, one last abelian application shall be discussed. This time, with a more physical background.

Example 5.10 (Hyperbolic Coordinates)**.** In particle physics, a (free) particle of mass m is interpreted as an elementary excitation of a (tensor-valued) relativistic quantum field, with each component fulfilling the Klein-Gordon equation

$$(-\hbar^2\partial_t^2 + \hbar^2 c^2 \sum_k \partial_k^2)\psi(t,x) - (mc^2)^2\psi(t,x) = 0, \quad (t,x) \in \mathbb{R}^{1+3},$$

with $1+3$ meaning that we have 1 temporal and 3 spatial variables, respectively

$$(-(\tfrac{1}{2\pi}\partial_t)^2 + \sum_k (\tfrac{1}{2\pi}\partial_k)^2)\psi - m^2\psi = 0,$$

if we set $h = c = 1$ and let the variables implicit [60, 69].

While we will not need and further discuss the Klein-Gordon equation, it is related to - and, via quantization, can in fact be derived from - the relativistic energy-momentum relation

$$E^2 - c^2p^2 = (mc^2)^2$$

respectively

$$\pm\sqrt{\nu^2 - k^2} = m,$$

if we use $E = h\nu$, $p = hk$ and finally set $h = c = 1$. This relation defines higher-dimensional hyperbolas - the *mass-shells* - in the energy-momentum (E, p_x, p_y, p_z), respectively the frequency-wavenumber (ν, k_x, k_y, k_z) domain and in fact tells us that a relativistic quantum field of mass m, and thus its quanta, the particles, are constrained to have their Fourier-domain support restricted to this mass-hyperbola.

Remark 5.22. Note that for a particle, in 1+3-dimensional spacetime, having spin and various charges, there are more degrees of freedom that need to be considered and the particle lives in a tensor product space, where only the spatiotemporal degrees of freedom are determined by the Klein-Gordon equation and the other factors are representation spaces of various Lie groups (resp. algebras), like $SU(2)$ for isospin and $SU(3)$ for the color-charge of quarks or

$$G_{SM} := SU(3) \otimes SU(2) \otimes U(1)$$

for the full gauge group of the standard model [76, 88]. ◆

Leaving the physical interpretation aside and restricting to 1+1 dimensions, we will decompose the two-dimensional Fourier domain into hyperbolas via

$$\sigma_{hyp}(x,y) := \left(\log\left(\sqrt{y^2-x^2}\right), \operatorname{artanh}(\tfrac{x}{y})\right)$$

and

$$\sigma_{hyp}^{-1}(m,\theta) := \left(e^m \sinh(\theta), e^m \cosh(\theta)\right),$$

where θ is the hyperbolic angle - sometimes referred to as *rapidity* in special relativity - and sinh, cosh and artanh are the *hyperbolic sine*, the *hyperbolic cosine* and the *area hyperbolic tangent*, respectively.

To illustrate this, we again define the indicator function,

$$\Psi(x,y) := \varphi\left(\log_2\left(\sqrt{y^2-x^2}\right)\right) \cdot \eta\left(\operatorname{artanh}(x/y)\right)$$

with

$$\varphi(m) := \begin{cases} 1 & , \ -1 < m \leq 0 \\ 0 & , \ \text{else} \end{cases} \quad \text{and} \quad \eta(\theta) := \begin{cases} 1 & , \ -\frac{1}{2} < \theta \leq \frac{1}{2} \\ 0 & , \ \text{else} \end{cases}.$$

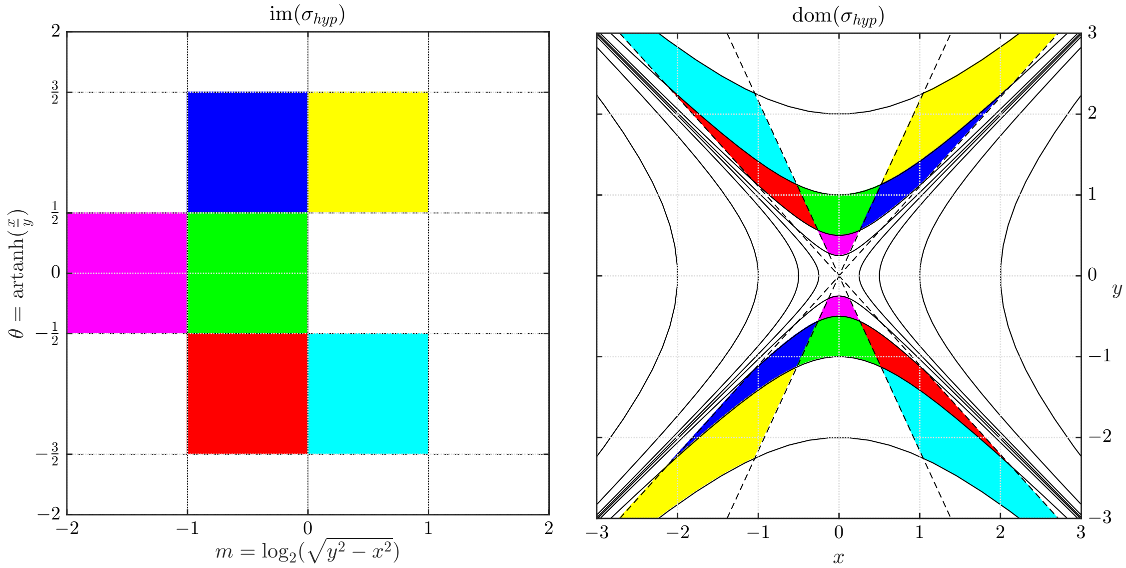
Translating in the image-domain, $\operatorname{im}(\sigma_{hyp})$, and pulling it back to the Cartesian coordinates in the Fourier domain leads to a hyperbolic rotation along with a scaling. This is shown in *Figure 5.16* for six exemplary shifting values, plotted in various colors for identification of corresponding pairs.

The coordinates and the arising diffeomorphism do correspond to the affine Poincaré group $\mathcal{P}_{aff}(1,1) := \mathbb{R}^{1+1} \rtimes (\mathbb{R} \times SO(1,1))$, where $\mathbb{R}^{1+1}$ acts via spatiotemporal translations on itself, and $\mathbb{R}$ and $SO(1,1)$ act via (exponential) scaling and hyperbolic rotations (Lorentz boosts), respectively [1]. This is the fourth case to which *Theorem 5.2 (Spectral Diffeomorphisms for Dual Orbits)* applies. To see this, let $H := \mathbb{R} \times SO(1,1)$, acting as

$$(\widehat{\mathbb{R}}^2, H) \ni (\xi, (m,\theta)) \mapsto e^m \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}^T . \xi \tag{5.185}$$

on $\widehat{\mathbb{R}}^2$ and set $\xi_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then

$$\begin{aligned} \sigma^{-1}(m,\theta) &:= e^m \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}^T \xi_0 \\ &= \begin{pmatrix} e^m \cosh(\theta) & e^m \sinh(\theta) \\ e^m \sinh(\theta) & e^m \cosh(\theta) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^m \sinh(\theta) \\ e^m \cosh(\theta) \end{pmatrix} \end{aligned} \tag{5.186}$$



(a) Image of σ_{hyp}.

(b) Domain of σ_{hyp}.

Figure 5.16: Translation vs. Hyperbolic-Scaling

and we have recovered the diffeomorphism.

The Jacobian and its transposed inverse are given by

$$J_\sigma(\xi_1,\xi_2) = \begin{pmatrix} -\frac{\xi_1}{\xi_2^2-\xi_1^2} & \frac{\xi_2}{\xi_2^2-\xi_1^2} \\ \frac{\xi_2}{\xi_2^2-\xi_1^2} & -\frac{\xi_1}{\xi_2^2-\xi_1^2} \end{pmatrix} \implies \frac{-1}{2\pi i} J_\sigma = \begin{pmatrix} \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^1, \widehat{\widehat{B}}_2\right] \\ \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_1\right] & \left[\widehat{\widehat{A}}^2, \widehat{\widehat{B}}_2\right] \end{pmatrix} \tag{5.187}$$

and

$$J_\sigma^{-T}(\xi_1,\xi_2) = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} \implies \frac{-1}{2\pi i} J_\sigma^{-T} = \begin{pmatrix} \left[\widehat{A}^1, \widehat{B}_1\right] & \left[\widehat{A}^1, \widehat{B}_2\right] \\ \left[\widehat{A}^2, \widehat{B}_1\right] & \left[\widehat{A}^2, \widehat{B}_2\right] \end{pmatrix} \tag{5.188}$$

Thus, the spectral cotangent lift is

$$\Sigma(\xi;x) = (\sigma(\xi); J_\sigma^{-T}(\xi)x) \tag{5.189}$$
$$= (\log(\sqrt{\xi_2^2 - \xi_1^2}), \operatorname{artanh}(\tfrac{\xi_1}{\xi_2}); \xi_1 x_1 + \xi_2 x_2, \xi_2 x_1 + \xi_1 x_2). \tag{5.190}$$

From this, we find that the Hamiltonians

$$A^i(\xi;x) := (J_\sigma^{-T}(\xi)x)^i \quad \text{and} \quad B_j(\xi;x) := \sigma_j(\xi), \tag{5.191}$$

are

$$A^1 := \xi_1 x_1 + \xi_2 x_2 \quad \text{and} \quad A^2 := \xi_2 x_1 + \xi_1 x_2 \tag{5.192}$$

and

$$B_1 := \log(\sqrt{\xi_2^2 - \xi_1^2}) \quad \text{and} \quad B_2 := \operatorname{artanh}(\tfrac{\xi_1}{\xi_2}), \tag{5.193}$$

where the latter induce the warped translations and are thus only interesting for the warped Gabor case. The former two Hamiltonians induce the Hamiltonian vector fields

$$X_{A^1} = [\xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2}] - [x_1\partial_{x_1} + x_2\partial_{x_2}] \tag{5.194}$$

and

$$X_{A^2} = [\xi_1\partial_{\xi_2} + \xi_2\partial_{\xi_1}] + [x_1\partial_{x_2} + x_2\partial_{x_1}], \tag{5.195}$$

where the first is an infinitesimal symplectic flow, circular symmetric "away from the origin" for the ξ coordinates and circular symmetric towards the origin in x coordinates. The second induces a hyperbolic rotation in both the x and ξ coordinates.

Quantizing the Hamiltonians, using the symmetric quantization rule, we get

$$\begin{aligned} \xi_1 x_1 + \xi_2 x_2 = A^1 \ \longmapsto \ \widehat{A}^1 &= \tfrac{1}{2}\left[\left(\xi_1 \tfrac{-1}{2\pi i}\partial_{\xi_1} + \tfrac{-1}{2\pi i}\partial_{\xi_1}\xi_1\right) + \left(\xi_2 \tfrac{-1}{2\pi i}\partial_{\xi_2} + \tfrac{-1}{2\pi i}\partial_{\xi_2}\xi_2\right)\right] \\ &= -\tfrac{1}{2\pi i}\left[\left(\tfrac{1}{2} + \xi_1\partial_{\xi_1}\right) + \left(\tfrac{1}{2} + \xi_2\partial_{\xi_2}\right)\right] \end{aligned} \tag{5.196}$$

and

$$\begin{aligned} \xi_2 x_1 + \xi_1 x_2 = A^2 \ \longmapsto \ \widehat{A}^2 &= \tfrac{1}{2}\left[\left(\xi_2 \tfrac{-1}{2\pi i}\partial_{\xi_1} + \tfrac{-1}{2\pi i}\partial_{\xi_1}\xi_2\right) + \left(\xi_1 \tfrac{-1}{2\pi i}\partial_{\xi_2} + \tfrac{-1}{2\pi i}\partial_{\xi_2}\xi_1\right)\right] \\ &= -\tfrac{1}{2\pi i}\left[\xi_2\partial_{\xi_1} + \xi_1\partial_{\xi_2}\right], \end{aligned} \tag{5.197}$$

where $\widehat{A}^1$, represented on the Fourier domain, is again the infinitesimal generator of unitary dilation in two dimensions,

$$e^{-2\pi i \widehat{A}^1 m} = e^{\left((\frac{1}{2}+\xi_1\partial_{\xi_1}) + (\frac{1}{2}+\xi_2\partial_{\xi_2})\right)m}, \tag{5.198}$$

which induces a two-dimensional unitary dilation on $\mathcal{S}_\sigma$. The Hamiltonian $\widehat{A}^2$ generates a hyperbolic rotation, that is,

$$e^{-2\pi i \widehat{A}^2 \theta} = e^{(\xi_2\partial_{\xi_1} + \xi_1\partial_{\xi_2})\theta}, \tag{5.199}$$

and is unitary on $\mathcal{F}\mathcal{S}_\sigma$.

The associated corresponding warping operator,

$$\mathcal{W}_{\sigma_{hyp}} : L^2(\mathrm{dom}(\sigma_{hyp})) \ \to \ L^2(\mathrm{im}(\sigma_{hyp})), \tag{5.200}$$

on the Fourier domain, acts as

$$\begin{aligned} (\mathcal{W}_{\sigma_{hyp}}\hat{f})(m,\theta) &= \sqrt{\frac{\mathrm{d}\lambda(\sigma_{hyp}^{-1}(m,\theta))}{\mathrm{d}m\ \mathrm{d}\theta}}\,\hat{f}(\sigma_{hyp}^{-1}(m,\theta)) \\ &= \sqrt{\left|\det(J_{\sigma_{hyp}^{-1}})(m,\theta)\right|}\cdot \hat{f}(\sigma_{hyp}^{-1}(m,\theta)) \\ &= e^m \hat{f}(e^m\sinh\theta, e^m\cosh\theta), \end{aligned} \tag{5.201}$$

and its pendant on the space of admissible windows is

$$\widetilde{\mathcal{W}}_{\sigma_{pol}} : \begin{cases} \mathcal{A}_{\sigma_{hyp}} & \to L^2(\mathrm{im}(\sigma_{hyp})) \\ \psi & \mapsto \hat{\psi}(e^m \sinh\theta, e^m \cosh\theta), \end{cases} \tag{5.202}$$

where the equalities hold almost everywhere. Conjugating a translation with this warping operator leads to the associated spectral dilation operator

$$(\widehat{\widetilde{\mathcal{D}}}{}^{\sigma_{hyp}}_{(m,\theta)} \hat{f})(x,y) = \hat{f}\left(e^m H_\theta(x,y)\right), \tag{5.203}$$

where $H_\theta := \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix}$ is the two-dimensional hyperbolic rotation in the Fourier plane. Composing with one of the translation operators and conjugating with the Fourier transform leads to

$$\widetilde{\pi}^{\sigma_{hyp}}(\beta_1, \beta_2; m, \theta) := T_{\vec{\beta}} \widetilde{\mathcal{D}}^{\sigma_{hyp}}_{(m,\theta)}, \tag{5.204}$$

for standard translation and

$$\widetilde{\pi} \circ \Sigma_{hyp}(\beta_1, \beta_2; m, \theta) := \mathcal{T}^{\sigma_{hyp}}_{\vec{\beta}} \widetilde{\mathcal{D}}^{\sigma_{hyp}}_{(m,\theta)}, \tag{5.205}$$

for the warped translation, which gives the following.

Definition 5.23 (The SIM$(1,1)$ transform)**.** Let $f \in \mathcal{S}_{\sigma_{hyp}}$ be real-valued and $\psi \in \mathcal{A}_{\sigma_{hyp}}$, then

$$\begin{aligned} f &\mapsto (\pi^{\sigma_{hyp}}_{\psi} f)(\vec{\beta}, m, \theta) \\ &= \iint_{\mathbb{R}^2} \overline{e^{-2\pi i \langle \vec{\xi}, \vec{\beta} \rangle} \widehat{\psi}\left(e^m(\xi_1 \cosh\theta + \xi_2 \sinh\theta), e^m(\xi_2 \cosh\theta + \xi_1 \sinh\theta)\right)} \\ &\qquad \times \widehat{f}(\xi_1, \xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \\ &= \iint_{\mathbb{R}^2} e^{-2m} f(x,y) \\ &\times \overline{\psi\left(e^{-m}((x-\beta_1)\cosh\theta + (y-\beta_2)\sinh\theta), e^m((y-\beta_2)\cosh\theta + (x-\beta_1)\sinh\theta)\right)} \\ &\times \mathrm{d}x \, \mathrm{d}y \end{aligned} \tag{5.206}$$

is *the SIM(*$1,1$*) transform.* ◆

Definition 5.24 (Hyperbolic-warped STFT in 2D)**.** Let $f \in \mathcal{A}_{\sigma_{hyp}}$ and $\psi \in \mathcal{A}_{\sigma_{hyp}}$, then

$$f \mapsto \left\langle \widetilde{\mathcal{T}}^{\sigma_{hyp}}_{\vec{b}} \widetilde{\mathcal{D}}^{\sigma_{hyp}}_{\alpha} \psi, f \right\rangle = \iint_{\mathbb{R}^2} \overline{e^{2\pi i \langle \vec{\alpha}, \vec{x} - \vec{b} \rangle} \widetilde{\psi}\left(\vec{x} - \vec{b}\right)} \, \widetilde{f}(x) \, \mathrm{d}x, \tag{5.207}$$

with $\widetilde{\psi} := \mathcal{W}_{\sigma_{hyp}} \psi$ and $\widetilde{f} := \mathcal{W}_{\sigma_{hyp}} f$, is the *hyperbolically-warped Short-Time Fourier transform in 2D.* ◆

Localization

Again, the notions of localization differ, the Jacobian is non-diagonal and thus we cannot expect to have perfect equalizers, given as tensor-products of any lower-dimensional sub-solutions. Therefore, the general principle applies and the optimal waveform, for the principle of optimal alignment, has to fulfill the following system of differential equations, with $\xi = (x, y)$, reading

$$\begin{pmatrix} \frac{\partial \widehat{\psi}}{\partial_x} \\ \frac{\partial \widehat{\psi}}{\partial_y} \end{pmatrix} = -2\pi \begin{pmatrix} -\frac{x}{y^2-x^2} & \frac{y}{y^2-x^2} \\ \frac{y}{y^2-x^2} & -\frac{x}{y^2-x^2} \end{pmatrix} \begin{pmatrix} (\mu_x x - \mu_x \beta_x + i\alpha_x)\widehat{\psi} \\ (\mu_y y - \mu_y \beta_y + i\alpha_y)\widehat{\psi} \end{pmatrix}. \tag{5.208}$$

Again, the domain is not simply-connected.

Choosing $\mu := \mu_x = \mu_y$, we get

$$\begin{pmatrix} \frac{\partial \widehat{\psi}}{\partial_x} \\ \frac{\partial \widehat{\psi}}{\partial_y} \end{pmatrix} = -2\pi\mu \begin{pmatrix} -\frac{x}{y^2-x^2} & \frac{y}{y^2-x^2} \\ \frac{y}{y^2-x^2} & -\frac{x}{y^2-x^2} \end{pmatrix} \begin{pmatrix} (x - \beta_x)\widehat{\psi} \\ (y - \beta_y)\widehat{\psi} \end{pmatrix} - 2\pi i \begin{pmatrix} -\frac{x}{y^2-x^2} & \frac{y}{y^2-x^2} \\ \frac{y}{y^2-x^2} & -\frac{x}{y^2-x^2} \end{pmatrix} \begin{pmatrix} \alpha_x \widehat{\psi} \\ \alpha_y \widehat{\psi} \end{pmatrix} \tag{5.209}$$

and thus, the (weighted) Jacobian is again symmetric. Setting $\mu_x = \mu_y = \mu$, $\alpha_x = \alpha_y = 0$, and $\beta_x = 1/2, \beta_y = 0$, leading to a solution, centered at the point $(0, 1/2)$, this system is solved by

$$\begin{aligned} \widehat{\psi}(\xi) &= e^{-2\pi(\int \mu \frac{y^2-x^2}{y^2-x^2} \, \mathrm{d}x + \int \mu \frac{xy-yx}{y^2-x^2} \, \mathrm{d}y)} e^{2\pi\mu \log(\sqrt{y^2-x^2})} \\ &= e^{-2\pi(\int \mu \, \mathrm{d}x)} e^{2\pi\mu \log(\sqrt{y^2-x^2})} \\ &= Ce^{-2\pi\mu x} \sqrt{y^2 - x^2}^{2\pi\mu}, \end{aligned} \tag{5.210}$$

respectively its counterpart

$$\widehat{f}(\xi) = \frac{1}{\sqrt{y^2-x^2}} Ce^{-2\pi\mu x} \sqrt{y^2 - x^2}^{2\pi\mu}, \tag{5.211}$$

on $\mathcal{S}_{\sigma_{hyp}}$.

Again, after setting the constants to zero - which is legitimate since the spectra of the operators are the full line -, the system of differential equations for the principle of optimal concentration is

$$\begin{pmatrix} \frac{\partial \widehat{f}}{\partial_x} \\ \frac{\partial \widehat{f}}{\partial_y} \end{pmatrix} = \begin{pmatrix} -2\pi\mu_x \log(\sqrt{y^2 - y^2})\widehat{f} \\ -2\pi\mu_y \operatorname{artanh}\left(\frac{x}{y}\right)\widehat{f} \end{pmatrix}, \tag{5.212}$$

which, for $\mu_x = \mu_y = \mu$, is solved by the waveform

$$\begin{aligned} \widehat{f}(\xi) &:= e^{-2\pi(\mu_x \int \log(\sqrt{y^2-x^2}) \, \mathrm{d}x + \mu_y \int \operatorname{artanh}(\frac{x}{y}) \, \mathrm{d}y)} \\ &= Ce^{-2\pi\mu(x \log(\sqrt{y^2-x^2}) - x + y \operatorname{artanh}(\frac{x}{y}))} e^{-2\pi\mu(x \log(\sqrt{x^2-y^2}) + y \operatorname{artanh}(\frac{x}{y}))} \\ &= Ce^{-2\pi(x(\log(\sqrt{y^2-x^2}) + \log(\sqrt{x^2-y^2}) - 1) + 2y \operatorname{artanh}(\frac{x}{y}))} \end{aligned} \tag{5.213}$$

respectively its admissible counterpart

$$\widehat{\psi}(\xi) = \sqrt{y^2 - x^2} \cdot Ce^{-2\pi(x(\log(\sqrt{y^2-x^2})+\log(\sqrt{x^2-y^2})-1)+2y\,\mathrm{artanh}(\frac{x}{y}))} \tag{5.214}$$

on $\mathcal{A}_{\sigma_{hyp}}$.

As before, due to the form of the equalizers above, one should stick to the individual equalizers, for which, again, we have

$$\sum_n \partial_{p_n}\left(\frac{\sigma_n^{-1}}{\partial\sigma_i}\right) = \sum_n \partial_{p_n}\left((J^{-T})_n^i\right) = \begin{pmatrix}2\\0\end{pmatrix}. \tag{5.215}$$

Thus, we get the following equalizing waveforms for the principle of optimal alignment:

$$\begin{aligned}\widehat{\psi}_{1,1}(\xi) &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_1}e^{2\pi(\mu\beta-i\alpha)\sigma_1}e^{-\frac{1}{2}\int 2\mathrm{d}\sigma_1}\\ &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_1}e^{2\pi(\mu\beta-i\alpha)\sigma_1}e^{-\sigma_1}\\ &= C\sqrt{y^2-x^2}e^{-2\pi\mu(x\log(\sqrt{y^2-x^2})-y\,\mathrm{artanh}(\frac{x}{y})+x)}e^{2\pi(\mu\beta-i\alpha)\log(\sqrt{y^2-x^2})}\end{aligned} \tag{5.216}$$

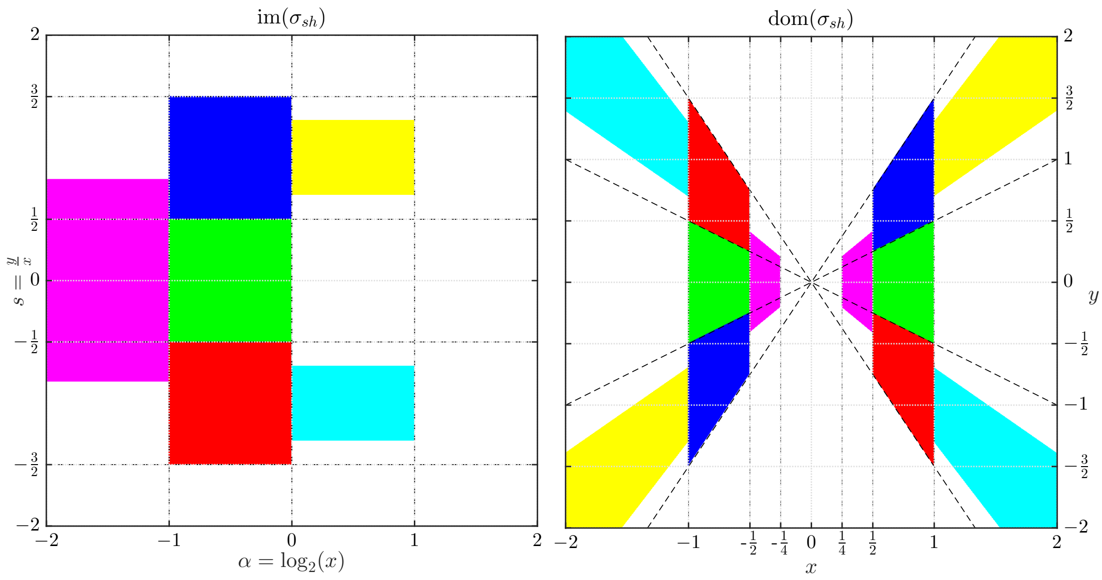
$$\widehat{\psi}_{1,2}(\xi) = C\sqrt{x^2+y^2}e^{-2\pi\mu(y\log(\sqrt{x^2-y^2})-x\,\mathrm{artanh}(\frac{y}{x})+y)}e^{2\pi(\mu\beta-i\alpha)\log(\sqrt{y^2-x^2})} \tag{5.217}$$

$$\begin{aligned}\widehat{\psi}_{2,1}(\xi) &= Ce^{-2\pi\mu\int x\mathrm{d}\sigma_2}e^{2\pi(\mu\beta-i\alpha)(\mathrm{artanh}(x/y))}\\ &= Ce^{-2\pi\mu(x\,\mathrm{artanh}(\frac{y}{x})-y\log(\sqrt{y^2-x^2}))}e^{2\pi(\mu\beta-i\alpha)(\mathrm{artanh}(x/y))}\end{aligned} \tag{5.218}$$

$$\begin{aligned}\widehat{\psi}_{2,2}(\xi) &= Ce^{-2\pi\mu\int y\mathrm{d}\sigma_2}e^{2\pi(\mu\beta-i\alpha)(\mathrm{artanh}(x/y))}\\ &= Ce^{-2\pi\mu(-y\,\mathrm{artanh}(\frac{x}{y})-x\log(\sqrt{y^2-x^2}))}e^{2\pi(\mu\beta-i\alpha)(\mathrm{artanh}(x/y))},\end{aligned} \tag{5.219}$$

all on $\mathcal{S}_\sigma$ and the usual alterations apply to find the associated functions on $\mathcal{S}_\sigma$.

For the principle of optimal concentration, we jump again straight to the



(a) Image of σ_{sh}.   **(b)** Domain of σ_{sh}.

Figure 5.17: Parabolic Translation vs. Parabolic Shear-Scaling

formulas:

$$
\begin{aligned}
\widehat{f_{1,1}}(x,y) &= Ce^{-2\pi\mu \int \log\sqrt{y^2-x^2}\ \mathrm{d}x} e^{2\pi x(\mu\alpha - i\beta)} \\
&= Ce^{-2\pi\mu(x(\log\sqrt{y^2-x^2}-1)+y\,\mathrm{artanh}(x/y))} e^{2\pi x(\mu\alpha - i\beta)}
\end{aligned} \tag{5.220}
$$

$$
\begin{aligned}
\widehat{f_{1,2}}(x,y) &= Ce^{-2\pi\mu \int \mathrm{artanh}(x/y)\ \mathrm{d}x} e^{2\pi x(\mu\alpha - i\beta)} \\
&= Ce^{-2\pi\mu(y\log\sqrt{y^2-x^2}+x\,\mathrm{artanh}(x/y))} e^{2\pi x(\mu\alpha - i\beta)}
\end{aligned} \tag{5.221}
$$

$$
\begin{aligned}
\widehat{f_{2,1}}(x,y) &= Ce^{-2\pi\mu \int \log\sqrt{y^2-x^2}\ \mathrm{d}y} e^{2\pi y(\mu\alpha - i\beta)} \\
&= Ce^{-2\pi\mu(y(\log\sqrt{y^2-x^2}-1)+x\,\mathrm{artanh}(y/x))} e^{2\pi y(\mu\alpha - i\beta)}
\end{aligned} \tag{5.222}
$$

$$
\begin{aligned}
\widehat{f_{2,2}}(x,y) &= Ce^{-2\pi\mu \int \mathrm{artanh}(x/y)\ \mathrm{d}y} e^{2\pi y(\mu\alpha - i\beta)} \\
&= Ce^{-2\pi\mu(x\log\sqrt{x^2-y^2}+y\,\mathrm{artanh}(x/y))} e^{2\pi x(\mu\alpha - i\beta)}\ .
\end{aligned} \tag{5.223}
$$

◆

5.2.5 A Non-Abelian Generalization

Above, we noted that the (non-parabolic) shearlet transform has a pendant for which the abelian translation in the warped domain is replaced by a non-commutative one. In order for the arising integrals to be invariant as in the abelian case, the warped domain needs to be equipped with a measure, invariant under this non-commutative translation, but it is possible to take care of this, by lifting the action to a unitary translation, having a normalizing constant, defined via a Radon-Nikodym derivative. We shall only consider the two-dimensional case for the shearlet transform, in which translation is defined as

$$\lambda_{\alpha',s'} : (\alpha, s) \mapsto (\alpha + \alpha', e^{-\alpha'/2}(s + s')). \tag{5.224}$$

It is straight-forward to check that a composition with σ_{sh} and its inverse gives

$$(x, y) \mapsto (e^{\alpha'} x, e^{\alpha'/2}(y + xs')) \tag{5.225}$$

and lifting this action to a unitary operator gives

$$\psi \mapsto e^{-\alpha/4} \psi \circ \sigma_{sh}^{-1} \circ \lambda_{\alpha',s'} \circ \sigma_{sh}. \tag{5.226}$$

As in the non-parabolic case, we define a rectangular window

$$\Psi(x, y) \coloneqq \varphi(\log_2(x)) \cdot \eta(y/x)$$

with

$$\varphi(\alpha) \coloneqq \begin{cases} 1 & , -1 < \alpha \leq 0 \\ 0 & , \text{else} \end{cases} \quad \text{and} \quad \eta(s) \coloneqq \begin{cases} 1 & , -\frac{1}{2} < s \leq \frac{1}{2} \\ 0 & , \text{else} \end{cases},$$

to illustrate this. Parabolically shifting this function in the image-domain, $\mathrm{im}(\sigma_{sh})$, corresponds to parabolic scaled shearing in Cartesian coordinates. An illustration of this is given by *Figure 5.17*, again exemplary for six shifting values, plotted in various colors for identification of corresponding pairs.

The following characterizes the arising (parabolic) Shearlet Transform.

Theorem 5.25 (Parabolic Shearlet Transform). *Let $f \in \mathcal{S}_{\sigma_{sh}}$ be real-valued and $\psi \in \mathcal{A}_{\sigma_{sh}}$. Let moreover*

$$\sigma_{sh}(x, y) \coloneqq (\log(x), y/x) \tag{5.227}$$

be the shear-diffeomorphism. Let furthermore

$$\lambda_{\alpha',s'} : (\alpha, s) \mapsto (\alpha + \alpha', e^{-\alpha'/2}(s + s')) \tag{5.228}$$

be non-abelian translation and let $\mathrm{d}\nu_{sh} \coloneqq \mathrm{d}\alpha \mathrm{d}s$ *denote the standard measure, with* $\mathrm{d}\sigma \coloneqq \mathrm{d}\nu_{sh} \circ \sigma$ *denoting its pullback measure,* Definition A.18 (Pullback of a measure), *to* $\mathrm{dom}(\sigma)$.

Then,

$$
\begin{aligned}
f &\mapsto \left\langle e^{-\alpha/4}\psi_{\alpha,s}\,,\, f\right\rangle \\
&= \iint_{\mathrm{dom}(\sigma_{sh})} \overline{e^{-2\pi i\langle \vec{\xi}\,,\,\vec{\beta}\,\rangle} e^{-\alpha/4}\widehat{\psi}\left(\sigma^{-1}(\lambda_{\alpha,s}(\sigma(\xi)))\right)} \widehat{f}(\xi_1,\xi_2)\ \mathrm{d}\xi_1\ \mathrm{d}\xi_2 \\
&= \iint_{\mathrm{dom}(\sigma_{sh})} \overline{e^{-2\pi i\langle \vec{\xi}\,,\,\vec{\beta}\,\rangle} e^{-\alpha/4}\widehat{\psi}\left(e^{\alpha}\xi_1, e^{\alpha/2}(\xi_2 + s\xi_1)\right)} \widehat{f}(\xi_1,\xi_2)\ \mathrm{d}\xi_1\ \mathrm{d}\xi_2,
\end{aligned}
\tag{5.229}
$$

the parabolic shearlet transform, *is a multiple of an isometry, that is,*

$$
\left\|\left\langle e^{-\alpha/4}\psi_{\alpha,s}\,,\, f\right\rangle\right\|^2_{L^2(\mathbb{R}^2\times\mathrm{im}(\sigma),\mathrm{d}x\mathrm{d}\nu_{sh})} = \|\psi\|^2_{\mathcal{A}_\sigma}\,\|f\|^2_{\mathcal{S}_\sigma}\,. \tag{5.230}
$$

◆

Proof. Only the claim about its isometric property needs to be proven, as the rest are definitions. Eventually dropping the integration domains to avoid clutter, we have

$$
\begin{aligned}
&\left\|\left\langle e^{-\alpha/4}\psi_{\alpha,s}\,,\, f\right\rangle\right\|^2_{\mathcal{H}_\sigma} \\
&:= \iiiint_{\mathbb{R}^2\times\mathrm{im}(\sigma)} \left|\left\langle e^{-\alpha/4}\psi_{\alpha,s}\,,\, f\right\rangle\right|^2\ \mathrm{d}b\ \mathrm{d}\nu_{sh} \\
&= \iiiint_{\mathbb{R}^2\times\mathrm{im}(\sigma)} \int_{\mathrm{dom}(\sigma)} \int_{\mathrm{dom}(\sigma)} e^{-\alpha/4}\widehat{\psi}\left(\sigma^{-1}(\lambda_{\alpha,s}(\sigma(\xi)))\right) \\
&\times \overline{e^{-\alpha/4}\widehat{\psi}\left(\sigma^{-1}(\lambda_{\alpha,s}(\sigma(\xi)))\right)\widehat{f}(\xi)}\widehat{f}(\xi')e^{-2\pi i\langle\,\xi-\xi'\,,\,b\,\rangle}\ \mathrm{d}\xi\ \mathrm{d}\xi'\ \mathrm{d}b\ \mathrm{d}\nu_{sh} \\
&= \iiiint \left|e^{-\alpha/4}\widehat{\psi}\left(\sigma^{-1}(\lambda_{\alpha,s}(\sigma(\xi)))\right)\right|^2 \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi\ \mathrm{d}\nu_{sh} \\
&= \iiiint e^{-\alpha/2}\left|\widehat{\psi}\left(\sigma^{-1}(\log(x)+\alpha, e^{-\alpha/2}(\tfrac{y}{x}+s))\right)\right|^2 \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi\mathrm{d}\alpha\mathrm{d}s \\
&= \iiiint e^{-\alpha/2}\left|\widehat{\psi}\left(\sigma^{-1}(\alpha, e^{-\alpha/2}s)\right)\right|^2 \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi\mathrm{d}\alpha\mathrm{d}s \\
&= \iiiint e^{-\alpha/2}\left|\widehat{\psi}\left(\sigma^{-1}(\alpha, s)\right)\right|^2 \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi e^{+\alpha/2}\mathrm{d}\alpha\mathrm{d}s \\
&= \iiiint \left|\widehat{\psi}\left(\sigma^{-1}(\alpha, s)\right)\right|^2 \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi\mathrm{d}\alpha\mathrm{d}s \\
&= \iint \left|\widehat{\psi}\left(\sigma^{-1}(\alpha, s)\right)\right|^2 \mathrm{d}\alpha\mathrm{d}s \int \left|\widehat{f}(\xi)\right|^2\ \mathrm{d}\xi \\
&= \iint \left|\widehat{\psi}\,(\xi)\right|^2 \mathrm{d}\sigma\,\|f\|^2_{\mathcal{S}_\sigma} \\
&= \|\psi\|^2_{\mathcal{A}_\sigma}\,\|f\|^2_{\mathcal{S}_\sigma}\,,
\end{aligned}
\tag{5.231}
$$

which was the claim. ■

Milk is for babies. When you grow up,
you have to drink beer.

— Arnold Schwarzenegger

6 Termination

HEREAFTER, as the program of spectral diffeomorphisms and its applications has been finally laid out, a short recapitulation of this thesis' main contents shall be presented.
Afterwards, to round out this monograph, a glance at possible extensions and enhancements of this program are given.

6.1 Conclusion

In this thesis, a theory of so-called spectral diffeomorphism has been elaborated. Starting from a diffeomorphism on the dual of the Euclidean space $\mathbb{R}^n$, a *spectral cotangent lift* assigns to this diffeomorphism a specific symplectomorphism on phase space $\mathbb{R}^n \times \widehat{\mathbb{R}}^n$, to which in turn a canonical coordinate system is assigned. These coordinates are used to define flows on phase space by means of which the phase space picture of a prototype function is shifted along the coordinate lines, such that each point of phase space is eventually reached by the shifted phase space picture of the prototype function.

Since the uncertainty principle restricts the amount to which the phase space picture of functions can be concentrated, with each such function is associated a "phase space cell". By integrating a signal against a prototype function, we get a number, interpreted as a quantum of information of that very function, characterizing the signal's content within the associated phase space cell. Using these phase space translates of such a probe, the notion of a quantum frame, associated with a classical frame in phase space is defined, by means of which a reservoir of interesting functions may be decomposed and - using a weighting function in phase space - altered. Since the phase space cells are images of symplectomorphism, the initial form of the phase space cell, assigned to the template function, is of relevance.

To define these, two complementing uncertainty principles - the "duo ottimale" -, associated with coordinate systems in phase space, are introduced, by means of which the specific form of the phase space cell can be optimized. The two principles are complementary in the sense that one of it - in this thesis referred to as *the principle of optimal alignment* - measures the deviation from the chosen coordinate lines and thus leads to waveforms, "snuggling" with the coordinate lines. The other principle, *the principle of optimal concentration*, optimizes with respect to the canonically conjugate coordinates and leads to more "concentrated" waveforms. Both uncertainty principles assign to the quantized Hamiltonians of the coordinate functions, respectively their canonically conjugate variables, an inequality. The lower bounds of these inequalities are dependent on the *Jacobian matrix*, respectively its inverse, of the spectral diffeomorphism.

Just as for the "classical uncertainty principle" for multi-dimensional configuration spaces assigned to (pairs of) generators of a Lie algebra, there do not necessarily exist waveforms, simultaneously optimizing all of the individual principles. There are, however, special cases - those, for which the Jacobian matrix is diagonal. In these special cases, the system of differential equations decouple and the principles restrict to factors of a tensor-product. Thus, the tensor product of the optimal waveforms for each principle is an optimal waveform for all principles simultaneously.

Finally, some examples are presented to exert the dull theory of spectral diffeomorphisms. The diffeomorphisms considered include, but are not limited to,

(i) the logarithm, giving rise to the wavelet transform,

(ii) the log-polar diffeomorphism, associated with the $SIM(2)$ transform,

(iii) the log-hyperbolic diffeomorphism, associated with the affine Poincare group in 1+1 dimensions, and

(iv) the log-shear diffeomorphism, giving rise to a non-parabolic variant of the shearlet transform.

6.2 Advancements

The obvious points to generalize the program of this thesis are

(i) the rather straight-forward generalization from $\mathbb{R}^n$ and its associated cotangent bundle - its phase space - to locally compact abelian groups and the associated Pontryagin duality, *Definition A.30 (Pontryagin duality)*, and its generalized Fourier transform, *Definition A.48 (Generalized Fourier transform)*, a.k.a., the Gelfand transform, and

(ii) the generalization of the affine uncertainty principle via a confirmation or a rejection of *Conjecture 4.35 on page 121.*

Further, the program may be generalized by not restricting the coordinate systems to those arising via cotangent lifts of a spectral diffeomorphism, but using arbitrary ones. This abstraction could shed more light on the merits and demerits of this program.

As far as practical applications are concerned, the appendix contains information on implementations of linear signal transforms, arising as generalized coherent state maps of spectral diffeomorphisms and it is these sample implementations which could be further refined, extended or put to use in fields of application not consider by, or going over the head of, the author.

Finally, it still seems to be an open problem whether there exist differential systems such that its - unique, normalized - solution is a simultaneous equalizer for a multi-dimensional uncertainty principle, associated with a coordinate system in phase space and hence a lot of work can be done on this compelling subject.

Never memorize something that you can look up.

— (attributed to) Albert Einstein

A
Mathematical Preliminaries

SCIENCE IS A CONSTRUCT, built upon the shoulders of giants and to vast for mortals to be conceivable in its entirety. For that very reason, it is vital to learn from those who have looked further ahead then oneself. In this regard, it is only natural to depend on various sources, serving as a point of departure for research.

Recommendable references are [43, 55] for differential geometry and its applications, [15, 61] for classical mechanics and symplectic geometry and [30, 64, 69] are good references for quantization and quantum mechanics in general. Further, [3, 30, 31, 44, 54, 72, 85] are highly recommendable treaties of functional analysis, topology, (linear) algebra and its applications and [13, 26, 27, 34, 39] for time-frequency analysis in particular.

A.1 Topology and Measure Theory

Definition A.1 (Topology)**.** A topology for a set X is a set, $\mathcal{T}$, consisting of subsets of X, referred to as *open sets*, such that

(i) $\varnothing, X \in \mathcal{T}$,

(ii) $(E_k)_{k\leq n} \subset \mathcal{T} \;\Rightarrow\; \bigcap_{k=1}^{n} E_k \in \mathcal{T}$,

(iii) $(E_k)_{k\in\mathbb{N}} \subset \mathcal{T} \;\Rightarrow\; \bigcup_{k\in\mathbb{N}} E_k \in \mathcal{T}$,

that is, it is a family of subsets containing the empty set and X itself as well as all finite intersections and countable unions. A set, which is the complement of an open set is defined to be a *closed set* and a set, along with a topology defined on it, is referred to as a *topological space*.

Just like the term *space* is borrowed from geometry, since now an abstract idea of closeness is at hand, the elements $x \in X$ of the space are referred to as *points*. Whenever it is clear from the context, the reference to the topology $\mathcal{T}$ is dropped and X is said to be the topological space. ◆

Definition A.2 (Nets and Convergence)**.** Let $(X, \mathcal{T}_X)$ be a topological space and let $(x_i)_{i \in I} \subseteq X$ be a *net*, that is, a subset of X which is parametrized by a set I, having a total ordering, $\leq$, that is,

$$\forall i, j \in I \text{ either } i \leq j \text{ or } j \leq i. \tag{A.1}$$

We may thus speak of limits and say that $(x_i)_i$ *converges* to x, written

$$x_i \to x \quad \text{or} \quad \lim_{i \to \infty} x_i = x \in X, \tag{A.2}$$

if for all open sets $O \in \mathcal{T}_X$ containing x, there exists an index $N_O \in I$, such that $x_i \in O$ for all $i > N_O$, to wit

$$x_i \to x \iff \forall \mathcal{T}_X \ni O \ni x \; \exists N_O \in I \text{ s. t.} \, x_i \in \mathcal{T}_X \quad \forall i > N_O. \tag{A.3}$$

Common domains are $I := \mathbb{R}$ and $I := \mathbb{N}$, where in the latter case one speaks of *converging sequences.* ◆

Definition A.3 (Countability)**.** Let $(X, \mathcal{T}_X)$ be a topological space. Then, X is said to be

(i) *first countable* if, for all $x \in X$, there exists a *countable* subset of its topology $B \subseteq \mathcal{T}_X$, such that each neighborhood N of x contains at least one set of B, that is,

$$\forall x \in X, \; \exists B_x \subseteq \mathcal{T}_X \quad \text{s. t.} \; \forall N \subseteq X, \; x \in N, \; \exists U \in B_x : U \subseteq N \tag{A.4}$$

(ii) *second countable* if there exists one *countable* subset of its topology $B \subseteq \mathcal{T}_X$ such that, every open set of X can be written as a union of elements of B, that is, if the topology has a countable basis.

Note that second countability implies first countability but not vice versa. ◆

Definition A.4 (Separability)**.** A topological space $(X, \mathcal{T}_X)$ is said to be

(i) T_0, or *Kolmogorov*, if each pair of points $x, y \in X$ are *topologically distinguishable*, meaning that there exists open sets, containing x but not y et vice versa.

(ii) T_2, or *Hausdorff*, if it is T_0 and each pair of in-equivalent points can be separated by *disjoint* open neighborhoods, that is,

$$\forall x, y \in X, x \neq y \ \exists N_x, N_y \in T_X \ \text{ s. t. } \ x \in N_x, \ y \in N_y \text{ and } N_x \cap N_y = \varnothing. \quad \text{(A.5)}$$

The importance of the first axiom is that if a space is not T_0, then there are points that cannot be distinguished from each other so that, from a topological point of view, they are identical. The Hausdorff axiom guarantees that limits of converging sequences are unique and thus that topological problems may be tackled by means of convergence of nets and sequences. ◆

Definition A.5 (Continuous and open mappings)**.** A mapping between two topological spaces, $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, is

(i) *continuous* if the inverse image of every open set is open

$$O \in \mathcal{T}_Y \implies f^{-1}(O) \in \mathcal{T}_X \quad \text{(A.6)}$$

(ii) *open* if the image of every open set is open

$$O \in \mathcal{T}_X \implies f(O) \in \mathcal{T}_Y. \quad \text{(A.7)}$$

By a slight abuse of language, we may say that f is *continuous* if and only if $\widetilde{f} : \mathcal{T}_X \to \mathcal{T}_Y$ is a“topological surjection” and *open* if and only if $\widetilde{f} : \mathcal{T}_X \to \mathcal{T}_Y$ is a “topological injection”. One speaks of a homeomorphism, if f is

(iii) continuous, open and bijective.

As is always the case with isomorphisms, from a topological point of view, spaces being homeomorphic to each other may be considered one and the same space and thus may be identified. ◆

Definition A.6 ((Local) Compactness)**.** Let $(X, \mathcal{T}_X)$ be a topological space. Then, a set $E \subseteq X$ is said to be

(i) *compact*, if every open cover has a finite subcover

$$(O_k)_{k \in \mathbb{N}} \subset \mathcal{T}_X, \ \bigcup_{k \in \mathbb{N}} O_k \supseteq E \implies \exists (k_i)_{i \in I}, \ |I| \in \mathbb{N} \ : \ \bigcup_{i \in I} O_{k_i} \supseteq E \quad \text{(A.8)}$$

(ii) *locally compact*, if every point of it has a compact neighborhood. ◆

Definition A.7 (Equivalence relation)**.** Let E be a set and $x, y, z \in E$. A relation $R \subset E \times E$, written $xRy =: x \sim y$, is an *equivalence* relation, if it is

(i) *reflexive*: $x \sim x$,

(ii) *transitive*: $x \sim y \wedge y \sim z \;\Rightarrow\; x \sim z$

(iii) *symmetric*: $x \sim y \;\Leftrightarrow\; y \sim x$.

The universe of all elements that are equivalent is referred to as an *equivalence class* and under the equivalence relation $\sim$, the set E partitions into a set of equivalence classes $E/\sim$, decomposing E into disjoint sets. The quotient map

$$q : E \to E/\sim,\; x \mapsto q(x) =: [x]_\sim \tag{A.9}$$

is therefore a surjection, identifying all elements of E that are equivalent with respect to $\sim$. ◆

Definition A.8 (σ-algebra and measurable spaces)**.** Let X be a set and Σ a family of subsets of X. Then, Σ is a *σ-algebra* if and only if

(i) $\varnothing, X \in \Sigma$,

(ii) $(E_k)_{k\in\mathbb{N}} \subset \Sigma \;\Rightarrow\; \bigcap_{k\in\mathbb{N}} E_k \in \Sigma$,

(iii) $(E_k)_{k\in\mathbb{N}} \subset \Sigma \;\Rightarrow\; \bigcup_{k\in\mathbb{N}} E_k \in \Sigma$,

(iv) $E \in \Sigma \;\Rightarrow\; E^c \in \Sigma$.

The set, along with a σ-algebra, is referred to as a *measurable space* (X, Σ) and the sets $E \in \Sigma$ as *measurable sets.* ◆

Measurable mappings are now defined to be those mappings between measurable spaces that preserve the measurable structure.

Definition A.9 (Measurable mappings)**.** A mapping

$$f : (X, \Sigma_X) \longrightarrow (Y, \Sigma_Y) \tag{A.10}$$

between measurable spaces is a *measurable mapping* if and only if the inverse image of every measurable subset of Y is a measurable subset of X, that is, iff

$$E \in \Sigma_Y \;\Rightarrow\; f^{-1}(E) \in \Sigma_X. \tag{A.11}$$

By a slight abuse of language, we may say that f is *measurable* if and only if $\widetilde{f} : \Sigma_X \to \Sigma_Y$ is a "measurable surjection" and *inverse measurable* if and only if $\widetilde{f} : \Sigma_X \to \Sigma_Y$ is a "measurable injection". One moreover speaks of a *measure-isomorphism*, if f is bijective and bi-measurable, i.e., measurable with measurable inverse.

As always with isomorphisms, measurable spaces that are measure-isomorphic to each other may be considered one and the same space, at least from a measure-theoretic point of view, and thus may be identified. ◆

Definition A.10 (Measure)**.** Let (X, Σ) be a measurable space. A mapping

$$\mu : \Sigma \longrightarrow \mathbb{R}_+ \cup \{\infty\},\ E \longmapsto \mu(E), \tag{A.12}$$

is a *measure*, if and only if

(i) the empty set has zero measure

$$\mu(\varnothing) = 0,$$

(ii) measurable sets have non-negative measure

$$E \in \Sigma \;\Rightarrow\; \mu(E) \geq 0,$$

(iii) and the measure is σ-additive, or countably additive, if

$$(E_k)_{k\in\mathbb{N}} \subset \Sigma,\ E_k \text{ pairwise disjoint} \;\Rightarrow\; \mu(\bigcup_{k\in\mathbb{N}} E_k) = \bigcup_{k\in\mathbb{N}} \mu(E_k).$$

If the measure maps to more general sets, it is renamed appropriately, like *signed, complex, vector-valued or operator-valued* measure. ◆

Definition A.11 (Borel sets)**.** Let X be a topological space. Then, the smallest σ-algebra, containing the open sets of X, is called its *Borel σ-algebra*, $\mathcal{B}$, and the measurable subsets are named *Borel sets.* Thus, the Borel sets are generated by the open sets by completing the set of open sets with countable sections and complements. The tuple $(X, \mathcal{B})$ is referred to as a *Borel space.* ◆

Definition A.12 (Borel measure)**.** A measure, defined on the Borel σ-algebra of some topological space is a *Borel measure.* ◆

Definition A.13 (Radon measure)**.** Let μ be a Borel measure on the measurable space (X, Σ).

(i) It is *inner regular* if measurable sets can be approximated by compact ones, that is, if

$$\mu(E) = \sup\{\, \mu(C) \,|\, C \subseteq E \text{ and } C \text{ is compact} \,\}, \tag{A.13}$$

for all $E \in \Sigma$.

(ii) It is *locally finite* if for all $x \in X$ there exists an open neighborhood $N_x \ni x$, such that $\mu(N_x) < \infty$.

A locally finite and inner regular Borel measure is referred to as a *Radon measure.* ◆

Definition A.14 (Absolute continuity of measures)**.** Let (X, Σ) be a measurable space and ν and μ measures on it. Then, ν is said to be absolutely continuous with respect to μ, written $\nu << \mu$, if and only if each zero-set of μ is also a zero-set of ν. That is, denoting the corresponding sets of zero-sets by Z_ν and Z_μ, we have

$$\nu << \mu \quad \Leftrightarrow \quad Z_\nu \supseteq Z_\mu. \tag{A.14}$$

◆

Since the fact that each zero-set of μ is also a zero-set of ν does not exclude the case that each zero-set of ν is also a zero-set of μ, it may very well be the case that both zero-sets coincide, which turns μ and ν into equivalent measures.

Definition A.15 (Equivalence of measures)**.** As before, let (X, Σ) be a measurable space and ν and μ measures on it. Then, ν and μ are said to be equivalent, $\nu \sim \mu$, if and only if the zero sets Z_ν and Z_μ coincide. That is, each zero-set of μ is also a zero-set of ν and vice versa

$$\nu \sim \mu \quad \Leftrightarrow \quad \mu << \nu << \mu \quad \Leftrightarrow \quad \nu << \mu << \nu \quad \Leftrightarrow \quad Z_\nu = Z_\mu. \tag{A.15}$$

It is easy to check that this is an equivalence relation and thus we may partition the set of all measures on (X, Σ) into equivalence classes, consisting of measures with identical zero-sets. ◆

Definition A.16 (Radon-Nikodym)**.** Let (X, Σ, μ) be a measure space, with σ-algebra Σ, and let μ be absolutely continuous with respect to another measure ν, that is, every set having zero-measure with respect to ν has zero-measure with respect to μ, usually written $\mu \ll \nu$. Then, the Radon-Nikodym theorem assures the existence of a ν-measurable density function Δ, such that

$$\mu(\Gamma) = \int_\Gamma \Delta(x) \ \ \mathrm{d}\nu(x), \tag{A.16}$$

for all measurable $\Gamma \in \Sigma$. Note that (i) both measures need to be defined on the same σ-algebraand (ii) that it is (pointwise) invertible if the measures are moreover equivalent. ◆

Definition A.17 (Push-forward of a measure)**.** Let (X, Σ_X, μ) be a measure space, (Y, Σ_Y) a measurable space and $\sigma : X \to Y$ a *measurable* mapping from X to Y. Then, the following holds:

(i) for every *measurable* set $A \in \Sigma_Y$, we have $\sigma^{-1}(A) \in \Sigma_X$, and

(ii) the pair (σ, μ) induces a *push-forward measure*

$$\nu := \mu \circ \sigma^{-1} : \Sigma_Y \to \mathbb{R}_+ \tag{A.17}$$

on Y, turning it into the measure space (Y, Σ_Y, ν). ◆

Definition A.18 (Pullback of a measure)**.** Let (X, Σ_X) be a measurable space, (Y, Σ_Y, ν) a measure space and $\sigma : X \to Y$ a mapping from X to Y, having a *measurable inverse*. Then, the following holds:

(i) for every *measurable* set $A \in \Sigma_X$, we have $\sigma(A) \in \Sigma_Y$, and

(ii) the pair (σ, ν) induces a *pullback measure*

$$\mu := \nu \circ \sigma : \Sigma_X \to \mathbb{R}_+ \tag{A.18}$$

on X, turning it into the measure space (X, Σ_X, μ). ◆

Definition A.19 (Lebesgue measure)**.** A *Lebesgue measure* is a "translation invariant" Radon measure on the locally compact group $\mathbb{R}^n$ and thus an example of a *Haar measure* from *Theorem A.61.* ◆

A.2 Algebra

Definition A.20 (Group)**.** Let X be a set, equipped with a binary operation $\circ : (x, y) \mapsto x \circ y$, such that

(i) $\circ$ is *closed*, $\circ : G \times G \to G$,

(ii) $\circ$ is *associative*, $(a \circ b) \circ c = a \circ (b \circ c) = a \circ b \circ c$,

(iii) X has an *identity*, $e \circ g = g \circ e = g$

(iv) each $g \in X$ has an *inverse*, $g^{-1}g = gg^{-1} = e$,

then $(X, \circ) =: X$ is a *group*. If the group multiplication is *commutative*, it is said to be an *abelian group*.

If the group is moreover equipped with a topology such that the *multiplication* and *inversion* are *continuous* maps, the group is a *topological group* and the product $G \times G$ is equipped with the product topology. ◆

Definition A.21 (Topological groups)**.** Let G be a topological group with $\mathcal{T}_G$ denoting its topology. If $\mathcal{T}_G$ is (locally) compact, first or second countable, Hausdorff, etc., the group G is said to be a (locally) compact, first or second countable, Hausdorff, etc. topological group. ◆

Definition A.22 (Group action)**.** Let G denote a topological group, X a topological space G acts on and $. : G \times X \to X, (g, x) \mapsto g.x$ the associated *continuous* group action. Then, the action is tagged to be

- *transitive*, if for all $x, y \in X$, there exists a $g \in G$ such that $g.x = y$,
- *faithful*, if for all $g, h \in G$, there exists a $x \in X$ such that $g.x \neq h.x$,
- *free*, if the stabilizer of each $x \in X$ is trivial, to wit, if the mapping $G \to X, g \mapsto g.x$ is injective, for all $x \in X$. ◆

If the action of the group is restricted to a specific element $x \in X$, the question arises what elements of X may be "reached" by the action of the group on the element x.

Definition A.23 (Orbits of group actions)**.** Let G act on X and $x \in X$, then

$$G.x := \{ y \in X \mid y = g.x, g \in G \} \tag{A.19}$$

is the *orbit* of x under G.

An orbit $\mathcal{O} \subseteq X$ is said to be

(i) *free*, whenever the corresponding group action is free, and

(ii) *open*, if the orbit is open in the topology of X. ◆

An important property of group actions and their orbits is that these decompose a G-space X into *disjoint orbits*.

Proposition A.24 (Partitions induced by group actions)**.** *Let G act on X as $x \mapsto g.x$, then X partitions into G-orbits. That is, the* orbit space *$G\backslash X$, or $X\backslash G$, decomposes X, such that*

$$\forall \mathcal{O} \in G\backslash X,\ \mathcal{O} \text{ is a an orbit} \qquad \text{and} \qquad \bigcup_{\mathcal{O} \in G\backslash X} \mathcal{O} = X\ . \tag{A.20}$$

Note that this defines an equivalence relation

$$x \sim y \iff y \in G.x \text{ and } x \in G.y\ . \tag{A.21}$$

◆

Definition A.25 (Group homomorphisms)**.** Let G and H be groups and $g, g_1, g_2 \in G$. A mapping $\sigma : G \to H,\ g \mapsto \sigma(g)$, with

$$\sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2) \quad \text{and} \quad \sigma(g^{-1}) = \sigma(g)^{-1} \tag{A.22}$$

is a *group homomorphism.* Thus, a group homomorphism respects the algebraic structure of the group and maps groups into homomorphic groups. ◆

As is the case for general mappings between sets, there is an important special case of homomorphisms, identifying two groups as a whole, rendering both - at least from an algebraic point of view - indistinguishable.

Definition A.26 (Group isomorphism)**.** A *group isomorphism* is a homomorphism, which is bijective.

Since the identical mapping is trivially a group isomorphism and since isomorphisms are invertible, being isomorphic is reflexive and symmetric. Moreover, since it is clear that if G is isomorphic to H and H is isomorphic to K, then also G is isomorphic to K, making it transitive. Being isomorphic is therefore an equivalence relation, partitioning all groups into equivalence classes of isomorphic ones. ◆

Definition A.27 (Characters)**.** Homomorphisms, $\sigma : G \to \mathbb{C}$, from a group G into the multiplicative group of complex numbers of modulus one, are referred to as *characters*. ◆

Those characters are interesting in their own right, since these are groups, too.

Theorem A.28 (Group of Characters)**.** *The set of all characters $\xi : G \to \mathbb{C}$ from G to the multiplicative group of complex numbers of modulus one is a* group. ◆

Proof. Let $\sigma, \xi : G \to \mathbb{C}$ be arbitrary characters from the group G to the torus. Then, since the torus is a multiplicative sub-group of the complex numbers

$$x \mapsto \xi(x) \cdot \sigma(x), \quad x \in G, \tag{A.23}$$

defines another character, since the product is again an element of the torus. Thus

$$\rho(x) := (\xi \circ \sigma)(x) := \xi(x) \cdot \sigma(x), \quad x \in G, \tag{A.24}$$

is a character. Moreover, the inverse $(\xi(x))^{-1}$ exists and is again a value of the torus, turning $\xi^{-1}(x) := (\xi(x))^{-1}$ into a character, too. Finally, since

$$(\xi^{-1} \circ \xi)(x) = (\xi(x))^{-1} \cdot \xi(x) = 1, \quad x \in G, \tag{A.25}$$

ξ^{-1} is the inverse of ξ and

$$e(x) := (\xi^{-1} \cdot \xi)(x), \quad x \in G, \tag{A.26}$$

is the group identity.

Note that the *modular function* of a locally compact group is necessarily a *character* of G. ■

Definition A.29 (Dual group)**.** Let G be a locally compact group, then the group of characters are dubbed as its *dual group* $\widehat{G}$. ◆

It is a well-known fact that a locally compact abelian groups G is its own "double dual". This means, that $\widehat{G}$ is the dual group of G and $\widehat{\widehat{G}} := G$ is the dual of its dual. This nexus became known as the

Definition A.30 (Pontryagin duality)**.** Let G be a locally compact abelian group. Then $\widehat{G}$ is its dual group, which is again a locally compact abelian group, and G itself is the dual group of its dual group. That is, $\widehat{\widehat{G}} = G$, making it an important special pair of groups. ◆

Definition A.31 (Homogeneous Spaces)**.** Let G be a locally compact σ-compact group and X a locally compact space on which G acts transitively via

$$x \mapsto g.x \ . \tag{A.27}$$

Then, all of X is given by a single orbit of some fixed fiducial point $x_0 \in X$ under G. If

$$S_0 := \{ g \in G \mid g.x_0 = x_0 \} \tag{A.28}$$

is the stabilizer of x_0 under G, then X and the quotient space G/S_0 are homeomorphic. Such spaces are called *homogeneous*.

The homogeneous space is *principal*, if the action is moreover free, meaning that its stabilizer is trivial, and thus G itself is homeomorphic to its orbit. ◆

Theorem A.32 (Lie groups)**.** (i) *A Lie group is a topological group which has the structure space of a differentiable manifold.*

(ii) *Each (finite-dimensional) Lie group is a locally compact group and thus is equipped with an invariant measure - its Haar measure.*

(iii) *To each Lie group we can assign a Lie algebra and to each Lie algebra we can associate a connected and simply-connected Lie group.*

(iv) *For each Lie group G, its universal cover, $\tilde{G}$, is isomorphic to its Lie algebra $\mathfrak{g}$ and the quotient $\tilde{G}/G$ is discrete.*

(v) *Each Lie group, G, has a connected component (of the identity), G_0, being a normal Lie group, with discrete (countable) quotient G/G_0.* ◆

A.3 Functional Analysis

Definition A.33 (Topological vector space)**.** Let V be a set. Then, whenever the set V is closed under addition and multiplication by complex numbers

$$\alpha v_1 + \beta v_2 \in V, \quad \forall \ v_1, v_2 \in V \text{ and } \alpha, \beta \in \mathbb{C}, \tag{A.29}$$

it is referred to as a $\mathbb{C}$-*vector space*. If V is further equipped with a topology, it is a *topological vector space* if and only if the above actions are continuous mappings.

Since $0 \in \mathbb{C}$ is a complex number, $0 \cdot v \in V$ is a vector, too, known as the zero-vector 0, and since $-v := (-1) \cdot v \in V$, each vector has an inverse, with

$$-v + v = 0 \in V. \tag{A.30}$$

◆

Definition A.34 (Topological vector spaces of continuous functions)**.** Let X be a topological space. Then, $C(X)$ denotes the topological vector space of continuous functions from X to $\mathbb{C}$. If X is moreover locally compact, we may define the subspace of continuous functions that vanish at infinity, $C_0(X)$, as well as the subspace of continuous functions with compact support, $C_c(X)$.

Finally, if $k \in \mathbb{N}$ and X is a differentiable manifold, then $C^k(X)$, $C^k_0(X)$ and $C^k_c(X)$ denote the corresponding vector subspaces of k-times continuously differentiable functions. If $k = \infty$, the functions are infinitely differentiable and we speak of smooth functions. ◆

Definition A.35 (Norm)**.** Let V be a $\mathbb{C}$-vector space, $x, y \in V$ and $\lambda \in \mathbb{C}$. A *norm*

$$\|\cdot\| : V \to \mathbb{R}_+, \quad x \mapsto \|x\|, \tag{A.31}$$

is a mapping with the following properties.

(i) It is non-negative

$$\|x\| \geq 0. \tag{A.32}$$

(ii) It is absolutely homogeneous

$$\|\lambda x\| = |\lambda| \, \|x\| . \tag{A.33}$$

(iii) It is sub-additive

$$\|x + y\| \leq \|x\| + \|y\| . \tag{A.34}$$

(iv) It is definite

$$\|x\| = 0 \quad \Leftrightarrow \quad x = 0. \tag{A.35}$$

If the last property only holds in the direction of

$$x = 0 \quad \Rightarrow \quad \|x\| = 0, \tag{A.36}$$

we call it a *semi-norm*. A vector space, equipped with a (semi-) norm is referred to as a (semi-) normed space. A normed space is a topological vector space, where the open balls are given by *open balls* of the form

$$B_\epsilon(x) := \{\, v \in V \mid \|v - x\| < \epsilon \,\} . \tag{A.37}$$

What makes normed spaces so useful is that these are all first countable Hausdorff topological vector spaces and thus we have an *algebraic structure* as well as a topological one and all continuity issues may be conveniently tackled with *(converging) sequences.* ◆

Since now there is a distance defined on a topological vector space - a *metric*, to be precise -, we may encounter the concept of a sequence which in a sense has some sort of "inner convergence". Meaning that the elements, $(f_n)_{n\in\mathbb{N}}$, "approach each other", as $n \to \infty$, which is precisely the definition of a *Cauchy sequence*.

Definition A.36 (Cauchy sequence in normed spaces)**.** Let V be a normed space. Let $(v_n)_{n\in\mathbb{N}} \subseteq V$ be a sequence, for which it holds that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$, such that the distance between v_n and v_m is smaller than ϵ, whenever $n, m > N_\epsilon$. That is, if

$$\forall \epsilon > 0 \ \exists N_\epsilon \text{ s.t. } \|v_n - v_m\| < \epsilon, \quad \forall n, m \ > N_\epsilon, \tag{A.38}$$

this sequence is a *Cauchy sequence*. ◆

As already pointed out above, these sequences have a certain kind of special convergence and a normed space that is considered to be some sort of "complete", in the sense of the vectors it contains, should contain the limits of these sequences, as these are very natural ones.

Definition A.37 (Completeness)**.** A normed space, in which every Cauchy sequence converges, is said to be *complete*. ◆

Definition A.38 (Banach space)**.** A *complete normed space* is said to be a *Banach space*. ◆

Definition A.39 (Inner product)**.** Let V be a $\mathbb{C}$-vector space, $x, y, z \in V$. Then, an *inner product*

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C} \tag{A.39}$$

is a mapping with the following properties.

(i) It is sesqui-linear, meaning complex-conjugated linear in the first and linear in the second argument. That is,

$$\langle \alpha x + \beta y , z \rangle = \overline{\alpha} \langle x , z \rangle + \overline{\beta} \langle y , z \rangle \tag{A.40}$$

and

$$\langle z , \alpha x + \beta y \rangle = \alpha \langle z , x \rangle + \beta \langle z , y \rangle . \tag{A.41}$$

(ii) It is Hermitian

$$\langle x , y \rangle = \overline{\langle y , x \rangle}. \tag{A.42}$$

(iii) It is positive definite

$$\langle x, x \rangle \geq 0 \tag{A.43}$$

and

$$\langle x, x \rangle = 0 \quad \Leftrightarrow \quad x = 0\,. \tag{A.44}$$

A vector space with an inner product is known as an *inner product space*, sometimes instead referred to as a *pre-Hilbert space*.

The inner product defines a norm

$$\|x\| = \sqrt{\langle x, x \rangle}, \tag{A.45}$$

turning every inner product space into a normed space. ◆

Definition A.40 (Hilbert space)**.** A complete vector space, equipped with an inner product, is a *Hilbert space*. ◆

Just like in the classical case of $\mathbb{C}^n$, the inner product is bounded.

Cauchy-Schwarz Theorem A.41. *Let V be an inner product space and $v, w \in V$, then*

$$|\langle v, w \rangle| \leq \|v\|\,\|w\|\,, \tag{A.46}$$

known as the Cauchy-Schwarz inequality*.* ◆

Proof. Let $v, w \in V$. Then, the standard proof, as found in almost every book about linear algebra and functional analysis, goes as follows. From

$$\begin{aligned} 0 &\leq \left\| w - \frac{\langle v, w \rangle v}{\|v\|^2} \right\|^2 \\ &= \|w\|^2 - 2\,|\langle v, w \rangle|^2 \frac{1}{\|v\|^2} + \left\| \frac{\langle v, w \rangle v}{\|v\|^2} \right\|^2 \\ &= \|w\|^2 - |\langle v, w \rangle|^2 \frac{1}{\|v\|^2} \end{aligned}$$

we find that

$$\begin{aligned} \frac{|\langle v, w \rangle|^2}{\|v\|^2} &\leq \|w\|^2 \\ \Leftrightarrow \ |\langle v, w \rangle|^2 &\leq \|w\|^2\,\|v\|^2 \end{aligned}$$

and taking square roots finishes the proof. ■

Plancherel's Theorem A.42. *Let $f, g \in L^2(\mathbb{R}^n)$, and $\widehat{f}, \widehat{g}$ the respective Fourier transforms. Then*

$$\left\langle \widehat{f}, \widehat{g} \right\rangle = \langle f, g \rangle \tag{A.47}$$

and in particular

$$\left\| \widehat{f} \right\| = \| f \| , \tag{A.48}$$

showing that the Fourier transform is an isometry - *in fact a* unitary operator - *on* $L^2(\mathbb{R}^n, \lambda)$. ◆

Definition A.43 (Linear operators)**.** Let X, Y be normed spaces. Then,

$$T : X \to Y, \quad x \mapsto Tx \tag{A.49}$$

is a bounded (linear) operator, if and only if it is linear and, for all $x \in X$, there exists M such that

$$\| Tx \|_Y \leq M \| x \|_X . \tag{A.50}$$

The infimum of all such M is defined to be the norm of T

$$\| T \|_{OP} := \sup_{x \in X} \frac{\| Tx \|_Y}{\| x \|_X} . \tag{A.51}$$

In particular, every bounded operator is *continuous* and, unless stated otherwise, an operator is assumed to be linear. ◆

Definition A.44 (Isometries and unitary operators)**.** Let X, Y be normed spaces and $T : X \to Y$ an operator. If

$$\| Tx \|_Y = \| x \|_X , \quad \forall x \in X, \tag{A.52}$$

the operator is said to be an *isometry*, which for inner product spaces X, Y is adapted to mean

$$\langle Tx , Ty \rangle_Y = \langle x , y \rangle_X , \tag{A.53}$$

i.e., it preserves angles (and distances). If, moreover, X and Y are Hilbert spaces and T is a bijection, it holds that

$$T^* T x = x, \quad \forall x \in X, \tag{A.54}$$

where T^* is the adjoint to T, defined as

$$\langle Tx , y \rangle_Y = \langle x , T^* y \rangle_X , \quad \forall x \in X, y \in Y. \tag{A.55}$$

A bijective isometry between Hilbert spaces is referred to as a *unitary operator.* ◆

Definition A.45 (Linear functionals)**.** A linear operator

$$T : X \to \mathbb{C}, \tag{A.56}$$

mapping into the underlying field of complex numbers, is a *(linear) functional.* ◆

Definition A.46 (Continuous functionals and the topological dual)**.** Let F be a topological vector space. Then, the set of all $\mathbb{C}$-linear and continuous functionals

$$l \;:\; F \to \mathbb{C}, \quad f \mapsto l(f) \tag{A.57}$$

is its *topological dual* F^*. ◆

Definition A.47 (Weak topology)**.** If $f \neq g$ in the topology of F, then there exists a $l \in F^*$, such that $l(f) \neq l(g)$. In other words, the map

$$F \longrightarrow (F^* \to \mathbb{C}), \quad f \longmapsto (l(f))_{l \in F^*} \tag{A.58}$$

is injective.

As far as measurements are concerned, after having defined the topological dual of F by means of the topology on F, we may forget the topology on F and speak of the weak topology of F, defined with respect to its topological dual F^*. That is, $(f_n)_n$ converges to f *in the weak sense*, if and only if $(l(f_n))_n$ converges to $l(f)$ in the sense of $\mathbb{C}$, for all functionals l:

$$f_n \xrightarrow{\mathrm{w}} f \quad \Leftrightarrow \quad l(f_n) \longrightarrow l(f), \quad \forall l \in F^*. \tag{A.59}$$

◆

Integrating a function f on a locally compact group against the characters, resembles a mapping from $L^1(G,\mu)$ to $L^\infty(\widehat{G},\mu_{\widehat{G}})$.

Definition A.48 (Generalized Fourier transform)**.** Let (G,μ) be a locally compact abelian group, with dual $(\widehat{G},\mu_{\widehat{G}})$ and $f \in L^1(G,\mu)$. Then,

$$L^1(G,\mu) \to L^\infty(\widehat{G},\mu_{\widehat{G}}), \quad f \mapsto \int_G f(x)\,\overline{\xi(x)}\,\mathrm{d}\mu(x) =: \widehat{f}(\xi), \quad \xi \in \widehat{G}, \tag{A.60}$$

exists and if $\widehat{f} \in (L^1 \cap L^\infty)(\widehat{G},\mu_{\widehat{G}})$, then there is an inverse

$$\widehat{f} \mapsto \int_{\widehat{G}} \widehat{f}(\xi)\,\widehat{x}(\xi)\,\mathrm{d}\mu_{\widehat{G}}(\xi), \qquad \widehat{x}(\xi) := \xi(x), \quad x \in G. \tag{A.61}$$

Note that if $(f,\widehat{f})$ is such a pair, then necessarily $f \in (L^1 \cap L^\infty)(G,\mu)$ and $\widehat{f} \in (L^1 \cap L^\infty)(\widehat{G},\mu_{\widehat{G}})$. ◆

Definition A.49 (Dirac Delta)**.** Dirac's delta distribution, δ_x, is defined as

$$f(x) = \int_E \delta_x f \, \mathrm{d}x, \quad x \in E, \tag{A.62}$$

where the integral is defined, whenever f is continuous in an open neighborhood of x. ◆

Definition A.50 (Types of convergences for operators)**.** Let X, Y be normed spaces, with Y' denoting the topological dual space of Y, and $T_n, T : X \to Y$, $n \in N$, be linear, bounded operators. Then $(T_n)_n$ is said to converge to T

(i) in *norm*, if $\|T_n - T\|_{OP} \to 0$, as $n \to \infty$.

(ii) *strongly*, if $\|T_n f - Tf\|_Y \to 0$, as $n \to \infty$, for all $f \in X$.

(iii) *weakly*, if $|l(T_n f) - l(Tf)| \to 0$, as $n \to \infty$, for all $f \in X$ and $l \in Y'$.

If Y is one-dimensional, the family of operators T_n become functionals and are said to converge to T with respect to

(i) *norm topology*, if $\sup\limits_{f \in X, \|f\|_X = 1} |T_n(f) - T(f)| \to 0$, as $n \to \infty$.

(ii) *weak* topology*, if, for all $f \in X$, $|T_n(f) - T(f)| \to 0$, as $n \to \infty$. ◆

Definition A.51 (Test functions)**.** The test functions are the smooth functions of compact support

$$\mathcal{D}(\mathbb{R}^n) := \{ \varphi \in C_0^\infty(\mathbb{R}^n) \mid \operatorname{supp} f \text{ is compact} \}, \tag{A.63}$$

i.e., as a set $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n)$, equipped with the topology induced by the family of semi-norms

$$\|\varphi\|_\alpha := \|D^\alpha \varphi\|_\infty, \qquad D^\alpha = D^{\alpha_1} \cdot \dots \cdot D^{\alpha_n}. \tag{A.64}$$

That is,

$$\varphi_n \to \varphi \text{ as } n \to \infty \iff \|\varphi_n - \varphi\|_\alpha \to 0 \text{ as } n \to \infty, \; \forall \alpha \in \mathbb{N}^n. \tag{A.65}$$

◆

Definition A.52 (Distributions)**.** Let $\mathcal{D}(\mathbb{R}^n)$ denote the test functions. Then, the topological dual of $\mathcal{D}(\mathbb{R}^n)$ is the topological vector space of *distributions* $\mathcal{D}'(\mathbb{R}^n)$, equipped with the *weak*-topology*. ◆

Definition A.53 (Schwartz space)**.** The Schwartz space consists of smooth functions vanishing at infinity and decay faster than the inverse of any polynomial. Let

$$x^\alpha := x^{\alpha_1} \cdot \cdots \cdot x^{\alpha_n} \qquad \text{and} \qquad D^\beta := D^{\beta_1} \cdot \cdots \cdot D^{\beta_n}, \tag{A.66}$$

then

$$\|\varphi\|_{\alpha,\beta} := \left\| x^\alpha D^\beta \varphi \right\|_\infty, \quad \alpha, \beta \in \mathbb{R}^n, \tag{A.67}$$

are the semi-norms that induce the locally convex topology of the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \varphi \in C_0^\infty(\mathbb{R}^n) \,\middle|\, \|\varphi\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{R}^n \right\}. \tag{A.68}$$

◆

Definition A.54 (Tempered distributions)**.** Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space. The topological dual of $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n)$, equipped with the weak*-topology.

A tempered distribution is sometimes referred to as a *generalized function*, if it is formally treated as (an extreme case of) a normal function, e.g., the limit of a sequence of $L^2(\mathbb{R}^n)$, merely converging in the sense of distributions. ◆

Definition A.55 (Self-Adjointness)**.** An operator $T : \mathcal{H} \to \mathcal{H}$ is self-adjoint if and only if it is *symmetric*,

$$\langle Tf \,,\, g \rangle_{\mathcal{H}} = \langle f \,,\, Tg \rangle_{\mathcal{H}}, \tag{A.69}$$

and its image coincides with its domain, $\mathrm{im}(T) = \mathrm{dom}(T)$. ◆

Definition A.56 (Projection-Valued Measure)**.** Let (X, Σ_X) be a measurable space and $P : \Sigma_X \to L(\mathcal{H})$ a measure, taking its values in the linear operators on some Hilbert space $\mathcal{H}$. Then, P is a *projection-valued measure*, or *PVM*, if it is self-adjoint, idempotent and $P(X) = \mathbb{1}_{\mathcal{H}}$. For a projection-valued measure and $f, g \in \mathcal{H}$, the map

$$\Sigma_X \ni \Delta \longmapsto \langle P(\Delta) g \,,\, f \rangle \in \mathbb{C} \tag{A.70}$$

defines a complex measure, which is a real and positive multiple of a probability measure, if $f = g$. ◆

Theorem A.57 (Spectral Theorem for Self-Adjoint Operators)**.** *Let* $T : \mathrm{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ *be an operator, unbounded or not but self-adjoint on its domain, and let* $\mathrm{spec}(T)$ *denote its spectrum, then there exists a* projection-valued spectral measure, *P, such that*

$$T := \int_{\mathrm{spec}(T)} \lambda \; dP_\lambda, \tag{A.71}$$

with convergence at least in the weak sense. ◆

Definition A.58 (Borel Functional Calculus)**.** Let T be a self-adjoint operator, E its projection-valued measure and $f : \mathrm{spec}(T) \to \mathbb{C}$ a Borel-measurable function, defined on the spectrum of T. Then

$$f(T) := \int_{\mathrm{spec}(T)} f(\lambda) \; dP_\lambda \tag{A.72}$$

defines the Borel functional calculus on the spectrum of T. ◆

Definition A.59 (Commutator)**.** Let S, T be operators, such that $\mathrm{im}(S) \subseteq \mathrm{dom}(T)$ and $\mathrm{im}(T) \subseteq \mathrm{dom}(S)$, then

$$[S,T]f := (ST - TS)f \;, \quad f \in \mathrm{dom}(S) \cap \mathrm{dom}(T) \cap \mathrm{dom}([S,T]) \tag{A.73}$$

defines its *commutator.* ◆

Definition A.60 (Unitary representations)**.** Let (G, μ_L) be a locally compact group and $\mathcal{H}$ a Hilbert space. A homomorphism

$$\pi : G \to \mathcal{U}(\mathcal{H}) \tag{A.74}$$

from G *into* the group of unitary operators over $\mathcal{H}$ is said to be a *unitary representation.* ◆

Theorem A.61 (Haar measure)**.** *Let G be a locally compact group and define Σ_G to be its Borel σ-algebra, built from its locally compact topology. Then, there exist non-negative Radon measures, μ_L and μ_R, that are invariant under the left, respectively right, action of the group on itself. That is, let $x \in G$ and $E \subset G$ measurable, then*

$$\mu_L(xE) = \mu_L(E) \quad \text{and} \quad \mu_R(Ex) = \mu_R(E). \tag{A.75}$$

Both of these measures are unique, up to a scalar, and known as the left *respectively* right Haar measure. ◆

Theorem A.62 (Modular function)**.** *Let G be a locally compact group, then there exist left and right invariant Haar measures, μ_L and μ_R, and these are equivalent. The Radon-Nikodym derivative of μ_L with respect to μ_R*

$$\frac{\mathrm{d}\mu_L}{\mathrm{d}\mu_R} = \Delta_G \tag{A.76}$$

is uniquely defined and dubbed the modular function *of G.*

Groups for which the modular function is trivial are called unimodular*, whereas all others are referred to as* non-unimodular *ones.* ◆

Definition A.63 (L^p spaces over locally compact groups)**.** Let G be topological group with left Haar measure μ_L. We define the Lebesgue spaces over the locally compact group G as

$$L^p(G, \mu_L) := \left\{ f : G \to \mathbb{C} \,\middle|\, \|f\|_p^p := \int_G |f|^p \,\mathrm{d}\mu_L < \infty \right\}, \tag{A.77}$$

for $1 \leq p < \infty$, respectively as

$$L^\infty(G, \mu_L) := \{ f : G \to \mathbb{C} \mid \|f\|_\infty := \operatorname{ess\,sup} |f| < \infty \}, \tag{A.78}$$

if $p = \infty$. ◆

The multiplication rule on the group itself lifts to another multiplication on the space of functions defined on it.

Definition A.64 (Convolution)**.** Let $f, g : (G, \mu_L) \to \mathbb{C}$, then

$$(f * g)(x) := \int_G f(y)\, g(y^{-1}x)\, \mathrm{d}\mu_L(y) \tag{A.79}$$

is the convolution of f and g and $f * g =: h$ is again a complex-valued function, $h : (G, \mu_L) \to \mathbb{C}$, defined on G. ◆

Lemma A.65 (Integrability)**.** *Let (G, μ_L) be a locally compact group and*

$$f, g : G \to \mathbb{C} \tag{A.80}$$

be absolutely integrable*, then $f * g$ is integrable, with*

$$\int_G f * g \,\mathrm{d}\mu_L = \int_G f \,\mathrm{d}\mu_L \int_G g \,\mathrm{d}\mu_L \,. \tag{A.81}$$

◆

Proof. We have

$$\begin{aligned}\int_G (f * g)(x)\ \mathrm{d}\mu_L(x) &:= \int_G \int_G f(y) g(y^{-1}x)\ \mathrm{d}\mu_L(y)\ \mathrm{d}\mu_L(x) \\ &= \int_G \int_G f(y) g(x)\ \mathrm{d}\mu_L(y)\ \mathrm{d}\mu_L(x) \\ &= \int_G f(y)\ \mathrm{d}\mu_L(y) \int_G g(x)\ \mathrm{d}\mu_L(x),\end{aligned} \tag{A.82}$$

by the left translation invariance of the Haar measure μ_L and since the absolute integrability allows us to exchange the integration, by Fubini's theorem. ∎

Theorem A.66 (Convolution algebra)**.** *If $f, g \in L^1(G, \mu_L)$, then even more can be said about the convolution of f and g, namely*

$$f,\, g \in L^1(G, \mu_L) \quad \Longrightarrow \quad f * g \in L^1(G, \mu_L)\ . \tag{A.83}$$

This turns $L^1(G, \mu)$ into an algebra*, that is, a vector space, along with a multiplication rule defined on it.* Note that this not necessarily includes inverses and a neutral element of multiplication. ◆

Since each group element has an inverse, it makes sense to define the "involution" - meaning self-inverse - of a function.

Proposition A.67 (Isometric Involution)**.** *Let G be a locally compact group, with left Haar measure μ_L and modular function Δ.*

Then,

$$* : L^p(G, \mu_L) \ \to\ L^p(G, \mu_L), \qquad f^*(x) := \Delta^{-1/p}(x)\overline{f(x^{-1})} \tag{A.84}$$

is the isometric involution *on $L^p(G, \mu_L)$, where $\Delta^{-1/p}(x)$ is understood to be trivial, $\Delta^{-1/p}(x) \equiv 1$, if $p = \infty$. The involution is*

(i) self-inverse, *that is,*

$$(f^*)^* = f, \tag{A.85}$$

(ii) *and* isometric

$$\|f^*\|_p = \|f\|_p\ . \tag{A.86}$$

◆

Proof. (i) is true, since Δ is a homomorphism and thus

$$f^{**}(x) = \overline{f^*(x^{-1})}\Delta(x)^{-1/p} = f(x)\Delta(x^{-1})^{-1/p}\Delta(x)^{-1/p} = f(x)\ . \tag{A.87}$$

(ii) follows from

$$\|f^*\|_p^p = \int_G \left|f(x^{-1})\right|^p \Delta(x)^{-1}\,\mathrm{d}\mu_L(x) = \int_G |f(x)|^p\ \mathrm{d}\mu_L(x)\ . \tag{A.88}$$

■

Corollary A.68 (Non-isometric Involution). *Let $f \in L^p(G,\mu_L)$ and define*

$$* : L^p(G,\mu_L)\ \to\ L^p(G,\mu_R), \qquad f^*(x) := \overline{f(x^{-1})}, \tag{A.89}$$

to be the involution, then

$$\|f^*\|_{L^p(G,\mu_L)} = \|f\|_{L^p(G,\mu_R)}\,, \qquad \textit{resp.} \qquad \|f^*\|_{L^p(G,\mu_L)} = \left\|f\Delta^{-1/p}\right\|_{L^p(G,\mu_L)}. \tag{A.90}$$

◆

Proof. Since $\mathrm{d}\mu_L(x^{-1}) = \mathrm{d}\mu_R(x)$, $x \in G$, we find

$$\begin{aligned}\|f^*\|^p_{L^p(G,\mu_L)} &= \int_G \left|\overline{f(x^{-1})}\right|^p\ \mathrm{d}\mu_L(x)\\ &= \int_G |f(x)|^p\ \mathrm{d}\mu_L(x^{-1})\\ &= \int_G |f(x)|^p\ \mathrm{d}\mu_R(x)\\ &= \|f\|^p_{L^p(G,\mu_R)}\,,\end{aligned} \tag{A.91}$$

and from $\Delta^{-1}\cdot\ \mathrm{d}\mu_L =\ \mathrm{d}\mu_R$ it follows that

$$\begin{aligned}\|f^*\|^p_{L^p(G,\mu_L)} &= \int_G |f(x)|^p\ \mathrm{d}\mu_R(x)\\ &= \int_G |f(x)|^p\,\Delta^{-1}(x)\ \mathrm{d}\mu_L(x)\\ &= \left\|f\Delta^{-1/p}\right\|^p_{L^p(G,\mu_L)}\ .\end{aligned} \tag{A.92}$$

■

Lemma A.69 (Isometries, induced by measure-equivalence). *Let (X,Σ,μ) be a measure space and let there be an equivalent measure ν, that is, $\mu \ll \nu \ll \mu$, with Radon-Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \phi$. Then, for any $f \in L^p(X,\Sigma,\mu)$ the map*

$$\iota_{\mu\mapsto\nu} : L^p(X,\Sigma,\mu)\ \to\ L^p(X,\Sigma,\nu)\ , \quad f\ \mapsto\ \phi^{\frac{1}{p}}\cdot f \tag{A.93}$$

is an isometrical bijection between $L^p(X,\Sigma,\mu)$ and $L^p(X,\Sigma,\nu)$ with its obvious inverse $\iota_{\nu\mapsto\mu}\ :\ f\ \mapsto\ \phi^{-\frac{1}{p}}\cdot f$ and the understanding that, for $p=\infty$, we mean $\phi^{\frac{1}{p}} = \phi^{-\frac{1}{p}} \equiv 1$. ◆

Proof. This fact follows from

$$\begin{aligned}\|\iota_{\mu\mapsto\nu}\, f\|^p_{L^p(X,\Sigma,\nu)} &= \int_X \left|\phi^{\frac{1}{p}}(x)\cdot f(x)\right|^p \; \mathrm{d}\nu(x) \\ &= \|f\|^p_{L^p(X,\Sigma,\mu)}\end{aligned}$$

and $\mathrm{d}\mu = \phi \;\, \mathrm{d}\nu$. ∎

Definition A.70 (Cyclicity)**.** A vector $\eta \in \mathcal{H}_\pi$ is cyclic for $(\pi, \mathcal{H}_\pi)$ if and only if the closure of its orbit under the group action is all of the representation space,

$$\overline{\operatorname{span}\{\, \pi(x)\eta \mid x \in G \,\}} = \mathcal{H}_\pi, \tag{A.94}$$

i.e., if the linear span of $\pi(G)\eta \subseteq \mathcal{H}_\pi$ is dense in $\mathcal{H}_\pi$. ◆

Definition A.71 (Irreducibility)**.** A representation $(\pi, \mathcal{H}_\pi)$ of G is irreducible, if and only if the only invariant subspaces of $\pi(G)$ are $\{0\}$ and $\mathcal{H}_\pi$. ◆

Schur's Lemma A.72. *Let G be locally compact group and $(\pi, \mathcal{H}_\pi)$ a unitary representation of G. Then, π is irreducible if and only if for all $T \in L(\mathcal{H}_\pi)$ which commute with π, i.e., $T\pi = \pi T$, it holds that T is a multiple of the identity, $T = c\mathbf{1}$, for some $c \in \mathbb{C}$.* ◆

A.4 Differential Geometry

Definition A.73 (Differentiable manifolds)**.** Let X be a second countable Hausdorff topological space. If X is *locally homeomorphic* to a subset of the n-dimensional Euclidean space, that is, for all $x \in X$ there exists a neighborhood $x \in N_x \subseteq X$ and a homeomorphism

$$\varphi_{N_x} : N_x \;\to\; \operatorname{im}(\varphi_{N_x}) \subseteq \mathbb{R}^n, \tag{A.95}$$

X is said to be a *manifold* of dimension n. A set of *charts* Φ, consisting of (local) homeomorphisms between a subset of the manifold and a subset of $\mathbb{R}^n$, is an *atlas* if the union of all pre-images cover the manifold and it is possible to transition between charts that have non-vanishing intersections on the manifold.

A manifold is a *differentiable manifold* if the atlas is differentiable, that is, if *all transitions* between charts like

$$\varphi_2 \circ \varphi_1^{-1}, \qquad \varphi_1, \varphi_2 \in \Phi, \tag{A.96}$$

are differentiable and have differentiable inverses. If the transitions are moreover smooth, it is called a *smooth manifold.* ◆

Definition A.74 (Path-connectedness)**.** A space is *path-connected* if each two points can be joined by a path. ◆

Definition A.75 (Simply connectedness)**.** A space is *simply connected*, if it is path-connected and every closed curve is contractible to a point. ◆

Definition A.76 (Differential Forms)**.** A differential form, ω, on a differentiable manifold is

(i) *exact* , iff there exists a potential, such that $\omega = \mathrm{d}\nu$, and

(ii) *closed*, iff its exterior derivative vanishes $\mathrm{d}\omega = 0$.

Every exact form is closed, since $\mathrm{d} \circ \mathrm{d} = 0$ and the opposite holds, if, and only if, the space is simply-connected. ◆

Definition A.77 (Hamiltonian)**.** The Hamiltonian of a classical mechanical system is the observable for its energy and is, in the simplest case, given by the sum of kinetic, T, and potential energy, V, that is

$$H := T + V. \tag{A.97}$$

◆

Definition A.78 (Lagrangian)**.** The Lagrangian of a classical system is a way to derive the dynamics of the system and in the simplest case defined to be the difference of kinetic, T, and potential energy, V,

$$L := T - V. \tag{A.98}$$

◆

Definition A.79 (Euler-Lagrange Equations)**.** Let $L(\vec{x}, \dot{\vec{x}})$ be a time-independent Lagrangian on the differentiable manifold X and $\vec{x}(t) \in X$ for all t in some interval. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0, \quad \mu = 1, \ldots, n, \tag{A.99}$$

are the *Euler-Lagrange equations.* ◆

Definition A.80 (Christoffel symbols)**.** Let X be a differentiable manifold and $(e^\mu)_\mu$ a local basis, with $(\partial_{e^\mu})_\mu$ denoting the associated vector field basis, then its *Christoffel symbols*, $\Gamma^\lambda_{\mu\nu}$, are defined as

$$\nabla_{\partial_{e^\mu}} \partial_{e^\nu} = \Gamma^\lambda_{\mu\nu} \partial_{e^\lambda}, \tag{A.100}$$

and are of relevance for describing the covariant derivative and arise, e.g., in the geodesic equations. ◆

B
Auxiliary Material

RESEARCH AND DEVELOPMENT go hand in hand and thus, this final addendum shall provide information about the MATLAB-based generation of the plots from this monograph, as well as exemplary implementations of the established *generalized coherent state maps*, a.k.a. "σ-transforms", of the form

$$f \mapsto \pi_g^\sigma f,$$

for some signal f, window g and (spectral) diffeomorphism σ.

Numerical Phase Space Decompositions and Plot generation

All numerical computations and plots, regarding phase space decompositions, presented in this monograph, were made with MATLAB [62]. The self-contained code can be found on

https://github.com/dlantzberg/AuxMatlabPlots

and should run smoothly on *MATLAB R2016a* without any additional toolboxes.

Implementations of the σ-Transform

In order to demonstrate the usability of the established σ-transforms, various exemplary implementations, written in MATLAB/Octave, Javascript, Java, Python and C/C++, are provided. The former is written and tested in *MATLAB R2016a* on a Windows 7 machine and the latter makes use of the *C++11 standard, STL containers* and the C-Library *FFTW3.3* [28, 33] and was compiled and tested with *g++ (Ubuntu/Linaro 5.4.0-6) on Ubuntu Linux 16.04.10,* as well as with *g++ (GCC) 4.8.1 on Windows 7.*

All implementations can be found on

https://github.com/dlantzberg

and specific implementations, like the MATLAB/Octave Version, can be directly accessed via

https://github.com/dlantzberg/SigmaTransform-Matlab ,

and the C++ implementation is located on

https://github.com/dlantzberg/SigmaTransform-Cpp .

For further information on how to use these, consult the above mentioned websites.

List of Definitions and Terminologies.

List of Lemmata.

List of Theorems.

List of Corollaries.

List of Examples.

List of Figures.

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