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A Primal-Dual Augmented Lagrangian Penalty-Interior-Point Algorithm for Nonlinear Programming

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Abstract

This thesis treats a new numerical solution method for large-scale nonlinear optimization problems. Nonlinear programs occur in a wide range of engineering and academic applications like discretized optimal control processes and parameter identification of physical systems. The most efficient and robust solution approaches for this problem class have been shown to be sequential quadratic programming and primal-dual interior-point methods.

The proposed algorithm combines a variant of the latter with a special penalty function to increase its robustness due to an automatic regularization of the nonlinear constraints caused by the penalty term. In detail, a modified barrier function and a primal-dual augmented Lagrangian approach with an exact $\ell 2$-penalty is used. Both share the property that for certain Lagrangian multiplier estimates the barrier and penalty parameter do not have to converge to zero or diverge, respectively. This improves the conditioning of the internal linear equation systems near the optimal solution, handles rank-deficiency of the constraint derivatives for all non-feasible iterates and helps with identifying infeasible problem formulations. Although the resulting merit function is non-smooth, a certain step direction is a guaranteed descent. The algorithm includes an adaptive update strategy for the barrier and penalty parameters as well as the Lagrangian multiplier estimates based on a sensitivity analysis. Global convergence is proven to yield a first-order optimal solution, a certificate of infeasibility or a Fritz-John point and is maintained by combining the merit function with a filter or piecewise linear penalty function. Unlike the majority of filter methods, no separate feasibility restoration phase is required. For a fixed barrier parameter the method has a quadratic order of convergence.

Furthermore, a sensitivity based iterative refinement strategy is developed to approximate the optimal solution of a parameter dependent nonlinear program under parameter changes. It exploits special sensitivity derivative approximations and converges locally with a linear convergence order to a feasible point that further satisfies the perturbed complementarity condition of the modified barrier method. Thereby, active-set changes from active to inactive can be handled. Due to a certain update of the Lagrangian multiplier estimate, the refinement is suitable in the context of warmstarting the penalty-interior-point approach.

A special focus of the thesis is the development of an algorithm with excellent performance in practice. Details on an implementation of the proposed primal-dual penalty-interior-point algorithm in the nonlinear programming solver `WORHP` and a numerical study based on the `CUTEst` test collection is provided. The efficiency and robustness of the algorithm is further compared to state-of-the-art nonlinear programming solvers, in particular the interior-point solvers `IPOPT` and `KNITRO` as well as the sequential quadratic programming solvers `SNOPT` and `WORHP`.

Keywords Nonlinear Programming · Large-Scale Optimization · Primal-Dual Penalty-Interior-Point Algorithm · Augmented Lagrangian Method · Modified Barrier Method · Parametric Sensitivity Analysis · WORHP

Zusammenfassung

Diese Arbeit behandelt eine neue numerische Lösungsmethode für hochdimensionale nichtlineare Optimierungsprobleme. Nichtlineare Optimierung tritt in einem weiten Spektrum an technischen und akademischen Anwendungen auf, wie beispielsweise in diskretisierten Optimalsteuerungsprozessen oder in der Parameteridentifikation von physikalischen Systemen. Als die effizientesten und robustesten Lösungsansätze für diese Problemklasse haben sich die Sequentielle-Quadratische-Programmierung und der primär-duale Innere-Punkte-Ansatz ergeben.

Der vorgeschlagene Algorithmus kombiniert eine Variante des Letzteren mit einer speziellen Bestrafungsfunktion, um seine Robustheit mittels der automatischen Regularisierung der nichtlinearen Nebenbedingungen durch den Bestrafungsterm zu erhöhen. Im Detail wird eine modifizierte Barrierefunktion und ein sogenannter primär-dualer erweiterter Lagrange Ansatz mit einer exakten $\ell 2$-Bestrafungsfunktion genutzt. Beide teilen die Eigenschaft, dass für bestimmte Lagrange-Multiplikator-Abschätzungen der Barriere- und der Bestrafungsprameter nicht gegen Null konvergieren, bzw. divergieren, müssen. Dies verbessert die Kondition des internen linearen Gleichungssystems nahe der optimalen Lösung, handhabt unzureichenden Rang der Ableitungen der Nebenbedingungen für alle nicht zulässigen Iterierten und hilft unzulässige Problemformulierungen zu identifizieren. Obwohl die resultierende Bewertungsfunktion nicht differenzierbar ist, führt eine spezielle Suchrichtung zu einem garantiertem Abstieg. Der Algorithmus verfügt über adaptive Aktualisierungsstrategien für den Barriere- und Bestrafungsparameter sowie die Lagrange-Multiplikator-Abschätzungen basierend auf einer Sensitivitätsanalyse. Globale Konvergenz zu einer optimalen Lösung ersten Grades, einer Garantie der Unzulässigkeit oder einem Fritz-John-Punkt wird erzeugt durch die Kombination der Bewertungsfunktion mit einem Filter oder einer stückweise linearen Bestrafungsfunktion. Anders als die Mehrzahl der Filtermethoden wird keine zusätzliche Zulässigkeitskorrekturphase benötigt. Für einen fixierten Barriereparameter ist die Methode lokal quadratisch konvergent.

Des Weiteren wird eine iterative sensitivitätsbasierte Verbesserungsstrategie entwickelt, um die optimale Lösung eines parameterabhängigen nichtlinearen Problems bei Änderungen des Parameters zu approximieren. Diese nutzt dabei spezielle Approximationen der Sensitivitätsableitungen aus und konvergiert lokal mit einer linearen Konvergenzordnung zu einem zulässigen Punkt, der zusätzlich die gestörte Komplementaritätsbedingung der modifizierten Barrieremethode erfüllt. Dabei können Änderungen der aktiven Menge in Form von aktiv zu inaktiv gehandhabt werden. Aufgrund besonderer Aktualisierungen der Lagrange-Multiplikator-Abschätzungen ist die iterative Verbesserungsstratgie bestens geeignet für den Warmstart des Bestrafungs-Innere-Punkte-Algorithmus.

Ein besonderer Fokus der Arbeit liegt auf der Entwicklung eines Algorithmus mit besonderer praktischer Performanz. Details einer Implementierung des vorgeschlagenen primär-dualen Bestrafungs-Innere-Punkte-Algorithmus in dem nichtlinearen Optimierungsproblemlöser `WORHP` und eine numerische Studie basierend auf der `CUTEst` Testkollektion werden ausgeführt. Die Effizienz und Robustheit des Algorithmus wird weiterhin verglichen mit hochmodernen nichtlinearen Lösungsroutinen, im Besonderen mit dem Innere-Punkte-Löser `IPOPT`

und KNITRO sowie mit dem Sequentielle-Quadratische-Programmierungs-Löser SNOPT und WORHP.

Schlüsselwörter Nichtlineare Optimierung · Hochdimensionale Optimierung · Primär-Dualer Bestrafungs-Innere-Punkte-Algorithmus · Erweiterte Lagrange-Methode · Modifizierte Barriere-Methode · Parametrische Sensitivitätsanalyse · WORHP

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Chapter 1

Introduction

> «Roughly speaking, local optimization methods are more art than technology. Local optimization is well developed art, and often very effective, but it is nevertheless an art.»
>
> Boyd and Vandenberghe [23, p. 9]

Optimization can be found almost everywhere. It is a fundamental principle in nature and a valuable tool for humans to improve their actions and making. No matter if an engineering application considers an automotive, a robot or a space rocket for example, they often share the endeavor to minimize energy consumption and environmental influences – which is often directly linked to the minimization of costs. If these costs can be specified or modeled as a function of decision variables, a mathematical optimization problem is defined. It usually contains some kind of restrictions for the decision variables, which are mathematically expressed as constraint functions. While it is the task of practitioners to model their real-world application as a set of these usually nonlinear cost and constraint functions, it is the goal of mathematical optimization to find the optimal decision variables that minimize the cost or objective function.

A subsequent scientific research question is how this optimal solution changes under perturbations of model parameters. These parameters appear in almost every optimization problem with a value that could be uncertain or for which different configurations need to be considered. A sensitivity analysis [60, 61] provides these insights and thus enables to approximate the optimal solution of the perturbed optimization problem.

Solving optimization problems with arbitrary nonlinear functions can be difficult both in theory and in practice. Complexity occurs due to non-convexity of functions, which implies existence of multiple solutions with different quality or – in other words – many local minima, and due to inequality constraints (cf., [172]). In particular the former may have motivated some researches (cf., [23]) to see nonlinear optimizers as artists because of the challenging task to compose practical algorithms to find good quality local solutions. Inequality constraints could be handled efficiently as equality constraints if the active set, i.e., the set of inequality constraints that are satisfied with equality, would be known for the optimal solution. Since this is usually not the case, numerical solution strategies like active-set [47, 58, 108], interior-point [37, 64, 77, 79], penalty [64, 116, 164] and sequential quadratic programming methods [19, 90, 192] have been developed, where the given references are just a very limited selection. While active-set approaches iteratively estimate the optimal active set when progressing towards the optimal solution and setting variables that are considered to be active to their bound value, interior-point methods add a sequentially decreasing barrier to the objective function to prevent constraints from becoming active during the process. Penalty methods are similar in the sense that they add a penalty to the objective function, but only if constraints are violated. Sequential quadratic programming is a different concept as it sequentially approximates the optimization problem using a quadratic model that is solved by either of the other methods – mainly active-set.

When comparing state-of-the-art nonlinear programming solvers, interior-point methods turn out to be the most efficient [13, 142, 143].[1] However, despite the development of many different practical interior-point algorithms [29, 185, 202] within the last two decades, some aspects still leave room for improvements: How to handle degeneracy of constraint functions, i.e., linear dependent gradients? How to quickly detect if a problem formulation is infeasible? And, how to warmstart an interior-point algorithm to solve a sequence of similar optimization problems efficiently? These research questions are of particular interest if interior-point algorithms shall serve as local solvers within global (mixed-integer) nonlinear programming methods, a field that is usually dominated by sequential quadratic programming methods [89, 144].

1.1 Thesis Aims and Contribution

The thesis firstly aims to survey the theory and existing numerical methods of derivative based optimization techniques to solve smooth nonlinear optimization problems. The main goal, however, is the design and development of a new primal-dual augmented Lagrangian penalty-interior-point algorithm that addresses the above research questions and is efficient in practice. A practical implementation of that algorithm within the nonlinear programming solver `WORHP` [36] is provided. The method is thoroughly studied theoretically and numerically.

The primal-dual augmented Lagrangian penalty-interior-point algorithm combines a modified barrier function [46, 97, 162] with an augmented Lagrangian penalty [116, 164] to solve the constrained nonlinear optimization problem as an unconstrained one. While the interior-point approach guarantees the high efficiency of the method, the additional penalty increases its robustness. This is due to an automatic dual regularization that handles degenerate constraint gradients similar to [2, 40, 43, 95, 97]. Unlike the majority of augmented Lagrangian based methods, an exact and non-smooth $\ell 2$-penalty is used [40, 42, 43] that includes a natural adaptive penalty parameter update strategy. A further penalty parameter multiplying the objective function follows [31, 67] and improves the quick detection of infeasibility. The special barrier-penalty combination benefits from barrier and penalty parameters that do not require

[1] The referenced benchmarks consider the one-time optimization of feasible nonlinear programs exploiting first and second-order derivatives when possible for a solver. The statement may change in favor of sequential quadratic programming methods if these assumptions are modified, cf., [92].

converging to zero. Whereas recent research trends [101, 136] try to avoid a merit function or filter approach as line search globalization, the proposed method combines the two. The merit function is essential for theoretical convergence and the filter, originally developed by [69], massively increases the step acceptance rates and thus improves the practical performance. However, most filter algorithms (e.g., [183, 202]) require a separate feasibility restoration phase. Due to the combination with the merit function, this is not required for the proposed method, which results in faster detection of infeasibility. A further advantage of the proposed filter is the independence on any of the involved barrier or penalty parameters.

A global convergence analysis proves that the algorithm converges for an arbitrary initial guess to either an optimal solution, a certificate of infeasibility or a Fritz-John point under standard assumptions. A vital element is the proof of a guaranteed descent direction for the non-smooth barrier-penalty merit function for a modified Newton step. Proofs for fast local convergence of the underlying penalty approach and asymptotic convergence orders when approaching an optimal solution or certificate of infeasibility conclude the theoretical analysis. Most of these proofs follow the presentations of similar $\ell 2$ [6, 40, 42, 43] or augmented Lagrangian [2, 3, 155] based penalty-interior-point algorithms, but are translated or extended to the barrier-penalty combination considered in this thesis.

Furthermore, the thesis provides an extensive study of applying sensitivity analysis as an internal tool to improve efficiency besides showing how to calculate sensitivity derivatives for classic post-optimality sensitivity analysis at low computational cost. Among them are complementarity refinement steps and adaptive barrier and penalty parameter updates. For the latter, sensitivity derivatives can indicate in every iteration of the algorithm which parameter update provides best progress towards an optimal solution and by that offers a highly flexible update scheme. This is similar to [48, 153], but has not been studied for a modified barrier function before, which requires further considerations.

A new warmstart approach for modified barrier based interior-point algorithms is proposed. It uses sensitivity derivatives in an iterative feasibility and complementarity refinement to approximate the optimal solution of the new perturbed optimization problem. It is proven that the method convergences to a point that satisfies the perturbed feasibility and complementarity condition of the barrier-penalty subproblem with a linear convergence order. This approach is a great advancement over classic real-time updates as it features certain active set changes. It can therefore be seen as the interior-point perspective on a task that is usually addressed by active-set approaches [123, 161, 193, 205]. Sensitivity information can be transferred to the Lagrangian multiplier estimates in the modified barrier function with a suitable projection to provide a good starting point for warmstarting a modified barrier based interior-point algorithm.

The aim of the numerical study is to provide insights that determine the algorithm components with the highest impact on the practical performance and to prove the high efficiency and robustness of the developed method by comparing it to the state-of-the-art interior-point and sequential quadratic programming solvers `IPOPT` [202], `KNITRO` [29], `SNOPT` [91] and `WORHP` [36] on the `CUTEst` test set [107]. A special emphasis is also put on a performance comparison on infeasible problem formulations showing the superiority of the proposed method over the other interior-point solvers. Finally, a crossover is designed that offers the possibility to switch from the penalty-interior-point algorithm to WORHP's sequential quadratic programming method at an arbitrary iteration of the optimization process.

Contributions to Publications

During the creation of this thesis, the author contributed to five publications, of which three are directly connected to the content of this work. A brief overview is given in the following.

[25] C. Buchheim, R. Kuhlmann, and C. Meyer. Combinatorial optimal control of semilinear elliptic PDEs. *Computational Optimization and Applications*, 70(3):641–675, 2018. doi:10.1007/s10589-018-9993-2

The paper considers a novel outer approximation approach for the efficient solution of optimal control problems with semilinear elliptic partial differential equations (PDEs) and static integer controls over arbitrary combinatorial structures. This problem class is difficult in practice and is usually addressed by a domain discretization, which leads to very large-scale mixed-integer nonlinear programs. The proposed algorithm, however, is based on a decomposition of the optimal control problem into an efficiently solvable integer linear programming master problem and a cutting plane generating subproblem. The latter relies on a pointwise concavity or submodularity of the PDE solution with respect to the integer controls. Such a sequential framework allows exploiting reoptimization techniques for solving the PDE. The paper includes a numerical study that shows the efficiency of the proposed approach.

Kuhlmann's main contribution is the development of reoptimization strategies for an efficient PDE solution, a `Matlab` implementation of the proposed algorithm and the numerical study in the paper. Although the publication is not directly linked to this thesis, it motivated many algorithmic considerations. Following the generic domain discretization approach for the solution of a PDE with inequality state constraints, an interior-point method would probably be the best choice for the resulting large-scale nonlinear program. The inclusion into a mixed-integer solution framework would then require certain features like fast detection of infeasibility and ability to warmstart that are usually considered to be a weakness of interior-point methods.

[129] R. Kuhlmann and C. Büskens. A primal–dual augmented Lagrangian penalty-interior-point filter line search algorithm. *Mathematical Methods of Operations Research*, 87(3): 451–483, 2018. doi:10.1007/s00186-017-0625-x

In this journal article a primal-dual penalty-interior-point algorithm based on the combination of a classic log-barrier and an augmented Lagrangian approach with an exact $\ell 2$-penalty is considered to solve generic nonlinear programs. Special emphasis is placed on the practical performance of the detection of infeasibility and of the line search strategy that combines a filter with a merit function. Unlike the majority of filter methods, this does not require a separate feasibility restoration phase. This publication is closely linked to this work, as one part of this thesis is the extension of the algorithm – among many smaller improvements – by a modified barrier function.

[130] R. Kuhlmann, S. Geffken, and C. Büskens. WORHP Zen: Parametric sensitivity analysis for the nonlinear programming solver WORHP. In N. Kliewer, J. F. Ehmke, and R. Borndörfer, editors, *Operations Research Proceedings 2017*, pages 649–654. Springer International Publishing, 2018. doi:10.1007/978-3-319-89920-6_86

The conference paper presents the practical parametric sensitivity analysis module `WORHP Zen` of the nonlinear programming solver `WORHP`. Sensitivity derivatives with respect to parameter data are of high interest because they improve the understanding of the optimal solution and allow the formulation of real-time capable update algorithms. Besides showing implementation details for the efficient calculation as well as sparse storage of parametric sensitivities and the real-time updates, the paper illustrates the application of `WORHP Zen` in the field of parameter identification.

As the development of `WORHP Zen` began with Schäfer [174] the author's main contribution was the efficient `Fortran` implementation in `WORHP` and the presentation of implementation details.

[144] B. Müller, R. Kuhlmann, and S. Vigerske. On the performance of NLP solvers within global MINLP solvers. In N. Kliewer, J. F. Ehmke, and R. Borndörfer, editors, *Operations Research Proceedings 2017*, pages 633–639. Springer International Publishing, 2018. doi:10.1007/978-3-319-89920-6_84

In this conference paper the performance of nonlinear programming solvers are studied when applied to the internal subproblems of the mixed-integer nonlinear programming solver `SCIP`. Among them are primal heuristics, convex relaxations and bound tightening methods.

Kuhlmann contributed in the development of the `WORHP` interface in `SCIP`, which included an adaptation of the warmstarting interface of `WORHP`, and extended the interior-point algorithm of `WORHP` with different warmstarting strategies.

[173] M. Schweinoch, R. Schäfer, A. Sacharow, D. Biermann, and C. Buchheim. A non-rigid registration method for the efficient analysis of shape deviations in production engineering applications. *Production Engineering*, 10(2):137–146, 2016. doi:10.1007/s11740-016-0660-0

The paper studies a new non-rigid registration method for the efficient calculation of correspondences of designed and as-built parts in production engineering applications. Non-rigid registration methods are based on a deformation of the one geometry onto the other. The proposed method combines an error-adaptive segmentation with rigid alignments of each segment and a restoration of connectivity by minimizing a mesh energy functional. The paper includes a numerical study where the method is applied to the problem of springback in sheet metal forming.

Kuhlmann's main contribution was the development and implementation of the energy functional optimization for restoring mesh connectivity.

1.2 Thesis Overview

The thesis is partitioned into four main chapters.

The Chapter 2 gives an overview of the theoretical foundations in nonlinear programming. After the definition of the problem task and its optimal solution, necessary and sufficient conditions for the characterization of an optimal solution are derived in Section 2.1 with a special emphasis on the first-order necessary conditions as these are extensively used by numerical algorithms. On this basis, the chapter continues with the theory of parametric sensitivity analysis in Section 2.2. This includes the derivation of first- and second-order sensitivity derivatives and the approximation of perturbed nonlinear programs.

Chapter 3 treats the question of how to solve nonlinear optimization problems numerically by studying the proposed approaches in the literature. The focus is on derivative based methods that apply Newton's method to the first-order necessary conditions as motivated in Section 3.1. This requires developing schemes for the globalization (Section 3.3) and regularization (Section 3.4) of this special variant of Newton's method. Nevertheless, inequality constraints cannot be handled by this approach directly and strategies to simplify these and to solve optimization problems with inequality constraints are presented in Section 3.2 and Section 3.5, respectively. Among them are active-set, interior-point or barrier and exterior-point or penalty methods. The final Section 3.6 of this chapter considers sensitivity analysis based techniques to increase the efficiency or robustness of numerical optimization algorithms.

The Chapter 4 presents the proposed primal-dual augmented Lagrangian penalty-interior-point algorithm. After a brief theoretical study of the combined penalty-barrier function in Section 4.1, the main ingredients of the algorithm, i.e., step computation, line search, parameter updates and a so called magic step, are introduced and discussed in Section 4.2. A convergence analysis studies the theoretical properties of the algorithm far away from the optimal solution (global convergence, Section 4.3) and very close to it (local convergence, Section 4.4). In the remainder of the chapter, the sensitivity analysis is widely applied to the proposed penalty-interior-point algorithm. In Section 4.5 sensitivity derivatives of optimization variables with respect to the original, the barrier sub- and the barrier-penalty subproblem are derived. Sensitivity derivatives of the step direction are the basis for adaptive barrier and penalty parameter updates in Section 4.6. Finally, Section 4.7 proposes an iterative refinement strategy using sensitivity information for an improved warmstart of a modified barrier function based interior-point algorithm.

In Chapter 5 a description of a practical implementation of the proposed penalty-interior-point algorithm within the nonlinear programming solver `WORHP` and numerical results are provided. After a brief introduction of the solver `WORHP` and the `CUTEst` test collection as well as benchmark metrics (Section 5.1), detailed algorithmic considerations and enhancements for a good practical performance are studied in Section 5.2. A comparison to state-of-the-art nonlinear programming solvers completes the chapter in Section 5.3.

A final conclusion of the thesis is given in Chapter 6.

1.3 Notation

Scalars and vectors are written in lowercase and matrices in uppercase. The letters are either Roman or Greek and squared brackets are used for its definition, e.g., $A := \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \in \mathbb{N}^{1\times 3}$. For a given vector $b \in \mathbb{R}^n$ with $n \in \mathbb{N}$ the uppercase version B is defined as a square diagonal matrix with b on its diagonal, i.e., $B := \operatorname{diag}(b) \in \mathbb{R}^{n\times n}$. The ith element of the vector b is b_i and, thus, the ith unit vector of appropriate size is e_i where $e_j = 1$ if $i = j$ and $e_j = 0$ otherwise. Following this approach, e is defined as a vector of ones and $E := \operatorname{diag}(e)$ is the identity matrix. As for the unit vector and the identity matrix the size is not specified for the zero vector or zero matrix 0, but will be evident from the context. A comparison of two vectors – e.g., lesser, greater or equal – is always defined to be element-wise, i.e., $a \leq b$ with two vectors $a, b \in \mathbb{R}^n$ is equivalent to $a_i \leq b_i$ for all $i = 1, \ldots, n$. A tuple of vectors $c = (a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ will also be accessed as the vector $c = \begin{bmatrix} a^\top & b^\top \end{bmatrix}^\top \in \mathbb{R}^{n+m}$. The norm of a vector or matrix is $\|\cdot\|$ and may be any of the possible vector or matrix norms unless specified, e.g., $\|\cdot\|_2$ for the Euclidean norm and $\|\cdot\|_\infty$ for the maximum norm. Analogously, $|\cdot|$ is the absolute value of a scalar.

Sets are written in calligraphic font and are defined using curly brackets, e.g., $\mathscr{A} := \{1, 2, 3\}$. The only exception to this rule are the number sets. The most important ones are $\mathbb{N}$ and $\mathbb{N}_0$ for natural numbers without or with zero as well as $\mathbb{R}$, $\mathbb{R}_{0+}$ and $\mathbb{R}_+$ for real, non-negative real and strictly positive real numbers. The empty set is $\emptyset$ and the number of elements of $\mathscr{A}$ is $|\mathscr{A}|$. A ball around a point $b \in \mathbb{R}^n$ with radius $\varepsilon > 0$ is defined as $\mathscr{B}_\varepsilon(b)$. If the radius is of no further relevance and just assumed to be sufficiently small or the shape is not necessarily a ball, a neighborhood written as $\mathscr{N}(b)$ is used. To simplify notation, the neighborhood around a tuple (a, b) is equivalently referred to as $\mathscr{N}((a, b)) = \mathscr{N}(a, b)$ and analogously for a ball.

Sequences of scalars are written as $\{a_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{R}$, of vectors as $\{a^k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{R}^n$ and of matrices as $\{A_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{R}^{n\times m}$ to avoid confusion with the ith element of a vector. If the index set is $\mathbb{N}_0$ the definition is abbreviated to $\{a_k\}_k$ and similar for vectors and matrices. For a given index set $\mathscr{K}$ the notation is also simplified to $\{a_k\}_{\mathscr{K}}$. Furthermore, the Landau notation is utilized.

Definition 1.1 (Landau Notation). *Let $\{a_k\}_k \subseteq \mathbb{R}_{0+}$ and $\{b_k\}_k \subseteq \mathbb{R}_{0+}$ be two sequences with non-negative elements. The Landau notation is defined as:*

- $a_k = \mathscr{O}(b_k)$ *if a_k is bounded above by b_k asymptotically, i.e.,* $\limsup_{k\to\infty} \frac{|a_k|}{|b_k|} < \infty$ *or – in other words – if there exists $c > 0$ such that $a_k \leq c b_k$ for $k \in \mathbb{N}$ large enough.*
- $a_k = \Omega(b_k)$ *if a_k is bounded below by b_k asymptotically, i.e., $b_k = \mathscr{O}(a_k)$.*
- $a_k = \Theta(b_k)$ *if a_k is bounded both above and below by b_k asymptotically, i.e., $a_k = \mathscr{O}(b_k)$ and $a_k = \Omega(b_k)$.*
- $a_k = o(b_k)$ *if a_k is dominated by b_k asymptotically, i.e.,* $\lim_{k\to\infty} \frac{|a_k|}{|b_k|} = 0$ *or if there exists a sequence $\{c_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ that converges to zero such that $a_k = c_k b_k$ for $k \in \mathbb{N}$ large enough.*

Special cases of the Landau notation are $a_k = \Theta(1)$ and $a_k = o(1)$ to state that a sequence $\{a_k\}_k \subseteq \mathbb{R}_{0+}$ is bounded away from zero – at least for large indices k – and bounded above or converges to zero, respectively.

Derivatives of sufficiently smooth functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ – also referred to as $g_i : \mathbb{R}^n \to \mathbb{R}$ with $i = 1, \dots, m$ – for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ evaluated at the points $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ are defined as

$$\nabla_x f(\bar{x}, \bar{y}) := \begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}, \bar{y}) & \dots & \frac{\partial f}{\partial x_n}(\bar{x}, \bar{y}) \end{bmatrix}^\top \in \mathbb{R}^n$$

$$\nabla_x g(\bar{x}) := \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(\bar{x}) & \dots & \frac{\partial g_m}{\partial x_1}(\bar{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n}(\bar{x}) & \dots & \frac{\partial g_m}{\partial x_n}(\bar{x}) \end{bmatrix} \in \mathbb{R}^{n \times m}$$

and consequently

$$\begin{aligned} \nabla^2_{xy} f(\bar{x}, \bar{y}) &:= \nabla_y \left(\begin{bmatrix} \frac{\partial f}{\partial x_1}(\bar{x}, \bar{y}) & \dots & \frac{\partial f}{\partial x_n}(\bar{x}, \bar{y}) \end{bmatrix}^\top \right) \\ &= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial y_1}(\bar{x}, \bar{y}) & \dots & \frac{\partial^2 f}{\partial x_n \partial y_1}(\bar{x}, \bar{y}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial y_m}(\bar{x}, \bar{y}) & \dots & \frac{\partial^2 f}{\partial x_n \partial y_m}(\bar{x}, \bar{y}) \end{bmatrix}, \end{aligned}$$

where $\frac{\partial}{\partial x_i}$ are partial derivatives with respect to $x_i, i = 1, \dots, n$. The short notation $\nabla g(\bar{x})$ is used for the Jacobian matrix $\nabla_x g(\bar{x})$, because it is the only derivable variable in this case. For a function $g_i(x)$ with $i = 1, \dots, m$ the derivative $\nabla g_i(\bar{x})$ is called the gradient and $\nabla^2 g_i(\bar{x})$ the Hessian matrix. For a non-smooth but convex function $h : \mathbb{R}^n \to \mathbb{R}$ the subdifferential evaluated at $\bar{x} \in \mathbb{R}^n$ is $\partial_x h(\bar{x})$.[2]

Finally, $(\cdot)_+$ is a short notation for $\max\{0, \cdot\}$. A list of all symbols defined throughout the thesis is provided at the end of the work.

[2]See also Definition A.15.

Chapter 2

Nonlinear Programming

The focus of attention in mathematical optimization is the minimization of an *objective function* $f(x)$ subject to *equality constraints* $g(x) = 0$ and *inequality constraints* $h(x) \leq 0$, where x are the so called *optimization variables*. In nonlinear programming the three functions $f(x)$, $g(x)$ or $h(x)$ may be nonlinear and possibly non-convex[1]. This work uses the formal definition of a nonlinear optimization problem or *nonlinear program* (NLP)

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & h(x) \leq 0 \end{aligned} \tag{NLP}$$

with twice continuously differentiable functions $f : \mathbb{R}^{n_x} \to \mathbb{R}$, $g : \mathbb{R}^{n_x} \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_h}$.[2] The term large-scale optimization refers to nonlinear programs (NLP) with a large number of optimization variables n_x or number of constraints n_g or n_h. It is possible to maximize a function $f(x)$ by considering the minimization of $-f(x)$.

For further expositions, the following basic definitions are necessary. A point x that satisfies the constraints $g(x) = 0$ and $h(x) \leq 0$ is called *feasible* and the *feasible set* is defined as $\mathscr{D} := \{x \in \mathbb{R}^{n_x} \mid g(x) = 0 \text{ and } h(x) \leq 0\}$. Accordingly, a point x is said to be *infeasible* if it is not feasible, i.e., $x \notin \mathscr{D}$. Furthermore, an inequality constraint is defined to be *active*, if it takes the value of its bound and *inactive* if it is bounded away from it. Consequently, the *active set* is defined as $\mathscr{A}(x) := \{i \mid h_i(x) = 0\}$ and the *inactive set* as $\mathscr{I}(x) := \{1, \ldots, n_h\} \setminus \mathscr{A}(x)$. The goal of the optimization is to find the *optimal solution* of (NLP) defined as follows.

Definition 2.1 (Optimal Solution). *A feasible point $x^* \in \mathscr{D}$ is called*

i. global optimal solution, if $f(x^*) \leq f(x)$ for all $x \in \mathscr{D}$.

ii. local optimal solution, if there exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in \mathscr{D} \cap \mathscr{B}_\varepsilon(x^*)$.

---

[1] For a formal definition of non-convexity of a function, see Definition A.11.

[2] The twice continuously differentiable condition of the functions $f(x)$, $g(x)$ and $h(x)$ will be assumed throughout the presentation. Although it will not be stated at all times, it will be clear from the usage of derivatives that this condition must hold.

If the condition is satisfied with $f(x^*) < f(x)$ for $x \neq x^*$, the point is called strict global or strict local optimal solution, respectively.

Finding the global optimal solution of the nonlinear program (NLP) is in general extremely difficult. In fact, it is NP-hard. This means that if P≠NP[3], the problem cannot be solved efficiently in polynomial time on a deterministic Turing machine. Sahni [172, Theorem 2.5.4], for example, proved this result for the special case of non-convex quadratic programming, which is a subset of nonlinear programming. Therefore, and because global solution algorithms often require an efficient local solver, this thesis aims to find local optimal solutions of (NLP). For surveys on global optimization, the reader is referred to Floudas [74], Hansen and Walster [115] and Pardalos and Rosen [158]. In the special case of convex functions $f(x)$, $g(x)$ and $h(x)$, local optimal solutions are always global optimal solutions (cf., Geiger and Kanzow [84, Theorem 2.46]). For convenience, the shorter term *optimal solution* is used to refer to a local optimal solution x^* and the definitions $f^* := f(x^*)$, $g^* := g(x^*)$, $h^* := h(x^*)$ – and analogously for variables and functions defined later on – are utilized.

It may not always be possible to find an optimal solution x^* of (NLP) since the equality constraints $g(x) = 0$ could be violated for all x satisfying the inequality constraints $h(x) \leq 0$ and thus $\mathscr{D} = \emptyset$.[4] In these cases it is desirable to find at least the point for which the *constraint violation* $\|g(x)\|$ is minimized, i.e., finding the optimal solution of the following *feasibility problem*:

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}} \quad & \|g(x)\|_2 \\ \text{subject to} \quad & h(x) \leq 0 \end{aligned} \tag{FeasNLP}$$

It has to be noted, that this definition does not satisfy the definition of the nonlinear optimization problem (NLP) since its objective function is not differentiable on the whole domain. However, the definition of the feasibility problem is only applied for infeasible points x with $\|g(x)\| > 0$ where the twice continuously differentiability condition holds. It is of course possible to formulate different feasibility problems, e.g., the smooth adaptation $\min_{x \in \mathbb{R}^{n_x}, h(x) \leq 0} \|g(x)\|_2^2$, but (FeasNLP) will harmonize well with the algorithm proposed in Chapter 4. Furthermore, for a definition of (FeasNLP) to make sense, the existence of a neighborhood has to be assumed for which the inequality constraints $h(x) \leq 0$ can be satisfied – actually an assumption of the just mentioned algorithm. In analogy to the optimal solution of (NLP), a *certificate of infeasibility* is defined.

Definition 2.2 (Certificate of Infeasibility). *A point x^* with $\|g(x^*)\| > 0$ and $h(x^*) \leq 0$ is called*

i. global certificate of infeasibility, if $\|g(x^*)\| \leq \|g(x)\|$ for all $x \in \mathbb{R}^{n_x}$ with $h(x) \leq 0$.

[3] P and NP are complexity classes and if P equals NP is an open question of complexity theory, but for this presentation just the following is relevant. If it was true, the difficult problems contained in NP would be solvable efficiently in polynomial time (similar to problems in P).

[4] It is of course also possible that there is no point x that satisfies the inequality constraints, i.e., $h(x) > 0$ for all x. That case will however not be considered in the presentation since it does not occur for the proposed algorithm due to a reformulation (cf., Section 3.2.1).

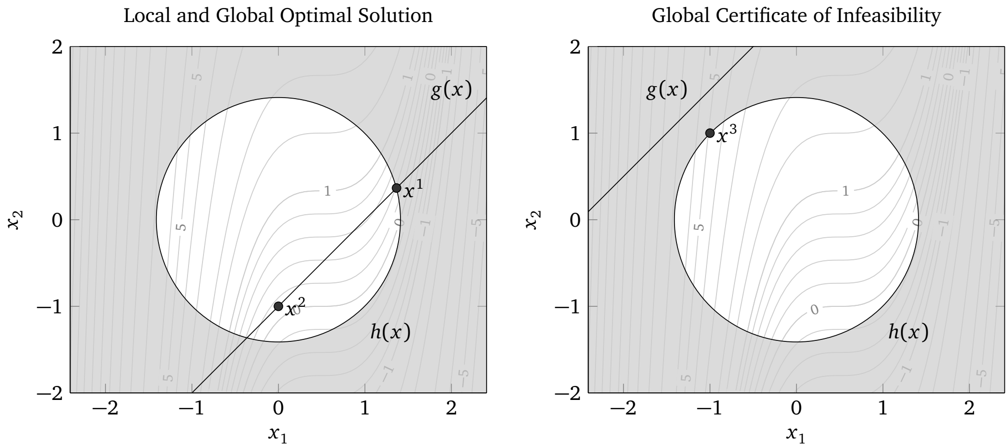

Figure 2.1: Local and global optimal solution and global certificate of infeasibility of Example 2.3. The objective function is plotted as level set. Left: Optimal solution for $p_g = 1$; Right: Certificate of infeasibility for $p_g = 2.5$. The infeasible region with respect to the inequality constraint is the light gray area.

ii. local certificate of infeasibility, if there exists $\varepsilon > 0$ *such that* $\|g(x^*)\| \leq \|g(x)\|$ *for all* $x \in \mathbb{R}^{n_x}$ *with* $h(x) \leq 0$ *and* $x \in \mathscr{B}_\varepsilon(x^*)$.

This section is closed by giving an illustrative example, which will further be studied throughout the thesis.

Example 2.3. *Consider the nonlinear program*

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & f(x) = -\left(x_1 - \frac{1}{2}\right)^3 + \frac{3}{4}(x_2+1) \\ \text{subject to} \quad & g(x) = x_1 - x_2 - p_g = 0 \\ & h(x) = x_1^2 + x_2^2 - 2 \leq 0 \end{aligned}$$

with a parameter $p_g \in \mathbb{R}$. *For the choice of* $p_g = 1$, *the problem has the local optimal solution* $x^1 = \left(\frac{1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$ *with an objective value* $f\left(x^1\right) = \frac{3}{8}$ *and the global optimal solution* $x^2 = (0, -1)$ *with* $f\left(x^2\right) = \frac{1}{8}$. *For* $p_g = -2.5$, *the problem is infeasible and the global certificate of infeasibility is* $x^3 = (-1, 1)$ *with a minimal constraint violation of* $\left\|g\left(x^3\right)\right\| = \frac{1}{2}$. *All three points are plotted in Figure 2.1.*

2.1 Optimality Conditions

In order to check if a given point x^* is *optimal*, i.e., it is an optimal solution of (NLP), first-order and second-order optimality conditions will be derived in this section. While the former utilizes first-order derivatives only and yields necessary conditions for an optimal solution, the latter also considers second-order derivatives and will be sufficient.

To motivate the first-order optimality conditions, first consider the special case of $n_g = 1$ and $n_h = 0$, meaning that (NLP) is constrained by just one equality constraint. Obviously, x^* is feasible and, thus, $g(x^*) = 0$ holds, because otherwise x^* cannot be optimal. Now, one would like to check if for all sufficiently small steps $d \in \mathbb{R}^{n_x}$ the point x^*+d is still feasible and does not improve the objective function, i.e., $g(x+d) = 0$ and $f(x^*+d) \geq f(x^*)$. Because otherwise, i.e., if there is a step such that x^*+d is feasible but decreases the objective function value, x^* again cannot be optimal. Applying a first-order Taylor approximation[5] to both conditions yields

$$0 = g(x^*+d) \approx g(x^*) + \nabla g(x^*)^\top d = \nabla g(x^*)^\top d \tag{2.1a}$$

$$0 \leq f(x^*+d) - f(x^*) \approx \nabla f(x^*)^\top d. \tag{2.1b}$$

Consequently, if for all sufficiently small directions d the conditions

$$\nabla f(x^*)^\top d \geq 0 \quad \text{and} \quad \nabla g(x^*)^\top d = 0 \tag{2.2}$$

are satisfied, it is likely that x^* is indeed an optimal solution.[6] If an inequality constraint is considered instead of an equality constraint, i.e., $n_g = 0$ and $n_h = 1$, (2.1a) changes to

$$0 \geq h(x^*+d) \approx h(x^*) + \nabla h(x^*)^\top d \tag{2.3}$$

and the situation gets slightly more complex since one has to distinguish between two cases: Either the constraint is active ($h(x^*) = 0$) or it is inactive ($h(x^*) < 0$). In the latter case x^* lies strictly inside the feasible region and one can find a sufficiently small step d such that this also holds for $h(x^*+d)$. If the constraint is active, (2.3) again simplifies to $\nabla h(x^*)^\top d \leq 0$. Together with the condition for the objective function (2.1b), one ends up with an analogue to (2.2):

$$\nabla f(x^*)^\top d \geq 0 \quad \text{and} \quad \nabla h(x^*)^\top d \leq 0, \text{ if } 1 \in \mathscr{A}(x^*). \tag{2.4}$$

Example 2.4. *Consider the nonlinear program of Example 2.3 with just the equality or just the inequality constraint, i.e.,*

$$\min_{x \in \mathbb{R}^2} \quad f(x) = -\left(x_1 - \frac{1}{2}\right)^3 + \frac{3}{4}(x_2+1) \quad \textit{subject to} \quad g(x) = x_1 - x_2 - p_g = 0, \tag{2.5}$$

or

$$\min_{x \in \mathbb{R}^2} \quad f(x) = -\left(x_1 - \frac{1}{2}\right)^3 + \frac{3}{4}(x_2+1) \quad \textit{subject to} \quad h(x) = x_1^2 + x_2^2 - 2 \leq 0. \tag{2.6}$$

Problem (2.5) *with* $p_g = 1$ *has a local optimal solution* $x^1 = (0, -1)$ *and is globally seen unbounded. The optimality condition* (2.2) *is illustrated in Figure 2.2 (left). Note, that the constraint gradient* $\nabla g(x)$ *is orthogonal to the constraint* $g(x) = 0$ *and, thus, possible directions* d *point in the direction of* $\{x \in \mathbb{R}^{n_x} \mid g(x) = 0\}$. *In other words,* d *is tangential to* $\{x \in \mathbb{R}^{n_x} \mid g(x) = 0\}$. *For just two points* x^1 *and* $x^4 = (1, 0)$ *the condition* (2.2) *is satisfied for all directions* d. *Because* x^4 *is actually a maximum of* (2.5)*, the condition* (2.2) *cannot be sufficient. Further note, that*

[5] See Theorem A.14.

[6] The point x^* is only likely to be an optimal solution here, because an approximation has been used in (2.1).

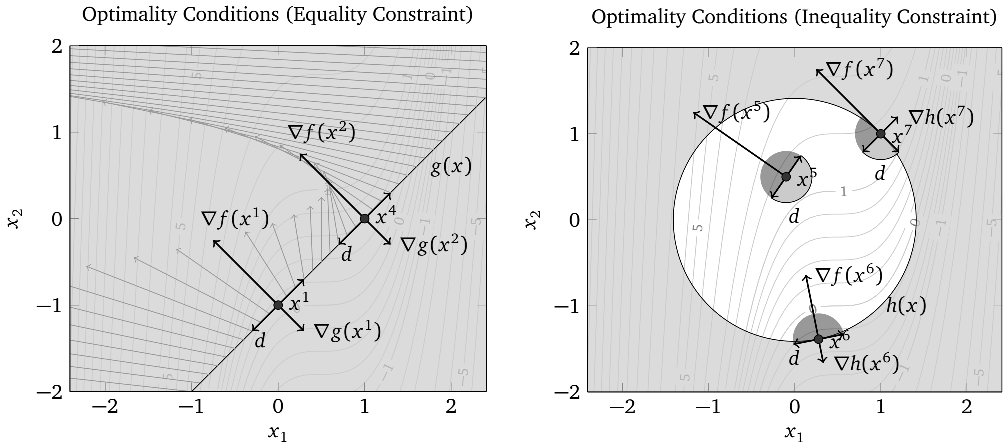

Figure 2.2: Geometric interpretation of optimality conditions for Example 2.4. Left: Conditions for one equality constraint (objective gradients are light gray; gradients and directions d for which (2.2) is satisfied are black; constraint gradients are scaled by 0.3); Right: Conditions for one inequality constraints (directions d satisfying the tangential inequality are contained in the dark gray sectors and those additionally are a descent direction for the objective function in the light gray sectors; constraint gradients are scaled by 0.1).

for these two points, the objective gradient $\nabla f(x)$ is parallel or – in other words – proportional to the constraint gradient $\nabla g(x)$, i.e., $\nabla f(x) = -\lambda \nabla g(x)$ for some $\lambda \in \mathbb{R}$ and $x \in \{x^1, x^4\}$.

Problem (2.6) *has the local and global optimal solution* $x^6 \approx (0.277, -1.387)$. *For this point, Figure 2.2 (right) reveals the same properties discussed above for problem* (2.5). *In particular, it holds that $\nabla f(x^6) = -\nu \nabla h(x^6)$ for some $\nu \in \mathbb{R}_{0+}$. For all other points x there exist descent directions d with $\nabla h(x)^\top d \leq 0$ if $1 \in \mathscr{A}(x)$, i.e., points violating $\nabla f(x)^\top d \geq 0$, which is indicated by the light gray sectors around x^5 and x^7 in Figure 2.2 (right).*

First-Order Necessary Optimality Conditions

Summarizing the above motivation, inactive inequality constraints can be neglected for the optimality conditions and only active inequality and equality constraints are of interest. These form the boundary of the feasible region $\mathscr{D}$. The motivation looked for vectors $d \in \mathbb{R}^{n_x}$ with two properties (cf., Figure 2.2):

i. d is tangential to $\mathscr{D}$.

ii. d is not a descent direction for the objective function $f(x)$.

While property (i) was easy to check for the case of just one constraint (cf., (2.1a) and (2.3)), the union of several constraints to form $\mathscr{D}$ requires special care. First, one has to define what it means for a vector to be tangent to the feasible region $\mathscr{D}$ of a general (NLP).

Definition 2.5 (Tangent Cone). *Let $\mathscr{D} \neq \emptyset$. A vector $d \in \mathbb{R}^{n_x}$ is called tangent to $\mathscr{D}$ at a point $x \in \mathscr{D}$, if there exist sequences $\{x^k\} \subseteq \mathscr{D}$ and $\{t_k\} \subseteq \mathbb{R}_+$ such that*

$$\lim_{k\to\infty} x^k = x, \quad \lim_{k\to\infty} t_k = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{x^k - x}{t_k} = d.$$

The set of all tangents to $\mathscr{D}$ at x is called the tangent cone $\mathscr{T}_{\mathscr{D}}(x)$.

It can be shown that the tangent cone $\mathscr{T}_{\mathscr{D}}(x)$ is a closed set (cf., Geiger and Kanzow [84, Lemma 2.29]). Using it, one can generalize the motivation at the beginning of this section to formulate a necessary optimality condition.

Theorem 2.6 (Optimality Condition). *Let x^* be a local optimal solution of* (NLP). *Then, $\nabla f(x^*)^\top d \geq 0$ for all $d \in \mathscr{T}_{\mathscr{D}}(x^*)$.*

Proof. See, for example, Fletcher [68, Lemma 9.2.3] or Spellucci [181, Theorem 2.1.1']. □

Unfortunately, the optimality condition of Theorem 2.6 is impractical, since it is difficult to determine the tangent cone $\mathscr{T}_{\mathscr{D}}(x^*)$ in general. Instead, one aims for a condition that resembles the motivation. Therefore, the *linearized tangent cone* is defined and its basic relation to the tangent cone is presented.

Definition 2.7 (Linearized Tangent Cone). *For a feasible point $x \in \mathscr{D}$, the linearized tangent cone is defined as*

$$\mathscr{T}_{lin}(x) := \left\{ d \in \mathbb{R}^{n_x} \mid \nabla g(x)^\top d = 0 \text{ and } \nabla h_i(x)^\top d \leq 0, i \in \mathscr{A}(x) \right\}.$$

Lemma 2.8. *Let $x \in \mathscr{D}$. Then, $\mathscr{T}_{\mathscr{D}}(x) \subseteq \mathscr{T}_{lin}(x)$.*

Proof. See, for example, Geiger and Kanzow [84, Lemma 2.32] or Nocedal and Wright [151, Lemma 12.2]. □

The linearized tangent cone $\mathscr{T}_{\text{lin}}(x)$ is also called set of linearized feasible directions in the literature. Using the first-order approximation $\mathscr{T}_{\text{lin}}(x)$ to the tangent cone $\mathscr{T}_{\mathscr{D}}(x)$, however, only makes sense if it captures its main geometric features at the point x. To guarantee this, the constraints have to fulfill some conditions, called *constraint qualification* (CQ). One of the most general is the *Abadie constraint qualification*, which simply requires the equivalence of the two cones.

Definition 2.9 (Abadie Constraint Qualification (Abadie CQ)). *The Abadie constraint qualification holds for a point x, if $\mathscr{T}_{\mathscr{D}}(x) = \mathscr{T}_{lin}(x)$.*

While the Abadie CQ is important in theory, it is highly impractical. Other commonly used constraint qualifications are the *linear independence constraint qualification* and the *Mangasarian-Fromovitz constraint qualification* [138], which both imply the Abadie CQ.

Definition 2.10 (Linear Independence Constraint Qualification (LICQ)). *The linear independence constraint qualification holds for a point x, if the gradients $\nabla g(x)$ and $\nabla h_i(x)$ with $i \in \mathscr{A}(x)$ are linearly independent.*

Definition 2.11 (Mangasarian-Fromovitz Constraint Qualification (MFCQ)). *The Mangasarian-Fromovitz constraint qualification holds for a point x, if the gradients $\nabla g(x)$ are linearly independent and there exists $d \in \mathbb{R}^{n_x} \setminus \{0\}$ such that $\nabla g(x)^\top d = 0$ and $\nabla h_i(x)^\top d < 0$ for $i \in \mathscr{A}(x)$.*

Lemma 2.12. *For a feasible point $x \in \mathscr{D}$ the following is true:*

i. If the LICQ is satisfied at x, then the MFCQ is satisfied at x.

ii. If the MFCQ is satisfied at x, then the Abadie CQ is satisfies at x.

Proof. See, for example, the proof of Geiger and Kanzow [84, Theorem 2.39 and Theorem 2.41]. □

For an overview of all the constraint qualifications and their relations, the reader is referred to Peterson [160]. Now assuming for instance the practical LICQ, the optimality condition of Theorem 2.6 becomes more tractable.

Corollary 2.13 (First-Order Optimality Condition). *Let x^* be a local optimal solution of* (NLP) *satisfying the LICQ. Then, $\nabla f(x^*)^\top d \geq 0$ for all $d \in \mathscr{T}_{lin}(x^*)$.*

Lagrangian Based First-Order Necessary Optimality Conditions

From a practitioners point of view, there is still one bothersome aspect of the first-order optimality condition of Corollary 2.13: The necessary condition $\nabla f(x^*)^\top d \geq 0$ has to be checked *for all $d \in \mathscr{T}_{\text{lin}}(x^*)$*. To avoid this, the observation of Example 2.4 – at the optimal solution x^* the gradient $\nabla f(x^*)$ was proportional to $\nabla g(x^*)$ or $\nabla h(x)$, respectively – is used. To generalize this, the *Lagrangian function* is defined.

Definition 2.14 (Lagrangian Function). *Let $\lambda \in \mathbb{R}^{n_g}$ and $\nu \in \mathbb{R}^{n_h}$. The Lagrangian function is defined as*

$$L(x, \lambda, \nu) := f(x) + \lambda^\top g(x) + \nu^\top h(x)$$

and λ and ν are called Lagrangian multipliers or dual variables.

By combining Theorem 2.6 and Farkas Lemma[7], one ends up at the *Karush-Kuhn-Tucker conditions* [126, 131]. Using the Lagrangian function they can be written compactly as follows.

Theorem 2.15 (Karush-Kuhn-Tucker (KKT) Conditions). *Let x^* be a local optimal solution of* (NLP) *satisfying the Abadie CQ. Then, there exist Lagrangian multipliers $\lambda^* \in \mathbb{R}^{n_g}$ and $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$ such that*

$$\nabla_x L(x^*, \lambda^*, \nu^*) = \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\nu^* = 0, \tag{2.7a}$$

$$\nabla_\lambda L(x^*, \lambda^*, \nu^*) = g(x^*) = 0, \tag{2.7b}$$

$$\nabla_\nu L(x^*, \lambda^*, \nu^*) = h(x^*) \leq 0, \tag{2.7c}$$

$$L(x^*, \lambda^*, \nu^*) - f(x^*) \overset{(2.7b)}{=} H(x^*)\nu^* = 0. \tag{2.7d}$$

Proof. See, for example, Geiger and Kanzow [84, Theorem 2.36] or Nocedal and Wright [151, Theorem 12.1]. □

[7] See Lemma A.9.

In the following a point (x^*, λ^*, ν^*) satisfying (2.7) is called a *KKT point* or *first-order optimal point*. The abbreviated notation

$$\Phi(x,\lambda,\nu) := \begin{bmatrix} \nabla_x L(x,\lambda,\nu) \\ g(x) \\ H(x)\nu \end{bmatrix} \tag{2.8}$$

will be used, which enables to measure the *KKT violation* $\|\Phi(x,\lambda,\nu)\|$ for a point x with $h(x) \leq 0$. Again, the inequality constraint plays a special role, because it cannot be included in $\Phi(x,\lambda,\nu)$ directly without using a non-differentiable maximum statement. In case of the LICQ, the result of Theorem 2.15 is even stronger as it guarantees the uniqueness of the Lagrangian multipliers and can be checked easily for a given primal-dual point (x,λ,ν).

Corollary 2.16. *Let x^* be a local optimal solution of* (NLP) *satisfying the LICQ. Then, there exist unique Lagrangian multipliers $\lambda^* \in \mathbb{R}^{n_g}$ and $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$ such that the KKT conditions* (2.7) *are satisfied.*

Proof. See, for example, Geiger and Kanzow [84, Theorem 2.41]. □

Using Corollary 2.16 one can also conclude optimality conditions for (FeasNLP) provided that the equality constraints are violated.

Corollary 2.17. *Let x^* with $\|g(x^*)\| > 0$ be a local optimal solution of* (FeasNLP) *satisfying the LICQ. Then, there exists a unique Lagrangian multiplier $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$ such that*

$$\begin{aligned} \|g(x^*)\|^{-1}\nabla g(x^*)g(x^*) + \nabla h(x^*)\nu^* &= 0 \\ h(x^*) &\leq 0 \\ H(x^*)\nu^* &= 0. \end{aligned}$$

Another form of first-order necessary optimality conditions are the *Fritz-John conditions* [121]. These are closely related to the KKT conditions but do not require a constraint qualification.

Theorem 2.18 (Fritz-John (FJ) Conditions). *Let x^* be a local optimal solution of* (NLP). *Then, there exist Lagrangian multipliers $\ell_0 \in \mathbb{R}$, $\lambda^* \in \mathbb{R}^{n_g}$ and $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$ with $(\ell_0, \lambda^*, \nu^*) \neq 0$ such that*

$$\ell_0 \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\nu^* = 0 \tag{2.9a}$$
$$g(x^*) = 0 \tag{2.9b}$$
$$h(x^*) \leq 0 \tag{2.9c}$$
$$H(x^*)\nu^* = 0 \tag{2.9d}$$

Proof. See, for instance, Geiger and Kanzow [84, Theorem 2.53] or John [121, Theorem 1]. □

Analogously to a KKT point, a point (x^*, λ^*, ν^*) is defined to be a *FJ point*, if (2.9) is satisfied. It can easily be checked, that the FJ conditions imply the KKT conditions for $\ell_0 > 0$. This is not true for the case $\ell_0 = 0$, for which the influence of the objective function on the optimality conditions vanishes. This case occurs if the constraint qualifications do not hold for this degenerate case, hence $\mathcal{T}_{\mathcal{D}}(x^*) \neq \mathcal{T}_{\text{lin}}(x^*)$, and a KKT point does not exist.

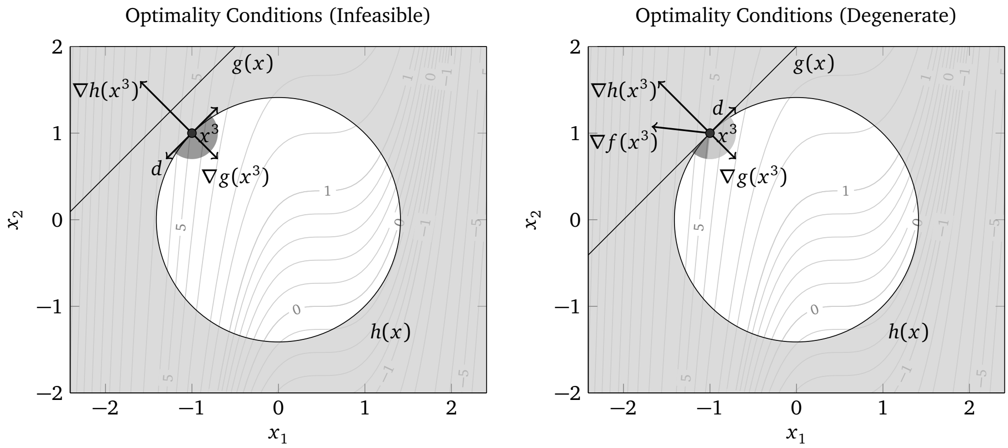

Figure 2.3: Geometric interpretation of optimality conditions for the infeasible (left) and degenerate (right) case of Example 2.19. For inequality constraints directions d satisfying the tangential inequality are in dark gray and those additionally are a descent for the objective function in light gray. Constraint gradients are scaled by 0.3 and objective gradients by 0.1.

Example 2.19. *Consider the nonlinear program of Example 2.3. For $p_g = 2.5$ the infeasibility of the problem was shown. The optimality conditions for the corresponding feasibility problem with optimal solution $x^3 = (-1, 1)$ are illustrated in Figure 2.3 (left). If p_g is changed to $p_g = 2$, the problem becomes degenerate, meaning that the LICQ and MFCQ are violated at the optimal solution x^3 as shown in Figure 2.3 (right). For this example the tangent cone has a significantly different geometry than the linearized tangent cone. It is $\mathcal{T}_{\mathcal{D}}(x^3) = \{0\}$ (single point) and the linearized tangent cone is $\mathcal{T}_{lin}(x^3) = \{ae \mid a \in \mathbb{R}\}$ (line), which also proves the violation of the Abadie CQ. Thus,* (2.7a) *cannot be satisfied and the Fritz-John conditions only hold with $\ell_0 = 0$. Note the similarity of the two problems in Figure 2.3 since the influence of the objective function vanishes for both.*

Second-Order Sufficient Optimality Conditions

In the remainder of this section sufficient conditions of an optimal solution are presented. Therefore, the linearized tangent cone $\mathcal{T}_{\text{lin}}(x)$ as a set of possible directions d has to be further reduced to the *critical cone*.

Definition 2.20 (Critical Cone). *For a feasible point $x \in \mathcal{D}$ and Lagrangian multiplier $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$, the critical cone is defined as*

$$\begin{aligned}\mathcal{T}_{crit}(x, \nu^*) := \{d \in \mathbb{R}^{n_x} \mid\ & \nabla g(x)^\top d = 0 \text{ and} \\ & \nabla h_i(x)^\top d = 0,\ \nu_i^* > 0, i \in \mathcal{A}(x) \text{ and} \\ & \nabla h_i(x)^\top d \leq 0,\ \nu_i^* = 0, i \in \mathcal{A}(x)\}.\end{aligned}$$

Corollary 2.21. *Let $x \in \mathcal{D}$ and $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$. Then, $\mathcal{T}_{crit}(x, \nu^*) \subseteq \mathcal{T}_{lin}(x)$.*

Recall the motivation at the beginning of this section where the linearized tangent cone was derived by sufficiently small steps $d \in \mathbb{R}^{n_x}$ for which the optimization variable stays feasible. The critical cone $\mathcal{T}_{\text{crit}}(x, \nu^*)$ further requires for a possible primal-dual optimal solution (x^*, λ^*, ν^*) that a certainly active inequality constraint – the case $\nu_i^* > 0$, which forces $h_i(x^*) = 0$ by (2.7d) and thus $i \in \mathcal{A}(x^*)$ – stays active for these sufficiently small steps d. When assuming *strict complementarity* of the inequality constraints as defined as follows, the definition of the critical cone $\mathcal{T}_{\text{crit}}(x, \nu^*)$ can be simplified by removing the dependence on the dual variable ν^*.

Definition 2.22 (Strict Complementarity Condition (SCC)). *The point x with Lagrangian multiplier ν fulfills the strict complementarity, if either $h_i(x) = 0$ and $\nu_i > 0$ or $h_i(x) < 0$ and $\nu_i = 0$ for all $i \in \{1, \ldots, n_h\}$.*

Corollary 2.23. *Let $x \in \mathcal{D}$ satisfy the SCC together with $\nu^* \in \mathbb{R}^{n_h}$, $\nu^* \geq 0$. Then,*

$$\mathcal{T}_{crit}(x, \nu^*) = \left\{ d \in \mathbb{R}^{n_x} \mid \nabla g(x)^\top d = 0 \text{ and } \nabla h_i(x)^\top d = 0, i \in \mathcal{A}(x) \right\}.$$

Using the critical cone $\mathcal{T}_{\text{crit}}(x, \nu^*)$, the second-order necessary and – later on – the sufficient conditions can be stated as follows.

Theorem 2.24 (Second-Order Necessary Condition). *Let x^* be a local optimal solution of* (NLP) *satisfying the LICQ. Then,*

$$d^\top \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) d \geq 0 \quad \text{for all } d \in \mathcal{T}_{crit}(x^*, \nu^*),$$

where λ^* and ν^* are the Lagrangian multipliers defined uniquely by Corollary 2.16.

Proof. See, for example, Fletcher [68, Theorem 9.3.1] or Geiger and Kanzow [84, Theorem 2.54]. □

The second-order necessary condition states that – in case the SCC holds at the primal-dual optimal solution of (NLP) – the Hessian matrix of the Lagrangian function $L(x^*, \lambda^*, \nu^*)$ must be positive semidefinite on the null space of the equality and active inequality constraints.[8] If, on the other hand, this matrix is even positive definite for the first-order optimal point (x, λ, ν), then it is in fact a strict local optimal solution of (NLP), which leads to the following sufficient optimality condition.

Theorem 2.25 (Second-Order Sufficient Condition (SOSC)). *Let (x^*, λ^*, ν^*) satisfy the KKT conditions and*

$$d^\top \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) d > 0 \quad \text{for all } d \in \mathcal{T}_{crit}(x^*, \nu^*), d \neq 0.$$

Then, x^* is a strict local optimal solution of (NLP).

Proof. See, for example, Fletcher [68, Theorem 9.3.2] or Geiger and Kanzow [84, Theorem 2.55]. □

[8] For a definition of positive and negative (semi-)definite matrices as well as the null space, see Definition A.2 and Definition A.7, respectively.

2.2 Parametric Sensitivity Analysis

Most nonlinear programs (NLP) depend on parameter choices, which of course influence the optimal solution of the problem. The parametric sensitivity analysis investigates the dependence of the optimal solution on these parameters to provide further useful information and tools. For general nonlinear parameters $p_n \in \mathbb{R}^{n_p}$, linear parameters $p_f \in \mathbb{R}^{n_x}$ and shift parameters $p_g \in \mathbb{R}^{n_g}$ and $p_h \in \mathbb{R}^{n_h}$, the *parameter dependent nonlinear program* is defined as

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x;p) := f_p(x;p_n) - p_f^\top x \\ \text{subject to} \quad & g(x;p) := g_p(x;p_n) - p_g = 0 \\ & h(x;p) := h_p(x;p_n) - p_h \leq 0 \end{aligned} \tag{NLPp}$$

with twice continuously differentiable functions $f_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}$, $g_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_g}$, $h_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_h}$ with respect to x and the parameter $p := (p_n, p_f, p_g, p_h)$. For a fixed *reference parameter* p^* the problem (NLPp) simplifies to the standard nonlinear program (NLP) and is called *unperturbed* or *reference problem*. In the presence of parameters, the definitions of the feasible set and the active set slightly change to $\mathscr{D}(p) := \{x \in \mathbb{R}^{n_x} \mid g(x;p) = 0 \text{ and } h(x;p) \leq 0\}$ and $\mathscr{A}(x;p) := \{i \mid h_i(x;p) = 0\}$, respectively. Analogously, the Lagrangian function is $L(x, \lambda, \nu; p) := f(x;p) + \lambda^\top g(x;p) + \nu^\top h(x;p)$. Since the analysis focuses on the influence of the parameters on the optimal solution x^* of the unperturbed problem, the abbreviated notation $f^*(p) := f(x^*;p)$, $g^*(p) := g(x^*;p)$, $h^*(p) := h(x^*;p)$ – and analogously for other functions – is introduced.

The sensitivity analysis was mainly developed by Fiacco [60, 61]. Fiacco and Kyparisis [59, 63] extended the research with special features of convex parameter dependent problems and construct underestimators for $f^*(p)$ (cf., additionally Boyd and Vandenberghe [23, Section 5.6]). Similar to the parameters p_f, p_g and p_h considered here, Geffken [81, Section 4.2.2 and Section 4.2.4] also derives special properties of quadratic parameters in the objective function and linear parameters in the constraints. For a general overview of the sensitivity analysis, the reader is referred to Fiacco and Ishizuka [62]. Here, the main theorem of the sensitivity analysis is directly provided, which guarantees the existence of an optimal solution of (NLPp) in a neighborhood around the reference parameter p^*, or – in other words – for sufficiently small perturbations $\Delta p := p - p^*$.

Theorem 2.26 (Sensitivity Theorem). *Let (x^*, λ^*, ν^*) satisfy the KKT conditions of the unperturbed problem* (NLPp)*, the LICQ, the SOSC and the SCC. Then, there exist a neighborhood of p^*, $\mathscr{P} \subseteq \mathbb{R}^{n_p+n_x+n_g+n_h}$, and continuously differentiable functions $x : \mathscr{P} \to \mathbb{R}^{n_x}$, $\lambda : \mathscr{P} \to \mathbb{R}^{n_g}$ and $\nu : \mathscr{P} \to \mathbb{R}^{n_h}$ that satisfy:*

i. *$x(p^*) = x^*$, $\lambda(p^*) = \lambda^*$ and $\nu(p^*) = \nu^*$.*

ii. *The active set does not change, i.e., $\mathscr{A}(x(p);p) = \mathscr{A}(x^*;p^*)$ for all $p \in \mathscr{P}$.*

iii. *$x(p)$ satisfies the LICQ for all $p \in \mathscr{P}$.*

iv. *$x(p)$ satisfies the SOSC together with $(\lambda(p), \nu(p))$ such that $\nu_i(p) > 0$, $i \in \mathscr{A}(x(p);p)$ for all $p \in \mathscr{P}$.*

In result, for $p \in \mathscr{P}$ the point $(x(p), \lambda(p), \nu(p))$ is a primal-dual optimal solution of (NLPp).

Proof. See, for example, Fiacco [60, Theorem 2.1] or the equivalent result developed independently by Robinson [168, Theorem 2.1]. □

Although Theorem 2.26 states the existence of an optimal solution $x(p)$ of (NLPp) with similar properties as the solution of the unperturbed problem x^* – in particular, the active set $\mathscr{A}(x(p); p)$ does not change – the exact representation of $x(p)$ is unfortunately generally unknown. However, the sensitivity analysis can provide first-order derivatives

$$\frac{\mathrm{d}x}{\mathrm{d}p}(p^*), \quad \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*), \quad \text{and} \quad \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \tag{2.10}$$

that indicate the dependence of the optimal solution on the parameters p. These derivatives are called *sensitivity derivatives* or just *sensitivities*. The next Section 2.2.1 explains how the sensitivity derivatives can be determined and, afterwards, Section 2.2.2 shows the application of sensitivity derivatives to approximate the optimal solution $(x(p), \lambda(p), \nu(p))$ of the perturbed nonlinear program (NLPp).

2.2.1 Calculation of Sensitivity Derivatives

The sensitivity derivatives of $(x(p), \lambda(p), \nu(p))$ can be generated by applying the implicit function theorem[9] on the KKT conditions of the reference problem. These are

$$\Phi(x, \lambda, \nu; p^*) = \begin{bmatrix} \nabla_x L(x, \lambda, \nu; p^*) \\ g(x; p^*) \\ \mathrm{diag}(\nu) h(x; p^*) \end{bmatrix}, \tag{2.11}$$

with a Jacobian matrix evaluated at the optimal solution

$$\nabla_{(x,\lambda,\nu)}\Phi(x^*, \lambda^*, \nu^*; p^*) = \begin{bmatrix} \nabla^2_{xx} L^* & \nabla g^* & \nabla h^* \\ (\nabla g^*)^\top & 0 & 0 \\ \mathrm{diag}(\nu^*)(\nabla h^*)^\top & 0 & H_* \end{bmatrix}. \tag{2.12}$$

Because of the combination of the LICQ, the SOSC and the SCC, $\nabla_{x,\lambda,\nu}\Phi(x^*, \lambda^*, \nu^*; p^*)$ is a regular matrix and the implicit function theorem can be applied. Besides the result of Theorem 2.26, it then also yields the sensitivity derivatives (2.10) of the primal-dual optimal solution.

Corollary 2.27 (Sensitivity of the Primal-Dual Optimal Solution). *Let the assumptions of Theorem 2.26 be satisfied. Then, first-order sensitivities of the optimal solution are given by*

$$\begin{bmatrix} \nabla^2_{xx} L^* & \nabla g^* & \nabla h^* \\ (\nabla g^*)^\top & 0 & 0 \\ \mathrm{diag}(\nu^*)(\nabla h^*)^\top & 0 & H_* \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p}(p^*) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*) \\ \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \end{bmatrix} = - \begin{bmatrix} \nabla^2_{xp} L^*(p^*)^\top \\ \nabla_p g^*(p^*)^\top \\ \mathrm{diag}(\nu^*)\nabla_p h^*(p^*)^\top \end{bmatrix}. \tag{2.13}$$

[9] See Theorem A.16.

Proof. See, for example, Spellucci [181, Theorem 2.5.1] or Fiacco [61, Section 3.2]. A compact proof can also be found in Büskens [35, Corollary 4.5]. □

To compute the sensitivities of the optimal solution, the linear equation system (2.13) has to be solved. Later on in Section 4.5.1, it will turn out, that a factorization of $\nabla_{(x,\lambda,\nu)}\Phi(x^*, \lambda^*, \nu^*; p^*)$ – or at least a good approximation of it – is already available from the numerical solution process of the nonlinear program (NLPp). Thus, the sensitivities of the optimal solution come at low computational cost. In addition, for the parameters p_f, p_g and p_h the right-hand-side of (2.13) reduces to $(E, 0, 0)$, $(0, E, 0)$ and $(0, 0, E)$, respectively, which avoids further function evaluations. Based on the active set and the special perturbations p_f, p_g and p_h further symmetry, similarity and sparsity properties can be derived.

Corollary 2.28 (Properties of the Sensitivities of the Primal-Dual Optimal Solution). *Let the assumptions of Theorem 2.26 be satisfied. The sensitivity derivatives* $\frac{\mathrm{d}x}{\mathrm{d}p_f}\left(p_f^*\right)$, $\frac{\mathrm{d}\lambda}{\mathrm{d}p_h}\left(p_h^*\right)$ *and* $\frac{\mathrm{d}\nu}{\mathrm{d}p_g}\left(p_g^*\right)$ *are symmetric matrices and the relations*

$$\frac{\mathrm{d}x}{\mathrm{d}p_g}\left(p_g^*\right) = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}p_f}\left(p_f^*\right)\right)^\top, \quad \frac{\mathrm{d}x}{\mathrm{d}p_h}\left(p_h^*\right) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}p_f}\left(p_f^*\right)\right)^\top, \quad \textit{and}$$
$$\frac{\mathrm{d}\lambda}{\mathrm{d}p_h}\left(p_h^*\right) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}p_g}\left(p_g^*\right)\right)^\top,$$

hold. Furthermore, the following sparsity properties are true:

i. *Inactive inequality constraint: If $i \in \mathscr{I}(x^*; p^*)$ then*

$$\frac{\mathrm{d}x}{\mathrm{d}(p_h)_i}\left(p_h^*\right) = 0, \quad \frac{\mathrm{d}\lambda}{\mathrm{d}(p_h)_i}\left(p_h^*\right) = 0, \quad \frac{\mathrm{d}\nu}{\mathrm{d}(p_h)_i}\left(p_h^*\right) = 0 \quad \textit{and} \quad \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) = 0.$$

ii. *Active inequality bound constraint: If there exists $i \in \{1, \ldots, n_x\}$ and $j \in \mathscr{A}(x^*; p^*)$ such that $h_j(x; p) = x_i - (p_h)_j$, then*

$$\frac{\mathrm{d}x_i}{\mathrm{d}p_n}\left(p_n^*\right) = 0, \quad \frac{\mathrm{d}x_i}{\mathrm{d}p_f}\left(p_f^*\right) = 0, \quad \frac{\mathrm{d}x_i}{\mathrm{d}p_g}\left(p_g^*\right) = 0, \quad \textit{and}$$
$$\left(\frac{\mathrm{d}x_i}{\mathrm{d}p_h}\left(p_h^*\right)\right)^\top = \frac{\mathrm{d}\nu}{\mathrm{d}\left(p_f\right)_i}\left(p_f^*\right) = e_j.$$

iii. *Equality bound constraint: If there exists $i \in \{1, \ldots, n_x\}$ and $j \in \left\{1, \ldots, n_g\right\}$ such that $g_j(x; p) = x_i - \left(p_g\right)_j$, then*

$$\frac{\mathrm{d}x_i}{\mathrm{d}p_n}\left(p_n^*\right) = 0, \quad \frac{\mathrm{d}x_i}{\mathrm{d}p_f}\left(p_f^*\right) = 0, \quad \frac{\mathrm{d}x_i}{\mathrm{d}p_h}\left(p_h^*\right) = 0, \quad \textit{and}$$
$$\left(\frac{\mathrm{d}x_i}{\mathrm{d}p_g}\left(p_g^*\right)\right)^\top = \frac{\mathrm{d}\lambda}{\mathrm{d}\left(p_f\right)_i}\left(p_f^*\right) = e_j.$$

Proof. Symmetry and relation between sensitivities of the special cases p_f, p_g and p_h are shown in Büskens [33, Section 3.5]. The sparsity properties are a direct implication of the active set property of Theorem 2.26. □

By utilizing the chain rule, sensitivities can also be obtained for the constraints $g(x;p)$ and $h(x;p)$ as well as the objective function $f(x;p)$.

Corollary 2.29 (Sensitivity of the Constraints). *Let the assumptions of Theorem 2.26 be satisfied. Then,*

$$\frac{\mathrm{d}h}{\mathrm{d}p}(x^*;p^*) = \left(\frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\right)^\top \nabla_x h(x^*;p^*) + \nabla_p h(x^*;p^*).$$

Moreover, the sparsity properties $\frac{\mathrm{d}g}{\mathrm{d}p}(x^*;p^*) = 0$ *and* $\frac{\mathrm{d}h_i}{\mathrm{d}p}(x^*;p^*) = 0$ *for* $i \in \mathscr{A}(x^*;p^*)$ *hold.*

Proof. See Büskens [35, Corollary 4.9]. □

Corollary 2.30 (Sensitivity of the Objective). *Let the assumptions of Theorem 2.26 be satisfied. Then,*

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}p}(x^*;p^*) &= \left(\frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\right)^\top \nabla_x f(x^*;p^*) + \nabla_p f(x^*;p^*)^\top \\ &= \nabla_p L(x^*,\lambda^*,\nu^*;p^*)^\top.\end{aligned}$$

In particular, $\frac{\mathrm{d}f}{\mathrm{d}p_f}\left(x^*;p_f^*\right)^\top = -x^*$, $\frac{\mathrm{d}f}{\mathrm{d}p_g}\left(x^*;p_h^*\right)^\top = -\lambda^*$ *and* $\frac{\mathrm{d}f}{\mathrm{d}p_h}\left(x^*;p_g^*\right)^\top = -\nu^*$. *Furthermore, the following sparsity properties hold:*

i. Inactive inequality constraint: If $i \in \mathscr{I}(x^*;p^*)$ *then*

$$\frac{\mathrm{d}f}{\mathrm{d}p_{hi}}\left(x^*;p_h^*\right) = 0$$

ii. Active bound constraint: If there exists $i \in \{1,\dots,n_x\}$ *and* $j \in \left\{1,\dots,n_g\right\}$ *such that* $g_j(x;p) = x_i - \left(p_g\right)_j$ *or* $j \in \mathscr{A}(x^*;p^*)$ *such that* $h_j(x;p) = x_i - (p_h)_j$*, then*

$$\frac{\mathrm{d}f}{\mathrm{d}\left(p_f\right)_i}\left(x^*;p_f^*\right) = 0$$

Proof. See Büskens [33, Theorem 3.2, Section 3.3 and Section 4.4]. □

For the objective function it turns out in Corollary 2.30 that the primal-dual solution (x^*,λ^*,ν^*) can be interpreted as sensitivity of the special perturbations p_f, p_g and p_h. To be more precise, if the dual variables are large, a shift of the constraints will have a huge impact on the objective function, which is why the dual variables are often interpreted as so called *shadow prices* in economics. Moreover, the special quality occurs that the sensitivity derivatives can be computed without the knowledge of the sensitivity derivatives of the optimal solution due to the relation $\frac{\mathrm{d}f}{\mathrm{d}p}(x^*;p^*) = \nabla_p L(x^*,\lambda^*,\nu^*;p^*)^\top$. This is a direct consequence of the KKT conditions and it is worth to recall two important properties: First, the Lagrangian function equals the objective function at the primal-dual optimal solution (cf., (2.7d)). Secondly, its first-order derivative measures the KKT violation, which is zero at the optimal solution (cf., (2.7a), (2.7b) and (2.7c)). It follows, that the sensitivity of the objective function is

$$\frac{\mathrm{d}f}{\mathrm{d}p}(x^*;p^*) \overset{(2.7d)}{=} \frac{\mathrm{d}L}{\mathrm{d}p}(x^*,\lambda^*,\nu^*;p^*)$$
$$= \underbrace{\left(\nabla_{(x,\lambda,\nu)}L(x^*,\lambda^*,\nu^*;p^*)\right)^\top}_{=0} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p}(p^*) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*) \\ \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \end{bmatrix} + \nabla_p L^*(p^*)^\top. \tag{2.14}$$

Based on this relation, it is also possible to specify second-order sensitivity derivatives for the objective function.

Corollary 2.31 (Second-Order Sensitivity of the Objective). *Let the assumptions of Theorem 2.26 be satisfied. Furthermore, suppose that $f(x;p)$, $g(x;p)$ and $h(x;p)$ are twice continuously differentiable with respect to p. Then,*

$$\frac{\mathrm{d}^2 f}{\mathrm{d}p\mathrm{d}p}(x^*;p^*)$$
$$= \nabla^2_{xp}L^*(p^*)\frac{\mathrm{d}x}{\mathrm{d}p}(p^*) + \nabla_p g^*(p^*)\frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*) + \nabla_p h^*(p^*)\frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) + \nabla^2_{pp}L^*(p^*)$$
$$= \left(\frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\right)^\top \nabla^2_{xx}L^*(p^*)\frac{\mathrm{d}x}{\mathrm{d}p}(p^*) + 2\nabla^2_{xp}L^*(p^*)\frac{\mathrm{d}x}{\mathrm{d}p}(p^*) + \nabla^2_{pp}L^*(p^*)$$

In particular, for the special parameters p_f, p_g and p_h the second-order sensitivity of the objective function simplifies to

$$\begin{aligned}
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_f\mathrm{d}p_f}\left(x^*;p_f^*\right) && = \frac{\mathrm{d}x}{\mathrm{d}p_f}\left(p_f^*\right),\\
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_f\mathrm{d}p_g}\left(x^*;\left(p_f^*,p_g^*\right)\right) = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}p_g\mathrm{d}p_f}\left(x^*;\left(p_g^*,p_f^*\right)\right)\right)^\top && = \left(\frac{\mathrm{d}x}{\mathrm{d}p_g}\left(p_g^*\right)\right)^\top = \frac{\mathrm{d}\lambda}{\mathrm{d}p_f}\left(p_f^*\right),\\
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_f\mathrm{d}p_h}\left(x^*;\left(p_f^*,p_h^*\right)\right) = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}p_h\mathrm{d}p_f}\left(x^*;\left(p_h^*,p_f^*\right)\right)\right)^\top && = \left(\frac{\mathrm{d}x}{\mathrm{d}p_h}\left(p_h^*\right)\right)^\top = \frac{\mathrm{d}\nu}{\mathrm{d}p_f}\left(p_f^*\right),\\
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_g\mathrm{d}p_g}\left(x^*;p_g^*\right) && = \frac{\mathrm{d}\lambda}{\mathrm{d}p_g}\left(p_g^*\right),\\
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_g\mathrm{d}p_h}\left(x^*;\left(p_g^*,p_h^*\right)\right) = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}p_h\mathrm{d}p_g}\left(x^*;\left(p_h^*,p_g^*\right)\right)\right)^\top && = \left(\frac{\mathrm{d}\nu}{\mathrm{d}p_g}\left(p_g^*\right)\right)^\top = \frac{\mathrm{d}\lambda}{\mathrm{d}p_h}\left(p_h^*\right),\\
&\frac{\mathrm{d}^2 f}{\mathrm{d}p_h\mathrm{d}p_h}\left(x^*;p_h^*\right) && = \frac{\mathrm{d}\nu}{\mathrm{d}p_h}\left(p_h^*\right),
\end{aligned}$$

and, thus, the sparsity properties of Corollary 2.28 apply.

Proof. See Büskens [33, Theorem 3.3 and Sections 3.3 - 3.5]. □

2.2.2 Approximation of Perturbed Problems

The information on the impact of the parameters on the optimal solution provided by the sensitivity derivatives can be used within first-order Taylor approximations of the perturbed primal-dual optimal solution, i.e.,

$$x(p) = x^* + \frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\Delta p + \mathcal{O}\big(\|\Delta p\|^2\big), \tag{2.15a}$$

$$\lambda(p) = \lambda^* + \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*)\Delta p + \mathcal{O}\big(\|\Delta p\|^2\big), \tag{2.15b}$$

$$\nu(p) = \nu^* + \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*)\Delta p + \mathcal{O}\big(\|\Delta p\|^2\big), \tag{2.15c}$$

as well as of the corresponding function values

$$\begin{aligned} f(x(p);p) = f(x^*;p^*) &+ \frac{\mathrm{d}f}{\mathrm{d}p}(x^*;p^*)\Delta p \\ &+ (\Delta p)^\top \frac{\mathrm{d}^2 f}{\mathrm{d}p\mathrm{d}p}(x^*;p^*)\Delta p + \mathcal{O}\big(\|\Delta p\|^3\big), \end{aligned} \tag{2.16a}$$

$$h(x(p);p) = h(x^*;p^*) + \frac{\mathrm{d}h}{\mathrm{d}p}(x^*;p^*)\Delta p + \mathcal{O}\big(\|\Delta p\|^2\big). \tag{2.16b}$$

These approximations have the property of being applicable in real-time, since the computational effort is relatively small. Only matrix-vector products have to be calculated compared to a whole re-optimization of the nonlinear program. Consequently, the approximations (2.15) and (2.16) are of special interest for applications where computational power is limited or the time limits for the optimization are extremely tight.

However, it has to be remembered that the sensitivity analysis is only valid in the neighborhood $\mathscr{P}$ of the reference parameter p^*. This neighborhood remains unknown. Büskens [33, Section 4.1.2] proposes to look at a linear approximation $\mathscr{P}_{\text{lin}}$ of $\mathscr{P}$ based on the active set property of Theorem 2.26. Since a change in the active set is not regarded by the theory, $\mathscr{P}$ can be approximated by the maximal parameter p for which a change in the active set occurs in the first-order approximations. For the moment, consider just one parameter $p \in \mathbb{R}$. An inactive constraint $h_i(x^*;p^*) < 0$ with $i \in \mathscr{I}(x^*;p^*)$ becomes active, if the sensitivity $\frac{\mathrm{d}h_i}{\mathrm{d}p}(x^*;p^*)$ is not zero and

$$0 = h_i(x(p);p) \overset{(2.16b)}{\approx} h_i(x^*;p^*) + \frac{\mathrm{d}h_i}{\mathrm{d}p}(x^*;p^*)\,(p-p^*)\,. \tag{2.17}$$

Analogously, an active constraint $h_i(x^*;p^*) = 0$ with $i \in \mathscr{A}(x^*;p^*)$ becomes inactive, if $\frac{\mathrm{d}\nu_i}{\mathrm{d}p}(p^*) \neq 0$ and

$$0 = \nu_i(p) \overset{(2.15c)}{\approx} \nu_i^* + \frac{\mathrm{d}\nu_i}{\mathrm{d}p}(p^*)\,(p-p^*)\,. \tag{2.18}$$

Both approximations can be reordered to

$$p \approx p^* - \frac{h_i(x^*;p^*)}{\frac{\mathrm{d}h_i}{\mathrm{d}p}(x^*;p^*)}, \quad \text{and} \quad p \approx p^* - \frac{\nu_i^*}{\frac{\mathrm{d}\nu_i}{\mathrm{d}p}(p^*)} \tag{2.19}$$

and the smallest such p defines $\mathscr{P}_{\text{lin}}$. Formally, for arbitrary parameter p, the neighborhood approximation $\mathscr{P}_{\text{lin}}$ can be expressed as

$$\mathscr{P}_{\text{lin}} := \mathscr{P}_{\text{lin}}^{1} \times \cdots \times \mathscr{P}_{\text{lin}}^{n_p+n_x+n_g+n_h}, \tag{2.20a}$$

$$\mathscr{P}_{\text{lin}}{}^{j} := \left[\max\left\{p_j \in \widetilde{\mathscr{P}_{\text{lin}}}^{j} \mid p_j < p_j^*\right\}, \min\left\{p_j \in \widetilde{\mathscr{P}_{\text{lin}}}^{j} \mid p_j > p_j^*\right\}\right], \tag{2.20b}$$

$$\widetilde{\mathscr{P}_{\text{lin}}}^{j} := \left\{p_j^i \mid i = 1, \ldots, n_h\right\} \cup \{-\infty, \infty\}, \tag{2.20c}$$

$$p_j^i := \begin{cases} p_j^* - \frac{h_i(x^*;p^*)}{\frac{\mathrm{d}h_i}{\mathrm{d}p_j}(x^*;p^*)} & \text{if } i \in \mathscr{I}(x^*;p^*) \\ p_j^* - \frac{\nu_i^*}{\frac{\mathrm{d}\nu_i}{\mathrm{d}p_j}(p^*)} & \text{if } i \in \mathscr{A}(x^*;p^*). \end{cases} \tag{2.20d}$$

The presentation is ended again by giving an example to illustrate sensitivity based approximations of a perturbed nonlinear program and the issue of active set changes.

Example 2.32. *Consider the nonlinear program*

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x;p) = -\left(x_1 - \frac{1}{2}\right)^3 + \frac{3}{4}(x_2+1) - p_f x_1 \\ \textit{subject to} \quad & g(x;p) = x_1 - x_2 - p_g = 0 \\ & h(x;p) = x_1^2 + x_2^2 - p_h \le 0 \end{aligned}$$

with $p_g^* = 1, p_h^* = 2$ and $p_f^* = 0$. Note, that the reference problem equals the nonlinear program of Example 2.3 with the global optimal solution $x^* = (0,-1)$, $\lambda^* = 3/4$, $\nu^* = 0$ and an inactive inequality constraint $h(x^*;p^*) = -1$. The sensitivity derivatives for the primal and dual optimization variables are:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}p_f}\left(p_f^*\right) &&&= \frac{\mathrm{d}^2 f}{\mathrm{d}p_f \mathrm{d}p_f}\left(x^*;p_f^*\right) &&= \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}, \\ \frac{\mathrm{d}x}{\mathrm{d}p_g}\left(p_g^*\right) = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}p_f}\left(p_f^*\right)\right)^{\top} &&&= \frac{\mathrm{d}^2 f}{\mathrm{d}p_f \mathrm{d}p_g}\left(x^*;\left(p_g^*,p_f^*\right)\right) &&= \begin{bmatrix}0\\-1\end{bmatrix}, \\ \frac{\mathrm{d}x}{\mathrm{d}p_h}\left(p_h^*\right) &&= \left(\frac{\mathrm{d}\nu}{\mathrm{d}p_f}\left(p_f^*\right)\right)^{\top} &= \frac{\mathrm{d}^2 f}{\mathrm{d}p_f \mathrm{d}p_h}\left(x^*;\left(p_h^*,p_g^*\right)\right) &&= \begin{bmatrix}0\\0\end{bmatrix}, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p_g}\left(p_g^*\right) &&&= \frac{\mathrm{d}^2 f}{\mathrm{d}p_g \mathrm{d}p_g}\left(x^*;p_g^*\right) &&= 0, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p_h}\left(p_h^*\right) &&= \left(\frac{\mathrm{d}\nu}{\mathrm{d}p_g}\left(p_g^*\right)\right)^{\top} &= \frac{\mathrm{d}^2 f}{\mathrm{d}p_h \mathrm{d}p_g}\left(x^*;\left(p_g^*,p_h^*\right)\right) &&= 0, \\ \frac{\mathrm{d}\nu}{\mathrm{d}p_h}\left(p_h^*\right) &&&= \frac{\mathrm{d}^2 f}{\mathrm{d}p_h \mathrm{d}p_h}\left(x^*;p_h^*\right) &&= 0. \end{aligned}$$

In addition, the sensitivities of the inequality constraint are

$$\frac{\mathrm{d}h}{\mathrm{d}p_f}\left(x^*;p_f^*\right) = -\frac{2}{3}, \quad \frac{\mathrm{d}h}{\mathrm{d}p_g}\left(x^*;p_g^*\right) = 2 \quad \textit{and} \quad \frac{\mathrm{d}h}{\mathrm{d}p_h}\left(x^*;p_h^*\right) = -1.$$

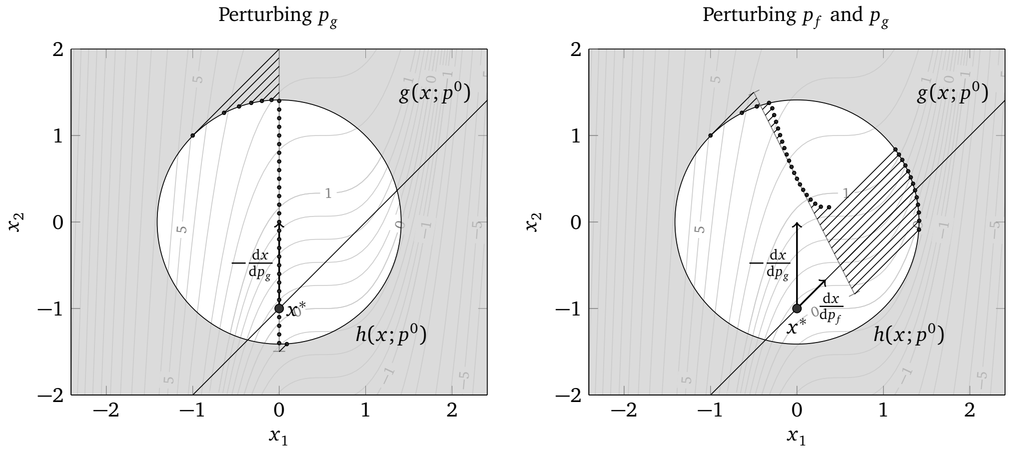

Figure 2.4: Sensitivities and first-order approximations for perturbations of parameters p_g (left) and (p_f, p_g) (right) of Example 2.32. First-order approximations are in light gray. Optimal solutions (small dark gray dots) are connected with their accompanying sensitivity based approximation. Level sets of the objective function are given for the reference problem.

Consequently, $\mathscr{P}_{lin} = [-3/2, \infty) \times (-\infty, 3/2] \times [1, \infty)$*. In Figure 2.4 (left) first-order approximations* (2.15) *for parameters* $p_g = \{-2.0, -1.9, \ldots, 1.5\}$ *and* $p_f = p_f^*, p_h = p_h^*$ *are illustrated. The figure shows that the approximations are of high quality. The upper bound of* p_g *in* $\mathscr{P}$ *does not deviate much from* $\mathscr{P}_{lin}$ *(bottom of feasible region in Figure 2.4), but – since curvature information is missing in the linear approximation* $\mathscr{P}_{lin}$ *– its lower bound is of no use here. For Figure 2.4 (right) parameters* $(p_f, p_g) = \{(-1.5, -2.0), (-1.4, -1.9), \ldots, (2.0, 1.5)\}$ *are considered. Although these perturbations are allowed by* $\mathscr{P}_{lin}$*, the approximation* $\mathscr{P}_{lin}$ *seems not to cope very well with the combination of the two perturbations. This, however, could have been expected from the definition of* $\mathscr{P}_{lin}$ *since the combination of two different perturbations has not been considered there. It can easily be changed if the relation between the perturbations is known a priori. Nevertheless, the approximations are of good quality for small perturbations of* p_f*. All in all, the example shows that the first-order approximations are a valuable tool for appropriate perturbations, but have to be handled with great care.*

Chapter 3

Numerical Solution Algorithms

After Chapter 2 has introduced the foundations of nonlinear programming, this chapter describes how to solve a nonlinear program. In practice this can rarely be done analytically, explaining the high demand for numerical solution algorithms. The key concept of these algorithms is to iteratively come closer to the optimal solution x^* starting at an initial guess x^0. This defines a sequence

$$x^0, x^1, x^2, x^3, \dots, \qquad \text{with (hopefully)} \qquad \lim_{k\to\infty} x^k = x^*. \tag{3.1}$$

The difference of two consecutive iterates is the step direction $\Delta x^k := x^{k+1} - x^k$, which has to be provided by the algorithm.[1]

The theoretical analysis of numerical algorithms divides into the following two: The *global convergence analysis* investigates the possible outcomes of the sequence $\left\{x^k\right\}_k$, which should be either a first-order optimal solution (preferably a KKT point but at least a FJ point) or a certificate of infeasibility, and the *local convergence analysis* studies the speed of convergence to these outcomes.[2] To formally classify the latter, the rate and order of convergence need to be defined.

Definition 3.1 (Rate of Convergence, Order of Convergence). *Let* $\left\{x^k\right\}_k \subseteq \mathbb{R}^{n_x}$ *be a sequence with* $\lim_{k\to\infty} x^k = x^*$ *and* $\{r_k\}_k \subseteq \mathbb{R}_{0+}$ *a convergent sequence. Furthermore, assume that it exists* $q \geq 1$ *and* $r \geq 0$ *such that*

$$\lim_{k\to\infty} \frac{\left\|x^{k+1} - x^*\right\|}{\|x^k - x^*\|^q} = \lim_{k\to\infty} r_k = r \tag{3.2}$$

holds. The constant q *is called order of convergence and* r *is the rate of convergence. The following cases are of special interest:*

i. If $q = 1$ *and* r_k *is fixed to* $r \in (0,1)$*, the sequence* $\left\{x^k\right\}_k$ *has a linear order of convergence, or is said to converge q-linearly.*

[1] The definition of the step direction Δx^k will be redefined to $\alpha_k \Delta x^k := x^{k+1} - x^k$ for some $\alpha_k \in (0,1]$ in case of line-search methods (cf., Section 3.3), but for now it is more convenient to assume $\alpha_k = 1$.

[2] As mentioned in Chapter 2, a first-order optimal solution stands for the local optimal solution defined in Definition 2.1. Global convergence does not imply the convergence to global optimal solutions.

ii. If $q = 1$ *and* $\lim_{k\to\infty} r_k = 1$*, the sequence* $\{x^k\}_k$ *converges q-sublinearly.*

iii. If $q = 1$ *and* $\lim_{k\to\infty} r_k = 0$*, the sequence* $\{x^k\}_k$ *converges q-superlinearly.*

iv. If $q = 2$ *and* r_k *is fixed to* $r > 0$*, the sequence* $\{x^k\}_k$ *converges q-quadratically.*

As the naming in Definition 3.1 suggests, an order of convergence $q \geq 2$ is a special case of a q-superlinear convergence order. To see this, simply consider multiplying (3.2) by

$$\lim_{k\to\infty} r_k := \lim_{k\to\infty} \left\|x^k - x^*\right\|^{q-1} = 0. \tag{3.3}$$

Using the Landau notation of Definition 1.1, the formulation of the q-quadratic order of convergence simplifies to $\left\|x^{k+1} - x^*\right\| = \mathcal{O}\left(\left\|x^k - x^*\right\|^2\right)$ and of the q-superlinear order to $\left\|x^{k+1} - x^*\right\| = o\left(\left\|x^k - x^*\right\|\right)$. In the remainder, q-quadratic convergence is also referred to as *fast local convergence*.

3.1 Lagrange-Newton Method for Equality Constrained Programs

To motivate the nonlinear programming algorithms, first the inequality constraints are neglected and the equality constrained optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \end{aligned} \tag{3.4}$$

is considered. For this special case of (NLP), finding a KKT point simplifies to the determination of a root for the nonlinear equation system

$$\Phi(x,\lambda) = \begin{bmatrix} \nabla_x L(x,\lambda) \\ g(x) \end{bmatrix} = 0, \tag{3.5}$$

(cf., Theorem 2.15). For this, it is well-known that under the assumption of $\nabla\Phi\left(x^k, \lambda^k\right)$ being a regular matrix, Newton's method can be applied, which yields

$$\begin{bmatrix} \Delta x^k \\ \Delta\lambda^k \end{bmatrix} = \begin{bmatrix} x^{k+1} - x^k \\ \lambda^{k+1} - \lambda^k \end{bmatrix} = -\nabla\Phi\left(x^k, \lambda^k\right)^{-1} \Phi\left(x^k, \lambda^k\right) \tag{3.6}$$

or the equivalent linear equation system

$$\begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k\right) & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta\lambda^k \end{bmatrix} = -\begin{bmatrix} \nabla_x L\left(x^k, \lambda^k\right) \\ g\left(x^k\right) \end{bmatrix}. \tag{3.7}$$

Summarizing the above, a rule for determining the step Δx^k has been developed in (3.7) and thus a first optimization algorithm, the so called *(primal-dual) Lagrange-Newton method*. The additional term *primal-dual* indicates that a dual step $\Delta\lambda^k$ is calculated simultaneously with the primal step Δx^k. A formal description is given by Algorithm A. Since it is basically Newton's method, it inherits its properties. In particular, the Lagrange-Newton method is q-quadratically convergent under standard assumptions.

Algorithm A Locally Convergent Lagrange-Newton Method for Equality Constrained Programs

A-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0\right)$. Choose a parameter $\varepsilon_{\text{tol}} > 0$.
A-2: *(Optimality check)* If $\left\|\Phi\left(x^k, \lambda^k\right)\right\| \leq \varepsilon_{\text{tol}}$, then STOP; x^k is a first-order optimal point of (3.4).
A-3: *(Step calculation)* Let $\left(\Delta x^k, \Delta\lambda^k\right)$ be the solution of the linear equation system (3.7).
A-4: *(Iterate update)* Set $\left(x^{k+1}, \lambda^{k+1}\right) \leftarrow \left(x^k, \lambda^k\right) + \left(\Delta x^k, \Delta\lambda^k\right)$.
A-5: *(k increment)* Set $k \leftarrow k+1$ and go to Step A-2.

Theorem 3.2 (Local Convergence of Lagrange-Newton Method). *Let (x^*, λ^*) be a first-order optimal point of* (3.4) *that satisfies the LICQ and SOSC. Furthermore, assume $\nabla^2_{xx} L(x, \lambda)$ to be Lipschitz-continuous. Then, there exists a neighborhood $\mathcal{N}(x^*, \lambda^*)$ such that for all $\left(x^0, \lambda^0\right) \in \mathcal{N}(x^*, \lambda^*)$ and sequences $\left\{\left(x^k, \lambda^k\right)\right\}_k$ of Algorithm A the matrix $\nabla\Phi\left(x^k, \lambda^k\right)$ is regular and $\left\{\left(x^k, \lambda^k\right)\right\}_k$ converges to (x^*, λ^*) q-quadratically.*

Proof. See Geiger and Kanzow [84, Theorem 5.26]. □

Unfortunately, the extension of the Lagrange-Newton method to inequality constrained problems is not straightforward. Because Newton's method can only be applied to equality constraints, it cannot directly include the additional KKT conditions

$$0 \leq \nu, \quad H(x)\nu = 0, \quad h(x) \leq 0. \tag{3.8}$$

Thus, and due to the local convergence statement of Theorem 3.2, the following central questions arise:

i. How to simplify the optimization problem, in particular regarding the handling of nonlinear inequality constraints?
ii. How to establish global convergence, i.e., convergence for initial guesses outside the neighborhood $\mathcal{N}(x^*, \lambda^*)$?
iii. Closely related: How to guarantee that $\nabla\Phi\left(x^k, \lambda^k\right)$ is a regular matrix and, thus, the linear equation system (3.7) is solvable outside the neighborhood $\mathcal{N}(x^*, \lambda^*)$?
iv. How to actually solve a problem with inequality constraints?
v. And, after all the former aspects have been settled: How to increase efficiency especially for large-scale nonlinear programs?

All these questions have intensively been studied in the literature, which will be surveyed in the following sections of this chapter.

3.2 Strategies for Simplifying Inequality Constraints

3.2.1 Reformulations

Reformulations convert the original nonlinear program (NLP) to an equivalent one – meaning that the two optimization problems share the same optimal solutions – but with improved properties for practical algorithms. In this subsection the focus is on reformulations that improve in particular the handling of inequality constraints.

Equality Constrained Reformulation

A good reformulation seems to transform inequality to equality constraints, which would cause the complicating complementarity condition (3.8) to vanish completely and allow the direct application of the Lagrange-Newton method. This is in fact possible by introducing so called *slack variables* $s \in \mathbb{R}^{n_h}$ and considering the nonlinear program

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}, s \in \mathbb{R}^{n_g}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & h(x) + Ss = 0, \end{aligned} \tag{3.9}$$

(cf., Bertsekas [18, Section 1.4]).[3] It is easy to see from Definition 2.1 that a local optimal solution x^* of (NLP) implies the local optimal solution (x^*, s^*) of (3.9) with $s_i^* = \sqrt{-h_i(x^*)}$ for $i \in \{1, \dots, n_h\}$. The following results explain this on the first-order level and in addition show, that this also holds the other way round.

Proposition 3.3. *Let* (x^*, λ^*, ν^*) *be a first-order optimal solution of* (NLP) *and* $s_i^* = \sqrt{-h_i(x^*)}$ *for* $i \in \{1, \dots, n_h\}$. *Then,* $(x^*, s^*, \lambda^*, \nu^*)$ *is a first-order optimal solution of* (3.9).

Proof. The satisfaction of the KKT conditions of (3.9)

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\nu^* = 0 \tag{3.10a}$$

$$2\nu^* s^* = 0 \tag{3.10b}$$

$$g(x^*) = 0 \tag{3.10c}$$

$$h(x^*) + S_* s^* = 0, \tag{3.10d}$$

follows directly from (2.7) and $s_i^* = \sqrt{-h_i(x^*)}$ for $i \in \{1, \dots, n_h\}$. □

Proposition 3.4. *Let* $(x^*, s^*, \lambda^*, \nu^*)$ *be a first-order optimal solution of* (3.9) *satisfying the second-order necessary condition of Theorem 2.24. Then,* (x^*, λ^*, ν^*) *is a first-order optimal solution of* (NLP) *satisfying the second-order necessary condition.*

Proof. The second-order necessary condition of (3.9) is

$$\begin{bmatrix} d \\ d_s \end{bmatrix}^\top \begin{bmatrix} \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) & 0 \\ 0 & 2\,\mathrm{diag}(\nu^*) \end{bmatrix} \begin{bmatrix} d \\ d_s \end{bmatrix} \geq 0 \tag{3.11}$$

for all (d, d_s) such that $\nabla g(x^*)^\top d = 0$ and $\nabla h(x^*)^\top d + 2(s^*)^\top d_s = 0$.

Now assume, that $(x^*, s^*, \lambda^*, \nu^*)$ is a first-order optimal solution of (3.9) satisfying the second-order necessary condition. Then (3.10d) is equivalent with $s_i^* = \sqrt{-h_i(x^*)}$ for $i \in \{1, \dots, n_h\}$ and together with (3.10b) it follows $\nu^* \perp h(x^*) \leq 0$. The non-negativity of ν^* follows from the second-order necessary condition by setting $d = 0$, $(d_s)_i = 1$ for $s_i^* = 0$ and $(d_s)_i = 0$ otherwise. In case of $s_i^* > 0$, (3.10b) directly yields $\nu_i^* = 0$. Finally, setting $d_s = 0$ (3.11) shows the second-order necessary condition of (NLP). □

[3] Recall that $S = \mathrm{diag}(s)$.

There are, however, considerable drawbacks of the reformulation (3.9). First, for large-scale nonlinear programs an enormous amount of slack variables may have to be added. Secondly, even for simple constraints of the form $h(x) = \operatorname{diag}(a)x + b$ with $a, b \in \mathbb{R}^{n_x}$ the reformulation has to be applied, transforming linear constraints to nonlinear ones. And finally, the KKT conditions include the complementarity constraint (3.10b), which can cause difficulties for the Lagrange-Newton method and is numerically studied, for instance, by Fletcher and Leyffer [70] for the very similar problem class of Mathematical Programs with Equilibrium Constraints (MPECs). The reason is that for an intermediate value $s^k = 0$ the Lagrange-Newton step (3.7) for problem (3.9) generates $\Delta s^k = 0$, which keeps $s^{k+1} = 0$ and would be problematic if $s^* \neq 0$. The analog situation arises for $\nu^k = 0$.

Infeasible Reformulation

At various places in Chapter 2 the possibility of finding x such that $h(x) \leq 0$ has been assumed, even when the existence of a feasible point of (NLP) is not guaranteed. This feasibility assumption on the inequality constraints will also be made in every iteration x^k by some algorithms presented in Section 3.5, i.e., $h\left(x^k\right) \leq 0$ for all k including $k = 0$. Finding such an initial point, however, can be very difficult for arbitrary nonlinear functions $h(x)$. Therefore, instead of solving (NLP) directly, it can be first reformulated to the equivalent nonlinear problem

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}, s \in \mathbb{R}^{n_h}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & h(x) - s = 0 \\ & s \leq 0 \end{aligned} \tag{3.12}$$

again by using slack variables $s \in \mathbb{R}^{n_h}$. The feasibility assumption now applies to $s \leq 0$ and not to $h(x) \leq 0$ anymore, which is why (3.12) is called *infeasible reformulation*. Determining $s^k \in \mathbb{R}^{n_h}$ such that $s^k \leq 0$ for all k, however, is trivial. The following lemma states the relation between first-order optimal points of (3.12) and the original nonlinear program (NLP).

Proposition 3.5. *The primal-dual point* (x^*, λ^*, ν^*) *is first-order optimal of* (NLP) *with* $s^* = h(x^*)$ *and* $\lambda_h^* = \nu^*$*, if and only if,* $\left(x^*, s^*, \lambda_h^*, \lambda^*, \nu^*\right)$ *is a first-order optimal point of* (3.12).

Proof. The proof directly follows from the KKT conditions of (3.12), which are:

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\lambda_{\mathrm{h}}^* = 0 \tag{3.13a}$$

$$-\lambda_{\mathrm{h}}^* + \nu^* = 0 \tag{3.13b}$$

$$g(x^*) = 0 \tag{3.13c}$$

$$h(x^*) - s^* = 0 \tag{3.13d}$$

$$0 \leq \nu^* \perp s^* \leq 0. \tag{3.13e}$$

□

In practical algorithms based on the first-order optimality conditions, the additional Lagrangian multiplier $\lambda_{\mathrm{h}}^* \in \mathbb{R}^{n_h}$ can be removed by fixing $\lambda_{\mathrm{h}}^* := \nu^*$. Then, condition (3.13b) can be neglected, (3.13a) changes to its original formulation $\nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\nu^* = 0$ and the infeasible reformulation appears to be just a simplification of the problematic complementarity condition (3.8). Note, that in practice, the infeasible reformulation needs less slack variables than the equality constrained reformulation, because constraints of the form $h(x) = \mathrm{diag}(a)x + b$ with $a, b \in \mathbb{R}^{n_x}$ do not have to be reformulated.

3.2.2 Sequential Quadratic Programming

The central element of the Lagrange-Newton method for equality constrained nonlinear programs (Algorithm A) was the solution of the linear equation system (3.7). With the definition $\lambda^{k+1} = \lambda^k + \Delta\lambda^k$, this system can equally be written as

$$\nabla^2_{xx} L\big(x^k, \lambda^k\big)\Delta x^k + \nabla g\big(x^k\big)\lambda^{k+1} = -\nabla f\big(x^k\big), \tag{3.14a}$$

$$\nabla g\big(x^k\big)^\top \Delta x^k = -g\big(x^k\big). \tag{3.14b}$$

A closer look offers a new interpretation of the Lagrange-Newton method. In fact, the linear equation system (3.14) corresponds to KKT conditions of the Quadratic Program (QP)

$$\begin{aligned} \min_{\Delta x^k \in \mathbb{R}^{n_x}} \quad & \frac{1}{2}\big(\Delta x^k\big)^\top \nabla^2_{xx} L\big(x^k, \lambda^k\big)\Delta x^k + \nabla f\big(x^k\big)^\top \Delta x^k \\ \text{subject to} \quad & g\big(x^k\big) + \nabla g\big(x^k\big)^\top \Delta x^k = 0 \end{aligned} \tag{3.15}$$

and, thus, solving (3.14) is equivalent to solving (3.15). Now, an extension to inequality constraints seems to be simple and intuitive by considering

$$\begin{aligned} \min_{\Delta x^k \in \mathbb{R}^{n_x}} \quad & \frac{1}{2}\big(\Delta x^k\big)^\top \nabla^2_{xx} L\big(x^k, \lambda^k, \nu^k\big)\Delta x^k + \nabla f\big(x^k\big)^\top \Delta x^k \\ \text{subject to} \quad & g\big(x^k\big) + \nabla g\big(x^k\big)^\top \Delta x^k = 0 \\ & h\big(x^k\big) + \nabla h\big(x^k\big)^\top \Delta x^k \leq 0 \end{aligned} \tag{QP}$$

for the determination of the step Δx^k. This strategy is called *Sequential Quadratic Programming (SQP)* since a quadratic program has to be solved in every iteration. It was introduced by Wilson [192] and a formal description is provided in Algorithm B. The SQP method is indeed fast locally convergent under similar assumptions as for the Lagrange-Newton method, because the inequality constraints have been moved to the QP level.

Theorem 3.6 (Local Convergence of SQP Method). *Let (x^*, λ^*, ν^*) be a first-order optimal point of* (NLP) *that satisfies the LICQ, SCC and SOSC. Furthermore, assume $\nabla^2_{xx} L(x, \lambda, \nu)$ to be Lipschitz-continuous. Then, there exists a neighborhood $\mathcal{N}(x^*, \lambda^*, \nu^*)$ such that for all $\big(x^0, \lambda^0, \nu^0\big) \in \mathcal{N}(x^*, \lambda^*, \nu^*)$ the sequence $\big\{\big(x^k, \lambda^k, \nu^k\big)\big\}_k$ of Algorithm B converges to (x^*, λ^*, ν^*) q-quadratically.*

Proof. See Theorem 3.7. □

Note, that the SQP method does not answer the question of how to solve a problem with inequality constraints, because a quadratic program with inequality constraints still has to be

Algorithm B Locally Convergent SQP Method

B-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0, \nu^0\right)$. Choose a parameter $\varepsilon_{\text{tol}} > 0$.

B-2: *(Optimality check)* If (2.7) is satisfied up to ε_{tol} for $\left(x^k, \lambda^k, \nu^k\right)$, then STOP; x^k is a first-order optimal point of (NLP).

B-3: *(Step calculation)* Solve the quadratic problem (QP) to get $\left(\Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$. If multiple solutions exist, choose the one such that $\left(x^k + \Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$ is closest to the last iterate $\left(x^k, \lambda^k, \nu^k\right)$.

B-4: *(Iterate update)* Set $x^{k+1} \leftarrow x^k + \Delta x^k$.

B-5: *(k increment)* Set $k \leftarrow k + 1$ and go to Step B-2.

solved. However, these inequality constraints are only linear now, which simplifies the complementarity condition (3.8). On the downside, this linearization reveals a new complication for globalization specially for SQP methods. The quadratic program (QP) can be infeasible (cf., Powell [163]), an aspect that is ignored for the moment. A comprehensive overview over the basic SQP algorithm is given among others by Boggs and Tolle [19], Conn et al. [45, Chapter 15], Gill et al. [90] and Nocedal and Wright [151, Chapter 18]. In the literature the Lagrange-Newton method is also referred to as SQP due to its equivalence stated in the beginning of this subsection.

Sensitivity Based Motivation of SQP

Although the motivation of the SQP method seems to be intuitive at first sight, it may actually be not directly clear why the addition of the linearized inequality constraint is valid. To see this, the SQP method can be understood as a special case of a sensitivity based recursive algorithm. Therefore, consider the parameter dependent program

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}} \quad & f(x;p) := \frac{1}{2}(x - p_x)^\top \nabla^2_{xx} L(p_x, p_\lambda, p_\nu)(x - p_x) \\ & \qquad + \nabla f(p_x)^\top (x - p_x) + f(p_x) \\ \text{subject to} \quad & g(x;p) := g(p_x) + \nabla g(p_x)(x - p_x) = 0 \\ & h(x;p) := h(p_x) + \nabla h(p_x)(x - p_x) \leq 0 \end{aligned} \tag{3.16}$$

with the unknown optimal solution of (NLP) as reference parameter, i.e., $p^* = (x^*, \lambda^*, \nu^*)$. Theoretically there are two possibilities to determine the solution of a perturbed program (3.16) with parameter $p = \left(x^k, \lambda^k, \nu^k\right)$. On the one hand, if p is close enough to p^*, the Sensitivity Theorem 2.26 showed the existence of functions $(x(p), \lambda(p), \nu(p))$ that are a primal-dual optimal solution of (3.16). On the other hand, the solution of (3.16) can simply be calculated by QP techniques. Let $\left(x^{k+1}, \lambda^{k+1}, \nu^{k+1}\right)$ be this solution and if multiple such solutions exist, choose the one that is closest to p. Now, if $\left(x^k, \lambda^k, \nu^k\right)$ is close enough to (x^*, λ^*, ν^*), these two possibilities coincide and it follows

$$x^{k+1} = x\left(x^k\right), \quad \lambda^{k+1} = \lambda\left(\lambda^k\right), \quad \text{and} \quad \nu^{k+1} = \nu\left(\nu^k\right), \tag{3.17}$$

which defines a sensitivity based recursive algorithm, formally described in Algorithm C. The local convergence then follows by a contraction condition on the KKT conditions $\Phi(x, \lambda, \nu; p)$ of (3.16) with respect to one recursive iteration of the form (3.17).

Algorithm C Locally Convergent Sensitivity Based Recursive Algorithm

C-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0, \nu^0\right)$. Choose a parameter $\varepsilon_{\text{tol}} > 0$.
C-2: *(Optimality check)* If (2.7) is satisfied up to ε_{tol} for $\left(x^k, \lambda^k, \nu^k\right)$, then STOP; x^k is a first-order optimal point of (NLP).
C-3: *(Step calculation)* Let $\left(x^{k+1}, \lambda^{k+1}, \nu^{k+1}\right)$ be a KKT point of (3.16) with $p = \left(x^k, \lambda^k, \nu^k\right)$. If multiple such points exist, choose the one that is closest to p.
C-4: *(k increment)* Set $k \leftarrow k + 1$ and go to Step C-2.

Theorem 3.7 (Local Convergence of Sensitivity Based Recursive Algorithm). *Let* (x^*, λ^*, ν^*) *be a first-order optimal point of* (3.16) *that satisfies the LICQ, SCC and SOSC and* $\omega(p) := (x(p), \lambda(p), \nu(p))$ *be the functions defined by the Sensitivity Theorem 2.26. Furthermore, assume*

$$\|\Phi(x(p), \lambda(p), \nu(p); \omega(p)) - \Phi(x(p), \lambda(p), \nu(p); p)\| = \mathcal{O}\left(\|\omega(p) - p\|^2\right)$$

for $p = (x, \lambda, \nu)$*, as well as*

$$\begin{aligned} \Phi(x, \lambda, \nu) &= \Phi(x, \lambda, \nu; p) \\ \nabla_{(x,\lambda,\nu)}\Phi(x, \lambda, \nu) &= \nabla_{(x,\lambda,\nu)}\Phi(x, \lambda, \nu; p) \end{aligned}$$

in a neighborhood around (x^*, λ^*, ν^*)*. Then, there exists a neighborhood* $\mathcal{N}(x^*, \lambda^*, \nu^*)$ *such that for all* $\left(x^0, \lambda^0, \nu^0\right) \in \mathcal{N}(x^*, \lambda^*, \nu^*)$ *the sequence* $\left\{\left(x^k, \lambda^k, \nu^k\right)\right\}_k$ *of Algorithm C converges to* (x^*, λ^*, ν^*) *q-quadratically.*

Proof. See Robinson [168, Theorem 3.1]. □

The conditions of Theorem 3.7 can easily be checked for problem (3.16) by using Taylor's Theorem. Because Theorem 3.7 is based on the Sensitivity Theorem 2.26, the active set $\mathcal{A}(x; p)$ is identical to $\mathcal{A}(x^*; p^*)$ for all $p = \left(x^k, \lambda^k, \nu^k\right) \in \mathcal{N}(x^*, \lambda^*, \nu^*)$. This justifies the inclusion of each inequality constraint in (QP): Either it has to be active and can be handled like an equality constraint, or inactive and strictly fulfilled anyway in the neighborhood $\mathcal{N}(x^*, \lambda^*, \nu^*)$ (cf., Section 2.1). Because of the equivalence of the active set, the SQP method for programs with inequality constraints is locally similar to the one for equality constraints, which leads to the fast local convergence. Finally, it is remarked, that the problem (3.16) has been defined to correspond to the SQP method, but this special definition has not been used for the convergence result. Robinson [167, 168] lists other possible q-quadratically convergent choices, but (3.16) for the SQP method seems to be the most practical one because it is only a quadratic program.

Sequential Linear Programming

Following the same motivation as for the SQP method, it is also possible to develop *Sequential Linear Programming (SLP)* methods, where the quadratic term in (QP) is neglected. Special care has to be taken that the linear problem is bounded and a step Δx^k can actually be calculated. The interested reader is referred to Fletcher et al. [71] and Palacios-Gomez et al. [156]. While solving a linear program is significantly more efficient than the quadratic program (QP), the local convergence properties of the SLP method, and thus its numerical performance, are lagging behind. To overcome these problems, a combination of the SLP and SQP method have been developed by Chin and Fletcher [44] and Waltz [188] more recently.

3.3 Globalization Strategies

Up to now the presented methods have been shown to be only locally convergent. In practice, it is however difficult to provide an initial guess close to the optimal solution. It is therefore desirable that the algorithm converges for any – maybe feasible – initial guess. The reason that Theorem 3.2 only ensures local convergence is the fact, that the Lagrange-Newton method is based on a linear model of the KKT conditions, e.g., the second equation of (3.7) is $h(x^k) + \nabla h(x^k)^\top \Delta x^k = 0$. For nonlinear functions, this approximation is locally sufficient but may be of poor quality for large steps Δx^k. This already motivates the two basic globalization strategies:

Line-Search Methods: When necessary, reduce the step length by scaling the step Δx^k afterwards with the *step size* $\alpha_k \in (0, 1]$. The new iterate is then determined by

$$x^{k+1} = x^k + \alpha_k \Delta x^k. \tag{3.18}$$

Newton's method with such a line-search is called *damped Newton's method*.

Trust-Region Methods: When necessary, reduce the step length by imposing the additional constraint $\left\|\Delta x^k\right\| \le \tau$ at the step calculation, forcing the step Δx^k to lay within a region defined by the *trust-region radius* $\tau \in (0, \infty)$. Since this additional constraint is an inequality constraint, the use of an SQP method is necessary for this strategy. The step could be determined for instance by the quadratic program (cf., Section 3.2.2)

$$\begin{aligned} \min_{\Delta x^k \in \mathbb{R}^{n_x}} \quad & \frac{1}{2} (\Delta x^k)^\top \nabla^2_{xx} L(x^k, \lambda^k, \nu^k) \Delta x^k + \nabla f(x^k)^\top \Delta x^k \\ \text{subject to} \quad & g(x^k) + \nabla g(x^k)^\top \Delta x^k = 0 \\ & h(x^k) + \nabla h(x^k)^\top \Delta x^k \le 0 \\ & -\tau e \le \Delta x^k \le \tau e. \end{aligned} \tag{3.19}$$

While for line-search methods a reduction of the step length (updating α_k) is considerably cheap, trust-region methods (updating τ) need to recalculate the quadratic program (3.19), which is the most expensive part of the optimization algorithm. On the other hand, the recalculation of (3.19) allows to compute a new orientation of the step, which can even be intensified by adding a scaling to the trust-region constraint, e.g., $\left\|D\Delta x^k\right\| \le \tau$ with a diagonal scaling matrix $D \in \mathbb{R}^{n_x \times n_x}$ (cf., Bellavia et al. [10]). For an overview of trust-region methods, the reader is referred to Sorensen [180] and Conn et al. [45] as this presentation focuses on the line-search approach. Nocedal and Yuan [152] and Waltz et al. [187] propose to combine both, the line-search and trust-region methods to benefit from the fast step length update of the line-search methods and, if these fail implying that the step size gets too small, update the step direction by resolving (3.19).

To decide when a reduction of the step Δx^k has to be performed, a measure for the quality of the step is necessary. Assessing the quality of Δx^k is always a balance between the two goals of the optimization: reduction of the objective function $f(x)$ and the constraint violation

$$\theta(x) := \|(g(x), \max\{h(x), 0\})\|. \tag{3.20}$$

For this, different strategies have been proposed in the literature. Among them are merit functions, filters and piecewise linear penalty functions, which are introduced exemplarily for the line-search method in the following subsections.[4]

3.3.1 Merit Function

The most straightforward idea for measuring progress is to combine the two goals – reduction of the objective function and constraint violation – into the so called *merit function* $\Psi : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$, e.g., defined by

$$\Psi(x;\tau) := f(x) + \tau\theta(x) \tag{3.21}$$

with a penalty parameter $\tau \in (0, \infty)$ that balances these two goals.[5] The measure of constraint violation $\theta(x)$ does not need to be defined as in (3.20), but a merit function has to fulfill two necessary conditions:

i. An optimal solution of $\min_{x\in\mathbb{R}^{n_x}} \Psi(x;\tau)$ for $\tau \to \infty$ must be an optimal solution of (NLP) and vice versa.
ii. A step Δx^k must produce a reduction in the merit function and therefore be a descent direction for it, i.e., $\nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k < 0$.

These two necessary conditions ensure that with every iteration k the merit function decreases monotonically until it ends up at an optimal solution x^*. So, optimizing (NLP) becomes equivalent to the unconstrained minimization of (3.21), but with the difference that the step calculation does not rely on the merit function directly.

Popular examples of merit functions are:

i. The ℓ_p merit functions (cf., Han [113]):

$$\Psi(x;\tau) = f(x) + \tau\,\|(g(x), \max\{h(x), 0\})\|_p, \quad p \in \{1, 2, \infty\} \tag{3.22}$$

ii. The differentiable ℓ_2 merit function for equality constrained problems (cf., Fiacco and McCormick [64, Chapter 4]):

$$\Psi(x;\tau) = f(x) + \frac{1}{2}\tau\,\|g(x)\|_2^2 \tag{3.23}$$

[4] Other globalization strategies like Gould and Toint [101] or Liu and Yuan [136] depend on different step calculations and are therefore not considered here.

[5] It is also possible to position the penalty parameter in front of the objective function, but (3.21) is the common definition in the literature.

iii. The *augmented Lagrangian* merit function for equality constrained problems (cf., Hestenes [116] and Powell [164]):

$$\Psi(x;\tau) = f(x) + \lambda^\top g(x) + \frac{1}{2}\tau \, \|g(x)\|_2^2 \tag{3.24}$$

iv. The augmented Lagrangian merit function for inequality constrained problems (cf., Arrow et al. [8] and Rockafellar [169]):

$$\Psi(x;\tau) = f(x) + \frac{1}{4\tau} \sum_{i=1}^{n_h} \left((\max\{\nu_i + \tau h_i(x), 0\})^2 - \nu_i^2 \right) \tag{3.25}$$

Exact Merit Functions

It can be impractical that the penalty parameter τ has to go to infinity in order to satisfy the necessary condition of merit functions, mentioned above. Instead, one wishes that there exists a finite penalty parameter $\bar{\tau} > 0$ such that this condition holds. For an implementation it would then be sufficient to choose this parameter $\bar{\tau}$ and never increase it. Merit functions having this additional property are called *exact* merit functions.

Definition 3.8 (Exact Merit Functions). *A merit function $\Psi(x;\tau)$ defined by* (3.21) *is called exact at an optimal solution x^*, if there exists a fixed parameter $\bar{\tau} > 0$ such that for all $\tau > \bar{\tau}$ the point x^* is also an optimal solution of* $\min_{x \in \mathbb{R}^{n_x}} \Psi(x;\tau)$.

It turns out that the ℓ_p and augmented Lagrangian merit functions are exact as stated in the following theorems, but unfortunately the differentiable ℓ_2 merit function is not.

Theorem 3.9. *Let x^* be an optimal solution of* (NLP) *satisfying the MFCQ and SOSC. Then, the merit function $\Psi(x;\tau) = f(x) + \tau \, \|(g(x), \max\{h(x), 0\})\|_p$ with $p \in [1, \infty]$ is exact.*

Proof. See Han and Mangasarian [114, Corollary 4.7]. □

Theorem 3.10. *Let x^* be an optimal solution of* (NLP) *satisfying the MFCQ and SOSC. Then, the merit function $\Psi(x;\tau) = f(x) + \lambda^\top g(x) + \frac{1}{2}\tau \, \|g(x)\|_2^2$ is exact.*

Proof. See Hestenes [116, Theorem 2.1]. □

The drawback of exact merit functions, however, is that the penalty parameter $\bar{\tau}$ is unknown a priori. This requires a strategy to update the penalty parameter during the optimization. Unfortunately, choosing a very large value from the beginning and hoping to be larger than $\bar{\tau}$ is not a good option as it can lead to very slow convergence. A very small penalty, on the other hand, can cause the attraction of unbounded infeasible points, if the objective function decreases much faster than the constraint violation increases. A survey on exact merit functions is given by Di Pillo [52], which also proposes to use penalty parameters that depend on the constraint violation to overcome the latter drawback.

Sufficient Decrease Condition

So far it has been neglected that the descent direction property, i.e., $\nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k < 0$, does not lead to a sufficient reduction of the merit function $\Psi(x;\tau)$ for nonlinear programming, since – similarly to the beginning of Section 3.3 – this property is based on local information only. This is the point, where the line-search method comes into play and the step Δx^k may have to be shortened. In the following it is assumed, that the merit function is differentiable[6] and compare the actual reduction

$$\Psi\big(x^k + \alpha_k \Delta x^k;\tau\big) - \Psi\big(x^k;\tau\big) \tag{3.26}$$

with the predicted reduction based on a linear or quadratic Taylor approximation

$$\begin{aligned} &\Psi\big(x^k;\tau\big) + \alpha_k \nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k + \alpha_k^2 \big(\Delta x^k\big)^\top \nabla_{xx}^2 \Psi\big(x^k;\tau\big)\Delta x^k - \Psi\big(x^k;\tau\big) \\ =&\alpha_k \nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k + \alpha_k^2 \big(\Delta x^k\big)^\top \nabla_{xx}^2 \Psi\big(x^k;\tau\big)\Delta x^k. \end{aligned} \tag{3.27}$$

If the actual reduction is at least a fraction of the predicted reduction, the step is said to be acceptable. In case of a linear model of reduction this yields the Armijo [7] condition

$$\Psi\big(x^k + \alpha_k \Delta x^k;\tau\big) - \Psi\big(x^k;\tau\big) \le \sigma \alpha_k \nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k \le 0 \tag{3.28}$$

with a parameter $\sigma \in (0,1)$ and which is illustrated in Figure 3.1 (left). Wolfe [194, 195] proposes to extend the Armijo condition by

$$\nabla_x \Psi\big(x^k + \alpha_k \Delta x^k;\tau\big)^\top \Delta x^k \ge \eta \nabla_x \Psi\big(x^k;\tau\big)^\top \Delta x^k, \tag{3.29}$$

$\eta \in (\sigma,1)$, to avoid arbitrarily small step sizes. In practice however, this further condition is often neglected and instead a value $\alpha_k \in (0,1]$ satisfying the Armijo condition and being as large as possible is selected. Note, that finding the optimal step size, e.g., solving $\min_{\alpha_k>0} \Psi\big(x^k + \alpha_k \Delta x^k;\tau\big)$, is not a practical option since it involves the solution of a (nonsmooth) nonlinear program.

Exemplary for the SQP method, Algorithm D presents a globally convergent version of Algorithm B under rather strong assumptions.

Theorem 3.11 (Global Convergence of SQP Method with a Merit Function). *Let $\big\{\big(x^k,\lambda^k,\nu^k\big)\big\}_k$ be a sequence generated by Algorithm D such that the tuple $\big(x^k,\lambda^k,\nu^k\big)$ lies in some compact set for all k, x^k satisfies the LICQ and, for all $d \in \mathbb{R}^{n_x}$,*

$$c_1 \|d\|^2 \le d^\top \nabla_{xx}^2 L\big(x^k,\lambda^k,\nu^k\big) d \le c_2 \|d\|^2 \tag{3.30}$$

with $c_1 > 0$ and $c_2 > 0$. Then, $\big\{\big(x^k,\lambda^k,\nu^k\big)\big\}_k$ converges to a first-order optimal point of (NLP).

Proof. See Boggs and Tolle [19, Theorem 4.3]. □

[6] If the merit function is not differentiable, then the linear or quadratic model for the predicted reduction has to be chosen differently (cf., Byrd et al. [28]).

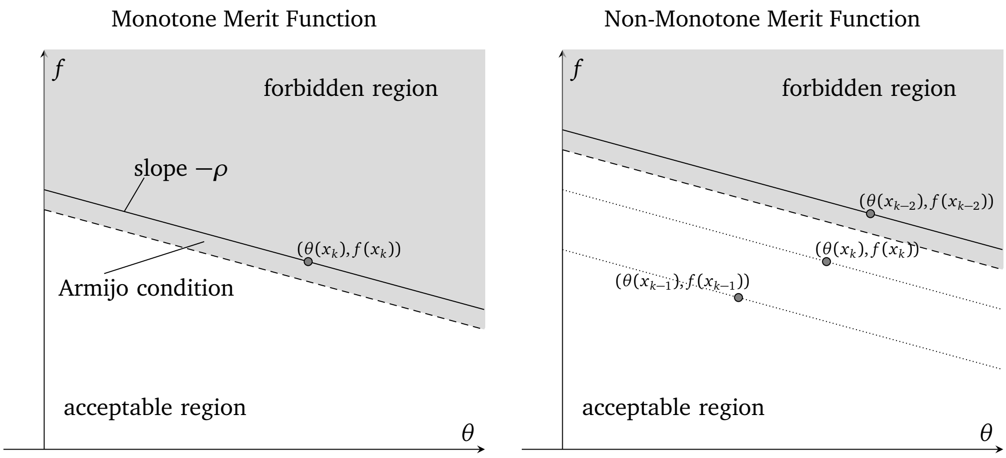

Figure 3.1: Monotone merit function (left) and non-monotone merit function (right). The non-monotonicity level on the right is $l = 2$.

Non-Monotone Merit Functions

Although Theorem 3.11 proofs global convergence for an optimization algorithm, the introduction of the merit function – the main extension done in Algorithm D – requires a new study of local convergence, since Theorem 3.6 is based on the full step ($\alpha_k = 1$). One could think that the same properties would hold, but this is actually not true. There exist examples (cf., Powell [165, Section 3]) that show search directions Δx^k yielding local q-quadratic convergence but increasing both, the objective function and the constraint violation, and, thus, would be rejected by the merit function. This is known as the *Maratos effect* [139]. But also in the unconstrained case, the step size can be reduced unnecessarily, for example when the step direction tries to follow a curvy valley. Possibilities to avoid this are the modification of the step Δx^k, in particular second-order-correction steps (cf., Conn et al. [45, Section 15.3.2.3] or Section 3.6.2), or the relaxation of the merit function acceptance criterion (3.28) to allow a non-monotone decrease of it. Examples include Chamberlain et al. [39], Panier and Tits [157] and Toint [182], which basically exchange (3.28) for

$$\Psi\big(x^k + \alpha_k \Delta x^k; \tau\big) - \max_{i=0,\dots,l_\mathrm{m}} \big\{\Psi\big(x^{(k-i)_+}; \tau\big)\big\} \leq \sigma \alpha_k \nabla_x \Psi\big(x^k; \tau\big)^\top \Delta x^k \tag{3.31}$$

and force a decrease with respect to the largest value of the former $l_\mathrm{m} \in \mathbb{N}$ merit function values, see Figure 3.1 (right). While non-monotone merit function techniques usually complicate the global convergence theory, overall efficiency gains can be reported (cf., Grippo et al. [112]).

3.3.2 Filter

As pointed out in Section 3.3.1, the choice and update strategy of the penalty parameter τ in the merit function can be difficult, but at the same time has a significant influence on the performance of an optimization algorithm. Recall the two sometimes contrary goals of the

Algorithm D Globally Convergent SQP Method (Merit Function)

D-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0, \nu^0\right)$. Choose parameters $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$, $\tau_0 > 0$ and $\varepsilon_{\text{tol}} > 0$.

D-2: *(Optimality check)* If (2.7) is satisfied up to ε_{tol} for $\left(x^k, \lambda^k, \nu^k\right)$, then STOP; x^k is a first-order optimal point of (NLP).

D-3: *(Step calculation)* Solve the quadratic problem (QP) to get $\left(\Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$. If multiple solutions exist, choose the one such that $\left(x^k + \Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$ is closest to the last iterate $\left(x^k, \lambda^k, \nu^k\right)$. If $\Delta x^k = 0$, then STOP; x^k is a first-order optimal point of (NLP).

D-4: *(Line search)*

D-4.1: *(Initialization)* Set $\alpha_k \leftarrow 1$ and increase τ_k if necessary such that $\tau_k > \left\|\left(\lambda^{k+1}, \nu^{k+1}\right)\right\|_\infty$.

D-4.2: *(Armijo)* If the Armijo condition (3.28) holds for

$$\Psi(x;\tau) = f(x) + \tau_k \left\|\left(g(x), \max\{h(x), 0\}\right)\right\|_1,$$

go to Step D-5.

D-4.3: *(Backtracking)* Set $\alpha_k \leftarrow \beta \alpha_k$ and go to Step D-4.2.

D-5: *(Iterate update)* Set $x^{k+1} \leftarrow x^k + \alpha_k \Delta x^k$ and $\tau_{k+1} \leftarrow \tau_k$.

D-6: *(k increment)* Set $k \leftarrow k+1$ and go to Step D-2.

globalization strategy, that a step Δx^k should suffice: Reduction of the objective function $f(x)$ and reduction of the constraint violation $\theta(x)$, which can formally be stated as the multi-objective optimization problem

$$\min_{x \in \mathbb{R}^{n_x}} \quad f(x) \qquad \text{and} \qquad \min_{x \in \mathbb{R}^{n_x}} \quad \theta(x), \tag{3.32}$$

with a slight emphasize on the latter as it must become zero in the end – its global optimal solution. This connection to multi-objective optimization motivated Fletcher and Leyffer [69] to propose the *filter*, for which a step Δx^k is acceptable if it reduces either the objective function $f(x)$ or the constraint violation $\theta(x)$ and not – as in the merit function approach – a combination of both, i.e., in the line-search context if

$$f\left(x^k + \alpha_k \Delta x^k\right) \le f\left(x^k\right), \quad \text{or} \tag{3.33a}$$

$$\theta\left(x^k + \alpha_k \Delta x^k\right) \le \theta\left(x^k\right). \tag{3.33b}$$

If both of the two conditions are satisfied, $x^k + \alpha_k \Delta x^k$ is said to *dominate* x^k in analogy to multi-objective optimization. Condition (3.33) is similar to force a reduction in the merit function $\Psi(x;\tau)$ for the specific penalty parameter choices $\tau = 0$ positioned either in front of the objective function or the constraint violation, respectively. Therefore, no penalty parameter has to be selected for the filter globalization.

Sufficient Decrease Condition

In order to produce a sufficient reduction, condition (3.33) has to be tightened to

$$f\left(x^k + \alpha_k \Delta x^k\right) + \gamma_f \delta_k \le f\left(x^k\right), \quad \text{or} \tag{3.34a}$$

$$\theta\left(x^k + \alpha_k \Delta x^k\right) + \gamma_f \delta_k \le \theta\left(x^k\right) \tag{3.34b}$$

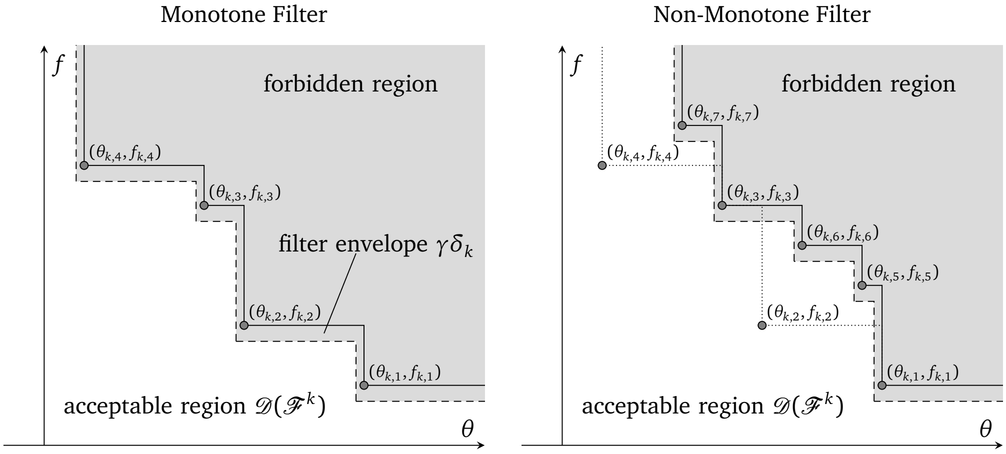

Figure 3.2: Monotone filter (left) and non-monotone filter (right). The filter envelope is dependent on current iterate and the non-monotonicity level on the right is $l = 1$.

with a filter envelope $\delta_k > 0$ and parameter $\gamma_f \in (0, 1)$. The filter envelope is often chosen to be $\delta_k = \theta\big(x^k\big)$, steering the iterates with a higher emphasize on the constraint violation and, similar to the Armijo condition (3.28), is generally necessary to promote global convergence. Unfortunately, with this choice of δ_k it is possible to reduce just the constraint violation in every iteration. Therefore, most filter approaches also include a separate sufficient reduction condition: If the step Δx^k is a descent direction of the objective function, specifically

$$\alpha_k \nabla f\big(x^k\big)^\top \Delta x^k < -\theta\big(x^k\big)^2 \leq 0 \tag{3.35}$$

– which will at least be the case close to the optimal solution – the Armijo condition

$$f\big(x^k + \alpha_k \Delta x^k\big) - f\big(x^k\big) \leq \sigma \alpha_k \nabla f\big(x^k\big)^\top \Delta x^k \overset{(3.35)}{<} -\sigma\theta\big(x^k\big)^2 \tag{3.36}$$

forces a decrease in the objective function sufficiently large compared to the current constraint violation. Conditions (3.35) and (3.36) can be formulated in a more sophisticated manner (cf., Fletcher et al. [72] and Wächter and Biegler [201]), but the above presentation increases the clarity. In addition, it would be sufficient to enforce (3.36) just if the constraint violation is below a certain threshold, which would increase the likelihood of the step being accepted.

To avoid cycling of the iterates, the reduction or domination condition (3.34) has to hold not just for the current iterate x^k, but also for some selected former iterates. This can be achieved by defining the filter $\mathscr{F}_k$ to be a set of these iterates specifying the acceptable region in the two-dimensional space of objective function and constraint violation values $\mathscr{D}(\mathscr{F}_k)$ to be

$$\mathscr{D}(\mathscr{F}_k) := \big\{(f, \theta) \in \mathbb{R}^2 \mid f \leq f' \text{ or } \theta \leq \theta' \text{ for all } \big(f', \theta'\big) \in \mathscr{F}_k\big\} . \tag{3.37}$$

Figure 3.2 (left) provides an illustration. The filter may be initially set to $\mathscr{F}_0 := \{(-\infty, \theta_{\max})\}$ with a maximum allowed constraint violation $\theta_{\max} > 0$ or simply to $\mathscr{F}_0 := \emptyset$ and, if necessary, is augmented by

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \big\{\big(f\big(x^k\big), \theta\big(x^k\big)\big)\big\} . \tag{3.38}$$

With the acceptable region defined in (3.37), the reduction condition for the former iterates can compactly be written as

$$\left(f\left(x^k + \alpha_k \Delta x^k\right) + \gamma_\mathrm{f}\delta_k, \theta\left(x^k + \alpha_k \Delta x^k\right) + \gamma_\mathrm{f}\delta_k\right) \in \mathscr{D}(\mathscr{F}_k), \tag{3.39}$$

and means that a new iterate must not be dominated by any element of the filter $\mathscr{F}_k$. The filter augmentation is required whenever the descent direction property (3.35) or the Armijo condition (3.36) do not hold and the acceptance only relies on (3.34) and (3.39). Otherwise, the objective function is strictly decreasing, which excludes the possibility of a cycle. Due to (3.39) iterates cannot return to points added to the filter previously or that lie in the filter's forbidden region. An alternative definition of the filter is to choose the envelope δ_k dependent on the former iterations, which would change the filter augmentation to

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \left\{\left(f\left(x^k\right) - \gamma_\mathrm{f}\delta_k, \theta\left(x^k\right) - \gamma_\mathrm{f}\delta_k\right)\right\} \tag{3.40}$$

and (3.39) to

$$\left(f\left(x^k + \alpha_k \Delta x^k\right), \theta\left(x^k + \alpha_k \Delta x^k\right)\right) \in \mathscr{D}(\mathscr{F}_k). \tag{3.41}$$

A severe disadvantage of the filter approach is that it can fail, meaning that no step size $\alpha_k \in (0,1]$ exists such that the reduction conditions (3.34), (3.35) and (3.36), and in particular (3.39) are satisfied. In that case a feasibility restoration must be invoked, that minimizes the constraint violation $\theta(x)$, e.g., by considering (FeasNLP), until a new acceptable iterate x^{k+1} is found such that $x^{k+1} \in \mathscr{D}(\mathscr{F}_k)$ and – similar to (3.34) –

$$f\left(x^{k+1}\right) + \gamma_\mathrm{f}\delta_k \le f\left(x^k\right), \quad \text{or} \tag{3.42a}$$

$$\theta\left(x^{k+1}\right) + \gamma_\mathrm{f}\delta_k \le \theta\left(x^k\right) \tag{3.42b}$$

holds. This internal optimization problem is again nonlinear and can be just as difficult as the original problem (NLP). However, if the feasibility restoration fails, it provides at least a certificate of infeasibility.

A formal description of a globally convergent SQP method combined with a filter line-search is presented in Algorithm E.

Theorem 3.12 (Global Convergence of SQP Method with a Filter). *Let $\left\{\left(x^k, \lambda^k, \nu^k\right)\right\}_k$ be a sequence generated by Algorithm E such that the tuple $\left(x^k, \lambda^k, \nu^k\right)$ lies in some compact set for all k, x^k satisfies the LICQ and, for all $d \in \mathbb{R}^{n_x}$,*

$$c_1 \|d\|^2 \le d^\top \nabla^2_{xx} L\left(x^k, \lambda^k, \nu^k\right) d \le c_2 \|d\|^2$$

with $c_1 > 0$ and $c_2 > 0$. Furthermore, assume that the feasibility restoration of Algorithm E in Step E-4.5 is always successful and is never invoked for sufficiently feasible points. Then, $\left\{\left(x^k, \lambda^k, \nu^k\right)\right\}_k$ converges to a first-order optimal point of (NLP).

Proof. See Wächter [198, Section 4.4.2].[7] □

[7] The assumptions in the proof of Wächter [198] are weaker but also more technical. However, the assumptions of Theorem 3.12 imply them.

Algorithm E Globally Convergent SQP Method (Filter)

E-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0, \nu^0\right)$. Choose parameters $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$, $\alpha_{\min} > 0$, $\gamma_{\mathrm{f}} \in (0,1)$, $\theta_{\max} > 0$, $\varepsilon_{\mathrm{tol}} > 0$ and initialize the filter by $\mathscr{F}_0 \leftarrow \{(-\infty, \theta_{\max})\}$.

E-2: *(Optimality check)* If (2.7) is satisfied up to $\varepsilon_{\mathrm{tol}}$ for $\left(x^k, \lambda^k, \nu^k\right)$, then STOP; x^k is a first-order optimal point of (NLP).

E-3: *(Step calculation)* Solve the quadratic problem (QP) to get $\left(\Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$. If multiple solutions exist, choose the one such that $\left(x^k + \Delta x^k, \lambda^{k+1}, \nu^{k+1}\right)$ is closest to the last iterate $\left(x^k, \lambda^k, \nu^k\right)$. If $\Delta x^k = 0$, then STOP; x^k is a first-order optimal point of (NLP).

E-4: *(Line search)*

- E-4.1: *(Initialization)* Set $\alpha_k \leftarrow 1$.
- E-4.2: *(Filter)* If $\left(f\left(x^k + \alpha_k \Delta x^k\right), \theta\left(x^k + \alpha_k \Delta x^k\right)\right) \notin \mathscr{D}(\mathscr{F}_k)$, go to Step E-4.4.
- E-4.3: *(Sufficient reduction)* If (3.35) and the Armijo condition (3.36) are satisfied, set $\mathscr{F}_{k+1} \leftarrow \mathscr{F}_k$ and go to Step E-5. Otherwise, if (3.34) holds, augment the filter by (3.40) to get $\mathscr{F}_{k+1}$ and go to Step E-5.
- E-4.4: *(Backtracking)* Set $\alpha_k \leftarrow \beta \alpha_k$. If (3.35) is satisfied or $\alpha_k \geq \alpha_{\min}$, go to Step E-4.2.
- E-4.5: *(Feasibility restoration)* Calculate x^{k+1} by minimizing the constraint violation $\theta(x)$ such that $\left(f\left(x^{k+1}\right), \theta\left(x^{k+1}\right)\right) \in \mathscr{D}(\mathscr{F}_k)$ and (3.42) is satisfied. Augment the filter by (3.40) to get $\mathscr{F}_{k+1}$ (using x^k) and go to Step E-6.

E-5: *(Iterate update)* Set $x^{k+1} \leftarrow x^k + \alpha_k \Delta x^k$.

E-6: *(k increment)* Set $k \leftarrow k + 1$ and go to Step E-2.

Filter Variants and Non-Monotone Filters

The filter has intensively been studied in the literature. Examples for approaches based on a trust-region globalization are Fletcher et al. [72, 73], Gould and Toint [104] and Ulbrich et al. [183]. Moreover, the filter allows different definitions of its two measures. Ulbrich [184] proposes to use the Lagrangian function $L(x, \lambda, \nu)$ instead of the objective function $f(x)$ combined with some other modifications to avoid the Maratos effect. Nie and Ma [147] use a merit function instead, which is surprising but can increase the acceptable region of the filter in the original (f, θ)-space. Another possibility are multi-dimensional filters, e.g., where every dimension of the KKT conditions corresponds to a filter dimension (cf., Gould et al. [109] for the case of unconstrained optimization).

Finally, similar to the extensions for merit functions, non-monotone filters have been proposed by Gould and Toint [104] trying to overcome the Maratos effect. The non-monotonicity is based on a relaxation of the acceptable region $\mathscr{D}(\mathscr{F}_k)$ to $\mathscr{D}(\mathscr{F}_k(l_{\mathrm{f}}))$ with $l_{\mathrm{f}} \geq 0$ and $\mathscr{F}_k(l_{\mathrm{f}}) \subset \mathscr{F}_k$ such that $\mathscr{F}_k \setminus \mathscr{F}_k(l_{\mathrm{f}})$ includes points that are at least one and maximally l_{f}-times dominated by other elements of $\mathscr{F}_k$. This is exemplarily shown in Figure 3.2 (right). In case of $l_{\mathrm{f}} = 0$, the non-monotone filter reduces to the monotone version presented in detail above. Finally, Shen et al. [177] proved fast local convergence of such a method in a trust-region context, but an important further modification has to be done: Instead of one, two filters – a monotone global and a non-monotone local – are considered. Close to an optimal solution the checks are performed with respect to the local filter, which is reset whenever the algorithm switches from local to global mode. This strategy avoids the prevention of fast local convergence due to outdated historic information stored in the global filter. Besides non-monotone approaches, it is again possible to use second-order-corrections modifying the step Δx^k to achieve the same local convergence properties (cf., Wächter and Biegler [200, 202]).

3.3.3 Piecewise Linear Penalty Function

An intermediate version between the merit function and the filter approach is the piecewise linearly defined merit function, also called Piecewise Linear Penalty Function (PLPF). It was introduced by Gomes [98]. The motivation behind it, is to find a point that reduces the merit function $\Psi(x;\tau)$ with respect to some – or maybe all – former iterates (stored in $\mathscr{F}_k$), but for an arbitrary penalty parameter $\tau \geq 0$. This again removes the dependence of the globalization strategy on a specific choice of penalty parameter.

More formally, for $\Psi(x;\tau) = f(x) + \tau\theta(x)$ the PLPF is defined as

$$\overline{\mathscr{D}}(\tau;\mathscr{F}_k) := \min\{f + \tau\theta \mid (f,\theta) \in \mathscr{F}_k\} \quad \forall \tau \geq 0 \tag{3.43}$$

and the acceptable region by

$$\mathscr{D}(\mathscr{F}_k) := \left\{(f,\theta) \in \mathbb{R}^2 \mid \exists \tau \geq 0 \text{ such that } f + \tau\theta + (1+\tau)\gamma\delta_k \leq \overline{\mathscr{D}}(\mathscr{F}_k;\tau)\right\}, \tag{3.44}$$

with some envelope $\delta_k \geq 0$ and a parameter $\gamma \in (0,1)$. More sophisticated definitions of the envelope are possible like in Gomes [98], but it has to be ensured that it does not become arbitrarily small with respect to the other terms in (3.44) for any $\tau \geq 0$. This is the reason why the penalty parameter is a factor of the envelope δ_k in (3.44). The augmentation can be equally performed as for the filter by $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \left\{\left(f\left(x^k\right), \theta\left(x^k\right)\right)\right\}$. Note, that the PLPF is indeed a piecewise linear function in τ. If no dominated points exist[8] in $\mathscr{F}_k$, i.e., points $(f,\theta) \in \mathscr{F}_k$ such that $f + \tau\theta > \overline{\mathscr{D}}(\tau;\mathscr{F}_k)$ for all $\tau \geq 0$, then its elements can be reordered by

$$f_{k,1} \leq f_{k,2} \leq \cdots \leq f_{k,|\mathscr{F}_k|} \quad \text{and} \quad \theta_{k,1} > \theta_{k,2} > \cdots > \theta_{k,|\mathscr{F}_k|}, \tag{3.45}$$

which changes (3.43) to

$$\overline{\mathscr{D}}(\tau;\mathscr{F}_k) = \min\left\{f_{k,i} + \tau\theta_{k,i} \mid i = \{1,\ldots,|\mathscr{F}_k|\}\right\}. \tag{3.46}$$

Furthermore, the breakpoints of two consecutive linear merit functions in the PLPF can be expressed by

$$\tau_{k,i} = \frac{f_{k,i} - f_{k,i-1}}{\theta_{k,i} - \theta_{k,i-1}}, \quad i = 2,\ldots,|\mathscr{F}_k|, \tag{3.47}$$

which finally gives an obvious piecewise linear definition

$$\overline{\mathscr{D}}(\tau;\mathscr{F}_k) = \begin{cases} f_{k,1} + \tau\theta_{k,1}, & 0 \leq \tau \leq \tau_{k,2} \\ f_{k,2} + \tau\theta_{k,2}, & \tau_{k,2} \leq \tau \leq \tau_{k,3} \\ \vdots & \vdots \\ f_{k,|\mathscr{F}_k|} + \tau\theta_{k,|\mathscr{F}_k|}, & \tau_{k,|\mathscr{F}_k|} \leq \tau. \end{cases} \tag{3.48}$$

Figure 3.3 (left) illustrates the PLPF with the same points as in Figure 3.2 for the filter. The difference to the filter is clearly visible. In fact, the forbidden region of the PLPF is a convex

[8] Dominated points can easily been removed from $\mathscr{F}_k$ to satisfy this assumption.

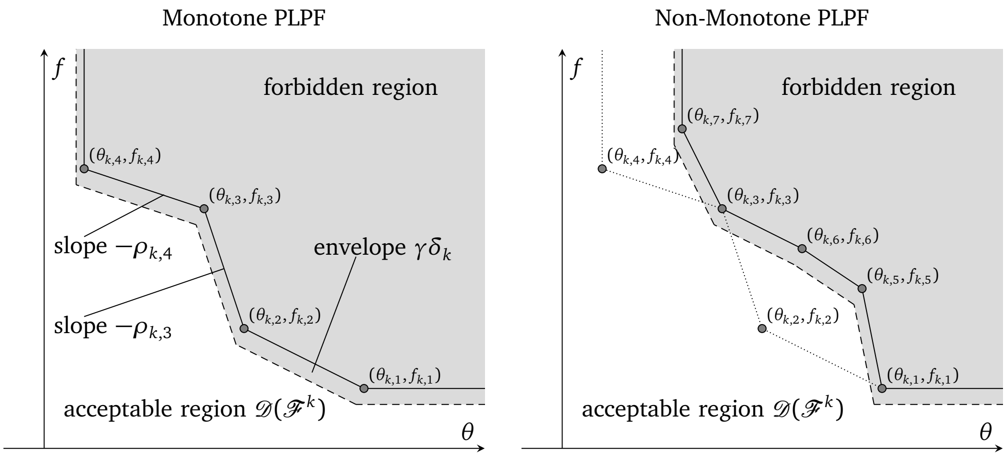

Figure 3.3: Monotone PLPF (left) and non-monotone PLPF (right). The PLPF envelope is dependent on current iterate and the non-monotonicity level on the right is $l = 1$.

envelope of the one for the filter. This makes the PLPF more restrictive in the sense that it is not possible to slightly improve one of the two measures – objective function $f(x)$ or constraint violation $\theta(x)$ – without considering the other. To the knowledge of the author, Chen and Goldfarb [43] published the only algorithm that utilizes a PLPF approach. A non-monotone variant of the PLPF can be generated by changing $\mathscr{F}_k$ to $\mathscr{F}_k(l)$ for $l \geq 0$ defined analogously to the non-monotone filter (cf., Figure 3.3 (right)), but has not been considered in the literature so far. Due to the similarity of the PLPF to the filter approach from a computational point of view, the formal presentation of an optimization algorithm based on a PLPF line-search is skipped.

3.4 Regularization Strategies

The central element of the Lagrange-Newton method was the solution of the linear equation system (3.7). In a neighborhood around the optimal solution Theorem 3.2 guaranteed the regularity of its matrix and, thus, the solvability of this linear equation system. In this section strategies are provided how to regularize the system matrix outside this neighborhood, which can be necessary for example in the case of linear dependent gradients of the constraints or of negative definite Hessians.

The matrix in (3.7) has the very special structure

$$K = \begin{bmatrix} Q & J \\ J^\top & -C \end{bmatrix} \tag{3.49}$$

with symmetric matrices $Q \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times m}$ ($C = 0$ in (3.7)) and a matrix $J \in \mathbb{R}^{n \times m}$. Matrices of the kind K are called *saddle point matrices*, since the solution of the corresponding linear equation system is equivalent with finding the saddle point – which is either a minimum or maximum – of an appropriate quadratic program, just as in (3.15). A comprehensive study of saddle-point matrices is given by Benzi et al. [17].

Two regularizations are of interest: Ensuring that (i) Q and (ii) C are positive (semi-)definite. In case of (3.7), (i) is related to the SOSC condition, i.e., $Q = \nabla^2_{xx} L(x, \lambda, \nu)$ is positive definite on the null space of $J^\top = \nabla g(x)^\top$, and (ii) to rank-deficient[9] Jacobian matrices $J = \nabla g(x)$ when the LICQ fails to be satisfied. The easiest way of regularizing K is by adding a multiple of the identity to it, i.e.,

$$\begin{bmatrix} Q & J \\ J^\top & -C \end{bmatrix} + \begin{bmatrix} \delta_{\mathrm{p}} E & 0 \\ 0 & -\delta_{\mathrm{d}} E \end{bmatrix} \tag{3.50}$$

with a *primal* and *dual regularization* $\delta_{\mathrm{p}} > 0$ and $\delta_{\mathrm{d}} > 0$, respectively. For the question if one of the two regularizations needs to be applied, the number of positive (λ_+), negative (λ_-) and zero eigenvalues (λ_0), which define the inertia $\mathrm{In}(K)$ of the matrix K by

$$\mathrm{In}(K) := (\lambda_+, \lambda_-, \lambda_0), \tag{3.51}$$

are of great help. The following two more technical results provide basic properties of the inertia of K for the two special cases C being positive semidefinite and $C = 0$, which will appear in most of the optimization algorithms.

Theorem 3.13. *Let K be given by* (3.49) *with a positive semidefinite matrix C and r be the rank of $\begin{bmatrix} J^\top & -C \end{bmatrix} \in \mathbb{R}^{m \times n+m}$. Let N_0 be the matrix whose columns form a basis*[10] *for the null space of C and N be a matrix whose columns form a basis for the null space of ${N_0}^\top J^\top$. Furthermore, let C^- be the pseudoinverse*[11] *of C and m_0 be the dimension of the null space of C. Then,*

$$\mathrm{In}(K) = \mathrm{In}\big(N^\top \big(Q + JC^- J^\top\big) N\big) + (m_0 - m + r, r, m - r)$$

and $\mathrm{rank}\big({N_0}^\top J^\top\big) = m_0 - m + r$.

Proof. See Forsgren [75, Proposition 2]. □

Corollary 3.14. *Let K be given by* (3.49) *with $C = 0$, r be the rank of $J^\top$ and N be a matrix whose columns form a basis for the null space of $J^\top$. Then,*

$$\mathrm{In}(K) = \mathrm{In}\big(N^\top Q N\big) + (r, r, m - r).$$

Proof. Follows from Theorem 3.13. An alternative proof is given by Gould [103, Lemma 3.4]. □

Under the assumption that K is regular, Theorem 3.13 and Corollary 3.14 state that K has at least m negative eigenvalues and at most n positive eigenvalues. The most important case is actually this bound of $\mathrm{In}(K) = (n, m, 0)$, which corresponds to the situation of J not being rank-deficient and Q being positive definite on the null space of $J^\top$ if $C = 0$, as shown by the next lemma.

[9] A matrix is rank-deficient if it does not have full rank (cf., Definition A.3).
[10] See Definition A.6.
[11] See Definition A.8.

Lemma 3.15. *Let K be given by* (3.49) *with $C = cE$, $c > 0$. Then,* $\text{In}(K) = (n, m, 0)$ *if, and only if, (i) either $c = 0$, J has rank m and $d^\top Qd > 0$ for all $d \neq 0$ such that $J^\top d = 0$, or (ii) $c > 0$ and $Q + \frac{1}{c}JJ^\top$ is positive definite.*

Proof. First consider $c > 0$. Then, the dimension of the null space of $C = cE$ is $m_0 = 0$. It follows $J_0 = 0$ and $N = E$. The rank of $\begin{bmatrix} J^\top & -cE \end{bmatrix}$ is $r = m$. Applying Theorem 3.13 yields:

$$\begin{aligned} (n, m, 0) = \text{In}(K) &= \text{In}\left(N^\top \left(Q + \frac{1}{c}JJ^\top\right) N\right) + (m_0 - m + r, r, m - r) \\ \Leftrightarrow \text{In}\left(Q + \frac{1}{c}JJ^\top\right) &= (n, 0, 0), \end{aligned}$$

which is equivalent with $Q + \frac{1}{c}JJ^\top$ being positive definite.

For $c = 0$, first assume $m > n$. Then $r \leq n$ and by Corollary 3.14 the number of zero eigenvalues is at least $m - r \geq m - n > 0$, which is a contradiction to $\text{In}(K) = (n, m, 0)$. For $m \leq n$ it follows from Corollary 3.14, that

$$\begin{aligned} &(n, m, 0) = \text{In}(K) = \text{In}\left(N^\top QN\right) + (r, r, m - r) \\ \Leftrightarrow\ &\text{In}\left(N^\top QN\right) = (n - r, m - r, r - m) \\ \Leftrightarrow\ &\text{In}\left(N^\top QN\right) = (n - m, 0, 0) \\ \Leftrightarrow\ &d^\top Qd > 0 \quad \text{for all} \quad d \neq 0 \quad \text{such that} \quad J^\top d = 0, \end{aligned}$$

since N is a matrix whose columns form a basis for the null space of $J^\top$. For the rank of $J^\top$ in the second equation the only valid value is $r = m$. □

A simple algorithm to regularize K is suggested by Vanderbei and Shanno [186, Section 3.1], which increases δ_p iteratively and maybe δ_d once until the desired inertia $\text{In}(K) = (n, m, 0)$ is achieved. A more sophisticated technique is the structured regularization of Wan and Biegler [189] that performs a dual regularization just with respect to the linear dependent rows to eliminate them from the system. The crucial point is to identify these rows, but the authors show that the pivots determined by the linear solver for the factorization of K can provide this information.

3.5 Solution Strategies

This section presents the most commonly used techniques to extend the Lagrange-Newton method to inequality constraints, either by modifying the right-hand-side of its linear equation system (3.7) appropriately or the problem formulation (NLP). The latter is no equivalent reformulation in the sense of Section 3.2 but rather a steadily improving approximation to (NLP). Special emphasis is also put on the quadratic program (QP), when the solution strategy is used within an SQP method, and the infeasible reformulation (3.12) of Section 3.2. Instead of (3.12), the nonlinear program

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & x \geq 0 \end{aligned} \tag{NLP+}$$

is considered for improved readability, as commonly done in the literature. This problem can also be interpreted as special case of (NLP) with $h(x) = -x$ and $n_h = n_x$. The presented algorithms are limited to locally convergent variants, which contain the key elements of the corresponding solution strategy. For global convergence, these methods are usually combined with a globalization approach of Section 3.3, but it has to be emphasized that a thorough global convergence analysis has to be performed, since several conditions must be met (e.g., the descent direction property for the merit function).

3.5.1 Active-Set Methods

Recalling the motivation of the optimality conditions in Section 2.1, inequality constraints have been handled in two different ways. Either they are inactive at the optimal solution and can be neglected, or they are active and behave analogously to equality constraints. Hence, if the active set $\mathscr{A}(x^*)$ of the optimal solution x^* is known and provided that an initial guess in a sufficiently small neighborhood of x^* is available, (NLP) can be solved efficiently by the Lagrange-Newton method by simply adding the active inequality as equality constraints, i.e., force $h_i(x) = 0$ for $i \in \mathscr{A}(x^*)$. But the crucial point is that $\mathscr{A}(x^*)$ is usually unknown and the number of possible active sets is exponential in the number of inequality constraints. The basic idea of active-set methods is to guess the optimal active set to be $\widetilde{\mathscr{A}}$ and apply the strategy above. Since this guess is likely to be wrong, it has to be updated in every iteration, e.g., by the common approach

$$\widetilde{\mathscr{A}_k} := \left\{ i \mid h_i\big(x^k\big) \geq -\tau\, \nu_i(x^k) \right\} \quad \text{and} \quad \widetilde{\mathscr{I}_k} := \{1, \ldots, n_h\} \setminus \widetilde{\mathscr{A}_k}, \tag{3.52}$$

where τ is a positive parameter and $\nu(x)$ is a multiplier function, or multiplier estimate depending on the current point x^k. The function $\nu(x)$ has to be continuous in x^* and at the optimal solution to have the value $\nu(x^*) = \nu^*$. Possible choices are simply $\nu\big(x^k\big) = \nu^k$, the solution of a SLP method (cf., the already mentioned references Chin and Fletcher [44] and Waltz [188]), or, following Glad and Polak [96], the solution of the linear equation system

$$\left(\nabla h(x)^\top \nabla h(x) + \gamma H(x)^2\right) \nu(x) = -\nabla h(x)^\top \left(\nabla f(x) + \nabla g(x)\lambda\right), \tag{3.53}$$

which is equivalent to the unconstrained program

$$\min_{\nu \in \mathbb{R}^{n_h}} \quad \frac{1}{2} \left\| \nabla f(x) + \nabla g(x)\lambda + \nabla h(x)\nu \right\|_2^2 + \frac{\gamma}{2} \left\| H(x)\nu \right\|_2^2, \tag{3.54}$$

thus, minimizing the KKT conditions (2.7a) and (2.7d) for given x and λ and weighting parameter $\gamma \geq 0$. A more comprehensive study on the active set prediction is done by Oberlin and Wright [154].

With the choice of $\widetilde{\mathscr{A}_k}$, the step determination is similar to the Lagrange-Newton method and can be done by the solution of the following linear equation system:

$$\begin{bmatrix} \nabla^2_{xx} L^k & \nabla g^k & \nabla h^k_{\widetilde{\mathscr{A}_k}} & \nabla h^k_{\widetilde{\mathscr{I}_k}} \\ \left(\nabla g^k\right)^\top & 0 & 0 & 0 \\ \left(\nabla h^k_{\widetilde{\mathscr{A}_k}}\right)^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta \nu^k_{\widetilde{\mathscr{A}_k}} \\ \Delta \nu^k_{\widetilde{\mathscr{I}_k}} \end{bmatrix} = - \begin{bmatrix} \nabla_x L\big(x^k, \lambda^k, \nu^k\big) \\ h\big(x^k\big) \\ h_{\widetilde{\mathscr{A}_k}}\big(x^k\big) \\ \nu^k_{\widetilde{\mathscr{I}_k}} \end{bmatrix} \tag{3.55}$$

Algorithm F Locally Convergent Active-Set Method

F-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point (x^0, λ^0, ν^0). Choose parameters $\tau > 0$ and $\varepsilon_{\text{tol}} > 0$.
F-2: *(Optimality check)* If (2.7) is satisfied up to ε_{tol} for (x^k, λ^k, ν^k), then STOP; x^k is a first-order optimal point of (NLP).
F-3: *(Active set identification)* Set the active set to be $\widetilde{\mathscr{A}_k} = \{i \mid h_i(x^k) \geq -\tau \nu_i^k\}$. Set $\nu_i^k \leftarrow 0$ for $i \notin \widetilde{\mathscr{A}_k}$.
F-4: *(Step calculation)* Solve the linear equation system (3.55) to get $(\Delta x^k, \Delta\lambda^k, \Delta\nu^k)$.
F-5: *(Iterate update)* Set $(x^{k+1}, \lambda^{k+1}, \nu^{k+1}) \leftarrow (x^k, \lambda^k, \nu^k) + (\Delta x^k, \Delta\lambda^k, \Delta\nu^k)$.
F-6: *(k increment)* Set $k \leftarrow k+1$ and go to Step F-2.

The last equation $\Delta\nu^k_{\widetilde{\mathscr{I}_k}} = -\nu^k_{\widetilde{\mathscr{I}_k}}$ results from the complementarity condition (2.7d) of the KKT conditions. By eliminating it, the system (3.55) can easily be symmetrized. It is also possible to set $\nu_i^k = 0$ for $i \in \widetilde{\mathscr{I}_k}$ before the solution of (3.55). An analog approach to choose x^k such that $h_i(x^k) = 0$ if $i \in \widetilde{\mathscr{A}_k}$ is, however, very difficult. Such a point x^k may not exist if the guess $\widetilde{\mathscr{A}_k}$ is inaccurate and, further, may increase the overall KKT violation. Both aspects are a big challenge for the global convergence analysis.

Locally, if the optimal active set has been identified, the step determined by the active-set method is equal to one of the sensitivity based recursive algorithm (Algorithm C) and, thus, to the SQP method (Algorithm B) requiring just one iteration for the step calculation. That is why, the active-set method is also referred to as equality constrained SQP method in contrast to the inequality constrained SQP of Algorithm B and it is not surprising that the active-set method (Algorithm F) is locally q-quadratically convergent under the same assumptions.

Theorem 3.16 (Local Convergence of Active-Set Method). *Let (x^*, λ^*, ν^*) be a first-order optimal point of* (NLP) *that satisfies the LICQ, SCC and SOSC. Furthermore, assume $\nabla^2_{xx} L(x, \lambda, \nu)$ to be Lipschitz-continuous. Then, there exists a neighborhood $\mathscr{N}(x^*, \lambda^*, \nu^*)$ such that for all $(x^0, \lambda^0, \nu^0) \in \mathscr{N}(x^*, \lambda^*, \nu^*)$ the sequence $\{(x^k, \lambda^k, \nu^k)\}_k$ of Algorithm F converges to (x^*, λ^*, ν^*) q-quadratically.*

Proof. See Facchinei and Lucidi [57, Theorem 4.1]. □

Active-Set Methods for Simplified NLPs

Applied to (NLP+), the active-set method simplifies enormously. The active set approximation then is

$$\widetilde{\mathscr{A}_k} = \{i \mid {x^k}_i \leq \tau \nu_i(x^k)\} \tag{3.56}$$

and the KKT based multiplier estimate $\nu(x)$ is given by

$$(E + \gamma X^2)\, \nu(x) = -\nabla f(x) - \nabla g(x)\lambda. \tag{3.57}$$

Furthermore, it is now trivial to set active constraints to its bounds, i.e., set $x_i^k = 0$ for $i \in \widetilde{\mathscr{A}_k}$, which however can still increase the KKT violation. Utilizing this projection together with $\nu(x^k) = \nu^k$, the active set identification can also be performed by adapting the current active set approximation $\widetilde{\mathscr{A}_k}$ and distinguishing between two simple cases:

i. If the constraint $x_i \geq 0$ is active at iteration k, it stays active, because the third equation of (3.55), $x_i^k + \Delta x_i^k = 0$, yields $\Delta x_i^k = 0$ for $x_i^k = 0$. In the next iteration $k+1$ the decision criterion (3.56), however, then removes constraint i from the active set $\widetilde{\mathscr{A}_k}$ if $0 \geq \nu_i^{k+1}$. Otherwise, it stays to be assumed active.

ii. Similarly, if the constraint $x_i \geq 0$ is inactive at iteration k, by the fourth equation of (3.55) it follows $\nu_i^{k+1} = 0$. Thus, the currently inactive constraint will be added to the active set $\widetilde{\mathscr{A}_k}$ in the next iteration $k+1$, if $x_i^k + \Delta x_i^k \leq 0$. Otherwise, it stays to be assumed inactive.

This active set strategy also transfers to quadratic programs (for example (QP) related to the SQP method) since the inequality constraints are also linear, but with the difference that an initial feasible point x^0 has to be determined. Due to this simple strategy and the difficulties for globalizing active set methods for nonlinear programs, it is rather used only for quadratic and linear optimization problems (cf., Gould et al. [108]). Examples for globally convergent active-set methods for nonlinear optimization problems similar to (NLP+) with $n_g = 0$ are Cristofari et al. [47] and Facchinei et al. [58].

3.5.2 Barrier or Interior-Point Methods

As explained in the last subsection, active-set methods focus on the active inequality constraints and neglect the inactive ones. This way inequality constrained problems can be reduced to problems with equality constraints only, for which an efficient algorithm – the Lagrange-Newton method – exists. There is also a more extreme methodology to achieve a similar reduction: Instead of separating active and inactive constraints, inequality constraints are forced to be inactive and, thus, to be strictly feasible. In other words, iterates x^k must lay within the interior of the feasible region $\mathscr{D}$, which motivates the name *interior-point* methods. This again allows the usage of the Lagrange-Newton method, but a strategy to keep the iterates strictly feasible is necessary. It is realized by adding a barrier to the objective function of (NLP):

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) + \sum_{i=1}^{n_g} \chi_{<0}^{\infty}(-h_i(x)) \\ \text{subject to} \quad & g(x) = 0 \end{aligned} \tag{3.58}$$

Here, $\chi_{<0}^{\infty}(-h_i(x))$ is an indicator function such that $\chi_{<0}^{\infty}(-h_i(x)) = \infty$ if $-h_i(x) < 0$ and zero otherwise. This way an optimization algorithm for (3.58) will not produce an iterate x^k such that $h_i\big(x^k\big) \geq 0$ for any $i = 1, \dots, n_h$ provided that the current iterate is within the feasible region.

Because (3.58) is not differentiable and a step based on first order derivatives close to the boundary of the feasible region would not be influenced by the barrier, (3.58) is approximated by

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) + \varphi_{\text{bar}}(x;\mu) \\ \text{subject to} \quad & g(x) = 0 \end{aligned} \tag{3.59}$$

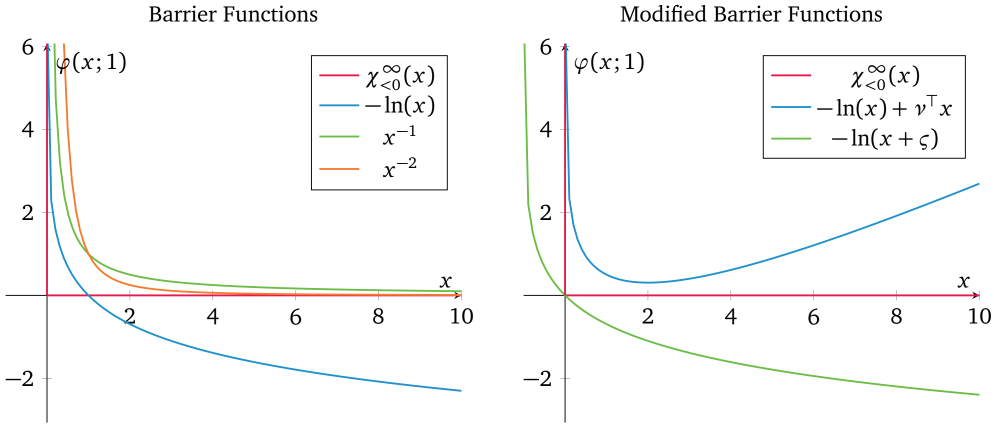

Figure 3.4: Different barrier functions (left) and modified barrier functions (right). The barrier parameter is chosen to be $\mu = 1$, the shift is $\varsigma = 1$ and the multipliers are $\nu = 0.5$.

with a barrier parameter $\mu > 0$ and a twice-continuously differentiable barrier function $\varphi_{\text{bar}} : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$ that goes to infinity whenever an inequality constraint approaches its boundary and converges to $\sum_{i=1}^{n_h} \chi^{\infty}_{<0}(-h_i(x))$ for feasible points of (NLP) and $\mu \to 0$. Then, in order to solve (NLP), interior-point methods solve a sequence of (3.59) for such a monotonically decreasing barrier parameter μ. Examples of $\varphi_{\text{bar}}(x;\mu)$, also illustrated in Figure 3.4 (left), are:

i. The log-barrier function (cf., Frisch [79]):

$$\varphi_{\text{bar}}(x;\mu) = -\mu \sum_{i=1}^{n_h} \ln(-h_i(x)) \tag{3.60}$$

ii. The rational-barrier function (cf., Carroll and Fiacco [37]):

$$\varphi_{\text{bar}}(x;\mu) = \mu \sum_{i=1}^{n_h} \frac{1}{(-h_i(x))^p}, \quad p \geq 1 \tag{3.61}$$

The nonlinear program (3.59) can also be interpreted as a parameter dependent nonlinear program (NLPp) with $p = \mu$. The optimal solutions of (3.59) then define a function $x(\mu)$, which is called the *central path*. Figure 3.5 illustrates this central path for Example 2.3, which starts with a large barrier parameter in the interior of the feasible region from where it moves to the optimal solution of (NLP) at the boundary. This is in contrast to the path of an active-set method that moves along active constraints. The convergence of the central path is formally stated in the next theorem.

Theorem 3.17 (Barrier Convergence Theorem). *Assume* (3.59) *has a solution for every* $\mu > 0$ *and in addition* (NLP) *has a first-order optimal solution satisfying the LICQ and the SOSC with a non-empty strictly feasible neighborhood around it. Let* $\{\mu_k\}_k$ *be a strictly monotone decreasing sequence with* $\{\mu_k\}_k \to 0$ *and* $x(\mu_k)$ *be an optimal solution of* (3.59) *for* $\mu = \mu_k$. *Then, there exists a limit point of* $\{x(\mu_k)\}_k$ *that is an optimal solution of* (NLP).

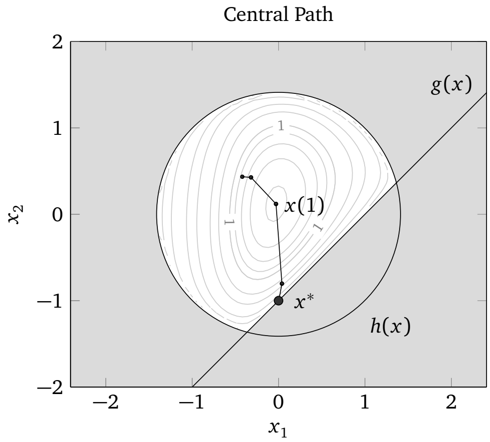

Figure 3.5: Central path for Example 2.3 based on log-barrier function. The equality constraint of Example 2.3 is relaxed to an inequality constraint. Nodes are plotted for solutions $x(\mu)$ with barrier parameters $\mu \in \{100, 10, 1, 0.1\}$ and contours for barrier function are plotted for $\mu = 1$.

Proof. See, for example, Forsgren et al. [77, Theorem 3.12 and Section 6.2.2]. □

Due to the barrier function, interior-point methods were initially called *barrier methods* and many of its techniques and theoretical foundations have been proposed by Fiacco and McCormick [64]. A comprehensive survey on interior-point methods is Forsgren et al. [77].

Primal-Dual Interior-Point Methods

Due to severe ill-conditioning of the Lagrange-Newton linear equation system[12]

$$\begin{bmatrix} Q_k & \nabla g(x^k) \\ \nabla g(x^k)^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = -\begin{bmatrix} \nabla f(x^k) - \mu \nabla h(x^k) H(x^k)^{-1} e + \nabla g(x^k) \lambda^k \\ h(x^k) \end{bmatrix}, \tag{3.62a}$$

$$Q_k = \nabla^2_{xx} L(x^k, \lambda^k) - \mu \sum_{i=1}^{n_h} \nabla^2 h_i(x^k) h_i(x^k)^{-1} + \mu \nabla h(x^k) H(x^k)^{-2} \nabla h(x^k)^\top, \tag{3.62b}$$

when μ converges to zero and iterates simultaneously converge to the boundary of the feasible region (cf., Wright [197] and especially Wright [196, Theorem 3.1]), SQP methods in connection with an active-set method for quadratic programming had been favored over barrier methods for a long time. After Karmarkar [125] established a polynomial-time algorithm for linear programming – which is an enormous advantage over the exponential worst-case complexity of active-set methods – that also performed well in practice and is closely related to the log-barrier method, the situation changed and the interest in interior-point methods increased. The key to overcome the ill-conditioning is the introduction of dual variables

$$z := -\mu H(x)^{-1} e, \tag{3.63}$$

[12] The linear equation system (3.62) is based on the log-barrier function. For rational-barrier methods this ill-conditioning is even more problematic due to the higher exponent of the inverse of $H(x^k)$.

which are positive for strictly feasible points and change the linear equation system (3.62) to

$$\begin{bmatrix} \nabla^2_{xx} L\big(x^k,\lambda^k,z^k\big) & \nabla g^k & \nabla h^k \\ \nabla {g^k}^\top & 0 & 0 \\ -Z_k \nabla {h^k}^\top & 0 & -H\big(x^k\big) \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta z^k \end{bmatrix} = - \underbrace{\begin{bmatrix} \nabla f\big(x^k\big) + \nabla g\big(x^k\big)\lambda^k + \nabla h\big(x^k\big) z^k \\ g\big(x^k\big) \\ -H\big(x^k\big) z^k - \mu e \end{bmatrix}}_{=:\Phi\left(x^k,\lambda^k,z^k;\mu\right)} . \tag{3.64}$$

The most important observation is that the right-hand-side of (3.64) equals the KKT conditions $\Phi\big(x^k,\lambda^k,\nu^k\big)$ of (NLP) where the complementarity condition (2.7d) is perturbed by μ.[13] Thus, a primal-dual interior-point method can be interpreted as a homotopy method for $\Phi(x,\lambda,z;\mu)$ converging to the primal-dual optimal solution of (NLP) with a homotopy parameter μ.

Theorem 3.18 (Primal-Dual Barrier Convergence Theorem). *Suppose the assumptions of Theorem 3.17 hold. Let $\{\mu_k\}_k$ be a strictly monotone decreasing sequence with $\{\mu_k\}_k \to 0$ and $(x(\mu_k),\lambda(\mu_k),z(\mu_k))$ be a primal-dual optimal solution of* (3.59) *for $\mu = \mu_k$ where $z(\mu) := -\mu_k H(x(\mu_k))^{-1} e$. Then, there exists a limit point of $\{(x(\mu_k),\lambda(\mu_k),z(\mu_k))\}_k$ that is a primal-dual optimal solution of* (NLP).

Proof. See, for example, Forsgren et al. [77, Theorem 3.12 and Section 6.2.2]. □

For an interior-point algorithm two further aspects have to be considered. First, the step Δx^k can violate the initially stated condition that the iterates x^k must be strictly feasible and may have to be shortened. This is usually done by the so called *fraction-to-the-boundary* rule that finds the maximal allowed step size

$$\alpha_{\max} := \max\left\{\alpha \in (0,1] \mid h\big(x^k + \alpha \Delta x^k\big) \le (1-\varepsilon_{\text{frac}})\, h\big(x^k\big)\right\} \tag{3.65}$$

with a parameter $\varepsilon_{\text{frac}} \in (0,1)$ and then updates the iterate by $x^{k+1} = x^k + \alpha_{\max}\Delta x^k$ or subject to an additional line-search as in Section 3.3. Second, a sequence of barrier parameters $\{\mu_k\}_k$ has to be chosen. While on the one hand sufficiently fast decreasing barrier parameters are necessary for fast local convergence (cf., Byrd and Liu [27]), too aggressive barrier updates can lead to failure because the solution of the last barrier problem may be a poor initial guess for the new one. Since this behavior is usually unknown a priori, the choice becomes even more complicated. To overcome these problems, adaptive barrier updates have been suggested by Armand et al. [5], El-Bakry et al. [54], Nocedal et al. [153] and Vanderbei and Shanno [186], which will be revised in more detail in Section 4.6.

For now, a classic monotone strategy, also called Fiacco-McCormick barrier update, is presented in the framework of Algorithm G as an example for a locally convergent interior-point method. Note, that in addition to Theorem 3.2 the satisfaction of the full step by the fraction-to-the-boundary rule has to be proven for fast local convergence with a fixed μ.

Theorem 3.19 (Local Convergence of Interior-Point Method). *Let $(x(\mu),\lambda(\mu),z(\mu))$ be a first-order optimal point of* (3.59) *that satisfies the LICQ, SCC and SOSC. Furthermore, assume*

[13] Also, note the similarity of the matrix of the interior-point linear equation system (3.64) and the one of the linear equation system used for calculating the first-order sensitivity derivatives of (NLPp) in Corollary 2.27.

Algorithm G Locally Convergent Primal-Dual Interior-Point Method

G-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(x^0, \lambda^0, z^0\right)$ such that $h\left(x^0\right) < 0$. Choose parameters $\mu_0 > 0$, $\varepsilon_{\text{frac}} \in (0,1)$ and $\varepsilon_{\text{tol}} > 0$.
G-2: *(Optimality check)* If $\left\|\Phi\left(x^k, \lambda^k, z^k\right)\right\| \leq \varepsilon_{\text{tol}}$, then STOP; x^k is a first-order optimal point of (NLP).
G-3: *(Barrier update)* If $\left\|\Phi\left(x^k, \lambda^k, z^k; \mu_k\right)\right\| \leq \varepsilon_{\text{tol}}$, choose $0 < \mu_{k+1} < \mu_k$. Otherwise, set $\mu_{k+1} \leftarrow \mu_k$.
G-4: *(Step calculation)* Solve the linear equation system (3.64) to get $\left(\Delta x^k, \Delta\lambda^k, \Delta z^k\right)$.
G-5: *(Line-search)* Determine $\alpha_{\max} \in (0,1]$ using the fraction-to-the-boundary rule (3.65).
G-6: *(Iterate update)* Set $\left(x^{k+1}, \lambda^{k+1}, z^{k+1}\right) \leftarrow \left(x^k, \lambda^k, z^k\right) + \left(\alpha_{\max}\Delta x^k, \Delta\lambda^k, \Delta z^k\right)$.
G-7: *(k increment)* Set $k \leftarrow k+1$ and go to Step G-2.

$\nabla^2_{xx} L(x, \lambda, \nu)$ *to be Lipschitz-continuous. Then, there exists a neighborhood* $\mathcal{N}(x(\mu), \lambda(\mu), z(\mu))$ *such that for all* $\left(x^0, \lambda^0, z^0\right) \in \mathcal{N}(x(\mu), \lambda(\mu), z(\mu))$ *the sequence* $\left\{\left(x^k, \lambda^k, z^k\right)\right\}_k$ *of Algorithm G converges to* $(x(\mu), \lambda(\mu), z(\mu))$ *q-quadratically.*

Proof. See Theorem 3.2. □

It has to be emphasized that the fast local convergence stated in Theorem 3.19 only holds for a fixed barrier parameter μ. The local convergence order of the overall algorithm is however further dependent on the reduction of μ.

Modified Barrier Functions

Similar to exact merit functions of Definition 3.8, modified barrier functions have been developed with a similar property. Instead of the necessity for the barrier parameter to converge to zero, it can be fixed at a sufficiently small value. These barrier functions are based on the idea, that its divergence does not occur for active constraints $h_i(x) = 0$, $i \in \{1, \ldots, n_h\}$, but slightly infeasible ones. It has the desirable side effect, that ill-conditioning of the Hessian matrix of the Lagrangian does not exist for points being in the feasible set $\mathcal{D}$. Polyak [162] motivated the modified barrier by the equivalence of the constraint $h_i(x) \leq 0$ with

$$-\mu \ln\left(1 - \frac{h_i(x)}{\mu}\right) \leq 0, \quad i \in \{1, \ldots, n_h\} \tag{3.66}$$

and then considering the optimization of the Lagrangian function

$$\varphi_{\text{bar}}(x; \mu) = f(x) - \mu \sum_{i=1}^{n_h} \nu_i \ln\left(1 - \frac{h_i(x)}{\mu}\right). \tag{3.67}$$

Comparable modified barrier functions have been considered by Conn et al. [46] and Goldfarb et al. [97]. Promising numerical studies are provided by Breitfeld and Shanno [24] and Nash et al. [146].

Forsgren and Gill [76] and Gertz and Gill [86] follow a different strategy based on a modified barrier function with respect to the Lagrangian multipliers. It allows a direct optimization in the primal-dual space (x, ν) by minimizing

$$\varphi_{\text{bar}}((x, \nu); \mu) = f(x) - \mu \sum_{i=1}^{n_h} \left(\ln(-h_i(x)) + \ln\left(-\frac{h_i(x)\nu_i}{\mu}\right) + 1 + \frac{h_i(x)\nu_i}{\mu}\right). \tag{3.68}$$

In particular, $\varphi_{\text{bar}}((x,\nu);\mu)$ is minimized for $\nu = z$ as defined above in (3.63) and $\varphi_{\text{bar}}((x,z);\mu) = \varphi_{\text{bar}}(x;\mu)$, where $\varphi_{\text{bar}}(x;\mu)$ is the classic log-barrier function and z is defined by (3.63).

By inspecting the different modified barrier functions closely, they provide the basic concept of a primal and dual shift of the complementarity condition (2.7d), since

$$\varphi_{\text{bar}}(x;\mu) = f(x) - \mu \sum_{i=1}^{n_h} \ln \underbrace{(-h_i(x) + \varsigma_i)}_{\text{primal shift by } \varsigma} - \underbrace{\nu^\top h(x)}_{\text{dual shift by } \nu} \tag{3.69}$$

together with the definition $z := \mu(-H(x) + \Sigma)^{-1} - \nu$ yields the same linear equation system as (3.64) but with the perturbed complementarity condition

$$\underbrace{\left(-H\left(x^k\right) + \Sigma_k\right)}_{\text{primal shift by } \varsigma} \underbrace{\left(z^k + \nu^k\right)}_{\text{dual shift by } \nu} - \mu e = 0 \tag{3.70}$$

instead of $-H\left(x^k\right)z^k - \mu e = 0$.[14] This allows intermediate negative values of z^k and positive values of $H\left(x^k\right)$ for non-negative shifts ν^k and ς^k, respectively, and is studied by Cartis and Yan [38] for linear programming. In particular, barrier methods with primal and dual shifts can start the optimization at the optimal solution of the original problem (NLP) or arbitrarily close by, which can be beneficial for the sequential optimization of similar nonlinear programs. In addition, Benson et al. [12] propose modified barrier functions as a strategy to avoid the so called *jamming*[15], which the author's identify as the reason of failure for many interior-point algorithms on the famous example of Wächter and Biegler [199]. A special case of the dual shift is also used by Wächter and Biegler [202] to overcome the disadvantage of the classic log-barrier function of being unbounded below. On the other hand, the drawback of modified barrier methods is that the performance strongly depends on the shifts, which have to be chosen and updated somehow. An illustration of shifted barrier functions in direct comparison to the classic barrier functions is provided by Figure 3.4.

Interior-Point Methods for Simplified NLPs

The reformulation as (NLP+) facilitates the interior-point algorithm because for nonlinear programs it is often difficult to keep $h(x)$ strictly feasible. In particular, an initial guess x^0 with that property has to be provided and for the fraction-to-the-boundary rule a line-search has to be applied. For (NLP+) in contrast, finding a strictly feasible initial guess $x^0 > 0$ is trivial and the maximal step size $\alpha_{\max}$ can be determined analytically. This is why most practical interior-point algorithms are based on (NLP+). In the literature these two options are referred to as *feasible* and *infeasible interior-point algorithm*.

The same also holds for (QP), but when considered within a SQP algorithm (for instance Büskens and Wassel [36] or Sachsenberg and Schittkowski [171]) special care has to be taken

[14] Recall that $\Sigma = \text{diag}(\varsigma)$.

[15] Jamming is a situation where an iterate crashes into the boundary of the feasible region with respect to the inequality constraints and is forced to stay there due to a search direction, that does not point backwards into the interior region.

for the convergence analysis of the overall algorithm. Theorem 3.6 assumes that the quadratic subproblem is solved exactly, which holds only in the limit for interior-point algorithms unfortunately. This requires the adaption of Algorithm B to accept inexact solutions of (QP) (cf., Leibfritz and Sachs [134]).

Boggs and Tolle [20] point out the interesting fact that the Lagrange-Newton systems of the interior-point algorithm applied to (NLP) and (QP) only differ in the perturbation of the complementarity condition, e.g., the third equation in (3.64). Practically however, this leads to a significant difference as for the SQP algorithm the perturbation – in particular the barrier parameter – has to be driven to zero in every major iteration repeatedly.

3.5.3 Penalty or Exterior-Point Methods

Penalty function based optimization methods are closely related to the merit functions introduced for globalization in Section 3.3.1. But instead of using it for measuring progress towards the optimal solution for an arbitrary optimization algorithm, penalty methods directly optimize the merit function leading to the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^{n_x}} \quad f(x) + \varphi_{\text{pen}}(x;\tau) \tag{3.71}$$

with a penalty function $\varphi_{\text{pen}} : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}$ measuring the constraint violation and a penalty parameter $\tau > 0$. Similar to merit functions, the penalty function must satisfy the condition that solving a sequence of (3.71) with monotonically decreasing penalty parameter $\tau > 0$, converging to zero, is equivalent to solving (NLP).[16] Moreover, a penalty function is called exact if this holds for some fixed penalty parameter $\bar{\tau} > 0$.

In order to meet the requirements of the definition of the smooth nonlinear program as given in Chapter 2 and, in addition, of the Lagrange-Newton method, the penalty function $\varphi_{\text{pen}}(x;\tau)$ must be at least twice continuously differentiable. Most merit functions presented in Section 3.3.1 do not fulfill this condition, in particular the ones that include inequality constraints. Therefore, penalty methods are usually just applied to equality constrained optimization problems using either of the following differentiable penalty functions:

i. The ℓ_2-penalty function (cf., Fiacco and McCormick [64, Chapter 4]):

$$\varphi_{\text{pen}}(x;\tau) := \frac{1}{2\tau} \|g(x)\|_2^2 \tag{3.72}$$

ii. The augmented Lagrangian penalty function (cf., Hestenes [116] and Powell [164]):

$$\varphi_{\text{pen}}(x;\tau) := \lambda^\top g(x) + \frac{1}{2\tau} \|g(x)\|_2^2 \tag{3.73}$$

Obviously, the ℓ_2-penalty function is a special case of the augmented Lagrangian. Other variants of it are studied by Gill and Robinson [87]. Analogously to barrier problems, the penalty problem (3.71) can be interpreted as parameter dependent nonlinear program (NLPp) with

[16] In contrast to the merit functions, the penalty parameter is chosen to converge to zero here, which is the most common definition in the literature.

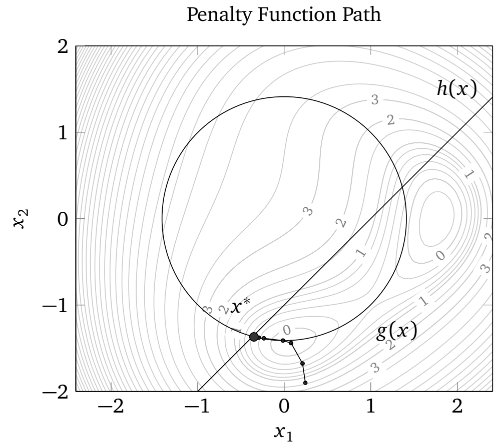

Figure 3.6: Penalty function path for Example 2.3 based on ℓ_2-penalty function. The inequality constraint of Example 2.3 is tightened to an equality constraint. Nodes are plotted for solutions $x(\tau)$ with penalty parameters $\tau \in \{10, 5, 1, 0.5, 0.1\}$ and contours for penalty function are plotted for $\tau = 1$. The original constraints are shown just for orientation.

parameter $p = \tau$. This defines functions $x(\tau)$, which are exemplarily shown in Figure 3.6 for Example 2.3 and the ℓ_2-penalty function. In contrast to the central path of interior-point methods (cf., Section 3.5.2 and Figure 3.5) the path $x(\tau)$ of penalty methods converges from the exterior of the feasible region $\mathscr{D}$, which is why Fiacco and McCormick [64] also call these optimization algorithms *exterior methods*. Convergence of the penalty method can be established similar to the interior-point algorithm.

Theorem 3.20 (Penalty Convergence Theorem). *Assume* (3.71) *has a solution for every* $\tau > 0$ *and in addition* (NLP) *has a first-order optimal solution satisfying the LICQ and the SOSC. Let* $\{\tau_k\}_k$ *be a strictly monotone decreasing sequence with* $\{\tau_k\}_k \to 0$ *and* $x(\tau_k)$ *be an optimal solution of* (3.71) *for* $\tau = \tau_k$. *Then, there exists a limit point of* $\{x(\tau_k)\}_k$ *that is an optimal solution of* (NLP).

Proof. See, for example, Geiger and Kanzow [84, Theorem 5.6]. □

Primal-Dual Penalty Methods

The problem of ill-conditioning of the Hessian matrix of the Lagrangian function is also present for penalty function methods when the penalty parameter converges to zero (cf., Murray [145]). The pure primal linear equation system for the ℓ_2-penalty function is

$$\left(\nabla^2_{xx} L\left(x^k, \frac{1}{\tau} g\left(x^k\right)\right) + \frac{1}{\tau} \nabla g\left(x^k\right) \nabla g\left(x^k\right)^\top\right) \Delta x^k = -\left(\nabla f\left(x^k\right) + \frac{1}{\tau} \nabla g\left(x^k\right) g\left(x^k\right)\right). \tag{3.74}$$

However, with the definition of the dual variables

$$y := \frac{1}{\tau} g\left(x^k\right) \tag{3.75}$$

Algorithm H Locally Convergent Primal-Dual Penalty Method for Equality Constrained Programs

H-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point (x^0, y^0). Choose parameters $\tau_0 > 0$ and $\varepsilon_{\text{tol}} > 0$.
H-2: *(Optimality check)* If $\|\Phi(x^k, y^k)\| \leq \varepsilon_{\text{tol}}$, then STOP; x^k is a first-order optimal point of (NLP).
H-3: *(Penalty update)* If $\|\Phi(x^k, y^k; \tau_k)\| \leq \varepsilon_{\text{tol}}$, choose $0 < \tau_{k+1} < \tau_k$. Otherwise, set $\tau_{k+1} \leftarrow \tau_k$.
H-4: *(Step calculation)* Solve the linear equation system (3.76) to get $(\Delta x^k, \Delta y^k)$.
H-5: *(Iterate update)* Set $(x^{k+1}, y^{k+1}) \leftarrow (x^k, y^k) + (\Delta x^k, \Delta y^k)$.
H-6: *(k increment)* Set $k \leftarrow k+1$ and go to Step H-2.

it changes to

$$\begin{bmatrix} \nabla^2_{xx} L(x^k, y^k) & \nabla g(x^k) \\ \nabla g(x^k)^\top & -\tau E \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla f(x^k) + \nabla g(x^k) y^k \\ g(x^k) - \tau y^k \end{bmatrix}}_{=:\Phi(x^k, y^k; \tau_k)} \tag{3.76}$$

and the ill-conditioning can be avoided.[17] The system (3.76) is the Lagrange-Newton system perturbed by τy^k, which yields a natural dual regularization $\delta_{\text{d}} = \tau > 0$ during the whole optimization process. Rank-deficiency caused by the gradients of the constraints at intermediate iterates x^k is therefore handled automatically. A direct consequence of Theorem 3.20 and (3.76) is the convergence of the primal-dual solution $(x(\tau_k), y(\tau_k))$ of (3.71) to the primal-dual optimal solution (x^*, λ^*) of (3.4), which will also become clearer in the convergence analysis in Section 4.3.

Corollary 3.21 (Primal-Dual Penalty Convergence Theorem). *Suppose the assumptions of Theorem 3.20 hold. Let $\{\tau_k\}_k$ be a strictly monotone decreasing sequence with $\{\tau_k\}_k \to 0$ and $(x(\tau_k), y(\tau_k))$ be a primal-dual optimal solution of* (3.71) *with the differentiable ℓ_2-penalty function and $\tau = \tau_k$ where $y(\tau_k) := \frac{1}{\tau_k} g(x(\tau_k))$. Then, there exists a limit point of $(x(\tau_k), y(\tau_k))$ that is a primal-dual optimal solution of* (3.4).

A locally convergent penalty method is presented in Algorithm H. Similar to the interior-point method, this algorithm converges q-quadratically for a fixed $\tau > 0$ (cf., Theorem 3.2), but its overall convergence order is dependent on the reduction of the penalty parameter (cf., Armand et al. [6]). This is different for instance for the augmented Lagrangian based penalty method, because its primal-dual linear equation system

$$\begin{bmatrix} \nabla^2_{xx} L(x^k, y^k) & \nabla g(x^k) \\ \nabla g(x^k)^\top & -\tau E \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \end{bmatrix} = -\begin{bmatrix} \nabla f(x^k) + \nabla g(x^k) y^k \\ g(x^k) + \tau (\lambda^k - y^k) \end{bmatrix} \tag{3.77}$$

reduces to a regularized Newton method if the Lagrangian multipliers are chosen to be $\lambda^k = y^k$. That is why, fast local convergence can be established for the augmented Lagrangian penalty algorithm (cf., Armand and Omheni [3] as well as Gill and Robinson [87]).

[17] Note, that a very similar definition of the dual variables (3.75) already appeared in the approximation of the active set in (3.52) for active-set methods (cf., Section 3.5.1).

Inclusion of Inequality Constraints using Penalty-Interior-Point Methods

Theoretically, penalty methods can directly be extended to inequality constraints, because Theorem 3.20 does not assume differentiability of the penalty function $\varphi_{\text{pen}}(g(x), h(x); \tau)$. However, the Lagrange-Newton method would not be applicable. A straightforward and widely utilized idea is to combine penalty with the interior-point methods of Section 3.5.2 to the *penalty-interior-point* methods[18] where the penalty function is responsible for the equality and the barrier function for the inequality constraints (cf., Fiacco and McCormick [64, Section 4.3]). Technically one would first apply a penalty function as in (3.71) to the equality constraints only, i.e.,

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}} \quad & f(x) + \varphi_{\text{pen}}(x; \tau) \\ \text{subject to} \quad & h(x) \leq 0 \end{aligned} \tag{3.78}$$

and afterwards an interior-point approach like (3.59) to get

$$\min_{x \in \mathbb{R}^{n_x}} \quad f(x) + \varphi_{\text{pen}}(x; \tau) + \varphi_{\text{bar}}(x; \mu). \tag{3.79}$$

Penalty-interior-point algorithms aim to benefit from the advantages of both solution strategies, the efficiency in the handling of inequality constraints of interior-point methods and the automatic dual regularization of the nonlinear constraints by penalty methods. The latter can circumvent the failure of global convergence of many interior-point methods discovered in the famous example by Wächter and Biegler [199]. In addition, penalty-interior-point algorithms avoid the loss of a strict relative interior for the barrier subproblems (3.59). To be more precise, it can easily happen in practice that the equality constraints $g(x) = 0$ imply an active inequality constraint $h_i(x) = 0$ that – of course feasible for (NLP) – would make the barrier subproblem (3.59) with the classic barrier function infeasible for $\mu > 0$ (cf., Wächter and Biegler [202, Section 3.5] or the description of jamming in Section 3.5.2). The main disadvantage of such a barrier-penalty combination is that not only one but two parameter sequences, the barrier and penalty parameter, have to be maintained. Nevertheless, penalty-interior-point algorithms have enjoyed an increased popularity lately. They have been studied for example by Armand and Omheni [2], Omheni [155] and Yamashita and Yabe [204] with a differentiable ℓ_2-penalty function and by Gertz and Gill [86] as well as Forsgren and Gill [76] using an augmented Lagrangian. Yamashita [203] investigates a non-differentiable ℓ_1-penalty function and Chen [42] as well as Chen and Goldfarb [40, 41, 43] propose to use the classic non-differentiable but exact ℓ_2-penalty function together with a modified Lagrange-Newton step calculation, an idea that will be picked up in Chapter 4.

Another common approach is to use a penalty-interior point algorithm with a smooth reformulation of the ℓ_1 or ℓ_∞-penalty function (cf., Gould et al. [110] and Boman [21]). In case of the ℓ_1-penalty function

$$\varphi_{\text{pen}}(x; \tau) = \frac{1}{\tau} \left(\sum_{i=1}^{n_g} |g_i(x)| + \sum_{i=1}^{n_h} \max\{0, h_i(x)\} \right), \tag{3.80}$$

[18] In the literature this combination is also referred to as *mixed interior-exterior point* and *mixed barrier-penalty* method.

a possible smooth reformulation can be attained by adding non-negative slack variables $s_{\mathrm{g,pos}}, s_{\mathrm{g,neg}} \in \mathbb{R}_{0+}^{n_g}$ and $s_{\mathrm{h,pos}}, s_{\mathrm{h,neg}} \in \mathbb{R}_{0+}^{n_h}$ to the nonlinear program that capture the positive and negative part of the nonlinear constraints $g(x)$ and $h(x)$, respectively, i.e.,

$$g(x) + s_{\mathrm{g,pos}} - s_{\mathrm{g,neg}} = 0 \quad \text{and} \tag{3.81a}$$

$$h(x) + s_{\mathrm{h,pos}} - s_{\mathrm{h,neg}} = 0, \tag{3.81b}$$

which simplifies the measure of the constraint violation in the penalty function to

$$\sum_{i=1}^{n_g} \left(s_{\mathrm{g,pos}}\right)_i + \left(s_{\mathrm{g,neg}}\right)_i, \qquad \text{and} \qquad \sum_{i=1}^{n_h} \left(s_{\mathrm{h,pos}}\right)_i. \tag{3.82}$$

The smooth ℓ_1-penalty nonlinear program then becomes:

$$\begin{aligned} \min_{\substack{x \in \mathbb{R}^{n_x}, \\ s_{\mathrm{g,pos}}, s_{\mathrm{g,neg}} \in \mathbb{R}^{n_g}, \\ s_{\mathrm{h,pos}}, s_{\mathrm{h,neg}} \in \mathbb{R}^{n_h}}} \quad & f(x) + \frac{1}{\tau}\left(s_{\mathrm{g,pos}} + s_{\mathrm{g,neg}} + s_{\mathrm{h,pos}}\right)^\top e \\ \text{subject to} \quad & g(x) + s_{\mathrm{g,pos}} - s_{\mathrm{g,neg}} = 0 \\ & h(x) + s_{\mathrm{h,pos}} - s_{\mathrm{h,neg}} = 0 \\ & s_{\mathrm{g,pos}} \geq 0 \\ & s_{\mathrm{g,neg}} \geq 0 \\ & s_{\mathrm{h,pos}} \geq 0 \\ & s_{\mathrm{h,neg}} \geq 0 \end{aligned} \tag{3.83}$$

and can then – as described above – be handled by an interior-point algorithm. The problem formulation (3.83) has some nice properties: First, due to the additional slack variables for every nonlinear constraint, this penalty subproblem does not suffer from rank-deficient constraint gradients. Secondly, it is easy to choose optimal slack variables that satisfy the nonlinear equality constraints while minimizing the penalty function and, thus, provide an initial guess or reset of the slack variables at an intermediate iteration of the optimization (cf., Gould et al. [110, Section 6.1]). Finally, the ℓ_1-penalty function is exact, which implies that τ does not necessarily converge to zero, in particular τ will be bounded away from it. Summarizing the above, the reformulation (3.83) is a very robust alternative to the direct optimization of (NLP). This is probably why Wächter and Biegler [202, Section 3.3] choose a variant of (3.83) for their feasibility restoration phase needed for the filter line-search (cf., Section 3.3.2). Other examples that study the ℓ_1-penalty-interior-point problem are Benson et al. [14], Curtis [48] and Fletcher [67]. However, for large-scale optimization the addition of $2n_g + 2n_h$ slack variables can be problematic, although it has to be mentioned that the resulting linear equation system is reducible due to the special structure of the slack variables as explained by Wächter and Biegler [202, Section 3.3].

Penalty Methods for Simplified NLPs

Penalty methods do not benefit from the simplification strategies of Section 3.2 as much as the former solution strategies because these strategies focus especially on the handling of inequality constraints. In contrast, penalty methods exploit their potential primarily for equality constraints. However, it was pointed out in Section 3.2.2, that the quadratic subproblem (QP) of the SQP method can be infeasible due to the linear approximation of the nonlinear constraints leading to failure of the SQP algorithm. This does not occur, if the SQP method would be applied for instance to the penalty problem (3.83), because its quadratic approximation is always feasible. Such strategies are called *stabilized SQP* methods and represent a natural alternative to the more technical constraint relaxation strategies usually applied in SQP algorithms such as Büskens and Wassel [36, Section 4.2.1.2], Geffken [81, Section 3.3.3.3] and Powell [163]. Stabilized SQP methods with different penalty functions and globalization strategies have recently been investigated by Gill and Robinson [88], Gill et al. [93, 94], Izmailov and Solodov [118, 119] and Shen et al. [178].

3.6 Parametric Sensitivity Based Inexact Methods

The application of the sensitivity analysis for the approximation of perturbed nonlinear programs in Section 2.2.2 has already lain the foundations of real-time approximations as an alternative to a full re-optimization of the perturbed program. This will be further exploited in this section to develop fast inexact algorithms to approximate solutions or to improve the calculated steps within a solution method.

3.6.1 Real-Time Approximation with Feasibility Corrections

Recall that x^* is the optimal solution of (NLPp), p^* is a reference parameter and, thus, $\Delta p = p - p^*$ is a perturbation or change of parameters (cf., Section 2.2). In case of the primal optimal solution $x(p)$ the real-time update is given by the first-order Taylor approximation

$$\widetilde{x}^0(p) = x^* + \frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\Delta p \tag{3.84}$$

with an error estimate $\left\|\widetilde{x}^0(p) - x(p)\right\| = \mathcal{O}\big(\|\Delta p\|^2\big)$, which is stated in (2.15).

While this approximation is very efficient as already pointed out in Section 2.2, there is a severe disadvantage: $\widetilde{x}^0(p)$ is likely to violate the feasibility condition that equality constraints $g\big(\widetilde{x}^0(p); p\big)$ and active inequality constraints $h_i\big(\widetilde{x}^0(p); p\big)$ with $i \in \mathscr{A}(x^*; p^*)$ evaluate to zero, i.e.,

$$g\big(\widetilde{x}^0(p); p\big) \neq 0, \tag{3.85a}$$

$$h_i\big(\widetilde{x}^0(p); p\big) \neq 0, \quad i \in \mathscr{A}(x^*; p^*). \tag{3.85b}$$

This must hold for $x(p)$, since Theorem 2.26 established the equivalence of the active sets $\mathscr{A}(x(p); p) = \mathscr{A}(x^*; p^*)$ for p within a neighborhood $\mathscr{P}$ – the one for which the parametric sensitivity analysis is valid. To remedy these shortcomings, Büskens [33, Section 4.2] proposes to interpret the constraint violations (3.85) as the special parameters $p_g \in \mathbb{R}^{n_g}$ and $p_h \in \mathbb{R}^{n_h}$ with reference values $p_g^* = 0$ and $p_h^* = 0$. Then, a further real-time update using the sensitivity

Algorithm I Real-Time Approximation with Feasibility Corrections

I-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\widetilde{x}^0(p)$ by (3.84) and tolerance $\varepsilon_{\text{tol}} > 0$.

I-2: *(Optimality check)* If $\left\|g\left(\widetilde{x}^k(p);p\right)\right\| \leq \varepsilon_{\text{tol}}$ and $\left|h_i\left(\widetilde{x}^k(p);p\right)\right| =\leq \varepsilon_{\text{tol}}$ for $i \in \mathscr{A}(x^*;p^*)$ is satisfied for $\widetilde{x}^k(p)$, then STOP.

I-3: *(Real-time update)* Set

$$\widetilde{x}^{k+1}(p) = \widetilde{x}^k(p) - \frac{\mathrm{d}x}{\mathrm{d}p_g}(0)g\left(\widetilde{x}^k(p);p\right) - \sum_{i\in\mathscr{A}(x^*;p^*)} \frac{\mathrm{d}x}{\mathrm{d}\left(p_h\right)_i}(0)h_i\left(\widetilde{x}^k(p);p\right)$$

I-4: *(k increment)* Set $k \leftarrow k+1$ and go to Step I-2.

derivatives $\frac{\mathrm{d}x}{\mathrm{d}p_g}(0)$ and $\frac{\mathrm{d}x}{\mathrm{d}(p_h)_i}(0)$ for $i \in \mathscr{A}(x^*;p^*)$ can be applied to compensate this constraint violation, i.e.,

$$\begin{aligned}\widetilde{x}^1(p) &= x^* + \frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\Delta p - \frac{\mathrm{d}x}{\mathrm{d}p_g}(0)g\left(\widetilde{x}^0(p);p\right) - \sum_{i\in\mathscr{A}(x^*;p^*)} \frac{\mathrm{d}x}{\mathrm{d}\left(p_h\right)_i}(0)h_i\left(\widetilde{x}^0(p);p\right) \\ &= \widetilde{x}^0(p) + \frac{\mathrm{d}x}{\mathrm{d}p_g}(0)g\left(\widetilde{x}^0(p);p\right) - \sum_{i\in\mathscr{A}(x^*;p^*)} \frac{\mathrm{d}x}{\mathrm{d}\left(p_h\right)_i}(0)h_i\left(\widetilde{x}^0(p);p\right).\end{aligned} \tag{3.86}$$

This approach not only works in practice (cf.„ for instance, Schäfer [175, Section 7.4] or Seelbinder [176, Section 5.4.2]) but significantly improves the error estimate of the constraint violation and also of the objective function over the standard real-time update, which was shown in (2.16).

Theorem 3.22 (Error Estimation of Feasibility Correction). *Let the assumptions of Theorem 2.26 be satisfied and $f(x;p)$, $g(x;p)$ and $h(x;p)$ be three times continuously differentiable. Then, it exists a neighborhood $\mathscr{P}$ around p^* such that for all $p \in \mathscr{P}$ the following error estimations hold:*

$$\begin{aligned}\left\|\widetilde{x}^1(p) - x(p)\right\| &= \mathscr{O}\left(\|\Delta p\|^2\right), \\ \left|f\left(\widetilde{x}^1(p);p\right) - f(x(p);p)\right| &= \mathscr{O}\left(\|\Delta p\|^3\right), \\ \left\|g\left(\widetilde{x}^1(p);p\right) - g(x(p);p)\right\| &= \mathscr{O}\left(\|\Delta p\|^3\right), \\ \left|h_i\left(\widetilde{x}^1(p);p\right) - h_i(x(p);p)\right| &= \mathscr{O}\left(\|\Delta p\|^3\right), \quad i \in \mathscr{A}(x^*;p^*).\end{aligned}$$

Proof. See Büskens [33, Theorem 4.3]. □

Since the approximation $\widetilde{x}^1(p)$ may again violate the feasibility condition similar to (3.85), this correction can be applied iteratively leading to Algorithm I. Büskens [33] further shows linear local convergence for this algorithm, which implies that eventually all constraints are satisfied for the approximated solution.[19] An extension to the primal-dual optimal solution $(x(p), \lambda(p), \nu(p))$ is straightforward.

[19] The inactive inequality constraints, which are not considered by the feasibility correction, stay inactive because $p \in \mathscr{P}$ as shown in Theorem 2.26.

3.6.2 Second-Order-Correction and Refinement Steps

Since the step Δx^k of an optimization algorithm is usually determined by the solution of a QP subproblem (cf., Section 3.2.2), the considerations made for feasibility corrections applied in the last subsection as a post-optimality strategy for (NLP) can be directly transferred to the step calculation. Such a strategy modifies the standard step Δx^k in order to handle the nonlinearities of the nonlinear program more accurately. The general aim of this modified step is to fulfill the constraint violation even better compared to the trial step $x^k + \alpha_k \Delta x^k$. Applications are then twofold: Either it is used as a backup strategy if the standard step fails to satisfy the line search acceptance criteria or it is applied more optimistically in every or most iterations to further improve the constraint violation trying to converge to a feasible point more quickly. The former case is usually referred to as *second-order-correction step* in the literature (cf., for instance, Conn et al. [45, Section 15.3.2.3] or Wächter and Biegler [200]), while the latter is called *feasibility refinement step* and has been studied by Geffken [81, Section 5.2], Geffken and Büskens [83], Nikolayzik [149] and Nikolayzik and Büskens [150]. In the following this step is introduced more formally and the extension to the whole KKT conditions will be demonstrated.

Recall that the QP subproblem for the step determination of an equality constrained[20] nonlinear program is

$$\begin{aligned} \min_{\Delta x^k \in \mathbb{R}^{n_x}} \quad & \frac{1}{2} \left(\Delta x^k\right)^\top \nabla^2_{xx} L\left(x^k, \lambda^k\right) \Delta x^k + \nabla f\left(x^k\right)^\top \Delta x^k \\ \text{subject to} \quad & g\left(x^k\right) + \nabla g\left(x^k\right)^\top \Delta x^k = 0 \end{aligned} \tag{3.87}$$

(cf., Section 3.2.2). Due to the linearization of the constraints, it is likely that the trial iterate $x^k + \alpha_k \Delta x^k$ violates the feasibility condition, i.e., $g\left(x^k + \alpha_k \Delta x^k\right) \neq 0$. Analogously to Section 3.6.1 this can be interpreted as a perturbation of the special parameter p_g for (3.87) and a real-time update

$$\widetilde{\Delta x}^{k,0} = -\frac{\mathrm{d}\Delta x^k}{\mathrm{d}p_g}(0) g\left(x^k + \alpha_k \Delta x^k\right) \tag{3.88}$$

may be performed to address this constraint violation. The next iterate would then be

$$x^{k,1} = x^k + \alpha_k \left(\Delta x^k + \widetilde{\Delta x}^{k,0}\right) \tag{3.89}$$

[20] The sensitivity analysis and thus the feasibility correction is based on equality and active inequality constraints only, because the sensitivities with respect to inactive constraints would be zero in any case (cf., Corollary 2.28 and Section 3.6.1). Therefore, just equality constraints are considered for simplicity.

Algorithm J Second-Order-Correction Steps

J-1: *(Initialization)* Choose $k \geq 0$. Set $j \leftarrow 0$. Choose a standard step Δx^k, iterate x^k and step size $\alpha_k \in (0,1]$. Set $x^{k,0} = x^k + \alpha_k \Delta x^k$.

J-2: *(Optimality check)* If some termination condition is satisfied, then STOP.

J-3: *(Second-Order-Correction Step)* Calculate $\widetilde{\Delta x}^{k,j}$ by (3.92).

J-4: *(Iterate update)* Set $x^{k,j+1} = x^{k,j} + \alpha_k \widetilde{\Delta x}^{k,j}$.

J-5: *(j increment)* Set $j \leftarrow j+1$ and go to Step J-2.

instead of $x^k + \alpha_k \Delta x^k$.[21] Again, this correction can be performed iteratively by

$$\begin{aligned} x^{k,j+1} &= x^{k,j} + \alpha_k \widetilde{\Delta x}^{k,j} \\ &= x^k + \alpha_k \left(\Delta x^k + \sum_{i=0}^{j} \widetilde{\Delta x}^{k,i} \right) \end{aligned} \tag{3.90a}$$

$$\widetilde{\Delta x}^{k,j} = -\frac{\mathrm{d}\Delta x^k}{\mathrm{d}p_g}(0) g\left(x^{k,j}\right) \tag{3.90b}$$

for $j \geq 1$, which leads to Algorithm J.[22] An important question is when to stop this iterative process. In case of second-order-corrections usually a small and fixed number of iterations is performed, often even at most one (cf., for example, Chen and Goldfarb [43] or Wächter and Biegler [202]). For the refinement steps, Geffken [81, Section 5.2.4] analyzed different strategies that are either based on a contraction factor, the constraint violation or the Lagrangian function. When the correction or refinement steps are stopped, the normal optimization, e.g., Algorithm B, continues with $x^{k+1} = x^{k,j}$.

While the local convergence order of a second-order-correction or feasibility refinement $\widetilde{\Delta x}^{k,j}$ is just linear as mentioned in the last subsection, the main advantage is its high efficiency. Because the sensitivity derivative $\frac{\mathrm{d}\Delta x^k}{\mathrm{d}(p_g)_i}(0)$ is given by

$$\begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k\right) & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}(p_g)_i}(0) \\ \frac{\mathrm{d}\Delta \lambda^k}{\mathrm{d}(p_g)_i}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix} \tag{3.91}$$

(cf., Corollary 2.27), the second-order-correction or feasibility refinement step can be determined by the linear equation system

$$\begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k\right) & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\Delta x}^{k,j} \\ \widetilde{\Delta \lambda}^{k,j} \end{bmatrix} = -\begin{bmatrix} 0 \\ g\left(x^{k,j}\right) \end{bmatrix}, \tag{3.92}$$

with the same system matrix as for the standard step $\left(\Delta x^k, \Delta \lambda^k\right)$ in (3.7), which by then is available in factored form.

[21] The use of a single step size α_k is for convenience only. In a practical algorithm a separate step size for each step Δx^k and $\widetilde{\Delta x}^{k,0}$ would be more suitable.

[22] Equations (3.88) and (3.90b) may be irritating on first sight because the first-order Taylor approximation, e.g., $\widetilde{\Delta x}^{k,j} = \widetilde{\Delta x}^{k,j-1} - \frac{\mathrm{d}\Delta x^k}{\mathrm{d}p_g}(0) g\left(x^{k,j}\right)$ is not directly visible. However, this has been substituted in the iterate update (3.90a).

Algorithm K Locally Convergent Modified Lagrange-Newton Method

K-1: *(Initialization)* Choose $k \geq 0$. Set $j \leftarrow 0$. Choose an initial guess $\left(x^{k,0}, \lambda^{k,0}\right)$ and tolerance $\varepsilon_{\text{tol}} > 0$.
K-2: *(Optimality check)* If (2.7) is satisfied up to ε_{tol}, then STOP; $\left(x^{k,j}, \lambda^{k,j}\right)$ is a first-order optimal solution of (3.87).
K-3: *(Second-Order-Correction Step)* Calculate $\left(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\right)$ by (3.93).
K-4: *(Iterate update)* Set $\left(x^{k,j+1}, \lambda^{k,j+1}\right) = \left(x^{k,j}, \lambda^{k,j}\right) + \left(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\right)$.
K-5: *(j increment)* Set $j \leftarrow j+1$ and go to Step K-2.

This strategy can of course be extended to violations of not just the feasibility but also the optimality condition (2.7a), i.e., violations of the form $\nabla_x L\left(x^{k,j}, \lambda^{k,j}\right) \neq 0$. Similar to the feasibility refinement, this can be interpreted as a perturbation of the special parameter p_f, which yields the refinement step

$$\widetilde{\Delta x}^{k,j} = -\frac{\mathrm{d}\Delta x^k}{\mathrm{d}p_g}(0) g\left(x^{k,j}\right) - \frac{\mathrm{d}\Delta x^k}{\mathrm{d}p_f}(0) \nabla_x L\left(x^{k,j}, \lambda^{k,j}\right) \tag{3.93a}$$

$$\widetilde{\Delta \lambda}^{k,j} = -\frac{\mathrm{d}\Delta \lambda^k}{\mathrm{d}p_g}(0) g\left(x^{k,j}\right) - \frac{\mathrm{d}\Delta \lambda^k}{\mathrm{d}p_f}(0) \nabla_x L\left(x^{k,j}, \lambda^{k,j}\right) \tag{3.93b}$$

or the equivalent linear equation system

$$\begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k\right) & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\Delta x}^{k,j} \\ \widetilde{\Delta \lambda}^{k,j} \end{bmatrix} = -\begin{bmatrix} \nabla_x L\left(x^{k,j}, \lambda^{k,j}\right) \\ g\left(x^{k,j}\right) \end{bmatrix}. \tag{3.94}$$

The solution process, formally stated in Algorithm K resembles the Lagrange-Newton method (Algorithm A), where the matrix of the linear equation system (3.94) stays fixed after a certain point or for a certain number of iterations to reduce the overall number of matrix factorizations. This variant is also known as *modified Newton's method* (cf., Ryaben'kii and Tsynkov [170, Section 8.3.3]) and is q-linearly locally convergent.

Theorem 3.23 (Local Convergence of Modified Newton's Method). *Let the assumptions of Theorem 2.26 be satisfied and assume that Algorithm K produces an infinite number of iterations. Then, there exists a neighborhood $\mathscr{P}$ of $\left(p_f^*, p_h^*\right) = 0$ such that $\left(x^{k,j}, \lambda^{k,j}\right)$ converges q-linearly to (x^*, λ^*).*

Proof. See, for example, Büskens [33, Theorem 4.7]. □

3.6.3 Inexact Newton Steps

For the modified Newton's method it should be clear that unless the nonlinear program (NLP) with equality constraints is only quadratic, the step $\left(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\right)$ differs from $\left(\Delta x^k, \Delta \lambda^k\right)$ because

$$\begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k\right) & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix} \neq \begin{bmatrix} \nabla^2_{xx} L\left(x^{k+1}, \lambda^{k+1}\right) & \nabla g\left(x^{k+1}\right) \\ \nabla g\left(x^{k+1}\right)^\top & 0 \end{bmatrix}, \tag{3.95}$$

where it has been assumed that $x^{k+1} = x^{k,j}$ and $\lambda^{k+1} = \lambda^{k,j}$ to avoid the introduction of a third iteration index in the following. Then, by exactly the same strategy as for the modified Newton's method

$$\begin{bmatrix} \nabla^2_{xx} L\big(x^{k+1}, \lambda^{k+1}\big) & \nabla g\big(x^{k+1}\big) \\ \nabla g\big(x^{k+1}\big)^\top & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\Delta x}^{k,j} \\ \widetilde{\Delta \lambda}^{k,j} \end{bmatrix} \tag{3.96}$$

can be interpreted as a further perturbation of the form p_f and p_g, respectively. As an extension – and mentioned here more like an outlook[23] – it is then possible to consider the altered refinement step

$$\begin{aligned} &\begin{bmatrix} \nabla^2_{xx} L\big(x^{k}, \lambda^{k}\big) & \nabla g\big(x^{k}\big) \\ \nabla g\big(x^{k}\big)^\top & 0 \end{bmatrix} \begin{bmatrix} \overline{\Delta x}^{k,j} \\ \overline{\Delta \lambda}^{k,j} \end{bmatrix} \\ &\quad = - \begin{bmatrix} \nabla_x L\big(x^{k+1}, \lambda^{k+1}\big) + \nabla^2_{xx} L\big(x^{k+1}, \lambda^{k+1}\big) \widetilde{\Delta x}^{k,j} + \nabla g\big(x^{k+1}\big) \widetilde{\Delta \lambda}^{k,j} \\ g\big(x^{k+1}\big) + \nabla g\big(x^{k+1}\big)^\top \widetilde{\Delta x}^{k,j} \end{bmatrix}, \end{aligned} \tag{3.97}$$

with an update procedure

$$\widetilde{\Delta x}^{k,j+1} = \widetilde{\Delta x}^{k,j} + \overline{\Delta x}^{k,j} \tag{3.98a}$$

$$\widetilde{\Delta \lambda}^{k,j+1} = \widetilde{\Delta \lambda}^{k,j} + \overline{\Delta \lambda}^{k,j} \tag{3.98b}$$

instead of (3.94). This refinement step is also known as *inexact Newton's method* (cf., for instance, Dembo et al. [51]). It can also be motivated by an iterative refinement for linear systems like in Johnson et al. [122, Section 3] or using quadratic perturbations of the objective function and linear perturbations of the equality constraints in a sensitivity analysis (cf., Geffken [81, Section 4.2]). It can be shown that under the assumption of a good initial guess and a certain contraction condition on the matrices in (3.95) – which basically ensures that they are sufficiently close to each other – the refinement steps $\big(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\big)$ eventually converge to the exact Newton step $\big(\Delta x^k, \Delta \lambda^k\big)$, but with the advantage of avoiding a matrix factorization and applying efficient matrix-vector products instead.

Theorem 3.24. *Let the assumptions of Theorem 2.26 be satisfied. Furthermore, assume that* $\big(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\big)$ *is sufficiently close to* $\big(\Delta x^k, \Delta \lambda^k\big)$ *and*

$$\left\| E - \begin{bmatrix} \nabla^2_{xx} L\big(x^{k-1}, \lambda^{k-1}\big) & \nabla g\big(x^{k-1}\big) \\ \nabla g\big(x^{k-1}\big)^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla^2_{xx} L\big(x^{k}, \lambda^{k}\big) & \nabla g\big(x^{k}\big) \\ \nabla g\big(x^{k}\big)^\top & 0 \end{bmatrix} \right\| < 1.$$

Then, $\big(\widetilde{\Delta x}^{k,j}, \widetilde{\Delta \lambda}^{k,j}\big)$ *converges to* $\big(\Delta x^k, \Delta \lambda^k\big)$ *q-linearly.*

Proof. See Johnson et al. [122, Section 3 and Theorem 4.3]. □

[23] In particular the application of specialized preconditioners and / or iterative solvers that generally improve convergence over the presented method are out of the scope of this thesis and are not discussed.

Chapter 4

A Primal-Dual Augmented Lagrangian Penalty-Interior-Point Algorithm

In this chapter a new algorithm for nonlinear programming will be proposed. While special emphasis will be put on the theoretical properties, the main goal is to develop an algorithm particularly intended for practical use when solving large-scale NLP. Therefore, it must meet the following requirements:

High efficiency: The algorithm should converge quickly to first-order optimal points of the nonlinear program (NLP).

High robustness: In practice, the nonlinear optimization problem may contain constraint redundancies leading to singularities in the Jacobian of the constraints[1] and hence to failure in the determination of a step direction (Section 3.4). The algorithm should handle these kinds of degeneracy.

Detection of infeasibility: In case (NLP) is (locally) infeasible, the algorithm should converge quickly to a certificate of infeasibility without the need to solve (FeasNLP) separately on top to the actual optimization. This information enables practitioners to either fix their optimization model or to provide a more reasonable initial guess.

Featuring sensitivity analysis and warmstarts: If multiple very similar nonlinear programs are solved sequentially, the algorithm should be able to utilize the optimal solution of a previous optimization to improve efficiency. This is called a *warmstart*. The sensitivity analysis (Section 2.2) is beneficial here to indicate how the solution may change.

While the first aspect is an obvious goal, the latter three are of special interest when the algorithm is applied in a solver for mixed-integer nonlinear programs. For example in a branch-and-bound context, many internal nonlinear programs with partly fixed and partly relaxed integrality constraints may be infeasible. In addition, the fixation of one integrality constraint

[1] For numerical algorithms it is even problematic if the Jacobian of the constraints is nearly singular. Such a matrix is called ill-conditioned as it has a large condition number (cf., Definition A.5).

within one branch-and-bound node may yield a very similar nonlinear program as the previous relaxed version and warmstarts become an important tool. Finally, cutting plane approaches like outer approximation in which linear constraints are successively added to the nonlinear program increase the risk of numerical degeneracy. For a general overview of mixed-integer nonlinear programming approaches, the reader is referred to Belotti et al. [11], Bonami et al. [22], Burer and Letchford [26] and D'Ambrosio and Lodi [49].

To improve the readability of the presentation, the proposed method uses the infeasible reformulation for (NLP) of Section 3.2.1 to simplify the handling of inequality constraints and considers the solution of (NLP+), i.e., the optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & x \geq 0. \end{aligned} \tag{NLP+}$$

4.1 The Penalty-Interior-Point Program

The four requirements above usually suggest the use of an SQP method, which has a q-quadratic local convergence order in contrast to interior-point methods (superlinear local convergence order), which is known to warmstart efficiently due to its internal active-set approach and which has shown to be able to handle degeneracy and quickly detect infeasible nonlinear programs (cf., Gill and Wong [89], Gill et al. [92]). However, motivated by the good practical performance of interior-point solvers especially on large-scale nonlinear programs (cf., Benson et al. [13], Mittelmann [142], Morales et al. [143]), the thesis aims to extend the classic interior-point method in order to improve with respect to the latter three requirements. To achieve this, the proposed method combines the interior-point with a penalty approach. Penalty methods provide a natural regularization of the constraints (Section 3.5.3), which is why they perform relatively well when applied for example to the degenerate MPECs (cf., Omheni [155, Section 4.6.6]) and can directly handle infeasible programs. In other words, if (NLP+) is feasible, the algorithm should focus on the interior-point approach and if it is infeasible or degenerate, the penalty approach takes the lead. The fourth aspect (featuring sensitivity analysis and warmstarts) requires the usage of a modified barrier method, which in analogy to the exact penalty methods does not require the barrier parameter μ to converge to zero, and offers the possibility to warmstart from a former optimal solution with active constraints.

Therefore, the modified barrier function of Polyak [162, Section 2]

$$\varphi_{\text{bar}}(x;\rho) := -\mu\pi\sum_{i=1}^{n_x}\varsigma_i\,\nu_i\ln\!\left(\frac{x_i}{\mu\varsigma_i}+1\right) \tag{4.1}$$

with a barrier parameter $\mu > 0$ and a primal shift $\varsigma \in \mathbb{R}_+^{n_x}$ is chosen. While the barrier parameter scales the primal shift, the parameter $\pi \in (0,1]$ scales the original Lagrangian multiplier ν. All these parameters are collected in ρ, which will be further extended and therefore defined later on. If the Lagrangian multiplier ν is set to one, (4.1) reduces to a standard log-barrier function with a shifted feasible region by $\mu\varsigma$. The corresponding barrier subproblem is

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) - \mu\pi\sum_{i=1}^{n_x}\varsigma_i\nu_i \ln\left(\frac{x_i}{\mu\varsigma_i}+1\right) \\ \text{subject to} \quad & g(x)=0. \end{aligned} \tag{NLPbar}$$

For the penalty function a non-smooth adaptation of the augmented Lagrangian penalty function

$$\varphi_{\text{pen}}(x;\rho) := \lambda^\top g(x) + \tau\,\|g(x)\|_2 \tag{4.2}$$

with a penalty parameter $\tau > 0$ is used. Although the standard augmented Lagrangian penalty is already exact (cf., Theorem 3.10), (4.2) would be so even if λ is fixed to zero throughout the optimization. This is a valuable feature if no accurate estimation of the Lagrangian multiplier λ exists, for example in case of a badly scaled optimal multiplier λ^*. Further benefits like better local convergence properties compared to the exact $\ell 2$-penalty and better penalty parameter handling compared to the classic augmented Lagrangian approach will be discussed later on. All in all, the penalty-interior-point approach yields the unconstrained and non-smooth optimization problem

$$\min_{x\in\mathbb{R}^{n_x}} \quad \Upsilon(x;\rho) := \pi\left(f(x) - \mu\sum_{i=1}^{n_x}\varsigma_i\nu_i\ln\left(\frac{x_i}{\mu\varsigma_i}+1\right) + \lambda^\top g(x)\right) + \tau\,\|g(x)\|_2\,, \tag{NLPpen}$$

with the barrier-penalty or merit function $\Upsilon : \mathbb{R}^{n_x} \times \mathbb{R}^{n_g+2n_x+3} \to \mathbb{R}$ and parameters $\rho := (\varsigma, \lambda, \nu, \mu, \pi, \tau)$. Here, $\pi \in (0,1]$ is a further penalty parameter serving as an alternative to τ. The problem formulation (NLPpen) is illustrated in the following example.

Example 4.1. *Consider the nonlinear program*

$$\begin{aligned} \min_{x\in\mathbb{R}^3} \quad & f(x) = -\left(x_1-\frac{1}{2}\right)^3 + \frac{3}{4}(x_2+1) \\ \textit{subject to} \quad & g_1(x) = x_1^2 + x_2^2 - x_3 - 2 = 0 \\ & g_2(x) = x_1 - x_2 - 1 = 0 \\ & h(x) = x_3 \le 0, \end{aligned}$$

which is an equivalent reformulation of Example 2.3. For the two local optimal solutions $x^1 = \left(\frac{1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}, 0\right)$, $\lambda^1 = \left(\frac{\sqrt{3}}{4}, \frac{6-\sqrt{3}}{4}\right)$, $\nu^1 = \frac{\sqrt{3}}{4}$ *and* $x^2 = (0,1,-1)$, $\lambda^2 = \left(\frac{3}{4}, 0\right)$, $\nu^2 = 0$ *the penalty-interior-point objective function of* (NLPpen) *is plotted in Figure 4.1 where the optimal* $\left(\lambda^1, \nu^1\right)$ *and* $\left(\lambda^2, \nu^2\right)$ *have been used as parameters for the Lagrangian multipliers and the barrier and penalty parameters are set to* $\mu = \pi = \tau = 1$. *The example illustrates that it is not necessary to drive* μ *and* π *to zero (or* τ *to infinity) in order to solve NLP+ due to the exactness of the merit function (cf., Figure 3.5 and Figure 3.6).*

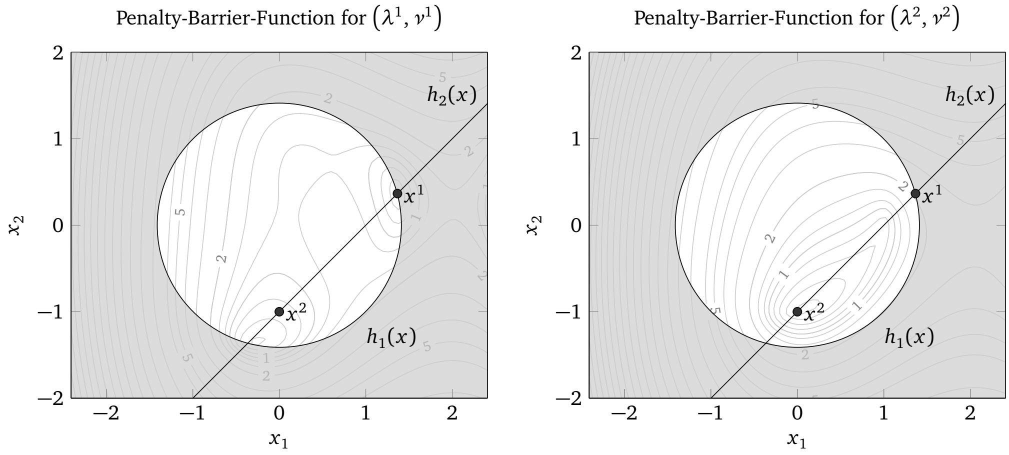

Figure 4.1: Optimal solution for (λ^1, ν^1) (left) and (λ^2, ν^2) (right) of Example 4.1 with corresponding penalty-interior-point objective function. The objective function is plotted as level set. Barrier and penalty parameter are set to $\mu = \pi = \tau = 1$ and x_3 is fixed to its optimal value. The infeasible region with respect to the inequality constraint is the light gray area.

Optimality Conditions of the Penalty-Interior-Point Subproblem

In the following, first-order optimality conditions for (NLPpen) will be derived. As a reminder, the KKT conditions of (NLP+) are

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* - \nu^* = 0 \tag{4.3a}$$
$$g(x^*) = 0 \tag{4.3b}$$
$$X_* \nu^* = 0. \tag{4.3c}$$

Analogously to Section 3.5.2, the primal-dual system for the barrier subproblem (NLPbar) is

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* - z^* = 0 \tag{4.4a}$$
$$g(x^*) = 0 \tag{4.4b}$$
$$X_* z^* = \mu \Sigma (\pi \nu - z^*), \tag{4.4c}$$

which equals the KKT conditions of (NLP+) except for the complementarity condition (4.3c), which is perturbed by $\mu\Sigma(\pi\nu - z^*)$.[2] If the Lagrangian multiplier ν is chosen to be $\nu = z^*/\pi$, this perturbation vanishes for all barrier parameters μ and all shifts ς.

Now following Fletcher [66, Section 4], first-order optimality conditions for the non-smooth (NLPpen) can be formulated by considering the subdifferential[3] of the penalty function with respect to the constraints $g(x)$, i.e., if a point x^* minimizes $\Upsilon(x;\rho)$, then there exist multipliers $y^* \in \mathbb{R}^{n_g}$ such that $y^* \in \partial_c \left(\pi \lambda^\top c + \tau \|c\|_2\right)$ with $c = g(x^*)$ and

$$\pi \nabla f(x^*) + \nabla_x \varphi_{\text{bar}}(x^*;\rho) + \nabla g(x^*) y^* = 0. \tag{4.5}$$

[2] Recall that $\Sigma = \text{diag}(\varsigma)$.
[3] See Definition A.15.

The subgradient is

$$\partial_c \left(\pi\lambda^\top c + \tau \|c\|_2 \right) = \pi\lambda + \tau \begin{cases} \|g(x^*)\|_2^{-1} g(x^*), & \text{if } \|g(x^*)\| > 0, \\ \{a \in \mathbb{R}^{n_g} \mid \|a\|_2 \leq 1\}, & \text{if } \|g(x^*)\| = 0, \end{cases} \tag{4.6}$$

which can be used to transform the condition $y^* \in \partial_c \left(\pi\lambda^\top c + \tau \|c\|_2 \right)$ in the first case to $g(x^*) - \tau^{-1} \|g(x^*)\|_2 (y^* - \pi\lambda) = 0$ and in the second case to $\|y^* - \pi\lambda\|_2 \leq \tau$. The maybe surprising fact is, that both transformed conditions are also satisfied for each of the cases and thus can be considered simultaneously. By applying the same primal-dual technique as for (NLPbar), the first-order optimality conditions for (NLPpen) are

$$\pi\nabla f(x^*) + \nabla g(x^*) y^* - z^* = 0 \tag{4.7a}$$

$$g(x^*) = \frac{\|g(x^*)\|_2}{\tau} (y^* - \pi\lambda) \tag{4.7b}$$

$$\tau \geq \|y^* - \pi\lambda\|_2 \tag{4.7c}$$

$$X_* z^* = \mu\Sigma(\pi\nu - z^*) \tag{4.7d}$$

As for (4.4) the primal-dual system (4.7) equals the KKT conditions where in addition to the perturbation of the complementarity condition (4.3c), the feasibility condition (4.3b) is perturbed by $\tau^{-1} \|g(x^*)\|_2 (y^* - \pi\lambda)$ with size equal or smaller to $\|g(x^*)\|_2$. Again, the perturbation vanishes if the Lagrangian multiplier λ is set to $\lambda = y^*/\pi$, but also if the constraint violation $\|g(x^*)\|_2$ is zero. The inequality (4.7c) can be interpreted as a dual trust-region condition: The Lagrangian multiplier y^* has to be found in a neighborhood around $\pi\lambda$ with size τ. If such a bounded y^*, additionally satisfying the other conditions of (4.7), does not exist, the neighborhood has to be changed, either by increasing its size (updating π or τ) or by moving its center (updating λ). Finally, the close relation of (4.7) to the first-order optimality conditions of (NLPpen) with the classic augmented Lagrangian penalty function (cf., Section 3.5.3) has to be pointed out, which are

$$\pi\nabla f(x^*) + \nabla g(x^*) y^* - z^* = 0 \tag{4.8a}$$

$$g(x^*) = \varrho (y^* - \pi\lambda) \tag{4.8b}$$

$$X_* z^* = \mu\Sigma(\pi\nu - z^*) \tag{4.8c}$$

with a penalty parameter $\varrho > 0$.

The connection between the original problem (NLP+), the barrier problem (NLPbar) and the penalty problem (NLPpen) is summarized in the following propositions.

Proposition 4.2. *The point (x^*, λ^*, ν^*) is first-order optimal solution for* (NLP+)*, if and only if,*

i. the point (x^*, λ^*, z^*) satisfying $z^* = \pi\nu^*$ is a first-order optimal solution of (NLPbar) *with $\varsigma > 0$, $\mu > 0$ and $\pi > 0$.*

ii. the point (x^*, y^*, z^*) satisfying $(y^*, z^*) = (\pi\lambda^*, \pi\nu^*)$ is a first-order optimal solution of (NLPpen) *with $\varsigma > 0$, $\mu > 0$, $\pi > 0$ and $\tau > 0$.*

Proof. The proof follows directly from comparing the KKT conditions by noting that all the perturbations of (4.4) and (4.7) vanish for the special choice of $(y^*, z^*) = (\pi\lambda^*, \pi\nu^*)$. □

Proposition 4.3. *Let $\mu > 0$, $\varsigma \in \mathbb{R}^{n_x}_+$, $\varrho > 0$ sufficiently small and (x^*, λ^*, ν^*) be an optimal solution of* (NLP+) *satisfying the LICQ, the SOSC and the SCC. Then,*

i. the point (x^*, λ^*) is a strict local optimal solution of (NLPbar) *with dual parameter ν^*.*

ii. the point x^* is a strict local optimal solution of

$$\min_{x \in \mathbb{R}^{n_x}} \quad \pi f(x) + \varphi_{bar}(x;\rho) + \left(\pi\lambda^* - \frac{1}{\varrho} g(x^*)\right)^\top g(x) + \frac{1}{2\varrho} \|g(x)\|_2^2, \tag{4.9}$$

with $\nu = \nu^*$ selected in ρ.

Proof. The proof follows the presentation in Polyak [162].

i. For $\mu > 0$ and $\varsigma \in \mathbb{R}^{n_x}_+$ the Hessian of the Lagrangian of (NLPbar) evaluated at (x^*, λ^*) is

$$\begin{aligned} &\nabla^2 f(x^*) + \mu\pi\Sigma(X_* + \mu\Sigma)^{-2} \operatorname{diag}(\nu^*) + \sum_{i=1}^{n_g} \lambda_i^* \nabla^2 g_i(x^*) \\ =&\nabla^2_{xx} L(x^*, \lambda^*, \nu^*) + \mu\pi\Sigma(X_* + \mu\Sigma)^{-2} \operatorname{diag}(\nu^*) \\ =&\nabla^2_{xx} L(x^*, \lambda^*, \nu^*) + \frac{\pi}{\mu}\Sigma^{-1} \operatorname{diag}(\nu^*) \end{aligned}$$

because due to the SCC either ν_i^* or x_i^* is zero for all $i = 1, \ldots, n_x$. Then, by Debreu [50, Theorem 3] (setting A to the Hessian of the Lagrangian of (NLP+), i.e., $A = \nabla^2_{xx} L(x^*, \lambda^*, \nu^*)$, and B to the active constraint gradients of (NLP+) scaled with the square root of ς, i.e., $B = \left[\frac{1}{\sqrt{\varsigma_{i_1}}} e_{i_1} \quad \cdots \quad \frac{1}{\sqrt{\varsigma_{i_n}}} e_{i_n}\right]$ with $i_j \in \mathscr{A}(x^*), j = 1, \ldots, n$ and $n = |\mathscr{A}(x^*)|$) and the SOSC, it follows, that there exists $\varepsilon > 0$ such that

$$d^\top \nabla^2_{xx} \left(f(x^*) + \varphi_{\mathrm{bar}}(x^*;\rho) + (\lambda^*)^\top g(x^*)\right) d \geq \varepsilon \|d\|_2^2,$$

for all $d \in \mathbb{R}^{n_x}$ with $d \neq 0$ and $\nabla g(x^*)^\top d = 0$. Hence, together with Proposition 4.2 and the choice $z^* = \pi\nu^*$, Theorem 2.25 implies that (x^*, λ^*) is a strict local optimal solution of (NLPbar).

ii. For $\mu > 0$, $\varsigma \in \mathbb{R}^{n_x}_+$ and $\varrho > 0$, the Hessian of (4.9) evaluated at x^* is

$$\begin{aligned} &\pi\nabla^2 f(x^*) + \frac{\pi}{\mu}\Sigma^{-1} \operatorname{diag}(\nu^*) + \pi\sum_{i=1}^{n_g} \lambda_i^* \nabla^2 g_i(x^*) + \frac{1}{\varrho}\nabla g(x^*)\nabla g(x^*)^\top \\ =&\pi\nabla^2_{xx} L(x^*, \lambda^*, \nu^*) + \frac{\pi}{\mu}\Sigma^{-1} \operatorname{diag}(\nu^*) + \frac{1}{\varrho}\nabla g(x^*)\nabla g(x^*)^\top, \end{aligned}$$

using (i). Then again by Debreu [50, Theorem 3], setting $A = \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) + \frac{\pi}{\mu}\Sigma^{-1} \operatorname{diag}(\nu^*)$ and $B = \nabla g(x^*)$, it follows, that there exists $\varepsilon > 0$ such that

$$\begin{aligned} &d^\top \nabla^2_{xx} \left(\pi f(x^*) + \varphi_{\mathrm{bar}}(x^*;\rho)\right) d \\ &+ d^\top \nabla^2_{xx} \left(\left(\pi\lambda^* - \frac{1}{\varrho} g(x^*)\right)^\top g(x^*) + \frac{1}{2\varrho} \|g(x^*)\|_2^2\right) d \geq \varepsilon \|d\|_2^2, \end{aligned}$$

for all $d \in \mathbb{R}^{n_x}$ with $d \neq 0$. Then, by (4.8) with choices $y^* = \pi\lambda^*$ and $z^* = \pi\nu^*$ Theorem 2.25 implies that x^* is a strict local optimal solution of

$$\min_{x\in\mathbb{R}^{n_x}} \quad \pi f(x) + \varphi_{\text{bar}}(x;\rho) + \left(\pi\lambda^* - \frac{1}{\varrho} g(x^*)\right)^\top g(x) + \frac{1}{2\varrho}\,\|g(x)\|_2^2.$$

□

Statement (ii) of Proposition 4.3 has been added to show the good local properties of the classic smooth $\ell 2$-penalty function approach. These will further be exploited for the step computation in Section 4.2.1.

Related Nonlinear Programs

The feasibility problem corresponding to (NLPpen) is a shifted variant of (FeasNLP), i.e.,

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & \|g(x)\|_2 \\ \text{subject to} \quad & x \geq -\mu\varsigma. \end{aligned} \tag{FeasNLP+}$$

Like in Chapter 2 its objective function is non-smooth for feasible points of (NLP+). However, (FeasNLP+) is just used to proof the convergence to a local certificate of infeasibility for which $\|g(x)\|_2 > 0$ holds. Closely related is also the shifted variant of (NLP+)

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & x \geq -\mu\varsigma. \end{aligned} \tag{ShiftNLP+}$$

Note, that for this nonlinear program the KKT conditions equal (4.4) with ν or π is set to zero.

4.2 Algorithm Description

Proposition 4.2 legitimizes the approach of optimizing (NLPpen) for solving (NLP+) when appropriate updates of the Lagrangian multipliers λ and ν are applied. The parameters ς, μ, π and τ have to be adapted during the optimization to balance the minimization between the objective function $f(x)$, the barrier function $\varphi_{\text{bar}}(x;\rho)$ and the penalty function $\varphi_{\text{pen}}(x;\rho)$, respectively.

As already pointed out in Section 3.5.3, the method is closely related to (i) the smooth ℓ_2 or augmented Lagrangian penalty-interior-point algorithm of Armand and Omheni [2], Armand et al. [6] and Omheni [155], (ii) the non-smooth ℓ_2-penalty-interior-point method of Chen [42] and Chen and Goldfarb [40, 41, 43] as well as (iii) the smooth version of it with a modified barrier function by Goldfarb et al. [97] or Gill et al. [95]. Similar is also the augmented Lagrangian penalty-interior-point algorithm of Kuhlmann and Büskens [129], which differs to this presentation – besides smaller technical changes – in the use of a modified barrier function. It resembles the barrier function in Conn et al. [46]. Because the approach (i) is based on a classic penalty function, the authors have developed an adaptive penalty update based on the current constraint violation to increase the practical performance of the method. In this section it will become clear that due to the exact and non-smooth penalty function, a comparable update is performed naturally in the proposed method without any further technical requirements. Since the approaches in (iii), which are also exact, do not consider such a penalty update, the practical performance – which is not reported at all – is probably lacking behind. While the approach (ii) does not face this problem, it has to calculate two different steps in every iteration in order to establish fast local convergence. This will not be the case for the presented method.

In the next sections the main elements of the algorithm, in particular the step computation (Section 4.2.1), the line search (Section 4.2.2) and the parameter updates (Section 4.2.1), will be presented. For increased readability the primal-dual iterates are abbreviated by $w := (x, y, z)$. Definitions of the current iteration, optimal solution or similar directly transfer to w, e.g., $w^k := \left(x^k, y^k, z^k\right)$ or $w^* := (x^*, y^*, z^*)$, respectively. Analogously and similar to Chapter 3, the definitions $\omega := (x, \lambda, \nu)$, $\omega^k := \left(x^k, \lambda^k, \nu^k\right)$ or $\omega^* := (x^*, \lambda^*, \nu^*)$ will also be used to refer to primal-dual variables of the original problem (NLP+).

4.2.1 Step Computation

Locally near the optimal solution x^*, Proposition 4.3 showed that the classic augmented Lagrangian method has better properties due to its differentiability at feasible points. The non-differentiability of the $\ell 2$-penalty function at feasible points and the inequality constraint (4.7c) imply that the first-order optimality conditions (4.7) cannot be used to derive equations analogous to those of the Lagrange-Newton method of Chapter 3. Thus, the step calculation will be based on the KKT conditions of the classic augmented Lagrangian penalty (4.8), but with an adaptive penalty parameter update

$$\varrho_k := \frac{\left\|g\left(x^k\right)\right\|_2}{\tau_k}. \tag{4.10}$$

Interestingly, for the classic augmented Lagrangian penalty function this parameter choice would be invalid for all feasible points due to a division by ϱ_k. However, in terms of the non-smooth penalty-interior-point function $\Upsilon(x;\rho)$, this adaptive update can be interpreted as setting $\left\|g\left(x^k\right)\right\|_2/\tau_k$ for ϱ_k in (4.8b) at every iteration, which is valid for all points. If in addition the dual trust-region condition (4.7c) is omitted, applying Newton's method to (4.7) (or equivalently to (4.8)) yields the linear equation system

$$\underbrace{\begin{bmatrix} Q_k & \nabla g\left(x^k\right) & -E \\ \nabla g\left(x^k\right)^\top & -\varrho_k E & 0 \\ Z_k & 0 & X_k + \mu_k \Sigma_k \end{bmatrix}}_{=:M_k} \underbrace{\begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{bmatrix}}_{=:\Delta w^k} = - \begin{bmatrix} \pi_k \nabla f\left(x^k\right) + \nabla g\left(x^k\right) y^k - z^k \\ g\left(x^k\right) + \varrho_k \left(\pi_k \lambda^k - y^k\right) \\ X_k z^k - \mu_k \Sigma_k \left(\pi_k \nu^k - z^k\right) \end{bmatrix} \tag{4.11}$$

with the Hessian of the Lagrangian function $Q_k := \pi_k \nabla^2 f\left(x^k\right) + \sum_{i=1}^{n_g} y_i^k \nabla^2 g_i\left(x^k\right)$ or an approximation to it. An important observation is the dependence of the Hessian on the dual variables y^k instead of the Lagrangian multipliers λ^k, which include a possible scaling of $\pi_k \in (0,1]$.

Because the iterates x^k will be bounded below by $-\mu_k \varsigma^k$ throughout the optimization, the last equation of the Newton system (4.11) can be eliminated, which leads to the smaller linear equation system

$$\widetilde{M}_k \begin{bmatrix} \Delta x^k \\ \Delta y^k \end{bmatrix} = -\begin{bmatrix} \pi_k \nabla f(x^k) + \nabla g(x^k) y^k - \mu_k \pi_k \Sigma_k (X_k + \mu_k \Sigma_k)^{-1} \nu^k \\ g(x^k) + \varrho_k (\pi_k \lambda^k - y^k) \end{bmatrix} \tag{4.12a}$$

$$\widetilde{M}_k := \begin{bmatrix} \widehat{H}_k & \nabla g(x^k) \\ \nabla g(x^k)^\top & -\varrho_k E \end{bmatrix} \tag{4.12b}$$

$$\widehat{H}_k := Q_k + (X_k + \mu_k \Sigma_k)^{-1} Z_k \tag{4.12c}$$

$$\Delta z^k = -z^k + (X_k + \mu_k \Sigma_k)^{-1} (\mu_k \pi_k \Sigma_k \nu^k - Z_k \Delta x^k). \tag{4.12d}$$

The next result shows, that despite the fixation (4.10) – or, in other words, the step calculation based on the classic augmented Lagrangian function – the step Δx^k is a descent direction for the exact and non-smooth merit function $\Upsilon(x;\rho)$.

Proposition 4.4 (Descent Direction). *Let $\varsigma^k \in \mathbb{R}_+^{n_x}$, $\mu_k > 0$, $\pi_k > 0$, $\tau_k > 0$ and $(\Delta x^k, \Delta y^k, \Delta z^k)$ be a solution of the linear system* (4.12). *Then, the directional derivative*[4] *$D^x_{\Delta x^k} \Upsilon(x^k;\rho^k)$ satisfies*

$$\begin{aligned}
&D^x_{\Delta x^k} \Upsilon(x^k;\rho^k) \\
&= \left(\pi_k \nabla f(x^k) + \nabla_x \varphi_{bar}(x^k;\rho^k)\right)^\top \Delta x^k - \pi_k {\lambda^k}^\top g(x^k) - \tau_k \|g(x^k)\|_2 \\
&\quad + \left(g(x^k) + \varrho_k \pi_k \lambda^k\right)^\top (y^k + \Delta y^k - \pi_k \lambda^k) \\
&= \left(\pi_k \nabla f(x^k) + \nabla_x \varphi_{bar}(x^k;\rho^k) + \pi_k \nabla g(x^k) \lambda^k\right)^\top \Delta x^k \\
&\quad + (y^k + \Delta y^k - \pi_k \lambda^k)^\top g(x^k) - \tau_k \|g(x^k)\|_2 \\
&= \begin{cases} -(\Delta x^k)^\top \widehat{H}_k \Delta x^k, & \text{if } \|g(x^k)\| = 0, \\ -(\Delta x^k)^\top \left(\widehat{H}_k + \frac{1}{\varrho_k} \nabla g(x^k) \nabla g(x^k)^\top\right) \Delta x^k, & \text{if } \|g(x^k)\| > 0 \end{cases}
\end{aligned}$$

Furthermore, if the inertia of (4.12) *satisfies* $\mathrm{In}(\widetilde{M}_k) = (n_x, n_g, 0)$ *and the primal step Δx^k is not zero, then Δx^k is a descent direction for the merit function $\Upsilon(x;\rho^k)$ at x^k, i.e., $D^x_{\Delta x^k} \Upsilon(x^k;\rho^k) < 0$.*

Proof. The proof is similar to the one of Chen and Goldfarb [40, Lemma 3.2] transferred to the augmented Lagrangian penalty and the modified log-barrier function. The proof is split into the two cases $\|g(x^k)\| > 0$ and $\|g(x^k)\| = 0$.

Case $\|g(x^k)\| > 0$. Then, $D^x_{\Delta x^k} \Upsilon(x^k;\rho^k) = \nabla_x \Upsilon(x^k;\rho^k)^\top \Delta x^k$ and

$$\begin{aligned}
&D^x_{\Delta x^k} \Upsilon(x^k;\rho^k) \\
&= \left(\pi_k \nabla f(x^k) + \nabla_x \varphi_{\mathrm{bar}}(x^k;\rho^k) + \pi_k \nabla g(x^k) \lambda^k\right)^\top \Delta x^k + \frac{1}{\varrho_k} g(x^k)^\top \nabla g(x^k)^\top \Delta x^k
\end{aligned}$$

[4] See Definition A.13.

$$
\begin{aligned}
&\overset{(4.12a)}{=} \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right) + \pi_k \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k - \frac{1}{\varrho_k} g\left(x^k\right)^\top g\left(x^k\right) \\
&\quad + g\left(x^k\right)^\top \left(y^k + \Delta y^k - \pi_k \lambda^k\right) \\
&= \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right) + \pi_k \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k \\
&\quad + \left(y^k + \Delta y^k - \pi_k \lambda^k\right)^\top g\left(x^k\right) - \tau_k \left\|g\left(x^k\right)\right\|_2
\end{aligned}
$$

where the second equality follows by applying the second equation of (4.12a). This proves the second equation in Proposition 4.4. Using (4.12a) again, yields

$$
\begin{aligned}
&D^x_{\Delta x^k} \Upsilon\left(x^k;\rho^k\right) \\
&\quad = \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right) + \pi_k \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k \\
&\quad\quad + \left(y^k + \Delta y^k - \pi_k \lambda^k\right)^\top g\left(x^k\right) - \tau_k \left\|g\left(x^k\right)\right\|_2 \\
&\quad \overset{(4.12a)}{=} \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)\right)^\top \Delta x^k - \pi_k {\lambda^k}^\top g\left(x^k\right) \\
&\quad\quad + \pi_k \varrho_k {\lambda^k}^\top \left(y^k + \Delta y^k - \pi_k \lambda^k\right) + \left(y^k + \Delta y^k - \pi_k \lambda^k\right)^\top g\left(x^k\right) - \tau_k \left\|g\left(x^k\right)\right\|_2 \\
&\quad = \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)\right)^\top \Delta x^k - \pi_k {\lambda^k}^\top g\left(x^k\right) - \tau_k \left\|g\left(x^k\right)\right\|_2 \\
&\quad\quad + \left(g\left(x^k\right) + \varrho_k \pi_k \lambda^k\right)^\top \left(y^k + \Delta y^k - \pi_k \lambda^k\right),
\end{aligned}
$$

which proves the first equation in Proposition 4.4. In addition,

$$
\begin{aligned}
&D^x_{\Delta x^k} \Upsilon\left(x^k;\rho^k\right) \\
&\quad = \left(\pi_k \nabla f\left(x^k\right) - \mu_k \pi_k \Sigma_k \left(X_k + \mu_k \Sigma_k\right)^{-1} \nu^k + \pi_k \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k \\
&\quad\quad + \frac{1}{\varrho_k} g\left(x^k\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad \overset{(4.12a)}{=} -\left(\Delta x^k\right)^\top \left(Q_k + \left(X_k + \mu_k \Sigma_k\right)^{-1} Z_k\right) \Delta x^k \\
&\quad\quad - \left(y^k + \Delta y^k - \pi_k \lambda^k - \frac{1}{\varrho_k} g\left(x^k\right)\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad \overset{(4.12c)}{=} -\left(\Delta x^k\right)^\top \widehat{H}_k \Delta x^k - \frac{1}{\varrho_k} \left(\Delta x^k\right)^\top \nabla g\left(x^k\right) \nabla g\left(x^k\right)^\top \Delta x^k
\end{aligned}
$$

where in the second equation the first equation of (4.12a) has been used.

Case $\left\|g\left(x^k\right)\right\| = 0$. Then, it directly follows $\nabla g\left(x^k\right)^\top \Delta x^k = 0$ from the second equation of (4.12a) and, thus,

$$
\begin{aligned}
\lim_{t \downarrow 0} \frac{\left\|g\left(x^k + t\Delta x^k\right)\right\|_2 - \left\|g\left(x^k\right)\right\|_2}{t} &= \lim_{t \downarrow 0} \frac{\left\|g\left(x^k + t\Delta x^k\right) - g\left(x^k\right)\right\|_2}{t} \\
&= \left\|\nabla g\left(x^k\right)^\top \Delta x^k\right\|_2 \\
&= 0,
\end{aligned}
$$

using the definition of the directional derivative. This, together with the fact that $g\big(x^k\big)=0$, $\varrho_k=0$ and, again, $\nabla g\big(x^k\big)^\top \Delta x^k=0$ yields

$$
\begin{aligned}
&D^x_{\Delta x^k}\Upsilon\big(x^k;\rho^k\big)\\
&\quad=\lim_{t\downarrow 0}\Bigg(\frac{\pi_k f\big(x^k+t\Delta x^k\big)+\varphi_{\mathrm{bar}}\big(x^k+t\Delta x^k;\rho^k\big)-\pi_k f\big(x^k\big)-\varphi_{\mathrm{bar}}\big(x^k;\rho^k\big)}{t}\\
&\qquad+\pi_k\frac{\lambda^{k\top}\big(g\big(x^k+t\Delta x^k\big)-g\big(x^k\big)\big)}{t}+\tau_k\frac{\big\|g\big(x^k+t\Delta x^k\big)\big\|_2-\big\|g\big(x^k\big)\big\|_2}{t}\Bigg)\\
&\quad=\big(\pi_k\nabla f\big(x^k\big)+\nabla_x\varphi_{\mathrm{bar}}\big(x^k;\rho^k\big)\big)^\top\Delta x^k+\pi_k\lambda^{k\top}\nabla g\big(x^k\big)^\top\Delta x^k\\
&\quad=\big(\pi_k\nabla f\big(x^k\big)+\nabla_x\varphi_{\mathrm{bar}}\big(x^k;\rho^k\big)+\pi_k\nabla g\big(x^k\big)\lambda^k\big)^\top\Delta x^k\\
&\qquad+\big(y^k+\Delta y^k-\pi_k\lambda^k\big)^\top g\big(x^k\big)-\tau_k\big\|g\big(x^k\big)\big\|_2
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&D^x_{\Delta x^k}\Upsilon\big(x^k;\rho^k\big)\\
&\quad=\big(\pi_k\nabla f\big(x^k\big)+\nabla_x\varphi_{\mathrm{bar}}\big(x^k;\rho^k\big)\big)^\top\Delta x^k-\pi_k\lambda^{k\top}g\big(x^k\big)-\tau_k\big\|g\big(x^k\big)\big\|_2\\
&\qquad+\big(g\big(x^k\big)+\varrho_k\pi_k\lambda^k\big)^\top\big(y^k+\Delta y^k-\pi_k\lambda^k\big).
\end{aligned}
$$

This proves the first and second equation of Proposition 4.4. Furthermore, using the first equation of (4.12a)

$$
\begin{aligned}
&D^x_{\Delta x^k}\Upsilon\big(x^k;\rho^k\big)\\
&\quad=\big(\pi_k\nabla f\big(x^k\big)+\nabla_x\varphi_{\mathrm{bar}}\big(x^k;\rho^k\big)\big)^\top\Delta x^k\\
&\quad=\big(\pi_k\nabla f\big(x^k\big)-\mu_k\pi_k\Sigma_k\left(X_k+\mu_k\Sigma_k\right)^{-1}\nu^k\big)^\top\Delta x^k\\
&\overset{(4.12a)}{=}-\big(\Delta x^k\big)^\top\big(Q_k+\left(X_k+\mu_k\Sigma_k\right)^{-1}Z_k\big)\Delta x^k-\big(y^k+\Delta y^k\big)\nabla g\big(x^k\big)^\top\Delta x^k\\
&\overset{(4.12c)}{=}-\big(\Delta x^k\big)^\top\widehat{H}_k\Delta x^k.
\end{aligned}
$$

Combining the two cases, the two equations of the proposition have been proven. Using Lemma 3.15, the inertia $\mathrm{In}\big(\widetilde{M}_k\big)=\big(n_x,n_g,0\big)$ implies the positive definiteness of the matrices $\widehat{H}_k$ or $\widehat{H}_k+\frac{1}{\varrho_k}\nabla g\big(x^k\big)\nabla g\big(x^k\big)^\top$ for $\big\|g\big(x^k\big)\big\|=0$ or $\big\|g\big(x^k\big)\big\|>0$, respectively. Hence, if $\Delta x^k\neq 0$, it follows $D^x_{\Delta x^k}\Upsilon\big(x^k;\rho^k\big)<0$. □

While the last equation of Proposition 4.4 is mainly used to show the descent direction property, the first two equalities can be used to calculate the descent. Especially the first equation is attractive for practical implementations as it goes without a matrix-vector product. In the following corollary these formulas are presented for the special case of $\lambda^k=y^k/\pi_k$.

Corollary 4.5. *Let $\varsigma^k\in\mathbb{R}^{n_x}_+$, $\mu_k>0$, $\pi_k>0$, $\tau_k>0$, $\big(\Delta x^k,\Delta y^k,\Delta z^k\big)$ be a solution of the* *linear system* (4.12) *and* $\lambda^k = y^k/\pi_k$. *Then,*

$$\begin{aligned} &D^x_{\Delta x^k}\Upsilon(x^k;\rho^k) \\ &\quad = \left(\pi_k \nabla f(x^k) + \nabla_x \varphi_{bar}(x^k;\rho^k)\right)^\top \Delta x^k - y^{k\top} g(x^k) - \tau_k \left\|g(x^k)\right\|_2 \\ &\qquad + \left(g(x^k) + \varrho_k y^k\right)^\top \Delta y^k \\ &\quad = \left(\pi_k \nabla f(x^k) + \nabla_x \varphi_{bar}(x^k;\rho^k) + \nabla g(x^k) y^k\right)^\top \Delta x^k + g(x^k)^\top \Delta y^k - \tau_k \left\|g(x^k)\right\|_2. \end{aligned}$$

Proposition 4.4 and Corollary 4.5 state that the descent direction property is satisfied as long as the inertia of the matrix in (4.12) is $\mathrm{In}(\widetilde{M}_k) = (n_x, n_g, 0)$. If this is not the case, the primal regularization of Section 3.4 with $\delta_p > 0$ large enough can be applied. Adding a further εE with $\varepsilon > 0$ to the Hessian then guarantees the conditions

$$\varepsilon \left\|\Delta x^k\right\|_2^2 \le \begin{cases} (\Delta x^k)^\top \widehat{H}_k \Delta x^k, & \left\|g(x^k)\right\| = 0 \\ (\Delta x^k)^\top \left(\widehat{H}_k + \frac{1}{\varrho_k} \nabla g(x^k) \nabla g(x^k)^\top\right) \Delta x^k, & \left\|g(x^k)\right\| > 0 \end{cases} \tag{4.13}$$

for all $\Delta x^k \in \mathbb{R}^{n_x}$ and in case of $\left\|g(x^k)\right\| = 0$ restricted to $\nabla g(x^k)^\top \Delta x^k = 0$ (cf., Chen and Goldfarb [40, Lemma 3.1]).

This strategy can also be interpreted as a proximal or trust-region algorithm (cf., Parikh and Boyd [159]). Failure can only occur if the current iterate x^k is feasible and the MFCQ fails to hold. In other words, the penalty-interior-point algorithm has the important property of handling problems with rank-deficient Jacobian matrices $\nabla g(x^k)$ at infeasible non-stationary points, because of the automatic dual regularization $\delta_d = \varrho_k > 0$ for $\left\|g(x^k)\right\| > 0$.

Moreover, another relevant feature has to be emphasized again. Choosing $\lambda^k = y^k/\pi_k$, or even $\nu^k = z^k/\pi_k$, reduces the linear equation system (4.12) to a regularized Newton's method for (NLPbar) or (NLP+), respectively. This allows computing steps adaptively that favor the minimization of the original optimization problem during the process rather than just focusing on (NLPpen) in a strict sequential framework.

4.2.2 Line Search

For the globalization of the penalty-interior-point algorithm a line search technique is applied, which updates the optimization variable x^k by

$$x^{k+1} = x^k + \alpha_k \Delta x^k, \tag{4.14}$$

where α_k is the primal step size such that $0 < \alpha_k \le \alpha_{\max,k} \le 1$ (cf., Section 3.3). The maximal allowed primal step size $\alpha_{\max,k}$ has to guarantee that the iterates x^k are bounded below by $-\mu_k$ to keep them valid for the merit function $\Upsilon(x;\rho^k)$ and, thus, is computed by the fraction-to-the-boundary rule

$$\alpha_{\max,k} = \max\left\{\alpha \in (0,1] \mid x^k + \mu_k \varsigma^k + \alpha \Delta x^k \ge (1 - \varepsilon_{\mathrm{frac},k})(x^k + \mu_k \varsigma^k)\right\} \tag{4.15}$$

with a parameter sequence $\{\varepsilon_{\mathrm{frac},k}\}_k$ satisfying $\varepsilon_{\mathrm{frac},k} \in (0,1)$ and $\{\varepsilon_{\mathrm{frac},k}\}_k \to 1$. To measure progress towards optimality, two approaches are combined: a non-monotone merit function (Section 3.3.1) with a non-monotone filter (Section 3.3.2) or a non-monotone PLPF (Section 3.3.3).

A Non-monotone Merit Function

Forcing a reduction in the merit function $\Upsilon(x;\rho^k)$ is a straightforward criterion for penalty-interior-point algorithms due to the descent direction property of Proposition 4.4. However, due to reasons of global convergence of the proposed algorithm, $\Upsilon(x;\rho^k)$ is further augmented to the primal-dual merit function

$$\begin{aligned}\Psi(w^k;\rho^k) = \Upsilon(x^k;\rho^k) + \frac{\tau_\mathrm{f}}{2}\left\|g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right\|_2^2 \\ + \tau_\mathrm{c}\left(\left(x^k + \mu_k\varsigma^k\right)^\top z^k - \mu_k\pi_k\sum_{i=1}^{n_x}\nu_i^k\varsigma_i^k\ln\left(\left(x_i^k + \mu_k\varsigma_i^k\right)z_i^k\right)\right)\end{aligned} \tag{4.16}$$

with penalty parameters $\tau_\mathrm{f} > 0$ and $\tau_\mathrm{c} > 0$. This merit function is adapted from [55, 56] and is again exact, because the penalty parameters τ_f and τ_c can be fixed to an arbitrary positive value throughout the whole optimization process. The descent direction property of $\Upsilon(x;\rho^k)$ directly transfers to the primal-dual merit function $\Psi(w;\rho^k)$ as shown by the following two results.

Proposition 4.6 (Descent Direction of Merit Function). *Let $\varsigma^k \in \mathbb{R}_+^{n_x}$, $\mu_k > 0$, $\pi_k > 0$, $\tau_k > 0$ and Δw^k be a solution of the linear system* (4.12)*. Then,*

$$\begin{aligned}D_{\Delta w^k}^x\Psi(w^k;\rho^k) = D_{\Delta x^k}^x\Upsilon(x^k;\rho^k) - \tau_f\left\|g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right\|_2^2 \\ - \tau_c\left\|\left(X_k + \mu_k\Sigma_k\right)^{-\frac{1}{2}} Z_k^{-\frac{1}{2}}\left(X_k z^k - \mu_k\Sigma_k\left(\pi_k\nu^k - z^k\right)\right)\right\|_2^2.\end{aligned}$$

Furthermore, if the inertia of (4.12) *satisfies* $\mathrm{In}(\widetilde{M}_k) = (n_x, n_g, 0)$ *and the step Δw^k is not zero, then Δw^k is a descent direction for the merit function $\Psi(w;\rho^k)$ at w^k, i.e., $D_{\Delta w^k}^x\Psi(w^k;\rho^k) < 0$.*

Proof. The proof is similar to Omheni [155, Proposition 4.2.1] but adapted to the modified barrier function. The directional derivative is given by

$$\begin{aligned}D_{\Delta w^k}^x\Psi(w^k;\rho^k) = D_{\Delta x^k}^x\Upsilon(x^k;\rho^k) + \tau_\mathrm{f}\nabla_w\left(\frac{1}{2}\left\|g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right\|_2^2\right)\Delta w^k \\ + \tau_\mathrm{c}\nabla_w\left(\left(x^k + \mu_k\varsigma^k\right)^\top z^k - \mu_k\pi_k\sum_{i=1}^{n_x}\nu_i^k\varsigma_i^k\ln\left(\left(x_i^k\mu_k\varsigma_i^k\right)z_i^k\right)\right)\Delta w^k\end{aligned}$$

Using (4.12a), it follows

$$\begin{aligned}&\nabla_w\left(\frac{1}{2}\left\|g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right\|_2^2\right)\Delta w^k \\ &\quad= \left(g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right)^\top\left(\nabla g(x^k)^\top\Delta x^k - \varrho_k\Delta y^k\right) \\ &\quad\overset{(4.12a)}{=} -\left(g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right)^\top\left(g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right) \\ &\quad= -\left\|g(x^k) + \varrho_k\left(\pi_k\lambda^k - y^k\right)\right\|_2^2.\end{aligned}$$

Furthermore, by (4.12d)

$$\begin{aligned}
&\nabla_w\left(\left(x^k+\mu_k\varsigma^k\right)^\top z^k-\mu_k\pi_k\sum_{i=1}^{n_x}\nu_i^k\varsigma_i^k\ln\left(\left(x_i^k\mu_k\varsigma_i^k\right)z_i^k\right)\right)\Delta w^k\\
&=\left(Z_k-\mu_k\pi_k\Sigma_k\left(X_k+\mu_k\Sigma_k\right)^{-1}\operatorname{diag}\left(\nu^k\right)\right)\Delta x^k\\
&\quad+\left(X_k+\mu_k\Sigma_k-\mu_k\pi_k\Sigma_k Z_k^{-1}\operatorname{diag}\left(\nu^k\right)\right)\Delta z^k\\
&=\left(\left(X_k+\mu_k\Sigma_k\right)Z_k-\mu_k\pi_k\Sigma_k\operatorname{diag}\left(\nu^k\right)\right)\left(\left(X_k+\mu_k\Sigma_k\right)^{-1}\Delta x^k+Z_k^{-1}\Delta z^k\right)\\
&=\left(\left(X_k+\mu_k\Sigma_k\right)Z_k-\mu_k\pi_k\Sigma_k\operatorname{diag}\left(\nu^k\right)\right)\left(X_k+\mu_k\Sigma_k\right)^{-1}Z_k^{-1}\\
&\quad\cdot\left(Z_k\Delta x^k+\left(X_k+\mu_k\Sigma_k\right)\Delta z^k\right)\\
&\overset{(4.12d)}{=}-\left(\left(X_k+\mu_k\Sigma_k\right)Z_k-\mu_k\pi_k\Sigma_k\operatorname{diag}\left(\nu^k\right)\right)\left(X_k+\mu_k\Sigma_k\right)^{-1}Z_k^{-1}\\
&\quad\cdot\left(\left(X_k+\mu_k\Sigma_k\right)z^k-\mu_k\pi_k\Sigma_k\nu^k\right)\\
&=-\left\|\left(X_k+\mu_k\Sigma_k\right)^{-\frac12}Z_k^{-\frac12}\left(X_kz^k-\mu_k\Sigma_k\left(\pi_k\nu^k-z^k\right)\right)\right\|_2^2.
\end{aligned}$$

Combining the above equations with Proposition 4.4 leads to the claim of this proposition. □

Corollary 4.7. *Let $\varsigma^k\in\mathbb{R}_+^{n_x}$, $\mu_k>0$, $\pi_k>0$, $\tau_k>0$, Δw^k be a solution of the linear system (4.12) and $\left(\lambda^k,\nu^k\right)=\left(y^k/\pi_k, z^k/\pi_k\right)$. Then,*

$$D^x_{\Delta w^k}\Psi\left(w^k;\rho^k\right)=D^x_{\Delta x^k}\Upsilon\left(x^k;\rho^k\right)-\tau_f\left\|g\left(x^k\right)\right\|_2^2-\tau_c\left\|\left(X_k+\mu_k\Sigma_k\right)^{-\frac12}Z_k^{\frac12}x^k\right\|_2^2.$$

For acceptance of a trial iterate $x^k+\alpha_k\Delta x^k$ the Armijo condition

$$z^k+\alpha_k\Delta z^k\geq\left(1-\varepsilon_{\text{frac},k}\right)z^k \tag{4.17a}$$

$$\Psi\left(w^k+\alpha_k\Delta w^k;\rho^k\right)-\max_{i=0,\dots,l_m}\left\{\Psi\left(w^{(k-i)_+};\rho^k\right)\right\}\leq\sigma\alpha_k\Psi\left(w^k;\rho^k\right)^\top\Delta w^k \tag{4.17b}$$

with $\sigma\in\left(0,\frac12\right)$ is checked. Due to the descent direction property for the merit function $\Upsilon\left(x;\rho^k\right)$ in Proposition 4.4 (for appropriate Hessian regularizations), the existence of a positive step size $\alpha_k\in\left(0,\alpha_{\max,k}\right]$ satisfying (4.17b) is guaranteed (cf., Section 4.3). The fraction-to-the-boundary rule (4.17a) ensures that the dual iterates stay strictly positive but has no influence on $\alpha_{\max,k}$, as it could reduce the step size α_k unnecessarily. Because of strict positivity of z^k there exists an $\alpha_k\in\left(0,\alpha_{\max,k}\right]$ such that (4.17a) and, thus, (4.17) is satisfied.

A Non-monotone Filter and PLPF

As pointed out in Section 3.3.1, the specific choice of parameters may have a huge impact on practical efficiency of the line search method. This is especially true for the proposed penalty-interior-point algorithm with its parameters μ_k, π_k, τ_k, ν^k and λ^k. Updating (at least) some of them within the line search method to promote larger step sizes α_k is not an option since the updated parameters may be a bad choice for the next iteration. To avoid this difficulty and to allow a higher flexibility of the step acceptance, the Armijo condition (4.17) is combined with

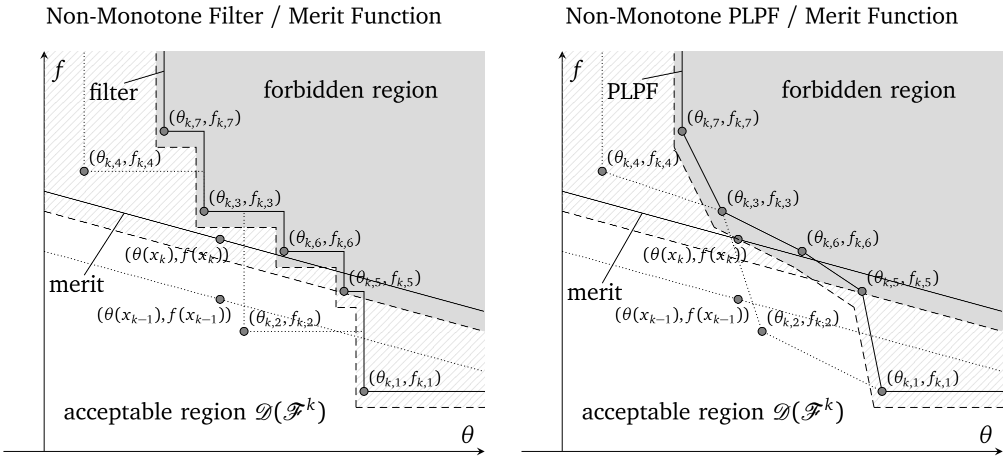

Figure 4.2: Non-monotone filter (left) and non-monotone PLPF (right) combined with non-monotone merit function. The filter and PLPF envelope is dependent on the current iterate and the non-monotonicity level is $l_\mathrm{f} = 1$. The shaded area is the part of the acceptable region that is added by the combination of the merit function with the filter or PLPF, respectively, in comparison to each method alone.

either a non-monotone filter or a non-monotone PLPF with non-monotonicity level $l_\mathrm{f} \in \mathbb{N}_0$. Figure 4.2 illustrates this combination. Like in Section 3.3.2 and Section 3.3.3 these work in the two-dimensional space of objective function values $f(x)$ and constraint violations $\theta(x) = \|g(x)\|_2$ of (NLP+) and not, for example, (NLPbar). This avoids the dependence of the filter or PLPF entries on parameter choices like the barrier parameter μ_k, which is often the case for interior-point filter line search methods (cf., for example, Benson et al. [16] or Wächter and Biegler [202]). The filter or PLPF envelope is defined as

$$\begin{aligned}\delta_k = {} & \left\|\pi_{k-1}\nabla f\left(x^k\right) + \nabla g\left(x^k\right)y^k - z^k\right\|_2^2 + \left\|g\left(x^k\right)\right\|_2^2 \\ & + \left\|X_k z^k - \mu_{k-1}\Sigma_{k-1}\left(\pi_{k-1}v^{k-1} - z^k\right)\right\|_2^2,\end{aligned} \tag{4.18}$$

Note, that depending on the specific choice – filter or PLPF – the set $\mathscr{F}_k(l_\mathrm{f})$ may and $\mathscr{D}(\mathscr{F}_k(l_\mathrm{f}))$ will certainly differ, but the augmentation, i.e.,

$$\mathscr{F}_{k+1}(l_\mathrm{f}) = \mathscr{F}_k(l_\mathrm{f}) \cup \left\{\left(f\left(x^k\right), \theta\left(x^k\right)\right)\right\}, \tag{4.19}$$

and acceptability check

$$\left(f\left(x^k + \alpha_k \Delta x^k\right) + \gamma_\mathrm{f}\delta_k, \theta\left(x^k + \alpha_k \Delta x^k\right) + \gamma_\mathrm{f}\delta_k\right) \in \mathscr{D}\left(\mathscr{F}_k(l_\mathrm{f}) \cup \left\{\left(f\left(x^k\right), \theta\left(x^k\right)\right)\right\}\right), \tag{4.20}$$

are defined equally. A very important feature of this line search combination is that there is no necessity for any further sufficient reduction conditions or a feasibility restoration phase, which separately optimizes (FeasNLP). Comparable line search combinations have been proposed by Chen and Goldfarb [43] (with a monotone PLPF) and Gould et al. [105, 106] (with a monotone or non-monotone filter, but applied to an SQP method).

Second-Order-Correction Steps

If all acceptance conditions – merit function, filter or PLPF – fail for a maximal step $\alpha_{\max,k}\Delta w^k$ but the current iteration is possibly close to the optimal solution, a second-order-correction step $\widehat{\Delta w}^k := \left(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k\right)$ using constraint information at the point $x^k + \alpha_{\max,k}\Delta x^k$ is calculated by

$$\widetilde{M}_k \begin{bmatrix} \widehat{\Delta x}^k \\ \widehat{\Delta y}^k \end{bmatrix} = -\begin{bmatrix} \pi_k \nabla f\left(x^k\right) + \nabla g\left(x^k\right) y^k - \mu_k \pi_k \Sigma_k \left(X_k + \mu_k \Sigma_k\right)^{-1} \nu^k \\ g\left(x^k + \alpha_{\max,k}\Delta x^k\right) - \alpha_{\max,k} \nabla g\left(x^k\right)^\top \Delta x^k \end{bmatrix} \tag{4.21a}$$

$$\widehat{\Delta z}^k = -z^k + \left(X_k + \mu_k \Sigma_k\right)^{-1} \left(\mu_k \pi_k \Sigma_k \nu^k - Z_k \widehat{\Delta x}^k\right), \tag{4.21b}$$

which is equivalent to

$$M_k \widehat{\Delta w}^k = -\begin{bmatrix} \pi_k \nabla f\left(x^k\right) + \nabla g\left(x^k\right) y^k - z^k \\ g\left(x^k + \alpha_{\max,k}\Delta x^k\right) - \alpha_{\max,k} \nabla g\left(x^k\right)^\top \Delta x^k \\ X_k z^k - \mu_k \Sigma_k \left(\pi_k \nu^k - z^k\right) \end{bmatrix}, \tag{4.22}$$

to improve the quality of the current step (cf., Section 3.6.2). Analogously to the standard line search, a fraction-to-the-boundary rule

$$\widehat{\alpha}_k = \max\left\{\alpha \in (0,1] \mid x^k + \mu_k \varsigma^k + \alpha \widehat{\Delta x}^k \geq \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \mu_k \varsigma^k\right)\right\} \tag{4.23}$$

is applied, followed by the Armijo condition

$$z^k + \widehat{\alpha}_k \widehat{\Delta z}^k \geq \left(1 - \varepsilon_{\text{frac},k}\right) z^k \tag{4.24a}$$

$$\Psi\left(w^k + \widehat{\alpha}_k \widehat{\Delta w}^k; \rho^k\right) - \max_{i=0,\dots,l_\text{m}} \left\{\Psi\left(w^{(k-i)_+}; \rho^k\right)\right\} \leq \sigma \alpha_k \Psi\left(w^k; \rho^k\right)^\top \Delta w^k \tag{4.24b}$$

and the filter or PLPF acceptance condition

$$\left(f\left(x^k + \widehat{\alpha}_k \widehat{\Delta x}^k\right) + \gamma_\text{f}\delta_k, \theta\left(x^k + \widehat{\alpha}_k \widehat{\Delta x}^k\right) + \gamma_\text{f}\delta_k\right) \in \mathscr{D}\left(\mathscr{F}_k(l_\text{f}) \cup \left\{\left(f\left(x^k\right), \theta\left(x^k\right)\right)\right\}\right). \tag{4.25}$$

If one of the two is satisfied, the second-order-correction is used to provide the next iterate by

$$x^{k+1} = x^k + \widehat{\alpha}_k \widehat{\Delta x}^k \tag{4.26}$$

instead of (4.14). Otherwise, the step $\widehat{\Delta w}^k$ is rejected, a backtracking line search $\alpha_k \leftarrow \beta \alpha_k$ with $\beta \in (0,1)$ reduces the step size α_k and the standard Armijo (4.17) and filter or PLPF condition (4.20) are checked again.

Since second-order-corrections help to improve local convergence behavior (cf., Section 4.4), it should – as mentioned in the beginning of this subsection – only be applied near the optimal solution: If the current dual variables y^k are of bad quality such that the Lagrangian multipliers λ^k cannot be updated (cf., the next Section 4.2.3), the current iterate may be too far away from the optimal solution and a second-order-correction is not used.

Update of Dual Variables

The dual iterates $\left(y^k, z^k\right)$ are updated by

$$y^{k+1} = y^k + \alpha_k \Delta y^k \tag{4.27a}$$
$$z^{k+1} = z^k + \min\left\{\alpha_k, \alpha_k^z\right\} \Delta z^k. \tag{4.27b}$$

The dual step size α_k^z is computed with a fraction-to-the-boundary rule

$$\alpha_k^z = \max\left\{\alpha \in (0,1] \mid z^k + \alpha \Delta z^k \geq \left(1 - \varepsilon_{\text{frac},k}\right) z^k\right\} \tag{4.28}$$

to ensure the strict positivity of the dual iterate z^{k+1}. A further projection of z_i^{k+1} into the interval

$$\frac{\pi_k \mu_k \varsigma_i^k \nu_i^k}{x_i^{k+1} + \mu_k \varsigma_i^k} \left[\kappa_z^{-1}, \kappa_z\right] \tag{4.29}$$

for all $i = 1, \ldots, n_x$ by

$$z_i^{k+1} \leftarrow \max\left\{\min\left\{z_i^{k+1}, \frac{\kappa_z \pi_k \mu_k \varsigma_i^k \nu_i^k}{x_i^{k+1} + \mu_k \varsigma_i^k}\right\}, \frac{\pi_k \mu_k \varsigma_i^k \nu_i^k}{\kappa_z \left(x_i^{k+1} + \mu_k \varsigma_i^k\right)}\right\}, \quad i = 1, \ldots, n_x \tag{4.30}$$

with $\kappa_z > 0$ prevents z^{k+1} from deviating too much from $\pi_k \mu_k \varsigma_i^k \nu_i^{k+1} / \left(x_i^{k+1} + \mu_k \varsigma_i^k\right)$, which must be satisfied for an optimal solution of (NLPbar).

4.2.3 Parameter Update

As described in Section 4.1 the transformation of (NLP+) into an unconstrained optimization problem using a combination of the interior-point and the penalty approach introduced parameters μ_k, ς^k and ν^k for (NLPbar) and π_k, τ_k and λ^k for (NLPpen). To eventually reach an optimal solution or certificate of infeasibility of the original program (NLP+), these parameters have to be controlled and carefully updated during the optimization process.

Update of Penalty Parameters and Lagrangian Multipliers λ

For the update of the penalty parameters $\pi_k \in (0,1]$ or $\tau_k > 0$ and the Lagrangian multiplier λ^k the first-order optimality conditions (4.7) of (NLPpen) are requested to be satisfied up to a certain tolerance $\varepsilon_{\rho,k} > 0$, i.e.,

$$\left\|\pi_{k-1} \nabla f\left(x^k\right) + \nabla g\left(x^k\right) y^k - z^k\right\|_\infty \leq \varepsilon_{\rho,k} \tag{4.31a}$$
$$\left\|g\left(x^k\right) - \frac{\left\|g\left(x^k\right)\right\|_2}{\tau_{k-1}} \left(y^k - \pi_{k-1} \lambda^{k-1}\right)\right\|_\infty \leq \varepsilon_{\rho,k} \tag{4.31b}$$
$$\left\|X_k z^k - \mu_{k-1} \Sigma_{k-1} \left(\pi_{k-1} \nu^{k-1} - z^k\right)\right\|_\infty \leq \varepsilon_{\rho,k} \tag{4.31c}$$

or abbreviated $\left\|\Phi_{\text{pen}}\left(w^k;\rho^{k-1}\right)\right\|_\infty \le \varepsilon_{\rho,k}$ with a function

$$\Phi_{\text{pen}}\left(w^k;\rho^{k-1}\right) := \begin{bmatrix} \pi_{k-1}\nabla f\left(x^k\right)+\nabla g\left(x^k\right)y^k - z^k \\ g\left(x^k\right) - \frac{\|g(x^k)\|_2}{\tau_{k-1}}\left(y^k - \pi_{k-1}\lambda^{k-1}\right) \\ X_k z^k - \mu_{k-1}\Sigma_{k-1}\left(\pi_{k-1}\nu^{k-1} - z^k\right) \end{bmatrix}. \tag{4.32}$$

Satisfying the conditions (4.31) is not sufficient to give convergence to a first-order optimal solution of (NLP+), (NLPbar) or (NLPpen). For the former two, primal feasibility has eventually to be improved – a goal, which can be controlled by the neglected dual trust-region condition (4.7c), i.e., $\left\|y^k - \pi_{k-1}\lambda^{k-1}\right\|_2 \le \tau_k$. From (4.6) it can be concluded, that if $\left\|y^k - \pi_{k-1}\lambda^{k-1}\right\|_2 < \tau_k$ the case $\left\|g\left(x^k\right)\right\| = 0$ in (4.6) would be true. Otherwise, no such implication can be made and therefore $\left\|g\left(x^k\right)\right\| \ge 0$. In the former case, Proposition 4.2 suggests updating the Lagrangian multipliers λ^k and in the other case the penalty parameters π_k or τ_k to try to avoid the situation of having $\left\|g\left(x^k\right)\right\| > 0$.

In particular, the Lagrangian multipliers λ^k are updated by

$$\lambda^k = \frac{y^k}{\pi_{k-1}} \tag{4.33}$$

if, in addition to (4.31), the dual trust-region conditions

$$\left\|y^k - \pi_{k-1}\lambda^{k-1}\right\|_2 \le \kappa_\lambda \tau_k, \tag{4.34a}$$

$$\left\|y^k\right\|_2 \le \kappa_\lambda \tau_k \tag{4.34b}$$

with $\kappa_\lambda \in (0,1)$ and

$$\left\|\Phi_{\text{bar}}\left(w^k;\rho^{k-1}\right)\right\|_\infty \le \varepsilon_{\lambda,k} \tag{4.35}$$

with $\varepsilon_{\lambda,k} > 0$ and

$$\Phi_{\text{bar}}\left(w^k;\rho^{k-1}\right) := \begin{bmatrix} \pi_{k-1}\nabla f\left(x^k\right)+\nabla g\left(x^k\right)y^k - z^k \\ g\left(x^k\right) \\ X_k z^k - \mu_{k-1}\Sigma_{k-1}\left(\pi_{k-1}\nu^{k-1} - z^k\right) \end{bmatrix} \tag{4.36}$$

hold. The condition (4.34b) is used to reduce the possibility of the complementarity term in the objective function of (NLPpen) becoming negative and larger in magnitude than the constraint violation penalty. Such a situation may cause the algorithm to increase the constraint violation even further, which would lead to divergence. If $\varepsilon_{\lambda,k} \le \varepsilon_{\rho,k}$, condition (4.35) ensures that the update of the parameter λ^k does not increase the violation of (4.31b). In fact, (4.31) and (4.34a) alone would imply that $\left\|g\left(x^k\right)\right\|_2$ and thus $\Phi_{\text{bar}}\left(w^k;\rho^k\right) = \Phi_{\text{pen}}\left(w^k;\rho^k\right)$ is bounded above by a multiple of $\left(1-\kappa_\lambda\right)^{-1}\varepsilon_{\rho,k}$ (cf., Lemma 4.8). But then, if κ_λ was chosen close to one – which is desirable for (4.34a) – the violation of (4.31b) could get extremely large after an update of λ^k. Condition (4.35) avoids this situation.

Lemma 4.8. *If* (4.31) *and* (4.34a) *are satisfied, then there exists a constant* $c > 0$ *such that* $\left\|g\left(x^k\right)\right\|_2 \le \frac{c}{1-\kappa_\lambda}\varepsilon_{\rho,k}$ *holds.*

Proof. From (4.31) and (4.34a) it follows

$$\begin{aligned} 0 &\leq (1-\kappa_\lambda)\left\|g\left(x^k\right)\right\|_2 \\ &\overset{(4.34a)}{\leq} \left\|g\left(x^k\right)\right\|_2 - \frac{\left\|g\left(x^k\right)\right\|_2}{\tau_k}\left\|y^k - \pi_{k-1}\lambda^{k-1}\right\|_2 \\ &\leq \left\|g\left(x^k\right) - \frac{\left\|g\left(x^k\right)\right\|_2}{\tau_k}\left(y^k - \pi_{k-1}\lambda^{k-1}\right)\right\|_2 \\ &\overset{(4.31)}{\leq} c\varepsilon_{\rho,k} \end{aligned}$$

for some constant $c > 0$, which leads to $\left\|g\left(x^k\right)\right\|_2 \leq \frac{c}{1-\kappa_\lambda}\varepsilon_{\rho,k}$. □

Otherwise, if (4.31) holds, but (4.34a), (4.34b) or (4.35) is violated, the penalty parameter is updated to increase the constraint penalization. For this, two possibilities exist: increasing τ or decreasing π. If a huge penalization is needed, i.e., τ has to be very large or π very small, both approaches have a disadvantage. In case of an infeasible (NLP+), τ will tend to infinity causing the multipliers y to diverge (cf., Chen and Goldfarb [40]) and complicate the check for a certificate of infeasibility in a practical algorithm. On the other hand, if (NLP+) is feasible, a penalty parameter π smaller than one scales the solver tolerances leading to possibly inaccurate solutions (cf., Curtis [48]). Therefore, the proposed algorithm makes use of both penalty parameters trying to avoid the disadvantages as much as possible. If τ_k is small, it is unlikely that (NLP+) is infeasible and τ_k will be updated instead of π_k. However, the possibility of infeasibility increases with τ_k and if τ_k reaches a certain threshold $\tau_{\max} > 0$, the method switches to update π_k, i.e.,

$$\begin{cases} \pi_k = \kappa_\pi \pi_{k-1} & \tau_{k-1} \geq \tau_{\max} \\ \tau_k = \min\{\tau_{\max}, \kappa_\tau \tau_{k-1}\} & \tau_{k-1} < \tau_{\max} \end{cases} \tag{4.37}$$

with parameters $\kappa_\pi \in (0,1)$ and $\kappa_\tau > 1$.

All iterations for which (4.31) holds are collected in an index set $\mathscr{K}_\pi$ or $\mathscr{K}_\lambda$ depending on whether the penalty parameter or the Lagrangian multiplier has been updated, i.e., if (4.31) and (4.34a) is satisfied, the set $\mathscr{K}_\lambda$ is augmented by $\mathscr{K}_\lambda \cup \{k\}$ and if (4.31) but not (4.34a) holds, $\mathscr{K}_\pi$ is augmented with the current iteration index k. Then, the definition

$$\varepsilon_{\rho,k} := \pi_{k-1}\left(\upsilon_1\left\|\Phi_{\text{pen}}\left(w^i;\rho^{i-1}\right)\right\|_\infty + \upsilon_2\varepsilon_{\lambda,k} + \xi_{\rho,k}\right) \tag{4.38a}$$

$$\varepsilon_{\lambda,k} := \upsilon_3\left\|\Phi_{\text{bar}}\left(w^i;\rho^{i-1}\right)\right\|_\infty + \xi_{\lambda,k} \tag{4.38b}$$

with $\upsilon_1 \in (0,1)$, $\upsilon_2 > 0$, $\upsilon_3 \in (0,1)$, two non-negative sequences $\left\{\xi_{\rho,k}\right\}_{\mathscr{K}_\lambda}$ and $\left\{\xi_{\lambda,k}\right\}_{\mathscr{K}_\lambda}$ both converging to zero and $i = \arg\max \mathscr{K}_\lambda$ guarantees that $\varepsilon_{\rho,k} \to 0$ if either the penalty parameter π_k (or τ_k) or the Lagrangian multiplier λ^k is updated infinitely many times. A formal proof is given in the next two lemmas.

Lemma 4.9. *Let $\{a_k\}_k$ be a sequence with $a_k \geq 0$ converging to zero, $c \in (0,1)$, $l_\varepsilon \in \mathbb{N}_0$ and $\mathscr{K}$ be an infinite index set. If $\{b_k\}_k$ is a sequence with $b_k > 0$ and*

$$b_k \begin{cases} \leq c\max_{i=0,\dots,l_\varepsilon}\left\{b_{(k-i)_+}\right\} + a_k, & \text{if } k \in \mathscr{K} \\ = b_{k-1} & \text{if } k \notin \mathscr{K} \end{cases}$$

then $\{b_k\}_k$ *converges to zero.*

Proof. See Armand and Omheni [3, Lemma 3.1]. □

Lemma 4.10. *Let* $\upsilon_1 \in (0,1)$*,* $\upsilon_2 > 0$*,* $\upsilon_3 \in (0,1)$*, two non-negative sequences* $\{\xi_{\rho,k}\}_{\mathcal{K}_\lambda}$ *and* $\{\xi_{\lambda,k}\}_{\mathcal{K}_\lambda}$ *both converging to zero. If* $\varepsilon_{\rho,k}$ *and* $\varepsilon_{\lambda,k}$ *are defined as in* (4.38)*, then*

i. $\varepsilon_{\rho,k}$ *converges to zero if the Lagrangian multipliers* λ^k *or the penalty parameter* π_k *are updated infinitely many times.*

ii. $\varepsilon_{\lambda,k}$ *converges to zero if the Lagrangian multipliers* λ^k *are updated infinitely many times.*

Proof. First, assume that the index set $\mathcal{K}_\lambda$ is finite and $\mathcal{K}_\pi$ is infinite, meaning that λ^k is updated finitely many times and π_k infinitely many times. Then, $\Phi_{\text{bar}}(w^i; \rho^{i-1})$ stays fixed for $k \in \mathcal{K}_\pi$ large enough and $\pi_k \to 0$ for $k \in \mathcal{K}_\pi \to \infty$ due to the update strategy (4.37). This directly implies $\varepsilon_{\rho,k} \to 0$ for $k \in \mathcal{K}_\pi \to \infty$.

Now assume that $\mathcal{K}_\lambda$ is infinite. Since $\pi_k \leq 1$, $0 \leq \xi_{\rho,k} \to 0$ and $0 \leq \xi_{\lambda,k} \to 0$ for $k \in \mathcal{K}_\lambda \to \infty$, Lemma 4.9 together with (4.31) and (4.35) implies $\varepsilon_{\rho,k} \to 0$ and $\varepsilon_{\lambda,k} \to 0$ for $k \in \mathcal{K}_\lambda \to \infty$. □

Update of Barrier Parameters and Lagrangian Multipliers ν

Similar to the previous section, the barrier parameter $\mu_k > 0$, the primal shift parameter $\varsigma^k > 0$ and the Lagrangian multipliers ν^k get updated if the barrier subproblem (NLPbar) has been solved to a certain tolerance $\varepsilon_{\mu,k} > 0$, i.e.,

$$\left\|\pi_{k-1}\nabla f(x^k) + \nabla g(x^k)y^k - z^k\right\|_\infty \leq \varepsilon_{\mu,k} \tag{4.39a}$$

$$\left\|g(x^k)\right\|_\infty \leq \varepsilon_{\mu,k} \tag{4.39b}$$

$$\left\|X_k z^k - \mu_{k-1}\Sigma_{k-1}\left(\pi_{k-1}\nu^{k-1} - z^k\right)\right\|_\infty \leq \varepsilon_{\mu,k} \tag{4.39c}$$

or compactly if $\left\|\Phi_{\text{bar}}(w^k; \rho^{k-1})\right\|_\infty \leq \varepsilon_{\mu,k}$ using the definition in (4.36).

In contrast to (NLPpen) there appears no fourth first-order optimality condition like (4.7c), that naturally leads to a criterion to either update the barrier parameter μ_k or the Lagrangian multiplier ν^k. Therefore, the Lagrangian multipliers ν^k are updated by

$$\left(\lambda^k, \nu^k\right) = \frac{1}{\pi_{k-1}}\left(y^k, z^k\right) \tag{4.40}$$

if good progress is made towards the original KKT conditions, i.e., if in addition to (4.39)

$$\left\|\Phi(w^k; \rho^{k-1})\right\|_\infty := \left\|\begin{bmatrix} \nabla f(x^k) + \frac{1}{\pi_{k-1}}\nabla g(x^k)y^k - \frac{1}{\pi_{k-1}}z^k \\ g(x^k) \\ \frac{1}{\pi_{k-1}}X_k z^k \end{bmatrix}\right\|_\infty \leq \varepsilon_{\nu,k} \tag{4.41}$$

holds.

The simultaneous update of the multipliers λ^k in (4.40) is performed because the satisfaction of (4.4) implies (4.35) if $\varepsilon_{\lambda,k} \leq \varepsilon_{\mu,k}$ and therefore indicates that y^k might already be a good approximation of the optimal Lagrangian multiplier. This strategy can also be motivated by Lemma 4.2 stating that an optimal solution of (NLP+) is an optimal solution of (NLPbar) and (NLPpen) and therefore good progress with respect to the former means good progress with respect to the latter. It has to be noted, that because z^k is kept strictly positive by (4.28), so will be ν^k. Thus, the barrier part of $\Upsilon\left(x;\rho^k\right)$ will not disappear for any constraint $x_i \geq 0$, $i = 1, \ldots, n_x$ of (NLP+).

The primal shift is defined to be

$$\varsigma_i^k = \left(\nu_i^k\right)^{\kappa_\varsigma} \tag{4.42}$$

for all $i = 1, \ldots, n_x$, with $\kappa_\varsigma \in (0,1)$ and therefore is updated together with the Lagrangian multipliers ν^k. This update strategy again may need a further primal projection. To reduce the occurrence of this case κ_ς may be set to a very small value.

For the barrier parameter μ_k two basic update criteria are considered. Update μ_k if both (4.39) is satisfied and (4.41) is not satisfied, or if (4.39) holds. While the former is similar to the penalty update strategy, the latter would also update the barrier parameter together with the Lagrangian multipliers ν^k. For the global convergence of the algorithm this will not make any difference. However, it will do for the local convergence as further discussed in Section 4.4. The following algorithm description focuses on the first choice because the other option can then be derived easily. When updating the barrier parameter it is important to remember that besides the primal shift ς^k it defines the size of the boundary shift of the inequality constraints (cf., Section 4.1) and therefore not all values in $(0, \mu_{k-1})$ may be valid as a new barrier parameter μ_k. To be more precise, if $x_i^k \in (-\mu_{k-1}, 0)$ for some $i \in \{1, \ldots, n_x\}$ meaning that the current iterate is infeasible for (NLP+) with respect to the constraint $x_i^k \geq 0$, the update of the barrier parameter must ensure that the iterate stays within some new and smaller interior region. Therefore, the new barrier parameter is requested to satisfy the fraction-to-the-boundary condition

$$x^k + \mu_k \varsigma^k \geq (1 - \tau_{k-1})\left(x^{k-1} + \mu_{k-1}\varsigma^{k-1}\right) \tag{4.43}$$

(cf., Section 4.2.2). This can either be achieved by choosing μ_k not to be smaller than the minimal allowed value, i.e., setting

$$\mu_k = \max\left\{\kappa_\mu \mu_{k-1}, \min\left\{\mu_k \in (0, \mu_{k-1}) \mid x^k + \mu_k \varsigma^k \geq (1 - \tau_{k-1})\left(x^{k-1} + \mu_{k-1}\varsigma^{k-1}\right)\right\}\right\} \tag{4.44}$$

or by applying a projection of the primal optimization variables afterwards, i.e., first choosing

$$\mu_k = \kappa_\mu \mu_{k-1} \tag{4.45}$$

and then adapting x^k by

$$x^k \leftarrow \max\left\{x^k, (1 - \tau_{k-1})\left(x^{k-1} + \mu_{k-1}\varsigma^{k-1}\right) - \mu_k \varsigma^k\right\}. \tag{4.46}$$

Although the latter strategy may lead to further function evaluations, for instance because the function values $f\left(x^{k-1} + \alpha_{k-1}\Delta x^{k-1}\right)$ and $g\left(x^{k-1} + \alpha_{k-1}\Delta x^{k-1}\right)$ that have been evaluated in the former line search may not equal $f(x^k)$ and $g(x^k)$, it is used here because of the possibility to yield a smaller barrier parameter.

The tolerances in (4.39) and (4.41) are chosen to be

$$\varepsilon_{\mu,k} := \pi_{k-1}\mu_{k-1}\left(\upsilon_4 \left\|\Phi_{\text{bar}}(w^i;\rho^{i-1})\right\|_\infty + \upsilon_5\varepsilon_{\nu,k} + \xi_{\mu,k}\right) \tag{4.47a}$$
$$\varepsilon_{\nu,k} := \upsilon_6 \left\|\Phi(w^i;\rho^{i-1})\right\|_\infty + \xi_{\nu,k} \tag{4.47b}$$

in analogy to (4.38), with $\upsilon_4 \in (0,1)$, $\upsilon_5 \in (0,1)$, $\upsilon_6 \in (0,1)$, non-negative sequences $\{\xi_{\mu,k}\}_{\mathscr{K}_\nu}$ and $\{\xi_{\nu,k}\}_{\mathscr{K}_\nu}$ converging to zero and $i = \arg\max \mathscr{K}_\nu$. As before, the index sets $\mathscr{K}_\mu$ and $\mathscr{K}_\nu$ are sets of iterations in which either the barrier parameter μ_k or the Lagrangian multiplier ν^k are updated, respectively. For infinitely many updates of either of them the tolerance $\varepsilon_{\mu,k}$ converges to zero.

Lemma 4.11. *Let $\upsilon_4 \in (0,1)$, $\upsilon_5 > 0$, $\upsilon_6 \in (0,1)$, two non-negative sequences $\{\xi_{\mu,k}\}_{\mathscr{K}_\nu}$ and $\{\xi_{\nu,k}\}_{\mathscr{K}_\nu}$ both converging to zero. If $\varepsilon_{\mu,k}$ and $\varepsilon_{\nu,k}$ are defined as in (4.47), then*

i. $\varepsilon_{\mu,k}$ converges to zero if the Lagrangian multipliers (λ^k, ν^k) or the barrier parameter μ_k are updated infinitely many times.

ii. $\varepsilon_{\nu,k}$ converges to zero if the Lagrangian multipliers (λ^k, ν^k) are updated infinitely many times.

Proof. The proof is analogous to that of Lemma 4.10. □

4.2.4 Magic Step

When developing practical solvers, it is often desirable to include heuristics to increase the efficiency of the practical method. Similarly, backup strategies can be used to increase robustness – for example, if the step calculation fails due to numerical difficulties. From a theoretical point of view these procedures can be seen as an oracle producing a new trial iterate $(x^{k+1}, y^{k+1}, z^{k+1})$. Here, this alternative to the step computation of Section 4.2.1 and the line search of Section 4.2.2 is called *magic step*. To establish global convergence, either a non-monotone filter or PLPF $\mathscr{F}_{\text{mag},k}(l_\text{f})$ with acceptable region $\mathscr{D}(\mathscr{F}_{\text{mag},k}(l_\text{f}))$ and non-monotonicity level $l_\text{f} \in \mathbb{N}_0$ is utilized. The envelope is set to

$$\delta_k = \left\|\nabla f(x^k) + \frac{1}{\pi_{k-1}}\nabla g(x^k)y^k - \frac{1}{\pi_{k-1}}z^k\right\|_2^2 + \left\|g(x^k)\right\|_2^2 + \left\|\frac{1}{\pi_{k-1}}X_k z^k\right\|_2^2. \tag{4.48}$$

Similar to the line search, the acceptability checks are

$$x^{k+1} + \mu_k\varsigma^k \geq (1-\varepsilon_{\text{frac},k})(x^k + \mu_k\varsigma^k), \tag{4.49a}$$
$$z^{k+1} \geq (1-\varepsilon_{\text{frac},k})z^k, \tag{4.49b}$$

which resembles the fraction-to-the-boundary rules (4.15) and (4.28), and

$$\left(f(x^{k+1}) + \gamma_\text{f}\delta_k, \theta(x^{k+1}) + \gamma_\text{f}\delta_k\right) \in \mathscr{D}\left(\mathscr{F}_{\text{mag},k}(l_\text{f}) \cup \{(f(x^k), \theta(x^k))\}\right). \tag{4.50}$$

The magic step can be executed in every iteration in which no parameter update of Section 4.2.3 occurs. In case the magic step is accepted, i.e., (4.49) and (4.50) hold, the filter or PLPF is augmented by

$$\mathscr{F}_{\mathrm{mag},k}(l_{\mathrm{f}}) = \mathscr{F}_{\mathrm{mag},k}(l_{\mathrm{f}}) \cup \left\{\left(f\left(x^k\right), \theta\left(x^k\right)\right)\right\}. \tag{4.51}$$

Otherwise, the standard procedure has to be used to provide a new iterate $\left(x^{k+1}, y^{k+1}, z^{k+1}\right)$.

The magic step offers a lot of flexibility. It would even be possible to run different optimization algorithms in parallel to produce a (somehow) good new iterate $\left(x^{k+1}, \lambda^{k+1}, \nu^{k+1}\right)$, which of course has to be acceptable to the filter or PLPF. In this thesis, it is vital for the adaptive parameter update scheme, which will be developed in Section 4.6.

4.2.5 The Overall Algorithm

This section formally states the exact penalty-interior-point algorithm. Although it can be split into three different algorithms of different hierarchy – an inner algorithm that creates iterates $\left(x^k, y^k, z^k\right)$ based on the step calculation (Section 4.2.1) together with the line search (Section 4.2.2) and two outer ones responsible for updating the penalty and barrier parameters (Section 4.2.3) and therefore the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ – the method is presented as a whole in Algorithm L.

After an initialization in Step L-1 Algorithm L enters the main optimization loop, which starts with optimality and infeasibility checks in Step L-2 and Step L-3, respectively. If one of these checks is fulfilled, the algorithm stops either with a first-order optimal solution or a certificate of infeasibility x^* of (NLP+). Otherwise, the optimization process continues with an optimality check of the barrier subproblem (NLPbar) in Step L-4, which involves updates of the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$, the primal shift ς^k (Step L-4.1) and the barrier parameter μ_k (Step L-4.2). Analogously, the penalty subproblem (NLPpen) is checked in Step L-5, which may induce Lagrangian multiplier λ^k updates (Step L-5.1) or penalty parameter updates for either π_k or τ_k (Step L-5.2).

Subsequently, a new iterate is computed either by an oracle within the magic step (Step L-6), if none of the parameter updates described above occurred, or by the standard procedure. The latter starts with a possible regularization in Step L-7 followed by the solution of the linear equation system of Newton's method in Step L-8 to generate the step direction. First, it is checked if a full step would yield an iterate for which a parameter update occurs. In that case, the full step is accepted unless the fraction-to-the-boundary rule is violated. Otherwise, the line search in Step L-10 with the possible help of a second-order-correction in Step L-11 checks acceptability of the trial step using a filter or PLPF (Step L-10.3 or Step L-11.3) and merit function (Step L-10.2 or Step L-11.2), which may require the reduction of the step length performed in Step L-10.4. After the primal update has been sorted in Step L-10.5 or L-11.4, respectively, the dual updates are executed in Step L-12. A dual projection is applied in Step L-13 for both possible generations of iterates – the magic step or the standard procedure. The iteration is finalized in Step L-14.

Algorithm L Primal-Dual Augmented Lagrangian Penalty-Interior-Point Algorithm

L-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point (x^0, y^0, z^0) and parameter (λ^0, ν^0) such that $x^0 \geq 0$, $z^0 > 0$ and $\nu^0 > 0$. Choose parameters $\mu_0 > 0$, $\tau_0 > 0$, $\sigma \in (0, 0.5)$, $\beta \in (0,1)$, $\{\varepsilon_{\text{frac},k}\}_k \subseteq (0,1)$, $l_\text{f} \in \mathbb{N}_0$, $l_\text{m} \in \mathbb{N}_0$, $l_\varepsilon \in \mathbb{N}_0$, $\gamma_\text{f} \in (0,1)$, $\kappa_z > 0$, $\kappa_\lambda > 0$, $\kappa_\pi \in (0,1)$, $\kappa_\tau > 1$, $\kappa_\mu \in (0,1)$, $\kappa_\varsigma \in (0,1]$, $\tau_{\max} > 0$, $\tau_\text{f} \in \mathbb{R}_+$, $\tau_\text{c} \in \mathbb{R}_+$, $\upsilon_1 \in (0,1)$, $\upsilon_2 \in (0,1)$, $\upsilon_3 \in (0,1)$, $\upsilon_4 \in (0,1)$, $\upsilon_5 \in (0,1)$, $\upsilon_6 \in (0,1)$, $\{\xi_{\lambda,k}\}_k \subseteq \mathbb{R}_{0+}$, $\{\xi_{\rho,k}\}_k \subseteq \mathbb{R}_{0+}$, $\{\xi_{\nu,k}\}_k \subseteq \mathbb{R}_{0+}$, $\{\xi_{\mu,k}\}_k \subseteq \mathbb{R}_{0+}$, $\varepsilon > 0$ and $\varepsilon_\text{tol} > 0$. Set $\pi_0 \leftarrow 1$ and initialize $\mathscr{F}_{\text{mag},k}(l_\text{f}) \leftarrow \emptyset$ and $\mathscr{F}_0(l_\text{f}) \leftarrow \emptyset$.

L-2: *(Optimality check)* If the KKT conditions (4.3) are satisfied, then STOP; x^k is a first-order optimal point of (NLP+).

L-3: *(Infeasibility check)* If the KKT conditions (4.7) are satisfied for $\pi = 0$, then STOP; x^k is a first-order optimal point of (FeasNLP+) that is infeasible for (NLP+) and (NLPbar).

L-4: *(Barrier subproblem check)*

- L-4.1: *(Multiplier update)* If (4.39) and (4.41) are satisfied, update $(\lambda^k, \nu^k) \leftarrow (y^k/\pi_{(k-1)_+}, z^k/\pi_{(k-1)_+})$, update ς^k by (4.42) and go to Step L-7. Otherwise, set $\nu^k \leftarrow \nu^{(k-1)_+}$ and $\varsigma^k \leftarrow \varsigma^{(k-1)_+}$.
- L-4.2: *(Barrier update)* If (4.39) and (4.41) are violated, update μ_k by (4.45), apply primal projection (4.46) and go to Step L-7. Otherwise, set $\mu_k \leftarrow \mu_{(k-1)_+}$.

L-5: *(Penalty subproblem check)*

- L-5.1: *(Equality multiplier update)* If (4.31), (4.34a), (4.34b) and (4.35) are satisfied, update $\lambda^k \leftarrow y^k/\pi_{(k-1)_+}$ and go to Step L-7. Otherwise, set $\lambda^k \leftarrow \lambda^{(k-1)_+}$.
- L-5.2: *(Penalty update)* If (4.31) is satisfied and (4.34a) is violated, update penalty parameter π_k (if $\tau_{(k-1)_+} \geq \tau_{\max}$) or τ_k (if $\tau_{(k-1)_+} < \tau_{\max}$) by (4.37) and go to Step L-7. Otherwise, set $\pi_k \leftarrow \pi_{(k-1)_+}$ and $\tau_k \leftarrow \tau_{(k-1)_+}$.

L-6: *(Magic step)* If the magic step is enabled, ask some oracle to provide $(x^{k+1}, y^{k+1}, z^{k+1})$ that satisfies the fraction-to-the-boundary rule (4.49). If (4.50) is satisfied, accept the trial iterate, augment the filter or PLPF $\mathscr{F}_{\text{mag},k}(l_\text{f})$ by (4.51) and go to Step L-13. Otherwise, set $\mathscr{F}_{\text{mag},k}(l_\text{f}) \leftarrow \mathscr{F}_{\text{mag},k}(l_\text{f})$.

L-7: *(Hessian regularization)* Modify the Hessian Q_k by adding $\delta_\text{p} E$ with iteratively increased $\delta_\text{p} > 0$ to it until $\widetilde{M}_k$ has the correct inertia $\text{In}(\widetilde{M}_k) = (n_x, n_g, 0)$. If necessary, add a further εE such that (4.13) holds. If the Hessian regularization fails, then STOP; x^k is feasible and the MFCQ fails to hold.

L-8: *(Step calculation)* Solve the linear equation system (4.12) to get $(\Delta x^k, \Delta y^k, \Delta z^k)$.

L-9: *(Check full step)* If the multipliers have been updated, i.e., $\lambda^k = y^k/\pi_k$ or $\nu^k = z^k/\pi_k$, $\alpha_{\max,k} = \alpha_k^z = 1$ in (4.15) and (4.28) and (4.31) is satisfied for $(x^k, y^k, z^k) + (\Delta x^k, \Delta y^k, \Delta z^k)$, go to Step L-13.

L-10: *(Line search)*

- L-10.1: *(Initialization)* Determine $\alpha_{\max,k} \in (0,1]$ using the fraction-to-the-boundary rule (4.15). Set $\alpha_k \leftarrow \alpha_{\max,k}$.
- L-10.2: *(Merit function check)* If the Armijo condition (4.17) is satisfied, set $\mathscr{F}_{k+1}(l_\text{f}) \leftarrow \mathscr{F}_k(l_\text{f})$ and go to Step L-10.5.
- L-10.3: *(Filter / PLPF check)* If (4.20) is satisfied, augment the filter or PLPF $\mathscr{F}_{k+1}(l_\text{f})$ by (4.19) and go to Step L-10.5. Otherwise, if $\alpha_k = \alpha_{\max,k}$ and $\lambda^k = y^k/\pi_k$, go to Step L-11.
- L-10.4: *(Backtracking)* Set $\alpha_k \leftarrow \beta \alpha_k$ and go to Step L-10.2.
- L-10.5: *(Primal update)* Set $x^{k+1} \leftarrow x^k + \alpha_k \Delta x^k$. Go to Step L-12.

L-11: *(Second-Order-Corrections)*

- L-11.1: *(Initialization)* Solve the linear equation system (4.21) to get $(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k)$. Determine $\widehat{\alpha}_k \in (0,1]$ using the fraction-to-the-boundary rule (4.23).
- L-11.2: *(Merit function check)* If the Armijo condition (4.24) is satisfied, set $\mathscr{F}_{k+1}(l_\text{f}) \leftarrow \mathscr{F}_k(l_\text{f})$ and go to Step L-11.4.
- L-11.3: *(Filter / PLPF check)* If (4.25) is satisfied, augment the filter or PLPF $\mathscr{F}_{k+1}(l_\text{f})$ by (4.19). Otherwise, go to Step L-10.4.
- L-11.4: *(Primal update)* Set $x^{k+1} \leftarrow x^k + \widehat{\alpha}_k \widehat{\Delta x}^k$.

L-12: *(Dual update)* Use fraction-to-the-boundary rule (4.28) to get step size $\alpha_k^z \in (0,1]$. Set $y^{k+1} \leftarrow y^k + \alpha_k \Delta y^k$ and $z^{k+1} \leftarrow z^k + \min\{\alpha_k, \alpha_k^z\} \Delta z^k$.

L-13: *(Dual projection)* Apply dual projection (4.30).

L-14: *(k increment)* Set $k \leftarrow k+1$ and go to Step L-2.

The most expensive part of Algorithm L is the factorization of the linear equation system (4.12) in Step L-7 or Step L-8 or the function evaluations including derivatives depending on the specific application. Once calculated, the factorization can be reused for the second-order-correction in Step L-11.

Although it is not considered in Algorithm L, a flushing of the filter, PLPF or the historic information in the non-monotone Armijo condition is possible whenever the update conditions of the Lagrangian multipliers (λ^k, ν^k) in Step L-4.1 or the barrier parameter μ_k in Step L-4.2 are satisfied. However, this does not hold for the filter or PLPF of the magic step (cf., Step L-6).

The stopping criteria in Step L-7 follows from (4.13) and will be investigated in the proof of Lemma 4.24 of the global convergence analysis in Section 4.3. In particular, it will be shown that due to condition (4.13) and Lemma 3.15 the regularization can only fail, i.e., the system matrix $\widetilde{M}_k$ is singular, if the current iterate is feasible and $\nabla g(x^k)$ consists of linear dependent columns (see also Armand and Benoist [1, Theorem 1]). This however implies that x^k is a Fritz-John point of problem (ShiftNLP+). If the regularization succeeds, a step $(\Delta x^k, \Delta y^k, \Delta z^k)$ will be available from Step L-8 and therefore also a new iterate $(x^{k+1}, y^{k+1}, z^{k+1})$.

An initial guess (x^0, y^0, z^0) with $x^0 \geq 0$ and $z^0 > 0$ can be found trivially. Unlike interior-point methods with the classic log-barrier function, it is also possible to begin with the optimal solution x^*.

4.3 Global Convergence Analysis

The global convergence analysis roughly follows the proof framework of Chen and Goldfarb [40, 43], but is extended or modified at many places mainly because of the use of an augmented Lagrangian penalty and a modified barrier function as well as a non-monotone line search. Throughout this section it is assumed that Algorithm L does not terminate at Step L-2 or Step L-3 and thus produces an infinite sequence of iterates (x^k, y^k, z^k). Moreover, the following standard assumptions are made.

Assumptions 4.12 (Global Convergence).

i. *The functions $f(x)$ and $g(x)$ are real valued and twice continuously differentiable.*

ii. *The primal iterates $\{x^k\}_k$ are bounded.*

iii. *The modified Hessians $\{Q_k\}_k$ are bounded.*

Since the penalty parameter τ_k is updated just finitely many times by definition, it is further assumed without loss of generality that $\tau_k = \bar{\tau} > 0$ for all iterations. During the analysis, index sets $\mathcal{K} \subseteq \mathbb{N}_0$ are extensively used. To avoid confusion that $k-1$, $k \in \mathcal{K}$ does not refer to the previous element in an ordered set $\mathcal{K}$, the notation $\lambda^{k_-}, \nu^{k_-}, \mu_{k_-}$ and π_{k_-} – and analogously for other variables – is utilized for variables whose update was delayed in iteration k.

The following statement regarding the MFCQ is useful because it is based on dual information instead of some arbitrary step $d \in \mathbb{R}^{n_x}$ (cf., Definition 2.11).

Lemma 4.13. *For a point $x \in \mathbb{R}^{n_x}$ the MFCQ is not satisfied, if there exists $y \in \mathbb{R}^{n_g}$ and $z \in \mathbb{R}^{n_x}$ with $z \geq 0$ and $(y,z) \neq 0$ such that $\nabla g(x)y - z = 0$.*

Proof. The proof follows directly from Farkas Lemma (cf., Lemma A.9). □

A final preliminary result shows that limit points of the primal and dual iterates are bounded below.

Lemma 4.14. *Let $\bar{x}$, $\bar{z}$ be any limit point of $\{x^k\}_k$ or $\{z^k\}_k$, respectively. Furthermore, assume that $\{\varsigma^k\}_k$ converges to $\bar{\varsigma}$. Then, there exists $\bar{\mu} \geq 0$ such that $\bar{x} \geq -\bar{\mu}\bar{\varsigma}$ and $\bar{z} \geq 0$.*

Proof. Since every updated iterate x^{k+1} satisfies the fraction-to-the-boundary rules (4.15), (4.23), (4.43) and (4.49), respectively, it follows

$$x^{k+1} + \mu_k \varsigma^k \geq \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \mu_k \varsigma^k\right) \geq \left(1 - \varepsilon_{\text{frac},k}\right)^{k+1}\left(x^0 + \mu_0 \varsigma^0\right).$$

Letting k tend to infinity yields $\bar{x} + \bar{\mu}\bar{\varsigma} \geq 0$ because $\varepsilon_{\text{frac},k} \in (0,1)$ and hence $\bar{x} \geq -\bar{\mu}\bar{\varsigma}$. The proof is analogue for z^k and $\bar{z}$. □

4.3.1 Global Convergence for Infinitely Many Magic Steps

The first two results show that the filter or PLPF envelope converges to zero, if the filter or PLPF is augmented infinitely many times.

Lemma 4.15. *Let the non-monotone filter $\mathscr{F}_{mag,k}(l_f)$ be augmented infinitely many times, $\{f(x^k)\}_k$ be bounded below and $\{\theta(x^k)\}_k$ be bounded above. Then, the sequence $\{\delta_k\}_k$ converges to zero.*

Proof. The statement is proven by contradiction and follows ideas of Nocedal et al. [153, Theorem 5.1] and Shen et al. [177, Lemma 3.1]. Assume there exists an infinite index set $\mathscr{K}$ such that the filter $\mathscr{F}_{\text{mag},k}(l_\text{f})$ is augmented (by assumption of this lemma) and $\delta_k \geq \varepsilon > 0$ for $k \in \mathscr{K}$.

Now assume that $\{f(x^k)\}_{\mathscr{K}}$ is not bounded above. Then, there exists an index set $\mathscr{K}' \subseteq \mathscr{K}$ such that the sequence $\{f(x^k)\}_{\mathscr{K}'}$ is monotonically increasing. Hence, since $\gamma_\text{f} > 0$ it follows

$$f\left(x^{k+1}\right) + \gamma_\text{f}\delta_k > f\left(x^k\right),$$

for all $k \in \mathscr{K}'$, i.e., all the iterates x^{k+1}, $k \in \mathscr{K}'$ are dominated with respect to the objective function. This is why, there must be one of the $x^k, x^{k-1}, \ldots, x^{k-l_\text{f}}$ iterates such that x^{k+1} is not dominated with respect to the constraint violation. Otherwise, x^{k+1} would not be acceptable to the non-monotone filter and therefore $k \notin \mathscr{K}$. Thus,

$$\theta\left(x^{k+1}\right) + \gamma_\text{f}\delta_k \leq \max_{i=0,\ldots,l_\text{f}} \theta\left(x^{(k-i)_+}\right).$$

By the same argument, this can be extended to the next l_f iterations, leading to

$$\max_{i=1,\ldots,l_\text{f}+1} \theta\left(x^{k+i}\right) \leq \max_{i=0,\ldots,l_\text{f}} \theta\left(x^{(k-i)_+}\right) - \gamma_\text{f}\delta_k \leq \max_{i=0,\ldots,l_\text{f}} \theta\left(x^{(k-i)_+}\right) - \varepsilon$$

and thus

$$\max_{i=1,\ldots,l_f+1} \theta\left(x^{k+i}\right) \to -\infty$$

for $k \in \mathcal{K}' \to \infty$, which is a contradiction to $\theta(x) = \|g(x)\|_2 \geq 0$. This implies that $\left\{f\left(x^k\right)\right\}_{\mathcal{K}}$ is bounded above.

Because the sequence $\left\{\left(\left\|f\left(x^k\right)\right\|, \left\|\theta\left(x^k\right)\right\|\right)\right\}_{\mathcal{K}}$ is bounded, it exists an index set $\mathcal{K}'' \subseteq \mathcal{K}$ and $\left(\bar{f}, \bar{\theta}\right)$ such that $\left(\left\|f\left(x^k\right)\right\|, \left\|\theta\left(x^k\right)\right\|\right) \to \left(\bar{f}, \bar{\theta}\right)$ for $k \in \mathcal{K}'' \to \infty$. For $k \in \mathcal{K}''$ large enough, it follows that

$$\left(\left\|f\left(x^{k+1-j}\right)\right\|, \left\|\theta\left(x^{k+1-j}\right)\right\|\right) \in \mathcal{B}_{\frac{1}{2}\varepsilon}\left(\bar{f}, \bar{\theta}\right)$$

for $j = 0, \ldots, l_f + 1$. Therefore,

$$\begin{aligned}
\left|\theta\left(x^{k+1}\right) - \theta\left(x^{k+1-j}\right)\right| &\leq \left|\theta\left(x^{k+1}\right) - \bar{\theta}\right| + \left|\theta\left(x^{k+1-j}\right) - \bar{\theta}\right| < \varepsilon, \\
\left|f\left(x^{k+1}\right) - f\left(x^{k+1-j}\right)\right| &\leq \left|f\left(x^{k+1}\right) - \bar{f}\right| + \left|f\left(x^{k+1-j}\right) - \bar{f}\right| < \varepsilon
\end{aligned}$$

and thus,

$$\begin{aligned}
\theta\left(x^{k+1}\right) &> \theta\left(x^{k+1-j}\right) - \varepsilon \geq \theta\left(x^{k+1-j}\right) - \delta_k, \\
f\left(x^{k+1}\right) &> f\left(x^{k+1-j}\right) - \varepsilon \geq f\left(x^{k+1-j}\right) - \delta_k
\end{aligned}$$

for $j = 1, \ldots, l_f + 1$ and $k \in \mathcal{K}''$, i.e., x^{k+1} is dominated by the $l_f + 1$ points $x^{k+1-j}, j = 1, \ldots, l_f + 1$. However, since $k \in \mathcal{K}$, the point x^{k+1} is acceptable to the non-monotone filter – a contradiction. All together, this implies that $\{\delta_k\}_k$ converges to zero. □

Lemma 4.16. *Let the non-monotone PLPF $\mathcal{F}_{mag,k}\left(l_f\right)$ be augmented infinitely many times, $\left\{f\left(x^k\right)\right\}_k$ be bounded below and $\left\{\theta\left(x^k\right)\right\}_k$ be bounded above. Then, the sequence $\{\delta_k\}_k$ converges to zero.*

Proof. Let $\mathcal{D}\left(\mathcal{F}_{\mathrm{mag},k}(l_f)\right)$ be the acceptable region of the PLPF and $\mathcal{D}'\left(\mathcal{F}_{\mathrm{mag},k}(l_f)\right)$ the one of a filter with the same points $\mathcal{F}_{\mathrm{mag},k}(l_f)$ and envelope δ_k for every k. In the following it will be shown, that if an iterate is acceptable to the PLPF, it will also be for the filter defined above and thus, that filter would be augmented with every PLPF augmentation.

Let $\mathcal{K}$ be an infinite index set of iterations in which the PLPF is augmented. Assume that there exists an index $k \in \mathcal{K}$ such that the non-monotone filter defined above would not be augmented. Then, there exists an index set $\mathcal{K}' \subseteq \{0, \ldots, k\}$ with $\left|\mathcal{K}'\right| = l_f + 1$ such that

$$\begin{aligned}
&f\left(x^{k+1}\right) + \gamma_f \delta_k > f\left(x^j\right), \text{ and} \\
&\theta\left(x^{k+1}\right) + \gamma_f \delta_k > \theta\left(x^j\right)
\end{aligned}$$

for all $j \in \mathcal{K}'$. This implies

$$f\left(x^{k+1}\right) + \tau\theta\left(x^{k+1}\right) + (1+\tau)\gamma_f\delta_k > f\left(x^j\right) + \tau\theta\left(x^j\right)$$

for all $j \in \mathcal{K}'$ and $\tau \geq 0$. Thus, $\left(f\left(x^{k+1}\right), \theta\left(x^{k+1}\right)\right)$ is dominated by $l_\mathrm{f}+1$ elements with respect to the PLPF condition, i.e.,

$$\left(f\left(x^{k+1}\right), \theta\left(x^{k+1}\right)\right) \notin \mathscr{D}\left(\mathscr{F}_{\mathrm{mag},k}(l_\mathrm{f})\right).$$

This is a contradiction to the assumption that the PLPF criterion was accepted for all $k \in \mathcal{K}$. It follows, that the filter is augmented infinitely many times. Then, Lemma 4.15 implies that $\{\delta_k\}_k$ converges to zero. □

Since the filter or PLPF envelope δ_k for the magic step is chosen to measure the KKT error of the problem (NLP+), the convergence of δ_k to zero directly leads to the following outcome for infinitely many performed magic steps.

Lemma 4.17. *Suppose the Assumptions 4.12 hold. If the magic step is executed infinitely many times in Step L-6, there exists an index set $\mathcal{K}$ such that the sequence $\left\{\left(x^k, y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}}$ converges to a first-order optimal solution of* (NLP+).

Proof. Let $\mathcal{K}$ be an index set of iterations in which the magic step is accepted, i.e., the filter or PLPF $\mathscr{F}_{\mathrm{mag},k}(l_\mathrm{f})$ is augmented, and for which $x^k \to x^*$ holds. The latter exists due to Assumption 4.12 (ii). Together with Assumption 4.12 (i) it follows that $\left\{\left|f\left(x^k\right)\right|\right\}_{\mathcal{K}}$ and $\left\{\left\|\theta\left(x^k\right)\right\|\right\}_{\mathcal{K}}$ are bounded. Then, by Lemma 4.15 or Lemma 4.16 it follows that

$$\delta_k = \left\|\nabla f\left(x^k\right) + \frac{1}{\pi_{k\_}}\nabla g\left(x^k\right)y^k - \frac{1}{\pi_{k\_}}z^k\right\|_2^2 + \left\|g\left(x^k\right)\right\|_2^2 + \left\|\frac{1}{\pi_{k\_}}X_k z^k\right\|_2^2 \to 0$$

for $k \in \mathcal{K} \to \infty$. □

4.3.2 Global Convergence for Infinitely Many Barrier Parameter Updates

In the following it is assumed that there are only finitely many magic steps, but infinitely many updates of the barrier parameter μ_k or Lagrangian multiplier ν^k.

Lemma 4.18. *Suppose the Assumptions 4.12 hold. If the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ are updated infinitely many times in Step L-4.1, there exists an index set $\mathcal{K}$ such that either of the following holds:*

i. The sequence $\left\{\left(\lambda^k, \nu^k\right)\right\}_{\mathcal{K}}$ is bounded and $\left\{\left(x^k, \lambda^k, \nu^k\right)\right\}_{\mathcal{K}}$ converges to a first-order optimal solution of (NLP+).

ii. The sequence $\left\{\left(\lambda^k, \nu^k\right)\right\}_{\mathcal{K}}$ is unbounded and $\left\{x^k\right\}_{\mathcal{K}}$ converges to a Fritz-John point of (NLP+) *that fails to satisfy the MFCQ.*

Proof. Let $\mathcal{K} \subseteq \mathcal{K}_\nu$ be an infinite index set of iterations at which (4.39) and (4.41) hold, the necessary condition for an update of the Lagrangian multiplier $\left(\lambda^k, \nu^k\right)$ in Step L-4.1, i.e., $\left(\lambda^k, \nu^k\right) = \left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)$ for $k \in \mathcal{K}$. Then, Lemma 4.11 implies $\varepsilon_{\mu,k} \to 0$ and $\varepsilon_{\nu,k} \to 0$ for $k \in \mathcal{K} \to \infty$. By Assumption 4.12 (ii) there exists an index set $\mathcal{K}' \subseteq \mathcal{K}$ and a point $x^* \in \mathbb{R}^{n_x}$ such that $x^k \to x^*$ for $k \in \mathcal{K}' \to \infty$. In the following, the two cases of $\left\{\left(\lambda^k, \nu^k\right)\right\}_{\mathcal{K}'}$ being bounded and unbounded will be considered:

i. Case $\{(\lambda^k, \nu^k)\}_{\mathcal{K}'}$ is bounded. Then, there exists an index set $\mathcal{K}'' \subseteq \mathcal{K}'$ and point (λ^*, ν^*) such that

$$\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right) = \left(\lambda^k, \nu^k\right) \to (\lambda^*, \nu^*)$$

for $k \in \mathcal{K}'' \to \infty$. Letting $k \in \mathcal{K}'' \to \infty$ in (4.41) then yields

$$\nabla f(x^*) + \nabla g(x^*)\lambda^* - \nu^* = 0, \quad g(x^*) = 0 \quad \text{and} \quad X_* \nu^* = 0.$$

By Lemma 4.14, it follows $x^* \geq -\bar{\mu}\bar{\varsigma}$ and $\nu^* \geq 0$. Since $\bar{\varsigma} = (\nu^*)^{\kappa_\varsigma}$ (cf., Step L-4.1 and (4.42)) it holds that $x_i^* \geq 0$ if $\nu_i^* = 0$ for $i = 1, \dots, n_x$. Otherwise, if $\nu_i^* > 0$ it follows from the complementarity condition above that $x_i^* = 0$. Hence, x^* is feasible for (NLP+). All together, (x^*, λ^*, ν^*) is a first-order optimal point of (NLP+).

ii. Case $\{(\lambda^k, \nu^k)\}_{\mathcal{K}'}$ is unbounded. Then it exists an index set $\mathcal{K}'' \subseteq \mathcal{K}'$ and point (λ^*, ν^*) such that for $k \in \mathcal{K}''$ the inequality $\left\|\left(\lambda^k, \nu^k\right)\right\| > 0$ holds,

$$\frac{\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)}{\left\|\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\|} = \frac{\left(\lambda^k, \nu^k\right)}{\|(\lambda^k, \nu^k)\|} \to (\lambda^*, \nu^*)$$

and $\|(\lambda^*, \nu^*)\| = 1$. Dividing (4.41) by $\left\|\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\|$ and letting $k \in \mathcal{K}$ tend to infinity yields

$$\nabla g(x^*)\lambda^* - \nu^* = 0, \quad g(x^*) = 0, \quad \text{and} \quad X_* \nu^* = 0.$$

By the same argument as in the other case it follows $x^* \geq 0$ and $\nu^* \geq 0$. Together with $\|(\lambda^*, \nu^*)\| = 1$ this implies that (x^*, λ^*, ν^*) is a Fritz-John point of problem (NLP+) that fails to satisfy the MFCQ (cf., Lemma 4.13).

□

Lemma 4.19. *Suppose the Assumptions 4.12 hold. Let the Lagrangian multiplier ν^k be updated finitely many times. If the barrier parameter μ_k is updated infinitely many times at Step L-4.2, there exists an index set $\mathcal{K}$ such that the sequence $\{(y^k/\pi_{k\_}, z^k/\pi_{k\_})\}_{\mathcal{K}}$ is unbounded and $\{x^k\}_{\mathcal{K}}$ converges to a Fritz-John point of* (NLP+) *that fails to satisfy the MFCQ.*

Proof. Let $\mathcal{K} \subseteq \mathcal{K}_\mu$ be an infinite index set of iterations for which (4.39) is satisfied, but assume that (4.41) is violated, the condition for an update of the barrier parameter μ_k in Step L-4.2. Hence, $\mu_k \to 0$ for $k \in \mathbb{N}_0 \to \infty$. By Assumption 4.12 (ii), there exists an index set $\mathcal{K}' \subseteq \mathcal{K}$ and points $x^* \in \mathbb{R}^{n_x}$ and $\bar{\nu} \in \mathbb{R}^{n_x}$ such that $x^k \to x^*$ and $\nu^k \to \bar{\nu}$ (by the assumption of ν^k being updated just finitely many times) and hence by (4.42) $\varsigma^k \to \bar{\varsigma}$ for $k \in \mathcal{K}' \to \infty$. Lemma 4.11 implies $\varepsilon_{\mu,k} \to 0$ for $k \in \mathcal{K}' \to \infty$. In the following the two cases of $\{(y^k/\pi_{k\_}, z^k/\pi_{k\_})\}_{\mathcal{K}'}$ being bounded or unbounded will be considered.

i. Case $\{(y^k/\pi_{k\_}, z^k/\pi_{k\_})\}_{\mathcal{K}'}$ is bounded. Then, it exists an index set $\mathcal{K}'' \subseteq \mathcal{K}'$ and point

(y^*, z^*) such that $\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right) \to (y^*, z^*)$. With (4.39) it follows

$$\begin{aligned}
0 &\leq \left\| \frac{1}{\pi_{k_}} X_k z^k \right\| \\
&\leq \left\| \frac{1}{\pi_{k_}} X_k z^k - \mu_{k_} \Sigma_{k_} \left(\nu^{k_} - \frac{z^k}{\pi_{k_}} \right) \right\| + \left\| \mu_{k_} \Sigma_{k_} \left(\nu^{k_} - \frac{z^k}{\pi_{k_}} \right) \right\| \\
&\overset{(4.39)}{\leq} \varepsilon_{\mu,k} + \mu_{k_} \left\| \Sigma_{k_} \left(\nu^{k_} - \frac{z^k}{\pi_{k_}} \right) \right\| \to 0
\end{aligned}$$

and thus $\left\| \frac{1}{\pi_{k\_}} X_k z^k \right\| \to 0$ for $k \in \mathscr{K}'' \to \infty$. But then, for every $\varepsilon_{\nu,k} > 0$ there exists an index $k' \in \mathscr{K}''$ such that (4.41) is satisfied. Since (4.39) is satisfied for $k \in \mathscr{K}$ by assumption, ν^k and not μ_k would be updated at iteration k in Step L-4.1. This is a contradiction, because this implies an update of ν^k infinitely many times. Hence, this case cannot occur.

ii. Case $\left\{\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\}_{\mathscr{K}'}$ is unbounded. Then, there exists an index set $\mathscr{K}'' \subseteq \mathscr{K}'$ and point (y^*, z^*) such that

$$\frac{\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)}{\left\|\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\|} \to (y^*, z^*),$$

because for $k \in \mathscr{K}''$ large enough $\left\|\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\| > 0$. Dividing (4.39a) and (4.39c) by $\left\|\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\|$ and letting $k \in \mathscr{K}'' \to \infty$ in (4.39) yields

$$g(x^*)y^* - z^* = 0, \quad g(x^*) = 0, \quad \text{and} \quad X_* z^* = 0$$

since μ_k and $\varepsilon_{\mu,k}$ tend to zero. It follows together with $x^* \geq -\bar{\mu}\bar{\varsigma} = 0$, $z^* \geq 0$ (cf., Lemma 4.14) and $\|(y^*, z^*)\| = 1$ that (x^*, y^*, z^*) is a Fritz-John point of (NLP+) that fails to satisfy the MFCQ (cf., Lemma 4.13).

□

4.3.3 Global Convergence for Infinitely Many Penalty Parameter Updates

In this subsection it is assumed that the barrier parameter μ_k and Lagrangian multipliers ν^k are updated finitely many times, i.e., for large iterations k their values $\mu_k = \bar{\mu} > 0$ and $\nu^k = \bar{\nu} > 0$ stay fixed. The following results investigate the influence of infinite many penalty parameter π_k or Lagrangian multiplier λ^k updates on the possible outcome of Algorithm L. Therefore, it is assumed, that no updates are performed in Step L-4 even if the conditions (4.39) and (4.41) are all satisfied.

Lemma 4.20. *Suppose the Assumptions 4.12 hold. Let the barrier parameter μ_k be bounded away from zero and the Lagrangian multiplier ν^k be updated finitely many times. If the Lagrangian multiplier λ^k is updated infinitely many times in Step L-5.1, there exists an index set $\mathscr{K}$ such that either of the following holds:*

i. *The sequence* $\left\{\left(\lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}}$ *is bounded and* $\left\{\left(x^k, \lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}}$ *converges to a first-order optimal point of* (NLPbar).

ii. *The sequence* $\left\{\left(\lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}}$ *is unbounded and* $\left\{x^k\right\}_{\mathcal{K}}$ *converges to a Fritz-John point of* (ShiftNLP+) *that fails to satisfy the MFCQ.*

Proof. Let $\mathcal{K} \subseteq \mathcal{K}_\lambda$ be an infinite index set such that (4.31), (4.34a) and (4.35) hold, which is the necessary condition for the update of $\lambda^k = y^k/\pi_{k\_}$ in Step L-5.1. By Assumption 4.12 (ii) and (4.42), there exists an index set $\mathcal{K}' \subseteq \mathcal{K}$ and point $x^* \in \mathbb{R}^{n_x}$ such that $x^k \to x^*$, $\mu_k \to \bar{\mu}$, $\nu^k \to \bar{\nu}$ and $\varsigma^k \to \bar{\varsigma}$ for $k \in \mathcal{K}' \to \infty$. With Lemma 4.10, it follows that $\varepsilon_{\rho,k} \to 0$ for $k \in \mathcal{K}' \to \infty$.

It will be shown that $\|g(x^*)\| = 0$ by contradiction. Assume, that $\|g(x^*)\| > 0$. Then, $\left\|g\left(x^k\right)\right\| > 0$ for $k \in \mathcal{K}'$ large enough and, thus, multiplying (4.31b) by $\frac{\bar{\tau}}{\|g(x^k)\|_2}$ for such k yields

$$0 \leq \left\|\bar{\tau}\frac{g\left(x^k\right)}{\|g(x^k)\|_2} - \left(y^k - \pi_{k\_}\lambda^{k\_}\right)\right\|_\infty \overset{(4.31b)}{\leq} \frac{\bar{\tau}}{\|g(x^k)\|_2}\varepsilon_{\rho,k} \leq c\varepsilon_{\rho,k}$$

with a constant $c = \bar{\tau}\inf_{k\in\mathcal{K}'}\left\{\left\|g\left(x^k\right)\right\|_2^{-1}\right\} \in \mathbb{R}_+$, because $\left\{\left\|g\left(x^k\right)\right\|\right\}_{\mathcal{K}'}$ is bounded due to Assumption 4.12 (i) and (ii). It follows $\left\|\bar{\tau}\frac{g\left(x^k\right)}{\|g(x^k)\|_2} - \left(y^k - \pi_{k\_}\lambda^{k\_}\right)\right\|_2 \to 0$ for $k \in \mathcal{K}' \to \infty$ since $\varepsilon_{\rho,k}$ converges to zero. This implies

$$\left\|y^k - \pi_{k\_}\lambda^{k\_}\right\|_2 \leq \left\|\bar{\tau}\frac{g\left(x^k\right)}{\|g(x^k)\|_2} - \left(y^k - \pi_{k\_}\lambda^{k\_}\right)\right\|_2 + \bar{\tau} \to \bar{\tau}.$$

For an index $k' \in \mathcal{K}'$ large enough, it then holds $\left\|y^k - \pi_{k\_}\lambda^{k\_}\right\|_2 \in (\kappa_\lambda\bar{\tau}, \bar{\tau}]$ for all $k \in \mathcal{K}'$, $k \geq k'$. Thus, (4.34a) is violated, a contradiction that λ^k is updated for $k \in \mathcal{K}'$. This implies $\|g(x^*)\| = 0$.[5]

In the following the two cases of $\left\{\left(\lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}'}$ being bounded or unbounded will be considered.

i. Case $\left\{\left(\lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathcal{K}'}$ is bounded. Then, it exists an index set $\mathcal{K}'' \subseteq \mathcal{K}'$ and point (λ^*, z^*) such that $\left(y^k/\pi_{k\_}, z^k/\pi_{k\_}\right) = \left(\lambda^k, z^k/\pi_{k\_}\right) \to (\lambda^*, z^*)$ for $k \in \mathcal{K}'' \to \infty$. Dividing (4.31a) and (4.31c) by $\pi_{k\_} > 0$ and letting $k \in \mathcal{K}''$ tend to infinity yields

$$\begin{aligned}\nabla f(x^*) + \nabla g(x^*)\lambda^* - z^* &= 0\\ X_* z^* &= \bar{\mu}\bar{\Sigma}(\bar{\nu} - z^*),\end{aligned}$$

due to the convergence of $\varepsilon_{\rho,k}$ to zero. Together with $g(x^*) = 0$ and Lemma 4.14 it follows that (x^*, λ^*, z^*) is a first-order optimal solution of (NLPbar).

[5] An alternative and direct proof can be formulated by using (4.35), however the given proof can be useful for future developments as it just assumes (4.31), (4.34a) and the convergence of $\varepsilon_{\rho,k}$ to zero.

ii. Case $\{(\lambda^k, z^k/\pi_{k\_})\}_{\mathcal{K}'}$ is unbounded. Then, there exists an infinite index set $\mathcal{K}'' \subseteq \mathcal{K}'$ and point (λ^*, z^*) such that $\|(\lambda^k, z^k/\pi_{k\_})\| > 0$ for $k \in \mathcal{K}''$ and

$$\frac{(y^k/\pi_{k\_}, z^k/\pi_{k\_})}{\|(y^k/\pi_{k\_}, z^k/\pi_{k\_})\|} = \frac{(\lambda^k, z^k/\pi_{k\_})}{\|(\lambda^k, z^k/\pi_{k\_})\|} \to (\lambda^*, z^*)$$

for $k \in \mathcal{K}'' \to \infty$ and $\|(\lambda^*, z^*)\| = 1$. Dividing (4.31a) and (4.31c) by $\pi_{k\_} > 0$ and $\|(\lambda^k, z^k/\pi_{k\_})\|$ and letting $k \in \mathcal{K}'' \to \infty$ then leads to

$$\begin{aligned} \nabla g(x^*)\lambda^* - z^* &= 0 \\ (X_* + \bar{\mu}\bar{\Sigma})z^* &= 0. \end{aligned}$$

Finally, $x^* \geq -\bar{\mu}\bar{\varsigma}$, $z^* \geq 0$ (cf., Lemma 4.14), $\|(\lambda^*, z^*)\| = 1$ and $g(x^*) = 0$ imply that (x^*, λ^*, z^*) is a Fritz-John point of (ShiftNLP+) that fails to satisfy the MFCQ (cf., Lemma 4.13).

□

Lemma 4.21. *Suppose the Assumptions 4.12 hold. Let the barrier parameter μ and Lagrangian multipliers (λ^k, ν^k) be updated finitely many times. If the penalty parameter π_k is decreased infinitely many times in Step L-5.2, there exists an index set $\mathcal{K}$ such that either of the following holds:*

i. The sequence $\{(x^k, z^k/\bar{\tau})\}_{\mathcal{K}}$ converges to a first-order optimal point $(x^*, z^*/\bar{\tau})$ of (FeasNLP+) *that is infeasible for* (NLP+) *and $\{y^k\}_{\mathcal{K}}$ converges to $\bar{\tau}g(x^*)/\|g(x^*)\|_2$.*

ii. The sequence $\{x^k\}_{\mathcal{K}}$ converges to a Fritz-John point of problem (ShiftNLP+) *that fails to satisfy the MFCQ.*

Proof. This proof is inspired by Chen and Goldfarb [43, Theorem 3.1] but extended to the augmented Lagrangian penalty and modified barrier function. Let $\mathcal{K} \subseteq \mathcal{K}_\pi$ be an infinite index set such that (4.31) holds and (4.34a) is violated, which is the necessary condition for the update of π_k in Step L-5.2. Because $\pi_k = \kappa_\pi \pi_{k\_}$ is executed infinitely many times and $\kappa_\pi \in (0,1)$, it follows $\pi_k \to 0$ for $k \in \mathbb{N}_0 \to \infty$. By Assumption 4.12 (ii) and the assumptions of this lemma, there exists an index set $\mathcal{K}' \subseteq \mathcal{K}$ such that $x^k \to x^*$, $\mu_k \to \bar{\mu}$, $\nu^k \to \bar{\nu}$, $\lambda^k \to \bar{\lambda}$ and $\varsigma^k \to \bar{\varsigma}$ for $k \in \mathcal{K}' \to \infty$. Lemma 4.9 yields that $\varepsilon_{\rho,k} \to 0$ for $k \in \mathcal{K}' \to \infty$. In the following the two cases $\|g(x^*)\| > 0$ and $\|g(x^*)\| = 0$ are considered.

i. Case $\|g(x^*)\| > 0$. Then, for $k \in \mathcal{K}'$ large enough $\|g(x^k)\|_2 > 0$ holds. Dividing (4.31b) by $\|g(x^k)\|_2/\bar{\tau}$ leads to

$$\begin{aligned} 0 &\leq \left\| \bar{\tau}\frac{g(x^k)}{\|g(x^k)\|_2} - y^k \right\|_\infty \\ &= \left\| \bar{\tau}\frac{g(x^k)}{\|g(x^k)\|_2} - y^k + \pi_{k\_}\lambda^{k\_} - \pi_{k\_}\lambda^{k\_} \right\|_\infty \\ &\leq \left\| \bar{\tau}\frac{g(x^k)}{\|g(x^k)\|_2} - y^k + \pi_{k\_}\lambda^{k\_} \right\|_\infty + \pi_{k\_}\|\bar{\lambda}\|_\infty \end{aligned}$$

$$\overset{(4.31b)}{\leq} \bar{\tau}\frac{\varepsilon_{\rho,k}}{\|g(x^k)\|_2} + \pi_{k\_}\left\|\bar{\lambda}\right\|_\infty$$

for $k \in \mathcal{K}'$ large enough. It follows that $y^k \to \bar{\tau}\frac{g(x^*)}{\|g(x^*)\|_2}$ since $\varepsilon_{\rho,k} \to 0$ and $\pi_k \to 0$ for $k \in \mathcal{K}' \to \infty$. Now, by letting $k \in \mathcal{K}' \to \infty$ in (4.31a) and (4.31c) and dividing by $\bar{\tau}$ leads to

$$\frac{1}{\|g(x^*)\|_2}\nabla g(x^*)g(x^*) - \frac{1}{\bar{\tau}}z^* = 0$$
$$\frac{1}{\bar{\tau}}\left(X_* + \bar{\mu}\bar{\Sigma}\right)z^* = 0.$$

Furthermore, $x^* \geq -\bar{\mu}\bar{\varsigma}$ and $z^* \geq 0$ (cf., Lemma 4.14). Thus, $(x^*, z^*/\bar{\tau})$ is a first-order optimal point of the feasibility problem (FeasNLP+) that is infeasible for (NLP+) and (NLPbar).

ii. Case $\|g(x^*)\| = 0$. For all $k \in \mathcal{K}'$ the condition (4.34a) is violated, i.e.,

$$\left\|y^k\right\|_2 + \pi_{k\_}\left\|\lambda^{k\_}\right\|_2 \geq \left\|y^k - \pi_{k\_}\lambda^{k\_}\right\|_2 \overset{(4.34a)}{>} \kappa_\lambda\bar{\tau} > 0.$$

For $k \in \mathcal{K}'$ large enough it follows $\left\|y^k\right\|_2 > 0$ because of $\pi_{k\_} \to 0$ and $\lambda^{k\_} \to \bar{\lambda}$ for $k \in \mathcal{K}' \to \infty$. Then there exists (y^*, z^*) with $\|(y^*, z^*)\| = 1$ such that

$$\frac{\left(y^k, z^k\right)}{\|(y^k, z^k)\|_2} \to (y^*, z^*). \tag{4.52}$$

Dividing (4.31a) and (4.31c) by $\left\|\left(y^k, z^k\right)\right\|_2$ and letting $k \in \mathcal{K}' \to \infty$ yields

$$\nabla g(x^*)y^* - z^* = 0$$
$$\left(X_* + \bar{\mu}\bar{\Sigma}\right)z^* = 0$$

since $\varepsilon_{\rho,k} \to 0$ and $\pi_k \to 0$ converge to zero for $k \in \mathcal{K}' \to \infty$. Because of $x^* \geq -\bar{\mu}\bar{\varsigma}$, $z^* \geq 0$ (cf., Lemma 4.14), $\|(y^*, z^*)\| = 1$ and $\|g(x^*)\| = 0$, it follows that (x^*, y^*, z^*) is a Fritz-John point of (ShiftNLP+) that fails to satisfy the MFCQ (cf., Lemma 4.13).

□

4.3.4 Global Convergence for Finitely Many Barrier and Penalty Parameter Updates

In the remainder, the global convergence will be analyzed in case all the barrier and penalty parameters except λ^k are fixed, which is the only possible further case. This means, that for sufficiently large iterations k the equations $\mu_k = \bar{\mu}$, $\varsigma^k = \bar{\varsigma}$, $\nu^k = \bar{\nu}$ and $\pi_k = \bar{\pi}$ hold. The first result shows the possible consequence if trial iterates are accepted in the line search Step L-10 by the filter or PLPF condition infinitely many times.

Lemma 4.22. *Suppose the Assumptions 4.12 hold. Furthermore, assume that the penalty parameter π_k and the barrier parameter μ_k are bounded away from zero and that the Lagrangian multiplier ν^k is updated finitely many times. If the filter or PLPF is augmented infinitely many times in Step L-10.3, the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ or the barrier parameter μ_k are updated infinitely many times in Step L-4.1 or Step L-4.2, respectively.*

Proof. The proof is analog to the proofs of Lemma 4.15, Lemma 4.16 and Lemma 4.17. □

The following two lemmas consider the case that also λ^k would be updated just finitely many times. It will further be assumed, that for infinite iterations k the non-monotone Armijo condition in Step L-10.2 is valid.

Lemma 4.23. *Suppose the Assumptions 4.12 hold. Furthermore, assume that the penalty parameter π_k and the barrier parameter μ_k are bounded away from zero and that the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ and filter or PLPF are updated or augmented finitely many times. Then $\left\{x^k\right\}_k$ is bounded below away from $-\mu_k \varsigma^k = -\bar{\mu}\bar{\varsigma}$ and $\left\{z^k\right\}_k$ is bounded below away from zero and bounded above.*

Proof. The proof is by contradiction and similar to Chen and Goldfarb [40, Lemma 3.7] since λ^k is assumed to be bounded, but with the extension to the non-monotone merit function. Let $\mathcal{K}$ be an infinite index set of iterations at which the trial step is accepted by the Armijo condition (4.17) and $\varsigma^k = \bar{\varsigma}$ and $\mu_k = \bar{\mu}$ holds. Assume, there exists an index $j \in \{1, \ldots, n_x\}$ such that $x_j^k \downarrow -\bar{\mu}\bar{\varsigma}$ for $k \in \mathcal{K} \to \infty$.

On the one hand, there exists an index k_0 such that $\lambda^k = \bar{\lambda}$, $\pi_k = \bar{\pi}$ and $\tau_k = \bar{\tau}$. Then, it follows from the Armijo conditions (4.17) and (4.24) together with Proposition 4.6 that

$$\Psi\left(w^{k+1}; \rho^k\right) = \Psi\left(w^{k+1}; \bar{\rho}\right) \overset{(4.17)}{<} \max_{i=0,\ldots,l_\mathrm{m}} \left\{\Psi\left(w^{k-i}; \bar{\rho}\right)\right\} \overset{(4.17)}{<} \max_{i=0,\ldots,l_\mathrm{m}} \left\{\Psi\left(w^{k_0+l_\mathrm{m}-i}; \bar{\rho}\right)\right\}, \tag{4.53}$$

for $k \geq k_0 + l_\mathrm{m}$.

On the other hand, $\left\{\left|f\left(x^k\right)\right|\right\}_{\mathcal{K}}$ and $\left\{\left\|g\left(x^k\right)\right\|\right\}_{\mathcal{K}}$ are bounded due to Assumption 4.12 (i) and (ii) and, thus, $\Psi\left(w^{k+1}; \bar{\rho}\right) \to \infty$ for $x_j^k \downarrow -\bar{\mu}\bar{\varsigma}$ due to the barrier term, a contradiction to (4.53). It follows, that $\left\{x^k\right\}_k$ is bounded below away from $-\bar{\mu}\bar{\varsigma}$. Consequently, by (4.30) $\left\{z^k\right\}_k$ is bounded. Analogously, it can be shown, that $\left\{z^k\right\}_k$ is bounded below away from zero. □

Lemma 4.24. *Suppose the Assumptions 4.12 hold. Furthermore, assume that the penalty parameter π_k and the barrier parameter μ_k are bounded away from zero and that the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ and filter or PLPF are updated or augmented finitely many times. Then, there exists an infinite index set $\mathcal{K}$ such that either of the following holds:*

i. *The sequences $\left\{\left(x^k, y^k, z^k\right)\right\}_{\mathcal{K}}$, $\left\{\left(\Delta x^k, \Delta y^k, \Delta z^k\right)\right\}_{\mathcal{K}}$ and $\left\{\left(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k\right)\right\}_{\mathcal{K}}$ are bounded.*

ii. *The sequence $\left\{x^k\right\}_{\mathcal{K}}$ converges to a Fritz-John point of problem* (ShiftNLP+) *that fails to satisfy the MFCQ.*

Proof. The proof is similar to Chen and Goldfarb [40, Lemma 3.8]. From Assumption 4.12 (ii) and (iii), the assumptions of this lemma and Lemma 4.23 it follows, that an index set $\mathscr{K}$ exists such that $x^k \to x^* > -\bar{\mu}$, $z^k \to z^* \geq 0$, $Q_k \to Q_*$, $\lambda^k \to \bar{\lambda}$, $\nu^k \to \bar{\nu}$, $\mu_k \to \bar{\mu}$, $\varsigma^k \to \bar{\varsigma}$ and $\pi_k \to \bar{\pi}$ for $k \in \mathscr{K} \to \infty$. Then,

$$\widetilde{M}_k \to \widetilde{M}_* := \begin{bmatrix} Q_* + \left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1} Z_* & \nabla g(x^*) \\ \nabla g(x^*)^\top & -\frac{\|g(x^*)\|_2}{\bar{\tau}} E \end{bmatrix}$$

for $k \in \mathscr{K} \to \infty$. The linear equation system (4.12a) can be equally formulated as

$$\widetilde{M}_k \begin{bmatrix} \Delta x^k \\ y^k + \Delta y^k \end{bmatrix} = -\begin{bmatrix} \pi_k \nabla f\left(x^k\right) - \mu_k \pi_k \Sigma_k \left(X_k + \mu_k \Sigma_k\right)^{-1} \nu^k \\ g\left(x^k\right) \end{bmatrix} \tag{4.54}$$

In the following the two cases of $\left\|\widetilde{M}_k^{-1}\right\|$ being bounded and unbounded will be considered.

i. Case $\left\{\left\|\widetilde{M}_k^{-1}\right\|\right\}_{\mathscr{K}}$ is bounded. Due to the convergence of $\left\{x^k\right\}_{\mathscr{K}}$ to $x^* \geq -\bar{\mu}\bar{\varsigma}$ (cf., Lemma 4.14) and all the other parameters, these sequences are bounded and thus, the right-hand-side of (4.54) is bounded. Together with the boundedness of $\left\{\left\|\widetilde{M}_k^{-1}\right\|\right\}_{\mathscr{K}}$ this implies that $\left\{y^k + \Delta y^k\right\}_{\mathscr{K}}$ is bounded. Then, (4.27) yields the boundedness of $\left\{y^k\right\}_{\mathscr{K}}$. Consequently, the right-hand-sides of (4.12a) and (4.12d) are bounded. Hence, by the same argument as above for (4.54), the sequence $\left\{\left(\Delta x^k, \Delta y^k, \Delta z^k\right)\right\}_{\mathscr{K}}$ is bounded. Finally, this implies that the right-hand-sides of (4.21) are bounded and, thus, due to the equivalence of the system matrices in (4.12) and (4.21) the boundedness of $\left\{\left(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k\right)\right\}_{\mathscr{K}}$.

ii. Case $\left\{\left\|\widetilde{M}_k^{-1}\right\|\right\}_{\mathscr{K}}$ is unbounded. This case is equivalent to the sequence $\left\{\widetilde{M}_k\right\}_{\mathscr{K}}$ converging to a singular matrix $\widetilde{M}_*$. Then, $g(x^*) = 0$. Otherwise, there would exist an index set $\mathscr{K}' \subseteq \mathscr{K}$ such that $\left\|g\left(x^k\right)\right\| > 0$ for $k \in \mathscr{K}'$. But then, by (4.13) and Lemma 3.15, the matrix $\widetilde{M}_*$ must be non-singular, which would contradict the assumption of this case.
In the following the rank-deficiency of the Jacobian $\nabla g(x^*)$ will be shown by contradiction. Assume, that $\nabla g(x^*)$ has full rank. Then it exists an index set $\mathscr{K}' \subseteq \mathscr{K}$ such that $\nabla g\left(x^k\right)$ has full rank for $k \in \mathscr{K}'$. For a fixed $d \in \mathbb{R}^{n_x}$ with $\nabla g(x^*)^\top d = 0$ let

$$d^k := \left(E - \nabla g\left(x^k\right)\left(\nabla g\left(x^k\right)^\top \nabla g\left(x^k\right)\right)^{-1} \nabla g\left(x^k\right)^\top\right) d$$

with $k \in \mathscr{K}'$. Obviously, $\nabla g\left(x^k\right)^\top d^k = 0$ and thus,

$$\begin{aligned} & \left(d^k\right)^\top \left(Q_k + \left(X_k + \mu_k \Sigma_k\right)^{-1} Z_k + \frac{1}{\varrho_k} \nabla g\left(x^k\right) \nabla g\left(x^k\right)^\top\right) d^k \\ = \; & \left(d^k\right)^\top \left(Q_k + \left(X_k + \mu_k \Sigma_k\right)^{-1} Z_k\right) d^k \\ \overset{(4.13)}{\geq} & \; \varepsilon \left\|d^k\right\|_2^2 \end{aligned} \tag{4.55}$$

Due to Assumption 4.12 (i), it follows

$$0 \leq \left\|d - d^k\right\| = \left\|\left(\nabla g\left(x^k\right)\left(\nabla g\left(x^k\right)^\top \nabla g\left(x^k\right)\right)^{-1} \nabla g\left(x^k\right)^\top\right) d\right\| \to 0$$

and consequently $d^k \to d$ for $k \in \mathcal{K}' \to \infty$. But then, letting $k \in \mathcal{K}' \to \infty$ in (4.55) yields $d^\top \left(Q_* + \left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1} Z_*\right) d \geq \varepsilon \|d\|_2^2$ and $\widetilde{M}_*$ cannot be singular due to Lemma 3.15, a contradiction. This implies, that $\nabla g(x^*)$ is rank-deficient and the feasible point x^* fails to satisfy the MFCQ. Then, by Lemma 4.13 Lagrangian multipliers $\lambda^* \neq 0$ exist and $\nu^* = 0$ such that $\nabla g(x^*)\lambda^* - \nu^* = 0$, hence, since $x^* \geq -\bar{\mu}\bar{\varsigma}$ is feasible and $\left(X_* + \bar{\mu}\bar{\Sigma}\right)\nu^* = 0$, (x^*, λ^*, ν^*) is a Fritz-John point of problem (ShiftNLP+) that fails to satisfy the MFCQ.

□

Finally, the last lemma shows that the case of λ^k being updated just finitely many times does actually not exist. In particular, the conditions for an update of λ^k will eventually be satisfied.

Lemma 4.25. *Suppose the Assumptions 4.12 hold. Furthermore, assume that the penalty parameter π_k and the barrier parameter μ_k are bounded away from zero and that the Lagrangian multipliers ν^k and filter or PLPF are updated or augmented finitely many times. Then, it exists an index set $\mathcal{K}$ such that either of the following holds:*

i. *The Lagrangian multipliers λ^k are updated infinitely many times in Step L-5.1.*
ii. *The sequence $\left\{x^k\right\}_{\mathcal{K}}$ converges to a Fritz-John point of problem* (ShiftNLP+) *that fails to satisfy the MFCQ.*

Proof. This proof is similar to Chen and Goldfarb [43, Lemma 3.8], but modified to the different merit function and parameter update rules. First, if the Lagrangian multiplier is updated infinitely many times, the second possible outcome follows directly from Lemma 4.20. Otherwise, if it is updated finitely many times, Lemma 4.24 has proven that either the second outcome of this lemma is true or it exists an index set $\mathcal{K}$ such that the sequences $\left\{\left(x^k, y^k, z^k\right)\right\}_{\mathcal{K}}$, $\left\{\left(\Delta x^k, \Delta y^k, \Delta z^k\right)\right\}_{\mathcal{K}}$ and $\left\{\left(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k\right)\right\}_{\mathcal{K}}$ are bounded.

In the following it will be proven that the case of λ^k being updated finitely many times and $\left\{\left(x^k, y^k, z^k\right)\right\}_{\mathcal{K}}$, $\left\{\left(\Delta x^k, \Delta y^k, \Delta z^k\right)\right\}_{\mathcal{K}}$ and $\left\{\left(\widehat{\Delta x}^k, \widehat{\Delta y}^k, \widehat{\Delta z}^k\right)\right\}_{\mathcal{K}}$ being bounded does actually not exist as it leads to a contradiction. So, assume this case is true. Then, together with Assumption 4.12 (ii), Lemma 4.24 and the assumptions of this lemma, it exists an index set $\mathcal{K}$ such that $x^k \to x^*$, $y^k \to y^*$, $z^k \to z^*$, $\Delta x^k \to \Delta x^*$, $\Delta y^k \to \Delta y^*$, $\Delta z^k \to \Delta z^*$, $\lambda^k \to \bar{\lambda}$, $\nu^k \to \bar{\nu}$, $\mu_k \to \bar{\mu}$, $\varsigma^k \to \bar{\varsigma}$ and $\pi_k \to \bar{\pi}$ for $k \in \mathcal{K} \to \infty$. Hence, it follows $\Delta w^k \to \Delta w^*$ and from Proposition 4.6 that

$$
\begin{aligned}
&D^x_{\Delta w^k}\Psi\left(x^k;\rho^k\right) \\
&\quad = \left(\pi_k \nabla f\left(x^k\right) + \nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)\right)^\top \Delta x^k - \pi_k {\lambda^k}^\top g\left(x^k\right) - \bar{\tau}\left\|g\left(x^k\right)\right\|_2 \\
&\qquad + \left(g\left(x^k\right) + \varrho_k \pi_k \lambda^k\right)^\top \left(y^k + \Delta y^k - \pi_k \lambda^k\right) - \tau_{\text{f}} \left\|g\left(x^k\right) + \varrho_k\left(\pi_k \lambda^k - y^k\right)\right\|_2^2 \\
&\qquad - \tau_{\text{c}} \left\|\left(X_k + \mu_k \Sigma_k\right)^{-\frac{1}{2}} Z_k^{-\frac{1}{2}} \left(X_k z^k - \mu_k \Sigma_k \left(\pi_k \nu^k - z^k\right)\right)\right\|_2^2
\end{aligned}
\tag{4.56}
$$

$$\begin{aligned}
&\to \left(\bar{\pi}\nabla f(x^*) + \nabla_x \varphi_{\mathrm{bar}}(x^*;\bar{\rho})\right)^\top \Delta x^* - \bar{\pi}\bar{\lambda}^\top g(x^*) - \bar{\tau}\,\|g(x^*)\|_2 \\
&\quad + \left(g(x^*) + \frac{\|g(x^*)\|_2}{\bar{\tau}}\bar{\pi}\bar{\lambda}\right)^\top \left(y^* + \Delta y^* - \bar{\pi}\bar{\lambda}\right) - \tau_{\mathrm{f}} \left\| g(x^*) + \frac{\|g(x^*)\|_2}{\bar{\tau}}\left(\bar{\pi}\bar{\lambda} - y^*\right)\right\|_2^2 \\
&\quad - \tau_{\mathrm{c}} \left\| \left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-\frac{1}{2}} Z_*^{-\frac{1}{2}} \left(X_* z^* - \bar{\mu}\bar{\Sigma}\left(\bar{\pi}\bar{\nu} - z^*\right)\right)\right\|_2^2 \\
&= D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) \qquad (4.57)
\end{aligned}$$

for $k \in \mathcal{K} \to \infty$.

In the following it will be shown, that $D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) = 0$ by contradiction. Assume, that $D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) < 0$, which implies $\Delta w^* \neq 0$.[6] Then, it exists $\widetilde{\alpha} > 0$ such that for all $\alpha \in (0,\widetilde{\alpha}]$

$$\Psi(w^* + \alpha\Delta w^*;\bar{\rho}) - \Psi(w^*;\bar{\rho}) \leq 2\sigma\alpha D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) \qquad (4.58)$$

holds because $\sigma \in \left(0,\frac{1}{2}\right)$. Due to $x^* > -\bar{\mu}\varsigma$ from Lemma 4.23, the fraction-to-the-boundary rule (4.15) implies that $\alpha_{\max,k} > 0$ for all k. Let $\mathcal{K}' \subseteq \mathcal{K}$ be an index set such that $\alpha_{\max,k} \to \alpha_{\max}$ and $j^* = \min\left\{j \mid \alpha_{\max}\beta^j \leq \widetilde{\alpha}\right\}$. With Assumption 4.12 (i) and the fact that (4.58) is satisfied for $\alpha_{\max}\beta^{j^*}$, it follows for $k \in \mathcal{K}'$ sufficiently large

$$\Psi\left(w^k + \alpha\Delta w^k;\rho^k\right) - \Psi\left(w^k;\rho^k\right) \leq \sigma\alpha D^x_{\Delta w^k}\Psi\left(w^k;\rho^k\right) \qquad (4.59)$$

for $\alpha \in (0,\widetilde{\alpha})$ with $\alpha \leq \alpha_{\max,k}\beta^{j^*}$ and, thus,

$$\Psi\left(w^k + \alpha\Delta w^k;\rho^k\right) - \max_{i=0,\ldots,l_{\mathrm{m}}} \Psi\left(w^{(k-i)_+};\rho^k\right) \leq \sigma\alpha D^x_{\Delta w^k}\Psi\left(w^k;\rho^k\right).$$

Moreover, due to $z^* > 0$ from Lemma 4.23 it follows for all $k \in \mathcal{K}$ that

$$0 < \alpha'_{\max} = \max\left\{\alpha' \in (0,1] \mid z^k + \alpha'\Delta z^k \geq \left(1 - \varepsilon_{\mathrm{frac},k}\right)z^k\right\}.$$

Therefore, the backtracking line search always finds a step size $\alpha_k \geq \min\left\{\alpha,\alpha'_{\max}\right\} > 0$ that is acceptable to (4.17) and is finite in every iteration. If the second-order-correction steps are executed infinitely many times, the fraction-to-the-boundary rule (4.23) implies that $\widehat{\alpha}_k \geq \widehat{\alpha}$ for some $\widehat{\alpha} > 0$ since $\widehat{\Delta x}^k$ is bounded. The inequality (4.59) together with (4.17), (4.24) and (4.56) then leads to

$$\Psi\left(w^{k+1};\rho^k\right) - \Psi\left(w^k;\rho^k\right) \leq \frac{1}{2}\sigma\min\left\{\alpha,\alpha'_{\max},\widehat{\alpha}\right\} D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) < 0$$

for $k \in \mathcal{K}'$ large enough. Hence, it follows $\Psi\left(w^k;\rho^k\right) \to -\infty$ for $k \in \mathcal{K}' \to \infty$, a contradiction. Finally, this implies $D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) = 0$.

Then, it follows from (4.13) that $-\varepsilon\,\|\Delta w^*\|_2^2 \geq D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) = 0$ and thus $\Delta w^* = 0$. Because $\Delta w^k \to \Delta w^*$ for $k \in \mathcal{K} \to \infty$ and the fact that all the components in (4.12) are bounded implies that for $k \in \mathcal{K}$

$$\begin{aligned}
\left\|\pi_k\nabla f\left(x^k\right) + \nabla g\left(x^k\right)y^k - z^k\right\| &\to 0, \\
\left\| g\left(x^k\right) - \frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\left(y^k - \bar{\pi}\bar{\lambda}\right)\right\| &\to 0, \\
\left\|X_k z^k - \bar{\mu}\bar{\Sigma}\left(\bar{\pi}\bar{\nu} - z^k\right)\right\| &\to 0.
\end{aligned}$$

[6] Recall that the case $D^x_{\Delta w^*}\Psi(w^*;\bar{\rho}) > 0$ does not occur due to Proposition 4.6.

But then, for every $\varepsilon_{\rho,k} > 0$ there exists an index $k' \in \mathscr{K}$ such that (4.31) is satisfied. Since the penalty parameter is updated just finitely many times by the assumption of this lemma, (4.34a), (4.34b) and (4.35) must be satisfied for these $k \in \mathscr{K}$. This implies that λ^k is updated infinitely many times in Step L-5.1. □

4.3.5 Global Convergence of the Overall Algorithm

Finally, by combining all the previous results, the global convergence properties of the exact penalty-interior-point algorithm (Algorithm L) can be summarized in the following Theorem.

Theorem 4.26. *Suppose the Assumptions 4.12 hold and Algorithm L generates an infinite sequence of iterates. If Step L-4 and Step L-6 are disabled, then the barrier parameter $\mu_k > 0$, Lagrangian multiplier $\nu^k > 0$ and primal shift $\varsigma^k > 0$ are fixed and it exists an index set $\mathscr{K}$ with either of the following:*

i. *The Lagrangian multiplier λ^k is updated infinitely many times and the sequence $\left\{\left(x^k, \lambda^k, z^k/\pi_{k\_}\right)\right\}_{\mathscr{K}}$ converges to a first-order optimal point of* (NLPbar).

ii. *The penalty parameter π_k tends to zero and $\left(x^k, z^k/\bar{\tau}\right)$ converges to a first-order optimal solution of* (FeasNLP+) *that is infeasible for* (NLP+). *The sequence $\left\{y^k\right\}_{\mathscr{K}}$ converges to $\bar{\tau} g(x^*)/\left\|g(x^*)\right\|_2$.*

iii. *The sequence $\left\{x^k\right\}_{\mathscr{K}}$ converges to a Fritz-John point of problem* (ShiftNLP+) *that fails to satisfy the MFCQ.*

If Step L-4 and Step L-6 are enabled, then (i) *does not occur but the following additional cases:*

iv. *The magic step is executed infinitely many times and the sequence $\left\{\left(x^k, y^k/\pi_{k\_}, z^k/\pi_{k\_}\right)\right\}_{\mathscr{K}}$ converges to a first-order optimal solution of* (NLP+).

v. *The Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ are updated infinitely many times and $\left\{\left(x^k, \lambda^k, \nu^k\right)\right\}_{\mathscr{K}}$ converges to a first-order optimal solution of* (NLP+).

vi. *The sequence $\left\{x^k\right\}_{\mathscr{K}}$ converges to a Fritz-John point of problem* (NLP+) *that fails to satisfy the MFCQ.*

Proof. The proof for (i), (ii) and (iii) follows from Lemma 4.20 to Lemma 4.25 and for (iv), (v) and (vi) from Lemma 4.15 to Lemma 4.19. It remains to show, that if Step L-4 and Step L-6 are enabled, the case (i) does not occur. Assume, $\mathscr{K}$ is the index set in which the Lagrangian multipliers $\lambda^k = y^k/\pi_{k\_}$ are updated in Step L-5.1. Then, by Lemma 4.20

$$
\begin{aligned}
\nabla f\left(x^k\right) + \frac{1}{\pi_{k\_}} \nabla g\left(x^k\right)\lambda^k - \frac{z^k}{\pi_{k\_}} &\to 0 \\
g\left(x^k\right) &\to 0 \\
\frac{1}{\pi_{k\_}} X_k z^k - \mu_{k\_} \Sigma_{k\_} \left(\nu^{k\_} - \frac{z^k}{\pi_{k\_}}\right) &\to 0
\end{aligned}
$$

for $k \in \mathscr{K} \to \infty$. Hence, for every $\varepsilon_{\mu,k} > 0$ it exists an index $k' \in \mathscr{K}$ such that (4.39) is satisfied. Then, either the Lagrangian multiplier ν^k is updated infinitely many times in Step L-4.1 or the barrier parameter μ_k is in Step L-4.2, which leads to the cases (v) and (vi) by Lemma 4.18 and Lemma 4.19. □

4.4 Local Convergence Analysis

The local convergence analysis studies the order of convergence of Algorithm L for the penalty subproblem (NLPpen), i.e., for fixed barrier parameter $\bar{\mu}$, primal boundary shift $\bar{\varsigma}$ and Lagrangian multiplier $\bar{\nu}$ and the overall algorithm when converging towards an optimal solution (x^*, λ^*, ν^*) as well as a certificate of infeasibility. The proof of the first two lemmas is mostly based on Chen and Goldfarb [40, 43] but extended to address the modified barrier and augmented Lagrangian penalty approach. The following standard assumptions are made.

Assumptions 4.27 (Local Convergence).

i. *The functions $f(x)$ and $g(x)$ are real valued and twice continuously differentiable. The Hessian matrices $\nabla^2 f(x)$ and $\nabla^2 g_i(x)$, $i \in \{1, \dots, n_g\}$, are locally Lipschitz continuous at x^*.*

ii. *The LICQ holds for x^*: the gradients $\nabla g_i(x^*)$, $i \in \{1, \dots, n_g\}$, and $-e_i$, $i \in \mathscr{A}(x^*)$, are linearly independent.*

iii. *The SOSC holds for (x^*, λ^*, ν^*): It exists $\varepsilon > 0$ such that*

$$d^\top \left(\nabla^2 f(x^*) + \sum_{i=1}^{n_g} \lambda_i^* \nabla^2 g_i(x^*) \right) d \geq \varepsilon \|d\|_2^2$$

for all $d \in \mathbb{R}^{n_x} \setminus \{0\}$ with $d_i = 0$ for all $i \in \mathscr{A}(x^*)$ and $\nabla g(x^*)^\top d = 0$.

iv. *The SCC holds for (x^*, λ^*, ν^*): $x_i^* + \nu_i^* > 0$, $i \in \{1, \dots, n_x\}$.*

Furthermore, it is assumed that Q_k is the exact Hessian, i.e., $Q_k = \nabla^2 f(x^k) + \sum_{i=1}^{n_g} \lambda_i^k \nabla^2 g_i(x^k)$. For convenience the primal-dual iterates and steps are abbreviated as $w^k := (x^k, y^k, z^k)$ and $\Delta w^k := (\Delta x^k, \Delta y^k, \Delta z^k)$, respectively.

4.4.1 Local Convergence for Penalty Subproblem

For convergence towards an optimal solution $w(\bar{\mu}, \bar{\nu}) = (x(\bar{\mu}, \bar{\nu}), y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu}))$ of (NLPbar) it is assumed that $\mu_k = \bar{\mu}$, $\varsigma^k = \bar{\varsigma}$ and $\nu^k = \bar{\nu}$ are fixed.[7] From Theorem 4.26, it holds that for convergence towards $w(\bar{\mu}, \bar{\nu})$ the penalty parameters are updated just finitely many times, i.e., are fixed as well to $\pi_k = \bar{\pi}$ and $\tau_k = \bar{\tau}$ for k sufficiently large, and $\lambda^k \to y(\bar{\mu}, \bar{\nu})/\bar{\pi}$. Furthermore, it is assumed that $\bar{\mu}$ is sufficiently small and $(x(\bar{\mu}, \bar{\nu}), y(\bar{\mu}, \bar{\nu})/\bar{\pi}, z(\bar{\mu}, \bar{\nu})/\bar{\pi})$ converges to an optimal solution (x^*, λ^*, ν^*) of (NLP+) for $\bar{\nu} \to \nu^*$. Then, $w(\bar{\mu}, \bar{\nu})$ is unique due to Assumption 4.27 (ii) and (4.4). In that case, the Assumption 4.27 (iii) holds also for the iterate (x^k, y^k, z^k), which implies that the regularization in Step L-7 is not applied and together with Assumption 4.27 (ii) the matrix of the linear equation system (4.11) is regular, i.e.,

$$M_k^{-1} = \mathcal{O}(1). \tag{4.60}$$

[7] The definition of $w(\bar{\mu}, \bar{\nu})$ neglects the dependence on $\bar{\varsigma}$ since it is defined by $\bar{\nu}$, cf., (4.42).

Moreover, from Assumption 4.27 (iv) and (4.4) it follows that

$$\begin{cases} x_i(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\varsigma}_i & = \Theta\big(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|\big), \\ z_i(\bar{\mu}, \bar{\nu}) & = \Theta(1) \end{cases} \quad \text{if } i \in \mathscr{A}(x^*) \tag{4.61a}$$

$$\begin{cases} x_i(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\varsigma}_i & = \Theta(1), \\ z_i(\bar{\mu}, \bar{\nu}) & = \Theta\big(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|\big) \end{cases} \quad \text{if } i \in \mathscr{I}(x^*) \tag{4.61b}$$

and subsequently if the current iterate is sufficiently close to the optimal solution of (NLPbar), i.e., if $\|w^k - w(\bar{\mu}, \bar{\nu})\| = o\big(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|\big)$, that

$$\begin{cases} x_i^k + \bar{\mu}\bar{\varsigma}_i & = \Theta\big(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|\big), \\ z_i^k & = \Theta(1) \end{cases} \quad \text{if } i \in \mathscr{A}(x^*) \tag{4.62a}$$

$$\begin{cases} x_i^k + \bar{\mu}\bar{\varsigma}_i & = \Theta(1), \\ z_i^k & = \Theta\big(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|\big) \end{cases} \quad \text{if } i \in \mathscr{I}(x^*). \tag{4.62b}$$

The first result shows the q-quadratic convergence order for a full (second-order-correction) step and provides an estimate for the difference of these two different steps.

Lemma 4.28. *Suppose Assumptions 4.27 hold. Let $w^k \in \mathscr{N}(w^*)$ and $\lambda^k = y^k/\bar{\pi}$. Then the following equalities hold:*

i. $\left\|w^k + \Delta w^k - w(\bar{\mu}, \bar{\nu})\right\| = \mathscr{O}\Big(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|^2\Big)$

ii. $\left\|w^k + \widehat{\Delta w}^k - w(\bar{\mu}, \bar{\nu})\right\| = \mathscr{O}\Big(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|^2\Big)$

iii. $\left\|\Delta w^k\right\| = \Theta\big(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|\big)$

iv. $\left\|\widehat{\Delta w}^k\right\| = \Theta\big(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|\big)$

v. $\left\|\Delta w^k - \widehat{\Delta w}^k\right\| = \mathscr{O}\Big(\left\|\Delta x^k\right\|^2\Big)$

Proof. The proof is similar to Chen and Goldfarb [43, Theorem 5.1]. Subtracting (4.22) from (4.11) and using Taylor's theorem yields

$$\begin{aligned} M_k\left(\Delta w^k - \widehat{\Delta w}^k\right) &= -\begin{bmatrix} 0 \\ g\big(x^k\big) - g\big(x^k + \alpha_{\max,k}\Delta x^k\big) + \alpha_{\max,k}\nabla g\big(x^k\big)^\top \Delta x^k \\ 0 \end{bmatrix} \\ &= \mathscr{O}\Big(\left\|\Delta x^k\right\|^2\Big). \end{aligned}$$

Taking the norm and applying (4.60) implies (v).

Without loss of generality it is assumed that $\bar{\pi} = 1$. Then from the linear equation system (4.11), it follows

$$\begin{aligned} & M_k\left(w^k + \Delta w^k - w(\bar{\mu}, \bar{\nu})\right) \\ &\quad = \quad M_k\left(w^k - w(\bar{\mu}, \bar{\nu})\right) + M_k\Delta w^k \end{aligned}$$

$$
\begin{aligned}
&\overset{(4.11)}{=} \begin{bmatrix} \nabla_{xx}^2 L\big(x^k, y^k, z^k\big)\big(x^k - x(\bar{\mu}, \bar{\nu})\big) + \nabla g\big(x^k\big)\big(y^k - y(\bar{\mu}, \bar{\nu})\big) - \big(z^k - z(\bar{\mu}, \bar{\nu})\big) \\ \nabla g\big(x^k\big)^\top \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \varrho_k \big(y^k - y(\bar{\mu}, \bar{\nu})\big) \\ Z_k \big(x^k - x(\bar{\mu}, \bar{\nu})\big) + \big(X_k + \bar{\mu}\bar{\Sigma}\big)\big(z^k - z(\bar{\mu}, \bar{\nu})\big) \end{bmatrix} \\
&\quad - \begin{bmatrix} \nabla_x L\big(x^k, y^k, z^k\big) \\ g\big(x^k\big) \\ X_k z^k - \bar{\mu}\bar{\Sigma}\big(\bar{\nu} - z^k\big) \end{bmatrix} \\
&= \begin{bmatrix} \nabla_{xx}^2 L\big(x^k, y^k, z^k\big)\big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \nabla_x L\big(x^k, y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu})\big) \\ -g\big(x^k\big) + \nabla g\big(x^k\big)^\top \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \varrho_k \big(y^k - y(\bar{\mu}, \bar{\nu})\big) \\ Z_k \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \big(X_k + \bar{\mu}\bar{\Sigma}\big) z(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\Sigma}\bar{\nu} \end{bmatrix}. \qquad (4.63)
\end{aligned}
$$

Then, by Assumption 4.27 (i), Taylor's theorem and $\nabla_x L(x(\bar{\mu}, \bar{\nu}), y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu})) = 0$ due to (4.4) the equation

$$
\begin{aligned}
&\nabla_{xx}^2 L\big(x^k, y^k, z^k\big)\big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \nabla_x L\big(x^k, y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu})\big) \\
&\overset{\text{A } 4.27}{=} \big(\nabla_{xx}^2 L\big(x^k, y^k, z^k\big) - \nabla_{xx}^2 L(x(\bar{\mu}, \bar{\nu}), y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu}))\big)\big(x^k - x(\bar{\mu}, \bar{\nu})\big) \\
&\quad - \nabla_x L(x(\bar{\mu}, \bar{\nu}), y(\bar{\mu}, \bar{\nu}), z(\bar{\mu}, \bar{\nu})) + \mathcal{O}\Big(\big\|x^k - x(\bar{\mu}, \bar{\nu})\big\|^2\Big) \\
&\overset{(4.4)}{=} \mathcal{O}\Big(\big\|w^k - w(\bar{\mu}, \bar{\nu})\big\|^2\Big) \qquad (4.64)
\end{aligned}
$$

holds. Moreover, using Taylor's theorem and $g(x(\bar{\mu}, \bar{\nu})) = 0$ (cf., (4.4)) leads to

$$
\begin{aligned}
&- g\big(x^k\big) + \nabla g\big(x^k\big)^\top \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \frac{\big\|g\big(x^k\big)\big\|_2}{\bar{\tau}} \big(y^k - y(\bar{\mu}, \bar{\nu})\big) \\
&\overset{(4.4)}{=} \mathcal{O}\big(x^k - x(\bar{\mu}, \bar{\nu})\big)\big(y^k - y(\bar{\mu}, \bar{\nu})\big) + \mathcal{O}\Big(\big\|x^k - x(\bar{\mu}, \bar{\nu})\big\|^2\Big) \\
&= \mathcal{O}\Big(\big\|w^k - w(\bar{\mu}, \bar{\nu})\big\|^2\Big). \qquad (4.65)
\end{aligned}
$$

For the third equation, it follows by $\big(X(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\Sigma}\big) z(\bar{\mu}, \bar{\nu}) = \bar{\mu}\bar{\Sigma}\bar{\nu}$ due to (4.4) that

$$
\begin{aligned}
&Z_k \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \big(X_k + \bar{\mu}\bar{\Sigma}\big) z(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\Sigma}\bar{\nu} \\
&\overset{(4.4)}{=} Z_k \big(x^k - x(\bar{\mu}, \bar{\nu})\big) - \big(X_k + \bar{\mu}\bar{\Sigma}\big) z(\bar{\mu}, \bar{\nu}) + \big(X(\bar{\mu}, \bar{\nu}) + \bar{\mu}\bar{\Sigma}\big) z(\bar{\mu}, \bar{\nu}) \\
&= (X_k - X(\bar{\mu}, \bar{\nu}))\big(z^k - z(\bar{\mu}, \bar{\nu})\big) \\
&= \mathcal{O}\Big(\big\|w^k - w(\bar{\mu}, \bar{\nu})\big\|^2\Big) \qquad (4.66)
\end{aligned}
$$

Combining the equations above, taking the norm and using (4.60) proves (i). Then, (iii) follows by

$$
\begin{aligned}
\big\|\Delta w^k\big\| &\leq \big\|w^k + \Delta w^k - w(\bar{\mu}, \bar{\nu})\big\| + \big\|w^k - w(\bar{\mu}, \bar{\nu})\big\| \\
&\overset{\text{(i)}}{=} \mathcal{O}\Big(\big\|w^k - w(\bar{\mu}, \bar{\nu})\big\|\Big),
\end{aligned}
$$

and

$$
\big\|\Delta w^k\big\| \geq \big\|w^k - w(\bar{\mu}, \bar{\nu})\big\| - \big\|w^k + \Delta w^k - w(\bar{\mu}, \bar{\nu})\big\|
$$

$$\overset{\text{(i)}}{\geq} \left\|w^k - w(\bar{\mu}, \bar{\nu})\right\| \left(1 - \mathcal{O}\left(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|\right)\right)$$
$$\geq \varepsilon \left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|$$

for some $\varepsilon > 0$ since w^k and $w(\bar{\mu}, \bar{\nu})$ are in a small neighborhood of w^*.

Finally, (v) and (iii) imply (ii) since

$$\begin{aligned}
\left\|w^k + \widehat{\Delta w}^k - w(\bar{\mu}, \bar{\nu})\right\| &\leq \left\|w^k + \Delta w^k - w(\bar{\mu}, \bar{\nu})\right\| + \left\|\Delta w^k - \widehat{\Delta w}^k\right\| \\
&\overset{\text{(i),(v)}}{=} \mathcal{O}\left(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|^2\right) + \mathcal{O}\left(\left\|\Delta x^k\right\|^2\right) \\
&\leq \mathcal{O}\left(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|^2\right) + \mathcal{O}\left(\left\|\Delta w^k\right\|^2\right) \\
&\overset{\text{(iii)}}{=} \mathcal{O}\left(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|^2\right)
\end{aligned}$$

and (iv) can then be shown analogously like (iii). □

In the second lemma, it is proven that if the iterates are sufficiently close to the optimal solution of (NLPbar), the fraction-to-the-boundary rules of Step L-6, Step L-10.1 and Step L-11.1 are satisfied for a full (second-order-correction) step and the dual projection of Step L-13 does not modify the dual iterate z^{k+1}.

Lemma 4.29. *Suppose Assumptions 4.27 hold. Let $\varepsilon_{frac,k} \in (0,1)$, the term $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ be sufficiently small, $\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ and λ^k be updated in Step L-5.1. Then, the following inequalities hold:*

i. $x^k + \bar{\mu}\bar{\varsigma} + \Delta x^k \geq \left(1 - \varepsilon_{frac,k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right)$

ii. $x^k + \bar{\mu}\bar{\varsigma} + \widehat{\Delta x}^k \geq \left(1 - \varepsilon_{frac,k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right)$

iii. $z^k + \Delta z^k \geq \left(1 - \varepsilon_{frac,k}\right) z^k$

iv. $z^k + \widehat{\Delta z}^k \geq \left(1 - \varepsilon_{frac,k}\right) z^k$

v. $z^k + \Delta z^k \in \bar{\pi}\bar{\mu}\bar{\Sigma}\left(X_k + \operatorname{diag}\left(\Delta x^k\right) + \bar{\mu}\bar{\Sigma}\right)^{-1} \bar{\nu}\left[\kappa_z^{-1}, \kappa_z\right]$

vi. $z^k + \widehat{\Delta z}^k \in \bar{\pi}\bar{\mu}\bar{\Sigma}\left(X_k + \operatorname{diag}\left(\widehat{\Delta x}^k\right) + \bar{\mu}\bar{\Sigma}\right)^{-1} \bar{\nu}\left[\kappa_z^{-1}, \kappa_z\right]$

Proof. The proof is similar to Chen and Goldfarb [43, Lemma 5.2] but extended to the modified barrier function. First, because of $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ being sufficiently small and $\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ it follows $w^k \in \mathcal{N}(w^*)$. Hence, $\left\|\Delta w^k\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ and $\left\|\widehat{\Delta w}^k\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ by Lemma 4.28. Together with (4.62), this implies

$$\begin{aligned}
x^k + \bar{\mu}\bar{\varsigma} + \Delta x^k &= \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right) + \varepsilon_{\text{frac},k}\left(x^k + \bar{\mu}\bar{\varsigma}\right) + \Delta x^k \\
&\geq \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right) + \varepsilon_{\text{frac},k}\left(x^k + \bar{\mu}\bar{\varsigma}\right) - \left\|\Delta x^k\right\| \\
&= \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right) + \Theta\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right) - o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right) \\
&\geq \left(1 - \varepsilon_{\text{frac},k}\right)\left(x^k + \bar{\mu}\bar{\varsigma}\right),
\end{aligned}$$

which proves (i). The proof for (ii), (iii) and (iv) is analogue. The cases (v) and (vi) follow by Lemma 4.28, since then

$$\begin{aligned}&\left(X_k + \operatorname{diag}\left(\Delta x^k\right) + \bar{\mu}\bar{\Sigma}\right)\left(z^k + \Delta z^k\right)\\&\quad = \left(X_k + \bar{\mu}\bar{\Sigma}\right)\left(z^k + \Delta z^k\right) + Z_k \Delta x^k + \mathcal{O}\left(\left\|\Delta w^k\right\|^2\right)\\&\quad = \bar{\mu}\bar{\pi}\bar{\Sigma}\bar{\nu} + o\left(\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)^2\right)\\&\quad \in \bar{\mu}\bar{\pi}\bar{\Sigma}\bar{\nu}\left[\kappa_z^{-1}, \kappa_z\right]\end{aligned}$$

and

$$\begin{aligned}&\left(X_k + \operatorname{diag}\left(\widehat{\Delta x}^k\right) + \bar{\mu}\bar{\Sigma}\right)\left(z^k + \Delta z^k\right)\\&\quad = \left(X_k + \bar{\mu}\bar{\Sigma}\right)\left(z^k + \Delta z^k\right) + Z_k \widehat{\Delta x}^k + \mathcal{O}\left(\left\|\widehat{\Delta w}^k\right\|\left\|\Delta w^k\right\|\right)\\&\quad = \left(X_k + \bar{\mu}\bar{\Sigma}\right)\left(z^k + \Delta z^k\right) + Z_k \Delta x^k + \mathcal{O}\left(\left\|\Delta w^k\right\|^2\right)\\&\quad \in \bar{\mu}\bar{\pi}\bar{\Sigma}\bar{\nu}\left[\kappa_z^{-1}, \kappa_z\right].\end{aligned}$$

□

In the following result, it will be shown that eventually the condition for an update of λ^k in the following iteration is fulfilled if a full (second-order-correction) step is applied. This also implies that a full step is accepted in Step L-9.

Lemma 4.30. *Suppose Assumptions 4.27 hold. Let $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ be sufficiently small, $\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$, $\left\|\lambda^k - \lambda^{k-1}\right\| = o(1)$ and λ^k be updated in Step L-5.1. Then for k sufficiently large, the following inequalities hold:*

i. $\left\|\Phi_{pen}\left(w^k + \Delta w^k; \rho^k\right)\right\| \le \varepsilon_{\rho,k+1}$

ii. $\left\|\Phi_{pen}\left(w^k + \widehat{\Delta w}^k; \rho^k\right)\right\| \le \varepsilon_{\rho,k+1}$

iii. $\left\|\Phi_{bar}\left(w^k + \Delta w^k; \rho^k\right)\right\| \le \varepsilon_{\lambda,k+1}$

iv. $\left\|\Phi_{bar}\left(w^k + \widehat{\Delta w}^k; \rho^k\right)\right\| \le \varepsilon_{\lambda,k+1}$

v. $\left\|y^k + \Delta y^k - \bar{\pi}\lambda^k\right\|_2 \le \kappa_\lambda \bar{\tau}$

vi. $\left\|y^k + \widehat{\Delta y}^k - \bar{\pi}\lambda^k\right\|_2 \le \kappa_\lambda \bar{\tau}$

Proof. First, the update of λ^k in Step L-5.1 implies that $\lambda^k = y^k/\bar{\pi}$ and (4.31), (4.34a) and (4.35) hold. Because $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ is sufficiently small and $\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ holds, it follows $w^k \in \mathcal{N}(w^*)$. Together with Lemma 4.28,

$$\left\|y^k + \Delta y^k - \bar{\pi}\lambda^k\right\|_2 = \left\|\Delta y^k\right\|_2 = \Theta\left(\left\|w^k - w(\bar{\mu}, \bar{\nu})\right\|\right) = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right) \le \kappa_\lambda \bar{\tau}$$

holds, which proves (v) for k sufficiently large and which can analogously be shown for (vi).

Without loss of generality, it is assumed that $\bar{\pi} = 1$ in the following. By Taylor's theorem, Lemma 4.28 together with $\|w^k - w(\bar{\mu}, \bar{\nu})\| = o(\bar{\mu}\bar{\pi}\,\|\bar{\Sigma}\bar{\nu}\|)$ and (4.11) it follows

$$\begin{aligned}
&\nabla f(x^k + \Delta x^k) + \nabla g(x^k + \Delta x^k)(y^k + \Delta y^k) - (z^k + \Delta z^k) \\
&\quad = \nabla f(x^k) + \nabla g(x^k) y^k - z^k + \nabla^2_{xx} L(x^k, y^k, z^k)\Delta x^k + \nabla g(x^k)\Delta y^k - \Delta z^k \\
&\qquad + \mathcal{O}\big(\|\Delta w^k\|^2\big) \\
&\quad \overset{(4.11)}{=} \mathcal{O}\big(\|\Delta w^k\|^2\big),
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
&g(x^k + \Delta x^k) - \frac{\|g(x^k + \Delta x^k)\|_2}{\bar{\tau}}(y^k + \Delta y^k - \lambda^k) \\
&\quad \le g(x^k) + \nabla g(x^k)^\top \Delta x^k - \frac{\|g(x^k)\|_2}{\bar{\tau}}\Delta y^k + \mathcal{O}\big(\|\Delta w^k\|^2\big) \\
&\quad \overset{(4.11)}{=} \mathcal{O}\big(\|\Delta w^k\|^2\big),
\end{aligned} \tag{4.68}$$

$$\begin{aligned}
&g(x^k + \Delta x^k) \\
&\quad = g(x^k) + \nabla g(x^k)^\top \Delta x^k + \mathcal{O}\big(\|\Delta x^k\|^2\big) \\
&\quad \overset{(4.11)}{=} \frac{g(x^k)}{\bar{\tau}}\Delta y^k + \mathcal{O}\big(\|\Delta x^k\|^2\big) \\
&\quad = o\big(\|g(x^k)\|\big) + \mathcal{O}\big(\|\Delta w^k\|^2\big)
\end{aligned} \tag{4.69}$$

and

$$\begin{aligned}
&\big(X_k + \operatorname{diag}(\Delta x^k)\big)(z^k + \Delta z^k) - \bar{\mu}\bar{\Sigma}(\bar{\nu} - z^k - \Delta z^k) \\
&\quad = X_k z^k - \bar{\mu}\bar{\Sigma}(\bar{\nu} - z^k) + Z_k \Delta x^k + (X_k + \bar{\mu}\bar{\Sigma})\Delta z^k + \mathcal{O}\big(\|\Delta w^k\|^2\big) \\
&\quad \overset{(4.11)}{=} \mathcal{O}\big(\|\Delta w^k\|^2\big).
\end{aligned} \tag{4.70}$$

The step $\|\Delta w^k\|$ can further be estimated by

$$\begin{aligned}
\|\Delta w^k\| &\overset{(4.60)}{=} \mathcal{O}\left(\left\|\begin{bmatrix} \nabla f(x^k) + \nabla g(x^k) y^k - z^k \\ g(x^k) \\ X_k z^k - \bar{\mu}\bar{\Sigma}(\bar{\nu} - z^k) \end{bmatrix}\right\|\right) \\
&= \mathcal{O}\big(\|\Phi_{\text{bar}}(w^k; \rho^k)\|\big) \\
&= \mathcal{O}\big(\|\Phi_{\text{bar}}(w^k; \rho^{k-1})\|\big),
\end{aligned} \tag{4.71}$$

since ρ^k equals ρ^{k-1} and $\bar{\rho}$ except for λ^k, λ^{k-1} or $\bar{\lambda}$, respectively, which do not appear in the equation above. Subsequently, using (4.32), (4.38) and $\|\lambda^k - \lambda^{k-1}\| = o(1)$ it follows

$$\begin{aligned}
\|\Phi_{\text{bar}}(w^k; \rho^{k-1})\| &\overset{(4.32)}{\le} \|\Phi_{\text{pen}}(w^k; \rho^{k-1})\| + \frac{\|g(x^k)\|_2}{\bar{\tau}}\|y^k - \lambda^{k-1}\|_\infty \\
&\le \|\Phi_{\text{pen}}(w^k; \rho^{k-1})\| + o\big(\|\Phi_{\text{bar}}(w^k; \rho^{k-1})\|\big)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(4.38a)}{\leq} \mathcal{O}\big(\varepsilon_{\rho,k+1}\big) - \frac{\upsilon_2}{\upsilon_1}\varepsilon_{\lambda,k+1} + o\big(\big\|\Phi_{\text{bar}}\big(w^k;\rho^{k-1}\big)\big\|\big) \\
&\overset{(4.38b)}{\leq} \mathcal{O}\big(\varepsilon_{\rho,k+1}\big) - \frac{\upsilon_2}{\upsilon_1}\varepsilon_{\lambda,k+1} + o\big(\varepsilon_{\lambda,k+1}\big) \\
&\leq \mathcal{O}\big(\varepsilon_{\rho,k+1}\big)
\end{aligned} \tag{4.72}
$$

for k large enough. Then, by combining (4.67), (4.69) and (4.70) with (4.38), (4.71) and Lemma 4.28 yields

$$
\begin{aligned}
\big\|\Phi_{\text{bar}}\big(w^k + \Delta w^k;\rho^k\big)\big\| &= o\big(\big\|g\big(x^k\big)\big\|\big) + \mathcal{O}\big(\big\|\Delta w^k\big\|^2\big) \\
&= o\big(\big\|\Phi_{\text{bar}}\big(w^k;\rho^{k-1}\big)\big\|\big) + o\big(\big\|\Delta w^k\big\|\big) \\
&\overset{(4.71)}{=} o\big(\big\|\Phi_{\text{bar}}\big(w^k;\rho^{k-1}\big)\big\|\big) \\
&\overset{(4.38b)}{\leq} o\big(\varepsilon_{\lambda,k+1}\big) \\
&\leq \varepsilon_{\lambda,k+1}
\end{aligned} \tag{4.73}
$$

for k sufficiently large, which proves (iii). Similarly, by combining (4.67), (4.68) and (4.70) with (4.71), (4.72) and Lemma 4.28 yields

$$
\begin{aligned}
\big\|\Phi_{\text{pen}}\big(w^k + \Delta w^k;\rho^k\big)\big\| &\leq \mathcal{O}\big(\big\|\Delta w^k\big\|^2\big) \\
&= o\big(\big\|\Delta w^k\big\|\big) \\
&\overset{(4.71)}{=} o\big(\big\|\Phi_{\text{bar}}\big(w^k;\rho^{k-1}\big)\big\|\big) \\
&\overset{(4.72)}{=} o\big(\varepsilon_{\rho,k+1}\big) \\
&\leq \varepsilon_{\rho,k+1},
\end{aligned} \tag{4.74}
$$

for k sufficiently large, which proves (i).

For the step $w^k + \widehat{\Delta w}^k$ the equations (4.67) and (4.70) hold equally, but (4.68) and (4.69) differ. These can be estimated by using Taylor's theorem, Lemma 4.28 and (4.22), which give

$$
\begin{aligned}
&g\Big(x^k + \widehat{\Delta x}^k\Big) - \frac{\Big\|g\Big(x^k + \widehat{\Delta x}^k\Big)\Big\|_2}{\bar{\tau}}\Big(y^k + \widehat{\Delta y}^k - \lambda^k\Big) \\
&\leq g\big(x^k\big) + \nabla g\big(x^k\big)^\top \widehat{\Delta x}^k - \frac{\big\|g\big(x^k\big)\big\|_2}{\bar{\tau}}\widehat{\Delta y}^k + \mathcal{O}\Big(\Big\|\widehat{\Delta w}^k\Big\|^2\Big) \\
&\overset{(4.22)}{=} g\big(x^k\big) - g\big(x^k + \alpha_{\max,k}\Delta x^k\big) + \alpha_{\max,k}\nabla g\big(x^k\big)^\top \Delta x^k + \mathcal{O}\Big(\Big\|\widehat{\Delta w}^k\Big\|^2\Big) \\
&= \mathcal{O}\Big(\Big\|\widehat{\Delta w}^k\Big\|^2\Big) + \mathcal{O}\big(\big\|\Delta w^k\big\|^2\big) \\
&= \mathcal{O}\big(\big\|\Delta w^k\big\|^2\big)
\end{aligned}
$$

and

$$\begin{aligned}
g\left(x^k+\widehat{\Delta x}^k\right) &\leq g\left(x^k\right)+\nabla g\left(x^k\right)^\top \widehat{\Delta x}^k+\mathcal{O}\left(\left\|\widehat{\Delta w}^k\right\|^2\right) \\
&\overset{(4.22)}{=} g\left(x^k\right)-g\left(x^k+\alpha_{\max,k}\Delta x^k\right)+\alpha_{\max,k}\nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad+\frac{\left\|g\left(x^k\right)\right\|}{\bar{\tau}}\widehat{\Delta y}^k+\mathcal{O}\left(\left\|\widehat{\Delta w}^k\right\|^2\right) \\
&= o\left(\left\|g\left(x^k\right)\right\|\right)+\mathcal{O}\left(\left\|\widehat{\Delta w}^k\right\|^2\right)+\mathcal{O}\left(\left\|\Delta w^k\right\|^2\right) \\
&= o\left(\left\|g\left(x^k\right)\right\|\right)+\mathcal{O}\left(\left\|\Delta w^k\right\|^2\right).
\end{aligned}$$

Then, analogously to (4.73) and (4.74) it follows (ii) and (iv) for k sufficiently large. □

Combining the results of Lemma 4.28, Lemma 4.29 and Lemma 4.30 shows the fast local convergence for Algorithm L towards a solution of (NLPbar).

Theorem 4.31. *Suppose the Assumptions 4.27 hold and Algorithm L with disabled steps Step L-4 and Step L-6 generates an infinite sequence of iterates $w^k=\left(x^k,y^k,z^k\right)$ with an accumulation point $w(\bar{\mu},\bar{\nu})=(x(\bar{\mu},\bar{\nu}),y(\bar{\mu},\bar{\nu}),z(\bar{\mu},\bar{\nu}))$. Assume that $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ is sufficiently small, $w(\bar{\mu},\bar{\nu})$ is close to w^* and $\pi_k=\bar{\pi}$ for k large enough. If $\left\|w^k-w(\bar{\mu},\bar{\nu})\right\|=o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$, then $\left\{w^k\right\}_k$ converges q-quadratically to $w(\bar{\mu},\bar{\nu})$, i.e., $\left\|w^{k+1}-w(\bar{\mu},\bar{\nu})\right\|=\mathcal{O}\left(\left\|w^k-w(\bar{\mu},\bar{\nu})\right\|^2\right)$.*

Proof. Because $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ is sufficiently small and $\left\|w^k-w(\bar{\mu},\bar{\nu})\right\|=o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ holds, Assumptions 4.27 imply $Q_k=\bar{\pi}\nabla^2_{xx}L\left(x^k,y^k/\bar{\pi},z^k/\bar{\pi}\right)$, i.e., no primal regularization is applied in Step L-7 and $\delta_{\mathrm{p}}=0$. Then, (4.60), (4.11) and (4.32) yield

$$\begin{aligned}
\left\|\Phi_{\mathrm{pen}}\left(w^k;\rho^{k-1}\right)\right\| &\leq \left\|\begin{bmatrix}\bar{\pi}\nabla f\left(x^k\right)+\nabla g\left(x^k\right)y^k-z^k \\ g\left(x^k\right)+\frac{\left\|g\left(x^k\right)\right\|}{\bar{\tau}}\left(\bar{\pi}\lambda^k-y^k\right) \\ X_k z^k-\bar{\mu}\bar{\Sigma}\left(\bar{\pi}\bar{\nu}-z^k\right)\end{bmatrix}\right\|+\bar{\pi}\frac{\left\|g\left(x^k\right)\right\|}{\bar{\tau}}\left\|\lambda^k-\lambda^{k-1}\right\| \\
&=\mathcal{O}\left(\left\|\Delta w^k\right\|\right)+\mathcal{O}\left(\left\|\lambda^k-\lambda^{k-1}\right\|\right) \\
&=o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)+\mathcal{O}\left(\left\|\lambda^k-\lambda^{k-1}\right\|\right).
\end{aligned} \tag{4.75}$$

In the following it is shown by induction that the Lagrangian multipliers λ^k are updated in every iteration $k\geq k_0>0$. The induction base case follows from (4.75) since for $k=k_0$ large enough it finally yields $\left\|\Phi_{\mathrm{pen}}\left(w^k;\rho^{k-1}\right)\right\|\leq\varepsilon_{\rho,k}$. Because π_k is updated finitely many times by assumption, (4.34a) and (4.35) must hold, which would lead to an update of λ^k. So assume that λ^k is updated in iteration k. Then, Lemma 4.29 implies that the fraction-to-the-boundary rules (4.15), (4.28), (4.23), (4.17a) and (4.24a) accept the full step size $\alpha_{\max,k}=1$ and $\widehat{\alpha}_k=1$, respectively. Thus, $\left(y^k,z^k\right)+\left(\Delta y^k,\Delta z^k\right)$ is the next dual iterate (cf., (4.27)). From Lemma 4.30, it follows that either $x^k+\Delta x^k$ or $x^k+\widehat{\Delta x}^k$ is accepted as next primal iterate. Subsequently, the Lagrangian multipliers are again updated in the next step, since (4.31), (4.34a) and (4.35) are satisfied by Lemma 4.30.

Finally, because λ^k is updated in every iteration for k sufficiently large, Lemma 4.28 implies the quadratic convergence order. □

It turns out, that for the fast local convergence in Theorem 4.31 neither the filter, the PLPF nor the merit function – in particular not the non-monotonicity of it – have been used. Furthermore, the second-order-correction steps have not been necessary so far. However, the step acceptance proved in Lemma 4.30 (cf., Step L-9) is not a good choice for a practical algorithm as it involves computationally expensive gradient and Jacobian evaluations. Proving fast local convergence based on the non-monotonicity of either the filter, PLPF or merit function would be beneficial, but is left as future research. Instead, it is proven in the following that an Armijo condition based on the primal merit function $\Upsilon\left(x^k;\rho^k\right)$ would always be accepted for the second-order-correction step close to the optimal solution and later in Section 5.2.4 it is motivated that this is sufficient in practice.

Proposition 4.32. *Suppose the Assumptions 4.27 hold. Let* $\|y(\bar{\mu},\bar{\nu})/\bar{\pi}\| \le c\bar{\tau}$ *for a* $c \in (0,1)$, $\sigma \in \left(0,\frac{1}{2}\right)$ *and assume the term* $\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|$ *to be sufficiently small,* $\left\|w^k - w(\bar{\mu},\bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ *and* λ^k *to be updated in Step L-5.1. Then,*

$$\Upsilon\left(x^k + \widehat{\Delta x}^k;\rho^k\right) - \Upsilon\left(x^k;\rho^k\right) \le \sigma D^x_{\Delta x^k}\Upsilon\left(x^k;\rho^k\right).$$

Proof. The proof is based on Chen and Goldfarb [43, Theorem 5.4] but extended to the modified barrier and augmented Lagrangian approach. Like in the proofs before, it is assumed without loss of generality, that $\bar{\pi} = 1$.

In the first part of the proof $\varphi_{\text{bar}}\left(x^k + \widehat{\Delta x}^k;\rho^k\right) + g\left(x^k + \widehat{\Delta x}^k\right)^\top \lambda^k$ will be estimated. Note, that $x^k > -\bar{\mu}\bar{\varsigma}$, $x^k + \widehat{\Delta x}^k > -\bar{\mu}\bar{\varsigma}$ due to (4.62) and Lemma 4.29. Then it follows from Taylor's theorem that

$$\begin{aligned}
&\varphi_{\text{bar}}\left(x^k + \widehat{\Delta x}^k;\rho^k\right) + g\left(x^k + \widehat{\Delta x}^k\right)^\top \lambda^k \\
&\quad = \varphi_{\text{bar}}\left(x^k;\rho^k\right) + g\left(x^k\right)^\top\lambda^k + \left(\nabla_x\varphi_{\text{bar}}\left(x^k;\rho^k\right) + \nabla g\left(x^k\right)\lambda^k\right)^\top \widehat{\Delta x}^k \\
&\qquad + \frac{1}{2}\left(\widehat{\Delta x}^k\right)^\top\left(\nabla^2 f\left(x^k + \xi\widehat{\Delta x}^k\right) + \sum_{i=1}^{n_g}\lambda_i^k\nabla^2 g_i\left(x^k + \xi\widehat{\Delta x}^k\right)\right)\widehat{\Delta x}^k \\
&\qquad + \frac{\bar{\mu}\bar{\Sigma}}{2}\left(\widehat{\Delta x}^k\right)^\top \operatorname{diag}\left(x^k + \xi\widehat{\Delta x}^k + \bar{\mu}\bar{\varsigma}\right)^{-2}\operatorname{diag}(\bar{\nu})\widehat{\Delta x}^k
\end{aligned} \tag{4.76}$$

for $\xi \in [0,1]$. By Lemma 4.28 $\left\|\widehat{\Delta x}^k\right\| = \mathcal{O}\left(\left\|w^k - w(\bar{\mu},\bar{\nu})\right\|\right) = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ holds and thus together with (4.62) $\frac{\left\|\widehat{\Delta x}^k\right\|}{x_i^k + \bar{\mu}\bar{\varsigma}_i} = o(1)$ for $i = 1,\ldots,n_x$. This implies

$$\begin{aligned}
\frac{\widehat{\Delta x}_i^k}{x_i^k + \xi\widehat{\Delta x}_i^k + \bar{\mu}\bar{\varsigma}_i} &= \frac{\widehat{\Delta x}_i^k}{x_i^k + \bar{\mu}\bar{\varsigma}_i}\left(1 - \frac{\xi\widehat{\Delta x}_i^k/\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)}{1 + \xi\widehat{\Delta x}_i^k/\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)}\right) \\
&= \frac{\widehat{\Delta x}_i^k}{x_i^k + \bar{\mu}\bar{\varsigma}_i} + \widehat{\Delta x}_i^k\,\mathcal{O}\left(\frac{\left\|\widehat{\Delta x}^k\right\|}{\left(x_i^k + \bar{\mu}\right)^2}\right) \\
&= \frac{\Delta x_i^k}{x_i^k + \bar{\mu}\bar{\varsigma}_i} + \Delta x_i^k\,\mathcal{O}\left(\frac{\left\|\Delta x^k\right\|}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) + \mathcal{O}\left(\frac{\left\|\Delta x^k\right\|^2}{x_i^k + \bar{\mu}\bar{\varsigma}_i}\right)
\end{aligned}$$

and subsequently

$$
\begin{aligned}
&\left(\frac{\widehat{\Delta x}_i^k}{x_i^k + \xi \widehat{\Delta x}_i^k + \bar{\mu}\bar{\varsigma}_i}\right)^2 \\
&\quad = \left(\frac{\Delta x_i^k}{x_i^k + \bar{\mu}\bar{\varsigma}_i} + \Delta x_i^k \mathcal{O}\left(\frac{\|\Delta x^k\|}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^2}{x_i^k + \bar{\mu}\bar{\varsigma}_i}\right)\right)^2 \\
&\quad = \frac{\left(\Delta x_i^k\right)^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2} + \left(\Delta x_i^k\right)^2 \mathcal{O}\left(\frac{\|\Delta x^k\|}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^3}\right) + \Delta x_i^k \mathcal{O}\left(\frac{\|\Delta x^k\|^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) \\
&\qquad + \left(\Delta x_i^k\right)^2 \mathcal{O}\left(\frac{\|\Delta x^k\|^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^4}\right) + \Delta x_i^k \mathcal{O}\left(\frac{\|\Delta x^k\|^3}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^3}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^4}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) \\
&\quad = \frac{\left(\Delta x_i^k\right)^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2} + \left(\Delta x_i^k\right)^2 \left(\mathcal{O}\left(\frac{\|\Delta x^k\|}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^3}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^4}\right)\right) \\
&\qquad + \mathcal{O}\left(\frac{\|\Delta x^k\|^3}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^4}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^3}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^4}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) \\
&\quad = \frac{\left(\Delta x_i^k\right)^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2} + \left(\Delta x_i^k\right)^2 \mathcal{O}\left(\frac{\|\Delta x^k\|}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^3}\right) + \mathcal{O}\left(\frac{\|\Delta x^k\|^3}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2}\right) \\
&\quad = \frac{\left(\Delta x_i^k\right)^2}{\left(x_i^k + \bar{\mu}\bar{\varsigma}_i\right)^2} + \left(\Delta x_i^k\right)^2 o\left(\frac{1}{\left(\bar{\mu}\|\bar{\Sigma}\bar{\nu}\|\right)^2}\right) + o\left(\frac{\|\Delta x^k\|^2}{\bar{\mu}\|\bar{\Sigma}\bar{\nu}\|}\right)
\end{aligned} \tag{4.77}
$$

Furthermore, it holds that

$$
\begin{aligned}
\left(\widehat{\Delta x}^k\right)^\top \nabla^2 f\left(x^k + \xi \widehat{\Delta x}^k\right)\widehat{\Delta x}^k &= \left(\widehat{\Delta x}^k\right)^\top \nabla^2 f\left(x^k\right)\widehat{\Delta x}^k + \mathcal{O}\left(\left\|\widehat{\Delta x}^k\right\|^3\right) \\
&= \left(\Delta x^k\right)^\top \nabla^2 f\left(x^k\right)\Delta x^k + \mathcal{O}\left(\|\Delta x^k\|^3\right),
\end{aligned} \tag{4.78a}
$$

$$
\begin{aligned}
\left(\widehat{\Delta x}^k\right)^\top \lambda_i^k \nabla^2 g_i\left(x^k + \xi \widehat{\Delta x}^k\right)\widehat{\Delta x}^k &= \left(\widehat{\Delta x}^k\right)^\top \lambda_i^k \nabla^2 g_i\left(x^k\right)\widehat{\Delta x}^k + \mathcal{O}\left(\left\|\widehat{\Delta x}^k\right\|^3\right) \\
&= \left(\Delta x^k\right)^\top \lambda_i^k \nabla^2 g_i\left(x^k\right)\Delta x^k + \mathcal{O}\left(\|\Delta x^k\|^3\right).
\end{aligned} \tag{4.78b}
$$

with $i \in \{1, \ldots, n_g\}$ due to Assumption 4.27 (i). Next, an estimate for $\nabla_x \varphi_{\mathrm{bar}}\left(x^k; \rho^k\right)^\top \widehat{\Delta x}^k$ in (4.76) will be derived. Let $\bar{\sigma} \in \left(0, \frac{1}{2} - \sigma\right)$ be a constant. Then, using (4.12) and Lemma 4.28 it follows

$$
\begin{aligned}
&\left(\nabla_x \varphi_{\mathrm{bar}}\left(x^k; \rho^k\right) + \nabla g\left(x^k\right)\lambda^k\right)^\top \widehat{\Delta x}^k \\
&\quad = \left(\frac{1}{2} - \bar{\sigma}\right)\left(\nabla_x \varphi_{\mathrm{bar}}\left(x^k; \rho^k\right) + \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k \\
&\qquad + \left(\frac{1}{2} + \bar{\sigma}\right)\left(\nabla_x \varphi_{\mathrm{bar}}\left(x^k; \rho^k\right) + \nabla g\left(x^k\right)\lambda^k\right)^\top \Delta x^k \\
&\qquad - \left(\nabla_x \varphi_{\mathrm{bar}}\left(x^k; \rho^k\right) + \nabla g\left(x^k\right)\lambda^k\right)^\top \left(\Delta x^k - \widehat{\Delta x}^k\right)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4.12)}{=} \left(\frac{1}{2}-\bar{\sigma}\right)\left(\nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)+\nabla g\left(x^k\right)y^k\right)^\top \Delta x^k \\
&\quad -\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta y^k\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad -\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta x^k\right)^\top \left(Q_k+(X_k+\bar{\mu}E)^{-1} Z_k\right)\Delta x^k \\
&\quad +\left(\left(\Delta x^k\right)^\top \left(Q_k+(X_k+\bar{\mu}E)^{-1} Z_k\right)+\left(\Delta y^k\right)^\top \nabla g\left(x^k\right)^\top\right)\left(\Delta x^k-\widehat{\Delta x}^k\right) \\
&= \left(\frac{1}{2}-\bar{\sigma}\right)\left(\nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)+\nabla g\left(x^k\right)y^k\right)^\top \Delta x^k \\
&\quad -\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta y^k\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad -\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta x^k\right)^\top \left(Q_k+(X_k+\bar{\mu}E)^{-1} Z_k\right)\Delta x^k+o\left(\left\|\Delta x^k\right\|^2\right) \qquad (4.79)
\end{aligned}
$$

where in the second equality $\lambda^k = y^k$ has been used. The second term in (4.79) is zero if $\left\|g\left(x^k\right)\right\| = 0$ because $\lambda^k = y^k$ and thus $\nabla g\left(x^k\right)^\top \Delta x^k = 0$ holds (cf., (4.12)). In the case $\left\|g\left(x^k\right)\right\| > 0$ it can be estimated by

$$
\begin{aligned}
&\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta y^k\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\overset{(4.12)}{=} \frac{1}{2}\left(\Delta y^k\right)^\top \left(\frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\Delta y^k-g\left(x^k\right)\right) \\
&\quad +\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2}\left(\nabla g\left(x^k\right)^\top \Delta x^k+g\left(x^k\right)\right)^\top \nabla g\left(x^k\right)^\top \Delta x^k \\
&\overset{(4.12)}{=} -\frac{1}{2}\left(\Delta y^k\right)^\top g\left(x^k\right)+\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2}\left(\Delta x^k\right)^\top \nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad +\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2} g\left(x^k\right)^\top \left(\frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\Delta y^k-g\left(x^k\right)\right)+o\left(\left\|g\left(x^k\right)\right\|\right) \\
&= -\frac{1}{2}\left(\Delta y^k\right)^\top g\left(x^k\right)+\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2}\left(\Delta x^k\right)^\top \nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top \Delta x^k \\
&\quad -\bar{\sigma}\bar{\tau}\left\|g\left(x^k\right)\right\|_2+o\left(\left\|g\left(x^k\right)\right\|\right) \qquad (4.80)
\end{aligned}
$$

In the following the two cases $\left\|g\left(x^k\right)\right\| > 0$ and $\left\|g\left(x^k\right)\right\| = 0$ are considered separately. So first assume $\left\|g\left(x^k\right)\right\| > 0$. Summarizing the above using (4.62), (4.76), (4.77), (4.78), (4.79) and (4.80), it follows for the estimation of $\varphi_{\text{bar}}\left(x^k+\widehat{\Delta x}^k;\rho^k\right)+g\left(x^k+\widehat{\Delta x}^k\right)^\top \lambda^k$ that

$$
\begin{aligned}
&\varphi_{\text{bar}}\left(x^k+\widehat{\Delta x}^k;\rho^k\right)+g\left(x^k+\widehat{\Delta x}^k\right)^\top y^k-\varphi_{\text{bar}}\left(x^k;\rho^k\right)-g\left(x^k\right)^\top y^k \\
&\le \left(\frac{1}{2}-\bar{\sigma}\right)\left(\nabla_x \varphi_{\text{bar}}\left(x^k;\rho^k\right)+\nabla g\left(x^k\right)y^k\right)^\top \Delta x^k+\frac{1}{2}\left(\Delta y^k\right)^\top g\left(x^k\right) \\
&\quad -\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2}\left(\Delta x^k\right)^\top \nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top \Delta x^k+\bar{\sigma}\bar{\tau}\left\|g\left(x^k\right)\right\|_2
\end{aligned}
$$

$$\begin{aligned}
&-\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta x^k\right)^\top \left(Q_k + \left(X_k+\bar{\mu}\bar{\Sigma}\right)^{-1} Z_k\right)\Delta x^k \\
&+\frac{1}{2}\left(\Delta x^k\right)^\top\left(\nabla^2 f\left(x^k\right)+\sum_{i=1}^{n_g} y_i^k \nabla^2 g_i\left(x^k\right)\right)\Delta x^k \\
&+\frac{1}{2}\left(\Delta x^k\right)^\top \sum_{i=1}^{n_x}\left(\left(\frac{\bar{\mu}\bar{\varsigma}_i\bar{\nu}_i}{\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right)^2}+o\left(\frac{1}{\bar{\mu}\left\|\bar{\Sigma}\bar{\nu}\right\|}\right)\right)e_i e_i{}^\top\right)\Delta x^k + o\left(\left\|\Delta x^k\right\|^2\right). \quad (4.81)
\end{aligned}$$

Next, the terms that are quadratic with respect to the step Δx^k will be estimated. Because of Assumption 4.27 (iv) and $\left\|w^k - w(\bar{\mu},\bar{\nu})\right\| = o\left(\bar{\mu}\bar{\pi}\left\|\bar{\Sigma}\bar{\nu}\right\|\right)$ the term $\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i}$ is sufficiently large for $i \in \mathscr{A}(x^*)$. Together with Assumption 4.27 (iii) it follows that

$$\left(\Delta x^k\right)^\top\left(Q_k + \frac{1}{2}\sum_{i\in\mathscr{A}(x^*)}\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i}e_i e_i{}^\top + \frac{\bar{\tau}}{\|g(x^k)\|_2}\nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top\right)\Delta x^k > 0. \quad (4.82)$$

In addition, (4.62) yields

$$\begin{aligned}
z_i^k - \frac{\bar{\mu}\bar{\varsigma}_i\bar{\nu}_i}{x_i^k+\bar{\mu}\bar{\varsigma}_i} &= \frac{1}{x_i^k+\bar{\mu}\bar{\varsigma}_i}\left(z_i^k\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right) - z_i(\bar{\mu},\bar{\nu})\left(x_i(\bar{\mu},\bar{\nu})+\bar{\mu}\bar{\varsigma}_i\right)\right) \\
&= \frac{1}{x_i^k+\bar{\mu}\bar{\varsigma}_i}\left(z_i^k\left(x_i^k - x_i(\bar{\mu},\bar{\nu})\right) + \left(x_i(\bar{\mu},\bar{\nu})+\bar{\mu}\bar{\varsigma}_i\right)\left(z_i^k - z_i(\bar{\mu},\bar{\nu})\right)\right) \\
&= \mathscr{O}\left(\frac{\left\|w^k - w(\bar{\mu},\bar{\nu})\right\|}{x_i^k+\bar{\mu}\bar{\varsigma}_i}\right) \\
&= o(1) \qquad (4.83)
\end{aligned}$$

for $i \in \mathscr{A}(x^*)$ and thus

$$\left(\frac{1}{2}+\frac{\bar{\sigma}}{2}\right)\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i} - \frac{\bar{\mu}\bar{\varsigma}_i\bar{\nu}_i}{2\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right)^2} = \frac{\bar{\sigma}z_i^k}{2\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right)} + o(1) > 0. \quad (4.84)$$

Combining (4.82) and (4.84) then leads to

$$\begin{aligned}
&-\frac{\bar{\sigma}\bar{\tau}}{\|g(x^k)\|_2}\left(\Delta x^k\right)^\top \nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top \Delta x^k \\
&-\left(\frac{1}{2}+\bar{\sigma}\right)\left(\Delta x^k\right)^\top \left(Q_k + \left(X_k+\bar{\mu}\bar{\Sigma}\right)^{-1} Z_k\right)\Delta x^k \\
&+\frac{1}{2}\left(\Delta x^k\right)^\top\left(\nabla^2 f\left(x^k\right)+\sum_{i=1}^{n_g} y_i^k \nabla^2 g_i\left(x^k\right)\right)\Delta x^k
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\Delta x^k\right)^\top \sum_{i=1}^{n_x}\left(\left(\frac{\bar{\mu}\bar{\varsigma}_i\bar{\nu}_i}{\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right)^2}+o\left(\frac{1}{\bar{\mu}\left\|\bar{\Sigma}\bar{\nu}\right\|}\right)\right)e_i{e_i}^\top\right)\Delta x^k \\
&=-\bar{\sigma}\left(\Delta x^k\right)^\top\left(Q_k+\frac{1}{2}\sum_{i\in\mathscr{A}(x^*)}\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i}e_i{e_i}^\top+\frac{\bar{\tau}}{\|g(x^k)\|_2}\nabla g\left(x^k\right)\nabla g\left(x^k\right)^\top\right)\Delta x^k \\
&\quad-\left(\Delta x^k\right)^\top\sum_{i\in\mathscr{A}(x^*)}\left(\left(\frac{1}{2}+\frac{\bar{\sigma}}{2}\right)\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i}-\frac{\bar{\mu}\bar{\varsigma}_i\bar{\nu}_i}{2\left(x_i^k+\bar{\mu}\bar{\varsigma}_i\right)^2}+o\left(\frac{1}{\bar{\mu}\left\|\bar{\Sigma}\bar{\nu}\right\|}\right)\right)\Delta x^k \\
&\quad-\left(\Delta x^k\right)^\top\frac{1+2\bar{\sigma}}{2}\sum_{i\in\mathscr{I}(x^*)}\frac{z_i^k}{x_i^k+\bar{\mu}\bar{\varsigma}_i}\Delta x^k<0,
\end{aligned}
\tag{4.85}
$$

where the fact $Q_k=\nabla^2 f\left(x^k\right)+\sum_{i=1}^{n_g}y_i^k\nabla^2 g_i\left(x^k\right)$ has been used. Then applying (4.85) to (4.81) yields

$$
\begin{aligned}
&\varphi_{\text{bar}}\left(x^k+\widehat{\Delta x}^k;\rho^k\right)+g\left(x^k+\widehat{\Delta x}^k\right)^\top y^k-\varphi_{\text{bar}}\left(x^k;\rho^k\right)-g\left(x^k\right)^\top y^k \\
&\overset{(4.85)}{\leq}\left(\frac{1}{2}-\bar{\sigma}\right)\left(\nabla_x\varphi_{\text{bar}}\left(x^k;\rho^k\right)+\nabla g\left(x^k\right)y^k\right)^\top\Delta x^k+\frac{1}{2}\left(\Delta y^k\right)^\top g\left(x^k\right) \\
&\qquad+\bar{\sigma}\bar{\tau}\left\|g\left(x^k\right)\right\|_2+o\left(\left\|\Delta x^k\right\|^2\right) \\
&=\quad\left(\frac{1}{2}-\bar{\sigma}\right)D_{\Delta x^k}^x\Upsilon\left(x^k;\rho^k\right)+\frac{1}{2}\left(\Delta y^k\right)^\top g\left(x^k\right)+\bar{\sigma}\bar{\tau}\left\|g\left(x^k\right)\right\|_2+o\left(\left\|\Delta x^k\right\|^2\right) \\
&\qquad+\left(\frac{1}{2}-\bar{\sigma}\right)\left(\left(y^k\right)^\top g\left(x^k\right)+\bar{\tau}\left\|g\left(x^k\right)\right\|_2-g\left(x^k\right)^\top\Delta y^k-\frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\left(y^k\right)^\top\Delta y^k\right) \\
&=\quad\left(\frac{1}{2}-\bar{\sigma}\right)D_{\Delta x^k}^x\Upsilon\left(x^k;\rho^k\right)+\frac{1}{2}\bar{\tau}\left\|g\left(x^k\right)\right\|_2+\left(\frac{1}{2}-\bar{\sigma}\right)\left(y^k\right)^\top g\left(x^k\right) \\
&\qquad+o\left(\left\|g\left(x^k\right)\right\|\right)+o\left(\left\|\Delta x^k\right\|^2\right)
\end{aligned}
\tag{4.86}
$$

In the following the result (4.86) is transferred to the primal merit function $\Upsilon\left(x^k;\rho^k\right)$. First note, that by Proposition 4.4, Assumption 4.27 (iii) and $\sigma\in\left(0,\frac{1}{2}\right)$ it follows for a ε that

$$
\begin{aligned}
&\left(\frac{1}{2}-\bar{\sigma}-\sigma\right)D_{\Delta x^k}^x\Upsilon\left(x^k;\rho^k\right)+o\left(\left\|\Delta x^k\right\|^2\right) \\
&\leq-\left(\frac{1}{2}-\bar{\sigma}-\sigma\right)\varepsilon\left\|\Delta x^k\right\|^2+o\left(\left\|\Delta x^k\right\|^2\right) \\
&\leq 0
\end{aligned}
\tag{4.87}
$$

In addition, by (4.34b) and the assumption that λ^k has been updated it holds that

$$
\begin{aligned}
\left(\frac{1}{2}-\bar{\sigma}\right)\left\|y^k\right\|\left\|g\left(x^k\right)\right\|-\frac{1}{2}\bar{\tau}\left\|g\left(x^k\right)\right\|&\leq\left(\frac{1}{2}-\bar{\sigma}\right)\bar{\tau}\left\|g\left(x^k\right)\right\|-\frac{1}{2}\bar{\tau}\left\|g\left(x^k\right)\right\| \\
&=-\bar{\sigma}\bar{\tau}\left\|g\left(x^k\right)\right\| \\
&<0.
\end{aligned}
\tag{4.88}
$$

For the constraint violation of a full second-order-correction step it follows by Taylor's theorem, Lemma 4.28, (4.21) and $\left\|w^k - w(\bar\mu, \bar\nu)\right\| = o\left(\bar\mu \left\|\bar\Sigma \bar\nu\right\|\right)$ that

$$\begin{aligned} g\left(x^k + \widehat{\Delta x}^k\right) &= g\left(x^k + \Delta x^k\right) + \nabla g\left(x^k + \widehat{\Delta x}^k\right)\left(\widehat{\Delta x}^k - \Delta x^k\right) + \mathcal{O}\left(\left\|\widehat{\Delta x}^k - \Delta x^k\right\|^2\right) \\ &= g\left(x^k + \Delta x^k\right) + \nabla g\left(x^k\right)\left(\widehat{\Delta x}^k - \Delta x^k\right) + \mathcal{O}\left(\left\|\Delta x^k\right\|^3\right) \\ &\overset{(4.21)}{=} \frac{\left\|g\left(x^k\right)\right\|_2}{\bar\tau}\widehat{\Delta y}^k + \mathcal{O}\left(\left\|\Delta x^k\right\|^3\right) \\ &= o\left(\left\|g\left(x^k\right)\right\|\right) + o\left(\left\|\Delta x^k\right\|^2\right). \end{aligned} \tag{4.89}$$

Finally, by combining (4.86), (4.87), (4.88) and (4.89) it follows that in case of $\left\|g\left(x^k\right)\right\| > 0$ a full second-order-correction step satisfies the Armijo condition of the primal merit function, i.e.,

$$\begin{aligned} &\Upsilon\left(x^k + \widehat{\Delta x}^k; \rho^k\right) - \Upsilon\left(x^k; \rho^k\right) \\ &\overset{(4.86)}{=} \left(\frac{1}{2} - \bar\sigma\right) D^x_{\Delta x^k}\Upsilon\left(x^k; \rho^k\right) + \frac{1}{2}\bar\tau\left\|g\left(x^k\right)\right\|_2 + \left(\frac{1}{2} - \bar\sigma\right)\left(y^k\right)^\top g\left(x^k\right) \\ &\quad + \bar\tau\left(\left\|g\left(x^k + \widehat{\Delta x}^k\right)\right\|_2 - \left\|g\left(x^k\right)\right\|_2\right) + o\left(\left\|g\left(x^k\right)\right\|\right) + o\left(\left\|\Delta x^k\right\|^2\right) \\ &\overset{(4.89)}{=} \left(\frac{1}{2} - \bar\sigma\right) D^x_{\Delta x^k}\Upsilon\left(x^k; \rho^k\right) - \frac{1}{2}\bar\tau\left\|g\left(x^k\right)\right\|_2 + \left(\frac{1}{2} - \bar\sigma\right)\left(y^k\right)^\top g\left(x^k\right) \\ &\quad + o\left(\left\|g\left(x^k\right)\right\|\right) + o\left(\left\|\Delta x^k\right\|^2\right) \\ &\overset{(4.88)}{\le} \left(\frac{1}{2} - \bar\sigma\right) D^x_{\Delta x^k}\Upsilon\left(x^k; \rho^k\right) + o\left(\left\|\Delta x^k\right\|^2\right) \\ &\overset{(4.87)}{\le} \sigma D^x_{\Delta x^k}\Upsilon\left(x^k; \rho^k\right) \end{aligned}$$

This proves the hypothesis of the lemma for the case $\left\|g\left(x^k\right)\right\| > 0$. It remains to show it for $\left\|g\left(x^k\right)\right\| = 0$. First note, that by the same argument as in (4.82) it holds that

$$\left(\Delta x^k\right)^\top \left(Q_k + \frac{1}{2}\sum_{i \in \mathcal{A}(x^*)} \frac{z_i^k}{x_i^k + \bar\mu\varsigma_i} e_i e_i^\top\right)\Delta x^k. \tag{4.90}$$

Then, together with (4.83) and (4.84) the same outcome as in (4.85) can be shown. This yields an estimate for the barrier objective function to be

$$\begin{aligned} &\varphi_{\text{bar}}\left(x^k + \widehat{\Delta x}^k; \rho^k\right) + g\left(x^k + \widehat{\Delta x}^k\right)^\top y^k - \varphi_{\text{bar}}\left(x^k; \rho^k\right) - g\left(x^k\right)^\top y^k \\ &\quad \le \left(\frac{1}{2} - \bar\sigma\right)\left(\nabla_x \varphi_{\text{bar}}\left(x^k; \rho^k\right)\right)^\top \Delta x^k + o\left(\left\|\Delta x^k\right\|^2\right) \end{aligned}$$

Then, applying (4.87) and (4.89) yields the desired outcome. □

4.4.2 Local Convergence for the Barrier Subproblem

In this section the local convergence of Algorithm L towards an optimal solution $\omega^* = (x^*, \lambda^*, \nu^*)$ of (NLP+) is analyzed. From Theorem 4.26 two possible outcomes would be relevant. However, since the magic step is intentionally constructed to be very general, a local convergence analysis is not applicable and the case of infinitely many updates of the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ remains for the study. The following proposition yields the local convergence order.

Proposition 4.33. *Suppose Assumptions 4.27 hold. Let* $\omega^k \in \mathcal{N}(\omega^*)$ *and* $\left(\lambda^k, \nu^k\right) = \left(y^k/\bar{\pi}, z^k/\bar{\pi}\right)$*. Then the following equalities hold:*

i. $\left\|\omega^k + \Delta\omega^k - \omega^*\right\| = \mathcal{O}\left(\mu_k \left\|\omega^k - \omega^*\right\|\right)$

ii. $\left\|\omega^k + \widehat{\Delta\omega}^k - \omega^*\right\| = \mathcal{O}\left(\mu_k \left\|\omega^k - \omega^*\right\|\right)$

Proof. Without loss of generality it is assumed that $\bar{\pi} = 1$ and thus, $\omega^k = w^k$ and $\Delta\omega^k = \Delta w^k$. It follows from Assumption 4.27 (i), Taylor's theorem and the linear equation system (4.11) similar to (4.63), (4.64) and (4.65) of the proof of Lemma 4.28 that

$$M_k\left(\omega^k + \Delta\omega^k - \omega^*\right) = \begin{bmatrix} \nabla^2_{xx} L\left(x^k, \lambda^k, \nu^k\right)\left(x^k - x^*\right) - \nabla_x L\left(x^k, \lambda^*, \nu^*\right) \\ -g\left(x^k\right) + \nabla g\left(x^k\right)^\top \left(x^k - x^*\right) - \varrho_k\left(y^k - \lambda^*\right) \\ \left(X_k - X_*\right)\nu^k - \left(X_k + \mu_k \Sigma_k\right)\nu^* + \mu_k \Sigma_k \nu^k \end{bmatrix}$$

and hence

$$\nabla^2_{xx} L\left(x^k, \lambda^k, \nu^k\right)\left(x^k - x^*\right) - \nabla_x L\left(x^k, \lambda^*, \nu^*\right) = \mathcal{O}\left(\left\|\omega^k - \omega^*\right\|^2\right)$$

$$-g\left(x^k\right) + \nabla g\left(x^k\right)^\top \left(x^k - x^*\right) - \frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\left(\lambda^k - \lambda^*\right) = \mathcal{O}\left(\left\|\omega^k - \omega^*\right\|^2\right).$$

Moreover, due to $X_* \nu^* = 0$ it holds that

$$\begin{aligned} &\left(X_k - X_*\right)\nu^k - \left(X_k + \mu_k \Sigma_k\right)\nu^* + \mu_k \Sigma_k \nu^k \\ &\quad = \left(X_k - X_*\right)\left(\nu^k - \nu^*\right) + \mu_k \Sigma_k\left(\nu^k - \nu^*\right) \\ &\quad = \mathcal{O}\left(\left\|\omega^k - \omega^*\right\|^2\right) + \mathcal{O}\left(\mu_k \left\|\nu^k - \nu^*\right\|\right) \\ &\quad = \mathcal{O}\left(\mu_k \left\|\omega^k - \omega^*\right\|\right). \end{aligned}$$

This proves (i) and, together with Lemma 4.28 (v), it follows (ii). □

As a result of Proposition 4.33 Algorithm L would be q-superlinearly locally convergent if the barrier parameter μ_k would tend to zero. Otherwise, if it remains fixed at some threshold $\bar{\mu}$ the convergence order of the algorithm would be only linear. It remains to prove that the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ are indeed updated in every iteration and a full step would be taken. In particular, this means to show that $\left\|\Phi_{\text{bar}}\left(\omega^k + \Delta\omega^k; \rho^k\right)\right\| \le \varepsilon_{\mu,k+1}$ and that the fraction-to-the-boundary rules accept the trial step. Because of Step L-9 and $\left(\lambda^k, \nu^k\right) = \left(y^k/\bar{\pi}, z^k/\bar{\pi}\right)$ no further full step Armijo condition would have to be proven. Both results are left as future research.

4.4.3 Local Convergence for Infeasible Programs

As pointed out by Byrd et al. [31], the detection of infeasibility plays an important role in many applications of nonlinear programming. Therefore, the local convergence of Algorithm L in such a situation is studied in this section. It is assumed, that $w^* = (x^*, y^*, z^*)$ with $y^* = \frac{\bar{\tau}}{\|g(x^*)\|_2} g(x^*)$ is a certificate of infeasibility or a first-order optimal point of (FeasNLP+), which is related to the outcome (ii) of Theorem 4.26. It then holds that the barrier parameter $\mu_k = \bar{\mu}$, the Lagrangian multiplier $\nu^k = \bar{\nu}$ as well as $\lambda^k = \bar{\lambda}$ and the penalty parameter $\tau_k = \bar{\tau}$ are fixed for sufficiently large iterations k while π_k tends to zero. From Step L-10.3, it follows that in this case the second-order-correction step will not be applied. Thus, the following result studies the convergence order of Algorithm L only for the standard step direction.

Proposition 4.34. *Suppose Assumptions 4.27 hold at a certificate of infeasibility $w^* = (x^*, y^*, z^*)$ with $y^* = \frac{\bar{\tau}}{\|g(x^*)\|_2} g(x^*)$. Let $w^k \in \mathcal{N}(w^*)$. Then,*

$$\left\|w^k + \Delta w^k - w^*\right\| = \mathcal{O}\left(\left\|w^k - w^*\right\|\right) + \mathcal{O}(\pi_k).$$

Proof. The first-order optimality conditions of (FeasNLP+) scaled by $\bar{\tau}$ are

$$\nabla g(x^*) y^* - z^* = 0 \tag{4.91a}$$

$$y^* - \frac{\bar{\tau}}{\|g(x^*)\|_2} g(x^*) = 0 \tag{4.91b}$$

$$\left(X_* + \bar{\mu}\bar{\Sigma}\right) z^* = 0. \tag{4.91c}$$

From the linear equation system (4.11) it follows

$$\begin{aligned}
& M_k\left(w^k + \Delta w^k - w^*\right) \\
&\quad = M_k\left(w^k - w^*\right) + M_k \Delta w^k \\
&\quad \overset{(4.11)}{=} \begin{bmatrix} \nabla^2_{xx} L\left(x^k, y^k, z^k\right)\left(x^k - x^*\right) + \nabla g\left(x^k\right)\left(y^k - y^*\right) - \left(z^k - z^*\right) \\ \nabla g\left(x^k\right)^\top \left(x^k - x^*\right) - \frac{\|g(x^k)\|_2}{\bar{\tau}}\left(y^k - y^*\right) \\ Z_k\left(x^k - x^*\right) + \left(X_k + \bar{\mu}\bar{\Sigma}\right)\left(z^k - z^*\right) \end{bmatrix} \\
&\qquad - \begin{bmatrix} \nabla_x L\left(x^k, y^k, z^k\right) \\ g\left(x^k\right) + \frac{\|g(x^k)\|_2}{\bar{\tau}}\left(\pi_k \bar{\lambda} - y^k\right) \\ X_k z^k - \bar{\mu}\bar{\Sigma}\left(\pi_k \bar{\nu} - z^k\right) \end{bmatrix} \\
&\quad = \begin{bmatrix} \nabla^2_{xx} L\left(x^k, y^k, z^k\right)\left(x^k - x^*\right) - \nabla_x L\left(x^k, y^*, z^*\right) \\ -g\left(x^k\right) + \nabla g\left(x^k\right)^\top \left(x^k - x^*\right) - \frac{\|g(x^k)\|_2}{\bar{\tau}}\left(\pi_k \bar{\lambda} - y^*\right) \\ Z_k\left(x^k - x^*\right) - \left(X_k + \bar{\mu}\bar{\Sigma}\right) z^* + \pi_k \bar{\mu}\bar{\Sigma}\bar{\nu} \end{bmatrix}
\end{aligned}$$

Then, by Assumption 4.12 (i), Taylor's theorem and $\nabla_x L(x^*, y^*, z^*) = \pi_k \nabla f(x^*)$ due to (4.91) the equation

$$\begin{aligned}
& \nabla^2_{xx} L\left(x^k, y^k, z^k\right)\left(x^k - x^*\right) - \nabla_x L\left(x^k, y^*, z^*\right) \\
&\quad \overset{\text{A } 4.27}{=} \left(\nabla^2_{xx} L\left(x^k, y^k, z^k\right) - \nabla^2_{xx} L(x^*, y^*, z^*)\right)\left(x^k - x^*\right) \\
&\qquad - \nabla_x L(x^*, y^*, z^*) + \mathcal{O}\left(\left\|x^k - x^*\right\|^2\right) \\
&\quad \overset{(4.91)}{=} \mathcal{O}(\pi_k) + \mathcal{O}\left(\left\|w^k - w^*\right\|^2\right)
\end{aligned} \tag{4.92}$$

holds. Using Taylor's theorem and (4.91b) yields

$$
\begin{aligned}
&-g\left(x^k\right)+\nabla g\left(x^k\right)^\top\left(x^k-x^*\right)-\frac{\left\|g\left(x^k\right)\right\|_2}{\bar{\tau}}\left(\pi_k\bar{\lambda}-y^*\right)\\
&\quad\leq\quad -g(x^*)+\frac{\|g(x^*)\|_2}{\bar{\tau}}y^*+\mathcal{O}(\pi_k)+\mathcal{O}\left(\left\|x^k-x^*\right\|\right)\\
&\quad\overset{(4.91b)}{=}\quad \mathcal{O}(\pi_k)+\mathcal{O}\left(\left\|w^k-w^*\right\|\right).
\end{aligned}
\tag{4.93}
$$

Furthermore, it follows by (4.91c) that

$$
\begin{aligned}
&Z_k\left(x^k-x^*\right)-\left(X_k+\bar{\mu}\bar{\Sigma}\right)z^*+\pi_k\bar{\mu}\bar{\Sigma}\bar{\nu}\\
&\quad\overset{(4.91c)}{=}\quad Z_k\left(x^k-x^*\right)-\left(X_k+\bar{\mu}\bar{\Sigma}\right)z^*+\left(X_*+\bar{\mu}\bar{\Sigma}\right)z^*+\mathcal{O}(\pi_k)\\
&\quad=\quad (X_k-X_*)\left(z^k-z^*\right)+\mathcal{O}(\pi_k)\\
&\quad=\quad \mathcal{O}(\pi_k)+\mathcal{O}\left(\left\|w^k-w^*\right\|^2\right).
\end{aligned}
\tag{4.94}
$$

Then, the combination of (4.92), (4.93) and (4.94) with a similar result as (4.60) yields the claimed convergence property. □

Proposition 4.34 shows that if π_k is chosen to be proportional to $\left\|w^k-w^*\right\|$ the convergence order would be linear. The reason that no superlinear order could be achieved with an appropriate choice of π_k is the fixation of $\left\|g\left(x^k\right)\right\|_2/\bar{\tau}$ in the linear equation system (4.11) when applying Newton's method (cf., Section 4.2.1).

4.5 Parametric Sensitivity Analysis

In this section the parametric sensitivity analysis (cf., Section 2.2) will be studied when applied to the three main nonlinear (sub-)programs of the penalty-interior-point approach, namely (NLP+), (NLPbar) and (NLPpen). It is of particular interest, how to efficiently calculate the sensitivity derivatives of the original optimization problem (NLP+) using already available information generated by the solution process of Algorithm L.

The presentation concentrates on sensitivity derivatives of the primal-dual optimal solution only (cf., Corollary 2.27) as these directly imply the sensitivity derivatives of the objective function and constraints using the chain rule (cf., Corollary 2.29 to Corollary 2.31).

4.5.1 Sensitivity Derivative Approximations of the Nonlinear Program

Sensitivity derivatives of (NLP+) are given by

$$
\begin{bmatrix}\nabla^2_{xx}L^* & \nabla g^* & -E\\ (\nabla g^*)^\top & 0 & 0\\ \operatorname{diag}(\nu^*) & 0 & X_*\end{bmatrix}\underbrace{\begin{bmatrix}\frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\\ \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*)\\ \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*)\end{bmatrix}}_{=:\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*)}=-\begin{bmatrix}\nabla^2_{xp}L^*(p^*)^\top\\ \nabla_p g^*(p^*)^\top\\ 0\end{bmatrix},
\tag{4.95}
$$

which is an application of Corollary 2.27. Usual penalty and interior-point methods cannot compute $\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*)$ directly because they approach the primal-dual optimal solution (x^*, λ^*, ν^*) just in the limit of a barrier and / or penalty parameter converging to zero. Instead, it is shown that the sensitivity derivatives of the subproblem eventually converge to $\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*)$ for the just mentioned parameter convergence (cf., Fiacco [61, Section 6.2]). This can be different for the proposed penalty-interior-point algorithm as it is exact. Nevertheless, instead of considering (4.95), it is preferred to calculate the sensitivity derivatives based on the matrix of the linear equation system (4.11) that is already factorized in Step L-8 and therefore saves significant computational costs. Such linear equation system is

$$M_* \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \bar{\rho}) = - \begin{bmatrix} \nabla^2_{xp} L^*(p^*)^\top \\ \nabla_p g^*(p^*)^\top \\ 0 \end{bmatrix} \tag{4.96a}$$

$$M_* := \begin{bmatrix} \bar{\pi} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \bar{\pi} E \end{bmatrix}^{-1} \begin{bmatrix} Q_* & \nabla g^* & -E \\ (\nabla g^*)^\top & -\bar{\varrho} E & 0 \\ Z_* & 0 & X_* + \bar{\mu} \bar{\Sigma} \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & \bar{\pi} E & 0 \\ 0 & 0 & \bar{\pi} E \end{bmatrix} \tag{4.96b}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \bar{\rho}) := \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p}(p^*, \bar{\rho}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*, \bar{\rho}) \\ \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*, \bar{\rho}) \end{bmatrix} \tag{4.96c}$$

where $Q_* = \bar{\pi} \nabla^2 f(x^*) + \sum_{i=1}^{n_g} y_i^* \nabla^2 g_i(x^*)$ and $\bar{\varrho} = \frac{\|g(x^*)\|_2}{\bar{\tau}} = 0$. Now comparing the two sensitivity derivative calculations it turns out that their difference is proportional to the primal shift $\bar{\mu}\,\|\varsigma\|$.

Proposition 4.35 (Estimation of Sensitivity Derivatives). *Let the assumptions of Theorem 2.26 applied to* (NLP+) *be satisfied in a neighborhood $\mathcal{N}(w^*)$ and $w(\mu, \nu) \in \mathcal{N}(w^*)$ be an optimal solution of* (NLPbar) *with $\lim_{\rho \to \bar{\rho}} w(\mu, \nu) = w^*$ for some $\bar{\rho}$. Assume $\bar{\pi} > 0$, $\bar{\tau} > 0$ and $(\lambda^*, \nu^*) = (y^*/\bar{\pi}, z^*/\bar{\pi})$. Then,*

$$\left\| \lim_{\rho \to \bar{\rho}} \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \rho) - \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*) \right\| = \mathcal{O}(\bar{\mu}\,\|\bar{\varsigma}\|).$$

In particular, $\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \bar{\rho}) \to \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*)$ for $\bar{\mu}\,\|\bar{\varsigma}\| \to 0$.

Proof. Let M_* be given by (4.96). Due to the LICQ, SOSC and SCC of (NLP+) the matrix M_* is nonsingular and its inverse is bounded (cf., Section 4.4). Furthermore, the limit $\lim_{\rho \to \bar{\rho}} \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \rho) = \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*, \bar{\rho})$ exists. Then, using $(\lambda^*, \nu^*) = (y^*/\bar{\pi}, z^*/\bar{\pi})$ – hence

$\frac{1}{\bar{\pi}}Q_* = \nabla^2_{xx}L^*$ – together with

$$M_*\left(\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*) - \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*,\bar{\rho})\right)$$
$$\overset{(4.96)}{=} \begin{bmatrix} \frac{1}{\bar{\pi}}Q_*\frac{\mathrm{d}x}{\mathrm{d}p}(p^*) + \nabla g^* \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*) - \frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \\ (\nabla g^*)^\top \frac{\mathrm{d}x}{\mathrm{d}p}(p^*) \\ \frac{1}{\bar{\pi}}Z_*\frac{\mathrm{d}x}{\mathrm{d}p}(p^*) + \left(X_* + \bar{\mu}\bar{\Sigma}\right)\frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \end{bmatrix} + \begin{bmatrix} \nabla^2_{xp}L^*(p^*)^\top \\ \nabla_p g^*(p^*)^\top \\ 0 \end{bmatrix}$$
$$\overset{(4.95)}{=} \begin{bmatrix} 0 \\ 0 \\ \bar{\mu}\bar{\Sigma}\frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*) \end{bmatrix}$$

implies $\left\|\frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*) - \frac{\mathrm{d}\omega}{\mathrm{d}p}(p^*,\bar{\rho})\right\| = \mathcal{O}(\bar{\mu}\,\|\bar{\varsigma}\|)$. □

As a result of Proposition 4.35, there is indeed the situation that (4.95) and (4.96) coincide, because if $z^* = \nu^* = 0$ – which means that all inequality constraints are inactive – then $\bar{\varsigma} = 0$ by (4.42). However, it has to be admitted that this case is very unlikely.

There is a further more significant advantage of the sensitivity derivative calculation based on (4.96) for the exact penalty-interior-point algorithm compared to classic interior-point methods. As it is common for a primal-dual interior-point approach, the linear equation system (4.11) is reduced to (4.12) and can analogously be applied to the sensitivity derivative determination, which leads to

$$\begin{bmatrix} \nabla^2_{xx}L^* + \left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}Z_* & \nabla g^* \\ (\nabla g^*)^\top & 0 \end{bmatrix}\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}p}(p^*,\bar{\rho}) \end{bmatrix} = -\begin{bmatrix} \nabla^2_{xp}L^*(p^*)^\top \\ \nabla_p g^*(p^*)^\top \end{bmatrix} \tag{4.97a}$$
$$\frac{\mathrm{d}\nu}{\mathrm{d}p}(p^*,\bar{\rho}) = -\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}Z_*\frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho}). \tag{4.97b}$$

The term $X_* + \bar{\mu}\bar{\Sigma}$ is bounded away from zero for an optimal solution $x^* \geq 0$ satisfying the SCC, which is a relevant assumption of the sensitivity analysis anyway (cf., Theorem 2.26). In particular, for active inequality constraints $x_i^* = 0$ for $i \in \mathscr{A}(x^*)$ it follows $\bar{\varsigma}_i > 0$ and therefore this bound can be controlled by the size of the barrier parameter $\bar{\mu}$. This way ill-conditioning of the matrix in (4.97a) can be avoided. For the classic log-barrier function, this is not the case as the analog matrix to (4.12) would be

$$\begin{bmatrix} \nabla^2_{xx}L^k + X_k^{-1}Z_k & \nabla g\left(x^k\right) \\ \nabla g\left(x^k\right)^\top & 0 \end{bmatrix}. \tag{4.98}$$

Although x^k would be kept strictly positive throughout the optimization, it can get arbitrarily close to zero leading to severe ill-conditioning (cf., Greif et al. [111]) and thus to inaccurate sensitivity derivatives.

Summarizing the above, the update strategy of the barrier parameter μ is of high importance for parametric sensitivity analysis. The size of it balances the theoretical quality of the sensitivity derivatives (error estimate) and the practical one (condition of linear equation system).

4.5.2 Sensitivity Derivatives of the Barrier Subprogram

Sensitivity derivatives can be provided for (NLPbar) by the direct application of Theorem 2.26 and Corollary 2.27 to Corollary 2.31.

The parameters of (NLPbar) are ρ with reference values $\bar{\rho}$. It is assumed that (x^*, λ^*, z^*) is an optimal solution of (NLPbar) where the Lagrangian multiplier $\bar{\nu}$ may violate $\bar{\nu} = z^*/\bar{\pi}$. The gradient of the Lagrangian function is

$$\nabla_x L(x, \lambda; \rho) = \nabla f(x) - \mu\pi\Sigma\,(X + \mu\Sigma)^{-1}\,\nu + \nabla g(x)\lambda \tag{4.99}$$

and the Hessian of it is given by

$$\nabla_{xx} L(x, \lambda; \rho) = \nabla^2 f(x) + \mu\pi\Sigma\,(X + \mu\Sigma)^{-2}\,\mathrm{diag}(\nu) + \sum_{i=1}^{n_g} \lambda_i \nabla^2 g_i(x). \tag{4.100}$$

From Corollary 2.27, it follows that the sensitivity derivatives of the optimal solution $\left(\frac{\mathrm{d}x}{\mathrm{d}\rho}(\bar{\rho}), \frac{\mathrm{d}\lambda}{\mathrm{d}\rho}(\bar{\rho})\right)$ can be determined by solving the linear equation system

$$\underbrace{\begin{bmatrix} \nabla^2_{xx} L^* + \bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu}) & \nabla g(x^*) \\ \nabla g(x^*)^\top & 0 \end{bmatrix}}_{=:K} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\rho}(\bar{\rho}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\rho}(\bar{\rho}) \end{bmatrix} = -\begin{bmatrix} \nabla_{x\rho} L(x^*, \lambda^*; \bar{\rho})^\top \\ 0 \end{bmatrix} \tag{4.101}$$

where $\nabla^2_{xx} L^* = \nabla^2 f(x^*) + \sum_{i=1}^{n_g} \lambda_i^* \nabla^2 g_i(x^*)$. This is a pure primal interpretation of the sensitivity information as Theorem 2.26 is not aware of the introduction of dual variables z.[8] To approximate the sensitivity of the dual variables z they can be defined as a function of x, i.e.,

$$z(x; \rho) = \mu\pi\Sigma\,(X + \mu\Sigma)^{-1}\,\nu \tag{4.102}$$

Then, their sensitivity derivative

$$\frac{\mathrm{d}z}{\mathrm{d}\rho}(x^*; \bar{\rho}) = -\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu})\frac{\mathrm{d}x}{\mathrm{d}\rho}(\bar{\rho}) + \nabla_\rho z(x; \rho)^\top \tag{4.103}$$

is given by the chain rule. The following corollary serves as an overview of the sensitivity derivatives $\frac{\mathrm{d}x}{\mathrm{d}\rho}(\bar{\rho})$, $\frac{\mathrm{d}\lambda}{\mathrm{d}\rho}(\bar{\rho})$ and $\frac{\mathrm{d}z}{\mathrm{d}\rho}(x^*; \bar{\rho})$ for the different parameters μ, π, ς and ν.

Corollary 4.36 (Sensitivity Derivatives of (NLPbar)**).** *Let the assumptions of Theorem 2.26 applied to* (NLPbar) *be satisfied and K be defined by* (4.101). *Then, the first-order sensitivity derivatives of the optimal solution* (x^*, λ^*) *with respect to* $(\mu, \pi, \varsigma, \nu)$ *are the following:*

i. *Sensitivity derivatives with respect to the barrier parameter μ are*

$$K \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\mu}(\bar{\mu}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\bar{\mu}) \end{bmatrix} = -\begin{bmatrix} \bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\left(-E + \bar{\mu}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\right)\bar{\nu} \\ 0 \end{bmatrix}$$

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\mu}(x^*; \bar{\mu}) = &-\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu})\frac{\mathrm{d}x}{\mathrm{d}\mu}(\bar{\mu}) \\ &+ \bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\left(E - \bar{\mu}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\right)\bar{\nu}. \end{aligned}$$

[8]The term *primal* refers to a method based on (3.62) instead of the primal-dual linear equation system (3.64). Nevertheless, this includes the handling of Lagrangian multipliers λ.

ii. *Sensitivity derivatives with respect to the primal shift parameter* ς_i *for* $i = 1, \dots, n_x$ *are*

$$K \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i) \end{bmatrix} = - \begin{bmatrix} \bar{\pi}\bar{\mu}\left(x_i^* + \bar{\mu}\bar{\varsigma}_i\right)^{-1}\left(-1 + \bar{\mu}\bar{\varsigma}_i\left(x_i^* + \bar{\mu}\bar{\varsigma}_i\right)^{-1}\right)\bar{\nu}_i e_i \\ 0 \end{bmatrix}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\varsigma_i}(x^*; \bar{\varsigma}_i) = -\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu})\frac{\mathrm{d}x}{\mathrm{d}\varsigma_i}(\bar{\varsigma}) + \bar{\pi}\bar{\mu}\left(x_i^* + \bar{\mu}\bar{\varsigma}_i\right)^{-1}\left(1 - \bar{\mu}\bar{\varsigma}_i\left(x_i^* + \bar{\mu}\bar{\varsigma}_i\right)^{-1}\right)\bar{\nu}_i e_i.$$

iii. *Sensitivity derivatives with respect to the scaling parameter* π *are*

$$K \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\pi}(\bar{\pi}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\pi}(\bar{\pi}) \end{bmatrix} = - \begin{bmatrix} -\bar{\mu}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\bar{\nu} \\ 0 \end{bmatrix}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\pi}(x^*; \bar{\pi}) = -\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu})\frac{\mathrm{d}x}{\mathrm{d}\pi}(\bar{\pi}) + \bar{\mu}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}\bar{\nu}.$$

iv. *Sensitivity derivatives with respect to the Lagrangian multiplier* ν_i *for* $i = 1, \dots, n_x$ *are*

$$K \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\nu_i}(\bar{\nu}_i) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_i}(\bar{\nu}_i) \end{bmatrix} = - \begin{bmatrix} -\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}e_i \\ 0 \end{bmatrix}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\nu_i}(x^*; \bar{\nu}_i) = -\bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-2}\mathrm{diag}(\bar{\nu})\frac{\mathrm{d}x}{\mathrm{d}\nu_i}(\bar{\nu}_i) + \bar{\mu}\bar{\pi}\bar{\Sigma}\left(X_* + \bar{\mu}\bar{\Sigma}\right)^{-1}e_i.$$

Primal-Dual Sensitivity Derivatives

Although it may not be clear on first sight of Corollary 4.36, the whole sensitivity information is captured in the part described by the function $z(x;\rho)$. In fact, combining (4.101) and (4.103) yields

$$\begin{bmatrix} \nabla^2_{xx}L^* & \nabla g(x^*) & -E \\ \nabla g(x^*)^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\rho}(\bar{\rho}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\rho}(\bar{\rho}) \\ \frac{\mathrm{d}z}{\mathrm{d}\rho}(x^*;\bar{\rho}) \end{bmatrix} = 0. \tag{4.104}$$

The matrix of (4.104) is actually a rectangular part of (4.11) ($\varrho_k = 0$), where a condition on the complementarity (4.4c) is missing, or – in other words – the definition of z (cf., (4.102)). However, it is straightforward to adapt the definition of sensitivity derivatives to the primal-dual context. Recall that the Corollary 4.36 follows by the application of the implicit function theorem on the KKT conditions of (NLPbar) as described in more detail in Section 2.2.1. This can analogously be done for the primal-dual system (4.4), which leads to the following primal-dual sensitivity derivatives.

Corollary 4.37 (Primal-Dual Sensitivity Derivatives of (NLPbar)**).** *Let the assumptions of Theorem 2.26 applied to* (NLPbar) *be satisfied and let* M_* *be defined by* (4.11) *evaluated at the optimal solution* (x^*, λ^*, z^*) *and reference parameter* $(\bar{\mu}, \bar{\pi}, \bar{\varsigma}, \bar{\nu})$*. Then, the primal-dual first-order sensitivity derivatives of the optimal solution* (x^*, λ^*, z^*) *with respect to* $(\mu, \pi, \varsigma, \nu)$ *based on the KKT conditions* (4.4) *are the following:*

i. Sensitivity derivatives with respect to the barrier parameter μ are

$$M_* \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\mu}(\bar{\mu}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\bar{\mu}) \\ \frac{\mathrm{d}z}{\mathrm{d}\mu}(\bar{\mu}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\Sigma}\left(\bar{\pi}\bar{\nu} - z^*\right) \end{bmatrix}.$$

ii. Sensitivity derivatives with respect to the primal shift parameter ς_i for $i = 1, \dots, n_x$ are

$$M_* \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i) \\ \frac{\mathrm{d}z}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu}\left(\bar{\pi}\bar{\nu}_i - z_i^*\right) e_i \end{bmatrix}.$$

iii. Sensitivity derivatives with respect to the scaling parameter π are

$$M_* \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\pi}(\bar{\pi}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\pi}(\bar{\pi}) \\ \frac{\mathrm{d}z}{\mathrm{d}\pi}(\bar{\pi}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu}\bar{\Sigma}\bar{\nu} \end{bmatrix}.$$

iv. Sensitivity derivatives with respect to the Lagrangian multiplier ν_i for $i = 1, \dots, n_x$ are

$$M_* \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\nu_i}(\bar{\nu}_i) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\nu_i}(\bar{\nu}_i) \\ \frac{\mathrm{d}z}{\mathrm{d}\nu_i}(\bar{\nu}_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu}\bar{\pi}\bar{\varsigma}_i e_i \end{bmatrix}.$$

In particular, it holds $\left(\frac{\mathrm{d}x}{\mathrm{d}\mu}(\bar{\mu}), \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}(\bar{\mu}), \frac{\mathrm{d}z}{\mathrm{d}\mu}(\bar{\mu})\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i), \frac{\mathrm{d}\lambda}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i), \frac{\mathrm{d}z}{\mathrm{d}\varsigma_i}(\bar{\varsigma}_i)\right) = 0$ if the Lagrangian multiplier is set to $\bar{\nu} = z^*/\bar{\pi}$.

In the literature, sensitivity derivatives of the barrier subprogram (NLPbar) are usually considered to be the primal-dual variant (cf., for example, Fiacco [61, Chapter 6], López-Negrete [137] or Pirnay et al. [161]). Further note, that the matrix M_* of Corollary 4.37 is indeed the matrix of (4.11) evaluated at the optimal solution (x^*, λ^*, z^*) and reference parameter $\bar{\rho} = (\bar{\mu}, \bar{\pi}, \bar{\varsigma})$ because due to $\|g(x^*)\| = 0$ the dual regularization $\delta_\mathrm{d} = \varrho = 0$ vanishes.

Real-Time Approximations for Parameter Updates

An application of such an internal[9] parametric sensitivity analysis is a real-time approximation of Section 2.2.2 after parameter updates. For instance, consider an update of the barrier parameter μ_k in Step L-4.2 (cf., Section 4.2.3). Then an extrapolation step

$$x^k \leftarrow x^k + \frac{\mathrm{d}x^k}{\mathrm{d}\mu}(\mu_{k-1})(\mu_k - \mu_{k-1}) \tag{4.105a}$$

$$\lambda^k \leftarrow \lambda^k + \frac{\mathrm{d}\lambda^k}{\mathrm{d}\mu}(\mu_{k-1})(\mu_k - \mu_{k-1}) \tag{4.105b}$$

$$z^k \leftarrow z^k + \frac{\mathrm{d}z^k}{\mathrm{d}\mu}(\mu_{k-1})(\mu_k - \mu_{k-1}) \tag{4.105c}$$

[9]Internal in contrast to a parametric sensitivity analysis of the original nonlinear program (NLP+) as in Section 4.5.1.

can approximate the primal-dual solution of (NLPbar) for this new barrier parameter and therefore potentially reduce the number of total iterations of Algorithm L. However, in an algorithmic framework like the one of Algorithm L the barrier subproblem (NLPbar) is solved just approximately, which makes the application of an extrapolation step (4.105) even more heuristic. The global convergence results of Section 4.3 are not affected as long as this strategy keeps the new iterate within the interior of the feasible region or – in other words – provides a valid initial guess for the updated barrier subprogram, e.g., by using a fraction-to-the-boundary rule.

4.5.3 Sensitivity Derivatives of the Penalty Subprogram

For the penalty subprogram (NLPpen) the non-differentiability of the objective function makes the application of the sensitivity analysis of Section 2.2 impossible, unless an optimal solution x^* of (NLPpen) is infeasible for (NLPbar) and (NLP+), i.e., $\|g(x^*)\| > 0$.[10] This case is however of no special interest for a sensitivity analysis. Nevertheless, it is possible to state primal-dual sensitivity derivatives of (NLPpen) that follow their derivation for (NLPbar) in Section 4.5.2. If $\left\|y^* - \bar{\pi}\bar{\lambda}\right\| < \bar{\tau}$, then at first this part of the first-order optimality conditions of (NLPpen) can be considered to be inactive and, thus, be neglected. Furthermore, it implies together with (4.6) that $\|g(x^*)\| = 0$. The primal-dual sensitivity derivatives – as shown in the following – then resemble the ones of (NLPbar). In particular due to $\|g(x^*)\| = 0$, an extrapolation step for $\bar{\lambda}$ and $\bar{\tau}$ is not available as their sensitivity derivatives are zero.

Corollary 4.38 (Primal-Dual Sensitivity Derivatives of (NLPpen)**).** *Let the assumptions of Theorem 2.26 applied to* (NLPbar) *be satisfied and let M_* be defined by* (4.11) *evaluated at the optimal solution (x^*, y^*, z^*) of* (NLPpen) *and reference parameter $\left(\bar{\mu}, \bar{\pi}, \bar{\tau}, \bar{\varsigma}, \bar{\nu}, \bar{\lambda}\right)$. Furthermore, assume that $\|g(x^*)\| = 0$. Then, the primal-dual first-order sensitivity derivatives of the optimal solution (x^*, y^*, z^*) with respect to $(\mu, \pi, \tau, \varsigma, \nu, \lambda)$ based on the KKT conditions* (4.7) *are the following:*

i. Sensitivity derivatives with respect to the penalty parameter τ and the Lagrangian multiplier λ are

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\tau}(\bar{\tau}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}(\bar{\tau}) \\ \frac{\mathrm{d}z}{\mathrm{d}\tau}(\bar{\tau}) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\lambda}(\bar{\lambda}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda}(\bar{\lambda}) \\ \frac{\mathrm{d}z}{\mathrm{d}\lambda}(\bar{\lambda}) \end{bmatrix} = 0.$$

ii. Sensitivity derivatives with respect to the penalty parameter π are

$$M_* \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\pi}(\bar{\pi}) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}\pi}(\bar{\pi}) \\ \frac{\mathrm{d}z}{\mathrm{d}\pi}(\bar{\pi}) \end{bmatrix} = - \begin{bmatrix} \nabla f(x^*) \\ 0 \\ -\bar{\mu}\bar{\Sigma}\bar{\nu} \end{bmatrix}.$$

iii. Sensitivity derivatives with respect to the barrier parameter μ, the Lagrangian multiplier ν and the primal shift ς equal the ones of Corollary 4.37.

[10] If $\|g(x^*)\| > 0$, it exists a neighborhood of x^* in which the objective function of (NLPpen) is differentiable and for which Theorem 2.26 with this further restriction could be applied.

4.6 Adaptive Parameter Updates

The proposed Algorithm L is a rather keen candidate of a penalty-interior-point algorithm that includes many very different parameters to steer the local and global convergence. These are the barrier parameter μ_k, the penalty parameters π_k and τ_k, the primal boundary shift ς^k and the Lagrangian multipliers ν^k and λ^k. The practical performance of such an algorithm however strongly depends on the specific parameter choice, which is even problem dependent. In the worst case the penalty parameter together with the Lagrangian multiplier λ^k may not prevent or even promote divergence of the algorithm. All parameters have in common that they may construct a merit function $\Upsilon(x^k; \rho^k)$ that is difficult to solve in practice and, thus, will lead to an unfortunate large number of iterations until a next parameter update can be executed in Step L-4 or Step L-5. This motivates a more flexible parameter update scheme that can adapt to these situations. In this section an adaptive parameter update framework will be developed for all parameters ρ^k.

In fact, the presented algorithm already includes some adaptability. Comparing the exact $\ell 2$-penalty function with its smooth squared version (with a penalty parameter ϱ), it turns out that the resulting linear equation system (4.11) is the same but the penalty parameter ϱ has adaptively been set to $\varrho_k = \left\| g(x^k) \right\|_2 / \tau_k$, i.e., proportional to the current constraint violation. Such a choice is intuitive as the algorithm should primarily focus on reducing the constraint violation by selecting a larger penalty parameter when it is far away from a feasible point. While this property appears rather naturally in this algorithm it is shown in Armand et al. [6] that such an adaptive update for the smooth $\ell 2$-penalty function can drastically reduce the number of iterations.

Apart from Armand et al. [6] adaptive penalty parameter updates have been studied by Byrd et al. [30, 32] and Curtis [48]. Instead of choosing the penalty parameter proportional to the current constraint violation, the latter choose it in such a way that the resulting descent of the step direction is proportional to the descent of a pure feasibility step, i.e., a step based on (FeasNLP+) from the current iterate. This ensures that sufficient progress is made towards feasibility with every step.

Similar strategies also exist for interior-point methods. In Silva et al. [179], Ulbrich et al. [183], Vanderbei [185] and Vanderbei and Shanno [186] the barrier parameter is set to the current complementarity measure $\left(x^k\right)^\top z^k / n_x$. In a Mehrotra-Predictor-Corrector scheme (cf., Mehrotra [141]), which has also been transferred to nonlinear programming by Nocedal et al. [153], the barrier parameter is chosen to be proportional to the complementarity measure after the application of a full step where the complementarity perturbation in (4.4c) is removed, i.e., where $\mu_k = 0$. Curtis [48] and Nocedal et al. [153] also investigate an adaptive update strategy that selects the barrier parameter such that a measure of the KKT conditions (4.3) is minimized for the next step. Armand et al. [4, 5] propose to augment the linear equation system for the step calculation by a barrier update function equation. This way the barrier parameter is adjusted automatically with every Newton step and can further be considered in a line search globalization framework.

The proposed adaptive updates for Algorithm L are based on the approaches where the parameter selection promotes best progress towards the optimal solution or sufficient progress with respect to (FeasNLP+) in the next primal-dual update. This implies that the step direction $\left(\Delta x^k(\rho), \Delta y^k(\rho), \Delta z^k(\rho)\right)$ is calculated in dependence on the parameter choice. Usually this strategy is based on the fact that for an algorithm with the classic log-barrier function and a Hessian approximation, the parameters μ_k and π_k only appear in the right-hand-side of the linear equation system and an evaluation of $\left(\Delta x^k(\rho), \Delta y^k(\rho), \Delta z^k(\rho)\right)$ is computationally cheap once the system matrix has been factorized. Unfortunately, this does not hold for Algorithm L. To avoid further matrix factorizations the adaptive update strategy will be based on sensitivity derivatives of the step direction.

4.6.1 Sensitivity Analysis of the Step Directions

The sensitivity derivatives of the step direction $\left(\Delta x^k(\rho), \Delta y^k(\rho), \Delta z^k(\rho)\right)$ can be derived by applying the implicit function theorem[11] on the linear equation system (4.11), which equals

$$\begin{bmatrix} Q_k & \nabla g\left(x^k\right) & -E \\ \nabla g\left(x^k\right)^\top & -\varrho_k E & 0 \\ Z_k & 0 & X_k + \mu_k \Sigma_k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{bmatrix} + \begin{bmatrix} \pi_k \nabla f\left(x^k\right) + \nabla g\left(x^k\right) y^k - z^k \\ g\left(x^k\right) + \varrho_k \left(\pi_k \lambda^k - y^k\right) \\ X_k z^k - \mu_k \Sigma_k \left(\pi_k \nu^k - z^k\right) \end{bmatrix} = 0 \tag{4.106}$$

with $Q_k = \pi_k \nabla^2 f\left(x^k\right) + \sum_{i=1}^{n_g} y_i^k \nabla^2 g_i\left(x^k\right)$ or an approximation to it and $\varrho_k = \frac{\|g(x^k)\|_2}{\tau_k}$. The following corollary lists the sensitivity derivatives with respect to all parameters ρ^k of Algorithm L.

Corollary 4.39 (Sensitivity Derivatives of the Step Directions). *Let $\left(x^k, y^k, z^k\right)$ be an iterate of Algorithm L where the Hessian Q_k may be regularized. If $\left\|g\left(x^k\right)\right\| = 0$ suppose that the LICQ holds. Then, the primal-dual first-order sensitivity derivatives of the step direction $\left(\Delta x^k, \Delta y^k, \Delta z^k\right)$ with respect to $(\mu, \pi, \tau, \varsigma, \nu, \lambda)$ are the following:*

i. Sensitivity derivatives with respect to the barrier parameter μ are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\mu}(\mu_k) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\mu}(\mu_k) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\mu}(\mu_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Sigma_k \left(\pi_k \nu^k - z^k - \Delta z^k\right) \end{bmatrix}.$$

ii. Sensitivity derivatives with respect to the penalty parameter π are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\pi}(\pi_k) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\pi}(\pi_k) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\pi}(\pi_k) \end{bmatrix} = -\begin{bmatrix} \nabla^2 f\left(x^k\right) \Delta x^k + \nabla f\left(x^k\right) \\ \frac{\|g(x^k)\|}{\tau_k} \lambda^k \\ -\mu_k \Sigma_k \nu^k \end{bmatrix}.$$

iii. Sensitivity derivatives with respect to the penalty parameter τ are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\tau}(\tau_k) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\tau}(\tau_k) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\tau}(\tau_k) \end{bmatrix} = -\begin{bmatrix} 0 \\ \frac{\|g(x^k)\|_2}{\tau_k^2} \left(y^k + \Delta y^k - \pi_k \lambda^k\right) \\ 0 \end{bmatrix}.$$

[11] See Theorem A.16.

iv. Sensitivity derivatives with respect to the primal shift ς are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\varsigma}\left(\varsigma^k\right) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\varsigma}\left(\varsigma^k\right) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\varsigma}\left(\varsigma^k\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu_k\left(\pi_k \nu^k - z^k - \Delta z^k\right) \end{bmatrix}.$$

v. Sensitivity derivatives with respect to the Lagrangian multiplier λ are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\lambda}\left(\lambda^k\right) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\lambda}\left(\lambda^k\right) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\lambda}\left(\lambda^k\right) \end{bmatrix} = - \begin{bmatrix} 0 \\ \frac{\|g(x^k)\|_2}{\tau_k}\pi_k e \\ 0 \end{bmatrix}.$$

vi. Sensitivity derivatives with respect to the Lagrangian multiplier ν are

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\nu}\left(\nu^k\right) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\nu}\left(\nu^k\right) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\nu}\left(\nu^k\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu_k \pi_k \varsigma^k \end{bmatrix}.$$

To approximate the step $\left(\Delta x^k(\rho), \Delta y^k(\rho), \Delta z^k(\rho)\right)$ for arbitrary parameters ρ the sensitivity derivatives of Corollary 4.39 can be used, i.e.,

$$\begin{bmatrix} \Delta x^k(\rho) \\ \Delta y^k(\rho) \\ \Delta z^k(\rho) \end{bmatrix} \approx \begin{bmatrix} \widetilde{\Delta x}^k(\rho) \\ \widetilde{\Delta y}^k(\rho) \\ \widetilde{\Delta z}^k(\rho) \end{bmatrix} := \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\rho}\left(\rho^k\right) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\rho}\left(\rho^k\right) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\rho}\left(\rho^k\right) \end{bmatrix} \left(\rho - \rho^k\right). \tag{4.107}$$

This linear approximation would be exact if the right-hand-sides in Corollary 4.39 would be independent of the standard step $\left(\Delta x^k, \Delta\lambda^k, \Delta \nu^k\right)$ like in Curtis [48] and Nocedal et al. [153]. For Algorithm L this holds only for the Lagrangian multipliers λ and ν and the penalty parameter π – the latter if a Hessian approximation is used.

The sensitivity derivatives further indicate that an adaptive update strategy is aimed to improve the global convergence of the algorithm. Locally, the barrier parameter μ_k and the constraint violation $\left\|g\left(x^k\right)\right\|_2$ are very small and the Lagrangian multipliers $\left(\lambda^k, \nu^k\right)$ are a very good approximation to $\left(y^k + \Delta y^k, z^k + \Delta z^k\right)/\pi_k$, which implies very small sensitivity derivatives with respect to the barrier parameter μ, the penalty parameter τ, the primal shift ς and the Lagrangian multipliers λ and ν.

4.6.2 Measuring Progress for Adaptive Parameter Updates

The question of how to adaptively select good parameters ρ based on sensitivity information is addressed in this section. To simplify the presentation an update of τ and ς is neglected. While the former is similar to the penalty parameter π, an update of the boundary shift ς is of primary concern in the limit of the global convergence (cf., Lemma 4.18) where the sensitivity derivative of Corollary 4.39 can be expected to be very small.

To assess the parameter selection two measures are necessary: a quality function for sufficient progress towards feasibility and a quality function for good progress towards optimality. These are defined as

$$Q_{\text{feas}}\big(w^k, \Delta w^k; \rho\big) := D^x_{\Delta w^k} \Psi\big(w^k; (\mu, \varepsilon, \lambda, \nu)\big), \tag{4.108a}$$

$$\begin{aligned} &Q_{\text{opt}}\big(w^k, \Delta w^k; \alpha, \pi\big) \\ &\quad := \left\| \pi \nabla f\big(x^k\big) + \nabla g\big(x^k\big)\big(y^k + \alpha_{\max,k} \Delta y^k\big) - \big(z^k + \alpha' \Delta z^k\big) + \alpha_{\max,k} Q_k \Delta x^k \right\|_2^2 \\ &\qquad + \left\| g\big(x^k\big) + \alpha_{\max,k} \nabla g\big(x^k\big)^\top \Delta x^k \right\|_2^2 + \left\| \big(X_k + \alpha_{\max,k} \operatorname{diag}\big(\Delta x^k\big)\big)\big(z^k + \alpha' \Delta z^k\big) \right\|_2^2 \end{aligned} \tag{4.108b}$$

where ε is a small positive number, $\alpha' = \min\left\{\alpha_{\max,k}, \alpha^z_k\right\}$ and $\alpha = \left(\alpha_{\max,k}, \alpha^z_k\right)$ is given by the fraction-to-the-boundary rules (4.15) and (4.28), respectively. The quality function for optimality $Q_{\text{opt}}\big(w^k, \Delta w^k; \alpha, \pi\big)$ is a linear approximation of the KKT conditions (4.3) of (NLP+) under a maximum step and follows the proposed strategy of Nocedal et al. [153]. However, relying on this quality function alone may lead to unnecessary many updates of the penalty parameter, which – in case of τ – should be as small as possible but large enough to yield sufficient progress towards feasibility. Therefore, the quality function $Q_{\text{feas}}\big(w^k, \Delta w^k; \rho\big)$ measures the descent of the step with respect to an approximation of (FeasNLP+) similarly to Curtis [48]. It is important to select a strictly positive number ε in (4.108a) to maintain a descent direction property.

Proposition 4.40 (Descent Direction of Approximate Feasibility Step). *Let $\mu_k, \mu > 0$, $\lambda^k \in \mathbb{R}^{n_g}$, $\nu^k \in \mathbb{R}^{n_x}_+$ and Q_k be an approximation of the Hessian independent of π such that $\operatorname{In}(M_k) = \big(n_x, n_g, 0\big)$. Then, $\Delta w^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big) = \widetilde{\Delta w}^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big)$ and*

$$Q_{feas}\left(w^k, \widetilde{\Delta w}^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big); \rho\right) < 0.$$

Proof. Due to the fixation of Q_k the sensitivity derivative $\frac{\mathrm{d}\Delta w^k}{\mathrm{d}\pi}(\pi_k)$ is

$$M_k \begin{bmatrix} \frac{\mathrm{d}\Delta x^k}{\mathrm{d}\pi}(\pi_k) \\ \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\pi}(\pi_k) \\ \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\pi}(\pi_k) \end{bmatrix} = - \begin{bmatrix} \nabla f\big(x^k\big) \\ \frac{\|g(x^k)\|}{\tau_k} \lambda^k \\ -\mu_k \Sigma_k \nu^k \end{bmatrix}.$$

Together with (4.11) and (4.107) it follows that

$$\begin{aligned} &M_k \left(\Delta w^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big) - \widetilde{\Delta w}^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big) \right) \\ &\quad = M_k \left(\Delta w^k\big(\mu_k, \varepsilon, \lambda^k, \nu^k\big) - \Delta w^k - \frac{\mathrm{d}\Delta w^k}{\mathrm{d}\pi}(\pi_k)(\varepsilon - \pi_k) \right) \\ &\quad = - \begin{bmatrix} \nabla f\big(x^k\big) \\ \frac{\|g(x^k)\|}{\tau_k} \lambda^k \\ -\mu_k \Sigma_k \nu^k \end{bmatrix} (\varepsilon - \pi_k) - M_k \left(\frac{\mathrm{d}\Delta w^k}{\mathrm{d}\pi}(\pi_k)(\varepsilon - \pi_k) \right) \\ &\quad = 0 \end{aligned}$$

and hence $\Delta w^k(\mu_k, \varepsilon, \lambda^k, \nu^k) = \widetilde{\Delta w}^k(\mu_k, \varepsilon, \lambda^k, \nu^k)$. Because of $\mathrm{In}(M_k) = (n_x, n_g, 0)$ and since the specific choice of Q_k was irrelevant for the proof of Proposition 4.4, it follows by Proposition 4.4 and Proposition 4.6 that $Q_{\text{feas}}\Big(w^k, \widetilde{\Delta w}^k(\mu_k, \varepsilon, \lambda^k, \nu^k); \rho\Big) < 0$. □

The assessment of the different parameters based on the two quality functions (4.108) is performed individually starting with $\rho = \rho^k$. Because for feasible and non-degenerate problems (NLP+) the least updates probably occur for π, it is selected first. For a sufficient decrease of the constraint violation, π is selected similar to Curtis [48] to be the largest value in $\{\pi, \kappa_\pi \pi, \kappa_\pi^2 \pi, \ldots, \kappa_\pi^{l_\rho} \pi\}$ with $l_\rho \in \mathbb{N}$ such that

$$Q_{\text{feas}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu); \rho\Big) \leq \kappa_1 Q_{\text{feas}}\Big(w^k, \widetilde{\Delta w}^k(\mu_k, \varepsilon, \lambda^k, \nu^k); \rho\Big) < 0 \tag{4.109}$$

where $\kappa_1 \in (0, 1)$. This criterion ensures a small enough penalty parameter to yield a step that is a descent direction for the approximate feasibility problem. If no penalty parameter π can be found that satisfies (4.109), the penalty parameter is left unchanged. Similar to Proposition 4.40 it can be shown that under this penalty update the step $\widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu)$ still yields a descent direction for the merit functions $\Upsilon(x^k; \rho^k)$ and $\Psi(w^k; \rho^k)$.

For the Lagrangian multiplier λ it is checked if an update $\lambda = y^k/\pi$ would yield a more beneficial step with respect to the optimality quality function by

$$Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, y^k/\pi, \nu); \alpha, \pi\Big) \leq \kappa_2 Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu); \alpha, \pi\Big), \tag{4.110a}$$

$$D^x_{\widetilde{\Delta w}^k(\mu, \pi, y^k \pi, \nu)} \Psi(w^k; \rho) \leq \kappa_3 Q_{\text{feas}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \varepsilon, \lambda, \nu); \rho\Big) < 0, \tag{4.110b}$$

with $\kappa_2, \kappa_3 \in (0, 1)$. The second condition (4.110b) is necessary because a combined update of the penalty parameter π and the Lagrangian multiplier λ can result in an inexact approximation $\widetilde{\Delta w}^k(\mu, \pi, y^k \pi, \nu)$ of the step $\Delta w^k(\mu, \pi, y^k \pi, \nu)$ and thus violate the descent direction property that is necessary for the line search. The update of the Lagrangian multiplier ν is handled analogously, i.e., it is updated to $\nu = z^k/\pi$ if

$$Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, \lambda, z/\pi); \alpha, \pi\Big) \leq \kappa_2 Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu); \alpha, \pi\Big), \tag{4.111a}$$

$$D^x_{\widetilde{\Delta w}^k(\mu, \pi, \lambda, z/\pi)} \Psi(w^k; \rho) \leq \kappa_3 Q_{\text{feas}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \varepsilon, \lambda, \nu); \rho\Big) < 0 \tag{4.111b}$$

holds. It has to be noted, that an adaptive update of ν_i for $i \in \{1, \ldots, n_x\}$ may be beneficial in reducing the number of iterations, but would be too expensive with respect to computation time in practice.

After the updates of the penalty parameter π and the Lagrangian multipliers (λ, ν) have been performed, the adaptive barrier update with the highest flexibility of all adaptive updates is applied. Here, one aims to find the best value of μ that minimizes the optimality quality function under the constraint that the resulting step still yields a descent direction for the primal-dual merit function $\Psi(w^k; \rho^k)$, i.e.,

$$\begin{aligned} \min_{\mu \in \mathcal{M}} \quad & Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu); \alpha, \pi\Big) \\ \text{subject to} \quad & D^x_{\widetilde{\Delta w}^k(\mu, \pi, \lambda, \nu)} \Psi(w^k; \rho) \leq \kappa_3 D^x_{\Delta w^k} \Psi(w^k; \rho^k). \end{aligned} \tag{4.112}$$

It is obviously too expensive in practice to solve (4.112), but an approximate solution found by a bi- or trisection rule is sufficient (cf., Nocedal et al. [153]). For the barrier parameter it is crucial to define a search interval $\mathscr{M}$, as not all values are valid in the modified barrier function. In particular, it is not allowed that $x_i^k + \mu\varsigma_i^k \leq 0$ for any $i \in \{1,\ldots,n_x\}$. A primal projection like in the update of Section 4.2.3 is not practical as this would imply to renew the factorization and step calculation – the most expensive part of Algorithm L. Thus, $\mathscr{M}$ is defined as

$$\mathscr{M} := \begin{cases} [\mu^+, \mu_k], & \text{if } Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k((1-\varepsilon)\mu_k, \pi, \lambda, \nu); \alpha, \pi\Big) \\ & \quad \leq Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(\mu_k, \pi, \lambda, \nu); \alpha, \pi\Big) \\ [\mu_k, \mu_0], & \text{otherwise} \end{cases} \tag{4.113}$$

Here, μ^+ is given by

$$\mu^+ := \min\left\{\mu_k \in (0, \mu_{k-1}) \mid x^k + \mu_k\varsigma^k \geq (1-\tau_{k-1})\left(x^{k-1} + \mu_{k-1}\varsigma^{k-1}\right)\right\} \tag{4.114}$$

similar to (4.44). A further benefit of the adaptive barrier update is the possibility of a sufficient reduction of the boundary shift throughout the iterations such that the number of primal projections (4.46), that require further function evaluations, is reduced.

Algorithm M formally states the adaptive parameter update strategy and would be applied to Algorithm L as a magic step in Step L-6. Therefore, it maintains the global convergence properties of Section 4.3. Due to Step M-4 the adaptive updates will not be applied locally near an optimal solution of (NLPbar) and the local convergence analysis of Section 4.4 stays valid. In Algorithm M a line search, second-order-corrections and dual update are included that equal the corresponding steps of Algorithm L.

4.7 Warmstarts

Many nonlinear programming applications require the sequential solution of similar nonlinear programs, which has already been explained in more detail at the beginning of this chapter. This setting can formally be expressed as solving the parameter dependent nonlinear program (NLPp), or according to (NLP+) the program

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x;p) \\ \text{subject to} \quad & g(x;p) = 0 \\ & x \geq 0. \end{aligned} \tag{NLPp+}$$

After an initial solve for a reference parameter $p^* \in \mathbb{R}^{n_p}$ Algorithm L should exploit this optimization run to improve efficiency for subsequent optimizations for different parameters p, or – in other words – should be able to warmstart.

Algorithm M One Iteration of Adaptive Updates for the Primal-Dual Penalty-Interior-Point Algorithm

M-1: *(Initialization)* Choose $k \in \mathbb{N}$, a point $\left(x^k, y^k, z^k\right)$ and parameter $\left(\lambda^k, \nu^k\right)$ such that $x^k \geq 0$, $z^k > 0$ and $\nu^k > 0$. Choose the parameters as in Algorithm L and $\kappa_1 \in (0,1)$, $\kappa_2 \in (0,1)$, $\kappa_3 \in (0,1)$ and $l_\rho \in \mathbb{N}$.
M-2: *(Hessian regularization)* Equals Step L-7.
M-3: *(Step calculation)* Solve the linear equation system (4.12) to get $\left(\Delta x^k, \Delta y^k, \Delta z^k\right)$.
M-4: *(Check full step)* If $\alpha_{\max,k} = \alpha_k^z = 1$ in (4.15) and (4.28) and (4.31) is satisfied for $\left(x^k, y^k, z^k\right) + \left(\Delta x^k, \Delta y^k, \Delta z^k\right)$, then RETURN FAILURE.
M-5: *(Sensitivity calculation)* Solve the linear equation systems of Corollary 4.39 to get $\left(\frac{\mathrm{d}\Delta x^k}{\mathrm{d}\rho}(\rho^k), \frac{\mathrm{d}\Delta y^k}{\mathrm{d}\rho}(\rho^k), \frac{\mathrm{d}\Delta z^k}{\mathrm{d}\rho}(\rho^k)\right)$.
M-6: *(Adaptive penalty update)* Choose π to be the largest value in $\left\{\pi, \kappa_\pi \pi, \kappa_\pi^2 \pi, \dots, \kappa_\pi^{l_\rho} \pi\right\}$ such that (4.109) holds. If none such value exists, $\pi = \pi_k$. Set $\pi_k \leftarrow \pi$.
M-7: *(Adaptive equality multiplier update)* If (4.110) is satisfied, set $\lambda^k \leftarrow y^k$.
M-8: *(Adaptive inequality multiplier update)* If (4.111) is satisfied, set $\nu^k \leftarrow z^k$.
M-9: *(Adaptive barrier update)* Find $\mu \in \mathscr{M}$ defined by (4.113) that approximately solves (4.112) and set $\mu_k \leftarrow \mu$.
M-10: *(Line search)* Equals Step L-10.
M-11: *(Second-Order-Corrections)* Equals Step L-11.
M-12: *(Dual update)* Equals Step L-12.
M-13: *(Dual projection)* Equals Step L-13.
M-14: *(Check adaptive update)* If (4.50) is satisfied, accept the trial iterate, augment the filter or PLPF $\mathscr{F}_{\mathrm{mag},k}(l_\mathrm{f})$ by (4.51) and RETURN OK. Otherwise, set $\mathscr{F}_{\mathrm{mag},k}(l_\mathrm{f}) \leftarrow \mathscr{F}_{\mathrm{mag},k}(l_\mathrm{f})$ and RETURN FAILURE.

4.7.1 Challenges for Interior-Point Warmstarts

For this warmstarting task, interior-point algorithms have a very bad reputation and SQP methods with an active-set QP solver are usually favored. The explanation is simple: If the perturbation $\Delta p = p - p^*$ is small, the active set $\mathscr{A}(x^*; p^*)$ is likely to be unchanged or to be at least a very good approximation to $\mathscr{A}(x(p); p)$ and the SQP method can be expected to converge quickly. In contrast, an interior-point method cannot exploit this. Even worse, the optimal solution (x^*, λ^*, ν^*) is an invalid initial guess for the new optimization if at least one inequality constraint is active – a very probable situation. An interior-point method must instead be initialized strictly inside the interior region, favorably at the central path of the new nonlinear program. The central path is however unknown and it can easily happen that the new barrier parameter forces the iterates to move even further into the interior even though the new optimal solution $x(p)$ is located at the boundary near x^*. This is illustrated in Figure 4.3 for a perturbed central path based on Example 2.3 (cf., Figure 3.5). For a classic log-barrier function method it is therefore usually advised, e.g., by Gondzio and González-Brevis [99] and Gondzio and Grothey [100], to warmstart not from the optimal solution (x^*, λ^*, ν^*), but from an intermediate iterate $\left(x^k, \lambda^k, \nu^k\right)$ that is sufficiently inside the interior region and close to the central path, i.e., from the optimal solution of the barrier subproblem (3.59) for some $\mu > 0$. Due to the boundary shift, this drawback does not occur for the modified barrier function (cf., Benson and Shanno [15]) and the method can warmstart from the previous optimal solution.

If the perturbation $\Delta p = p - p^*$ is explicitly known, the parametric sensitivity information can be used to approximate the solution $(x(p), \lambda(p), \nu(p))$ of (NLPp+) by

$$w^* + \frac{\mathrm{d}w}{\mathrm{d}p}(p^*)\Delta p. \tag{4.115}$$

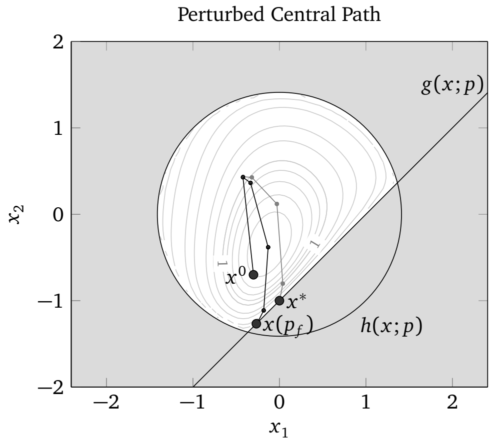

Figure 4.3: Perturbed central path for Example 2.3 and $p_f = (0,1)$ based on log-barrier function. The equality constraint of Example 2.3 is relaxed to an inequality constraint. Nodes are plotted for solutions $x\big(p_f^*,\mu\big)$ (gray) and $x\big(p_f,\mu\big)$ (black) with barrier parameters $\mu \in \{100, 10, 1, 0.1\}$ and contours for barrier function are plotted for p_f^* and $\mu = 1$. The point x^0 may be an initial point shifted into interior from x^*.

As described in Section 4.5.1 the calculation of the sensitivity derivatives is efficient and therefore suitable for a warmstart procedure. At this point the importance of the modified barrier function has again to be emphasized, because the following does not hold for it. As the classic log-barrier function methods are warmstarted from an intermediate point $\big(x^k, \lambda^k, \nu^k\big)$, the sensitivity analysis cannot be applied. Even if the sensitivity derivatives would be approximated by the application of Corollary 2.27 evaluated at $\big(x^k, \lambda^k, \nu^k\big)$, the computation would be too expensive because the matrix factorization is not available at the end of the optimization. The reason is that for large-scale nonlinear programming it is not a practical option to store it in addition to the one of the current iteration.

A further challenge are active set changes evoked by the first-order approximation (4.115). To handle this, Kadam and Marquardt [123] and Wolbert et al. [193] propose to solve the update $d = \frac{\mathrm{d}x}{\mathrm{d}p}(p^*)\Delta p$ directly by the quadratic program

$$\begin{aligned} \min_{d\in\mathbb{R}^{n_x}} \quad & \frac{1}{2} d^\top \nabla^2_{xx} L(x^*, \lambda^*, \nu^*; p^*) d + d^\top \nabla^2_{xp} L(x^*, \lambda^*, \nu^*; p^*)^\top (p - p^*) \\ \text{subject to} \quad & \nabla_x g(x;p)^\top d + \nabla_p g(x;p)^\top (p - p^*) = 0 \\ & x + d \geq 0, \end{aligned} \tag{4.116}$$

instead of using the sensitivity derivatives. While this approach can handle active set changes due to the further inequality constraint, it is computationally more expensive. In particular, the quadratic program (4.116) has to be solved for every perturbation $\Delta p = p - p^*$. Moreover, it can easily happen for nonlinear programming that the Hessian matrix $\nabla^2_{xx} L(x^*, \lambda^*, \nu^*; p^*)$ is not positive definite on the null space of the active constraints of (4.116) or that the linearized constraints are inconsistent, both leading to failure of this approach. A second proposal by Pirnay et al. [161] and Zavala [205, Section 3.2.3] is the addition of a further equality constraint to the linear equation system of Corollary 2.27 that forces the optimization variable $x_j^* + d_j$ to zero if the jth inactive constraint becomes active, i.e.,

$$\begin{bmatrix} \nabla^2_{xx}L^* & \nabla g^* & -E & 0 \\ (\nabla g^*)^\top & 0 & 0 & 0 \\ \operatorname{diag}(\nu^*) & 0 & X_* & e_j \\ e_j^\top & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ d_\lambda \\ d_\nu \\ d'_\nu \end{bmatrix} = - \begin{bmatrix} \nabla^2_{xp}L^*(p^*)^\top (p-p^*) \\ \nabla_p g^*(p^*)^\top (p-p^*) \\ 0 \\ x_j^* \end{bmatrix}, \tag{4.117}$$

(in case of a classic log-barrier function, cf., (4.95)) or analogously for the Lagrangian multipliers for active constraints becoming inactive. The linear equation system (4.117) can be solved by using a *Schur complement*[12] and therefore can benefit from the factorization of the linear equation system for the step calculation (cf., for instance Pirnay et al. [161, Section 2.5]) – if available at the warmstarting point. In the end, (4.117) is similar of one step of an active-set QP solver applied to (4.116) (cf., Section 3.5.1 and Bartlett and Biegler [9]).

4.7.2 Warmstarts Based on Iterative Real-Time Updates

In the context of warmstarting the penalty-interior-point algorithm the aim is to approximate not only the solution of (NLPp+) but also of

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x;p) - \sum_{i=1}^{n_x} (\mu\pi\varsigma_i \nu_i + p_c) \ln\left(\frac{x_i}{\mu\varsigma_i} + 1\right) \\ \text{subject to} \quad & g(x;p) - p_g = 0, \end{aligned} \tag{NLPpbar}$$

for some updated barrier parameter μ, primal shift parameter ς and Lagrangian multipliers ν to provide a good initial guess for Algorithm L. A former optimization of (NLPpbar) serves as a reference problem with reference parameters $\bar{\rho}$, $p_c^* = 0$ and $p_g^* = 0$, which is equivalent to solving (NLPbar). The parameters μ, ς and ν must be updated for the new optimization run because the barrier term might be invalid otherwise, e.g., if $\bar{\nu}_i = 0$ holds for some $i \in \{1, \dots, n_x\}$ in the previous optimization. Since (NLPbar) and (NLPpen) yield equivalent first-order optimality conditions for a sufficiently large penalty parameter $\tau > 0$ (cf., (4.4) and (4.7)), one can concentrate on (NLPpbar). In the following it will be derived how the sensitivity derivatives of (NLPpbar) can be exploited to generate good initial starting points for Algorithm L even if active constraints become inactive. The method will rely on the approximated sensitivity derivatives $\frac{\mathrm{d}w}{\mathrm{d}p}(p,\rho)$ of Section 4.5.1[13] and matrix-vector products only and is therefore a computationally very cheap warmstarting procedure.

To approximate $\nu(p)$ for the choice of ν in (NLPpbar) a projected real-time update is utilized, i.e.,

$$\widetilde{\nu}(p) = \frac{1}{\bar{\pi}} \max\left\{\varepsilon e, z^* + \frac{\mathrm{d}z}{\mathrm{d}p}(p^*, \bar{\rho})\Delta p\right\} \tag{4.118}$$

[12] See Definition A.10.

[13] In Section 4.5.1 only the approximated sensitivity derivatives $\frac{\mathrm{d}\omega}{\mathrm{d}p}(p,\rho)$ are defined in (4.96), but $\frac{\mathrm{d}w}{\mathrm{d}p}(p,\rho)$ directly follow by an appropriate scaling. In particular, the scaling matrices in (4.96b) need to be removed.

with $\varepsilon > 0$, to keep the Lagrangian multipliers ν strictly positive. Its perturbation is $\Delta\nu = \widetilde{\nu}(p) - \bar{\nu}$. If the penalty parameter τ stays unchanged, the primal-dual optimal solution can be approximated for the perturbations $\Delta p = p - p^*$ and $\Delta\rho = \rho - \bar{\rho}$ by

$$\begin{aligned}\widetilde{x}^0(p) &= x^* + \frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + \frac{\mathrm{d}x}{\mathrm{d}\rho}(p^*,\bar{\rho})\Delta\rho \\ &= x^* + \frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + \frac{\mathrm{d}x}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu \end{aligned} \tag{4.119a}$$

$$\widetilde{y}^0(p) = y^* + \frac{\mathrm{d}y}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + \frac{\mathrm{d}y}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu \tag{4.119b}$$

$$\widetilde{z}^0(p) = z^* + \frac{\mathrm{d}z}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + \frac{\mathrm{d}z}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu \tag{4.119c}$$

since from Corollary 4.36 it is known that most sensitivity derivatives evaluate to zero if $\left(\bar{\lambda},\bar{\nu}\right) = (y^*/\bar{\pi}, z^*/\bar{\pi})$, a condition that holds for limit points of Algorithm L. The first-order approximation (4.119) is however likely to violate the complementarity condition (4.3c), i.e.,

$$\widetilde{X}_0(p)\widetilde{z}^0(p) \neq 0. \tag{4.120}$$

This holds in particular for active set changes. The complementarity error can be estimated by

$$\begin{aligned}&\widetilde{X}_0(p)\widetilde{z}^0(p) \\ &\quad = X_* z^* + X_*\frac{\mathrm{d}z}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + Z_*\frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + X_*\frac{\mathrm{d}z}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu + Z_*\frac{\mathrm{d}x}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu \\ &\qquad + \mathcal{O}\left(\|\Delta p\|^2\right) + \mathcal{O}\left(\|\Delta\nu\|^2\right) \\ &\quad = X_* z^* - \bar{\mu}\bar{\Sigma}\frac{\mathrm{d}z}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p - \bar{\mu}\bar{\Sigma}\frac{\mathrm{d}z}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu + \bar{\mu}\bar{\pi}\bar{\Sigma}\Delta\nu + \mathcal{O}\left(\|\Delta p\|^2\right) + \mathcal{O}\left(\|\Delta\nu\|^2\right) \\ &\quad = \bar{\mu}\bar{\pi}\bar{\Sigma}\bar{\nu} - \bar{\mu}\bar{\Sigma}\left(z^* + \frac{\mathrm{d}z}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p + \frac{\mathrm{d}z}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right) + \bar{\mu}\bar{\pi}\bar{\Sigma}\Delta\nu \\ &\qquad + \mathcal{O}\left(\|\Delta p\|^2\right) + \mathcal{O}\left(\|\Delta\nu\|^2\right) \\ &\quad = \bar{\mu}\bar{\Sigma}\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right) + \mathcal{O}\left(\|\Delta p\|^2\right) + \mathcal{O}\left(\|\Delta\nu\|^2\right).\end{aligned} \tag{4.121}$$

Due to $\bar{\varsigma}_i = 0$, $z_i^* = 0$ and $\frac{\mathrm{d}z_i}{\mathrm{d}p}(p^*,\bar{\rho}) = 0$ for $i \notin \mathcal{A}(x^*;p^*)$ (cf., Section 4.2.3 and (4.96b)) it holds $\widetilde{x}_i^0(p)\widetilde{z}_i^0(p) = 0$ for inactive constraints. In addition, $\widetilde{x}_i^0(p)\widetilde{z}_i^0(p) = \mathcal{O}\left(\|\Delta p\|^2\right) + \mathcal{O}\left(\|\Delta\nu\|^2\right)$ if $\widetilde{z}_i^0(p)$ is positive since then $\bar{\pi}\widetilde{\nu}(p) = \widetilde{z}^0(p)$ holds. Both are expected results from sensitivity analysis, because it relies on the fact that the active set does not change (cf., Theorem 2.26). This is reflected by the sensitivity derivatives when computed by (4.95) and has been stated in Corollary 2.28. The following method exploits the fact, that the complementarity violation does however not vanish for the sensitivity derivative approximations (4.96) if $i \in \mathcal{A}(x^*;p^*)$, which will make the handling of active set changes (at least active to inactive) possible.

A direct consequence of (4.121) is the error of the perturbed complementarity condition (4.4c), which is for small perturbations Δp and $\Delta\nu$

$$\begin{aligned}\widetilde{X}_0(p)\widetilde{z}^0(p) - \mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right) &\approx \bar{\mu}\bar{\Sigma}\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right) - \mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right) \\ &= \left(\bar{\mu}\bar{\Sigma} - \mu\Sigma\right)\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right).\end{aligned} \tag{4.122}$$

Algorithm N Iterative Real-Time Update Based Warmstart

N-1: *(Initialization)* Set $k \leftarrow 0$. Choose a starting point $\left(\widetilde{x}^0(p), \widetilde{y}^0(p), \widetilde{z}^0(p)\right)$ by (4.119). Select $\varepsilon > 0$, $\mu > 0$ and $\varsigma > 0$. Compute $\widetilde{\nu}(p)$ by (4.118).

N-2: *(Termination check)* If some termination condition holds, then STOP.

N-3: *(Real-time update)* Update $\left(\widetilde{x}^{k+1}(p), \widetilde{y}^{k+1}(p), \widetilde{z}^{k+1}(p)\right)$ by (4.125) and (4.126).

N-4: *(k increment)* Set $k \leftarrow k+1$ and go to Step N-2.

As before, it vanishes if the multiplier $\widetilde{z}^0(p)$ stays positive, which is related to the case that the active set would not change. The violation of (4.122) can be interpreted as a perturbation of the parameter p_c. Analogously to Section 3.6.1 this perturbation can be compensated by

$$\widetilde{x}^1(p) = \widetilde{x}^0(p) - \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*, \bar{\rho}, 0)\left(\bar{\mu}\bar{\Sigma} - \mu\Sigma\right)\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^0(p)\right) \tag{4.123}$$

and similar for the dual variables. As this approximation will probably again violate the complementarity condition as in (4.120) this procedure can be applied iteratively, i.e.,

$$\widetilde{x}^{k+1}(p) = \widetilde{x}^k(p) - \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*, \bar{\rho}, 0)\left(\widetilde{X}_k(p)\widetilde{z}^k(p) - \mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^k(p)\right)\right). \tag{4.124}$$

Finally, the combination of (4.124) with the feasibility refinement of Section 3.6.1 yields the iterative process

$$\begin{bmatrix} \widetilde{x}^{k+1}(p) \\ \widetilde{z}^{k+1}(p) \end{bmatrix} = \begin{bmatrix} \widetilde{x}^{k}(p) \\ \widetilde{z}^{k}(p) \end{bmatrix} - \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*, \bar{\rho}, 0) & \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*, \bar{\rho}, 0) \\ \frac{\mathrm{d}z}{\mathrm{d}p_g}(p^*, \bar{\rho}, 0) & \frac{\mathrm{d}z}{\mathrm{d}p_c}(p^*, \bar{\rho}, 0) \end{bmatrix} \begin{bmatrix} \Delta p_g^k \\ \Delta p_c^k \end{bmatrix}, \tag{4.125a}$$

$$\begin{bmatrix} \Delta p_g^k \\ \Delta p_c^k \end{bmatrix} = \begin{bmatrix} g\left(\widetilde{x}^k(p); p\right) \\ \widetilde{X}_k(p)\widetilde{z}^k(p) - \mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p) - \widetilde{z}^k(p)\right) \end{bmatrix}, \tag{4.125b}$$

an approach that is closely related to the higher order corrections developed for linear and quadratic programming (cf., Gondzio and Grothey [100]). The dual variables y do not have to, but can be updated analogously by

$$\widetilde{y}^{k+1}(p) = \widetilde{y}^k(p) - \begin{bmatrix} \frac{\mathrm{d}y}{\mathrm{d}p_g}(p^*, \bar{\rho}, 0) & \frac{\mathrm{d}y}{\mathrm{d}p_c}(p^*, \bar{\rho}, 0) \end{bmatrix} \begin{bmatrix} \Delta p_g^k \\ \Delta p_c^k \end{bmatrix}. \tag{4.126}$$

Algorithm N formally states the warmstart procedure based on iterative real-time updates. This iterative algorithm converges linearly to a limit point that satisfies the feasibility and perturbed complementarity condition of (NLPpbar), has an acceptably small perturbation of the optimal objective function value and is therefore an excellent starting point for a new optimization run of the penalty-interior-point algorithm.

Theorem 4.41 (Local Convergence of Iterative Real-Time Update Based Warmstart). *Let the assumptions of Theorem 2.26 be satisfied in a neighborhood $\mathcal{N}(w^*)$ and $w(\mu,\nu) \in \mathcal{N}(w^*)$ be an optimal solution of* (NLPbar) *with $\lim_{\rho \to \bar{\rho}} w(\mu,\nu) = w^*$ for some $\bar{\rho}$. Let $f(x;p)$ as well as $g(x;p)$ be three times continuously differentiable. Additionally, assume that Algorithm N produces an infinite number of iterations and $\left\|\mu\Sigma - \bar{\mu}\bar{\Sigma}\right\| = \mathcal{O}(\|\Delta p\|)$ as well* *as* $\|\Delta\nu\| = \mathcal{O}(\|\Delta p\|)$.[14] *Then it exists a neighborhood* $\mathscr{P}$ *around* p^* *such that for all* $p \in \mathscr{P}$ *the iterates* $\big(\tilde{x}^k(p,\rho), \tilde{y}^k(p,\rho), \tilde{z}^k(p,\rho)\big)$ *converge q-linearly to a unique limit point* $(\tilde{x}(p,\rho), \tilde{y}(p,\rho), \tilde{z}(p,\rho))$ *and the following error estimations hold:*

$$\begin{aligned}\|\tilde{w}(p,\rho) - w(p,\rho)\| &= \mathcal{O}\big(\|\Delta p\|^2\big)\\ |f(\tilde{x}(p,\rho);p) - f(x(p,\rho);p)| &= \mathcal{O}\big(\|\Delta p\|^2\big)\\ \|g(\tilde{x}(p,\rho);p)\| &= 0\\ \big\|\tilde{X}(p,\rho)\tilde{z}(p,\rho) - \mu\Sigma(\bar{\pi}\tilde{\nu}(p) - \tilde{z}(p,\rho))\big\| &= 0.\end{aligned}$$

Furthermore, if $\bar{\pi}\tilde{\nu}(p) = z(p,\rho)$, *then* $\big\|\tilde{X}(p,\rho)\tilde{z}(p,\rho)\big\| = \mathcal{O}\big(\|\Delta p\|^2\big)$. *After the first iteration the following error estimates hold:*

$$\begin{aligned}\big\|g\big(\tilde{x}^1(p,\rho);p\big)\big\| &= \mathcal{O}\big(\|\Delta p\|^3\big)\\ \big\|\tilde{X}_1(p,\rho)\tilde{z}^1(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^1(p,\rho)\big)\big\| &= \mathcal{O}\big(\|\Delta p\|^3\big).\end{aligned}$$

Proof. The proof is an extension of the feasibility refinement convergence proof of Büskens [33, Theorem 4.4]. Due to assumption $w(\mu,\nu) \in \mathcal{N}(w^*)$ for whose elements the LICQ and SOSC hold, the used sensitivity derivatives exist. First, it will be shown by induction that

$$\left\|\tilde{w}^k(p,\rho) - w^* - \frac{\mathrm{d}w}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p - \frac{\mathrm{d}w}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right\| \le \mathcal{O}\big(\|\Delta p\|^2\big). \tag{4.127}$$

The induction base case is straightforward because (4.119) implies

$$\left\|\tilde{w}^0(p,\rho) - w^* - \frac{\mathrm{d}w}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p - \frac{\mathrm{d}w}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right\| = 0$$

So assume that the induction hypothesis (4.127) is true for some $k \in \mathbb{N}$. For the induction step case the violation of the feasibility and complementarity condition appearing in (4.125) will be estimated. Using the induction hypothesis, a first-order Taylor approximation in both, the primal-dual optimization variables and the parameters, Corollary 2.29 and $\nabla_x g(x^*;p^*)^\top \frac{\mathrm{d}x}{\mathrm{d}\nu}(p^*,\bar{\rho}) = 0$ it follows for the constraint violation

$$\begin{aligned}&\big\|g\big(\tilde{x}^k(p,\rho);p\big)\big\|\\ &\quad\le \Big\|g(x^*;p) + \nabla_x g(x^*;p)^\top\big(\tilde{x}^k(p,\rho) - x^*\big)\Big\| + \mathcal{O}\Big(\big\|\tilde{x}^k(p,\rho) - x^*\big\|^2\Big)\\ &\quad\le \Big\|\nabla_p g(x^*;p^*)^\top\Delta p + \nabla_x g(x^*;p^*)^\top\big(\tilde{x}^k(p,\rho) - x^*\big)\Big\| + \mathcal{O}\Big(\big\|\tilde{x}^k(p,\rho) - x^*\big\|^2\Big)\\ &\qquad + \mathcal{O}\Big(\big\|\tilde{x}^k(p,\rho) - x^*\big\|\,\|\Delta p\|\Big) + \mathcal{O}\big(\|\Delta p\|^2\big)\\ &\quad\overset{(4.127)}{\le} \left\|\nabla_p g(x^*;p^*)^\top\Delta p + \nabla_x g(x^*;p^*)^\top\frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p\right\|\\ &\qquad + \left\|\nabla_x g(x^*;p^*)^\top\frac{\mathrm{d}x}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right\| + \mathcal{O}\big(\|\Delta p\|^2\big)\\ &\quad= \mathcal{O}\big(\|\Delta p\|^2\big)\end{aligned} \tag{4.128}$$

[14]The assumptions on the perturbation of the algorithmic parameters of (NLPpbar) are not very restrictive and made for reasons of clarity only – in particular for the proof.

and for the violation of the complementarity condition

$$
\begin{aligned}
&\left\|\widetilde{X}_k(p,\rho)\widetilde{z}^k(p,\rho)-\mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p)-\widetilde{z}^k(p,\rho)\right)\right\| \\
&\quad\le \left\|-\mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p)-z^*\right)+Z_*\left(\widetilde{x}^k(p,\rho)-x^*\right)+\left(X_*+\mu\Sigma\right)\left(\widetilde{z}^k(p,\rho)-z^*\right)\right\| \\
&\qquad+\mathcal{O}\left(\left\|\widetilde{w}^k(p,\rho)-w^*\right\|^2\right) \\
&\quad\le \left\|Z_*\left(\widetilde{x}^k(p,\rho)-x^*\right)+\left(X_*+\bar{\mu}\bar{\Sigma}\right)\left(\widetilde{z}^k(p,\rho)-z^*\right)-\mu\bar{\pi}\Sigma\Delta\nu\right\| \\
&\qquad+\left\|\left(\widetilde{z}^k(p,\rho)-z^*\right)\left(\mu\Sigma-\bar{\mu}\bar{\Sigma}\right)\right\|+\mathcal{O}\left(\left\|\widetilde{w}^k(p,\rho)-w^*\right\|^2\right) \\
&\quad\overset{(4.127)}{\le} \left\|Z_*\frac{\mathrm{d}x}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p+\left(X_*+\bar{\mu}\bar{\Sigma}\right)\frac{\mathrm{d}z}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p\right\| \\
&\qquad+\left\|Z_*\frac{\mathrm{d}x}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu+\left(X_*+\bar{\mu}\bar{\Sigma}\right)\frac{\mathrm{d}z}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu-\mu\bar{\pi}\Sigma\Delta\nu\right\|+\mathcal{O}\left(\|\Delta p\|^2\right) \\
&\quad=\mathcal{O}\left(\|\Delta p\|^2\right).
\end{aligned}
\tag{4.129}
$$

From (4.128) and (4.129) it then follows $\left\|\Delta p_g^k\right\|=\mathcal{O}\left(\|\Delta p\|^2\right)$ and $\left\|\Delta p_c^k\right\|=\mathcal{O}\left(\|\Delta p\|^2\right)$ and for the step case

$$
\begin{aligned}
&\left\|\widetilde{w}^{k+1}(p,\rho)-w^*-\frac{\mathrm{d}w}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p-\frac{\mathrm{d}w}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right\| \\
&\quad\le\left\|\widetilde{w}^k(p,\rho)-w^*-\frac{\mathrm{d}w}{\mathrm{d}p}(p^*,\bar{\rho})\Delta p-\frac{\mathrm{d}w}{\mathrm{d}\nu}(p^*,\bar{\rho})\Delta\nu\right\|+\mathcal{O}\left(\left\|\left(\Delta p_g^k,\Delta p_c^k\right)\right\|\right) \\
&\quad\overset{(4.127)}{\le}\mathcal{O}\left(\|\Delta p\|^2\right),
\end{aligned}
$$

which proves the hypothesis (4.127).

Next, it will be shown that the conditions for the application of the Banach fixed-point theorem[15] are satisfied. The iterative update (4.125) and (4.126) can be formulated equivalently as

$$
\underbrace{\begin{bmatrix}\widetilde{x}^{k+1}(p,\rho)-\widetilde{x}^k(p,\rho)\\ \widetilde{y}^{k+1}(p,\rho)-\widetilde{y}^k(p,\rho)\\ \widetilde{z}^{k+1}(p,\rho)-\widetilde{z}^k(p,\rho)\end{bmatrix}}_{=\widetilde{w}^{k+1}(p,\rho)-\widetilde{w}^k(p,\rho)}=-\underbrace{\begin{bmatrix}\frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0)\\ \frac{\mathrm{d}y}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}y}{\mathrm{d}p_c}(p^*,\bar{\rho},0)\\ \frac{\mathrm{d}z}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}z}{\mathrm{d}p_c}(p^*,\bar{\rho},0)\end{bmatrix}}_{=\frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)}\begin{bmatrix}\Delta p_g^k\\ \Delta p_c^k\end{bmatrix}.
\tag{4.130}
$$

By Corollary 2.27 and Corollary 4.37 it holds that

$$
(\nabla g^*)^\top\frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0)=E,
\tag{4.131a}
$$

$$
(\nabla g^*)^\top\frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0)=0,
\tag{4.131b}
$$

$$
Z_*\frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0)+(X_*+\bar{\mu})\frac{\mathrm{d}z}{\mathrm{d}p_g}(p^*,\bar{\rho},0)=0,\quad\text{and}
\tag{4.131c}
$$

$$
Z_*\frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0)+(X_*+\bar{\mu})\frac{\mathrm{d}z}{\mathrm{d}p_c}(p^*,\bar{\rho},0)=E.
\tag{4.131d}
$$

[15]See Theorem A.17.

Using this and multiplying (4.130) from the left with $\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}$ leads to

$$\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix} \left(\widetilde{w}^{k+1}(p,\rho) - \widetilde{w}^k(p,\rho)\right) = -\begin{bmatrix} \Delta p_g^k \\ \Delta p_c^k \end{bmatrix}.$$

This can then be substituted in (4.130), which yields

$$0 = \left(E - \frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0) \begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right) \left(\widetilde{w}^{k+1}(p,\rho) - \widetilde{w}^k(p,\rho)\right)$$

It is easy to show by induction that in general

$$\left(E - \frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0) \begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right) \left(\widetilde{w}^{k}(p,\rho) - \widetilde{w}^j(p,\rho)\right) = 0 \tag{4.132}$$

holds for all $k, j \in \mathbb{N}_0$ and therefore in particular for $j = 0$. This means, that for a fixed j (e.g., $j = 0$) the equation (4.132) describes a linear equation system with variables $\left(\widetilde{x}^k(p,\rho), \widetilde{y}^k(p,\rho), \widetilde{z}^k(p,\rho)\right) \in \mathbb{R}^{2n_x+n_g}$, where the kernel of the system matrix is invariant with respect to the iteration update (4.125). Because of

$$\begin{aligned} n_x + n_g &\geq \operatorname{rank}\left(\frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0)\right) \\ &\geq \operatorname{rank}\left(\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix} \frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0)\right) \\ &= n_x + n_g, \end{aligned}$$

which follows by (4.131) and Sylvester's rank inequality[16] and means that the matrix of sensitivity derivatives has full column rank, and

$$\left(E - \frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0) \begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right) \frac{\mathrm{d}w}{\mathrm{d}(p_g, p_c)}(p^*, \bar{\rho}, 0) = 0,$$

it follows that the matrix of the linear equation system in (4.132) has $n_x + n_g$ zero eigenvalues. It is easy to show that there are n_x eigenvectors with an eigenvalue of one, e.g., choose $\frac{\mathrm{d}x}{\mathrm{d}p_f}(p^*, \bar{\rho}, 0)$. This implies that the matrix has rank n_x and it exists a decomposition

$$\begin{aligned} \widetilde{x}^k(p,\rho) &= P_x \begin{bmatrix} a_x^k \\ b_x^k \end{bmatrix} = P_x \begin{bmatrix} a_x^k \\ b_x^k\left(a^k\right) \end{bmatrix} \\ \widetilde{y}^k(p,\rho) &= P_y \begin{bmatrix} a_y^k \\ b_y^k \end{bmatrix} = P_y \begin{bmatrix} a_y^k \\ b_y^k\left(a^k\right) \end{bmatrix} \end{aligned}$$

[16] See Lemma A.4.

$$\widetilde{z}^k(p,\rho) = P_z \begin{bmatrix} a_z^k \\ b_z^k \end{bmatrix} = P_z \begin{bmatrix} a_z^k \\ b_z^k\left(a^k\right) \end{bmatrix}$$
$$\widetilde{w}^k(p,\rho) = P \begin{bmatrix} a^k \\ b^k \end{bmatrix} = P \begin{bmatrix} a^k \\ b^k\left(a^k\right) \end{bmatrix}$$

with $a^k = \left(a_x^k, a_y^k, a_z^k\right) \in \mathbb{R}^{n_x+n_g}$, $b^k = \left(b_x^k, b_y^k, b_z^k\right) \in \mathbb{R}^{n_x}$ and appropriate permutation matrices P, P_x, P_y and P_z. Then, the system (4.132) is

$$\left(E - \frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right)\left(P\begin{bmatrix} a^k \\ b^k\left(a^k\right) \end{bmatrix} - \widetilde{w}^j(p,\rho)\right) = 0. \tag{4.133}$$

Because (4.132) is a linear equation system, $b^k\left(a^k\right)$ is an affine linear function and therefore differentiable with respect to a^k. The iteration (4.130) is reformulated to

$$P\begin{bmatrix} a^{k+1} \\ b^{k+1}\left(a^{k+1}\right) \end{bmatrix} = P\begin{bmatrix} a^k \\ b^k\left(a^k\right) \end{bmatrix} - \frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)B \tag{4.134a}$$

$$B = \begin{bmatrix} g\left(P_x\begin{bmatrix} a_x^k \\ b_x^k\left(a^k\right) \end{bmatrix};p\right) \\ P_x\begin{bmatrix} a_x^k \\ b_x^k\left(a^k\right) \end{bmatrix} P_z\begin{bmatrix} a_z^k \\ b_z^k\left(a^k\right) \end{bmatrix} - \mu\Sigma\left(\bar{\pi}\widetilde{\nu}(p) - P_z\begin{bmatrix} a_z^k \\ b_z^k\left(a^k\right) \end{bmatrix}\right) \end{bmatrix}. \tag{4.134b}$$

The Banach fixed-point theorem can then be applied to the update procedure with respect to the a^k if it is a contraction mapping[17]. In the following this missing piece will be shown. The derivative of (4.133) with respect to a^k for a fixed $j \in \mathbb{N}_0$ is

$$\left(E - \frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right) P\begin{bmatrix} E \\ \frac{\mathrm{d}b^k}{\mathrm{d}a^k}\left(a^k\right) \end{bmatrix} = 0.$$

It then follows for the derivative of (4.134) with respect to a^k that

$$\begin{aligned} &\left\|\left(E - \frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)\begin{bmatrix} \nabla_x g\left(\widetilde{x}^k(p,\rho);p\right)^\top & 0 & 0 \\ \widetilde{Z}_k(p,\rho) & 0 & \widetilde{X}_k(p,\rho) + \mu\Sigma \end{bmatrix}\right) P\begin{bmatrix} E \\ \frac{\mathrm{d}b^k}{\mathrm{d}a^k}\left(a^k\right) \end{bmatrix}\right\| \\ &\quad\le \left\|\left(E - \frac{\mathrm{d}w}{\mathrm{d}\left(p_g,p_c\right)}(p^*,\bar{\rho},0)\begin{bmatrix} \nabla_x g^*(p^*)^\top & 0 & 0 \\ Z_* & 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix}\right) P\begin{bmatrix} E \\ \frac{\mathrm{d}b^k}{\mathrm{d}a^k}\left(a^k\right) \end{bmatrix}\right\| + \mathcal{O}(\|\Delta p\|) \\ &\quad= \mathcal{O}(\|\Delta p\|) \\ &\quad< 1 \end{aligned}$$

for sufficiently small perturbations Δp. This shows that the update procedure is a contraction mapping. From the Banach fixed-point theorem it then follows that a^k converges q-linearly to a unique limit point. This transfers to the original variable

[17] See Definition A.12.

space, i.e., $(\tilde{x}_k(p,\rho), \tilde{y}_k(p,\rho), \tilde{z}_k(p,\rho))$ converges q-linearly to a unique limit point $(\tilde{x}(p,\rho), \tilde{y}(p,\rho), \tilde{z}(p,\rho))$. Furthermore, it follows that

$$g(\tilde{x}(p,\rho);p) = 0, \quad \text{and} \quad \tilde{X}(p,\rho)\tilde{z}(p,\rho) - \mu\Sigma(\bar{\pi}\tilde{\nu}(p) - \tilde{z}(p,\rho)) = 0.$$

Next, the proof is concerned with the error estimates after the first iteration. For the constraint violation it follows by Taylor's theorem, (4.128) as well as (4.131) that

$$\begin{aligned}
&\left\|g\big(\tilde{x}^1(p,\rho);p\big)\right\| \\
&\overset{(4.125)}{\leq} \left\|g\big(\tilde{x}^0(p,\rho);p\big) - \nabla_x g\big(\tilde{x}^0(p,\rho);p\big)^\top \frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0)\Delta p_g^0\right\| \\
&\quad + \left\|\nabla_x g\big(\tilde{x}^0(p,\rho);p\big)^\top \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0)\Delta p_c^0\right\| + \mathcal{O}\Big(\left\|\Delta p_g^0\right\|^2\Big) \\
&\overset{(4.131)}{\leq} \left\|g\big(\tilde{x}^0(p,\rho);p\big) - (\nabla_x g(x^*;p^*) + \mathcal{O}(\|\Delta p\|))^\top \frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0)\Delta p_g^0\right\| \\
&\quad + \mathcal{O}\Big(\left\|\Delta p_g^0\right\|^2\Big) \\
&\overset{(4.131)}{=} \left\|g\big(\tilde{x}^0(p,\rho);p\big) - \Delta p_g^0\right\| + \mathcal{O}\Big(\left\|\Delta p_g^0\right\| \|\Delta p\|\Big) + \mathcal{O}\Big(\left\|\Delta p_g^0\right\|^2\Big) \\
&\overset{(4.128)}{=} \mathcal{O}\big(\|\Delta p\|^3\big).
\end{aligned} \tag{4.135}$$

For the perturbed complementarity condition it follows analogously by (4.128), (4.129) and (4.131) that

$$\begin{aligned}
&\left\|\tilde{X}_1(p,\rho)\tilde{z}^1(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^1(p,\rho)\big)\right\| \\
&\overset{(4.128)}{\leq} \Bigg\|\tilde{X}_0(p,\rho)\tilde{z}^0(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^0(p,\rho)\big) \\
&\quad - \begin{bmatrix} \tilde{Z}_0(p,\rho) & 0 \\ 0 & \tilde{X}_0(p,\rho) + \mu\Sigma \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0) \\ \frac{\mathrm{d}z}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}z}{\mathrm{d}p_c}(p^*,\bar{\rho},0) \end{bmatrix} \begin{bmatrix} \Delta p_g^0 \\ \Delta p_c^0 \end{bmatrix}\Bigg\| + \mathcal{O}\big(\|\Delta p\|^4\big) \\
&\overset{(4.128)}{=} \Bigg\|\tilde{X}_0(p,\rho)\tilde{z}^0(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^0(p,\rho)\big) \\
&\quad - \begin{bmatrix} Z_* & 0 \\ 0 & X_* + \bar{\mu}\bar{\Sigma} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}x}{\mathrm{d}p_c}(p^*,\bar{\rho},0) \\ \frac{\mathrm{d}z}{\mathrm{d}p_g}(p^*,\bar{\rho},0) & \frac{\mathrm{d}z}{\mathrm{d}p_c}(p^*,\bar{\rho},0) \end{bmatrix} \begin{bmatrix} \Delta p_g^0 \\ \Delta p_c^0 \end{bmatrix}\Bigg\| + \mathcal{O}\big(\|\Delta p\|^3\big) \\
&\overset{(4.131)}{=} \left\|\tilde{X}_0(p,\rho)\tilde{z}^0(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^0(p,\rho)\big) - \Delta p_c^0\right\| + \mathcal{O}\big(\|\Delta p\|^3\big) \\
&= \mathcal{O}\big(\|\Delta p\|^3\big).
\end{aligned} \tag{4.136}$$

The error estimates of the first iteration can then be used to show the error estimate of the optimization variables. The Banach fixed-point theorem yields

$$\left\|\tilde{w}(p,\rho) - \tilde{w}^2(p,\rho)\right\| = \mathcal{O}\big(\left\|\tilde{w}^2(p,\rho) - \tilde{w}^1(p,\rho)\right\|\big). \tag{4.137}$$

The inequalities (4.135) and (4.136) imply

$$\begin{aligned}
&\left\|\tilde{w}^2(p,\rho) - \tilde{w}^1(p,\rho)\right\| \\
&\quad \leq \mathcal{O}\big(\left\|g\big(\tilde{x}^1(p,\rho);p\big)\right\|\big) + \mathcal{O}\big(\left\|\big(\tilde{X}_1(p,\rho)\tilde{z}^1(p,\rho) - \mu\Sigma\big(\bar{\pi}\tilde{\nu}(p) - \tilde{z}^1(p,\rho)\big)\big)\right\|\big) \\
&\quad = \mathcal{O}\big(\|\Delta p\|^3\big).
\end{aligned} \tag{4.138}$$

Similar it can be shown that

$$\left\|\widetilde{w}^1(p,\rho)-w(p,\rho)\right\| = \mathcal{O}\big(\|\Delta p\|^2\big) \tag{4.139}$$

because $\left\|\widetilde{w}^0(p,\rho)-w(p,\rho)\right\| = \mathcal{O}\big(\|\Delta p\|^2\big)$ follows directly by Taylor's theorem (cf., Section 2.2.2). Finally, combining (4.137), (4.138) and (4.139) shows

$$\begin{aligned}
&\|\widetilde{w}(p,\rho)-w(p,\rho)\| \\
&\quad \le \left\|\widetilde{w}(p,\rho)-\widetilde{w}^2(p,\rho)\right\| + \left\|\widetilde{w}^2(p,\rho)-\widetilde{w}^1(p,\rho)\right\| + \left\|\widetilde{w}^1(p,\rho)-w(p,\rho)\right\| \\
&\quad = \mathcal{O}\big(\|\Delta p\|^2\big),
\end{aligned}$$

which also implies $|f(\widetilde{x}(p,\rho);p)-f(x(p,\rho);p)| = \mathcal{O}\big(\|\Delta p\|^2\big)$.

The statement $\left\|\widetilde{X}(p,\rho)\widetilde{z}(p,\rho)\right\| = \mathcal{O}\big(\|\Delta p\|^2\big)$ if $\bar{\pi}\widetilde{\nu}(p) = z(p,\rho)$ is a direct implication of the error estimate of the primal-dual optimization variables $\|\widetilde{w}(p,\rho)-w(p,\rho)\| = \mathcal{O}\big(\|\Delta p\|^2\big)$. □

In contrast to the feasibility refinement of Section 3.6.1 the error estimate of the objective function does unfortunately not improve regarding the standard real-time update. The reason is that this iterative strategy is based on sensitivity approximations (cf., Section 4.5.1), which differ in the complementarity handling. In detail, $Z_* \frac{\mathrm{d}x}{\mathrm{d}p}(p,\rho) = 0$ is usually violated, a property that is however true for the exact sensitivity derivatives and that is used implicitly in the proof of Büskens [33, Theorem 4.4].[18]

However, an advantageous consequence of Theorem 4.41 is that in the case of an active set change in the direction of an active constraint becoming inactive, the multiplier $\widetilde{\nu}(p)$ can be easily identified to the optimal Lagrangian multiplier, which would be zero. Thus, the refinement strategy would be able to well approximate the new optimal solution despite this active set change – a feature that is not present in standard real-time updates. In the context of an interior-point warmstart, the Lagrangian multiplier $\widetilde{\nu}(p)$ has been chosen to be bounded away from zero in (4.118) to produce a valid initial point for Algorithm L. The effectiveness of this warmstart strategy based on iterative real-time updates in the situation of an active-set change is illustrated in the following example.

Example 4.42. *Consider the equivalent version of the nonlinear program in Example 2.3*

$$\begin{aligned}
\min_{x\in\mathbb{R}^2} \quad & f(x;p) = -\left(x_1-\frac{1}{2}\right)^3 + \frac{3}{4}(x_1-p+1) \\
\textit{subject to} \quad & g(x;p) = x_1^2 + (x_1-p)^2 + x_2 - 2 = 0 \\
& h(x;p) = -x_2 \le 0
\end{aligned}$$

where the former second optimization variable is substituted using the linear equality constraint and the infeasible reformulation (cf., Section 3.2.1) is applied. The constant parameter of the constraint in Example 2.3 now appears as the nonlinear parameter p. From Example 2.32, it is

[18] The complementarity property is used in Büskens [33, Theorem 4.4] when substituting the objective function for the constraints using the Lagrangian function and (4.3a) to benefit from the higher order of the constraint violation error, i.e., $\left\|g\big(\widetilde{x}^1(p,\rho);p\big)\right\| = \mathcal{O}\big(\|\Delta p\|^3\big)$.

known that for varying parameters p an active set change occurs near $p \approx 1.5$ *(to be more precise, it is* $p = \sqrt{2}$*), which leads to inaccurate real-time approximations for larger values. This difficult situation is the objective of this example. In particular, it aims to optimize a sequence of nonlinear programs with* $p \in \{1, 1.75\}$ *where the warmstart of Algorithm N shall provide a good initial guess despite the active set change.*

The optimal solutions

$$x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad x(1.75) = \left(\frac{1}{8}\left(7 - \sqrt{15}\right), 0\right) \approx \begin{bmatrix} 0.39 \\ 0 \end{bmatrix},$$
$$\lambda(1) = z(1) = 0, \qquad \lambda(1.75) = z(1.75) = \frac{1}{4}\left(\frac{9}{4} - \sqrt{\frac{3}{5}}\right) \approx 0.37$$

together with their sensitivity derivatives

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}p}(1,\bar{\rho}) &\approx -\begin{bmatrix} 0.00 \\ 2.00 \end{bmatrix}, & \frac{\mathrm{d}x}{\mathrm{d}p}(1.75,\bar{\rho}) &\approx \begin{bmatrix} 1.40 \\ 0.00 \end{bmatrix}, \\
\frac{\mathrm{d}x}{\mathrm{d}\nu}(1,\bar{\rho}) &\approx \begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}, & \frac{\mathrm{d}x}{\mathrm{d}\nu}(1.75,\bar{\rho}) &\approx \begin{bmatrix} 0.00 \\ 0.00 \end{bmatrix}, \\
\frac{\mathrm{d}x}{\mathrm{d}p_g}(1,\bar{\rho},0) &\approx \begin{bmatrix} 0.00 \\ 1.00 \end{bmatrix}, & \frac{\mathrm{d}x}{\mathrm{d}p_g}(1.75,\bar{\rho},0) &\approx -\begin{bmatrix} 0.52 \\ 0.00 \end{bmatrix}, \\
\frac{\mathrm{d}x}{\mathrm{d}p_c}(1,\bar{\rho},0) &\approx \begin{bmatrix} 0.67 \\ 1.33 \end{bmatrix}, & \frac{\mathrm{d}x}{\mathrm{d}p_c}(1.75,\bar{\rho},0) &\approx \begin{bmatrix} 1.40 \\ 2.71 \end{bmatrix}
\end{aligned}$$

are plotted in Figure 4.4. Obviously, $x_2(1)$ *is inactive and* $x_2(1.75)$ *is active. Assume the parameters of the solution process (Algorithm L) converge to* $\bar{\mu} = 10^{-6}$, $\bar{\pi} = 1$ *and the others have been updated as described in Section 4.2.3. They shall be initialized for the new optimization to* $\pi = 1$, $\tau = 1$, $\varsigma = (1, 1)$ *and* $\mu = 0.001$. *In addition,* $\varepsilon = 10^{-4}$.

If the parameter p is changed from 1.75 to 1, the active inequality constraint becomes inactive. The standard real-time update (4.119) *and the iterative refinement* (4.125) *are illustrated in Figure 4.5 (left). While the real-time update cannot cope with the active set change and leaves* $\widetilde{x}_2^0(1)$ *numerically at the boundary zero, the iterative refinement decouples from the boundary and converges successfully to a feasible point (up to* 10^{-5}*) that additionally satisfies the complementarity condition (up to* 10^{-5}*) of* (NLPp+) *and* (NLPpbar) *as shown in Table 4.1. The approximated point after 22 iterations* $\widetilde{x}^{22}(1) = (0.0556, 1.1049)$ *is indeed very close to the optimal solution* $x(1)$. *In particular much closer than* $\widetilde{x}^0(1)$.

If the warmstart should be carried out the other way round setting p to 1.75 from 1, the procedure returns an approximation $\widetilde{x}^{27}(1.75) = (0.3903, -0.001)$ *(cf., Table 4.2). However, it has to be noted, that* $\widetilde{x}_2^{27}(1.75)$ *violates its original inequality constraint and lays on the boundary of* (NLPpbar). *In order to overcome this problem, a warmstart could choose a larger barrier parameter* μ *after the refinement, e.g.,* $\mu = 0.1$.

iter	$\widetilde{x}_1^k(1)$	$\widetilde{x}_2^k(1)$	$\widetilde{z}^k(1)$	opti	feas	comp	comp-bar
0	-0.6619	0.0000	-0.7892	4.04e+00	1.20e+00	1.69e-06	1.58e-01
1	-0.0411	0.0021	-0.1063	3.32e-01	9.12e-01	2.28e-04	6.69e-04
2	-0.5117	0.0031	-0.6240	2.73e+00	5.50e-01	1.91e-03	5.06e-03
3	-0.2240	0.0099	-0.3076	9.59e-01	4.42e-01	3.05e-03	6.71e-03
4	-0.4474	0.0190	-0.5532	2.25e+00	3.14e-01	1.05e-02	2.21e-02
5	-0.2697	0.0490	-0.3578	1.18e+00	2.66e-01	1.75e-02	3.58e-02
6	-0.3821	0.0975	-0.4814	1.81e+00	1.54e-01	4.69e-02	9.49e-02
7	-0.2363	0.2261	-0.3211	1.01e+00	1.90e-01	7.26e-02	1.46e-01
8	-0.2321	0.4238	-0.3164	9.95e-01	4.24e-03	1.34e-01	2.69e-01
9	-0.0461	0.7883	-0.1118	3.44e-01	1.15e-01	8.81e-02	1.77e-01
10	0.0179	1.0275	-0.0414	2.12e-01	7.67e-03	4.25e-02	8.52e-02
11	0.0736	1.1430	0.0198	2.04e-01	6.65e-03	2.27e-02	4.53e-02
12	0.0453	1.0815	-0.0113	1.71e-01	4.93e-03	1.22e-02	2.45e-02
13	0.0599	1.1147	0.0048	1.69e-01	2.14e-03	5.30e-03	1.06e-02
14	0.0536	1.1004	-0.0022	1.60e-01	1.03e-03	2.41e-03	4.83e-03
15	0.0564	1.1069	0.0009	1.60e-01	4.46e-04	1.04e-03	2.09e-03
16	0.0552	1.1041	-0.0004	1.58e-01	1.96e-04	4.54e-04	9.09e-04
17	0.0557	1.1053	0.0002	1.58e-01	8.47e-05	1.96e-04	3.92e-04
18	0.0555	1.1048	-0.0001	1.58e-01	3.67e-05	8.45e-05	1.69e-04
19	0.0556	1.1050	0.0000	1.57e-01	1.58e-05	3.66e-05	7.31e-05
20	0.0555	1.1049	-0.0000	1.57e-01	6.82e-06	1.56e-05	3.14e-05
21	0.0556	1.1050	0.0000	1.57e-01	2.94e-06	6.88e-06	1.37e-05
22	0.0556	1.1049	-0.0000	1.57e-01	1.27e-06	2.82e-06	5.75e-06

Table 4.1: Iterations of feasibility and complementarity refinement for Example 4.42 with active set change from active to inactive. The columns contain for every iteration (iter) the point $(\widetilde{x}(p), \widetilde{z}(p))$ and the KKT conditions (4.3), i.e., optimality (opti), feasibility (feas) and complementarity (comp), as well as the KKT conditions (4.4) (opti, feas, comp-bar).

iter	$\widetilde{x}_1^k(1)$	$\widetilde{x}_2^k(1)$	$\widetilde{z}^k(1)$	opti	feas	comp	comp-bar
0	0.0000	-0.5000	0.0000	5.29e-08	5.63e-01	5.00e-19	1.00e-05
1	0.0000	-1.0625	0.0000	4.47e-07	1.00e-07	1.06e-07	3.12e-07
2	0.0000	-1.0625	0.0000	1.48e-06	2.06e-07	3.25e-07	7.50e-07
3	0.0000	-1.0625	0.0000	3.60e-06	4.25e-07	7.77e-07	1.65e-06
4	0.0000	-1.0625	0.0000	7.98e-06	8.76e-07	1.71e-06	3.51e-06
5	0.0000	-1.0625	0.0000	1.70e-05	1.81e-06	3.63e-06	7.35e-06
6	0.0000	-1.0625	0.0000	3.56e-05	3.72e-06	7.58e-06	1.53e-05
7	0.0000	-1.0625	0.0000	7.40e-05	7.68e-06	1.57e-05	3.15e-05
8	0.0000	-1.0624	0.0000	1.53e-04	1.58e-05	3.25e-05	6.51e-05
9	0.0000	-1.0624	0.0001	3.16e-04	3.26e-05	6.72e-05	1.34e-04
10	0.0001	-1.0623	0.0001	6.52e-04	6.72e-05	1.39e-04	2.77e-04
11	0.0002	-1.0620	0.0003	1.34e-03	1.39e-04	2.86e-04	5.71e-04
12	0.0004	-1.0615	0.0006	2.77e-03	2.85e-04	5.89e-04	1.18e-03
13	0.0008	-1.0604	0.0011	5.71e-03	5.87e-04	1.21e-03	2.42e-03
14	0.0016	-1.0582	0.0024	1.17e-02	1.21e-03	2.49e-03	4.98e-03
15	0.0032	-1.0537	0.0048	2.41e-02	2.47e-03	5.10e-03	1.02e-02
16	0.0066	-1.0444	0.0099	4.93e-02	5.03e-03	1.04e-02	2.07e-02
17	0.0135	-1.0256	0.0203	9.99e-02	1.01e-02	2.08e-02	4.16e-02
18	0.0274	-0.9877	0.0411	1.99e-01	1.97e-02	4.06e-02	8.12e-02
19	0.0545	-0.9139	0.0817	3.82e-01	3.61e-02	7.47e-02	1.49e-01
20	0.1042	-0.7783	0.1563	6.84e-01	5.88e-02	1.22e-01	2.43e-01
21	0.1852	-0.5575	0.2778	1.08e+00	7.46e-02	1.55e-01	3.09e-01
22	0.2883	-0.2768	0.4324	1.41e+00	5.70e-02	1.20e-01	2.39e-01
23	0.3678	-0.0608	0.5516	1.54e+00	1.49e-02	3.35e-02	6.60e-02
24	0.3897	-0.0019	0.5846	1.56e+00	3.30e-04	1.09e-03	1.02e-03
25	0.3901	-0.0015	0.5851	1.56e+00	2.02e-05	8.90e-04	6.10e-04
26	0.3903	-0.0011	0.5854	1.56e+00	1.24e-05	6.64e-04	1.58e-04
27	0.3903	-0.0010	0.5855	1.56e+00	3.22e-06	6.10e-04	4.90e-05

Table 4.2: Iterations of feasibility and complementarity refinement for Example 4.42 with active set change from inactive to active. The columns contain for every iteration (`iter`) the point $(\widetilde{x}(p), \widetilde{z}(p))$ and the KKT conditions (4.3), i.e., optimality (`opti`), feasibility (`feas`) and complementarity (`comp`), as well as the KKT conditions (4.4) (`opti`, `feas`, `comp-bar`).

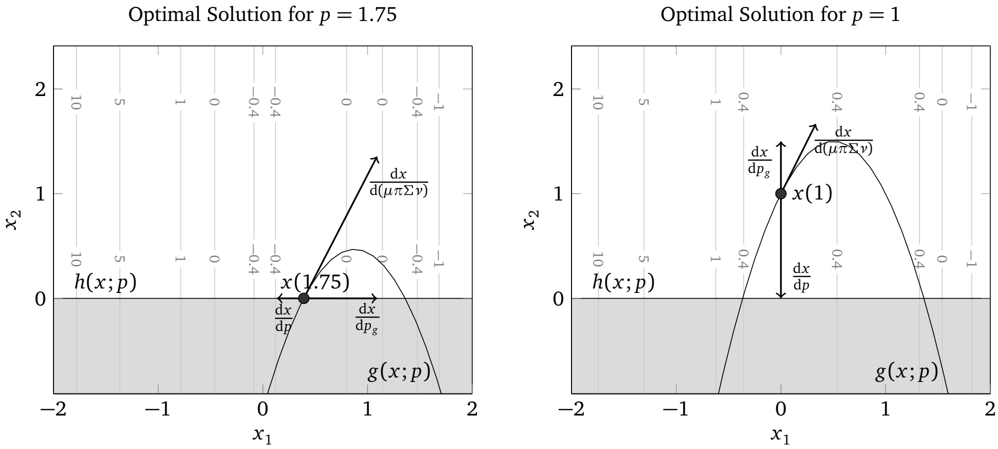

Figure 4.4: Optimal solution and sensitivity derivatives of the nonlinear program of Example 4.42 for $p = 1.75$ (left) and $p = 1$ (right). The objective function is plotted as level set. Sensitivity derivatives are written without argument and are scaled by 0.5.

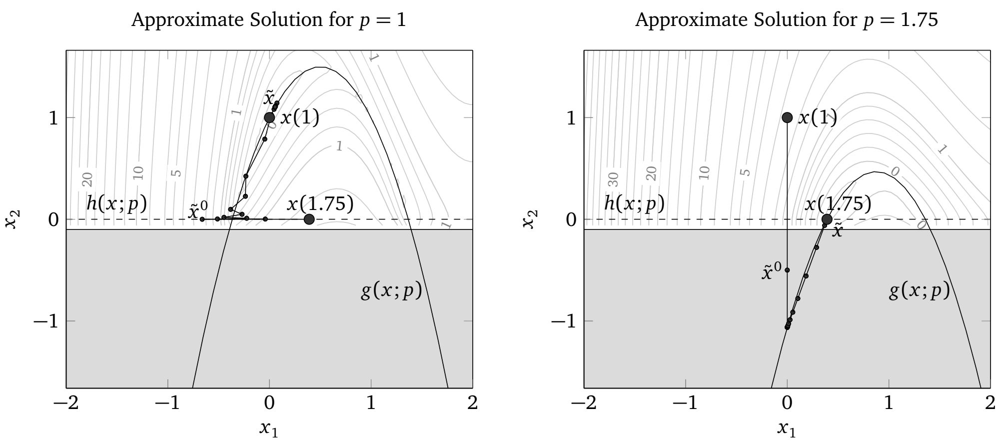

Figure 4.5: Approximation of the nonlinear program of Example 4.42 for $p = 1$ (left) and $p = 1.75$ (right) generated by warmstart based on iterative real-time updates. The objective function of (NLPpbar) is plotted as level set.

Chapter 5

Performance of the NLP Solver WORHP

The primal-dual augmented Lagrangian penalty-interior-point algorithm of Chapter 4 is designed with a special focus on the practical performance, because it has been implemented within the nonlinear programming solver `WORHP`. In this chapter, further algorithmic considerations and enhancements for increased practical efficiency and robustness are reported and numerically studied.

The software library `WORHP`[1] is designed to solve large-scale continuous nonlinear optimization problems and its development was initiated by the European Space Agency. Its name `WORHP` is an acronym for "We Optimize Really Huge Problems" and the software implementation is a hybrid of `C` and `Fortran` – the former mainly for the data management and the latter for numerical calculations. `WORHP` provides two different solution approaches: an SQP method (cf., Büskens and Wassel [36]) and the new penalty-interior-point algorithm discussed in this thesis (see also Kuhlmann and Büskens [129]). During the last decade many researches have successfully applied `WORHP` to real-world optimization applications but also contributed to the development of it, which is briefly summarized in the following.

The SQP method of `WORHP` (see also Linke et al. [135]) solves the QPs using the interior-point solver for quadratic programming `QPSOL` (cf., Gerdts [85]). However, in a future release this may be replaced by the promising developments of Jacobse [120]. For the line search a merit function and a filter approach are available (cf., Kemper [128]). A sensitivity based feasibility refinement step for the SQP method has been studied by Geffken [81], Geffken and Büskens [83], Nikolayzik [149] and Nikolayzik and Büskens [150]. The SQP method furthermore profits by different termination conditions, e.g., *scaled KKT* conditions or a so called *lowpass filter*, and recovery strategies, e.g., gradient based steps, a *dual feasibility mode* or a non-monotone merit function, which make it more robust (cf., Nikolayzik [149]). A study to use different precisions in the floating point arithmetic for the step calculation is given by Geffken [80].

Function derivatives can be provided by the user or calculated using efficient finite differences, which exploit *group strategies* to evaluate different dimensions simultaneously (cf.,

[1] For more information see www.worhp.de.

Kalmbach [124]). For the Hessian matrix different *Quasi-Newton* approximation have been studied, among them *sparse* or *limited-memory* adaptations (cf., Kalmbach [124] and Rauski [166]). In addition, parallel executions are supported by `WORHP` as described in Geffken [81] and Geffken and Büskens [82]. The module `WORHP Zen` (cf., Kuhlmann et al. [130] and Schäfer [174]) provides easy access to sensitivity derivatives and efficiently performs real-time updates.

`WORHP` is designed to allow as much user interaction as desired and therefore is based on the *reverse communication* paradigm (cf., Büskens et al. [34], Nikolayzik et al. [148] and Wassel et al. [190]). For more information about the technical implementation details of `WORHP` the reader is referred to Wassel [191].

Nonlinear Programming Formulation in WORHP

The general problem formulation considered by `WORHP` is

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & h_\mathrm{L} \le h(x) \le h_\mathrm{U} \\ & x_\mathrm{L} \le x \le x_\mathrm{U}. \end{aligned} \tag{5.1}$$

Single sided constraints can be specified by setting the bound to positive or negative infinity, e.g., $h_\mathrm{L} = -\infty$ implies $h(x) \le h_\mathrm{U}$, and equality constraints by setting lower and upper bounds to an equal value, e.g., $h_\mathrm{L} = h_\mathrm{U}$ implies $g(x) := h(x) - h_\mathrm{U} = 0$. For the penalty-interior-point algorithm this is transformed to

$$\begin{aligned} \min_{x\in\mathbb{R}^{n_x}} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \\ & h(x) - s = 0 \\ & h_\mathrm{L} \le s \le h_\mathrm{U} \\ & x_\mathrm{L} \le x \le x_\mathrm{U} \end{aligned} \tag{5.2}$$

(cf., Section 3.2.1), in which it is assumed that $h_{\mathrm{L}_i} < h_{\mathrm{U}_i}$ and $x_{\mathrm{L}_j} < x_{\mathrm{U}_j}$ for all $i \in \{1, \dots, n_h\}$ and $j \in \{1, \dots, n_x\}$. While general equality constraints have been transformed to the constraint function $g(x)$, equality box constraints are removed in a preprocessing phase. In the following, the sets

$$\mathscr{X}_\mathrm{L} := \left\{ i \in \{1, \dots, n_x\} \mid x_{\mathrm{L}_i} > -\infty \right\}, \tag{5.3a}$$

$$\mathscr{X}_\mathrm{U} := \left\{ i \in \{1, \dots, n_x\} \mid x_{\mathrm{U}_i} < \infty \right\}, \tag{5.3b}$$

$$\mathscr{H}_\mathrm{L} := \left\{ i \in \{1, \dots, n_h\} \mid h_{\mathrm{L}_i} > -\infty \right\}, \tag{5.3c}$$

$$\mathscr{H}_\mathrm{U} := \left\{ i \in \{1, \dots, n_h\} \mid h_{\mathrm{U}_i} < \infty \right\}, \tag{5.3d}$$

are used to refer to lower or upper bounded box or general constraints, respectively. Dual variables have to be assigned to every constraint, which are $z_\mathrm{XL} \in \mathbb{R}^{n_x}$, $z_\mathrm{XU} \in \mathbb{R}^{n_x}$, $z_\mathrm{HL} \in \mathbb{R}^{n_h}$

and $z_{\mathrm{HU}} \in \mathbb{R}^{n_h}$ for the inequality box and $y_{\mathrm{G}} \in \mathbb{R}^{n_g}$ and $y_{\mathrm{H}} \in \mathbb{R}^{n_h}$ for the equality constraints. Then, the KKT conditions of (5.2) are

$$\nabla f(x) + \nabla g(x) y_{\mathrm{G}} + \nabla h(x) y_{\mathrm{H}} - \sum_{i \in \mathscr{X}_{\mathrm{L}}} z_{\mathrm{XL}_i} e_i + \sum_{i \in \mathscr{X}_{\mathrm{U}}} z_{\mathrm{XU}_i} e_i = 0 \tag{5.4a}$$

$$-y_{\mathrm{H}} - \sum_{i \in \mathscr{H}_{\mathrm{L}}} z_{\mathrm{HL}_i} e_i + \sum_{i \in \mathscr{H}_{\mathrm{U}}} z_{\mathrm{HU}_i} e_i = 0 \tag{5.4b}$$

$$g(x) = 0 \tag{5.4c}$$

$$h(x) - s = 0 \tag{5.4d}$$

$$\left(x_i - x_{\mathrm{L}_i}\right) z_{\mathrm{XL}_i} = 0, \quad i \in \mathscr{X}_{\mathrm{L}} \tag{5.4e}$$

$$\left(x_{\mathrm{U}_i} - x_i\right) z_{\mathrm{XU}_i} = 0, \quad i \in \mathscr{X}_{\mathrm{U}} \tag{5.4f}$$

$$\left(s_i - h_{\mathrm{L}_i}\right) z_{\mathrm{HL}_i} = 0, \quad i \in \mathscr{H}_{\mathrm{L}} \tag{5.4g}$$

$$\left(h_{\mathrm{U}_i} - s_i\right) z_{\mathrm{HU}_i} = 0, \quad i \in \mathscr{H}_{\mathrm{U}}. \tag{5.4h}$$

The barrier-penalty subproblem considered by Algorithm L corresponding to (5.2) is

$$\begin{aligned} \min_{x \in \mathbb{R}^{n_x}, s \in \mathbb{R}^{n_h}} \quad & \Upsilon(x;\rho) := \pi f(x) + \pi {\lambda_{\mathrm{G}}}^\top g(x) + \pi {\lambda_{\mathrm{H}}}^\top (h(x) - s) + \tau \left\| \begin{bmatrix} g(x) \\ h(x) - s \end{bmatrix} \right\|_2 \\ & - \pi\mu \sum_{i \in \mathscr{X}_{\mathrm{L}}} \varsigma_{\mathrm{XL}_i} \nu_{\mathrm{XL}_i} \ln\left(\frac{x_i - x_{\mathrm{L}_i}}{\mu \varsigma_{\mathrm{XL}_i}} + 1 \right) - \pi\mu \sum_{i \in \mathscr{X}_{\mathrm{U}}} \varsigma_{\mathrm{XU}_i} \nu_{\mathrm{XU}_i} \ln\left(\frac{x_{\mathrm{U}_i} - x_i}{\mu \varsigma_{\mathrm{XU}_i}} + 1 \right) \\ & - \pi\mu \sum_{i \in \mathscr{H}_{\mathrm{L}}} \varsigma_{\mathrm{HL}_i} \nu_{\mathrm{HL}_i} \ln\left(\frac{s_i - h_{\mathrm{L}_i}}{\mu \varsigma_{\mathrm{HL}_i}} + 1 \right) - \pi\mu \sum_{i \in \mathscr{H}_{\mathrm{U}}} \varsigma_{\mathrm{HU}_i} \nu_{\mathrm{HU}_i} \ln\left(\frac{h_{\mathrm{U}_i} - s_i}{\mu \varsigma_{\mathrm{HU}_i}} + 1 \right), \end{aligned} \tag{5.5}$$

where $\varsigma_{\mathrm{XL}} \in \mathbb{R}^{n_x}$, $\varsigma_{\mathrm{XU}} \in \mathbb{R}^{n_x}$, $\varsigma_{\mathrm{HL}} \in \mathbb{R}^{n_h}$ and $\varsigma_{\mathrm{HU}} \in \mathbb{R}^{n_h}$ are the boundary shifts, $\nu_{\mathrm{XL}} \in \mathbb{R}^{n_x}$, $\nu_{\mathrm{XU}} \in \mathbb{R}^{n_x}$, $\nu_{\mathrm{HL}} \in \mathbb{R}^{n_h}$ and $\nu_{\mathrm{HU}} \in \mathbb{R}^{n_h}$ are the Lagrangian multiplier parameters for the inequality constraints and $\lambda_{\mathrm{G}} \in \mathbb{R}^{n_g}$ and $\lambda_{\mathrm{H}} \in \mathbb{R}^{n_h}$ the ones of the equality constraints. In the following the primal-dual variables w and ω are redefined to correspond to (5.4).

5.1 Benchmark Environment

This section presents the used benchmark test set and the profiles to evaluate the performance of the algorithm. All benchmarks have disabled parallelization of software components (e.g., linear solver or linear algebra) and are executed on a machine with two `Intel`® `Xeon`® `CPU E5-2637 v3` (3.50 GHz, 8 cores, 16 threads) running `Ubuntu 16.04`. Ten test instances are evaluated in parallel at every time. CPU time is measured in seconds using the `Linux` command `time` (user time). `WORHP` is compiled using `gfortran` 5.4 and `gcc` 5.4.

The CUTEst Test Set

The used benchmark test set is `CUTEst` (cf., Gould et al. [107]). The used version[2] contains $n_{\mathrm{p}} = 1305$ academic and real-world application based optimization problems from small to

[2] `CUTEst` version of February 19, 2018 (git commit: 6c7af0a). All programs are used with standard configuration.

large scale. The objective and constraint functions range from general nonlinear, quadratic, least-squares type to linear and even constant. This large variety makes CUTEst a challenging test collection for general purpose nonlinear programming solvers. For a detailed analysis of different instance types, the reader is referred to Geffken [81] and Wassel [191].

The CUTEst test collection also contains infeasible optimization problems. However, to better assess the performance of the nonlinear programming solvers in the infeasible case, a modified version of CUTEst is used, which provides guaranteed globally infeasible problems. For this modification, the subset of constrained CUTEst instances is considered. For every constraint a new constraint is added such that the combination is a contradiction, i.e., for every

i. lower bounded constraint $h_{\mathrm{L}_i} \leq h_i(x)$ with $i \in \mathscr{H}_\mathrm{L} \setminus \mathscr{H}_\mathrm{U}$ the constraint $h_i(x) \leq h_{\mathrm{L}_i} - 1$ is added.

ii. upper bounded constraint $h_i(x) \leq h_{\mathrm{U}_i}$ with $i \in \mathscr{H}_\mathrm{U} \setminus \mathscr{H}_\mathrm{L}$ the constraint $h_{\mathrm{U}_i} + 1 \leq h_i(x)$ is added.

iii. lower and upper bounded constraint $h_{\mathrm{L}_i} \leq h_i(x) \leq h_{\mathrm{U}_i}$ with $i \in \mathscr{H}_\mathrm{L} \cap \mathscr{H}_\mathrm{U}$ the constraint $h_{\mathrm{U}_i} + 1 \leq h_i(x) \leq 2h_{\mathrm{U}_i} - h_{\mathrm{L}_i} + 1$ is added.

Furthermore, problems that contain more equality constraints than optimization variables after this modification are removed from the infeasible CUTEst test set, which results in $n_\mathrm{p} = 517$ infeasible instances. In the following this test set is called CUTEst-infeas.

A further modification of the CUTEst test set aims to evaluate the warmstart performance. Therefore, for all CUTEst instances five variants with altered lower and upper bound values are solved in a row, where the initial solve of the original CUTEst instance is neglected. The bounds are modified by selecting 30% of box or general constraints at random[3], where each bound selection is shifted randomly, but at maximum by 10% of the original value, e.g., x_{L_i} is altered to $x_{\mathrm{L}_i} + t \max\left\{1, \left|x_{\mathrm{L}_i}\right|\right\}$ where $t \in [-0.1, 0.1]$ is selected at random. If a solver run fails in a previous warmstart or returns a different objective function value up to a relative error of 0.1, the sequential warmstart sequence is stopped. Each warmstarted instance is counted as individual optimization problem leading to at maximum $n_\mathrm{p} = 5 \cdot 1305 = 6525$ instances in this test set, where the exact number is unknown a priori. This test set is called CUTEst-warm.

Performance Profiles

To evaluate the performance of n_s different solver options or different solvers on a test set with n_p problem instances, the performance profiles proposed by Dolan and Moré [53] are used. These provide a graphical comparison of a quantity $Q_{i,j}$, e.g., number of iterations, number of function evaluations or CPU time, which are evaluated for every solver run on every problem instance, i.e., $i \in \left\{1, \ldots, n_\mathrm{p}\right\}$ and $j \in \{1, \ldots, n_\mathrm{s}\}$. First, the ratio

$$R_{i,j} := \frac{Q_{i,j}}{\min\left\{Q_{i,j} \mid j \in \{1, \ldots, n_\mathrm{s}\}\right\}} \tag{5.6}$$

[3]Random numbers are generated for every benchmark using the same random seed to reproduce equal test instances.

is calculated, which compares a solver run j to the best run on a specific problem instance. In case a solver run j fails on problem instance i, the ratio is defined to be $R_{i,j} := \infty$. Then, the function

$$F_j(t) := \frac{100}{n_\mathrm{p}} \left| \left\{ i \in \left\{1, \ldots, n_\mathrm{p}\right\} \mid R_{i,j} \leq t \right\} \right| \tag{5.7}$$

with $j \in \{1, \ldots, n_\mathrm{s}\}$ defines for every solver run a fraction of test problems that were solved up to a factor $t \geq 1$ worse than the best solver. Thus, $F_j(1)$ is the percentage of problems a solver run is better or equal compared to the others (measuring efficiency) and $\lim_{t\to\infty} F_j(t)$ is the percentage of problems a solver run can solve (measuring robustness). The performance profiles additionally show virtual best and worst solver runs, which are computed for each problem instance i by $\min_{j\in\{1,\ldots,n_\mathrm{s}\}} Q_{i,j}$ and $\max_{j\in\{1,\ldots,n_\mathrm{s}\}} Q_{i,j}$, respectively.

The above performance profile has one major drawback. If more than two solver runs are compared, the profile may be misleading. A graph being higher than another does not necessarily imply a better solver performance as described by Gould and Scott [102]. Therefore, a second type of performance profile is proposed and used that compares two solver runs directly. The ratio is defined to be

$$R_{i,1} := \begin{cases} 1 & \text{if } Q_{i,2} \text{ refers to failure} \\ 1 - \frac{Q_{i,1}}{Q_{i,2}} & \text{if } Q_{i,1} < Q_{i,2} \\ 0 & \text{if } Q_{i,1} = Q_{i,2} \\ -1 + \frac{Q_{i,2}}{Q_{i,1}} & \text{if } Q_{i,1} > Q_{i,2} \\ -1 & \text{if } Q_{i,1} \text{ refers to failure.} \end{cases} \tag{5.8}$$

These ratios can be plotted as bar plot over the percentage of problem instances. In this thesis the plot is divided into six sections where the first five refer to the cases above and the last one indicates the percentage of problems none of the two solver runs could solve (see for example Figure 5.2). These sections directly allow to see the percentage of problem instances for which the first solver is more robust (first section or case) or more efficient (second section or case). These numbers will be plotted within the profile. To reduce the required space, the fourth and fifth section is shifted up by one. The ratios in (5.8) have a further advantage as they allow to calculate a single score for every comparison by

$$\frac{100}{n_\mathrm{p}} \sum_{i=1}^{n_\mathrm{p}} R_{i,1} \in [-100, 100]. \tag{5.9}$$

If this score is positive, the first solver run can be considered to be better and worse otherwise. The score will be printed to the right of the profile.

For the performance profiles in this thesis, a non-failed outcome is defined to be either a first-order optimal solution or a certificate of infeasibility. To avoid divisions by zero, the quantities $Q_{i,j}$ are shifted up by 10^{-6}.

5.2 Algorithmic Considerations and Enhancements

In this section a practical parameter choice for Algorithm L in Step L-1 is presented as well as further modifications and extensions for increased practical efficiency. It has to be noted that this presentation is restricted to the most important features in the standard configuration. Many others have been implemented and, thus, are available in WORHP. The numerical studies in this section are based on the original CUTEst test set unless specified differently and compare different solver components to the standard configuration (WORHP IPm) of Algorithm L defined by the whole Section 5.2.

5.2.1 Termination

The algorithm terminates in Step L-2 with an optimal solution if the KKT conditions (5.4) are satisfied in the infinity norm up to a tolerance $\varepsilon_{\text{tol}} = 10^{-6}$. In case the KKT conditions derived after the application of the gradient based scaling procedure (cf., Section 5.2.2) are fulfilled, the scaling is removed and the optimization continued. In particular, the scaling factors D_{f} (scaling of the objective function), D_{g} (scaling of equality constraints), D_{h} (scaling of inequality constraints) are set to the identity matrix E and the penalty parameter π_k to one after the Lagrangian multipliers are updated to

$$z_{\text{XL}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} z_{\text{XL}}^k, \qquad z_{\text{XU}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} z_{\text{XU}}^k, \tag{5.10a}$$

$$z_{\text{HL}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} z_{\text{HL}}^k, \qquad z_{\text{HU}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} z_{\text{HU}}^k, \tag{5.10b}$$

$$y_{\text{G}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} D_{\text{g}}^{-1} y_{\text{G}}^k, \quad y_{\text{H}}^k \leftarrow \frac{1}{\pi_k D_{\text{f}}} D_{\text{h}}^{-1} y_{\text{H}}^k. \tag{5.10c}$$

If the current iterate is feasible up to the tolerance ε_{tol} and the objective function value is less than -10^{20} the problem is considered to be unbounded.

The IEEE numbers Inf and NaN, that may occur in the gradients of the objective function, Jacobians of the constraints or Hessians of the Lagrangian function, do not lead to failure. Instead, they are replaced by 10^{20} or 1, respectively.

Failure occurs if the infinity norm of the step direction or primal-dual update relatively to the current primal-dual iterate is less than $\varepsilon^{\frac{3}{4}}$ (ε is the machine precision), the step size in the line search is less than 10^{-12}, the Hessian regularization gets larger than 10^{20} or if the process seems to diverge. In particular, it is checked if the constraint violation becomes larger than 10^{25}, the optimality error larger than 10^{30} or the infinity norms of primal or dual iterates larger than 10^{20}. In addition, the algorithm fails if a maximum number of iterations (10000) or a time limit (1800s) is reached.

Detection of Infeasibility or Degeneracy

Deduced from the global convergence analysis in Section 4.3, Algorithm L terminates with a Fritz-John point of the original or shifted problem (cf., Section 4.1) if the current iterate is feasible up to the tolerance ε_{tol}, i.e., if $\left\|g(x^k)\right\|_\infty \le \varepsilon_{\text{tol}}$ and $\left\|h(x^k)-s^k\right\|_\infty \le \varepsilon_{\text{tol}}$, and if either the Hessian regularization fails or the penalty parameter π_k becomes smaller than 10^{-12}.

For detection of infeasibility, the KKT conditions of the shifted feasibility problem

$$\frac{1}{\tau_k}\nabla g(x^k)y_{\text{G}}^k + \frac{1}{\tau_k}\nabla h(x^k)y_{\text{H}}^k - \sum_{i\in\mathscr{X}_\text{L}}\frac{1}{\tau_k}z_{\text{XL}_i}e_i + \sum_{i\in\mathscr{X}_\text{U}}\frac{1}{\tau_k}z_{\text{XU}_i}e_i = 0 \tag{5.11a}$$

$$-\frac{1}{\tau_k}y_{\text{H}}^k - \sum_{i\in\mathscr{H}_\text{L}}\frac{1}{\tau_k}z_{\text{HL}_i}e_i + \sum_{i\in\mathscr{H}_\text{U}}\frac{1}{\tau_k}z_{\text{HU}_i}e_i = 0 \tag{5.11b}$$

$$g(x^k) - \varrho_k y_{\text{G}}^k = 0 \tag{5.11c}$$

$$h(x^k) - s^k - \varrho_k y_{\text{H}}^k = 0 \tag{5.11d}$$

$$\frac{1}{\tau_k}\left(x_i^k - x_{\text{L}_i} + \mu_k\varsigma_{\text{XL}_i}^k\right)z_{\text{XL}_i} = 0, \quad i\in\mathscr{X}_\text{L} \tag{5.11e}$$

$$\frac{1}{\tau_k}\left(x_{\text{U}_i} - x_i^k + \mu_k\varsigma_{\text{XU}_i}^k\right)z_{\text{XU}_i} = 0, \quad i\in\mathscr{X}_\text{U} \tag{5.11f}$$

$$\frac{1}{\tau_k}\left(s_i^k - h_{\text{L}_i} + \mu_k\varsigma_{\text{HL}_i}^k\right)z_{\text{HL}_i} = 0, \quad i\in\mathscr{H}_\text{L} \tag{5.11g}$$

$$\frac{1}{\tau_k}\left(h_{\text{U}_i} - s_i^k + \mu_k\varsigma_{\text{HU}_i}^k\right)z_{\text{HU}_i} = 0, \quad i\in\mathscr{H}_\text{U}. \tag{5.11h}$$

with

$$\varrho_k = \frac{1}{\tau_k}\left\|\begin{bmatrix} g(x^k) \\ h(x^k)-s^k \end{bmatrix}\right\|_2 \tag{5.12}$$

are checked (cf., Section 4.1). If these are fulfilled in the infinity norm up to a tolerance ε_{tol} and, furthermore, either the penalty parameter π_k becomes smaller than 10^{-12}, the infinity norm of the step direction relatively to the current iterate is smaller than 10^{-6} or a maximum step size in the line search fails, the algorithm terminates with a certificate of infeasibility. The additional conditions help to avoid a false positive infeasibility detection, i.e., a termination with a local certificate of infeasibility for a globally seen feasible problem formulation. Although this is an unwanted behavior, it cannot be fully prevented in local nonlinear programming solvers. Nevertheless, the outcome can be very helpful for practitioners even in that case, as it suggests to provide an initial guess that is closer to the feasible region.

5.2.2 Initialization

At the beginning of the optimization, the primal variables $x^0 \in \mathbb{R}^{n_x}$ (provided as initial guess by the user) and $s \in \mathbb{R}^{n_h}$ (set to the initial constraint function value $h(x^0)$) are projected into the feasible region, i.e.,

$$x_i^0 \leftarrow \max\left\{x_{\text{L}_i}, \min\left\{x_{\text{U}_i}, x_i^0\right\}\right\}, \qquad i = 1,\dots,n_x \tag{5.13a}$$

$$s_i^0 \leftarrow \max\left\{h_{\text{L}_i}, \min\left\{h_{\text{U}_i}, h(x_i^0)\right\}\right\}, \quad i = 1,\dots,n_h. \tag{5.13b}$$

A further shift into the interior region, usually performed for interior-point algorithms, is not necessary due to the modified barrier function. The dual variables are initialized as

$$z^0_{\mathrm{XL}_i} = \begin{cases} 1 & \text{if } i \in \mathscr{X}_\mathrm{L} \\ 0 & \text{otherwise} \end{cases} \qquad z^0_{\mathrm{XU}_i} = \begin{cases} 1 & \text{if } i \in \mathscr{X}_\mathrm{U} \\ 0 & \text{otherwise} \end{cases} \tag{5.14a}$$

$$z^0_{\mathrm{HL}_i} = \begin{cases} 1 & \text{if } i \in \mathscr{H}_\mathrm{L} \\ 0 & \text{otherwise} \end{cases} \qquad z^0_{\mathrm{HU}_i} = \begin{cases} 1 & \text{if } i \in \mathscr{H}_\mathrm{U} \\ 0 & \text{otherwise} \end{cases} \tag{5.14b}$$

and

$$y^0_\mathrm{G} = 0 \tag{5.15a}$$

$$y^0_{\mathrm{H}_i} = \begin{cases} -z^0_{\mathrm{HL}_i} & \text{if } i \in \mathscr{H}_\mathrm{L} \setminus \mathscr{H}_\mathrm{U} \\ z^0_{\mathrm{HU}_i} & \text{if } i \in \mathscr{H}_\mathrm{U} \setminus \mathscr{H}_\mathrm{L} \\ 0 & \text{if } i \in \mathscr{H}_\mathrm{L} \cap \mathscr{H}_\mathrm{U} \end{cases}, \quad i = 1, \ldots, n_h. \tag{5.15b}$$

The initialization of y^0_H ensures that the optimality condition (5.4b) is satisfied.

Dual Adjustment

In practice, it is often the optimality condition (5.4a) that is difficult to satisfy. Therefore, the dual variables for the box constraints are further adjusted to minimize the optimality error. For a parameter $\kappa_\mathrm{adj} > 1$ keeping the dual variables bounded away from zero, the *dual adjustment* is

$$z^0_{\mathrm{XL}_i} \leftarrow \max\left\{ \frac{z^0_{\mathrm{XL}_i}}{\kappa_\mathrm{adj}}, \left(\nabla f\left(x^0\right) + \nabla g\left(x^0\right) y^0_\mathrm{G} + \nabla h\left(x^0\right) y^0_\mathrm{H}\right)^\top e_i \right\}, \quad i \in \mathscr{X}_\mathrm{L} \setminus \mathscr{X}_\mathrm{U} \tag{5.16a}$$

$$z^0_{\mathrm{XU}_i} \leftarrow \max\left\{ \frac{z^0_{\mathrm{XU}_i}}{\kappa_\mathrm{adj}}, -\left(\nabla f\left(x^0\right) + \nabla g\left(x^0\right) y^0_\mathrm{G} + \nabla h\left(x^0\right) y^0_\mathrm{H}\right)^\top e_i \right\}, \quad i \in \mathscr{X}_\mathrm{U} \setminus \mathscr{X}_\mathrm{L}. \tag{5.16b}$$

In case of a lower and upper bounded box constraint, i.e., $i \in \mathscr{X}_\mathrm{L} \cap \mathscr{X}_\mathrm{U}$, it is checked if $(\nabla f(x) + \nabla g(x) y_\mathrm{G} + \nabla h(x) y_\mathrm{H})^\top e_i - z_{\mathrm{XL}_i} + z_{\mathrm{XU}_i}$ is positive or negative. In the former case, $z^0_{\mathrm{XL}_i}$ is adjusted by

$$z^0_{\mathrm{XL}_i} \leftarrow \max\left\{ \frac{z^0_{\mathrm{XL}_i}}{\kappa_\mathrm{adj}}, \left(\nabla f\left(x^0\right) + \nabla g\left(x^0\right) y^0_\mathrm{G} + \nabla h\left(x^0\right) y^0_\mathrm{H}\right)^\top e_i + z^0_{\mathrm{XU}_i} \right\} \tag{5.17}$$

and in the latter, $z^0_{\mathrm{XU}_i}$ is updated to

$$z^0_{\mathrm{XU}_i} \leftarrow \max\left\{ \frac{z^0_{\mathrm{XU}_i}}{\kappa_\mathrm{adj}}, -\left(\nabla f\left(x^0\right) + \nabla g\left(x^0\right) y^0_\mathrm{G} + \nabla h\left(x^0\right) y^0_\mathrm{H}\right)^\top e_i - z^0_{\mathrm{XL}_i} \right\}. \tag{5.18}$$

This strategy is inspired by Gondzio and Grothey [100], who applied a similar dual adjustment in the context of quadratic programming. The parameter κ_adj is chosen to be $\kappa_\mathrm{adj} = 2$.

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
`-dualadj`	1073	93	10	1	27	33	68
`-scaling`	1067	88	4	1	38	43	64
`WORHP IPm`	1080	90	7	1	29	35	63

Table 5.1: Numbers of termination statuses for the numerical study of the initialization strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without gradient based scaling (`-scaling`) and without dual adjustment (`-dualadj`).

Scaling

A big challenge for numerical solution algorithms is a badly scaled problem formulation, where function values are of different order of magnitude. While a scaling of the objective and constraint functions is invariant for the Newton step, it can have a significant impact on the barrier and penalty parameter update conditions as well as the line search globalization. Therefore, an automatic gradient based scaling procedure, proposed by Wächter and Biegler [202], is applied. The diagonals of the scaling matrices $D_\mathrm{f} \in \mathbb{R}$, $D_\mathrm{g} \in \mathbb{R}^{n_g \times n_g}$ and $D_\mathrm{h} \in \mathbb{R}^{n_h \times n_h}$ are set to

$$D_\mathrm{f} := \frac{\kappa_\mathrm{sc}}{\max\left\{\kappa_\mathrm{sc}, \|\nabla f(x^0)\|_\infty\right\}} \tag{5.19a}$$

$$D_{\mathrm{g}_{ii}} := \frac{\kappa_\mathrm{sc}}{\max\left\{\kappa_\mathrm{sc}, \|\nabla g_i(x^0)\|_\infty\right\}}, \quad i = 1, \dots, n_g, \tag{5.19b}$$

$$D_{\mathrm{h}_{ii}} := \frac{\kappa_\mathrm{sc}}{\max\left\{\kappa_\mathrm{sc}, \|\nabla h_i(x^0)\|_\infty\right\}}, \quad i = 1, \dots, n_h, \tag{5.19c}$$

respectively. It holds, that $D_\mathrm{f} \leq 1$, $D_{\mathrm{g}_{ii}} \leq 1$ and $D_{\mathrm{h}_{ii}} \leq 1$ and, hence, function values are never increased. This scaling procedure ensures that all gradient components are smaller or equal to κ_sc at the beginning of the optimization. The threshold value $\kappa_\mathrm{sc} > 0$ is chosen to be $\kappa_\mathrm{sc} = 100$.

Numerical Results

The following numerical study shows the influence of the two initialization strategies scaling (referred to as `-scaling`) and dual adjustment (`-dualadj`) by analyzing how the penalty-interior-point algorithm performs if they are disabled. The resulting number of different termination statuses are listed in Table 5.1 and performance profiles are shown in Figure 5.1 and Figure 5.2. While `WORHP IPm` solves 1080 (82.76%) instances to optimality, the numbers reduce to 1073 (82.22%) and 1067 (81.76%) when dual adjustment or scaling is disabled. In particular, without dual adjustment the number of infeasibility and Fritz-John terminations increases (+3 `infeas`, +3 `fritzjohn`) and without scaling the resource limitations (+9 `maxtime`, +8 `maxiter`). The performance profiles in Figure 5.2 illustrate that the penalty-interior-point algorithm needs less iterations with dual adjustment on 16% of the problem instances and more iterations on 11%. The majority of instances is not affected (62%). This leads to an iteration score of +1.6 in favor of `WORHP IPm`. The benefit from scaling is even higher as 15% can be solved within less iterations (9% with more, 63% with equal). The score is therefore +3.1. All

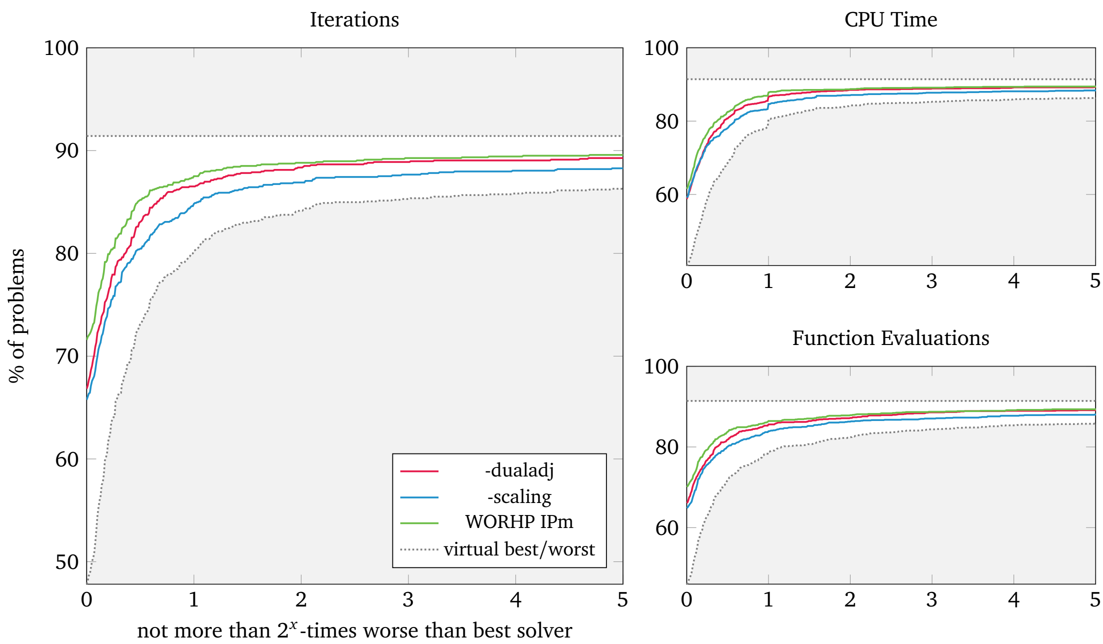

Figure 5.1: Performance profile for the numerical study of the initialization strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without gradient based scaling (`-scaling`) and without dual adjustment (`-dualadj`).

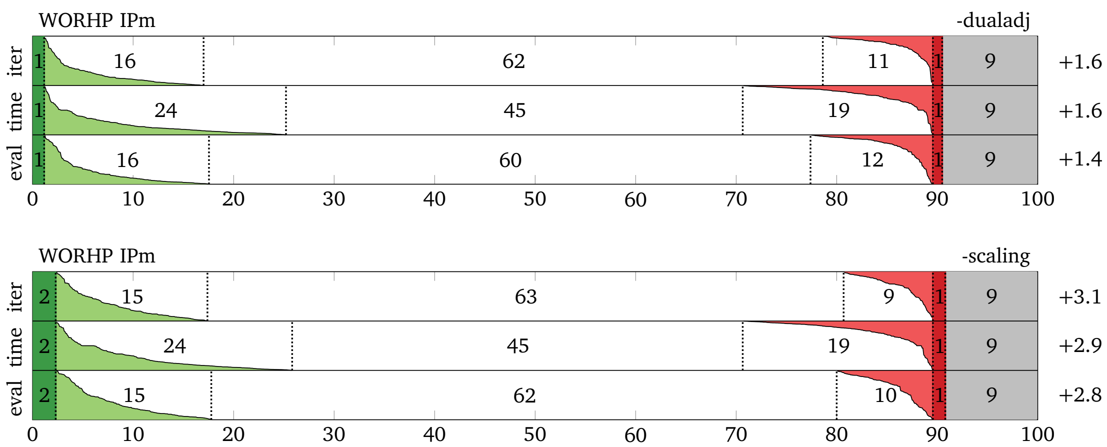

Figure 5.2: Individual performance profiles for the numerical study of the initialization strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without gradient based scaling (`-scaling`) and without dual adjustment (`-dualadj`).

in all, the results show that the two initialization enhancements slightly improve the overall performance.

5.2.3 Solving the Linear Equation System

The central element of the penalty-interior-point algorithm is the calculation of the step direction by solving the sparse and symmetric linear equation system

$$\begin{bmatrix} Q_k + A_1 & 0 & \nabla g(x^k) & \nabla h(x^k) \\ 0 & A_2 & 0 & -E \\ \nabla g(x^k)^\top & 0 & -\varrho_k E & 0 \\ \nabla h(x^k)^\top & -E & 0 & -\varrho_k E \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta s^k \\ \Delta y_{\mathrm{G}}^k \\ \Delta y_{\mathrm{H}}^k \end{bmatrix}$$

$$= - \begin{bmatrix} b_1 + \sum_{i \in \mathscr{X}_{\mathrm{L}}} \frac{b_{5,i}}{x_i^k - x_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{XL}_i}^k} - \sum_{i \in \mathscr{X}_{\mathrm{U}}} \frac{b_{6,i}}{x_{\mathrm{U}_i} - x_i^k + \mu_k \varsigma_{\mathrm{XU}_i}^k} \\ b_2 + \sum_{i \in \mathscr{H}_{\mathrm{L}}} \frac{b_{7,i}}{s_i^k - h_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{HL}_i}^k} - \sum_{i \in \mathscr{H}_{\mathrm{U}}} \frac{b_{8,i}}{h_{\mathrm{U}_i} - s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k} \\ b_3 \\ b_4 \end{bmatrix}, \tag{5.20a}$$

$$\Delta z_{\mathrm{XL}_i}^k = \frac{-b_{5,i} - z_{\mathrm{XL}_i}^k \Delta x_i^k}{x_i^k - x_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{XL}_i}^k}, \quad i \in \mathscr{X}_{\mathrm{L}} \tag{5.20b}$$

$$\Delta z_{\mathrm{XU}_i}^k = \frac{-b_{6,i} + z_{\mathrm{XU}_i}^k \Delta x_i^k}{x_{\mathrm{U}_i} - x_i^k + \mu_k \varsigma_{\mathrm{XU}_i}^k}, \quad i \in \mathscr{X}_{\mathrm{U}} \tag{5.20c}$$

$$\Delta z_{\mathrm{HL}_i}^k = \frac{-b_{7,i} - z_{\mathrm{HL}_i}^k \Delta s_i^k}{s_i^k - h_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{HL}_i}^k}, \quad i \in \mathscr{H}_{\mathrm{L}} \tag{5.20d}$$

$$\Delta z_{\mathrm{HU}_i}^k = \frac{-b_{8,i} + z_{\mathrm{HL}_i}^k \Delta s_i^k}{h_{\mathrm{U}_i} - s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k}, \quad i \in \mathscr{H}_{\mathrm{U}} \tag{5.20e}$$

with

$$Q_k := \pi_k \nabla^2 f(x^k) + \sum_{i=1}^{n_g} y_{\mathrm{G}_i}^k \nabla^2 g(x^k) + \sum_{i=1}^{n_h} y_{\mathrm{H}_i}^k \nabla^2 h(x^k), \tag{5.21a}$$

$$b_1 := \pi_k \nabla f(x^k) + \nabla g(x^k) y^k + \nabla h(x^k) y_{\mathrm{H}} - \sum_{i \in \mathscr{X}_{\mathrm{L}}} z_{\mathrm{XL}_i}^k e_i + \sum_{i \in \mathscr{X}_{\mathrm{U}}} z_{\mathrm{XU}_i}^k e_i, \tag{5.21b}$$

$$b_2 := -y_{\mathrm{H}}^k - \sum_{i \in \mathscr{H}_{\mathrm{L}}} z_{\mathrm{HL}_i}^k e_i + \sum_{i \in \mathscr{H}_{\mathrm{U}}} z_{\mathrm{HU}_i}^k e_i, \tag{5.21c}$$

$$b_3 := g(x^k) - \varrho_k \left(y_{\mathrm{G}}^k - \pi_k \lambda_{\mathrm{G}}^k\right), \tag{5.21d}$$

$$b_4 := h(x^k) - s^k - \varrho_k \left(y_{\mathrm{H}}^k - \pi_k \lambda_{\mathrm{H}}^k\right) \tag{5.21e}$$

$$b_{5,i} = \left(x_i^k - x_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{XL}_i}^k\right) z_{\mathrm{XL}_i}^k - \pi_k \mu_k \varsigma_{\mathrm{XL}_i}^k \nu_{\mathrm{XL}_i}^k, \quad i \in \mathscr{X}_{\mathrm{L}} \tag{5.21f}$$

$$b_{6,i} = \left(x_{\mathrm{U}_i} - x_i^k + \mu_k \varsigma_{\mathrm{XU}_i}^k\right) z_{\mathrm{XU}_i}^k - \pi_k \mu_k \varsigma_{\mathrm{XU}_i}^k \nu_{\mathrm{XU}_i}^k, \quad i \in \mathscr{X}_{\mathrm{U}} \tag{5.21g}$$

$$b_{7,i} = \left(s_i^k - h_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{HL}_i}^k\right) z_{\mathrm{HL}_i}^k - \pi_k \mu_k \varsigma_{\mathrm{HL}_i}^k \nu_{\mathrm{HL}_i}^k, \quad i \in \mathscr{H}_\mathrm{L} \tag{5.21h}$$

$$b_{8,i} = \left(h_{\mathrm{U}_i} - s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k\right) z_{\mathrm{HU}_i}^k - \pi_k \mu_k \varsigma_{\mathrm{HU}_i}^k \nu_{\mathrm{HU}_i}^k, \quad i \in \mathscr{H}_\mathrm{U}, \tag{5.21i}$$

$$(A_1)_{ij} := \begin{cases} \frac{z_{\mathrm{XL}_i}^k}{x_i^k - x_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{XL}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{X}_\mathrm{L} \setminus \mathscr{X}_\mathrm{U} \\ \frac{z_{\mathrm{XU}_i}^k}{x_{\mathrm{U}_i} - x_i^k + \mu_k \varsigma_{\mathrm{XU}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{X}_\mathrm{U} \setminus \mathscr{X}_\mathrm{L} \\ \frac{z_{\mathrm{XL}_i}^k}{x_i^k - x_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{XL}_i}^k} + \frac{z_{\mathrm{XU}_i}^k}{x_{\mathrm{U}_i} - x_i^k + \mu_k \varsigma_{\mathrm{XU}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{X}_\mathrm{L} \cap \mathscr{X}_\mathrm{U} \\ 0 & \text{otherwise} \end{cases} \quad i, j = 1, \ldots, n_x, \tag{5.21j}$$

$$(A_2)_{ij} := \begin{cases} \frac{z_{\mathrm{HL}_i}^k}{s_i^k - h_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{HL}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{H}_\mathrm{L} \setminus \mathscr{H}_\mathrm{U} \\ \frac{z_{\mathrm{HU}_i}^k}{h_{\mathrm{U}_i} - s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{H}_\mathrm{U} \setminus \mathscr{H}_\mathrm{L} \\ \frac{z_{\mathrm{HL}_i}^k}{s_i^k - h_{\mathrm{L}_i} + \mu_k \varsigma_{\mathrm{HL}_i}^k} + \frac{z_{\mathrm{HU}_i}^k}{h_{\mathrm{U}_i} - s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k} & \text{if } i = j \text{ and } i \in \mathscr{H}_\mathrm{L} \cap \mathscr{H}_\mathrm{U} \\ 0 & \text{otherwise} \end{cases} \quad i, j = 1, \ldots, n_h. \tag{5.21k}$$

It corresponds to the reduced linear equation system (4.12) described in Section 4.2.1. It would further be possible to eliminate the second equation in (5.20) with no dense fill-in, but numerical studies have not shown any benefit in practice.

Regularization

A primal regularization is invoked if either the inertia of the system matrix in (5.20) is incorrect, i.e., it differs from $\left(n_x + n_h, n_g + n_h, 0\right)$, or if a sufficient descent is not achieved. The latter is checked for a small parameter $\varepsilon_{\mathrm{desc}} > 0$ by

$$\Psi\left(w^k; \rho^k\right) \geq -\varepsilon_{\mathrm{desc}} \left\|\Delta w^k\right\|^2. \tag{5.22}$$

Unfortunately, it is unknown a priori, which primal regularization value δ_p would satisfy these conditions and, thus, a heuristic is applied. It is crucial to minimize the number of tryouts as for every δ_p the system matrix of (5.20) has to be factorized. Therefore, the strategy of Wächter and Biegler [202] is used, which chooses δ_p based on previous iterations. It is described in Algorithm O. The parameters are set to $\varepsilon_{\mathrm{desc}} = 10^{-12}$, $\delta_\mathrm{p}^0 = 10^{-4}$, $\delta_\mathrm{p}^{\min} = 10^{-12}$, $\kappa_{\delta,\mathrm{dec}} = 3.33 \cdot 10^{-1}$, $\kappa_{\delta,\mathrm{inc}} = 10^2$ and $\kappa_{\delta,\mathrm{inc}}^0 = 8$.

Linear Solver

The linear equation system (5.20) is solved numerically by the linear solver `HSL MA97` (cf., Hogg and Scott [117]), with a pivot tolerance of 10^{-10} and a threshold for small numbers counted as zero of 10^{-12}. Because the structure of the linear equation system does not change,

Algorithm O Primal Regularization

O-1: *(Initialization)* If $k = 0$ (Iteration index of Algorithm L), set $\bar{\delta}_\text{p} = 0$ and choose $\delta_\text{p}^0 > \delta_\text{p}^\text{min} > 0$, $\kappa_{\delta,\text{dec}} \in (0,1)$, $\kappa_{\delta,\text{inc}} > \kappa_{\delta,\text{inc}}^0 > 1$.

O-2: *(Check unmodified)* Attempt to factorize the modified matrix in (5.20). If the inertia is $(n_x + n_h, n_g + n_h, 0)$ and (5.22) is satisfied, STOP.

O-3: *(First regularization)* If $\bar{\delta}_\text{p} = 0$, set $\delta_\text{p} \leftarrow \delta_\text{p}^0$. Otherwise, set $\delta_\text{p} \leftarrow \min\left\{\delta_\text{p}^\text{min}, \kappa_{\delta,\text{dec}}\bar{\delta}_\text{p}\right\}$.

O-4: *(Check modified)* Attempt to factorize the modified matrix in (5.20). If the inertia is $(n_x + n_h, n_g + n_h, 0)$ and (5.22) is satisfied, set $\bar{\delta}_\text{p} \leftarrow \delta_\text{p}$ and STOP.

O-5: *(Regularization)* If $\bar{\delta}_\text{p} = 0$, set $\delta_\text{p} \leftarrow \kappa_{\delta,\text{inc}}^0 \delta_\text{p}$. Otherwise, set $\delta_\text{p} \leftarrow \kappa_{\delta,\text{inc}}\delta_\text{p}$. Go to Step O-4.

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
`-inertia`	958	92	13	0	42	63	137
`-itref`	1081	91	12	1	28	32	60
`-ma97tuning`	1059	87	15	1	27	27	89
`WORHP IPm`	1080	90	7	1	29	35	63

Table 5.2: Numbers of termination statuses for the numerical study of the linear equation system solution strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without inertia based regularization (`-inertia`), without iterative refinement (`-itref`) and without `HSL MA97` tuning (`-ma97tuning`).

only one *symbolic factorization* using either an *approximate minimum degree* or `METIS` (cf., Karypis and Kumar [127]) *ordering* is performed per optimization.

It is important to calculate the step directions with a sufficiently high numerical accuracy. Thus, *iterative refinement* is applied until the relative residual is smaller than 10^{-10}. It is further stopped after 10 iterations or if the iterative refinement fails to reduce the residual. The computation of the residual is based on the original linear equation system and then transformed back to the reduced system (5.20) instead of directly applying it to the latter. This is due to the fact that the original system is directly related to the termination conditions of Section 5.2.1 (cf., Wächter and Biegler [202]). If the iterative refinement fails to achieve a step with a residual smaller than 10^{-10}, the optimization problem is likely to be badly scaled and the `MC64` scaling of `HSL MA97` is activated. A strategy to adaptively increase the pivot tolerance in this situation like in Wächter and Biegler [202] has lead to worse practical performance together with an overall increase of memory usage by `HSL MA97` and is therefore not used.

Numerical Results

The numerical study of this linear equation system section includes iterative refinement (`-itref`), tuning of the linear solver `HSL MA97` (`-ma97tuning`), i.e., its parameter changes (apart from the pivot tolerance) and activating the scaling in case of iterative refinement failure, and inertia based regularization (`-inertia`). The latter uses the fact that condition (5.22) alone guarantees a sufficient descent direction of the merit function. As before, the strategies are evaluated by disabling and comparing them to the standard configuration `WORHP IPm`. Results are given by Table 5.2, Figure 5.3 and Figure 5.4.

A significantly worse performance is observed by deactivating inertia based regularization.

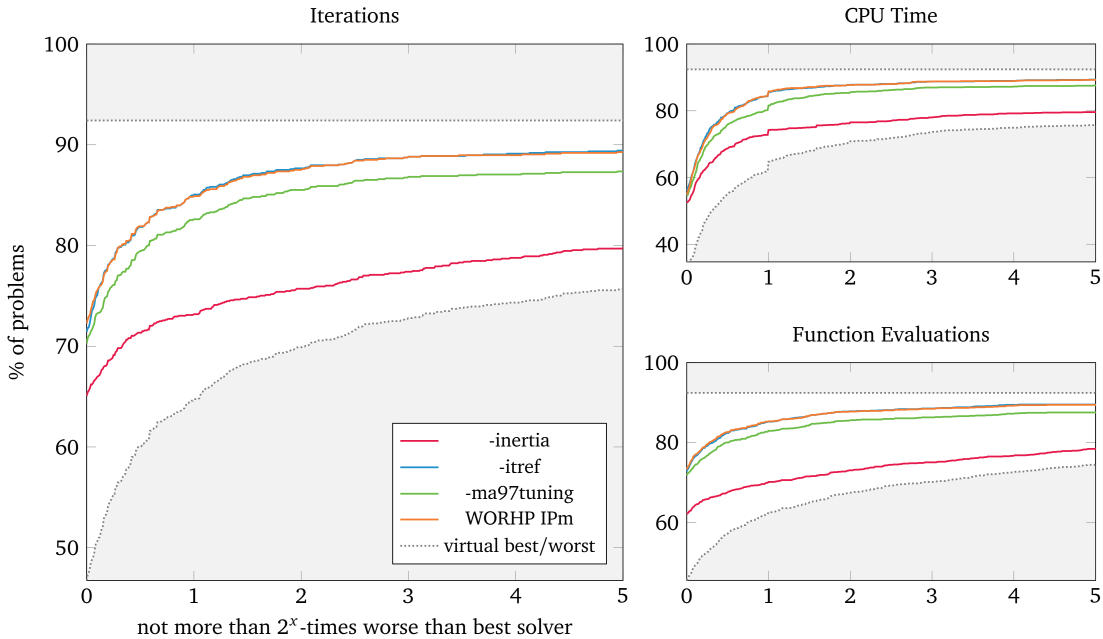

Figure 5.3: Performance profile for the numerical study of the linear equation system solution strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without inertia based regularization (`-inertia`), without iterative refinement (`-itref`) and without HSL MA97 tuning (`-ma97tuning`).

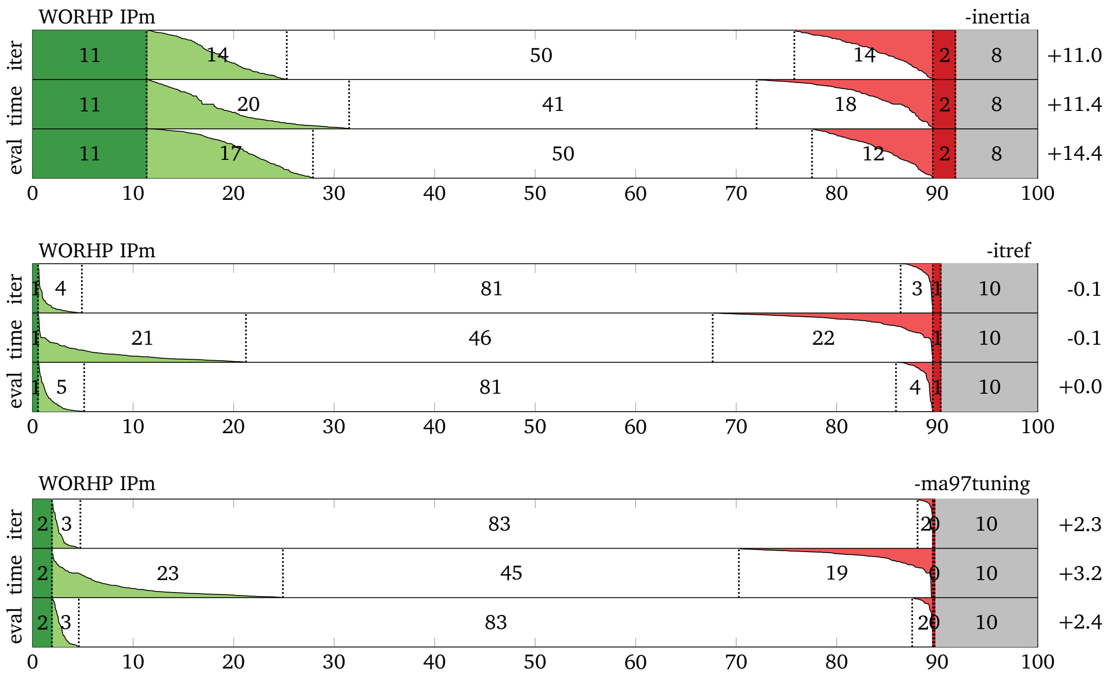

Figure 5.4: Individual performance profiles for the numerical study of the linear equation system solution strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without inertia based regularization (`-inertia`), without iterative refinement (`-itref`) and without HSL MA97 tuning (`-ma97tuning`).

Only 958 (73.41%) problem instances could be solved to optimality compared to 1080 (82.76%) for the default configuration `WORHP IPm`. However, detection of infeasibility does not seem to be influenced, as in 92 (7.05%) compared to 90 (6.90%) cases a certificate of infeasibility is returned. Additionally, the performance profile in Figure 5.4 indicates that the two configurations are balanced on the problem instances that both solve. `WORHP IPm` and `-inertia` both require less iterations on 14% of problem instances compared to each other and equally many on 50%. Only function evaluations are slightly reduced for `WORHP IPm` (17% less, 50% equal, 12% more). Thus, the scores of +11.0 (iterations), +11.4 (CPU time) and +14.4 (function evaluations) result from the higher robustness of inertia based regularization.

Similarly, the tuning of `HSL MA97` improves the robustness of the penalty-interior-point algorithm. 1059 (81.15%) problems can be solved to optimality without it, which are 21 (1.61% points) less. Figure 5.4 shows an additional minimal improvement on all metrics resulting in scores from +2.3 (iterations) to +3.2 (CPU time) in favor of `WORHP IPm`.

Disabling the iterative refinement has no considerable effect on the practical performance, which is surprising but evident from the performance profiles in Figure 5.3 and Figure 5.4. However, it has to be noted, that if the linear equation system solution has a too large residual norm, the `HSL MA97` scaling will still be activated in `-itref` for subsequent iterations to handle badly scaled systems.

5.2.4 Line Search

The implemented line search is the combination of the non-monotone filter and non-monotone merit function approach. To increase the potential dual steps, the acceptance criteria are first checked with a full dual step, i.e.,

$$x^{k+1} = x^k + \alpha_k \Delta x^k, \qquad s^{k+1} = s^k + \alpha_k \Delta s^k, \tag{5.23a}$$

$$y_{\mathrm{G}}^{k+1} = y_{\mathrm{G}}^k + \Delta y_{\mathrm{G}}^k, \qquad y_{\mathrm{H}}^{k+1} = y_{\mathrm{H}}^k + \Delta y_{\mathrm{H}}^k, \tag{5.23b}$$

$$z_{\mathrm{XL}}^{k+1} = z_{\mathrm{XL}}^k + \alpha_k^z \Delta z_{\mathrm{XL}}^k, \qquad z_{\mathrm{XU}}^{k+1} = z_{\mathrm{XU}}^k + \alpha_k^z \Delta z_{\mathrm{XU}}^k, \tag{5.23c}$$

$$z_{\mathrm{HL}}^{k+1} = z_{\mathrm{HL}}^k + \alpha_k^z \Delta z_{\mathrm{HL}}^k, \qquad z_{\mathrm{HU}}^{k+1} = z_{\mathrm{HU}}^k + \alpha_k^z \Delta z_{\mathrm{HU}}^k, \tag{5.23d}$$

before an update as in (4.14) and (4.27) is applied. The line search parameters are set to $\sigma = 10^{-8}$, $\beta = 0.5$, $l_{\mathrm{f}} = 1, l_{\mathrm{m}} = 3$, $\gamma_{\mathrm{f}} = 10^{-5}$, $\kappa_z = 10^{10}$, $\tau_{\mathrm{f}} = 10^{-8}$, $\tau_{\mathrm{c}} = 10^{-8}$ and $\varepsilon_{\mathrm{frac},k} = \max\left\{0.99, 1 - {\mu_k}^{0.5}\right\}$.

Slack Reset

Because of the special structure of the constraint $h(x) - s = 0$, it may be possible that the constraint function value of the trial variable $h\left(x^k + \alpha_k \Delta x^k\right)$ is a better and more accurate choice for the trial slack variable $s^k + \alpha_k \Delta s^k$. A so called *slack reset*

$$s_i^{k+1} = h_i\left(x^k + \alpha_k \Delta x^k\right), \quad i = 1, \ldots, n_h \tag{5.24}$$

is applied instead of $s_i^{k+1} = s_i^k + \alpha_k \Delta s_i^k$ if

i. index i belongs to a lower bounded constraint, i.e., $i \in \mathscr{H}_\mathrm{L} \setminus \mathscr{H}_\mathrm{U}$, and

$$h_i\big(x^k + \alpha_k \Delta x^k\big) - h_{\mathrm{L}_i} > -\mu_k \varsigma^k_{\mathrm{HL}_i}, \tag{5.25a}$$

$$-\ln\left(\frac{h_i\big(x^k + \alpha_k \Delta x^k\big) - h_{\mathrm{L}_i}}{\mu_k \varsigma^k_{\mathrm{HL}_i}} + 1\right) < -\ln\left(\frac{s_i^{k+1} - h_{\mathrm{L}_i}}{\mu_k \varsigma^k_{\mathrm{HL}_i}} + 1\right), \tag{5.25b}$$

ii. index i belongs to an upper bounded constraint, i.e., $i \in \mathscr{H}_\mathrm{U} \setminus \mathscr{H}_\mathrm{L}$, and

$$h_{\mathrm{U}_i} - h_i\big(x^k + \alpha_k \Delta x^k\big) > -\mu_k \varsigma^k_{\mathrm{HU}_i}, \tag{5.26a}$$

$$-\ln\left(\frac{h_{\mathrm{U}_i} - h_i\big(x^k + \alpha_k \Delta x^k\big)}{\mu_k \varsigma^k_{\mathrm{HU}_i}} + 1\right) < -\ln\left(\frac{h_{\mathrm{U}_i} - s_i^{k+1}}{\mu_k \varsigma^k_{\mathrm{HU}_i}} + 1\right), \tag{5.26b}$$

iii. index i belongs to a lower and upper bounded constraint, i.e., $i \in \mathscr{H}_\mathrm{L} \cap \mathscr{H}_\mathrm{U}$, and

$$h_i\big(x^k + \alpha_k \Delta x^k\big) - h_{\mathrm{L}_i} > -\mu_k \varsigma^k_{\mathrm{HL}_i}, \tag{5.27a}$$

$$h_{\mathrm{U}_i} - h_i\big(x^k + \alpha_k \Delta x^k\big) > -\mu_k \varsigma^k_{\mathrm{HU}_i} \tag{5.27b}$$

and

$$\begin{aligned} &-\ln\left(\frac{h_i\big(x^k + \alpha_k \Delta x^k\big) - h_{\mathrm{L}_i}}{\mu_k \varsigma^k_{\mathrm{HL}_i}} + 1\right) - \ln\left(\frac{h_{\mathrm{U}_i} - h_i\big(x^k + \alpha_k \Delta x^k\big)}{\mu_k \varsigma^k_{\mathrm{HU}_i}} + 1\right) \\ &< -\ln\left(\frac{s_i^{k+1} - h_{\mathrm{L}_i}}{\mu_k \varsigma^k_{\mathrm{HL}_i}} + 1\right) - \ln\left(\frac{h_{\mathrm{U}_i} - s_i^{k+1}}{\mu_k \varsigma^k_{\mathrm{HU}_i}} + 1\right). \end{aligned} \tag{5.27c}$$

Conditions (5.25b), (5.26b) and (5.27c) ensure that the slack reset results in a necessary descent of the merit function $\Psi\big(w^k; \rho^k\big)$. This further merit function decrease increases the possibility of step acceptance by the Armijo condition in Step L-10.2.

Complementarity Refinement

Similar to the feasibility refinement in Section 3.6.1 the step direction is further improved by one iteration of a *complementarity refinement*. For this strategy the linear equation system (5.20) is solved again with the same matrix – which allows a resuse of the factorization – and the right-hand-sides

$$b_1 = b_2 = b_3 = b_4 = 0 \tag{5.28a}$$

$$b_{5,i} = \left(x_i^k - x_{\mathrm{L}_i} + \alpha_k \Delta x_i^k + \mu_k \varsigma^k_{\mathrm{XL}_i}\right)\left(z^k_{\mathrm{XL}_i} + \alpha^z_k \Delta z^k_{\mathrm{XL}_i}\right) - \pi_k \mu_k \varsigma^k_{\mathrm{XL}_i} \nu^k_{\mathrm{XL}_i}, \qquad i \in \mathscr{X}_\mathrm{L} \tag{5.28b}$$

$$b_{6,i} = \left(x_{\mathrm{U}_i} - x_i^k - \alpha_k \Delta x_i^k + \mu_k \varsigma^k_{\mathrm{XU}_i}\right)\left(z^k_{\mathrm{XU}_i} + \alpha^z_k \Delta z^k_{\mathrm{XU}_i}\right) - \pi_k \mu_k \varsigma^k_{\mathrm{XU}_i} \nu^k_{\mathrm{XU}_i}, \qquad i \in \mathscr{X}_\mathrm{U} \tag{5.28c}$$

$$b_{7,i} = \left(s_i^k - h_{\mathrm{L}_i} + \alpha_k \Delta s_i^k + \mu_k \varsigma^k_{\mathrm{HL}_i}\right)\left(z^k_{\mathrm{HL}_i} + \alpha^z_k \Delta z^k_{\mathrm{HL}_i}\right) - \pi_k \mu_k \varsigma^k_{\mathrm{HL}_i} \nu^k_{\mathrm{HL}_i}, \qquad i \in \mathscr{H}_\mathrm{L} \tag{5.28d}$$

$$b_{8,i} = \left(h_{\mathrm{U}_i} - s_i^k - \alpha_k \Delta s_i^k + \mu_k \varsigma^k_{\mathrm{HU}_i}\right)\left(z^k_{\mathrm{HU}_i} + \alpha^z_k \Delta z^k_{\mathrm{HU}_i}\right) - \pi_k \mu_k \varsigma^k_{\mathrm{HU}_i} \nu^k_{\mathrm{HU}_i}, \qquad i \in \mathscr{H}_\mathrm{U} \tag{5.28e}$$

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
`-complref`	1077	92	8	1	28	32	67
`-filter`	1031	92	10	1	38	67	66
`-nonmon`	1078	90	9	1	30	32	65
`-sreset`	1081	89	11	1	28	33	62
`WORHP IPm`	1080	90	7	1	29	35	63

Table 5.3: Numbers of termination statuses for the numerical study of the line search strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without filter (`-filter`), without non-monotonicity (`-nonmon`), without slack reset (`-sreset`) and without complementarity refinement (`-complref`).

to get the refinement steps $\widetilde{\Delta x}^k$, $\widetilde{\Delta s}^k$, $\widetilde{\Delta y}_{\mathrm{G}}^k$, $\widetilde{\Delta y}_{\mathrm{H}}^k$, $\widetilde{\Delta z}_{\mathrm{XL}}^k$, $\widetilde{\Delta z}_{\mathrm{XU}}^k$, $\widetilde{\Delta z}_{\mathrm{HL}}^k$ and $\widetilde{\Delta z}_{\mathrm{HU}}^k$. In contrast to the feasibility refinement, no further function evaluations are necessary. Maximum step sizes $\widehat{\alpha}_k \in (0,1]$ and $\widehat{\alpha}_k^z \in (0,1]$ are computed by a fraction-to-the-boundary rule similar to (4.15) and (4.28). The combined (or refined) step $\widehat{\alpha}_k\left(\Delta x^k + \widetilde{\Delta x}^k\right)$, $\widehat{\alpha}_k\left(\Delta s^k + \widetilde{\Delta s}^k\right)$, $\Delta y_{\mathrm{G}}^k + \widetilde{\Delta y}_{\mathrm{G}}^k$, $\Delta y_{\mathrm{H}}^k + \widetilde{\Delta y}_{\mathrm{H}}^k$, $\widehat{\alpha}_k^z\left(\Delta z_{\mathrm{XL}}^k + \widetilde{\Delta z}_{\mathrm{XL}}^k\right)$, $\widehat{\alpha}_k^z\left(\Delta z_{\mathrm{XU}}^k + \widetilde{\Delta z}_{\mathrm{XU}}^k\right)$, $\widehat{\alpha}_k^z\left(\Delta z_{\mathrm{HL}}^k + \widetilde{\Delta z}_{\mathrm{HL}}^k\right)$ and $\widehat{\alpha}_k^z\left(\Delta z_{\mathrm{HU}}^k + \widetilde{\Delta z}_{\mathrm{HU}}^k\right)$ replaces the standard step if it is accepted by the filter or merit function criterion, if $\widehat{\alpha}_k \geq \alpha_k$ as well as $\widehat{\alpha}_k^z \geq \alpha_k^z$ and if it yields a smaller shifted complementarity error than the standard step, which for the latter is defined as

$$\begin{aligned}
\max\Bigg\{ & \max_{i\in\mathscr{X}_{\mathrm{L}}}\left(x_i^k - x_{\mathrm{L}_i} + \alpha_k \Delta x_i^k + \mu_k \varsigma_{\mathrm{XL}_i}^k\right)\left(z_{\mathrm{XL}_i}^k + \alpha_k^z \Delta z_{\mathrm{XL}_i}^k\right), \\
& \max_{i\in\mathscr{X}_{\mathrm{U}}}\left(s_i^k - h_{\mathrm{L}_i} + \alpha_k \Delta s_i^k + \mu_k \varsigma_{\mathrm{HL}_i}^k\right)\left(z_{\mathrm{HL}_i}^k + \alpha_k^z \Delta z_{\mathrm{HL}_i}^k\right), \\
& \max_{i\in\mathscr{H}_{\mathrm{L}}}\left(s_i^k - h_{\mathrm{L}_i} + \alpha_k \Delta s_i^k + \mu_k \varsigma_{\mathrm{HL}_i}^k\right)\left(z_{\mathrm{HL}_i}^k + \alpha_k^z \Delta z_{\mathrm{HL}_i}^k\right), \\
& \max_{i\in\mathscr{H}_{\mathrm{U}}}\left(h_{\mathrm{U}_i} - s_i^k - \alpha_k \Delta s_i^k + \mu_k \varsigma_{\mathrm{HU}_i}^k\right)\left(z_{\mathrm{HU}_i}^k + \alpha_k^z \Delta z_{\mathrm{HU}_i}^k\right)\Bigg\}.
\end{aligned} \tag{5.29}$$

If these conditions are not met, the line search continues with the standard step, which then includes a possible reduction of the step size by backtracking.

Numerical Results

Besides assessing the deactivation of the slack reset (`-sreset`) and the complementarity refinement (`-complref`), it is evaluated how the filter (`-filter`) and line search non-monotonicity (`-nonmon`) influences the practical performance. Table 5.3 lists the number of different termination statuses and Figure 5.5 and Figure 5.6 show the resulting performance profiles.

Firstly, the line search non-monotonicity and slack reset do not impact the performance much. The percentage of problems with an equal number of iterations, CPU time or number of function evaluations dominates the performance profile in Figure 5.6. Nevertheless, the strategies slightly improve the algorithm as indicated by the scores. These are for the non-monotonicity +0.9 (iterations), +0.7 (CPU time), +1.8 (function evaluations) and +0.4 (iterations), +0.7 (CPU time) and +0.3 (function evaluations) for the slack reset.

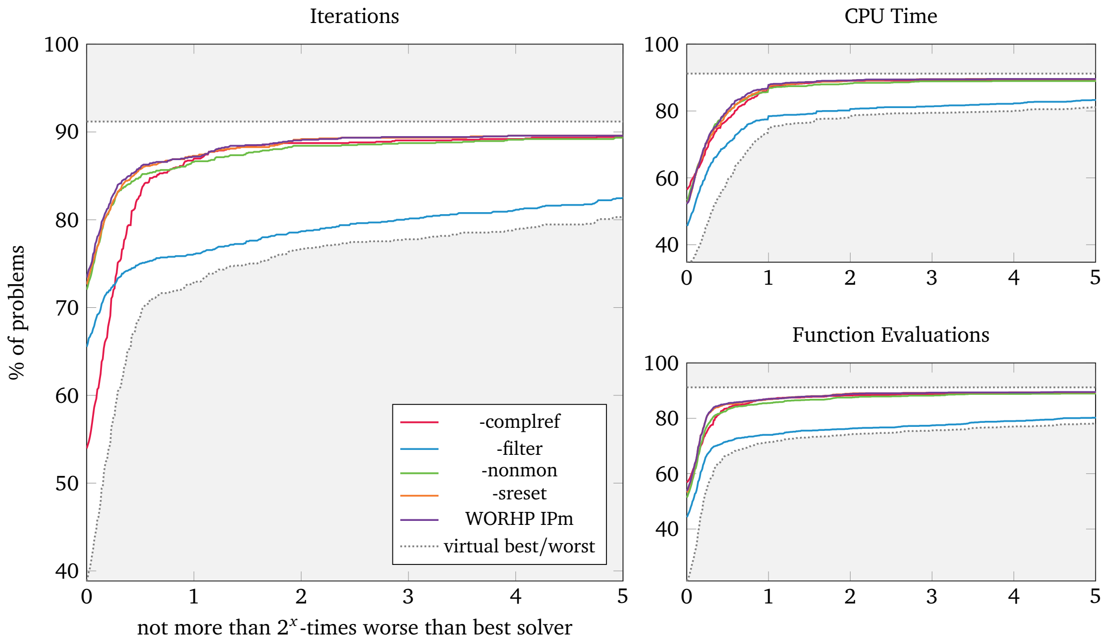

Figure 5.5: Performance profile for the numerical study of the line search strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without filter (`-filter`), without non-monotonicity (`-nonmon`), without slack reset (`-sreset`) and without complementarity refinement (`-complref`).

The most important line search component is the combination of the merit function with the filter approach. Disabling the filter leads to only 1031 (79.00%) solved problem instances and an increase of terminations with reached resource limits of +32 (`maxiter`) and +9 (`maxtime`). In addition to the robustness, the efficiency of the line search with a filter is greatly improved as shown by the performance profiles. While the number of problem instances for which `WORHP IPm` needs less resources is not large (13% for iterations, 25% for CPU time and 17% for function evaluations), the resource savings are significant. This is the reason for the high scores of +10.8 (iterations), +9.6 (CPU time) and +13.4 (function evaluations) and the clear domination of `WORHP IPm` over `-filter` in the performance profile of Figure 5.5.

An interesting computational result is given by the complementarity refinement. While Table 5.3 lists similar numbers of different termination statuses, in particular 1077 (82.53%) solved instances compared to 1080 (82.76%) of `WORHP IPm`, the performance profile in Figure 5.5 reports a huge improvement in efficiency. Figure 5.6 shows, that `WORHP IPm` needs fewer iterations on 31% of the test problems and more on only 10%. Unfortunately, the individual savings in the number of iterations are rather low such that the iteration score is only +3.8. The other metrics are also influenced on many problem instances but more balanced with scores of +1.0 (CPU time) and +0.9 (function evaluations).

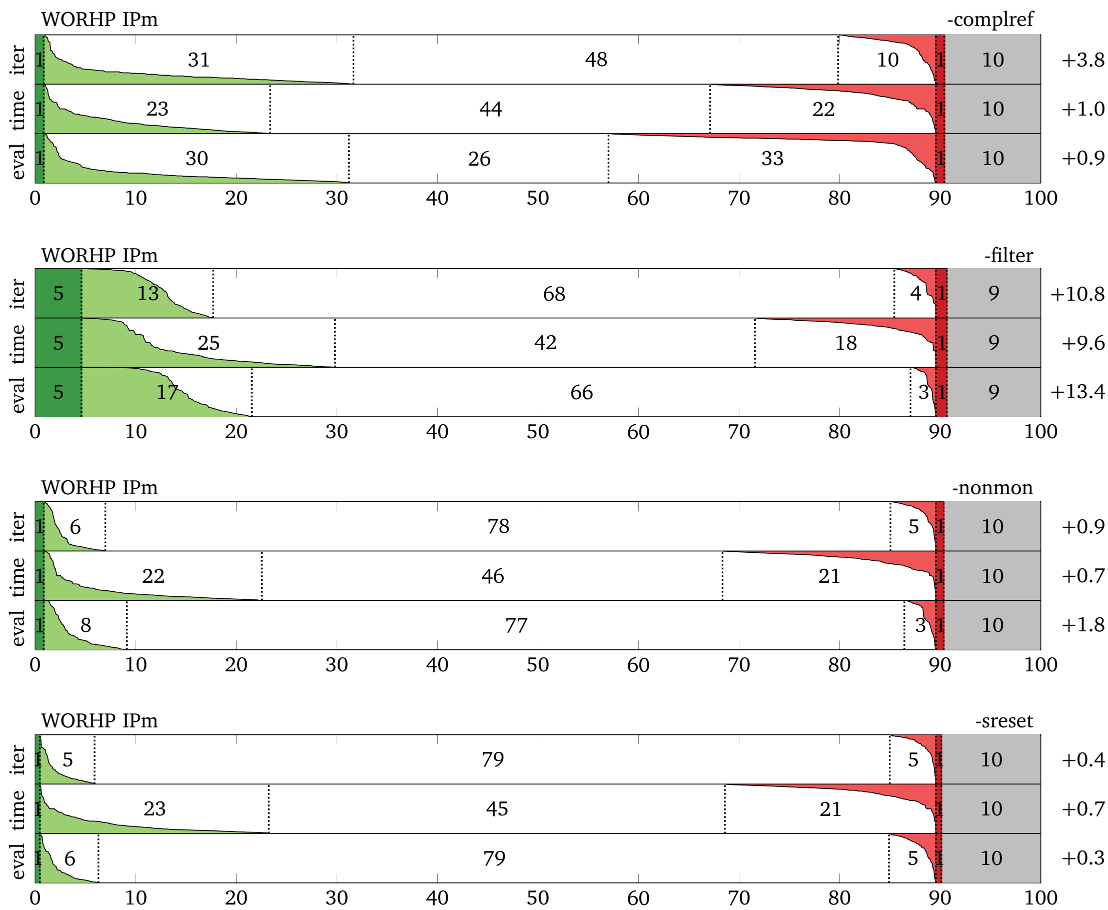

Figure 5.6: Individual performance profiles for numerical study of the line search strategies. The default configuration (`WORHP IPm`) is compared to `WORHP IPm` without filter (`-filter`), without non-monotonicity (`-nonmon`), without slack reset (`-sreset`) and without complementarity refinement (`-complref`).

5.2.5 Parameter Handling

The implementation of the penalty-interior-point algorithm in `WORHP` offers different barrier and penalty functions. Besides the modified barrier and the augmented Lagrangian exact $\ell 2$-penalty function defined in Section 4.1, the classic log-barrier and shifted barrier function as well as the pure exact $\ell 2$-penalty function are available. For Algorithm L, the Lagrangian multiplier parameters are initialized as $\lambda_{\text{G}}^0 = 0$, $\lambda_{\text{H}}^0 = 0$, $\nu_{\text{XL}}^0 = 1$, $\nu_{\text{XU}}^0 = 1$, $\nu_{\text{HL}}^0 = 1$ and $\nu_{\text{HU}}^0 = 1$. Therefore, the algorithm equals a shifted barrier and exact $\ell 2$-penalty function version up to the first parameter update. The initial boundary shifts are $\varsigma_{\text{XL}}^0 = 1$, $\varsigma_{\text{XU}}^0 = 1$, $\varsigma_{\text{HL}}^0 = 1$ as well as $\varsigma_{\text{HU}}^0 = 1$ and initial barrier and penalty parameters are $\mu_0 = 0.1$, $\pi_0 = 1$ and

$$\tau_0 = \min\left\{10^5, 10^2 \max\left\{1, \frac{|f(x^0)|}{1 + \left\| \begin{bmatrix} g(x^0) \\ h(x^0) - s^0 \end{bmatrix} \right\|_2}, \frac{\|\nabla f(x^0)\|_\infty}{1 + \left\| \begin{bmatrix} \nabla g(x^0)^\top \\ \nabla h(x^0)^\top \end{bmatrix} \right\|_\infty} \right\}\right\}. \tag{5.30}$$

The latter aims to balance the minimization of the objective function and constraint violation. The penalty parameter in front of the measure of constraint violation τ_k is increased using the parameters $\kappa_\lambda = 10^{-2}$ and $\kappa_\tau = 20$ until a threshold of $\tau_{\max} = 10^5$ is reached and the procedure switches to updates of π_k with $\kappa_\pi = 0.2$. For the update of the barrier parameter μ_k it turns out to be beneficial to decrease it at a higher speed if the update conditions (4.39) are satisfied within a few iterations after the last update (cf., Waltz et al. [187]), i.e.,

$$\mu_k = \begin{cases} \min\left\{0.01\mu_k, {\mu_k}^{1.8}\right\} & \text{if fast barrier update is activated} \\ \min\left\{0.2\mu_k, {\mu_k}^{1.5}\right\} & \text{otherwise.} \end{cases} \tag{5.31}$$

In this implementation an iteration limit of 3 is used for the fast barrier updates. Differently to the presentation in Section 4.2.3, the barrier parameter is updated also together with the Lagrangian multiplier parameters, i.e., whenever (4.39) holds. The boundary shifts are kept numerically close to one by setting $\kappa_\varsigma = 10^{-10}$. The tolerances of the update conditions are defined by the parameters $\upsilon_1 = 0.9$, $\upsilon_2 = 10^{-12}$, $\upsilon_3 = 0.99$, $\upsilon_4 = 0.5$, $\upsilon_5 = 0.99$, $\upsilon_6 = 0.5$, $\xi_{\lambda,k} = \xi_{\rho,k} = \varepsilon_{\text{tol}}/(1+k)$ and $\xi_{\nu,k} = \xi_{\mu,k} = 0.25/(1+k)$. To avoid arbitrarily large tolerances in (4.38) and (4.47), the quantities $\left\|\Phi_{\text{bar}}(w^i;\rho^{i-1})\right\|_\infty$, $\left\|\Phi_{\text{pen}}(w^i;\rho^{i-1})\right\|_\infty$ and $\left\|\Phi(w^i;\rho^{i-1})\right\|_\infty$ are limited to at maximum 10. Finally, after every decrease of the penalty parameter π_k the dual variables are updated to

$$z_{\text{XL}}^k \leftarrow \frac{\pi_k}{\pi_{k-1}} z_{\text{XL}}^k, \quad z_{\text{XU}}^k \leftarrow \frac{\pi_k}{\pi_{k-1}} z_{\text{XU}}^k, \quad z_{\text{HL}}^k \leftarrow \frac{\pi_k}{\pi_{k-1}} z_{\text{HL}}^k, \quad \text{and} \quad z_{\text{HU}}^k \leftarrow \frac{\pi_k}{\pi_{k-1}} z_{\text{HU}}^k \tag{5.32}$$

to keep an equal scaling of the dual variables and the Lagrangian multiplier parameters ν_{XL}^k, ν_{XU}^k, ν_{HL}^k and ν_{HU}^k.

Lowpass Filter

In practice, it may happen that the KKT based update conditions (4.31) and (4.39) cannot be satisfied by the solver due to numerical difficulties, e.g., bad scaling. This prevents the solver from converging to the optimal solution. To overcome this situation, similar to Nikolayzik [149] two so called *lowpass filters* Γ^k_{bar} and Γ^k_{pen} are used to measure the progress towards the barrier and penalty subprogram. They are initialized by $\Gamma^0_{\text{bar}} = \Gamma^0_{\text{pen}} = 0$ and then iteratively updated by

$$\Gamma^k_{\text{bar}} := 0.45 \left\|\Phi_{\text{bar}}\left(w^k;\rho^k\right)\right\|_\infty + 0.55\Gamma^{k-1}_{\text{bar}}, \tag{5.33a}$$

$$\Gamma^k_{\text{pen}} := 0.45 \left\|\Phi_{\text{pen}}\left(w^k;\rho^k\right)\right\|_\infty + 0.55\Gamma^{k-1}_{\text{pen}}. \tag{5.33b}$$

If the current iterate is feasible with respect to the barrier subprogram but the lowpass filter Γ^k_{bar} indicates no further progress towards a solution of it, i.e.,

$$\frac{\left|\Gamma^k_{\text{bar}} - \Gamma^{k-1}_{\text{bar}}\right|}{\max\left\{1, \Gamma^k_{\text{bar}}\right\}} \le 10^{-9}, \tag{5.34a}$$

$$g\left(x^k\right) \le \varepsilon_{\mu,k}, \tag{5.34b}$$

$$h\left(x^k\right) - s^k \le \varepsilon_{\mu,k}, \tag{5.34c}$$

the KKT based update conditions (4.39) are considered to be true. This leads to an update of the barrier parameter μ_k or Lagrangian multipliers ν^k_{XL}, ν^k_{XU}, ν^k_{HL} as well as ν^k_{HU} and, by that, can create a new barrier subprogram that may be easier solvable. Analogously, an update of the penalty parameters is invoked if the conditions

$$\frac{\left|\Gamma^k_{\text{pen}} - \Gamma^{k-1}_{\text{pen}}\right|}{\max\left\{1, \Gamma^k_{\text{pen}}\right\}} \le 10^{-9}, \tag{5.35a}$$

$$g\left(x^k\right) - \varrho_k\left(y^k_{\text{G}} - \pi_k\lambda^k_{\text{G}}\right) \le \varepsilon_{\rho,k}, \tag{5.35b}$$

$$h\left(x^k\right) - s^k - \varrho_k\left(y^k_{\text{H}} - \pi_k\lambda^k_{\text{H}}\right) \le \varepsilon_{\rho,k} \tag{5.35c}$$

hold.

Adaptive Updates

The implementation of adaptive parameter updates follows the description of Algorithm M, where the question of how to approximately solve (4.112) – the minimization of the optimality quality function over the possible barrier parameters in $\mathscr{M}$ – was left open. For this purpose the following trisection procedure is applied similarly to the adaptive update strategy in Nocedal et al. [153]. Let $[a,b]$ be the search interval for the barrier parameter, e.g., $a = \mu^+$ and $b = \mu_k$ (cf., Section 4.6). Then, two midpoints

$$c_1 := a - \frac{2}{1+\sqrt{5}}(b-a), \qquad c_2 := b + \frac{2}{1+\sqrt{5}}(b-a), \tag{5.36}$$

where $c_1 < c_2$ holds, are selected. Step sizes α_1 and α_2 are determined and the optimality quality functions are evaluated for both points. If

$$\begin{aligned} &Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(c_1,\pi,(\lambda_{\text{G}},\lambda_{\text{H}}),(\nu_{\text{XL}},\nu_{\text{XU}},\nu_{\text{HL}},\nu_{\text{HU}}));\alpha_1,\pi\Big) \\ <&Q_{\text{opt}}\Big(w^k, \widetilde{\Delta w}^k(c_2,\pi,(\lambda_{\text{G}},\lambda_{\text{H}}),(\nu_{\text{XL}},\nu_{\text{XU}},\nu_{\text{HL}},\nu_{\text{HU}}));\alpha_2,\pi\Big), \end{aligned} \tag{5.37}$$

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
`WORHP IP`	1090	99	18	3	29	25	41
`WORHP IP+adapt`	1086	101	12	3	27	30	46
`WORHP IPm+adapt`	1078	92	9	2	34	26	64
`WORHP IPm`	1080	90	7	1	29	35	63

Table 5.4: Numbers of termination statuses for the numerical study of the modified (`WORHP IPm`) and classic (`WORHP IP`) barrier function and adaptive parameter updates (`WORHP IPm+adapt`, `WORHP IP+adapt`).

the search interval is updated to $[a,b] \leftarrow [a,c_2]$ and $[a,b] \leftarrow [c_1,b]$ otherwise. The procedure repeats the process until 12 iterations are reached or if the search interval is sufficiently small, i.e., $b - a \leq 0.01b$. Finally, the barrier parameter is updated to $\mu_k = \frac{1}{2}(a+b)$ if the sufficient descent constraint of (4.112) is satisfied and left unchanged otherwise. The additional parameters for the adaptive updates are chosen to be $\kappa_1 = 10^{-12}$, $\kappa_2 = 0.99$, $\kappa_3 = 10^{-12}$ and $l_\rho = 5$.

Numerical Results

The numerical study focuses on the comparison of the modified barrier (`WORHP IPm`) with the classic log-barrier interior-point approach (`WORHP IP`) and the adaptive parameter updates (`+adapt`), which are disabled by default. The method `WORHP IP` is described in detail in Kuhlmann and Büskens [129] and differs from Algorithm L – apart from the barrier function – mainly in the updates of the barrier parameter. These are invoked whenever (4.39) is satisfied. Lagrangian multiplier parameters ν_{XL}, ν_{XU}, ν_{HL}, ν_{HU} and boundary shifts ς_{XL}, ς_{XU}, ς_{HL}, ς_{HU} do not have to be considered for `WORHP IP`. The numerical results are given by Table 5.4, Figure 5.7 and Figure 5.8.

It turns out that the interior-point method `WORHP IP` is more robust than `WORHP IPm` as it can solve 1090 (83.52%) problem instances to optimality, 99 (7.59%) to a certificate of local infeasibility and can detect 18 (1.38%) Fritz-John points compared to 1080 (82.76%), 90 (6.90%) and 7 (0.54%), respectively. Furthermore, `WORHP IP` needs in general less function evaluations, while the efficiency on the solved instances with respect to number of iterations and CPU time is balanced. The resulting scores are +1.8 (iterations), +1.3 (CPU time) and +4.3 (function evaluations) in favor of `WORHP IP`. If adaptive updates are activated, the results become even better for `WORHP IP`. Fewer iterations and function evaluations are needed. However, due to the more complex step calculation, `WORHP IP+adapt` performs slightly worse with respect to CPU time. The scores then update to +2.8 (iterations), -0.2 (CPU time) and +4.9 (function evaluations) for `WORHP IP` with enabled adaptive parameter updates.

Unfortunately, such an improvement by the adaptive parameter updates cannot be reported for `WORHP IPm`, which however is not unexpected from theory. Adaptive barrier updates are more restricted in the modified barrier function case, because they must keep the boundary shift as large as required for a feasible iterate. Furthermore, the sensitivity derivatives of the step direction with respect to the parameters become very small close to the optimal solution

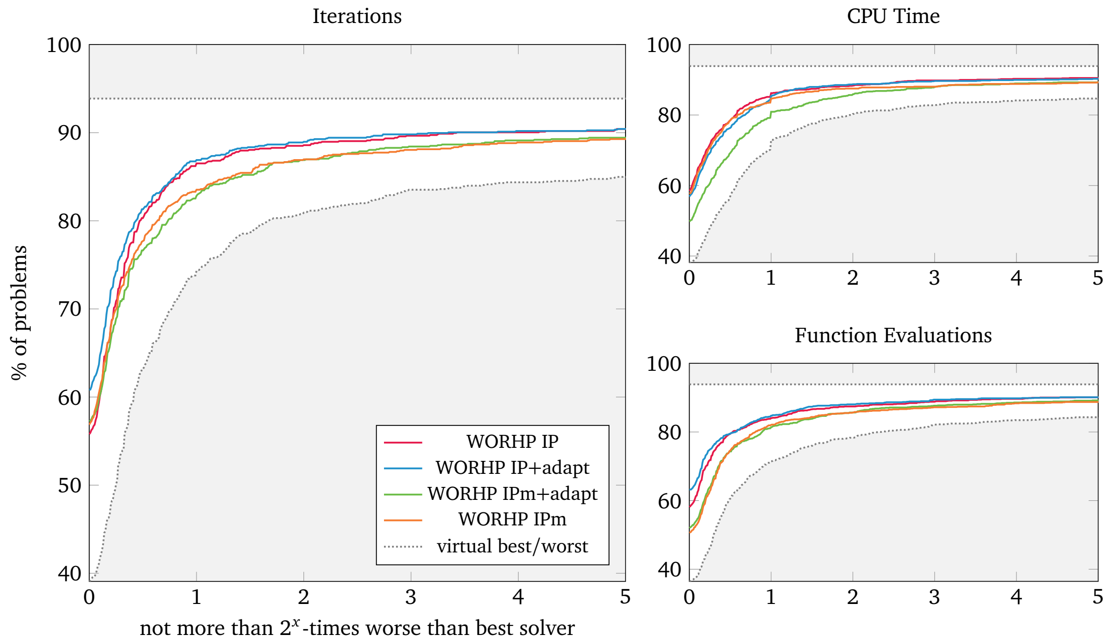

Figure 5.7: Performance profile for the numerical study of the modified (`WORHP IPm`) and classic barrier function (`WORHP IP`) and adaptive parameter updates (`WORHP IPm+adapt`, `WORHP IP+adapt`).

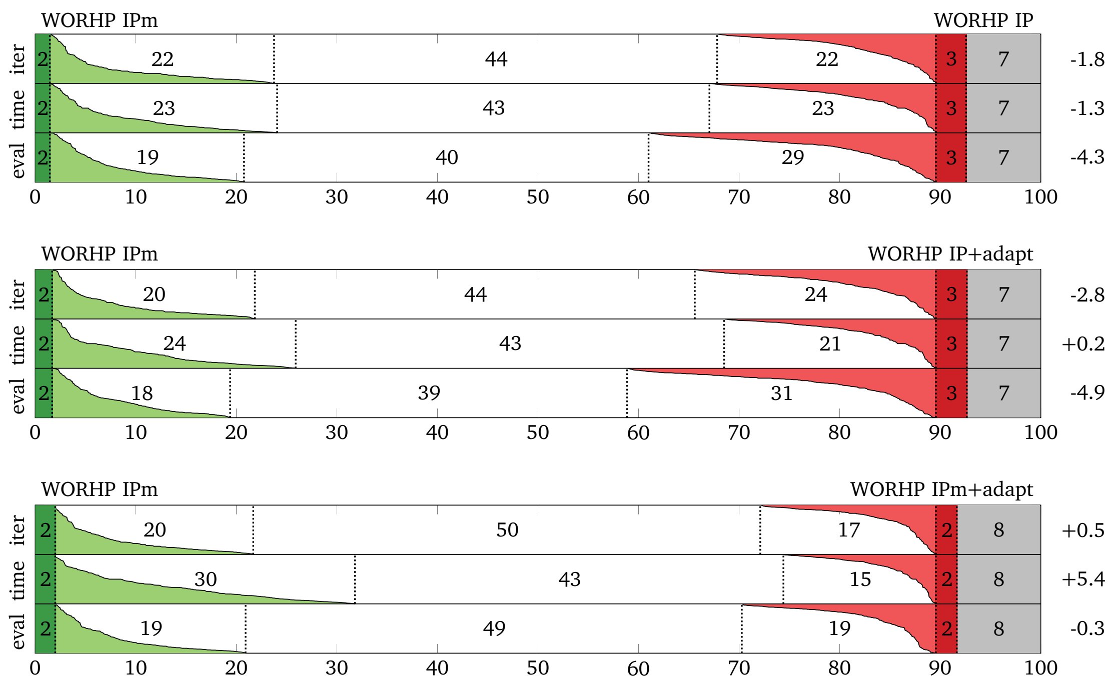

Figure 5.8: Individual performance profiles for the numerical study of the modified (`WORHP IPm`) and classic barrier function (`WORHP IP`) and adaptive parameter updates (`WORHP IPm+adapt`, `WORHP IP+adapt`).

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
`WORHP IP`	4527	113	8	5	1	27	827
`WORHP IPm`	4565	114	5	5	5	16	798

Table 5.5: Numbers of termination statuses for the numerical study of the modified (`WORHP IPm`) and classic barrier function (`WORHP IP`) when warmstarting.

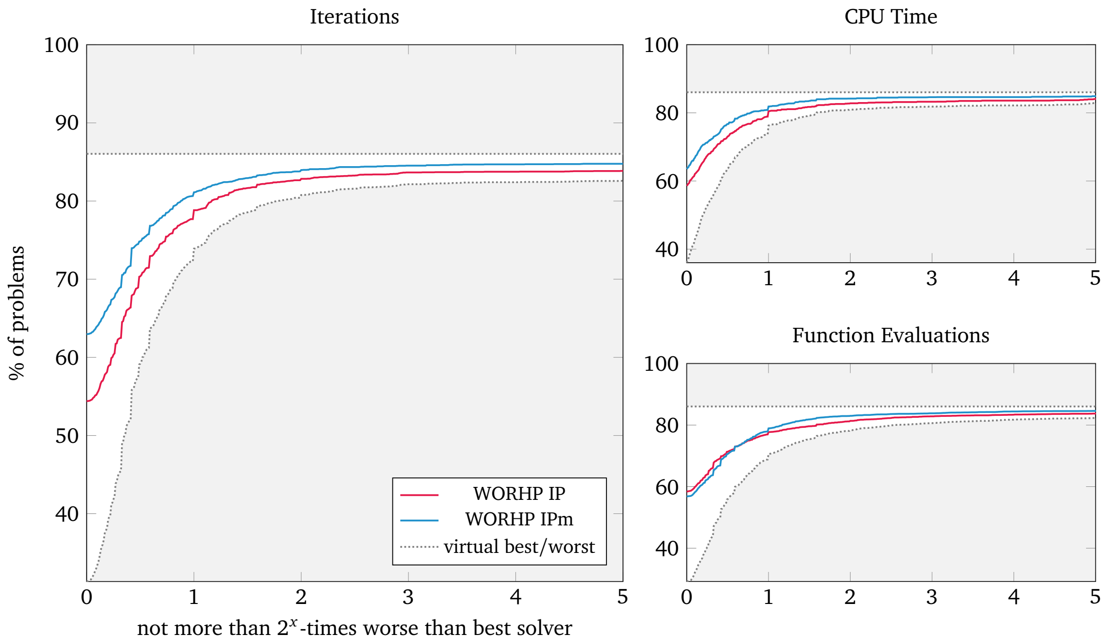

Figure 5.9: Performance profile for the numerical study of the modified (`WORHP IPm`) and classic barrier function (`WORHP IP`) when warmstarting.

(cf., Section 4.6.1), which reduces the impact of adaptive parameter updates. Both aspects do not occur for `WORHP IP`.

Numerical Results on `CUTEst-warm`

The penalty-interior-point algorithm has been designed with a modified barrier function to improve the warmstart performance. In a first step, this should be achieved simply by being able to start the optimization from the former optimal solution. This would not be possible for classic interior-point algorithms since the optimal solution could have active constraints, which would have to be shifted into the interior-region before the start. In this numerical study based on `CUTEst-warm` with $n_p = 5508$ test instances additional warmstart enhancements like the iterative refinement of Section 4.7 are therefore not used. Instead, the standard configurations of `WORHP IP` and `WORHP IPm` are compared in Table 5.5 and Figure 5.9. While `WORHP IP` solves 4527 (82.19%) test instances to optimality, `WORHP IPm` can solve 4565 (82.88%), which is an improvement of only 38 (0.69% points). However, the performance profile in Figure 5.9 shows a better efficiency of `WORHP IPm` in terms of number of iterations and CPU time although the

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
IPOPT	1052	27	0	4	15	30	177
KNITRO	1108	36	0	5	34	19	103
SNOPT	911	92	0	16	23	32	231
WORHP IP	1090	99	18	3	29	25	41
WORHP IPm	1080	90	7	1	29	35	63
WORHP SQP	1017	26	0	3	51	54	154

Table 5.7: Numbers of termination statuses for the numerical study of the nonlinear programming solvers IPOPT, KNITRO, SNOPT, WORHP IP, WORHP IPm and WORHP SQP.

former numerical study showed WORHP IP to be better on the original CUTEst test set. This can be seen as an indication that WORHP IPm indeed has a natural advantage for warmstarting.

5.3 Comparison to State-Of-The-Art NLP Solvers

In this section the proposed penalty-interior-point algorithm is compared to the SQP method SNOPT (cf., Gill et al. [91]), the one of WORHP 1.12 (cf., Büskens and Wassel [36]), denoted as WORHP SQP, and the interior-point algorithms IPOPT 3.12 (cf., Wächter and Biegler [202]) as well as KNITRO 11.0 (cf., Byrd et al. [29]). All four are state-of-the-art NLP solvers and belong to the best worldwide (cf., Mittelmann [142]). For a fair comparison, IPOPT also uses the linear solver HSL MA97 and optimized Basic Linear Algebra Subroutines (BLAS) are disabled for all solvers. In addition, acceptable and scaled termination conditions are deactivated, because the implementations differ. Instead, an absolute KKT error tolerance of $\varepsilon_{\text{tol}} = 10^{-6}$ is requested. Because the internal scaling of KNITRO influences the optimality conditions, it is disabled. All parameter changes for the different solvers are listed in Table 5.6 and Figure 5.10 shows the altered performance.[4] It turns out that while WORHP SQP and KNITRO perform worse due to the stricter termination conditions, IPOPT improved.

Numerical Results

A first numerical study compares the solver performances on the original CUTEst test set. The number of different termination statuses are given by Table 5.7 and a comprehensive result list by Table B.2. The most robust solver is KNITRO, which solves 1108 (84.90%) problem instances to optimality followed by WORHP IP (1090, 83.52%), WORHP IPm (1080, 82.76%), IPOPT (1052, 80.61%), WORHP SQP (1017, 77.93%) and SNOPT (911, 69.81%). Furthermore, the table shows that more certificates of infeasibility can be found by SNOPT (92, 7.05%), WORHP IP (99, 7.57%) and WORHP IPm (90, 6.70%) compared to IPOPT (27, 2.07%), KNITRO (36, 2.76%) and WORHP SQP (26, 1.99%), which – at least in the case of WORHP IP and WORHP IPm – leads

[4]For the default solver configuration time and iteration limits have also been changed to 1800 and 10000, respectively.

IPOPT

parameter	value	default
`linear_solver`	`ma97`	`(ma27)`
`max_cpu_time`	`1.8e+3`	`(1e+6)`
`max_iter`	`10000`	`(3000)`
`tol`	`1e-6`	`(1e-8)`
`dual_inf_tol`	`1e-6`	`(1e+0)`
`constr_viol_tol`	`1e-6`	`(1e-4)`
`compl_inf_tol`	`1e-6`	`(1e-4)`
`acceptable_iter`	`0`	`(15)`

KNITRO

parameter	value	default
`maxtime_cpu`	`1.8e+3`	`(1e+8)`
`opttol_abs`	`1e-6`	`(1e-3)`
`feastol_abs`	`1e-6`	`(1e-3)`
`infeastol`	`1e-6`	`(1e-8)`
`ftol_iters`	`10000`	`(5)`
`blasoption`	`0`	`(1)`
`scale`	`0`	`(1)`
`datacheck`	`no`	`(yes)`

SNOPT

parameter	value	default
`Major iterations`	`10000`	`(1000)`
`Time limit`	`1.8e+3`	`(0)`

WORHP SQP

parameter	value	default
`AcceptTolFeas`	`1e-6`	`(1e-3)`
`AcceptTolOpti`	`1e-6`	`(1e-3)`
`ScaledKKT`	`false`	`(true)`
`LowPassFilter`	`false`	`(true)`
`KeepAcceptableSol`	`false`	`(true)`

Table 5.6: Altered parameter configuration of NLP solvers `IPOPT`, `KNITRO`, `SNOPT` and `WORHP SQP`.

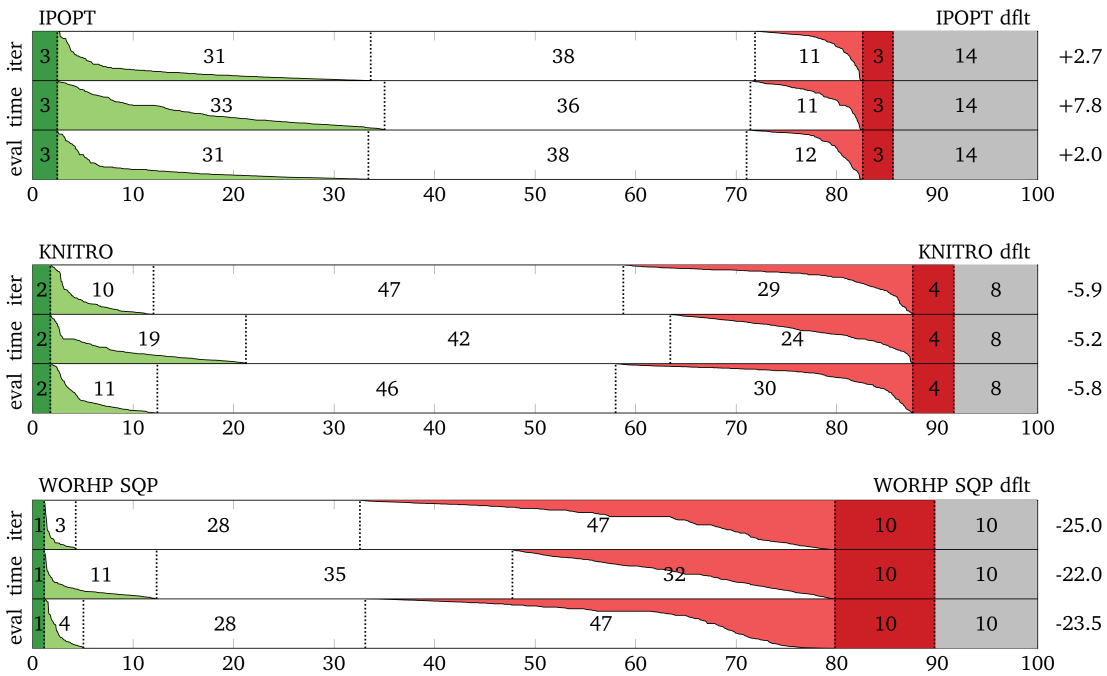

Figure 5.10: Individual performance profiles comparing the altered performance of NLP solvers `IPOPT`, `KNITRO` and `WORHP SQP` due to configuration changes. Left: Modified parameters; Right: Default parameters.

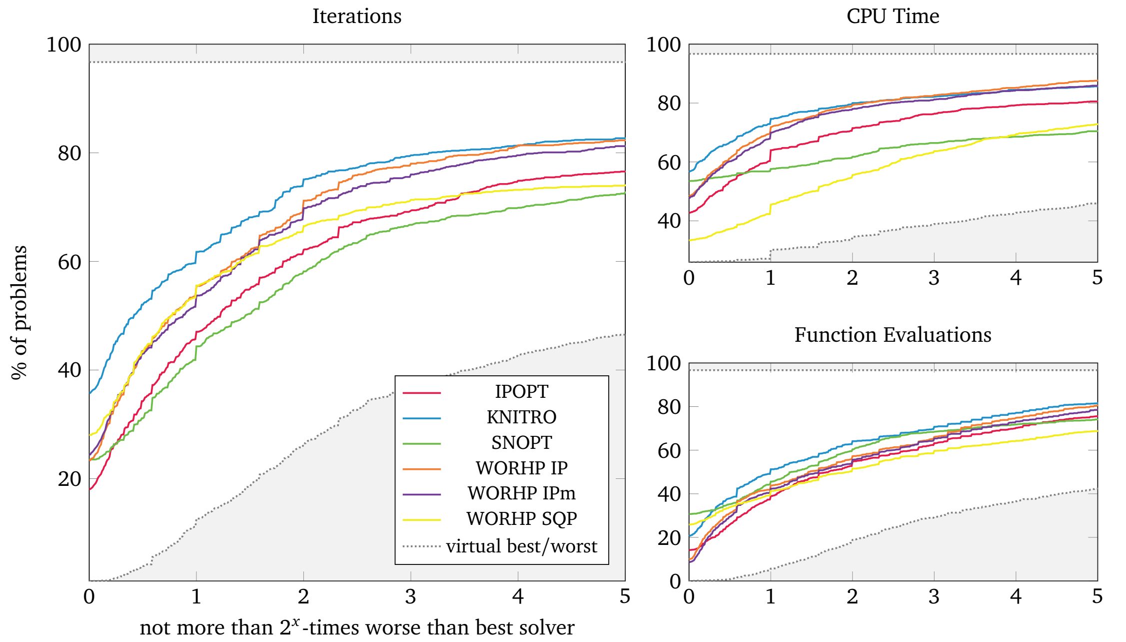

Figure 5.11: Performance profile for the numerical study of the nonlinear programming solvers `IPOPT`, `KNITRO`, `SNOPT`, `WORHP IP`, `WORHP IPm` and `WORHP SQP`.

to the least general failures compared to the other solvers. As described in Section 5.2.1, infeasibility detection may be affected by false positives. A lower bound on this false positive detections is given by the number of instances one solver returned with a certificate of infeasibility while another with an optimal solution. Table B.2 reports these values to be 40.74% (`IPOPT`), 22.22% (`KNITRO`), 20.65% (`SNOPT`), 18.18% (`WORHP IP`), 14.44% (`WORHP IPm`) and 20.65% (`WORHP SQP`). The performance on infeasible problem instances will be evaluated in more detail later on.

Performance profiles, comparing also the efficiency of the different solvers, are given by Figure 5.11 and Figure 5.12. Although the termination conditions have been brought into line, their actual implementation cannot be checked for all solvers. Therefore, the latter figure also includes profiles for the objective function to assess the quality of the returned solutions. The objective function values have been shifted up by the best solution found for a problem instance (if negative) to avoid comparisons of negative values. Since six solvers are compared in the performance profile of Figure 5.11, the line order has to be handled with great care (cf., Gould and Scott [102]), but the following general trends can be made out. `KNITRO` needs the least amount of iterations in general. The two SQP methods `SNOPT` and `WORHP SQP` cannot profit from fewer major iterations, although their step computation per iteration is more complex. This results in significantly more required CPU time. However, both SQP methods need fewer function evaluations in general. Thus, they might be better suited for real-world applications with computationally expensive function evaluations.

In the following, the state-of-the-art solvers `IPOPT`, `KNITRO`, `SNOPT` and `WORHP SQP` are compared to `WORHP IPm` in more detail using Figure 5.12. Studying `IPOPT` and `WORHP IPm`, it turns

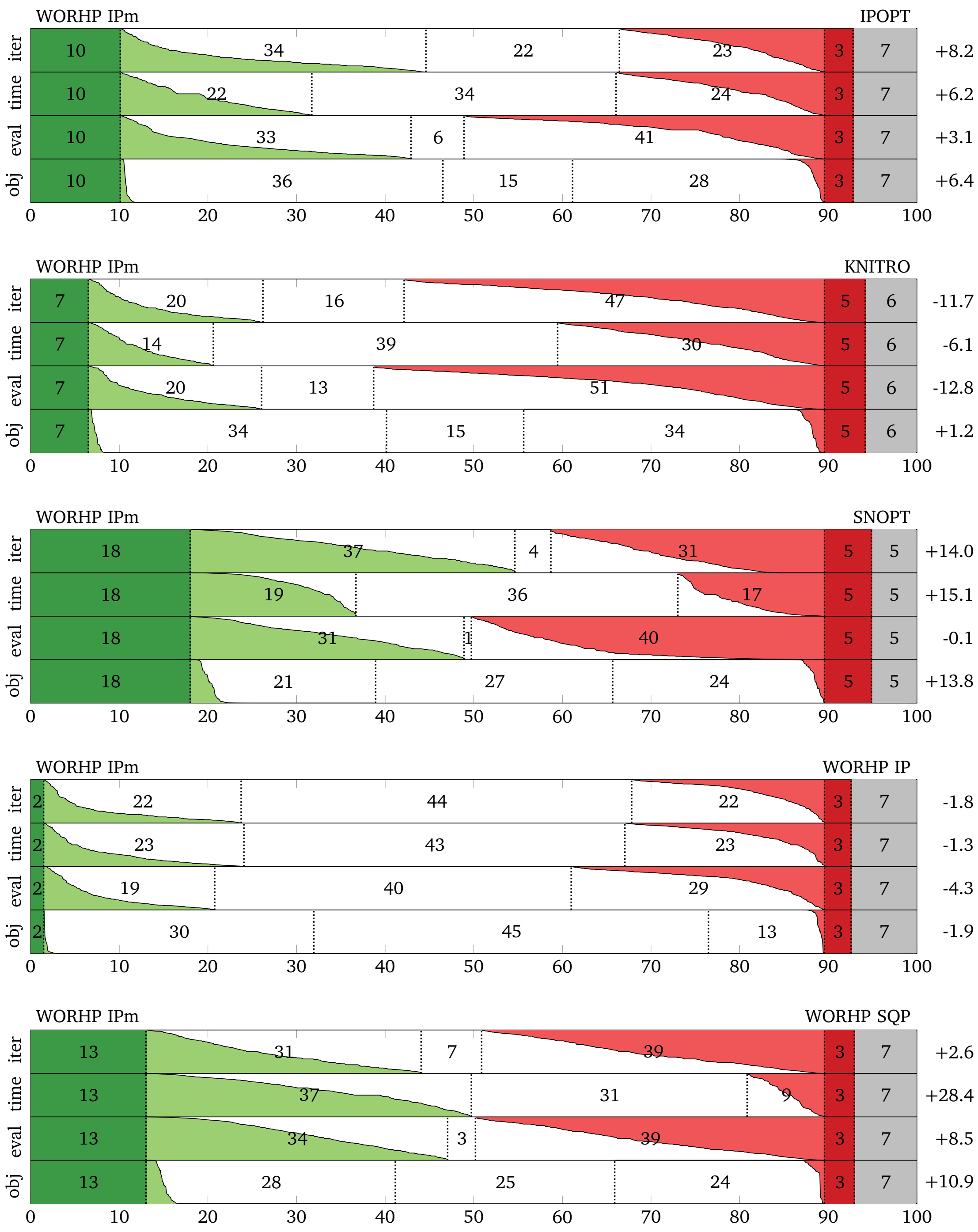

Figure 5.12: Individual performance profiles for the numerical study of the nonlinear programming solvers `IPOPT`, `KNITRO`, `SNOPT`, `WORHP IP`, `WORHP IPm` and `WORHP SQP`.

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
IPOPT	0	320	0	0	5	5	187
KNITRO	0	441	0	0	5	9	62
SNOPT	1	429	0	2	12	31	42
WORHP IP	0	486	0	0	6	5	20
WORHP IPm	0	477	0	0	9	8	23
WORHP SQP	0	33	0	0	45	2	437

Table 5.8: Numbers of termination statuses for the numerical study of the nonlinear programming solvers IPOPT, KNITRO, SNOPT, WORHP IP, WORHP IPm and WORHP SQP on infeasible CUTEst version.

out that WORHP IPm is more robust with 10% of problem instances WORHP IPm but not IPOPT can solve and only 3% the other way round. On the problem instances both software packages solved, WORHP IPm needs fewer iterations (34% less, 22% equal, 23% more). However, IPOPT is slightly more sparing with function evaluations (33% less, 6% equal, 41% more). The resulting scores are +8.2 (iterations), +6.2 (CPU time) and +3.1 (function evaluations) in favor of WORHP IPm. The picture changes when WORHP IPm is compared to KNITRO. While both solvers are comparably robust, WORHP IPm requires more number of iterations (20% less, 16% equal, 47% more), CPU time (14% less, 39% equal, 30% more) and function evaluations (20% less, 13% equal, 51% more) with scores of -11.7, -6.1 and -12.8, respectively. Comparing WORHP IPm to the two SQP methods, the outcome is as described above, leading to scores of +14.0 (iterations), +15.1 (CPU time) and -0.1 (function evaluations) in case of SNOPT and +2.6 (iterations), +28.4 (CPU time) and +8.5 (function evaluations) in case of WORHP SQP in favor of WORHP IPm. All in all, WORHP IPm is a very efficient and robust nonlinear programming solver that can compete with state-of-the-art solvers.

Numerical Results on CUTEst-infeas

In a second numerical study the performance of the nonlinear programming solvers is analyzed on the infeasible CUTEst version CUTEst-infeas. It has to be noted that detection of infeasibility and therefore termination criteria for a certificate of infeasibility differ for the different solvers. IPOPT switches to a feasibility restoration phase, that optimizes the constraint violation in an $\ell 1$-norm, if progress towards optimality in the line search is not possible (cf., Wächter and Biegler [202]). A certificate of infeasibility is reported if the feasibility restoration phase converges to a stationary point. This is similar to the elastic mode of SNOPT, which is invoked as soon as a quadratic subprogram is infeasible or if the multipliers become too large (cf., Gill et al. [91]). However, the elastic mode adds an $\ell 1$-penalty to the original problem formulation similar to Section 3.5.3 and does not enter a completely separate optimization phase. It terminates if the penalty parameter is sufficiently large. KNITRO monitors stationarity conditions for a feasibility problem without optimizing it directly (cf., Waltz et al. [187]). WORHP SQP reports infeasibility if the search direction becomes numerically zero and either the algorithm is in a dual feasibility mode (cf., Nikolayzik [149]) or if progress towards a smaller constraint violation measured by a lowpass filter variant cannot be made. Because detailed

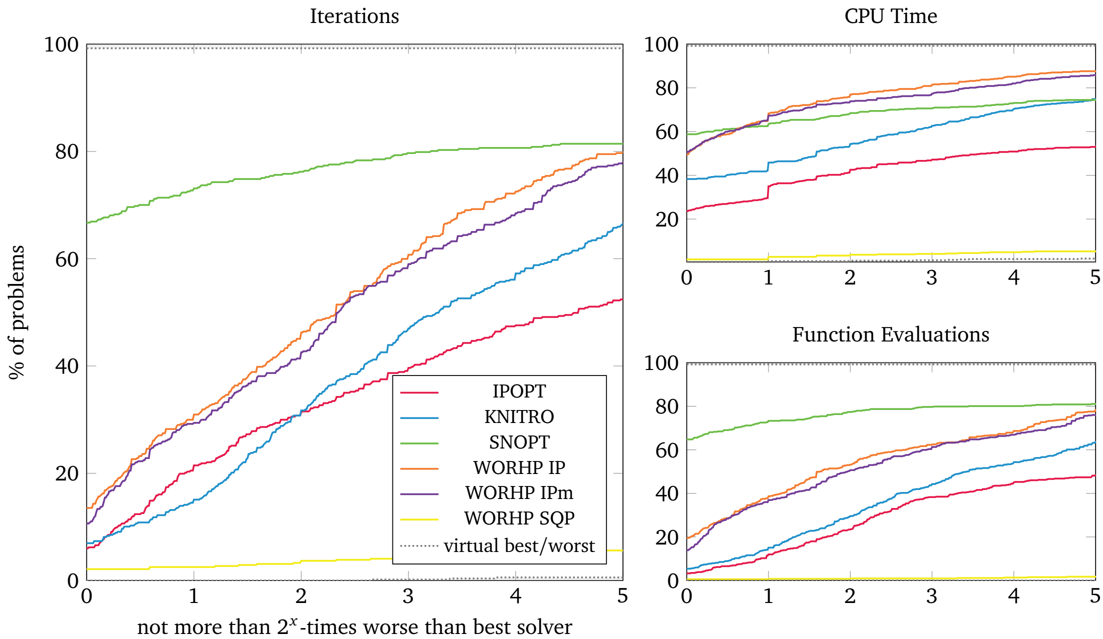

Figure 5.13: Performance profile for the numerical study of the nonlinear programming solvers `IPOPT`, `KNITRO`, `SNOPT`, `WORHP IP`, `WORHP IPm` and `WORHP SQP`.

parameter settings for termination conditions are not available for all solvers, the final constraint violations are additionally compared in this study. The results are listed in Table 5.8 and performance profiles are shown in Figure 5.13 and Figure 5.14.

The most problem instances are correctly classified by `WORHP IP` (486, 94.00%), followed by `WORHP IPm` (477, 92.26%), `KNITRO` (441, 85.30%), `SNOPT` (435, 84.14%), `IPOPT` (320, 61.90%) and `WORHP SQP` (33, 6.38%). In one case `SNOPT` erroneously reports an optimal solution although the final constraint violation is numerically one. The performance profiles show `SNOPT` to outperform the other solvers when detecting infeasibility. The most efficient interior-point algorithm is `WORHP IPm` (as well as `WORHP IPm`) with clear scores of +37.0 (iterations), +37.6 (CPU time) and +51.5 (function evaluations) in the case of `IPOPT` and +21.4 (iterations), +21.8 (CPU time) and +30.6 (function evaluations) compared to `KNITRO`. Additionally, `WORHP IPm` finds the most points with the least constraint violation, i.e., 48% of the problem instances `WORHP IPm` and `IPOPT` can solve `WORHP IPm` has a smaller constraint violation and 9% a larger one, in case of `KNITRO` its 37% (smaller) and 35% (larger) and in case of `SNOPT` 69% (smaller) and 7% (larger). Thus, `SNOPT`'s faster detection of infeasibility may be due to a weaker termination condition.

5.4 Crossover

From the numerical results of the previous Section 5.3 it seems that interior-point methods are generally superior SQP methods. This may be surprising at the first sight, because SQP

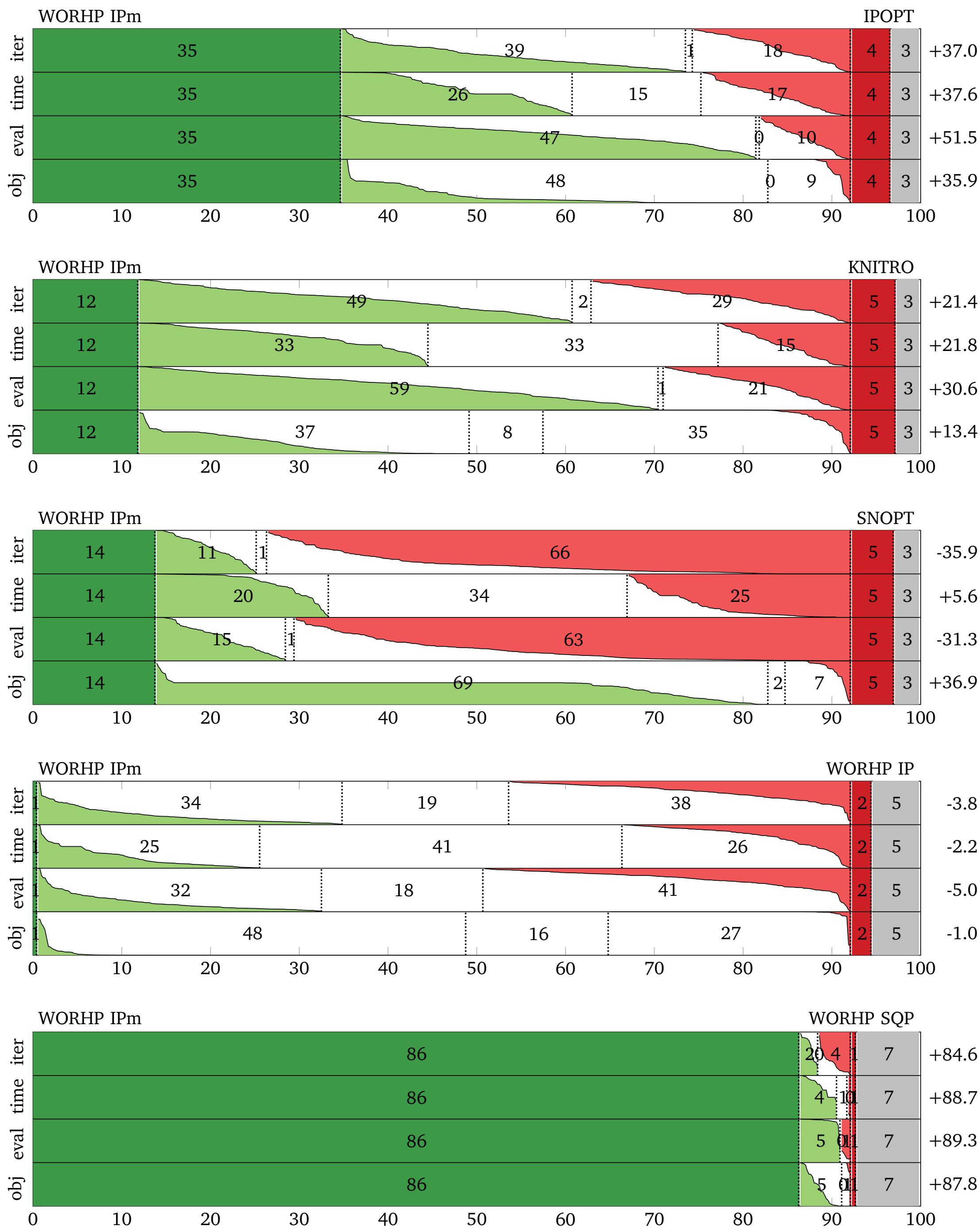

Figure 5.14: Individual performance profiles for the numerical study of the nonlinear programming solvers `IPOPT`, `KNITRO`, `SNOPT`, `WORHP IP`, `WORHP IPm` and `WORHP SQP`.

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
CO101	1061	90	5	1	31	40	77
CO103	1050	87	4	2	33	42	87
CO106	1035	58	3	3	38	49	119
WORHP IPm	1080	90	7	1	29	35	63
WORHP SQP	1017	26	0	3	51	54	154

Table 5.9: Numbers of termination statuses for the numerical study of the crossover with switch tolerances $10\varepsilon_{\text{tol}}$ (CO101), $10^3\varepsilon_{\text{tol}}$ (CO103) and $10^6\varepsilon_{\text{tol}}$ (CO106) compared to the penalty-interior-point (WORHP IPm) and the SQP algorithm (WORHP SQP).

approaches usually enjoy a q-quadratic local convergence rate while the one of interior-point methods is only q-superlinear (cf., Section 3.2.2). However, SQP methods face a significant drawback during the globalization phase, as they have to maintain a positive definite Hessian or Hessian approximation at every iteration to efficiently and uniquely solve the quadratic subprogram. Algorithm L for example needs to satisfy the positive definiteness condition (4.13), which is slightly weaker. In practice one can observe in the beginning of the optimization that – due to bad Lagrangian multiplier estimations – SQP methods regularize the Hessian very much and that the number of iterations in the QP solver is very large. A popular strategy to overcome this problem is to use Quasi-Newton methods. Because these lose the property of being q-quadratically local convergent, Geffken [81] proposed to switch locally to the exact Hessian.

Since WORHP now offers two algorithms, an SQP method and a penalty-interior-point approach, a second option becomes available. Instead of beginning the optimization with a Quasi-Newton based SQP method the penalty-interior-point algorithm is used, i.e., a so called *crossover* from interior-point to SQP. Such a strategy is in particular interesting for the modified barrier based interior-point algorithm, as it can provide – after a projection back into the original feasible region defined by the box constraints – an initial guess with a correct active-set estimation. This can beneficially be exploited by an SQP algorithm. Furthermore, Section 4.5.1 showed that the sensitivity derivatives calculated with the final matrix factorization of the penalty-interior-point algorithm will only be approximate. A crossover can therefore increase the accuracy of the sensitivity analysis performed by WORHP Zen.

Numerical Results

For this numerical study the crossover is executed if the current primal-dual iterate of the penalty-interior-point algorithm satisfies the KKT based termination conditions (5.4) up to a threshold of $10\varepsilon_{\text{tol}}$ (CO101), $10^3\varepsilon_{\text{tol}}$ (CO103) and $10^6\varepsilon_{\text{tol}}$ (CO106). Table 5.9, Figure 5.15 and Figure 5.16 show the results. Due to the higher robustness of WORHP IPm with 1080 (82.76%) solved problems, the crossover runs solve more instances than WORHP SQP (1017, 77.93%), in detail CO101 solves 1061 (81.30%) to optimality, CO103 1050 (80.46%) and CO106 1035 (79.31%). While the crossover improves the overall robustness and required CPU time of WORHP SQP, it does not perform better than WORHP IPm. For WORHP SQP – as expected from the

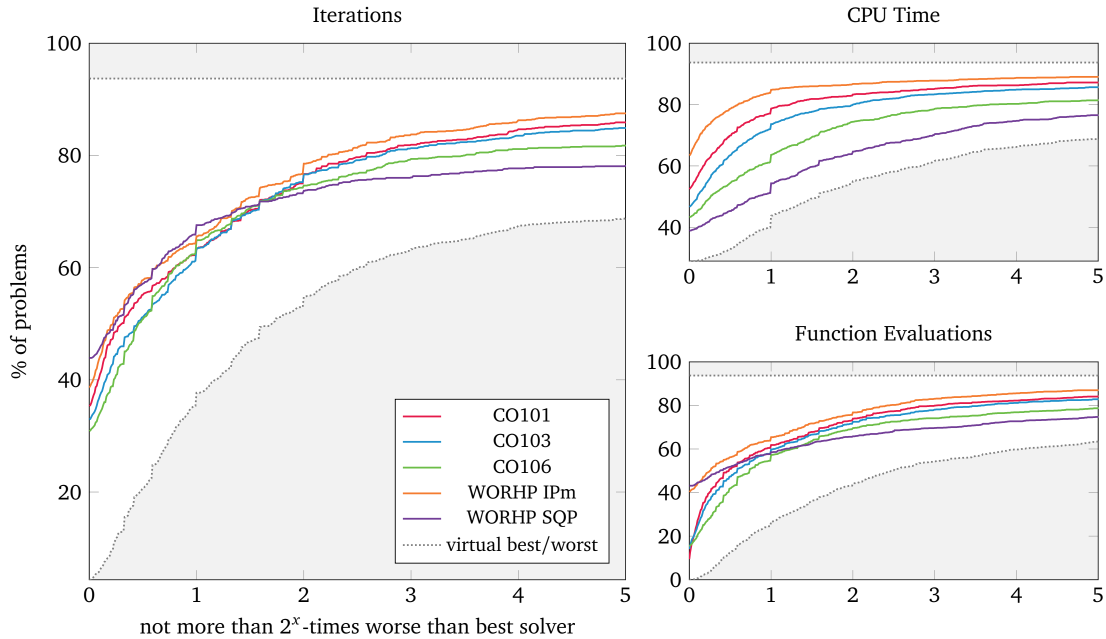

Figure 5.15: Performance profile for the numerical study of the crossover with switch tolerances $10\varepsilon_{\text{tol}}$ (CO101), $10^3\varepsilon_{\text{tol}}$ (CO103) and $10^6\varepsilon_{\text{tol}}$ (CO106) with the penalty-interior-point (WORHP IPm) and the SQP algorithm (WORHP SQP).

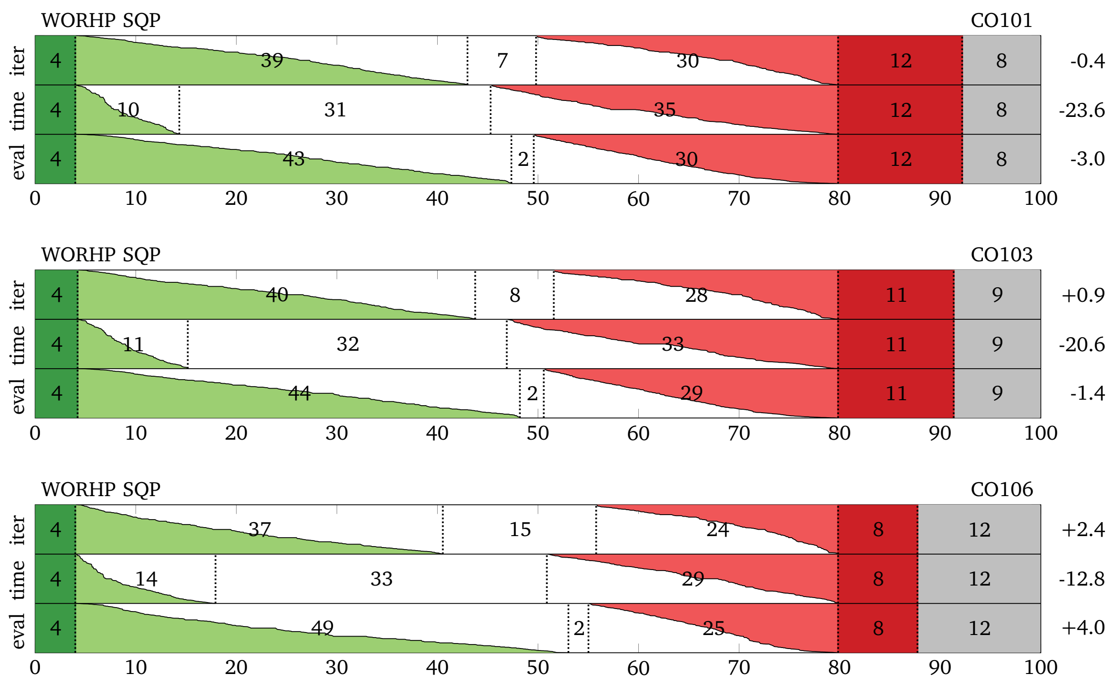

Figure 5.16: Individual performance profiles for the numerical study of the crossover with switch tolerances $10\varepsilon_{\text{tol}}$ (CO101), $10^3\varepsilon_{\text{tol}}$ (CO103) and $10^6\varepsilon_{\text{tol}}$ (CO106) with the penalty-interior-point (WORHP IPm) and the SQP algorithm (WORHP SQP).

last section – all performance profiles report fewer number of iterations and function evaluations and more required CPU time compared to the crossovers. For example `WORHP SQP` needed fewer iterations on 40%, equal on 8% and more on 28% of problem instances compared to `CO103`.

Chapter 6

Conclusions

The present thesis aimed to develop and analyze a practical penalty-interior-point algorithm for the solution of nonlinear optimization problems. The presentation began with the theory of nonlinear programming including post-optimality sensitivity analysis and continued with a broad overview over existing numerical solution strategies. These have been discussed and reviewed from the perspective of a practical implementation and led to the proposed primal-dual augmented Lagrangian penalty-interior-point algorithm that uses a modified barrier and an exact $\ell 2$-penalty function. This combination is new and challenging as it incorporates many parameters that have to be adapted throughout the optimization process. Key features beside the special barrier and penalty function are a special step calculation that yields a descent direction for the non-smooth merit function, automatic handling of rank-deficiency of the constraint derivatives for all non-feasible iterates, a combination of a non-monotone filter with a non-monotone merit function that avoids the necessity of a separate feasibility restoration phase and a magic step that offers a globalization framework for additional feature prototypes or internal heuristics.

A theoretical analysis of convergence properties has been given. Global convergence was proven to yield a first-order optimal solution, a certificate of infeasibility or a Fritz-John point and the local convergence order for a fixed barrier parameter has been shown to be quadratic. The overall algorithm can be superlinearly local convergent for certain barrier updates. In addition, an extensive study of the sensitivity analysis applied to the penalty-interior-point approach has been presented. This included the calculation of sensitivity derivatives for the original optimization problem as well as the barrier and penalty subprogram. Sensitivities have been used for the post-optimality sensitivity analysis module `WORHP Zen`, an adaptive parameter update strategy, a complementarity refinement step and a warmstart procedure. The latter is able to well approximate the optimal solution of a perturbed problem even under certain active-set changes – a property that standard real-time updates lack.

The proposed algorithm has been implemented within the nonlinear programming solver `WORHP` and is therefore accessible to academics and practitioners. To the knowledge of the author, `WORHP` is now the first state-of-the-art nonlinear programming solver that offers a shifted and modified barrier based interior-point algorithm. The implementation includes the option to crossover to `WORHP`'s SQP method at an arbitrary iteration. An intensive numerical study evaluated different solver options and showed the generally good performance of the penalty-interior-point algorithm on the CUTEst test set. While it performs slightly worse than the well-developed commercial solver KNITRO, it is superior the state-of-the-art solvers IPOPT, SNOPT and WORHP SQP. Moreover, it is the best interior-point algorithm in the study for detecting infeasibility of a problem formulation.

Recommendations for Future Work

Nevertheless, the numerical comparison of WORHP's SQP method and the penalty-interior-point algorithm indicated the existence of many problem instances that WORHP SQP solved with fewer iterations. Future developments could focus on the identification of nonlinear programming classes with that property to help designing an automatic algorithm selection. This could be based on machine learning techniques and has the potential to greatly improve the performance of WORHP. Currently, it is advised to use WORHP's multi-core interface (cf., Geffken and Büskens [82]) to run the different solvers in parallel.

While the crossover is able to increase the robustness of WORHP SQP, the hoped performance gain for the penalty-interior-point algorithm due to better theoretical local convergence properties of SQP methods failed to appear. The reason could be that WORHP SQP is based on an interior-point quadratic programming solver and that an early good approximation of the active-set due to the modified barrier function cannot be fully exploited. An extension of WORHP with an active-set based SQP method as well as further improvements of the WORHP SQP initialization after a crossover could be beneficial.

Future research on the penalty-interior-point algorithm should improve the current local convergence proof, which may also imply small changes to the presented method. However, these additional insights have the potential to further improve the practical performance. Another important aspect is the Hessian regularization. While the current inertia based strategy works sufficiently well, the numerical studies showed that changes on the Hessian regularization can have a huge impact on the practical performance. Disadvantages of the inertia based Hessian regularization are that they are computationally expensive as for every update the linear equation system has to be re-factorized, that they may regularize too much leading to unnecessary small search directions and that they restrict the linear solver choice to a small subset of solvers that provide eigenvalue information. To overcome this, a combination of a line search and trust-region globalization could be developed similar to the strategy of KNITRO (cf., Waltz et al. [187]).

A very interesting feature of the exact $\ell 2$-penalty function is the existence of a special modification of the Newton system for the step that yields a descent direction for the merit function. It was shown that the resulting Newton system is similar to the one of a smooth $\ell 2$-penalty function where the penalty parameter is adaptively updated to be proportional to the constraint violation (cf., Chen and Goldfarb [40]). The analyzed adaptive barrier update resembles this using the complementarity error, but lacks the descent direction property. That is why a magic step with a switching strategy between adaptive and monotone updates is needed for global convergence. However, it is often difficult in practice to choose a good barrier parameter for the switch to the monotone phase. To find a merit function and Newton system modification that establishes the descent direction property would not need the phase switch for the adaptive barrier updates and is therefore of great interest. Unfortunately, it is not clear if it exists.

Finally, the proposed complementarity and feasibility refinement for warmstarting a modified barrier function based interior-point algorithm could further be improved. An open question is, if it actually improves the warmstart performance in practice. While the current version successfully handles active-set changes from active to inactive, its approximation is highly dependent on the chosen barrier parameter when an inactive inequality constraint becomes active during the process. The reason is that the iterative refinement converges to the optimal solution of the barrier subprogram and thus would require further modifications of the barrier parameter in order to warmstart. It is desirable that the refinement would instead converge to the optimal solution of the original nonlinear program, which could be addressed by a modified perturbation within the refinement.

Appendix A

Theoretical Foundations

This appendix is a collection of some fundamental definitions and theorems that are used or referred to in the main part of the thesis.

Linear Algebra

Lemma A.1 (Cauchy-Schwarz Inequality). *Let $a, b \in \mathbb{R}^n$. Then, $\left|a^\top b\right| \leq \|a\|_2 \|b\|_2$ and $\left|a^\top b\right| = \|a\|_2 \|b\|_2$ if, and only if, a and b are linear dependent.*

Proof. See, for example, Fischer [65, Section 5.1]. □

Definition A.2 (Definite Matrix). *Let $A \in \mathbb{R}^{n \times n}$. Then, A is*

i. *positive definite, if $v^\top A v > 0$*
ii. *positive semi-definite, if $v^\top A v \geq 0$*
iii. *negative definite, if $v^\top A v < 0$*
iv. *negative semi-definite, if $v^\top A v \leq 0$*

for all $v \in \mathbb{R}^n \setminus \{0\}$.

Definition A.3 (Matrix Rank). *The rank of a matrix $A \in \mathbb{R}^{n \times m}$ is the dimension of the column or row space of A. The matrix is said to have full rank if $\operatorname{rank}(A) = \min\{n, m\}$ and rank-deficient otherwise.*

Lemma A.4 (Sylvester's Rank Inequality). *Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$. Then,*

$$\operatorname{rank}(A) + \operatorname{rank}(B) - m \leq \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Proof. See, for example, Matsaglia and Styan [140]. □

Definition A.5 (Matrix Condition). *Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $m \geq n$. If A is regular, then the condition of A is $\|A\| \left\|A^{-1}\right\|$. If B has full rank, the condition of B is $\max_{\|v\|=1} \|Bv\| \left(\min_{\|v\|=1} \|Bv\|\right)^{-1}$.*

Definition A.6 (Basis). *A basis of a vector space is linearly independent subset of the vector space and every other element can be formulated as a linear combination of the basis vectors.*

Definition A.7 (Null Space). *Let $A \in \mathbb{R}^{m \times n}$. The null space (or kernel) of the linear function defined by A is the set $\{v \in \mathbb{R}^n \mid Av = 0\}$.*

Definition A.8 (Pseudo Inverse). *Let $A \in \mathbb{R}^{m \times n}$. The pseudo inverse of A is the uniquely defined matrix A^- with the following properties:*

i. $AA^-A = A$.

ii. $A^-AA^- = A^-$.

iii. $\left(AA^-\right)^\top = AA^-$.

iv. $\left(A^-A\right)^\top = A^-A$.

Lemma A.9 (Farkas Lemma). *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ for $n, m \in \mathbb{N}$. Then the following two statements are equivalent:*

i. *The linear system $A^\top x = b, x \geq 0$ has a solution.*

ii. *The inequality $b^\top d \geq 0$ is satisfied for all $d \in \mathbb{R}^n$ with $Ad \geq 0$.*

Proof. See, for example, Geiger and Kanzow [84, Lemma 2.27]. □

Definition A.10 (Schur Complement). *Let A be a block matrix, i.e. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and A_{11} be regular. Then, the Schur complement of A_{11} in A is defined as $A/A_{11} := A_{22} - A_{21}{A_{11}}^{-1}A_{12}$.*

Analysis

Definition A.11 (Convex and Concave Function). *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then, f is*

i. *convex, if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$*

ii. *strictly convex, if $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$*

iii. *concave, if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$*

iv. *strictly concave, if $f(tx + (1-t)y) > tf(x) + (1-t)f(y)$*

for all $x, y \in \mathbb{R}^n, x \neq y$ and $t \in [0, 1]$.

Definition A.12 (Contraction Mapping). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $c \in [0, 1)$. Then, f is a contraction mapping if for all $x, y \in \mathbb{R}^n$ the inequality $\|f(x) - f(y)\| \leq c\,\|x - y\|$ holds.*

Definition A.13 (Directional Derivative). *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$. Then, the directional derivative of f along v is*

$$D_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

If f is differentiable, then $D_v f(x) = \nabla f(x)^\top v$.

Theorem A.14 (Taylor's Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a k-times differentiable function at $v \in \mathbb{R}^n$. Then, there exists $h_a : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x) = \sum_{|a|\leq k} \frac{1}{a_1! \cdots a_n!} \frac{\partial^{|a|}}{\partial_{x_1}^{a_1} \cdots \partial_{x_n}^{a_n}} f(v)(x_1 - a_1)^{a_1} \cdots (x_n - a_n)^{a_n}$$
$$+ \sum_{|a|=k} h_a(x)(x_1 - a_1)^{a_1} \cdots (x_n - a_n)^{a_n}$$

and $\lim_{x\to a} h_a(x) = 0$, where $|a| = a_1 + \cdots + a_n$.

Proof. See, for example, Königsberger [133, Section 2.4]. □

Definition A.15 (Subdifferential). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, v is a subgradient of f at $x' \in \mathbb{R}^n$ if for all $x \in \mathbb{R}^n$*

$$f(x) - f(x') \geq v^\top (x - x').$$

The subdifferential $\partial f(x')$ is the set of all subgradients at x'.

Theorem A.16 (Implicit Function Theorem). *Let $\mathcal{U}_1 \subseteq \mathbb{R}^n$ and $\mathcal{U}_2 \subseteq \mathbb{R}^m$ be open sets and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable. Assume that it exists $(a,b) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that $f(a,b) = 0$ and that the matrix $\nabla_y f(x,y)$ is regular at (a,b). Then, there exist open neighborhoods $\mathcal{N}_1(a) \subseteq \mathcal{U}_1$ and $\mathcal{N}_2(b) \subseteq \mathcal{U}_2$ and a continuously differentiable function $g : \mathcal{N}_1(a) \to \mathcal{N}_2(b)$ with $g(a) = b$ such that $f(x, g(x)) = 0$ for all $x \in \mathcal{N}_1(a)$. If $(x,y) \in \mathcal{N}_1(a) \times \mathcal{N}_2(b)$ is a point for which $f(x,y) = 0$ holds, then $y = g(x)$. In addition, if $\nabla_y f(x,y)$ is regular at $(x, g(x))$, then*

$$\nabla g(x) = \left(\nabla_y f(x, g(x))\right)^{-1} \nabla_x f(x, g(x)).$$

Proof. See, for example, Forster [78, Section 8 Theorem 2]. □

Theorem A.17 (Banach Fixed-Point Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction mapping. Then, f has a unique fixed-point, i.e. it exists $x^* \in \mathbb{R}^n$ such that $f(x^*) = x^*$. In addition, the sequence $\{x^k\}_k$ with $k := f(x^{k-1})$ converges to x^* for all $x^0 \in \mathbb{R}^n$.*

Proof. See, for example, Forster [78, Section 8 Theorem 1]. □

Theorem A.18 (Sandwich Theorem). *Let $\{a_k\}_k, \{b_k\}_k, \{c_k\}_k$ be sequences with $\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k$ and $a_k \leq b_k \leq c_k$ for almost all k. Then, $\{b_k\}_k$ is convergent and $\lim_{k\to\infty} b_k = \lim_{k\to\infty} a_k$.*

Proof. See, for example, Königsberger [132, Section 5.2]. □

Appendix B

CUTEst Results

Table B.1 and Table B.2 list the results on the CUTEst test set for the nonlinear programming solvers:

IPOPT	Interior-Point Solver, Wächter and Biegler [202], v3.12.
KNITRO	Interior-Point Solver, Byrd et al. [29], v11.0.
SNOPT	SQP Solver, Gill et al. [91], v7.7.
WORHP IP	Penalty-Interior-Point Solver, Kuhlmann and Büskens [129], like Algorithm L with log-barrier.
WORHP IPm	Penalty-Interior-Point Solver, Algorithm L.
WORHP SQP	SQP Solver, Büskens and Wassel [36], v1.12.

The possible status outcomes are:

optimal	Optimal solution found.
infeas	Certificate of infeasibility found.
fritzjohn	Fritz-John point found.
unbound	Problem seemed to be unbounded.
maxiter	Maximum number of iterations reached.
maxtime	Maximum time reached.
minalpha	Minimum step size reached in line search.
zerostep	Step direction became numerically zero.
smallstep	Step direction became very small.
degree	Problem instance has too few degrees of freedom to be solved.
resto	Feasibility restoration failed.
toobig	Some quantities (e.g. KKT violation) became too big.
diverge	Iterates diverged.
regular	Hessian regularization failed.
nan	Function evaluation error (e.g. NaN or Inf).
noimpr	No further improvement possible.
degen	Problem instance seem to be degenerate or ill-conditioned.

solver	optimal	infeas	fritzjohn	unbound	maxtime	maxiter	other
IPOPT	1052	27	0	4	15	30	177
KNITRO	1108	36	0	5	34	19	103
SNOPT	911	92	0	16	23	32	231
WORHP IP	1090	99	18	3	29	25	41
WORHP IPm	1080	90	7	1	29	35	63
WORHP SQP	1017	26	0	3	51	54	154

Table B.1: Overview of solver status outcomes of the nonlinear programming solvers IPOPT, KNITRO, WORHP IP, WORHP IPm and WORHP SQP on the CUTEst test set.

memory	Not enough memory.
sbasics	Superbasics limit too small.
killed	Extreme memory or time usage in linear solver → process had to be killed.

The table columns are:

instance	Name of CUTEst instance.
solver	Name of solver.
status	Returned status.
iter	Number of Iterations.
time	CPU time in seconds.
obj	Objective function value.
nf	Number of objective function evaluations.
ng	Number of constraint function evaluations.
ndf	Number of gradient evaluations.
ndg	Number of Jacobian evaluations.
nhm	Number of Hessian evaluations.

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
10FOLDTR	IPOPT	optimal	54	2.28e+01	0.00e+00	55	55	55	55	54
	KNITRO	optimal	55	6.55e+01	0.00e+00	60	61	57	58	55
	SNOPT	optimal	28	7.31e+00	0.00e+00	1	31	1	30	0
	WORHP IP	optimal	55	5.84e+01	0.00e+00	61	61	57	57	55
	WORHP IPm	regular	49	7.88e+01	0.00e+00	55	55	50	50	50
	WORHP SQP	optimal	55	1.31e+02	0.00e+00	56	56	56	56	55
3PK	IPOPT	optimal	11	1.00e-02	1.72e+00	12	0	12	0	11
	KNITRO	optimal	7	1.00e-02	1.72e+00	9	0	8	0	7
	SNOPT	optimal	438	5.00e-02	1.72e+00	446	0	445	0	0
	WORHP IP	optimal	8	1.00e-02	1.72e+00	10	0	10	0	8
	WORHP IPm	optimal	8	1.00e-02	1.72e+00	10	0	9	0	8
	WORHP SQP	optimal	28	2.00e-02	1.72e+00	29	0	29	0	28
A0ENDNDL	IPOPT	optimal	10	4.20e-01	1.81e-04	11	11	11	11	10
	KNITRO	optimal	7	4.90e-01	2.35e-06	10	11	9	10	7
	SNOPT	optimal	4	1.22e+01	0.00e+00	8	1	7	1	0
	WORHP IP	optimal	8	8.10e-01	1.10e-03	10	10	9	1	8
	WORHP IPm	optimal	8	6.60e-01	-3.29e-07	13	13	12	1	8
	WORHP SQP	optimal	2	1.69e+00	1.16e-09	2	2	3	3	2
A0ENINDL	IPOPT	optimal	10	4.30e-01	1.81e-04	11	11	11	11	10
	KNITRO	optimal	7	5.20e-01	3.45e-06	10	11	9	10	7
	SNOPT	optimal	4	1.53e+01	0.00e+00	8	1	7	1	0
	WORHP IP	optimal	8	8.10e-01	1.10e-03	10	10	9	1	8
	WORHP IPm	optimal	8	8.10e-01	-3.29e-07	13	13	12	1	8
	WORHP SQP	optimal	2	2.49e+00	7.36e-09	2	2	3	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
AOENSNDL	IPOPT	optimal	25	5.03e+00	9.18e-05	26	26	26	26	25
	KNITRO	optimal	8	6.20e-01	1.05e-08	11	12	10	11	8
	SNOPT	optimal	6	1.15e+01	8.15e-11	10	1	9	1	0
	WORHP IP	optimal	11	1.25e+00	9.12e-04	20	20	13	1	11
	WORHP IPm	optimal	8	7.90e-01	-2.73e-07	13	13	12	1	8
	WORHP SQP	optimal	2	3.51e+00	1.84e-14	3	3	3	2	2
AOESDNDL	IPOPT	optimal	10	6.10e-01	1.81e-04	11	11	11	11	10
	KNITRO	optimal	7	6.70e-01	2.83e-06	10	11	9	10	7
	SNOPT	optimal	7	1.61e+01	0.00e+00	11	1	10	1	0
	WORHP IP	optimal	8	8.00e-01	1.10e-03	10	10	9	1	8
	WORHP IPm	optimal	8	7.90e-01	-3.29e-07	13	13	12	1	8
	WORHP SQP	optimal	2	1.59e+00	8.99e-10	2	2	3	3	2
AOESINDL	IPOPT	optimal	10	4.50e-01	1.81e-04	11	11	11	11	10
	KNITRO	optimal	7	4.60e-01	3.12e-06	10	11	9	10	7
	SNOPT	optimal	5	1.23e+01	0.00e+00	9	1	8	1	0
	WORHP IP	optimal	8	5.00e-01	1.10e-03	10	10	9	1	8
	WORHP IPm	optimal	8	4.70e-01	-3.29e-07	13	13	12	1	8
	WORHP SQP	optimal	2	2.24e+00	9.58e-10	2	2	3	3	2
AOESSNDL	IPOPT	optimal	47	6.33e+00	1.47e-04	67	67	37	48	47
	KNITRO	optimal	8	6.80e-01	1.04e-08	11	12	10	11	8
	SNOPT	optimal	8	1.08e+01	-3.43e-12	11	1	10	1	0
	WORHP IP	optimal	11	1.58e+00	9.12e-04	19	19	13	1	11
	WORHP IPm	optimal	8	6.30e-01	-2.73e-07	13	13	12	1	8
	WORHP SQP	optimal	2	2.98e+00	1.73e-14	3	3	3	2	2
AONNDNDL	IPOPT	optimal	23	1.02e+00	1.81e-04	24	24	24	24	23
	KNITRO	optimal	12	1.05e+00	3.00e-08	15	16	14	15	12
	SNOPT	optimal	148	7.59e+01	-1.51e-06	162	1	161	1	0
	WORHP IP	optimal	15	1.07e+00	1.31e-03	17	17	16	1	15
	WORHP IPm	optimal	17	1.61e+00	5.00e-09	24	24	23	1	17
	WORHP SQP	optimal	2	4.71e+00	8.19e-12	3	3	4	3	2
AONNDNIL	IPOPT	optimal	97	6.42e+00	1.95e-04	98	98	98	98	97
	KNITRO	optimal	131	1.10e+01	2.19e-08	134	135	133	134	131
	SNOPT	optimal	84	3.49e+01	5.81e-05	117	1	116	1	0
	WORHP IP	optimal	151	1.27e+01	2.39e-07	153	153	153	1	151
	WORHP IPm	optimal	165	1.22e+01	1.70e-08	174	174	171	1	165
	WORHP SQP	optimal	104	2.82e+02	2.10e-10	104	104	104	2	104
AONNDNSL	IPOPT	optimal	54	3.64e+00	-7.65e-04	60	60	55	55	54
	KNITRO	optimal	14	1.42e+00	1.14e-05	17	18	16	17	14
	SNOPT	optimal	77	7.01e+01	-6.42e-13	99	1	98	1	0
	WORHP IP	optimal	25	2.51e+00	3.91e-06	27	27	27	1	25
	WORHP IPm	regular	250	1.98e+01	-1.16e+05	256	256	251	1	251
	WORHP SQP	optimal	3	1.41e+01	1.70e-15	368	368	4	2	3
AONNSNSL	IPOPT	accept	33	3.81e+00	-1.68e-04	61	61	35	35	34
	KNITRO	optimal	16	1.46e+00	1.73e-06	19	20	18	19	16
	SNOPT	optimal	17	4.66e+01	-2.85e-12	24	1	23	1	0
	WORHP IP	optimal	22	1.61e+00	3.79e-05	24	24	24	1	22
	WORHP IPm	optimal	23	2.27e+00	1.40e-08	30	30	29	1	23
	WORHP SQP	optimal	4	1.89e+01	1.80e-15	827	827	5	2	4
AONSDSDL	IPOPT	optimal	20	1.05e+00	1.81e-04	21	21	21	21	20
	KNITRO	optimal	11	7.70e-01	1.20e-05	14	15	13	14	11
	SNOPT	optimal	26	7.32e+01	0.00e+00	29	1	28	1	0
	WORHP IP	optimal	10	8.10e-01	1.10e-03	12	12	11	1	10
	WORHP IPm	optimal	11	8.50e-01	-4.18e-07	16	16	15	1	11
	WORHP SQP	optimal	2	5.14e+00	2.13e-14	3	3	4	3	2
AONSDSDS	IPOPT	optimal	29	2.70e-01	-1.53e-05	30	30	30	30	29
	KNITRO	optimal	14	1.70e-01	3.07e-07	17	18	16	17	14
	SNOPT	optimal	16	7.50e-01	0.00e+00	24	1	23	1	0
	WORHP IP	optimal	25	2.80e-01	8.11e-05	31	31	27	1	25
	WORHP IPm	optimal	20	1.70e-01	4.73e-10	26	26	25	1	20
	WORHP SQP	optimal	5	1.95e+00	9.56e-12	6	6	6	2	5
AONSDSIL	IPOPT	optimal	100	6.66e+00	1.95e-04	101	101	101	101	100
	KNITRO	optimal	141	1.00e+01	2.43e-08	144	145	143	144	141
	SNOPT	optimal	46	3.27e+01	5.76e+01	64	1	63	1	0
	WORHP IP	optimal	145	1.29e+01	2.39e-07	147	147	147	1	145
	WORHP IPm	optimal	128	8.25e+00	4.98e-09	135	135	134	1	128
	WORHP SQP	optimal	59	1.28e+02	1.49e-15	60	60	60	2	59
AONSDSSL	IPOPT	optimal	39	1.88e+00	-5.99e-03	43	43	40	40	39
	KNITRO	optimal	13	9.20e-01	9.15e-07	16	17	15	16	13
	SNOPT	optimal	21	6.11e+01	-5.61e-06	30	1	29	1	0
	WORHP IP	optimal	19	1.36e+00	4.70e-07	21	21	21	1	19
	WORHP IPm	optimal	15	1.41e+00	4.53e-09	21	21	20	1	15
	WORHP SQP	optimal	3	1.38e+01	6.00e-15	4	4	4	2	3
AONSSSSL	IPOPT	optimal	69	7.21e+00	-2.57e-03	278	278	60	71	69
	KNITRO	optimal	14	1.51e+00	8.26e-07	17	18	16	17	14
	SNOPT	optimal	24	7.52e+01	-1.25e-06	32	1	31	1	0
	WORHP IP	optimal	23	2.90e+00	1.03e-06	25	25	24	1	23
	WORHP IPm	optimal	19	2.04e+00	2.97e-09	25	25	24	1	19
	WORHP SQP	optimal	5	1.57e+01	2.73e-11	6	6	6	2	5

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
A2ENDNDL	IPOPT	optimal	15	8.60e-01	9.86e-04	16	16	16	16	15
	KNITRO	optimal	10	6.80e-01	7.58e-04	13	14	12	13	10
	SNOPT	optimal	9	2.81e+01	2.28e-05	13	1	12	1	0
	WORHP IP	optimal	12	9.00e-01	2.77e-04	14	14	13	1	12
	WORHP IPm	optimal	11	5.10e-01	6.79e-04	16	16	15	1	11
	WORHP SQP	optimal	3	1.88e+00	6.24e-06	4	4	5	3	3
A2ENINDL	IPOPT	optimal	15	5.10e-01	9.71e-04	16	16	16	16	15
	KNITRO	optimal	10	9.20e-01	8.06e-04	13	14	12	13	10
	SNOPT	optimal	10	1.76e+01	9.08e-06	14	1	13	1	0
	WORHP IP	optimal	12	4.90e-01	2.72e-04	14	14	13	1	12
	WORHP IPm	optimal	11	8.70e-01	6.66e-04	16	16	15	1	11
	WORHP SQP	optimal	3	2.35e+00	5.95e-06	4	4	5	3	3
A2ENSNDL	IPOPT	optimal	30	1.24e+01	1.31e-04	85	85	31	32	30
	KNITRO	optimal	11	9.20e-01	6.50e-04	14	15	13	14	11
	SNOPT	optimal	8	1.70e+01	2.23e-06	11	1	10	1	0
	WORHP IP	optimal	13	7.70e-01	6.00e-04	15	15	14	1	13
	WORHP IPm	optimal	12	7.40e-01	1.31e-04	19	19	18	1	12
	WORHP SQP	optimal	3	3.51e+00	2.47e-05	4	4	5	3	3
A2ESDNDL	IPOPT	optimal	15	6.00e-01	9.86e-04	16	16	16	16	15
	KNITRO	optimal	10	7.60e-01	7.92e-04	13	14	12	13	10
	SNOPT	optimal	30	6.72e+01	1.73e-05	33	1	32	1	0
	WORHP IP	optimal	12	8.50e-01	2.77e-04	14	14	13	1	12
	WORHP IPm	optimal	11	9.00e-01	6.79e-04	16	16	15	1	11
	WORHP SQP	optimal	3	2.56e+00	6.28e-06	4	4	5	3	3
A2ESINDL	IPOPT	optimal	15	8.90e-01	9.71e-04	16	16	16	16	15
	KNITRO	optimal	10	4.90e-01	8.03e-04	13	14	12	13	10
	SNOPT	optimal	10	2.98e+01	1.52e-05	14	1	13	1	0
	WORHP IP	optimal	12	8.70e-01	2.72e-04	14	14	13	1	12
	WORHP IPm	optimal	11	6.30e-01	6.66e-04	16	16	15	1	11
	WORHP SQP	optimal	3	2.69e+00	6.22e-06	4	4	5	3	3
A2ESSNDL	IPOPT	optimal	53	6.42e+00	6.86e-04	61	61	54	54	53
	KNITRO	optimal	13	9.40e-01	8.94e-05	16	17	15	16	13
	SNOPT	optimal	10	2.29e+01	5.55e-06	13	1	12	1	0
	WORHP IP	optimal	13	7.80e-01	6.00e-04	15	15	14	1	13
	WORHP IPm	optimal	12	8.20e-01	1.31e-04	19	19	18	1	12
	WORHP SQP	optimal	3	4.99e+00	1.45e-05	4	4	5	3	3
A2NNDNDL	IPOPT	optimal	29	2.22e+00	3.01e-04	30	30	30	30	29
	KNITRO	optimal	16	1.36e+00	1.46e-04	19	20	18	19	16
	SNOPT	optimal	266	1.10e+02	3.66e-04	290	1	289	1	0
	WORHP IP	optimal	18	1.13e+00	1.42e-04	20	20	19	1	18
	WORHP IPm	optimal	20	2.13e+00	5.32e-05	27	27	26	1	20
	WORHP SQP	optimal	42	1.05e+02	1.41e-08	43	43	44	3	42
A2NNDNIL	IPOPT	infeas	107	7.69e+00	1.91e+03	134	134	86	117	108
	KNITRO	smallstep	545	3.55e+02	8.57e+08	1404	1405	547	548	546
	SNOPT	infeas	0	2.19e+01	6.00e+04	1	1	1	1	0
	WORHP IP	infeas	135	1.28e+01	9.78e+00	137	137	136	1	136
	WORHP IPm	infeas	155	9.06e+00	9.52e+00	162	162	160	1	155
	WORHP SQP	minalpha	49	2.34e+02	2.04e+00	5836	5842	45	2	45
A2NNDNSL	IPOPT	optimal	45	3.15e+00	1.54e-04	53	53	46	46	45
	KNITRO	optimal	15	2.36e+00	3.03e-04	18	19	17	18	15
	SNOPT	optimal	97	9.45e+01	8.18e-13	135	1	134	1	0
	WORHP IP	optimal	31	2.50e+00	9.69e-04	58	58	33	1	31
	WORHP IPm	regular	674	4.30e+01	-2.37e+03	679	679	677	1	675
	WORHP SQP	maxtime	1604	1.73e+03	1.68e-05	95629	95779	1605	2	1605
A2NNSNSL	IPOPT	optimal	54	6.03e+00	-5.65e-03	92	92	49	56	54
	KNITRO	optimal	14	1.20e+00	1.03e-04	17	18	16	17	14
	SNOPT	optimal	15	4.12e+01	-5.14e-12	20	1	19	1	0
	WORHP IP	optimal	18	1.93e+00	8.58e-04	20	20	20	1	18
	WORHP IPm	optimal	29	3.05e+00	8.68e-09	36	36	35	1	29
	WORHP SQP	minalpha	406	7.00e+02	3.25e-05	61573	61805	379	2	379
A2NSDSDL	IPOPT	optimal	25	1.57e+00	7.74e-04	26	26	26	26	25
	KNITRO	optimal	13	9.50e-01	7.17e-04	16	17	15	16	13
	SNOPT	optimal	48	3.07e+02	2.37e-07	51	1	50	1	0
	WORHP IP	optimal	14	1.49e+00	1.44e-03	16	16	15	1	14
	WORHP IPm	optimal	14	9.80e-01	1.79e-04	20	20	19	1	14
	WORHP SQP	optimal	3	7.01e+00	2.14e-05	4	4	5	3	3
A2NSDSIL	IPOPT	maxtime	449	1.80e+03	1.91e+03	689	689	394	458	449
	KNITRO	optimal	203	2.10e+01	1.85e+00	207	208	205	206	203
	SNOPT	optimal	25	5.98e+01	3.19e+00	40	1	39	1	0
	WORHP IP	optimal	139	1.32e+01	1.15e+01	292	292	141	1	139
	WORHP IPm	fritzjohn	306	2.14e+01	2.70e+01	338	338	313	1	306
	WORHP SQP	minalpha	82	2.94e+02	5.61e-01	6267	6279	78	2	78
A2NSDSSL	IPOPT	optimal	51	3.81e+00	-1.11e-04	54	54	52	52	51
	KNITRO	optimal	16	1.95e+00	3.79e-05	19	20	18	19	16
	SNOPT	optimal	24	6.87e+01	-2.53e-06	34	1	33	1	0
	WORHP IP	optimal	25	2.17e+00	2.02e-04	27	27	27	1	25
	WORHP IPm	regular	352	2.44e+01	-8.32e+05	355	355	354	1	353
	WORHP SQP	optimal	4	1.47e+01	1.56e-09	445	445	5	2	4

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
A2NSSSSL	IPOPT	resto	31	5.38e+00	4.09e-04	55	55	33	33	32
	KNITRO	optimal	17	1.56e+00	2.05e-06	20	21	19	20	17
	SNOPT	optimal	23	4.34e+01	-5.87e-12	30	1	29	1	0
	WORHP IP	optimal	20	4.27e+00	2.72e-05	22	22	22	1	20
	WORHP IPm	optimal	27	4.48e+00	1.00e-08	33	33	32	1	27
	WORHP SQP	optimal	17	9.68e+01	3.78e-12	5333	5346	18	2	17
A4X12	IPOPT	optimal	316	8.40e-01	6.81e-01	377	754	285	636	316
	KNITRO	optimal	357	2.29e+00	6.82e-01	1851	1852	359	360	358
	SNOPT	maxiter	2441	4.09e+01	6.99e-01	1	5663	1	5662	0
	WORHP IP	optimal	493	2.82e+00	6.82e-01	3639	3639	495	495	493
	WORHP IPm	optimal	5203	2.38e+01	6.82e-01	24046	24046	5210	5210	5203
	WORHP SQP	minalpha	2590	6.10e+01	-2.58e-01	78252	78317	1764	2599	1763
A5ENDNDL	IPOPT	optimal	15	7.90e-01	2.19e-03	16	16	16	16	15
	KNITRO	optimal	10	5.00e-01	1.66e-03	13	14	12	13	10
	SNOPT	optimal	12	3.14e+01	4.22e-06	16	1	15	1	0
	WORHP IP	optimal	11	8.90e-01	1.97e-03	13	13	12	1	11
	WORHP IPm	optimal	11	8.70e-01	1.70e-03	16	16	15	1	11
	WORHP SQP	optimal	3	2.61e+00	1.52e-05	4	4	5	3	3
A5ENINDL	IPOPT	optimal	15	8.40e-01	2.25e-03	16	16	16	16	15
	KNITRO	optimal	10	7.00e-01	1.75e-03	13	14	12	13	10
	SNOPT	optimal	11	3.82e+01	2.58e-04	15	1	14	1	0
	WORHP IP	optimal	11	4.70e-01	2.00e-03	13	13	12	1	11
	WORHP IPm	optimal	11	5.00e-01	1.74e-03	16	16	15	1	11
	WORHP SQP	optimal	4	2.62e+00	1.56e-05	5	5	6	3	4
A5ENSNDL	IPOPT	optimal	23	1.23e+01	1.08e-03	24	24	24	24	23
	KNITRO	optimal	11	8.90e-01	1.15e-03	14	15	13	14	11
	SNOPT	optimal	8	1.52e+01	3.33e-07	12	1	11	1	0
	WORHP IP	optimal	15	9.70e-01	3.26e-04	30	30	17	1	15
	WORHP IPm	optimal	12	9.90e-01	3.44e-04	19	19	18	1	12
	WORHP SQP	optimal	3	6.08e+00	2.71e-05	4	4	5	3	3
A5ESDNDL	IPOPT	optimal	15	6.90e-01	2.19e-03	16	16	16	16	15
	KNITRO	optimal	10	6.30e-01	1.71e-03	13	14	12	13	10
	SNOPT	optimal	20	3.71e+01	1.55e-04	24	1	23	1	0
	WORHP IP	optimal	11	8.10e-01	1.97e-03	13	13	12	1	11
	WORHP IPm	optimal	11	8.70e-01	1.70e-03	16	16	15	1	11
	WORHP SQP	optimal	3	2.61e+00	1.52e-05	4	4	5	3	3
A5ESINDL	IPOPT	optimal	15	4.70e-01	2.25e-03	16	16	16	16	15
	KNITRO	optimal	10	8.80e-01	1.77e-03	13	14	12	13	10
	SNOPT	optimal	13	2.94e+01	2.24e-05	17	1	16	1	0
	WORHP IP	optimal	11	7.00e-01	2.00e-03	13	13	12	1	11
	WORHP IPm	optimal	11	9.00e-01	1.74e-03	16	16	15	1	11
	WORHP SQP	optimal	4	2.62e+00	1.56e-05	5	5	6	3	4
A5ESSNDL	IPOPT	optimal	48	4.74e+00	1.00e-03	59	59	49	49	48
	KNITRO	optimal	11	7.80e-01	1.15e-03	14	15	13	14	11
	SNOPT	optimal	10	2.08e+01	9.33e-07	13	1	12	1	0
	WORHP IP	optimal	15	9.50e-01	3.26e-04	29	29	17	1	15
	WORHP IPm	optimal	12	1.42e+00	3.44e-04	19	19	18	1	12
	WORHP SQP	optimal	3	6.18e+00	3.74e-05	4	4	5	3	3
A5NNDNDL	IPOPT	optimal	37	1.63e+00	1.86e-03	38	38	38	38	37
	KNITRO	optimal	19	1.61e+00	8.31e-06	22	23	21	22	19
	SNOPT	optimal	538	1.28e+02	2.84e+00	746	1	745	1	0
	WORHP IP	optimal	15	1.09e+00	1.68e-03	17	17	16	1	15
	WORHP IPm	optimal	26	2.74e+00	1.29e-04	32	32	31	1	26
	WORHP SQP	optimal	142	4.25e+02	3.37e-08	1104	1104	144	3	142
A5NNDNIL	IPOPT	infeas	84	6.23e+00	1.33e+03	112	112	57	88	85
	KNITRO	maxtime	6741	1.80e+03	7.39e+07	28052	28053	6743	6744	6741
	SNOPT	infeas	0	4.80e+01	6.00e+04	1	1	1	1	0
	WORHP IP	infeas	116	8.89e+00	9.62e+00	118	118	117	1	116
	WORHP IPm	infeas	134	9.76e+00	1.73e+01	140	140	139	1	134
	WORHP SQP	minalpha	58	2.48e+02	1.78e+00	4901	4907	54	2	54
A5NNDNSL	IPOPT	optimal	52	2.41e+00	-3.70e-03	68	68	53	53	52
	KNITRO	optimal	24	2.69e+00	4.21e-04	27	28	26	27	24
	SNOPT	optimal	52	1.07e+02	-9.23e-13	66	1	65	1	0
	WORHP IP	optimal	29	3.72e+00	4.42e-07	31	31	31	1	29
	WORHP IPm	optimal	30	3.62e+00	6.40e-09	37	37	36	1	30
	WORHP SQP	optimal	3	1.74e+01	5.85e-08	4	4	4	2	3
A5NNSNSL	IPOPT	optimal	49	1.28e+01	-4.46e-03	98	98	44	51	49
	KNITRO	optimal	15	1.90e+00	1.41e-04	18	19	17	18	15
	SNOPT	optimal	15	5.12e+01	-4.17e-12	21	1	20	1	0
	WORHP IP	optimal	21	6.53e+00	3.81e-07	23	23	23	1	21
	WORHP IPm	optimal	23	3.39e+00	7.48e-09	29	29	28	1	23
	WORHP SQP	optimal	4	3.07e+01	1.07e-06	5	5	5	2	4
A5NSDSDL	IPOPT	optimal	24	1.06e+00	1.86e-03	25	25	25	25	24
	KNITRO	optimal	13	9.40e-01	8.87e-04	16	17	15	16	13
	SNOPT	optimal	52	3.27e+02	1.21e-07	55	1	54	1	0
	WORHP IP	optimal	14	1.57e+00	1.96e-03	16	16	15	1	14
	WORHP IPm	optimal	14	1.50e+00	4.29e-04	20	20	19	1	14
	WORHP SQP	optimal	5	1.03e+01	4.77e-06	6	6	7	3	5

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
A5NSDSDM	IPOPT	optimal	29	3.00e-01	-1.53e-05	30	30	30	30	29
	KNITRO	optimal	14	2.20e-01	3.07e-07	17	18	16	17	14
	SNOPT	optimal	16	7.40e-01	0.00e+00	24	1	23	1	0
	WORHP IP	optimal	25	2.70e-01	8.11e-05	31	31	27	1	25
	WORHP IPm	optimal	20	2.20e-01	4.73e-10	26	26	25	1	20
	WORHP SQP	optimal	5	2.58e+00	9.56e-12	6	6	6	2	5
A5NSDSIL	IPOPT	optimal	369	8.87e+01	1.27e+00	875	875	365	377	369
	KNITRO	optimal	108	1.35e+01	6.52e-01	112	113	110	111	108
	SNOPT	optimal	22	8.00e+01	1.17e+01	36	1	35	1	0
	WORHP IP	optimal	130	1.32e+01	2.27e+00	669	669	136	1	130
	WORHP IPm	fritzjohn	178	1.20e+01	1.14e+01	208	208	185	1	178
	WORHP SQP	minalpha	73	2.70e+02	1.36e+00	4372	4377	74	2	74
A5NSDSSL	IPOPT	optimal	45	2.51e+00	-1.65e-02	54	54	46	46	45
	KNITRO	optimal	17	2.05e+00	1.87e-06	20	21	19	20	17
	SNOPT	optimal	27	1.08e+02	-3.19e-06	36	1	35	1	0
	WORHP IP	optimal	17	1.87e+00	7.14e-05	19	19	19	1	17
	WORHP IPm	optimal	21	2.79e+00	1.22e-06	28	28	27	1	21
	WORHP SQP	optimal	3	1.81e+01	5.63e-06	4	4	4	2	3
A5NSSNSM	IPOPT	optimal	29	3.10e-01	-1.53e-05	30	30	30	30	29
	KNITRO	optimal	14	1.90e-01	3.07e-07	17	18	16	17	14
	SNOPT	optimal	16	7.50e-01	0.00e+00	24	1	23	1	0
	WORHP IP	optimal	25	1.60e-01	8.11e-05	31	31	27	1	25
	WORHP IPm	optimal	20	2.20e-01	4.73e-10	26	26	25	1	20
	WORHP SQP	optimal	5	2.62e+00	9.56e-12	6	6	6	2	5
A5NSSSSL	IPOPT	optimal	33	1.07e+01	-3.77e-04	34	34	34	34	33
	KNITRO	optimal	17	3.02e+00	2.30e-05	20	21	19	20	17
	SNOPT	optimal	17	4.70e+01	-4.59e-12	24	1	23	1	0
	WORHP IP	optimal	30	1.50e+01	2.56e-05	33	33	32	1	30
	WORHP IPm	optimal	22	3.03e+00	2.06e-07	31	31	27	1	22
	WORHP SQP	optimal	32	8.80e+01	8.17e-08	4505	4520	33	2	32
ACOPP118	IPOPT	optimal	16	1.10e-01	1.30e+05	17	34	17	34	16
	KNITRO	optimal	12	8.00e-02	1.30e+05	15	16	14	15	12
	SNOPT	optimal	58	3.30e-01	1.30e+05	72	72	71	71	0
	WORHP IP	optimal	17	1.50e-01	1.30e+05	19	19	19	19	17
	WORHP IPm	optimal	16	1.50e-01	1.30e+05	22	22	21	21	16
	WORHP SQP	optimal	6	1.10e-01	1.30e+05	7	7	8	8	6
ACOPP14	IPOPT	optimal	11	2.00e-02	8.08e+03	12	24	12	24	11
	KNITRO	optimal	10	1.00e-02	8.08e+03	13	14	12	13	10
	SNOPT	optimal	16	1.00e-02	8.08e+03	22	22	21	21	0
	WORHP IP	optimal	12	1.00e-02	8.08e+03	14	14	14	14	12
	WORHP IPm	optimal	12	1.00e-02	8.08e+03	18	18	17	17	12
	WORHP SQP	optimal	3	1.00e-02	8.08e+03	4	4	5	5	3
ACOPP30	IPOPT	optimal	21	3.00e-02	5.77e+02	22	44	22	44	21
	KNITRO	optimal	11	2.00e-02	5.77e+02	13	14	12	13	11
	SNOPT	optimal	23	1.00e-02	5.77e+02	30	30	29	29	0
	WORHP IP	optimal	31	6.00e-02	5.77e+02	93	93	33	33	31
	WORHP IPm	optimal	12	2.00e-02	5.77e+02	16	16	15	15	12
	WORHP SQP	minalpha	1662	7.96e+00	6.38e+02	110921	110987	1596	1668	1595
ACOPP300	IPOPT	optimal	21	2.80e-01	7.20e+05	22	44	22	44	21
	KNITRO	optimal	15	2.10e-01	7.20e+05	18	19	17	18	15
	SNOPT	optimal	45	8.80e-01	7.20e+05	61	61	60	60	0
	WORHP IP	optimal	32	8.80e-01	7.20e+05	162	162	34	34	32
	WORHP IPm	optimal	36	8.20e-01	7.20e+05	62	62	42	42	36
	WORHP SQP	optimal	9	4.50e-01	7.20e+05	10	10	11	11	9
ACOPP57	IPOPT	optimal	13	4.00e-02	4.17e+04	15	30	14	28	13
	KNITRO	optimal	10	3.00e-02	4.17e+04	13	14	12	13	10
	SNOPT	optimal	19	3.00e-02	4.17e+04	32	32	31	31	0
	WORHP IP	optimal	12	4.00e-02	4.17e+04	14	14	14	14	12
	WORHP IPm	optimal	18	7.00e-02	4.17e+04	33	33	23	23	18
	WORHP SQP	optimal	4	5.00e-02	4.17e+04	5	5	6	6	4
ACOPR118	IPOPT	optimal	24	1.40e-01	1.30e+05	26	52	25	50	24
	KNITRO	optimal	21	1.30e-01	1.30e+05	36	37	23	24	21
	SNOPT	optimal	61	3.80e-01	1.30e+05	79	79	78	78	0
	WORHP IP	optimal	84	6.20e-01	1.30e+05	449	449	86	86	84
	WORHP IPm	optimal	67	4.60e-01	1.30e+05	224	224	73	73	67
	WORHP SQP	optimal	1734	1.53e+01	1.30e+05	12551	12551	1736	1736	1734
ACOPR14	IPOPT	optimal	16	1.00e-02	8.08e+03	17	34	17	34	16
	KNITRO	optimal	15	1.00e-02	8.08e+03	31	32	17	18	15
	SNOPT	optimal	16	1.00e-02	8.08e+03	22	22	21	21	0
	WORHP IP	optimal	950	6.10e-01	8.08e+03	10734	10734	957	957	950
	WORHP IPm	optimal	52	3.00e-02	8.08e+03	95	95	55	55	52
	WORHP SQP	optimal	4	1.00e-02	8.08e+03	5	5	6	6	4
ACOPR30	IPOPT	optimal	41	6.00e-02	5.77e+02	92	184	42	84	41
	KNITRO	optimal	20	2.00e-02	5.77e+02	28	29	21	22	20
	SNOPT	optimal	23	2.00e-02	5.77e+02	30	30	29	29	0
	WORHP IP	optimal	1281	3.32e+00	5.77e+02	18016	18016	1352	1352	1281
	WORHP IPm	optimal	28	4.00e-02	5.77e+02	48	48	29	29	28
	WORHP SQP	optimal	9	3.00e-02	5.77e+02	10	10	11	11	9

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ACOPR300	IPOPT	optimal	36	4.40e-01	7.20e+05	92	184	37	74	36
	KNITRO	optimal	23	3.80e-01	7.20e+05	39	40	25	26	23
	SNOPT	optimal	49	1.06e+00	7.20e+05	66	66	65	65	0
	WORHP IP	optimal	1187	2.21e+01	7.20e+05	12853	12853	1189	1189	1187
	WORHP IPm	optimal	177	2.24e+00	7.20e+05	452	452	183	183	177
	WORHP SQP	optimal	24	8.90e-01	7.20e+05	348	346	26	28	24
ACOPR57	IPOPT	optimal	23	7.00e-02	4.17e+04	27	54	24	48	23
	KNITRO	optimal	18	4.00e-02	4.17e+04	26	27	20	21	18
	SNOPT	optimal	21	4.00e-02	4.17e+04	38	38	37	37	0
	WORHP IP	optimal	219	4.90e-01	4.17e+04	374	374	229	229	219
	WORHP IPm	optimal	32	5.00e-02	4.17e+04	76	76	33	33	32
	WORHP SQP	optimal	6	5.00e-02	4.17e+04	7	7	8	8	6
AGG	IPOPT	optimal	190	3.00e-01	-3.60e+07	201	402	189	384	190
	KNITRO	optimal	74	9.00e-02	-3.60e+07	77	78	76	77	74
	SNOPT	optimal	0	1.00e-02	-3.60e+07	1	1	1	1	0
	WORHP IP	optimal	286	7.10e-01	-3.60e+07	288	288	287	1	286
	WORHP IPm	optimal	277	4.40e-01	-3.60e+07	284	284	283	1	277
	WORHP SQP	minalpha	7055	6.12e+01	1.88e+09	11215	11224	6993	2	6993
AIRCRFTA	IPOPT	optimal	3	1.00e-02	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.00e-02	0.00e+00	5	6	4	5	3
	SNOPT	optimal	3	1.00e-02	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	3	1.00e-02	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	1.00e-02	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	3	1.00e-02	0.00e+00	4	4	5	5	3
AIRCRFTB	IPOPT	optimal	15	1.00e-02	4.79e-25	26	0	16	0	15
	KNITRO	optimal	15	1.00e-02	4.24e-17	34	0	16	0	15
	SNOPT	optimal	49	1.00e-02	5.08e-17	61	0	60	0	0
	WORHP IP	optimal	12	1.00e-02	2.87e-22	25	0	13	0	12
	WORHP IPm	optimal	12	1.00e-02	2.87e-22	25	0	13	0	12
	WORHP SQP	optimal	17	1.00e-02	6.76e-14	26	0	18	0	17
AIRPORT	IPOPT	optimal	15	2.00e-02	4.80e+04	16	16	16	16	15
	KNITRO	optimal	12	2.00e-02	4.80e+04	14	15	13	14	12
	SNOPT	optimal	34	2.00e-02	4.80e+04	59	59	58	58	0
	WORHP IP	optimal	21	3.00e-02	4.80e+04	23	23	22	22	21
	WORHP IPm	optimal	25	4.00e-02	4.80e+04	27	27	26	26	25
	WORHP SQP	optimal	11	6.00e-02	4.80e+04	12	12	13	13	11
AKIVA	IPOPT	optimal	6	1.00e-02	6.17e+00	7	0	7	0	6
	KNITRO	optimal	6	1.00e-02	6.17e+00	8	0	7	0	6
	SNOPT	optimal	17	1.00e-02	6.17e+00	24	0	23	0	0
	WORHP IP	optimal	6	1.00e-02	6.17e+00	8	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	6.17e+00	8	0	7	0	6
	WORHP SQP	optimal	6	1.00e-02	6.17e+00	7	0	7	0	6
ALJAZZAF	IPOPT	optimal	40	7.00e-02	3.74e+04	128	128	41	41	40
	KNITRO	optimal	60	1.50e-01	3.74e+04	142	143	62	63	64
	SNOPT	optimal	111	3.55e+00	3.74e+04	298	298	297	297	0
	WORHP IP	optimal	37	9.00e-02	3.74e+04	94	94	39	39	37
	WORHP IPm	optimal	46	6.00e-02	3.74e+04	63	63	51	51	46
	WORHP SQP	optimal	27	2.00e-01	3.74e+04	150	150	28	28	27
ALLINIT	IPOPT	optimal	11	1.00e-02	1.67e+01	19	0	12	0	11
	KNITRO	optimal	9	1.00e-02	1.67e+01	12	0	11	0	9
	SNOPT	optimal	11	1.00e-02	1.67e+01	19	0	18	0	0
	WORHP IP	optimal	9	1.00e-02	1.67e+01	12	0	10	0	9
	WORHP IPm	optimal	7	1.00e-02	1.67e+01	9	0	8	0	7
	WORHP SQP	optimal	8	1.00e-02	1.67e+01	12	0	9	0	8
ALLINITA	IPOPT	optimal	23	1.00e-02	3.33e+01	25	50	24	48	23
	KNITRO	optimal	28	1.00e-02	3.33e+01	32	33	30	31	28
	SNOPT	noimpr	39	1.00e-02	3.33e+01	78	78	77	77	0
	WORHP IP	optimal	22	1.00e-02	3.33e+01	97	97	24	24	22
	WORHP IPm	optimal	45	1.00e-02	3.33e+01	445	445	52	52	45
	WORHP SQP	optimal	649	1.10e-01	3.33e+01	1210	1252	650	650	649
ALLINITC	IPOPT	optimal	23	1.00e-02	3.05e+01	33	33	24	24	23
	KNITRO	optimal	11	1.00e-02	3.05e+01	14	15	13	14	11
	SNOPT	noimpr	52	1.00e-02	3.05e+01	106	106	105	105	0
	WORHP IP	optimal	23	1.00e-02	3.05e+01	136	136	25	25	23
	WORHP IPm	optimal	45	1.00e-02	3.05e+01	418	418	55	55	45
	WORHP SQP	optimal	539	1.10e-01	3.05e+01	1300	1413	540	540	539
ALLINITU	IPOPT	optimal	13	1.00e-02	5.74e+00	14	0	14	0	13
	KNITRO	optimal	7	1.00e-02	5.74e+00	10	0	8	0	7
	SNOPT	optimal	10	1.00e-02	5.74e+00	15	0	14	0	0
	WORHP IP	optimal	13	1.00e-02	5.74e+00	15	0	14	0	13
	WORHP IPm	optimal	13	1.00e-02	5.74e+00	15	0	14	0	13
	WORHP SQP	optimal	9	1.00e-02	5.74e+00	17	0	10	0	9
ALLINQP	IPOPT	optimal	19	1.32e+00	-5.48e+03	20	40	20	40	19
	KNITRO	optimal	15	2.09e+00	-5.48e+03	18	19	17	18	15
	SNOPT	sbasics	10000	9.22e+02	-5.48e+03	11235	1	11234	1	0
	WORHP IP	optimal	22	2.59e+00	-5.48e+03	24	24	23	1	22
	WORHP IPm	optimal	23	2.73e+00	-5.48e+03	26	26	25	1	23
	WORHP SQP	optimal	9	4.61e+00	-5.48e+03	10	10	11	3	9

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ALSOTAME	IPOPT	optimal	8	1.00e-02	8.21e-02	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	8.21e-02	8	9	7	8	6
	SNOPT	optimal	4	1.00e-02	8.21e-02	7	7	6	6	0
	WORHP IP	optimal	7	1.00e-02	8.21e-02	9	9	8	8	7
	WORHP IPm	optimal	6	1.00e-02	8.21e-02	8	8	7	7	6
	WORHP SQP	optimal	4	1.00e-02	8.21e-02	5	5	6	6	4
ANTWERP	IPOPT	optimal	150	6.00e-02	3.25e+03	221	442	151	302	150
	KNITRO	optimal	81	1.00e-02	3.25e+03	103	104	83	84	81
	SNOPT	maxiter	10000	2.73e+00	3.59e+03	24527	1	24526	1	0
	WORHP IP	fritzjohn	107	2.00e-02	3.25e+03	168	168	109	1	108
	WORHP IPm	fritzjohn	69	1.00e-02	3.54e+03	87	87	70	1	70
	WORHP SQP	optimal	2745	8.50e-01	3.25e+03	2758	2758	2747	3	2745
ARGAUSS	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	17	1.00e-02	0.00e+00	19	20	18	19	15
	SNOPT	infeas	3	1.00e-02	0.00e+00	1	11	1	10	0
	WORHP IP	infeas	2	1.00e-02	0.00e+00	4	4	3	3	3
	WORHP IPm	infeas	2	1.00e-02	0.00e+00	4	4	3	3	3
	WORHP SQP	minalpha	34	3.00e-02	0.00e+00	3062	3068	30	36	29
ARGLALE	IPOPT	degree	0	6.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	24	1.64e+00	0.00e+00	83	84	24	25	22
	SNOPT	infeas	0	1.50e-01	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	2	2.30e-01	0.00e+00	4	4	3	1	3
	WORHP IPm	infeas	2	2.80e-01	0.00e+00	4	4	3	1	3
	WORHP SQP	minalpha	5	8.60e+00	0.00e+00	2448	2454	7	3	6
ARGLBLE	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	47	3.49e+00	0.00e+00	210	211	48	49	46
	SNOPT	infeas	0	7.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	2	4.00e-01	0.00e+00	4	4	3	1	3
	WORHP IPm	infeas	2	5.60e-01	0.00e+00	4	4	3	1	3
	WORHP SQP	minalpha	12	9.72e+00	0.00e+00	2497	2503	11	3	10
ARGLCLE	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	63	7.40e+00	-1.00e+00	249	250	63	64	62
	SNOPT	infeas	0	9.00e-02	1.00e+00	1	1	1	1	0
	WORHP IP	infeas	2	5.50e-01	-1.00e+00	4	4	3	1	3
	WORHP IPm	infeas	2	4.60e-01	-1.00e+00	4	4	3	1	3
	WORHP SQP	minalpha	8	7.58e+00	-1.00e+00	2451	2457	8	3	7
ARGLINA	IPOPT	optimal	1	7.20e-01	2.00e+02	2	0	2	0	1
	KNITRO	optimal	1	4.60e-01	2.00e+02	3	0	2	0	1
	SNOPT	optimal	3	7.00e-02	2.00e+02	7	0	6	0	0
	WORHP IP	optimal	1	6.20e-01	2.00e+02	3	0	2	0	1
	WORHP IPm	optimal	1	5.40e-01	2.00e+02	3	0	2	0	1
	WORHP SQP	optimal	3	1.15e+00	2.00e+02	4	0	4	0	3
ARGLINB	IPOPT	optimal	3418	5.42e+02	9.96e+01	94423	0	3419	0	3418
	KNITRO	noimpr	3	9.00e-01	9.96e+01	9	0	4	0	3
	SNOPT	toobig	4	6.00e-02	9.96e+01	7	0	6	0	0
	WORHP IP	optimal	33	7.17e+00	9.96e+01	772	0	106	0	33
	WORHP IPm	minalpha	429	5.91e+01	9.96e+01	10669	0	582	0	430
	WORHP SQP	maxtime	9449	1.80e+03	9.96e+01	616772	0	9450	0	9450
ARGLINC	IPOPT	optimal	985	1.29e+02	1.01e+02	26304	0	986	0	985
	KNITRO	noimpr	3	8.70e-01	1.01e+02	11	0	4	0	4
	SNOPT	toobig	4	6.00e-02	1.01e+02	7	0	6	0	0
	WORHP IP	accept	193	2.60e+01	1.01e+02	4541	0	311	0	194
	WORHP IPm	optimal	76	1.56e+01	1.01e+02	1676	0	94	0	76
	WORHP SQP	maxiter	10000	1.57e+03	1.01e+02	1170246	0	10001	0	10001
ARGTRIG	IPOPT	optimal	3	9.00e-02	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.40e-01	0.00e+00	5	6	4	5	3
	SNOPT	optimal	3	9.00e-02	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	3	8.00e-02	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	1.40e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	3	5.40e-01	0.00e+00	4	4	5	5	3
ARGTRIGLS	IPOPT	optimal	8	1.00e-02	7.05e-25	14	0	9	0	8
	KNITRO	optimal	5	1.00e-02	8.00e-17	10	0	6	0	5
	SNOPT	optimal	24	1.00e-02	6.82e-14	29	0	28	0	0
	WORHP IP	optimal	8	1.00e-02	7.06e-25	12	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	7.06e-25	12	0	9	0	8
	WORHP SQP	optimal	6	1.00e-02	7.50e-20	12	0	7	0	6
ARTIF	IPOPT	infeas	122	1.18e+00	0.00e+00	273	273	24	125	123
	KNITRO	optimal	20	4.10e-01	0.00e+00	69	70	22	23	20
	SNOPT	optimal	6	1.80e-01	0.00e+00	1	12	1	11	0
	WORHP IP	infeas	22	5.00e-01	0.00e+00	73	73	23	23	23
	WORHP IPm	infeas	22	5.20e-01	0.00e+00	73	73	23	23	23
	WORHP SQP	optimal	8	2.18e+00	0.00e+00	9	9	10	10	8
ARWHDNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	41	1.20e-01	0.00e+00	125	126	42	43	39
	SNOPT	infeas	147	2.10e-01	0.00e+00	1	550	1	549	0
	WORHP IP	infeas	10	2.00e-02	0.00e+00	53	53	11	11	11
	WORHP IPm	infeas	10	2.00e-02	0.00e+00	53	53	11	11	11
	WORHP SQP	minalpha	46	7.30e-01	0.00e+00	2831	2837	48	48	47

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ARWHEAD	IPOPT	optimal	6	1.50e-01	0.00e+00	7	0	7	0	6
	KNITRO	optimal	6	2.20e-01	0.00e+00	8	0	7	0	6
	SNOPT	optimal	3	1.30e-01	0.00e+00	7	0	6	0	0
	WORHP IP	optimal	6	1.90e-01	0.00e+00	8	0	8	0	6
	WORHP IPm	optimal	6	1.60e-01	0.00e+00	8	0	7	0	6
	WORHP SQP	optimal	6	1.70e-01	0.00e+00	7	0	7	0	6
AUG2D	IPOPT	optimal	2	2.90e-01	1.69e+06	3	3	3	3	2
	KNITRO	optimal	2	5.40e-01	1.69e+06	4	5	3	4	2
	SNOPT	sbasics	21	1.46e+01	2.39e+06	24	1	23	1	0
	WORHP IP	optimal	9	6.60e-01	1.69e+06	11	11	10	1	9
	WORHP IPm	optimal	9	8.20e-01	1.69e+06	11	11	10	1	9
	WORHP SQP	optimal	10	1.13e+00	1.69e+06	11	11	12	3	10
AUG2DC	IPOPT	optimal	1	2.50e-01	1.82e+06	2	2	2	2	1
	KNITRO	optimal	1	1.60e-01	1.82e+06	3	4	2	3	1
	SNOPT	sbasics	21	1.75e+01	2.53e+06	24	1	23	1	0
	WORHP IP	optimal	8	5.70e-01	1.82e+06	10	10	9	1	8
	WORHP IPm	optimal	8	5.60e-01	1.82e+06	10	10	9	1	8
	WORHP SQP	optimal	7	8.80e-01	1.82e+06	8	8	9	3	7
AUG2DCQP	IPOPT	optimal	26	1.16e+00	6.50e+06	27	27	27	27	26
	KNITRO	optimal	20	9.30e-01	6.50e+06	23	24	22	23	20
	SNOPT	sbasics	17	2.85e+01	7.32e+06	26	1	25	1	0
	WORHP IP	optimal	22	1.29e+00	6.50e+06	24	24	24	1	22
	WORHP IPm	optimal	22	1.55e+00	6.50e+06	28	28	27	1	22
	WORHP SQP	optimal	22	1.31e+01	6.50e+06	23	23	23	2	22
AUG2DQP	IPOPT	optimal	26	1.23e+00	6.24e+06	27	27	27	27	26
	KNITRO	optimal	20	9.00e-01	6.24e+06	23	24	22	23	20
	SNOPT	sbasics	15	3.56e+01	7.36e+06	24	1	23	1	0
	WORHP IP	optimal	21	1.26e+00	6.24e+06	23	23	23	1	21
	WORHP IPm	optimal	24	1.45e+00	6.24e+06	30	30	29	1	24
	WORHP SQP	optimal	18	1.01e+01	6.24e+06	19	19	19	2	18
AUG3D	IPOPT	optimal	2	8.10e-01	2.46e+04	3	3	3	3	2
	KNITRO	optimal	1	1.75e+00	2.46e+04	3	4	2	3	1
	SNOPT	sbasics	21	1.90e+01	6.11e+04	24	1	23	1	0
	WORHP IP	optimal	4	2.56e+00	2.46e+04	6	6	5	1	4
	WORHP IPm	optimal	4	3.36e+00	2.46e+04	6	6	5	1	4
	WORHP SQP	optimal	6	5.60e+00	2.46e+04	7	7	8	3	6
AUG3DC	IPOPT	optimal	1	7.00e-01	2.77e+04	2	2	2	2	1
	KNITRO	optimal	1	7.40e-01	2.77e+04	3	4	2	3	1
	SNOPT	sbasics	21	1.45e+01	7.74e+04	24	1	23	1	0
	WORHP IP	optimal	4	1.36e+00	2.77e+04	6	6	5	1	4
	WORHP IPm	optimal	4	1.64e+00	2.77e+04	6	6	5	1	4
	WORHP SQP	optimal	5	5.11e+00	2.77e+04	6	6	7	3	5
AUG3DCQP	IPOPT	optimal	20	3.17e+00	6.16e+04	21	21	21	21	20
	KNITRO	optimal	14	4.48e+00	6.16e+04	17	18	16	17	14
	SNOPT	sbasics	17	2.49e+01	9.52e+04	21	1	20	1	0
	WORHP IP	optimal	16	3.82e+00	6.16e+04	18	18	18	1	16
	WORHP IPm	optimal	16	3.73e+00	6.16e+04	22	22	21	1	16
	WORHP SQP	optimal	10	6.31e+01	6.16e+04	11	11	11	2	10
AUG3DQP	IPOPT	optimal	20	4.23e+00	5.42e+04	21	21	21	21	20
	KNITRO	optimal	14	3.04e+00	5.42e+04	17	18	16	17	14
	SNOPT	sbasics	16	2.46e+01	7.60e+04	20	1	19	1	0
	WORHP IP	optimal	15	5.40e+00	5.42e+04	17	17	17	1	15
	WORHP IPm	optimal	14	5.13e+00	5.42e+04	20	20	19	1	14
	WORHP SQP	optimal	11	6.33e+01	5.42e+04	12	12	12	2	11
AVGASA	IPOPT	optimal	10	1.00e-02	-4.63e+00	11	11	11	11	10
	KNITRO	optimal	6	1.00e-02	-4.63e+00	8	9	7	8	6
	SNOPT	optimal	9	1.00e-02	-4.63e+00	12	1	11	1	0
	WORHP IP	optimal	7	1.00e-02	-4.63e+00	9	9	8	1	7
	WORHP IPm	optimal	6	1.00e-02	-4.63e+00	10	10	9	1	6
	WORHP SQP	optimal	2	1.00e-02	-4.63e+00	2	2	3	3	2
AVGASB	IPOPT	optimal	11	1.00e-02	-4.48e+00	12	12	12	12	11
	KNITRO	optimal	7	1.00e-02	-4.48e+00	9	10	8	9	7
	SNOPT	optimal	7	1.00e-02	-4.48e+00	10	1	9	1	0
	WORHP IP	optimal	9	1.00e-02	-4.48e+00	11	11	10	1	9
	WORHP IPm	optimal	7	1.00e-02	-4.48e+00	12	12	11	1	7
	WORHP SQP	optimal	2	1.00e-02	-4.48e+00	2	2	3	3	2
AVION2	IPOPT	optimal	205	1.00e-01	9.47e+07	242	242	206	206	205
	KNITRO	optimal	31	2.00e-02	9.47e+07	34	35	33	34	31
	SNOPT	noimpr	13	1.00e-02	9.47e+07	29	1	28	1	0
	WORHP IP	optimal	447	6.00e-02	9.47e+07	504	504	449	1	447
	WORHP IPm	maxiter	10000	1.22e+00	9.47e+07	10110	10110	10006	1	10000
	WORHP SQP	optimal	4	1.00e-02	9.47e+07	5	5	6	3	4
BA-L1	IPOPT	optimal	5	1.00e-02	0.00e+00	6	6	6	6	5
	KNITRO	optimal	4	1.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	optimal	22	1.00e-02	0.00e+00	1	62	1	61	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	7	7	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	7	7	6	6	5
	WORHP SQP	optimal	8	1.00e-02	0.00e+00	45	45	10	10	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BA-L16	IPOPT	degree	0	2.13e+02	0.00e+00	0	0	0	0	0
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	killed	-	-	-	-	-	-	-	-
	WORHP IP	infeas	81	5.72e+02	0.00e+00	132	132	82	82	81
	WORHP IPm	infeas	81	5.25e+02	0.00e+00	132	132	82	82	81
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L16LS	IPOPT	optimal	163	8.48e+02	8.49e+05	668	0	164	0	163
	KNITRO	noimpr	103	5.27e+02	8.49e+05	153	0	104	0	103
	SNOPT	killed	-	-	-	-	-	-	-	-
	WORHP IP	optimal	132	8.95e+02	8.49e+05	863	0	136	0	132
	WORHP IPm	optimal	123	6.91e+02	8.49e+05	738	0	153	0	123
	WORHP SQP	optimal	111	5.08e+02	8.49e+05	220	0	112	0	111
BA-L1LS	IPOPT	optimal	10	1.00e-02	7.65e-21	11	0	11	0	10
	KNITRO	optimal	11	1.00e-02	1.22e-23	22	0	12	0	11
	SNOPT	optimal	24	1.00e-02	1.77e-18	40	0	39	0	0
	WORHP IP	optimal	10	1.00e-02	2.28e-25	12	0	12	0	10
	WORHP IPm	optimal	10	1.00e-02	7.65e-21	12	0	11	0	10
	WORHP SQP	optimal	29	1.00e-02	9.19e-19	43	0	30	0	29
BA-L1SP	IPOPT	optimal	5	2.00e-02	0.00e+00	6	6	6	6	5
	KNITRO	optimal	5	1.00e-02	0.00e+00	7	8	6	7	5
	SNOPT	optimal	5	1.00e-02	0.00e+00	1	9	1	8	0
	WORHP IP	optimal	5	2.00e-02	0.00e+00	7	7	6	6	5
	WORHP IPm	optimal	5	2.00e-02	0.00e+00	7	7	6	6	5
	WORHP SQP	optimal	8	8.00e-02	0.00e+00	49	49	10	10	8
BA-L1SPLS	IPOPT	optimal	10	2.00e-02	2.17e-23	11	0	11	0	10
	KNITRO	optimal	10	3.00e-02	3.86e-21	21	0	11	0	10
	SNOPT	optimal	28	2.00e-02	7.69e-19	34	0	33	0	0
	WORHP IP	optimal	10	3.00e-02	6.80e-27	12	0	12	0	10
	WORHP IPm	optimal	10	3.00e-02	2.14e-23	12	0	11	0	10
	WORHP SQP	optimal	20	5.00e-02	5.61e-19	21	0	21	0	20
BA-L21	IPOPT	degree	0	1.84e+01	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	589	1.82e+03	0.00e+00	607	608	590	591	590
	SNOPT	noimpr	3	4.21e+02	0.00e+00	1	19	1	18	0
	WORHP IP	infeas	286	4.71e+02	0.00e+00	500	500	287	287	287
	WORHP IPm	infeas	286	4.20e+02	0.00e+00	500	500	287	287	287
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L21LS	IPOPT	smallstep	400	7.59e+02	2.21e+06	1942	0	401	0	401
	KNITRO	noimpr	1222	1.48e+03	3.95e+05	1391	0	1223	0	1222
	SNOPT	toobig	3588	1.53e+03	1.77e+07	4029	0	4028	0	0
	WORHP IP	minalpha	176	3.57e+02	2.14e+06	583	0	238	0	177
	WORHP IPm	minalpha	176	3.58e+02	2.14e+06	583	0	238	0	177
	WORHP SQP	maxtime	1154	1.75e+03	3.99e+05	2871	0	1155	0	1155
BA-L49	IPOPT	degree	0	5.98e+00	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	686	1.81e+03	0.00e+00	797	798	687	688	686
	SNOPT	maxtime	23	1.81e+03	0.00e+00	1	125	1	124	0
	WORHP IP	maxtime	744	1.65e+03	0.00e+00	2757	2757	745	745	745
	WORHP IPm	maxtime	813	1.65e+03	0.00e+00	3784	3784	821	821	814
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L49LS	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	maxtime	2165	1.81e+03	1.93e+06	2334	0	2166	0	2165
	SNOPT	maxtime	4730	1.81e+03	1.66e+07	5320	0	5319	0	0
	WORHP IP	maxtime	1494	1.59e+03	2.97e+05	1796	0	1496	0	1495
	WORHP IPm	maxtime	1110	1.63e+03	2.97e+05	1351	0	1111	0	1111
	WORHP SQP	maxtime	1296	1.71e+03	1.44e+07	2193	0	1297	0	1297
BA-L52	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	killed	-	-	-	-	-	-	-	-
	WORHP IP	killed	-	-	-	-	-	-	-	-
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L52LS	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	killed	-	-	-	-	-	-	-	-
	WORHP IP	killed	-	-	-	-	-	-	-	-
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L73	IPOPT	degree	0	9.01e+00	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	485	1.80e+03	0.00e+00	1097	1098	486	487	485
	SNOPT	maxtime	15	1.83e+03	0.00e+00	1	92	1	91	0
	WORHP IP	maxtime	506	1.66e+03	0.00e+00	2625	2625	507	507	507
	WORHP IPm	maxtime	584	1.65e+03	0.00e+00	3115	3115	585	585	585
	WORHP SQP	killed	-	-	-	-	-	-	-	-
BA-L73LS	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	maxtime	1240	1.81e+03	1.53e+06	1875	0	1241	0	1240
	SNOPT	maxtime	3624	1.81e+03	5.44e+07	4008	0	4007	0	0
	WORHP IP	maxtime	774	1.62e+03	9.48e+07	825	0	775	0	775
	WORHP IPm	maxtime	1027	1.59e+03	9.47e+07	1077	0	1028	0	1028
	WORHP SQP	maxtime	1088	1.69e+03	1.63e+08	2028	0	1089	0	1089

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BARD	IPOPT	optimal	7	1.00e-02	8.21e-03	8	0	8	0	7
	KNITRO	optimal	7	1.00e-02	8.21e-03	9	0	8	0	7
	SNOPT	optimal	20	1.00e-02	8.21e-03	24	0	23	0	0
	WORHP IP	optimal	7	1.00e-02	8.21e-03	9	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	8.21e-03	9	0	8	0	7
	WORHP SQP	optimal	9	1.00e-02	8.21e-03	10	0	10	0	9
BARDNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	21	1.00e-02	0.00e+00	25	26	22	23	20
	SNOPT	maxiter	10000	5.00e-01	0.00e+00	1	20003	1	20002	0
	WORHP IP	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP SQP	minalpha	32	3.00e-02	0.00e+00	3817	3823	32	34	31
BATCH	IPOPT	optimal	54	2.00e-02	2.59e+05	55	110	55	110	54
	KNITRO	optimal	14	1.00e-02	2.59e+05	17	18	16	17	14
	SNOPT	optimal	23	1.00e-02	2.59e+05	39	39	38	38	0
	WORHP IP	optimal	48	1.00e-02	2.59e+05	90	90	50	50	48
	WORHP IPm	optimal	68	2.00e-02	2.59e+05	181	181	74	74	68
	WORHP SQP	optimal	8	2.00e-02	2.59e+05	9	9	10	10	8
BDEXP	IPOPT	optimal	13	1.80e-01	3.04e-04	14	0	14	0	13
	KNITRO	optimal	12	1.80e-01	2.11e-04	14	0	13	0	12
	SNOPT	toobig	48	6.96e+00	8.00e+02	60	0	59	0	0
	WORHP IP	optimal	13	2.10e-01	1.25e-04	15	0	14	0	13
	WORHP IPm	optimal	13	1.90e-01	1.37e-04	15	0	14	0	13
	WORHP SQP	optimal	13	3.00e-01	1.57e-04	14	0	14	0	13
BDQRTIC	IPOPT	optimal	10	2.60e-01	2.00e+04	11	0	11	0	10
	KNITRO	optimal	10	2.20e-01	2.00e+04	12	0	11	0	10
	SNOPT	toobig	43	4.60e+01	3.09e+05	46	0	45	0	0
	WORHP IP	optimal	12	3.30e-01	2.00e+04	76	0	14	0	12
	WORHP IPm	optimal	12	3.10e-01	2.00e+04	76	0	13	0	12
	WORHP SQP	optimal	9	3.10e-01	2.00e+04	10	0	10	0	9
BDRY2	IPOPT	maxtime	36	1.82e+03	1.48e-01	37	37	37	37	36
	KNITRO	optimal	27	1.81e+02	1.44e-01	30	31	29	30	27
	SNOPT	memory	0	1.03e+03	6.35e+04	1	1	1	1	0
	WORHP IP	maxtime	114	1.84e+03	1.98e-01	206	206	116	1	115
	WORHP IPm	maxtime	184	1.85e+03	1.63e-02	187	187	185	1	185
	WORHP SQP	maxtime	6	1.85e+03	2.04e-01	7	7	8	3	7
BDVALUE	IPOPT	optimal	0	6.00e-02	0.00e+00	1	1	1	1	0
	KNITRO	optimal	0	7.00e-02	0.00e+00	2	3	1	2	0
	SNOPT	optimal	0	6.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	optimal	0	6.00e-02	0.00e+00	2	2	1	1	0
	WORHP IPm	optimal	0	7.00e-02	0.00e+00	2	2	1	1	0
	WORHP SQP	optimal	0	6.00e-02	0.00e+00	1	1	1	1	0
BDVALUES	IPOPT	optimal	10	2.50e-01	0.00e+00	11	11	11	11	10
	KNITRO	optimal	10	2.70e-01	0.00e+00	12	13	11	12	10
	SNOPT	optimal	12	9.80e-01	0.00e+00	1	15	1	14	0
	WORHP IP	optimal	10	2.70e-01	0.00e+00	12	12	11	11	10
	WORHP IPm	optimal	10	3.10e-01	0.00e+00	12	12	11	11	10
	WORHP SQP	optimal	14	1.06e+00	0.00e+00	15	15	16	16	14
BEALE	IPOPT	optimal	8	1.00e-02	4.34e-18	19	0	9	0	8
	KNITRO	optimal	7	1.00e-02	1.31e-16	9	0	8	0	7
	SNOPT	optimal	13	1.00e-02	8.97e-18	16	0	15	0	0
	WORHP IP	optimal	8	1.00e-02	4.34e-18	14	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	4.34e-18	14	0	9	0	8
	WORHP SQP	optimal	14	1.00e-02	4.88e-15	15	0	15	0	14
BENNETT5	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	33	4.00e-02	0.00e+00	74	75	33	34	32
	SNOPT	infeas	10	1.00e-02	0.00e+00	1	19	1	18	0
	WORHP IP	infeas	8	1.00e-02	0.00e+00	15	15	9	9	9
	WORHP IPm	infeas	8	1.00e-02	0.00e+00	15	15	9	9	9
	WORHP SQP	minalpha	14	3.80e-01	0.00e+00	2816	2822	8	16	7
BENNETT5LS	IPOPT	optimal	655	4.70e-01	5.24e-04	1750	0	656	0	655
	KNITRO	optimal	12	1.00e-02	5.39e-04	21	0	13	0	12
	SNOPT	optimal	29	1.00e-02	5.39e-04	40	0	39	0	0
	WORHP IP	optimal	654	2.50e-01	5.24e-04	1158	0	656	0	654
	WORHP IPm	optimal	22	1.00e-02	5.56e-04	30	0	23	0	22
	WORHP SQP	optimal	25	1.00e-02	5.39e-04	72	0	26	0	25
BIGBANK	IPOPT	optimal	24	1.00e-01	-4.21e+06	25	25	25	25	24
	KNITRO	optimal	23	2.20e-01	-4.21e+06	27	28	25	26	23
	SNOPT	optimal	730	7.26e+01	-4.21e+06	758	1	757	1	0
	WORHP IP	optimal	21	2.00e-01	-4.21e+06	23	23	22	1	21
	WORHP IPm	optimal	19	1.90e-01	-4.21e+06	21	21	20	1	19
	WORHP SQP	optimal	21	1.44e+00	-4.21e+06	22	22	23	3	21
BIGGS3	IPOPT	optimal	8	1.00e-02	9.99e-14	27	0	9	0	8
	KNITRO	optimal	9	1.00e-02	4.08e-17	16	0	10	0	9
	SNOPT	optimal	21	1.00e-02	7.84e-14	26	0	25	0	0
	WORHP IP	optimal	8	1.00e-02	9.99e-14	19	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	9.99e-14	19	0	9	0	8
	WORHP SQP	optimal	11	1.00e-02	6.33e-11	22	0	12	0	11

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BIGGS5	IPOPT	optimal	20	1.00e-02	4.02e-17	36	0	21	0	20
	KNITRO	optimal	23	1.00e-02	5.43e-15	36	0	24	0	23
	SNOPT	optimal	89	1.00e-02	5.66e-03	114	0	113	0	0
	WORHP IP	optimal	20	1.00e-02	4.02e-17	28	0	21	0	20
	WORHP IPm	optimal	20	1.00e-02	4.02e-17	28	0	21	0	20
	WORHP SQP	optimal	21	1.00e-02	8.21e-12	37	0	22	0	21
BIGGS6	IPOPT	optimal	75	2.00e-02	1.81e-13	117	0	76	0	75
	KNITRO	optimal	68	1.00e-02	2.66e-15	84	0	69	0	68
	SNOPT	optimal	41	1.00e-02	5.66e-03	48	0	47	0	0
	WORHP IP	optimal	78	1.00e-02	4.25e-14	105	0	79	0	78
	WORHP IPm	optimal	78	1.00e-02	4.25e-14	105	0	79	0	78
	WORHP SQP	optimal	58	1.00e-02	2.71e-12	99	0	59	0	58
BIGGS6NE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	156	2.00e-02	0.00e+00	265	266	157	158	156
	SNOPT	infeas	224	2.00e-02	0.00e+00	1	1055	1	1054	0
	WORHP IP	optimal	53	1.00e-02	0.00e+00	108	108	54	54	53
	WORHP IPm	optimal	53	1.00e-02	0.00e+00	108	108	54	54	53
	WORHP SQP	optimal	23	1.00e-02	0.00e+00	539	540	25	25	23
BIGGSB1	IPOPT	optimal	13	9.00e-02	1.54e-02	14	0	14	0	13
	KNITRO	optimal	8	1.30e-01	1.51e-02	11	0	10	0	8
	SNOPT	sbasics	10000	5.32e+02	1.74e-02	11057	0	11056	0	0
	WORHP IP	optimal	11	1.20e-01	1.51e-02	13	0	12	0	11
	WORHP IPm	optimal	16	1.70e-01	1.52e-02	20	0	17	0	16
	WORHP SQP	optimal	9	1.50e-01	1.58e-02	10	0	10	0	9
BIGGSC4	IPOPT	optimal	23	1.00e-02	-2.45e+01	31	31	24	24	23
	KNITRO	optimal	11	1.00e-02	-2.45e+01	14	15	13	14	11
	SNOPT	optimal	5	1.00e-02	-2.44e+01	10	1	9	1	0
	WORHP IP	optimal	17	1.00e-02	-2.45e+01	19	19	18	1	17
	WORHP IPm	optimal	18	1.00e-02	-2.45e+01	20	20	19	1	18
	WORHP SQP	optimal	7	1.00e-02	-2.45e+01	8	8	9	3	7
BLEACHNG	IPOPT	optimal	16	9.67e+01	9.18e+03	78	0	17	0	16
	KNITRO	optimal	6	3.50e+01	9.18e+03	8	0	7	0	6
	SNOPT	optimal	7	1.56e+01	9.18e+03	14	0	13	0	0
	WORHP IP	optimal	13	5.93e+01	9.18e+03	153	0	15	0	13
	WORHP IPm	optimal	27	1.85e+02	9.18e+03	109	0	33	0	27
	WORHP SQP	optimal	6	1.03e+02	9.18e+03	275	0	7	0	6
BLOCKQP1	IPOPT	optimal	73	2.62e+00	-4.99e+03	77	154	74	148	73
	KNITRO	optimal	64	4.04e+00	-4.99e+03	66	67	65	66	64
	SNOPT	optimal	4	3.40e+00	-4.90e+03	11	1	10	1	0
	WORHP IP	optimal	64	3.46e+00	-4.99e+03	66	66	65	1	64
	WORHP IPm	optimal	72	3.22e+00	-4.99e+03	76	76	75	1	72
	WORHP SQP	optimal	3	6.70e-01	5.00e+00	4	4	5	3	3
BLOCKQP2	IPOPT	optimal	18	6.90e-01	-4.99e+03	19	38	19	38	18
	KNITRO	optimal	17	1.48e+00	-4.99e+03	19	20	18	19	17
	SNOPT	optimal	2	3.03e+00	-4.99e+03	7	1	6	1	0
	WORHP IP	optimal	20	8.80e-01	-4.99e+03	22	22	21	1	20
	WORHP IPm	optimal	23	1.14e+00	-4.99e+03	27	27	26	1	23
	WORHP SQP	optimal	3	8.70e-01	5.00e+00	4	4	5	3	3
BLOCKQP3	IPOPT	maxiter	10000	2.58e+02	-1.79e+03	10002	20004	10001	20002	10000
	KNITRO	optimal	1592	8.40e+01	-2.49e+03	1594	1595	1593	1594	1592
	SNOPT	optimal	23	3.02e+00	-2.49e+03	38	1	37	1	0
	WORHP IP	maxiter	10000	2.85e+02	-1.41e+03	10002	10002	10001	1	10000
	WORHP IPm	maxiter	10000	4.14e+02	-1.07e+03	10003	10003	10002	1	10000
	WORHP SQP	optimal	3	6.40e-01	5.00e+00	4	4	5	3	3
BLOCKQP4	IPOPT	optimal	31	8.80e-01	-2.50e+03	32	64	32	64	31
	KNITRO	optimal	14	9.40e-01	-2.50e+03	16	17	15	16	14
	SNOPT	optimal	3	3.02e+00	-2.50e+03	7	1	6	1	0
	WORHP IP	optimal	48	1.35e+00	-2.50e+03	50	50	49	1	48
	WORHP IPm	optimal	33	1.51e+00	-2.50e+03	37	37	36	1	33
	WORHP SQP	optimal	36	5.47e+00	-2.50e+03	36	36	37	3	36
BLOCKQP5	IPOPT	maxiter	10000	2.43e+02	-1.57e+03	10006	20012	10001	20002	10000
	KNITRO	optimal	1694	6.53e+01	-2.49e+03	1696	1697	1695	1696	1694
	SNOPT	optimal	3	2.86e+00	-2.47e+03	9	1	8	1	0
	WORHP IP	maxiter	10000	3.39e+02	-1.33e+03	10002	10002	10001	1	10000
	WORHP IPm	maxiter	10000	3.59e+02	-1.16e+03	10004	10004	10003	1	10000
	WORHP SQP	optimal	3	8.60e-01	5.00e+00	4	4	5	3	3
BLOWEYA	IPOPT	optimal	7	1.10e-01	-2.27e-02	8	8	8	8	7
	KNITRO	optimal	8	1.40e-01	-2.28e-02	11	12	10	11	8
	SNOPT	optimal	0	4.00e-02	-2.00e-05	3	1	2	1	0
	WORHP IP	optimal	6	3.05e+01	-6.64e-03	8	8	7	1	6
	WORHP IPm	optimal	6	3.63e+01	-5.36e-03	8	8	7	1	6
	WORHP SQP	optimal	2	2.00e-01	-4.95e-05	3	3	4	3	2
BLOWEYB	IPOPT	optimal	6	1.20e-01	-1.52e-02	7	7	7	7	6
	KNITRO	optimal	8	1.40e-01	-1.52e-02	11	12	10	11	8
	SNOPT	optimal	0	4.00e-02	3.09e-16	3	1	2	1	0
	WORHP IP	optimal	6	2.26e+01	-8.91e-03	8	8	8	1	6
	WORHP IPm	optimal	5	3.27e+01	-2.81e-03	7	7	6	1	5
	WORHP SQP	optimal	2	1.80e-01	-2.67e-05	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BLOWEYC	IPOPT	optimal	8	1.00e-01	-1.52e-02	9	9	9	9	8
	KNITRO	optimal	7	1.60e-01	-1.52e-02	10	11	9	10	7
	SNOPT	optimal	0	3.00e-02	-8.01e-05	3	1	2	1	0
	WORHP IP	optimal	6	2.67e+01	-4.74e-03	8	8	7	1	6
	WORHP IPm	optimal	6	3.73e+01	-2.85e-03	8	8	7	1	6
	WORHP SQP	optimal	12	3.40e-01	-1.55e-03	13	13	14	3	12
BOOTH	IPOPT	optimal	1	1.00e-02	0.00e+00	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	0.00e+00	3	4	2	3	1
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP SQP	optimal	1	1.00e-02	0.00e+00	2	2	3	3	1
BOX	IPOPT	optimal	4	3.80e-01	-1.86e+03	16	0	5	0	4
	KNITRO	optimal	3	4.50e-01	-1.86e+03	10	0	4	0	3
	SNOPT	toobig	39	7.70e+00	-5.17e+01	95	0	94	0	0
	WORHP IP	optimal	4	3.10e-01	-1.86e+03	14	0	5	0	4
	WORHP IPm	optimal	4	3.20e-01	-1.86e+03	14	0	5	0	4
	WORHP SQP	optimal	5	4.50e-01	-1.86e+03	46	0	6	0	5
BOX2	IPOPT	optimal	7	1.00e-02	7.34e-15	8	0	8	0	7
	KNITRO	optimal	9	1.00e-02	4.57e-15	13	0	10	0	9
	SNOPT	optimal	12	1.00e-02	7.74e-12	16	0	15	0	0
	WORHP IP	optimal	7	1.00e-02	7.34e-15	9	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	7.34e-15	9	0	8	0	7
	WORHP SQP	optimal	9	1.00e-02	2.00e-13	10	0	10	0	9
BOX3	IPOPT	optimal	8	1.00e-02	2.87e-14	14	0	9	0	8
	KNITRO	optimal	8	1.00e-02	2.87e-14	11	0	9	0	8
	SNOPT	optimal	25	1.00e-02	5.23e-13	28	0	27	0	0
	WORHP IP	optimal	8	1.00e-02	2.87e-14	12	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	2.87e-14	12	0	9	0	8
	WORHP SQP	optimal	8	1.00e-02	8.65e-12	9	0	9	0	8
BOX3NE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	5	1.00e-02	0.00e+00	7	8	6	7	5
	SNOPT	optimal	9	1.00e-02	0.00e+00	1	16	1	15	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	7	7	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	7	7	6	6	5
	WORHP SQP	optimal	6	1.00e-02	0.00e+00	118	117	3	8	2
BOXBOD	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	26	1.00e-02	0.00e+00	34	35	26	27	24
	SNOPT	toobig	6	1.00e-02	0.00e+00	1	35	1	34	0
	WORHP IP	infeas	15	1.00e-02	0.00e+00	37	37	16	16	16
	WORHP IPm	infeas	15	1.00e-02	0.00e+00	37	37	16	16	16
	WORHP SQP	infeas	1490	1.70e-01	0.00e+00	3821	3825	1488	1491	1487
BOXBODLS	IPOPT	optimal	12	1.00e-02	9.77e+03	18	0	13	0	12
	KNITRO	optimal	14	1.00e-02	1.17e+03	24	0	15	0	14
	SNOPT	optimal	16	1.00e-02	1.17e+03	33	0	32	0	0
	WORHP IP	optimal	12	1.00e-02	9.77e+03	16	0	13	0	12
	WORHP IPm	optimal	12	1.00e-02	9.77e+03	16	0	13	0	12
	WORHP SQP	optimal	33	1.00e-02	1.17e+03	35	0	34	0	33
BOXPOWER	IPOPT	optimal	17	6.40e-01	8.50e-09	18	0	18	0	17
	KNITRO	optimal	17	9.10e-01	8.49e-09	19	0	18	0	17
	SNOPT	toobig	103	1.24e+01	4.72e-02	117	0	116	0	0
	WORHP IP	optimal	17	6.20e-01	8.50e-09	19	0	19	0	17
	WORHP IPm	optimal	17	5.50e-01	8.49e-09	19	0	18	0	17
	WORHP SQP	optimal	39	1.44e+00	4.29e-08	40	0	40	0	39
BQP1VAR	IPOPT	optimal	5	1.00e-02	8.10e-08	6	0	6	0	5
	KNITRO	optimal	4	1.00e-02	9.63e-11	6	0	5	0	4
	SNOPT	optimal	1	1.00e-02	0.00e+00	4	0	3	0	0
	WORHP IP	optimal	5	1.00e-02	1.91e-10	7	0	6	0	5
	WORHP IPm	optimal	4	1.00e-02	4.79e-08	6	0	5	0	4
	WORHP SQP	optimal	1	1.00e-02	2.40e-10	2	0	2	0	1
BQPGABIM	IPOPT	optimal	12	1.00e-02	-3.53e-05	18	0	13	0	12
	KNITRO	optimal	9	1.00e-02	-3.78e-05	12	0	11	0	9
	SNOPT	optimal	19	1.00e-02	-3.79e-05	24	0	23	0	0
	WORHP IP	optimal	5	1.00e-02	-3.68e-05	7	0	6	0	5
	WORHP IPm	optimal	6	1.00e-02	-3.78e-05	8	0	7	0	6
	WORHP SQP	optimal	3	1.00e-02	-3.79e-05	4	0	4	0	3
BQPGASIM	IPOPT	optimal	12	1.00e-02	-5.25e-05	18	0	13	0	12
	KNITRO	optimal	9	1.00e-02	-5.50e-05	12	0	11	0	9
	SNOPT	optimal	21	1.00e-02	-5.52e-05	26	0	25	0	0
	WORHP IP	optimal	5	1.00e-02	-5.41e-05	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	-5.49e-05	7	0	6	0	5
	WORHP SQP	optimal	23	2.00e-02	-5.52e-05	1519	0	24	0	23
BQPGAUSS	IPOPT	optimal	21	1.40e-01	-3.63e-01	22	0	22	0	21
	KNITRO	optimal	14	1.20e-01	-3.63e-01	17	0	16	0	14
	SNOPT	optimal	7121	1.95e+02	-3.63e-01	7966	0	7965	0	0
	WORHP IP	optimal	16	1.80e-01	-3.63e-01	18	0	17	0	16
	WORHP IPm	optimal	16	2.20e-01	-3.63e-01	20	0	19	0	16
	WORHP SQP	optimal	16	8.10e-01	-3.63e-01	17	0	17	0	16

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BRAINPC0	IPOPT	resto	1122	2.25e+01	8.56e+03	1270	8012	1117	1125	1123
	KNITRO	optimal	22	1.67e+00	1.50e-03	32	33	23	24	22
	SNOPT	optimal	37	1.60e+00	1.50e-03	48	48	47	47	0
	WORHP IP	optimal	926	3.86e+01	3.40e-01	9061	9061	930	930	926
	WORHP IPm	optimal	371	1.94e+01	2.61e-03	2576	2576	374	374	371
	WORHP SQP	optimal	15	3.24e+00	1.50e-03	16	16	17	17	15
BRAINPC1	IPOPT	optimal	73	2.13e+00	4.17e-04	98	164	74	74	73
	KNITRO	optimal	54	4.17e+00	4.07e-04	118	119	55	56	55
	SNOPT	optimal	12	6.80e-01	8.33e-08	15	15	14	14	0
	WORHP IP	optimal	668	5.32e+01	4.33e-04	13015	13015	670	670	668
	WORHP IPm	optimal	160	7.85e+00	4.37e-04	1011	1011	162	162	160
	WORHP SQP	optimal	42	4.81e+00	6.39e-08	43	43	44	44	42
BRAINPC2	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	27	3.84e+00	4.11e-08	40	41	28	29	27
	SNOPT	optimal	51	1.29e+01	4.11e-08	58	58	57	57	0
	WORHP IP	fritzjohn	3744	4.71e+02	4.07e-01	50270	50270	3750	3750	3745
	WORHP IPm	optimal	4140	3.82e+02	4.12e-01	39796	39796	4146	4146	4140
	WORHP SQP	optimal	17	1.00e+01	7.06e-04	18	18	19	19	17
BRAINPC3	IPOPT	resto	228	8.86e+00	3.92e+05	317	1187	205	232	229
	KNITRO	optimal	27	1.76e+00	1.69e-04	43	44	28	29	27
	SNOPT	optimal	96	2.08e+00	1.69e-04	110	110	109	109	0
	WORHP IP	optimal	2971	1.49e+02	3.66e-01	36947	36947	2973	2973	2971
	WORHP IPm	optimal	215	9.69e+00	3.65e-01	1426	1426	218	218	215
	WORHP SQP	optimal	103	9.65e+00	1.69e-04	226	225	104	106	102
BRAINPC4	IPOPT	resto	202	7.39e+00	6.52e+03	424	994	200	206	203
	KNITRO	optimal	34	2.90e+00	1.29e-03	68	69	35	36	34
	SNOPT	optimal	53	1.70e+00	1.29e-03	58	58	57	57	0
	WORHP IP	fritzjohn	5232	3.21e+02	3.51e-01	70813	70813	5246	5246	5233
	WORHP IPm	optimal	886	5.58e+01	3.52e-01	15197	15197	916	916	886
	WORHP SQP	optimal	16	3.56e+00	1.29e-03	17	17	18	18	16
BRAINPC5	IPOPT	resto	198	6.13e+00	1.63e+06	847	1219	174	202	199
	KNITRO	optimal	31	1.41e+00	1.36e-03	41	42	32	33	31
	SNOPT	optimal	100	3.45e+00	1.36e-03	170	170	169	169	0
	WORHP IP	maxiter	10000	5.99e+02	2.34e-03	125432	125432	10015	10015	10000
	WORHP IPm	infeas	3539	2.21e+02	3.34e-01	62089	62089	3540	3540	3540
	WORHP SQP	optimal	21	4.32e+00	1.36e-03	22	22	23	23	21
BRAINPC6	IPOPT	resto	427	1.63e+03	7.57e+03	732	2678	406	432	428
	KNITRO	optimal	77	3.20e+00	5.93e-05	101	102	78	79	77
	SNOPT	noimpr	81	6.84e+00	5.93e-05	140	140	139	139	0
	WORHP IP	accept	930	7.52e+01	3.78e-01	13277	13277	947	947	931
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	optimal	19	3.25e+00	5.93e-05	20	20	21	21	19
BRAINPC7	IPOPT	resto	224	8.96e+00	8.22e+03	327	521	147	228	225
	KNITRO	optimal	23	1.22e+00	3.84e-05	34	35	24	25	23
	SNOPT	optimal	61	5.11e+00	3.82e-05	83	83	82	82	0
	WORHP IP	accept	5803	3.66e+02	3.94e-01	75476	75476	5820	5820	5804
	WORHP IPm	maxiter	10000	4.91e+02	7.63e-04	148479	148479	10002	10002	10000
	WORHP SQP	optimal	19	3.24e+00	3.82e-05	20	20	21	21	19
BRAINPC8	IPOPT	resto	191	8.70e+00	7.68e+03	246	867	188	194	192
	KNITRO	optimal	25	1.60e+00	1.66e-04	36	37	26	27	25
	SNOPT	noimpr	70	3.05e+00	1.65e-04	137	137	136	136	0
	WORHP IP	killed	-	-	-	-	-	-	-	-
	WORHP IPm	optimal	200	1.08e+01	3.55e-01	1314	1314	203	203	200
	WORHP SQP	optimal	17	4.47e+00	1.65e-04	18	18	19	19	17
BRAINPC9	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	25	1.20e+00	8.25e-04	48	49	26	27	25
	SNOPT	optimal	78	1.73e+00	8.23e-04	90	90	89	89	0
	WORHP IP	fritzjohn	1440	8.14e+01	4.24e-01	16106	16106	1455	1455	1441
	WORHP IPm	maxiter	10000	7.32e+02	3.50e-01	148200	148200	10001	10001	10000
	WORHP SQP	optimal	23	3.70e+00	8.23e-04	136	136	25	25	23
BRATU1D	IPOPT	smallstep	58	1.53e+00	-6.82e+00	821	0	59	0	59
	KNITRO	unbound	2	1.40e-01	-5.28e+162	6	0	3	0	2
	SNOPT	sbasics	10000	2.63e+02	1.15e+06	11234	0	11233	0	0
	WORHP IP	accept	24	8.90e-01	-7.94e+00	378	0	30	0	25
	WORHP IPm	minalpha	40	1.52e+00	-8.52e+00	659	0	91	0	41
	WORHP SQP	unbound	65	1.10e+01	-2.68e+20	6653	0	66	0	65
BRATU2D	IPOPT	optimal	2	5.70e-01	0.00e+00	3	3	3	3	2
	KNITRO	optimal	2	8.30e-01	0.00e+00	12	13	3	4	2
	SNOPT	optimal	2	1.08e+00	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	2	1.60e-01	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.60e-01	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	3	2.36e+00	0.00e+00	4	4	5	5	3
BRATU2DT	IPOPT	optimal	6	1.02e+00	0.00e+00	7	7	7	7	6
	KNITRO	optimal	6	6.00e-01	0.00e+00	12	13	7	8	6
	SNOPT	optimal	12	1.67e+00	0.00e+00	1	34	1	33	0
	WORHP IP	optimal	6	3.90e-01	0.00e+00	8	8	7	7	6
	WORHP IPm	optimal	6	3.40e-01	0.00e+00	8	8	7	7	6
	WORHP SQP	optimal	7	2.60e+00	0.00e+00	8	8	9	9	7

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BRATU3D	IPOPT	optimal	3	1.23e+00	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	5.61e+00	0.00e+00	13	14	4	5	3
	SNOPT	optimal	3	6.10e+00	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	3	9.70e-01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	9.80e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	3	8.82e+00	0.00e+00	4	4	5	5	3
BRIDGEND	IPOPT	optimal	52	4.40e-01	5.38e+01	53	53	53	53	52
	KNITRO	optimal	41	3.80e-01	5.38e+01	44	45	43	44	41
	SNOPT	optimal	11	1.27e+00	6.02e+01	1	16	1	15	0
	WORHP IP	optimal	43	4.40e-01	5.38e+01	53	53	44	44	43
	WORHP IPm	optimal	72	7.00e-01	5.38e+01	79	79	76	76	72
	WORHP SQP	optimal	41	4.96e+00	5.38e+01	205	205	43	43	41
BRITGAS	IPOPT	maxiter	10000	2.05e+01	2.48e+00	40564	147549	10001	10001	10000
	KNITRO	optimal	7	3.00e-02	2.05e-08	10	11	9	10	7
	SNOPT	optimal	8	6.00e-02	0.00e+00	17	17	16	16	0
	WORHP IP	regular	1112	6.50e+00	6.79e-05	17643	17643	1141	1141	1113
	WORHP IPm	regular	2374	1.22e+01	-6.72e-02	37801	37801	2479	2479	2375
	WORHP SQP	optimal	13	3.20e-01	1.01e-10	17	17	15	15	13
BRKMCC	IPOPT	optimal	3	1.00e-02	1.69e-01	4	0	4	0	3
	KNITRO	optimal	3	1.00e-02	1.69e-01	5	0	4	0	3
	SNOPT	optimal	5	1.00e-02	1.69e-01	11	0	10	0	0
	WORHP IP	optimal	3	1.00e-02	1.69e-01	5	0	4	0	3
	WORHP IPm	optimal	3	1.00e-02	1.69e-01	5	0	4	0	3
	WORHP SQP	optimal	3	1.00e-02	1.69e-01	4	0	4	0	3
BROWNAL	IPOPT	optimal	5	4.80e-01	1.15e-21	6	0	6	0	5
	KNITRO	optimal	8	1.38e+00	2.08e-18	10	0	9	0	8
	SNOPT	optimal	104	8.00e-02	5.67e-12	138	0	137	0	0
	WORHP IP	optimal	5	6.80e-01	1.10e-21	7	0	7	0	5
	WORHP IPm	optimal	5	8.50e-01	1.10e-21	7	0	6	0	5
	WORHP SQP	optimal	17	2.21e+00	9.90e-10	18	0	18	0	17
BROWNALE	IPOPT	optimal	7	2.00e-01	0.00e+00	10	15	8	8	7
	KNITRO	optimal	5	1.60e-01	0.00e+00	11	12	6	7	5
	SNOPT	optimal	9	1.10e-01	0.00e+00	1	22	1	21	0
	WORHP IP	optimal	7	3.60e-01	0.00e+00	16	16	8	8	7
	WORHP IPm	optimal	7	3.80e-01	0.00e+00	16	16	8	8	7
	WORHP SQP	optimal	9	1.16e+01	0.00e+00	142	141	4	12	2
BROWNBS	IPOPT	optimal	7	1.00e-02	0.00e+00	8	0	8	0	7
	KNITRO	optimal	5	1.00e-02	0.00e+00	7	0	6	0	5
	SNOPT	optimal	26	1.00e-02	1.60e-25	33	0	32	0	0
	WORHP IP	optimal	7	1.00e-02	0.00e+00	9	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	0.00e+00	9	0	8	0	7
	WORHP SQP	optimal	39	1.00e-02	1.36e-20	40	0	40	0	39
BROWNDEN	IPOPT	optimal	8	1.00e-02	8.58e+04	9	0	9	0	8
	KNITRO	optimal	8	1.00e-02	8.58e+04	10	0	9	0	8
	SNOPT	optimal	38	1.00e-02	8.58e+04	41	0	40	0	0
	WORHP IP	optimal	8	1.00e-02	8.58e+04	10	0	10	0	8
	WORHP IPm	optimal	8	1.00e-02	8.58e+04	10	0	9	0	8
	WORHP SQP	optimal	8	1.00e-02	8.58e+04	9	0	9	0	8
BROWNDENE	IPOPT	optimal	1	1.00e-02	9.03e+02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	9.03e+02	3	0	2	0	1
	SNOPT	optimal	22	1.00e-02	9.03e+02	25	0	24	0	0
	WORHP IP	optimal	1	1.00e-02	9.03e+02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	9.03e+02	3	0	2	0	1
	WORHP SQP	optimal	4	1.00e-02	9.03e+02	5	0	5	0	4
BROYDN3D	IPOPT	optimal	4	8.00e-02	0.00e+00	5	5	5	5	4
	KNITRO	optimal	4	9.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	optimal	4	7.00e-02	0.00e+00	1	7	1	6	0
	WORHP IP	optimal	4	8.00e-02	0.00e+00	6	6	5	5	4
	WORHP IPm	optimal	4	8.00e-02	0.00e+00	6	6	5	5	4
	WORHP SQP	optimal	4	2.20e-01	0.00e+00	5	5	6	6	4
BROYDN3DLS	IPOPT	optimal	5	1.00e-02	3.11e-17	6	0	6	0	5
	KNITRO	optimal	5	1.00e-02	3.11e-17	7	0	6	0	5
	SNOPT	optimal	22	1.00e-02	2.96e-14	26	0	25	0	0
	WORHP IP	optimal	5	1.00e-02	3.11e-17	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	3.11e-17	7	0	6	0	5
	WORHP SQP	optimal	5	1.00e-02	5.01e-17	6	0	6	0	5
BROYDN7D	IPOPT	optimal	273	4.20e+00	1.69e+03	360	0	274	0	273
	KNITRO	optimal	120	1.36e+00	1.50e+03	142	0	121	0	120
	SNOPT	toobig	272	4.58e+01	5.79e+03	308	0	307	0	0
	WORHP IP	optimal	296	3.44e+00	1.87e+03	325	0	297	0	296
	WORHP IPm	optimal	296	3.52e+00	1.87e+03	325	0	297	0	296
	WORHP SQP	optimal	34	6.60e-01	2.40e+03	117	0	35	0	34
BROYDNBD	IPOPT	optimal	17	5.00e-01	0.00e+00	18	18	18	18	17
	KNITRO	optimal	11	3.70e-01	0.00e+00	13	14	12	13	11
	SNOPT	optimal	8	2.50e-01	0.00e+00	1	15	1	14	0
	WORHP IP	optimal	6	1.90e-01	0.00e+00	8	8	7	7	6
	WORHP IPm	optimal	6	2.00e-01	0.00e+00	8	8	7	7	6
	WORHP SQP	optimal	7	1.10e+00	0.00e+00	119	118	3	9	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BROYDNBDLS	IPOPT	optimal	11	1.00e-02	7.96e-18	17	0	12	0	11
	KNITRO	optimal	11	1.00e-02	4.08e-17	16	0	12	0	11
	SNOPT	optimal	40	1.00e-02	8.58e-15	45	0	44	0	0
	WORHP IP	optimal	11	1.00e-02	7.96e-18	15	0	12	0	11
	WORHP IPm	optimal	11	1.00e-02	7.96e-18	15	0	12	0	11
	WORHP SQP	optimal	41	1.00e-02	8.11e-14	60	0	42	0	41
BRYBND	IPOPT	optimal	11	2.40e-01	4.30e-21	17	0	12	0	11
	KNITRO	optimal	13	1.60e-01	3.83e-24	28	0	14	0	13
	SNOPT	toobig	431	4.72e+01	7.50e+04	476	0	475	0	0
	WORHP IP	optimal	11	2.30e-01	4.28e-21	15	0	12	0	11
	WORHP IPm	optimal	11	2.50e-01	4.28e-21	15	0	12	0	11
	WORHP SQP	optimal	9	2.90e-01	6.03e-14	10	0	10	0	9
BT1	IPOPT	optimal	6	1.00e-02	-1.00e+00	10	11	7	7	6
	KNITRO	optimal	6	1.00e-02	-1.00e+00	12	13	7	8	6
	SNOPT	optimal	14	1.00e-02	-1.00e+00	67	67	66	66	0
	WORHP IP	optimal	131	1.00e-02	-1.00e+00	1033	1033	132	132	131
	WORHP IPm	optimal	131	1.00e-02	-1.00e+00	1033	1033	132	132	131
	WORHP SQP	optimal	40	1.00e-02	-1.00e+00	1050	1050	42	42	40
BT10	IPOPT	optimal	6	1.00e-02	-1.00e+00	7	7	7	7	6
	KNITRO	optimal	6	1.00e-02	-1.00e+00	8	9	7	8	6
	SNOPT	optimal	13	1.00e-02	-1.00e+00	1	25	1	24	0
	WORHP IP	optimal	6	1.00e-02	-1.00e+00	8	8	7	7	6
	WORHP IPm	optimal	6	1.00e-02	-1.00e+00	8	8	7	7	6
	WORHP SQP	optimal	9	1.00e-02	-1.00e+00	10	10	11	11	9
BT11	IPOPT	optimal	7	1.00e-02	8.25e-01	8	8	8	8	7
	KNITRO	optimal	7	1.00e-02	8.25e-01	9	10	8	9	7
	SNOPT	optimal	14	1.00e-02	8.25e-01	20	20	19	19	0
	WORHP IP	optimal	7	1.00e-02	8.25e-01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	8.25e-01	9	9	8	8	7
	WORHP SQP	optimal	7	1.00e-02	8.25e-01	8	8	9	9	7
BT12	IPOPT	optimal	4	1.00e-02	6.19e+00	5	5	5	5	4
	KNITRO	optimal	4	1.00e-02	6.19e+00	6	7	5	6	4
	SNOPT	optimal	8	1.00e-02	6.19e+00	11	11	10	10	0
	WORHP IP	optimal	4	1.00e-02	6.19e+00	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	6.19e+00	6	6	5	5	4
	WORHP SQP	optimal	6	1.00e-02	6.19e+00	7	7	8	8	6
BT13	IPOPT	optimal	22	1.00e-02	8.09e-08	23	23	23	23	22
	KNITRO	optimal	22	1.00e-02	1.32e-12	26	27	23	24	22
	SNOPT	optimal	28	1.00e-02	0.00e+00	1	35	1	34	0
	WORHP IP	optimal	23	1.00e-02	1.03e-10	25	25	25	25	23
	WORHP IPm	optimal	23	1.00e-02	1.00e-12	26	26	25	25	23
	WORHP SQP	optimal	21	1.00e-02	9.79e-31	25	25	23	23	21
BT2	IPOPT	optimal	12	1.00e-02	3.26e-02	13	13	13	13	12
	KNITRO	optimal	12	1.00e-02	3.26e-02	14	15	13	14	12
	SNOPT	optimal	15	1.00e-02	3.26e-02	18	18	17	17	0
	WORHP IP	optimal	12	1.00e-02	3.26e-02	14	14	13	13	12
	WORHP IPm	optimal	12	1.00e-02	3.26e-02	14	14	13	13	12
	WORHP SQP	optimal	12	1.00e-02	3.26e-02	13	13	14	14	12
BT3	IPOPT	optimal	1	1.00e-02	4.09e+00	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	4.09e+00	3	4	2	3	1
	SNOPT	optimal	6	1.00e-02	4.09e+00	9	1	8	1	0
	WORHP IP	optimal	3	1.00e-02	4.09e+00	5	5	4	1	3
	WORHP IPm	optimal	3	1.00e-02	4.09e+00	5	5	4	1	3
	WORHP SQP	optimal	3	1.00e-02	4.09e+00	4	4	5	3	3
BT4	IPOPT	optimal	9	1.00e-02	-3.70e+00	10	12	10	10	9
	KNITRO	optimal	11	1.00e-02	-4.55e+01	16	17	12	13	11
	SNOPT	optimal	7	1.00e-02	-4.55e+01	11	11	10	10	0
	WORHP IP	optimal	11	1.00e-02	-4.55e+01	13	13	12	12	11
	WORHP IPm	optimal	11	1.00e-02	-4.55e+01	13	13	12	12	11
	WORHP SQP	optimal	10	1.00e-02	-4.55e+01	567	566	12	13	10
BT5	IPOPT	optimal	6	1.00e-02	9.62e+02	7	7	7	7	6
	KNITRO	optimal	6	1.00e-02	9.62e+02	8	9	7	8	6
	SNOPT	optimal	7	1.00e-02	9.62e+02	12	12	11	11	0
	WORHP IP	optimal	7	1.00e-02	9.62e+02	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	9.62e+02	9	9	8	8	7
	WORHP SQP	optimal	13	1.00e-02	9.62e+02	14	14	15	15	13
BT6	IPOPT	optimal	13	1.00e-02	2.77e-01	18	18	14	14	13
	KNITRO	optimal	9	1.00e-02	2.77e-01	12	13	10	11	9
	SNOPT	optimal	11	1.00e-02	2.77e-01	15	15	14	14	0
	WORHP IP	optimal	9	1.00e-02	2.77e-01	13	13	10	10	9
	WORHP IPm	optimal	9	1.00e-02	2.77e-01	13	13	10	10	9
	WORHP SQP	optimal	10	1.00e-02	2.77e-01	15	15	12	12	10
BT7	IPOPT	optimal	15	1.00e-02	3.06e+02	29	29	16	16	15
	KNITRO	optimal	19	1.00e-02	3.06e+02	29	30	20	21	19
	SNOPT	optimal	23	1.00e-02	3.60e+02	37	37	36	36	0
	WORHP IP	optimal	39	1.00e-02	3.06e+02	103	103	40	40	39
	WORHP IPm	optimal	39	1.00e-02	3.06e+02	103	103	40	40	39
	WORHP SQP	optimal	23	1.00e-02	3.60e+02	159	159	25	25	23

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
BT8	IPOPT	optimal	79	3.00e-02	1.00e+00	80	80	80	80	79
	KNITRO	optimal	10	1.00e-02	1.00e+00	12	13	11	12	10
	SNOPT	optimal	12	1.00e-02	1.00e+00	15	15	14	14	0
	WORHP IP	maxiter	10000	5.80e-01	1.00e+00	10545	10545	10002	10002	10000
	WORHP IPm	maxiter	10000	5.50e-01	1.00e+00	10545	10545	10001	10001	10000
	WORHP SQP	optimal	22	1.00e-02	1.00e+00	23	23	24	24	22
BT9	IPOPT	optimal	13	1.00e-02	-1.00e+00	14	14	14	14	13
	KNITRO	optimal	13	1.00e-02	-1.00e+00	15	16	14	15	13
	SNOPT	optimal	17	1.00e-02	-1.00e+00	1	30	1	29	0
	WORHP IP	optimal	22	1.00e-02	-1.00e+00	38	38	23	23	22
	WORHP IPm	optimal	22	1.00e-02	-1.00e+00	38	38	23	23	22
	WORHP SQP	optimal	12	1.00e-02	-1.00e+00	13	13	14	14	12
BTS4	IPOPT	optimal	32	2.85e+01	1.61e+04	34	34	33	33	32
	KNITRO	optimal	15	3.61e+01	1.61e+04	18	19	17	18	15
	SNOPT	sbasics	193	1.23e+02	2.08e+04	203	1	202	1	0
	WORHP IP	optimal	16	1.76e+01	1.61e+04	18	18	17	1	16
	WORHP IPm	optimal	15	2.40e+01	1.61e+04	19	19	18	1	15
	WORHP SQP	maxtime	799	1.75e+03	1.61e+04	800	800	801	3	800
BURKEHAN	IPOPT	infeas	13	1.00e-02	-1.74e-05	17	17	6	16	14
	KNITRO	infeas	28	1.00e-02	-7.04e-04	41	42	30	31	29
	SNOPT	infeas	5	1.00e-02	-4.21e-06	1	19	1	18	0
	WORHP IP	infeas	18	1.00e-02	-1.07e-06	30	30	19	19	19
	WORHP IPm	infeas	18	1.00e-02	-1.29e-15	22	22	19	19	19
	WORHP SQP	minalpha	27	1.00e-02	0.00e+00	2535	2546	28	29	27
BYRDSPHR	IPOPT	optimal	12	1.00e-02	-4.68e+00	13	29	13	13	12
	KNITRO	optimal	12	1.00e-02	-4.68e+00	18	19	13	14	12
	SNOPT	optimal	51	1.00e-02	-4.68e+00	1	202	1	201	0
	WORHP IP	optimal	11	1.00e-02	-4.68e+00	20	20	12	12	11
	WORHP IPm	optimal	11	1.00e-02	-4.68e+00	20	20	12	12	11
	WORHP SQP	optimal	12	1.00e-02	-4.68e+00	722	723	14	14	12
C-RELOAD	IPOPT	accept	5084	3.03e+01	-1.01e+00	5218	10436	5086	10192	5085
	KNITRO	optimal	86	6.90e-01	-1.02e+00	89	90	88	89	86
	SNOPT	optimal	37	1.10e-01	-1.03e+00	1	72	1	71	0
	WORHP IP	optimal	142	1.15e+00	-1.03e+00	146	146	144	144	142
	WORHP IPm	optimal	58	4.70e-01	-1.02e+00	89	89	60	60	58
	WORHP SQP	optimal	1524	6.10e+01	-1.03e+00	1525	1525	1525	1525	1524
CAMEL6	IPOPT	optimal	10	1.00e-02	-1.03e+00	11	0	11	0	10
	KNITRO	optimal	7	1.00e-02	-2.15e-01	9	0	8	0	7
	SNOPT	optimal	11	1.00e-02	-1.03e+00	20	0	19	0	0
	WORHP IP	optimal	9	1.00e-02	-1.03e+00	11	0	10	0	9
	WORHP IPm	optimal	9	1.00e-02	-1.03e+00	11	0	10	0	9
	WORHP SQP	optimal	7	1.00e-02	2.23e+00	8	0	8	0	7
CAMSHAPE	IPOPT	optimal	61	2.50e-01	-4.28e+00	80	80	62	66	61
	KNITRO	optimal	70	3.30e-01	-4.27e+00	73	74	72	73	70
	SNOPT	optimal	7	6.50e-01	-4.24e+00	1	16	1	15	0
	WORHP IP	optimal	41	2.50e-01	-4.27e+00	43	43	43	43	41
	WORHP IPm	optimal	1039	5.62e+00	-4.27e+00	1042	1042	1041	1041	1039
	WORHP SQP	optimal	36	3.98e+00	-4.27e+00	593	595	37	37	36
CANTILVR	IPOPT	optimal	11	1.00e-02	1.34e+00	12	12	12	12	11
	KNITRO	optimal	12	1.00e-02	1.34e+00	14	15	13	14	12
	SNOPT	optimal	19	1.00e-02	1.34e+00	1	29	1	28	0
	WORHP IP	optimal	12	1.00e-02	1.34e+00	14	14	13	13	12
	WORHP IPm	optimal	12	1.00e-02	1.34e+00	14	14	13	13	12
	WORHP SQP	optimal	15	1.00e-02	1.34e+00	19	19	8	18	6
CAR2	IPOPT	optimal	27	1.05e+00	2.67e+00	34	68	28	56	27
	KNITRO	optimal	25	9.20e-01	2.67e+00	29	30	27	28	25
	SNOPT	optimal	47	9.97e+01	2.67e+00	1	62	1	61	0
	WORHP IP	optimal	35	1.53e+00	2.67e+00	44	44	36	36	35
	WORHP IPm	optimal	168	8.68e+00	2.67e+00	1112	1112	169	169	168
	WORHP SQP	optimal	10	1.67e+00	2.67e+00	11	11	12	12	10
CATENA	IPOPT	optimal	55	3.90e-01	-2.10e+06	71	82	56	56	55
	KNITRO	optimal	13	7.00e-02	-2.10e+06	15	16	14	15	13
	SNOPT	maxtime	4960	1.80e+03	-2.00e+06	1	16421	1	16420	0
	WORHP IP	optimal	22	1.40e-01	-2.10e+06	35	35	23	23	22
	WORHP IPm	optimal	22	1.50e-01	-2.10e+06	35	35	23	23	22
	WORHP SQP	optimal	103	1.58e+00	-2.10e+06	466	466	105	105	103
CATENARY	IPOPT	optimal	2889	1.77e+01	-2.10e+06	7095	7403	1487	2904	2889
	KNITRO	maxiter	10000	5.16e+01	-6.54e+09	38757	38758	10002	10003	10000
	SNOPT	sbasics	739	1.27e+03	-1.96e+08	1	3020	1	3019	0
	WORHP IP	optimal	1224	9.48e+00	-2.10e+06	3437	3437	1225	1225	1224
	WORHP IPm	optimal	1224	5.52e+00	-2.10e+06	3437	3437	1225	1225	1224
	WORHP SQP	minalpha	1628	1.82e+01	-1.33e+06	49151	49157	246	1630	245
CATMIX	IPOPT	optimal	14	1.90e-01	-4.78e-02	15	15	15	15	14
	KNITRO	optimal	8	1.70e-01	-4.78e-02	11	12	10	11	8
	SNOPT	optimal	37	3.13e+00	-4.64e-02	45	45	44	44	0
	WORHP IP	optimal	16	2.30e-01	-4.79e-02	18	18	17	17	16
	WORHP IPm	optimal	72	5.90e-01	-4.79e-02	76	76	75	75	72
	WORHP SQP	optimal	15	7.10e-01	-4.64e-02	16	16	17	17	15

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
CB2	IPOPT	optimal	8	1.00e-02	1.95e+00	9	9	9	9	8
	KNITRO	optimal	9	1.00e-02	1.95e+00	11	12	10	11	9
	SNOPT	optimal	8	1.00e-02	1.95e+00	1	14	1	13	0
	WORHP IP	optimal	8	1.00e-02	1.95e+00	10	10	9	9	8
	WORHP IPm	optimal	7	1.00e-02	1.95e+00	9	9	8	8	7
	WORHP SQP	optimal	18	1.00e-02	1.95e+00	19	19	20	20	18
CB3	IPOPT	optimal	9	1.00e-02	2.00e+00	10	10	10	10	9
	KNITRO	optimal	6	1.00e-02	2.00e+00	8	9	7	8	6
	SNOPT	optimal	10	1.00e-02	2.00e+00	1	18	1	17	0
	WORHP IP	optimal	8	1.00e-02	2.00e+00	10	10	9	9	8
	WORHP IPm	optimal	6	1.00e-02	2.00e+00	8	8	7	7	6
	WORHP SQP	optimal	7	1.00e-02	2.00e+00	8	8	9	9	7
CBRATU2D	IPOPT	optimal	3	2.70e-01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	4.40e-01	0.00e+00	13	14	4	5	3
	SNOPT	optimal	3	3.10e-01	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	3	2.00e-01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	1.90e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	3	2.11e+00	0.00e+00	4	4	5	5	3
CBRATU3D	IPOPT	optimal	3	1.13e+00	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	2.10e+00	0.00e+00	5	6	4	5	3
	SNOPT	optimal	3	3.20e-01	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	3	5.60e-01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	5.50e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	3	6.73e+00	0.00e+00	4	4	5	5	3
CBS	IPOPT	optimal	133	4.83e+00	8.38e+04	146	146	134	134	133
	KNITRO	optimal	26	7.70e-01	8.38e+04	29	30	28	29	26
	SNOPT	sbasics	380	8.71e+01	9.97e+04	409	1	408	1	0
	WORHP IP	optimal	19	7.30e-01	8.38e+04	21	21	20	1	19
	WORHP IPm	optimal	14	5.60e-01	8.38e+04	19	19	18	1	14
	WORHP SQP	optimal	730	5.44e+01	8.38e+04	761	761	732	3	730
CHACONN1	IPOPT	optimal	6	1.00e-02	1.95e+00	7	7	7	7	6
	KNITRO	optimal	5	1.00e-02	1.95e+00	7	8	6	7	5
	SNOPT	optimal	8	1.00e-02	1.95e+00	1	14	1	13	0
	WORHP IP	optimal	5	1.00e-02	1.95e+00	7	7	6	6	5
	WORHP IPm	optimal	4	1.00e-02	1.95e+00	6	6	5	5	4
	WORHP SQP	optimal	5	1.00e-02	1.95e+00	6	6	7	7	5
CHACONN2	IPOPT	optimal	9	1.00e-02	2.00e+00	11	11	10	10	9
	KNITRO	optimal	7	1.00e-02	2.00e+00	9	10	8	9	7
	SNOPT	optimal	10	1.00e-02	2.00e+00	1	18	1	17	0
	WORHP IP	optimal	8	1.00e-02	2.00e+00	10	10	9	9	8
	WORHP IPm	optimal	7	1.00e-02	2.00e+00	9	9	8	8	7
	WORHP SQP	optimal	9	1.00e-02	2.00e+00	10	10	11	11	9
CHAIN	IPOPT	optimal	7	1.00e-02	5.07e+00	8	8	8	8	7
	KNITRO	optimal	7	2.00e-02	5.07e+00	9	10	8	9	7
	SNOPT	optimal	71	2.45e+00	5.07e+00	95	95	94	94	0
	WORHP IP	optimal	12	2.00e-02	5.07e+00	14	14	13	13	12
	WORHP IPm	optimal	12	2.00e-02	5.07e+00	14	14	13	13	12
	WORHP SQP	optimal	13	4.00e-02	5.07e+00	14	14	15	15	13
CHAINWOO	IPOPT	optimal	187	2.06e+00	7.93e+01	728	0	188	0	187
	KNITRO	optimal	236	1.87e+00	8.15e+00	433	0	237	0	236
	SNOPT	toobig	39	7.85e+01	2.24e+06	42	0	41	0	0
	WORHP IP	optimal	185	1.31e+00	7.93e+01	454	0	187	0	185
	WORHP IPm	optimal	185	1.26e+00	7.93e+01	454	0	186	0	185
	WORHP SQP	optimal	104	1.34e+00	2.99e+03	752	0	105	0	104
CHANDHEQ	IPOPT	optimal	10	9.00e-02	0.00e+00	11	11	11	11	10
	KNITRO	optimal	10	1.20e-01	0.00e+00	12	13	11	12	10
	SNOPT	optimal	9	5.00e-02	0.00e+00	1	12	1	11	0
	WORHP IP	optimal	10	1.00e-01	0.00e+00	12	12	11	11	10
	WORHP IPm	optimal	10	9.00e-02	0.00e+00	12	12	11	11	10
	WORHP SQP	optimal	11	2.70e-01	0.00e+00	12	12	13	13	11
CHANDHEU	IPOPT	optimal	10	1.14e+01	0.00e+00	11	11	11	11	10
	KNITRO	optimal	10	4.80e+00	0.00e+00	12	13	11	12	10
	SNOPT	optimal	9	3.94e+00	0.00e+00	1	12	1	11	0
	WORHP IP	optimal	10	7.97e+00	0.00e+00	12	12	11	11	10
	WORHP IPm	optimal	10	1.05e+01	0.00e+00	12	12	11	11	10
	WORHP SQP	optimal	11	2.85e+01	0.00e+00	12	12	13	13	11
CHANNEL	IPOPT	optimal	3	2.30e-01	1.00e+00	5	5	4	4	3
	KNITRO	optimal	2	2.90e-01	1.00e+00	5	6	3	4	2
	SNOPT	optimal	5	1.80e-01	-1.00e+00	1	8	1	7	0
	WORHP IP	optimal	7	3.90e-01	1.00e+00	9	9	8	8	7
	WORHP IPm	optimal	7	3.80e-01	1.00e+00	9	9	8	8	7
	WORHP SQP	optimal	15	4.79e+00	1.00e+00	17	17	16	16	15
CHARDIS0	IPOPT	optimal	7	3.22e+01	0.00e+00	8	0	8	0	7
	KNITRO	optimal	5	2.18e+01	6.86e-26	7	0	6	0	5
	SNOPT	optimal	16	2.72e+01	2.11e-21	19	0	18	0	0
	WORHP IP	optimal	6	2.44e+01	0.00e+00	8	0	8	0	6
	WORHP IPm	optimal	6	2.20e+01	0.00e+00	8	0	7	0	6
	WORHP SQP	optimal	27	5.54e+01	2.54e-25	1141	0	28	0	27

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
CHARDIS1	IPOPT	optimal	19	7.48e+01	4.23e-04	21	22	20	20	19
	KNITRO	optimal	8	2.93e+01	1.49e-05	10	11	9	10	8
	SNOPT	optimal	46	9.93e+01	1.05e-12	49	49	48	48	0
	WORHP IP	optimal	16	6.62e+01	7.07e-06	18	18	18	18	16
	WORHP IPm	optimal	12	4.75e+01	8.59e-05	14	14	13	13	12
	WORHP SQP	maxtime	493	1.81e+03	1.76e-09	33611	33611	495	495	494
CHEBYQAD	IPOPT	optimal	264	7.55e+00	4.88e-03	496	0	265	0	264
	KNITRO	optimal	213	5.61e+00	1.76e-02	223	0	214	0	213
	SNOPT	noimpr	10000	5.49e+01	9.47e-03	11159	0	11158	0	0
	WORHP IP	optimal	235	3.81e+00	4.88e-03	302	0	236	0	235
	WORHP IPm	optimal	244	5.87e+00	4.88e-03	310	0	245	0	244
	WORHP SQP	optimal	77	2.73e+00	4.51e-03	264	0	78	0	77
CHEMRCTA	IPOPT	optimal	3	1.10e-01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	83	4.52e+00	0.00e+00	116	117	84	85	84
	SNOPT	infeas	0	5.05e+01	0.00e+00	1	3	1	2	0
	WORHP IP	infeas	33	6.90e-01	0.00e+00	60	60	34	34	34
	WORHP IPm	infeas	24	6.10e-01	0.00e+00	57	57	25	25	25
	WORHP SQP	minalpha	14	7.47e+00	0.00e+00	2765	2771	7	16	6
CHEMRCTB	IPOPT	optimal	3	1.00e-01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	2331	4.58e+01	0.00e+00	2366	2367	2332	2333	2332
	SNOPT	optimal	3	1.00e-01	0.00e+00	1	7	1	6	0
	WORHP IP	infeas	39	7.10e-01	0.00e+00	87	87	40	40	40
	WORHP IPm	infeas	29	5.70e-01	0.00e+00	75	75	30	30	30
	WORHP SQP	optimal	4	5.30e-01	0.00e+00	11	11	6	6	4
CHENHARK	IPOPT	optimal	16	1.40e-01	-2.00e+00	17	0	17	0	16
	KNITRO	optimal	8	6.00e-02	-2.00e+00	10	0	9	0	8
	SNOPT	optimal	1553	2.84e+01	-2.00e+00	1775	0	1774	0	0
	WORHP IP	optimal	14	1.30e-01	-2.00e+00	16	0	15	0	14
	WORHP IPm	optimal	12	1.10e-01	-2.00e+00	17	0	16	0	12
	WORHP SQP	optimal	14	2.30e-01	-2.00e+00	15	0	15	0	14
CHNROSNB	IPOPT	optimal	42	2.00e-02	1.54e-22	92	0	43	0	42
	KNITRO	optimal	43	1.00e-02	1.82e-26	56	0	44	0	43
	SNOPT	optimal	152	1.00e-02	3.40e-14	171	0	170	0	0
	WORHP IP	optimal	42	1.00e-02	1.54e-22	69	0	43	0	42
	WORHP IPm	optimal	42	1.00e-02	1.54e-22	69	0	43	0	42
	WORHP SQP	optimal	41	1.00e-02	1.27e-23	103	0	42	0	41
CHNRSBNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	1	1.00e-02	0.00e+00	4	5	3	4	1
	SNOPT	optimal	2	1.00e-02	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	46	1.00e-02	0.00e+00	137	137	47	47	46
	WORHP IPm	optimal	46	1.00e-02	0.00e+00	137	137	47	47	46
	WORHP SQP	optimal	35	1.20e-01	0.00e+00	1637	1639	20	36	20
CHNRSNBM	IPOPT	optimal	52	3.00e-02	8.49e-16	153	0	53	0	52
	KNITRO	optimal	55	1.00e-02	1.57e-24	99	0	56	0	55
	SNOPT	optimal	171	1.00e-02	1.34e-14	196	0	195	0	0
	WORHP IP	optimal	52	1.00e-02	8.51e-16	109	0	53	0	52
	WORHP IPm	optimal	52	1.00e-02	8.51e-16	109	0	53	0	52
	WORHP SQP	optimal	49	1.00e-02	6.90e-17	266	0	50	0	49
CHWIRUT1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	19	2.00e-02	0.00e+00	23	24	20	21	19
	SNOPT	infeas	4	1.00e-02	0.00e+00	1	15	1	14	0
	WORHP IP	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP IPm	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP SQP	infeas	31	2.20e-01	0.00e+00	1212	1214	24	32	23
CHWIRUT1LS	IPOPT	optimal	6	1.00e-02	2.38e+03	12	0	7	0	6
	KNITRO	optimal	8	1.00e-02	2.38e+03	17	0	9	0	8
	SNOPT	toobig	29	1.00e-02	2.38e+03	40	0	39	0	0
	WORHP IP	optimal	6	1.00e-02	2.38e+03	52	0	8	0	6
	WORHP IPm	optimal	6	1.00e-02	2.38e+03	52	0	7	0	6
	WORHP SQP	optimal	23	1.00e-02	2.38e+03	59	0	24	0	23
CHWIRUT2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	21	1.00e-02	0.00e+00	53	54	21	22	20
	SNOPT	infeas	5	1.00e-02	0.00e+00	1	14	1	13	0
	WORHP IP	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP IPm	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP SQP	minalpha	17	6.00e-02	0.00e+00	3354	3360	16	19	15
CHWIRUT2LS	IPOPT	optimal	6	1.00e-02	5.13e+02	12	0	7	0	6
	KNITRO	optimal	7	1.00e-02	5.13e+02	16	0	8	0	7
	SNOPT	optimal	29	1.00e-02	5.13e+02	40	0	39	0	0
	WORHP IP	optimal	6	1.00e-02	5.13e+02	10	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	5.13e+02	10	0	7	0	6
	WORHP SQP	optimal	23	1.00e-02	5.13e+02	56	0	24	0	23
CLIFF	IPOPT	optimal	27	1.00e-02	2.00e-01	28	0	28	0	27
	KNITRO	optimal	27	1.00e-02	2.00e-01	29	0	28	0	27
	SNOPT	optimal	21	1.00e-02	2.00e-01	29	0	28	0	0
	WORHP IP	optimal	27	1.00e-02	2.00e-01	29	0	29	0	27
	WORHP IPm	optimal	27	1.00e-02	2.00e-01	29	0	28	0	27
	WORHP SQP	optimal	27	1.00e-02	2.00e-01	28	0	28	0	27

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
CLNLBEAM	IPOPT	optimal	524	6.75e+00	3.45e+02	568	568	525	525	524
	KNITRO	optimal	63	1.48e+00	3.46e+02	68	69	65	66	63
	SNOPT	optimal	1	3.30e-01	3.50e+02	4	4	3	3	0
	WORHP IP	optimal	857	1.07e+01	3.45e+02	892	892	858	858	857
	WORHP IPm	optimal	894	1.13e+01	3.45e+02	1021	1021	896	896	894
	WORHP SQP	optimal	11	7.10e-01	3.48e+02	12	12	13	13	11
CLPLATEA	IPOPT	optimal	5	1.70e-01	-1.26e-02	11	0	6	0	5
	KNITRO	optimal	5	2.00e-01	-1.26e-02	8	0	6	0	5
	SNOPT	toobig	1838	4.82e+01	-1.02e-02	2054	0	2053	0	0
	WORHP IP	optimal	5	1.80e-01	-1.26e-02	9	0	6	0	5
	WORHP IPm	optimal	5	1.50e-01	-1.26e-02	9	0	6	0	5
	WORHP SQP	optimal	6	2.20e-01	-1.26e-02	10	0	7	0	6
CLPLATEB	IPOPT	optimal	2	1.60e-01	-5.09e-03	3	0	3	0	2
	KNITRO	optimal	2	1.60e-01	-5.09e-03	4	0	3	0	2
	SNOPT	toobig	1288	2.86e+01	-2.04e-03	1458	0	1457	0	0
	WORHP IP	optimal	2	1.20e-01	-5.09e-03	4	0	3	0	2
	WORHP IPm	optimal	2	1.50e-01	-5.09e-03	4	0	3	0	2
	WORHP SQP	optimal	5	2.00e-01	-5.09e-03	6	0	6	0	5
CLPLATEC	IPOPT	optimal	1	1.40e-01	-5.02e-03	2	0	2	0	1
	KNITRO	optimal	1	1.40e-01	-5.02e-03	3	0	2	0	1
	SNOPT	sbasics	10000	1.77e+02	-4.58e-04	11164	0	11163	0	0
	WORHP IP	optimal	1	1.00e-01	-5.02e-03	3	0	2	0	1
	WORHP IPm	optimal	1	1.20e-01	-5.02e-03	3	0	2	0	1
	WORHP SQP	optimal	5	1.90e-01	-5.02e-03	6	0	6	0	5
CLUSTER	IPOPT	optimal	8	1.00e-02	0.00e+00	9	9	9	9	8
	KNITRO	optimal	8	1.00e-02	0.00e+00	10	11	9	10	8
	SNOPT	optimal	8	1.00e-02	0.00e+00	1	11	1	10	0
	WORHP IP	optimal	8	1.00e-02	0.00e+00	10	10	9	9	8
	WORHP IPm	optimal	8	1.00e-02	0.00e+00	10	10	9	9	8
	WORHP SQP	optimal	8	1.00e-02	0.00e+00	9	9	10	10	8
CONCON	IPOPT	optimal	9	1.00e-02	-6.23e+03	10	10	10	10	9
	KNITRO	optimal	6	1.00e-02	-6.23e+03	9	10	8	9	6
	SNOPT	optimal	7	1.00e-02	-6.23e+03	1	10	1	9	0
	WORHP IP	optimal	9	1.00e-02	-6.23e+03	11	11	10	10	9
	WORHP IPm	optimal	10	1.00e-02	-6.23e+03	15	15	14	14	10
	WORHP SQP	optimal	6	1.00e-02	-6.23e+03	7	7	7	7	6
CONGIGMZ	IPOPT	optimal	25	1.00e-02	2.80e+01	30	30	26	26	25
	KNITRO	optimal	16	1.00e-02	2.80e+01	18	19	17	18	16
	SNOPT	optimal	6	1.00e-02	2.80e+01	1	9	1	8	0
	WORHP IP	optimal	20	1.00e-02	2.80e+01	24	24	21	21	20
	WORHP IPm	optimal	21	1.00e-02	2.80e+01	30	30	22	22	21
	WORHP SQP	optimal	6	1.00e-02	2.80e+01	7	7	7	7	6
CONT5-QP	IPOPT	optimal	50	1.44e+02	6.40e-03	51	51	51	51	50
	KNITRO	optimal	42	2.91e+01	6.52e-03	45	46	44	45	42
	SNOPT	optimal	16	6.43e+02	6.36e-03	30	1	29	1	0
	WORHP IP	minalpha	27	8.38e+01	5.68e-02	101	101	64	1	28
	WORHP IPm	minalpha	302	7.11e+02	6.02e-02	504	504	344	1	303
	WORHP SQP	optimal	89	3.21e+02	6.41e-03	90	90	91	3	89
CONT6-QQ	IPOPT	infeas	270	1.18e+02	-4.32e+00	415	415	110	294	271
	KNITRO	infeas	23	3.46e+00	-4.33e+00	25	26	24	25	23
	SNOPT	infeas	4	1.07e+01	-4.17e+00	10	10	9	9	0
	WORHP IP	infeas	60	8.11e+00	-4.33e+00	63	63	61	61	61
	WORHP IPm	infeas	25	5.53e+00	-4.33e+00	29	29	28	28	26
	WORHP SQP	minalpha	30	2.61e+02	-4.33e+00	5396	5415	32	32	31
COOLHANS	IPOPT	optimal	8	1.00e-02	0.00e+00	9	9	9	9	8
	KNITRO	optimal	8	1.00e-02	0.00e+00	10	11	9	10	8
	SNOPT	optimal	17	1.00e-02	0.00e+00	1	26	1	25	0
	WORHP IP	optimal	8	1.00e-02	0.00e+00	10	10	9	9	8
	WORHP IPm	optimal	8	1.00e-02	0.00e+00	10	10	9	9	8
	WORHP SQP	optimal	27	1.00e-02	0.00e+00	28	28	29	29	27
CORE1	IPOPT	optimal	103	5.00e-02	9.11e+01	109	218	74	210	103
	KNITRO	optimal	17	1.00e-02	9.11e+01	20	21	19	20	17
	SNOPT	optimal	4	1.00e-02	9.11e+01	1	9	1	8	0
	WORHP IP	optimal	346	1.00e-01	9.11e+01	525	525	347	347	346
	WORHP IPm	optimal	267	7.00e-02	9.11e+01	403	403	269	269	267
	WORHP SQP	optimal	9	3.00e-02	9.11e+01	35	35	10	10	9
CORE2	IPOPT	optimal	242	1.90e-01	7.29e+01	250	500	156	488	242
	KNITRO	optimal	54	5.00e-02	7.29e+01	62	63	56	57	54
	SNOPT	optimal	3	1.00e-02	7.29e+01	1	6	1	5	0
	WORHP IP	optimal	3847	1.36e+02	7.29e+01	3954	3954	3848	3848	3847
	WORHP IPm	optimal	705	5.30e-01	7.29e+01	3104	3104	709	709	705
	WORHP SQP	optimal	162	3.10e-01	7.29e+01	3193	3191	31	165	30
CORKSCRW	IPOPT	optimal	231	2.15e+00	8.19e+01	243	486	223	468	231
	KNITRO	optimal	30	4.80e-01	8.19e+01	33	34	32	33	30
	SNOPT	optimal	16	4.00e+00	8.19e+01	20	20	19	19	0
	WORHP IP	optimal	192	1.65e+00	8.19e+01	197	197	194	194	192
	WORHP IPm	optimal	502	5.70e+00	8.19e+01	548	548	508	508	502
	WORHP SQP	optimal	15	1.44e+01	8.19e+01	16	16	16	16	15

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
COSHFUN	IPOPT	resto	268	7.35e+02	-5.75e+04	701	1907	252	276	269
	KNITRO	noimpr	93	2.04e+00	-5.78e+15	312	313	92	93	93
	SNOPT	infeas	263	5.42e+02	-1.52e+00	1	1326	1	1325	0
	WORHP IP	optimal	207	2.78e+00	-7.81e-01	729	729	208	208	207
	WORHP IPm	maxiter	10000	2.43e+02	-7.81e-01	119387	119387	10001	10001	10000
	WORHP SQP	maxiter	10000	2.29e+02	-7.99e-01	11567	11526	9988	10043	9987
COSINE	IPOPT	optimal	12	2.80e-01	-1.00e+04	13	0	13	0	12
	KNITRO	optimal	6	1.90e-01	-1.00e+04	9	0	7	0	6
	SNOPT	toobig	148	6.07e+01	5.02e+03	194	0	193	0	0
	WORHP IP	optimal	12	2.90e-01	-1.00e+04	14	0	13	0	12
	WORHP IPm	optimal	12	2.80e-01	-1.00e+04	14	0	13	0	12
	WORHP SQP	optimal	8	2.30e-01	-1.00e+04	9	0	9	0	8
CRAGGLVY	IPOPT	optimal	14	2.80e-01	1.69e+03	15	0	15	0	14
	KNITRO	optimal	14	2.30e-01	1.69e+03	16	0	15	0	14
	SNOPT	toobig	28	4.11e+01	6.16e+05	31	0	30	0	0
	WORHP IP	optimal	14	2.40e-01	1.69e+03	16	0	16	0	14
	WORHP IPm	optimal	14	2.40e-01	1.69e+03	16	0	15	0	14
	WORHP SQP	optimal	12	2.50e-01	1.69e+03	13	0	13	0	12
CRESC100	IPOPT	resto	5249	1.06e+01	1.27e+01	17498	18402	1053	5320	5250
	KNITRO	optimal	108	9.00e-02	5.70e-01	143	144	109	110	109
	SNOPT	optimal	93	3.00e-02	5.68e-01	222	222	221	221	0
	WORHP IP	minalpha	4328	4.81e+00	2.80e+08	12744	12744	4346	4346	4329
	WORHP IPm	regular	2158	2.85e+00	8.22e+07	6095	6095	2159	2159	2159
	WORHP SQP	maxiter	10000	1.42e+01	5.68e-01	107890	107870	10001	10022	10000
CRESC132	IPOPT	resto	2046	3.19e+02	5.67e+00	6048	8247	686	2103	2047
	KNITRO	optimal	513	8.57e+00	6.85e-01	1127	1128	514	515	518
	SNOPT	noimpr	822	1.08e+01	6.85e-01	1984	1984	1983	1983	0
	WORHP IP	regular	1816	4.65e+02	8.88e+07	5157	5157	1817	1817	1817
	WORHP IPm	maxtime	4429	1.79e+03	2.26e+08	11147	11147	4432	4432	4430
	WORHP SQP	optimal	2796	9.80e+01	6.85e-01	14218	13726	2719	3290	2717
CRESC4	IPOPT	optimal	131	5.00e-02	8.72e-01	298	298	129	136	131
	KNITRO	optimal	61	1.00e-02	8.72e-01	133	134	62	63	61
	SNOPT	optimal	51	1.00e-02	8.72e-01	96	96	95	95	0
	WORHP IP	maxiter	10000	1.67e+00	3.50e+08	37756	37756	10017	10017	10000
	WORHP IPm	regular	7262	1.23e+00	8.51e+07	29405	29405	7263	7263	7263
	WORHP SQP	optimal	29	2.00e-02	8.72e-01	52	52	31	31	29
CRESC50	IPOPT	resto	2485	2.84e+00	1.14e-03	14522	14771	1035	2521	2486
	KNITRO	maxiter	10000	5.09e+00	-7.66e-07	41688	41689	10001	10002	10001
	SNOPT	noimpr	1713	5.70e-01	5.93e-01	5085	5085	5084	5084	0
	WORHP IP	regular	2445	2.00e+00	1.59e+08	6824	6824	2446	2446	2446
	WORHP IPm	maxiter	10000	2.04e+01	4.92e+08	111939	111939	10001	10001	10000
	WORHP SQP	optimal	54	1.80e-01	7.86e-01	257	257	56	56	54
CSFI1	IPOPT	optimal	15	1.00e-02	-4.91e+01	17	34	16	32	15
	KNITRO	optimal	9	1.00e-02	-4.91e+01	12	13	11	12	9
	SNOPT	optimal	20	1.00e-02	-4.91e+01	1	50	1	49	0
	WORHP IP	optimal	27	1.00e-02	-4.91e+01	32	32	28	28	27
	WORHP IPm	optimal	26	1.00e-02	-4.91e+01	30	30	28	28	26
	WORHP SQP	optimal	13	1.00e-02	-4.91e+01	16	16	14	14	13
CSFI2	IPOPT	optimal	27	1.00e-02	5.50e+01	67	134	27	60	27
	KNITRO	optimal	14	1.00e-02	5.50e+01	17	18	16	17	14
	SNOPT	optimal	20	1.00e-02	5.50e+01	1	34	1	33	0
	WORHP IP	optimal	60	1.00e-02	5.50e+01	214	214	61	61	60
	WORHP IPm	optimal	62	1.00e-02	5.50e+01	223	223	67	67	62
	WORHP SQP	optimal	23	1.00e-02	5.50e+01	24	24	25	25	23
CUBE	IPOPT	optimal	27	1.00e-02	1.75e-24	58	0	28	0	27
	KNITRO	optimal	27	1.00e-02	5.38e-25	45	0	28	0	27
	SNOPT	optimal	30	1.00e-02	9.18e-16	43	0	42	0	0
	WORHP IP	optimal	27	1.00e-02	1.75e-24	44	0	29	0	27
	WORHP IPm	optimal	27	1.00e-02	1.75e-24	44	0	28	0	27
	WORHP SQP	optimal	26	1.00e-02	3.47e-18	88	0	27	0	26
CUBENE	IPOPT	optimal	1	1.00e-02	0.00e+00	3	3	2	2	1
	KNITRO	optimal	0	1.00e-02	0.00e+00	3	4	2	3	0
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	optimal	4	1.00e-02	0.00e+00	13	13	5	5	4
	WORHP IPm	optimal	4	1.00e-02	0.00e+00	13	13	5	5	4
	WORHP SQP	optimal	4	1.00e-02	0.00e+00	21	21	6	6	4
CURLY10	IPOPT	optimal	21	8.10e-01	-1.00e+06	22	0	22	0	21
	KNITRO	optimal	9	4.00e-01	-1.00e+06	17	0	10	0	9
	SNOPT	sbasics	10000	3.08e+02	-2.02e+05	11288	0	11287	0	0
	WORHP IP	optimal	21	9.60e-01	-1.00e+06	23	0	22	0	21
	WORHP IPm	optimal	21	9.00e-01	-1.00e+06	23	0	22	0	21
	WORHP SQP	optimal	39	2.14e+00	-1.00e+06	86	0	40	0	39
CURLY20	IPOPT	optimal	26	2.27e+00	-1.00e+06	27	0	27	0	26
	KNITRO	optimal	11	1.04e+00	-1.00e+06	23	0	12	0	11
	SNOPT	sbasics	10000	2.87e+02	-2.03e+05	11228	0	11227	0	0
	WORHP IP	optimal	26	2.55e+00	-1.00e+06	70	0	27	0	26
	WORHP IPm	optimal	26	2.38e+00	-1.00e+06	70	0	27	0	26
	WORHP SQP	optimal	46	6.05e+00	-1.00e+06	136	0	47	0	46

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
CURLY30	IPOPT	optimal	29	6.53e+00	-1.00e+06	30	0	30	0	29
	KNITRO	optimal	10	1.29e+00	-1.00e+06	17	0	11	0	10
	SNOPT	sbasics	10000	3.42e+02	-2.04e+05	11188	0	11187	0	0
	WORHP IP	optimal	29	5.57e+00	-1.00e+06	31	0	30	0	29
	WORHP IPm	optimal	29	5.60e+00	-1.00e+06	31	0	30	0	29
	WORHP SQP	optimal	40	1.26e+01	-1.00e+06	95	0	41	0	40
CVXBQP1	IPOPT	optimal	13	9.40e-01	2.25e+06	14	0	14	0	13
	KNITRO	optimal	8	6.60e-01	2.25e+06	10	0	9	0	8
	SNOPT	optimal	39	7.05e+00	2.25e+06	44	0	43	0	0
	WORHP IP	optimal	8	7.70e-01	2.25e+06	10	0	10	0	8
	WORHP IPm	optimal	9	9.60e-01	2.25e+06	15	0	14	0	9
	WORHP SQP	optimal	8	3.44e+00	2.25e+06	9	0	9	0	8
CVXQP1	IPOPT	optimal	21	2.34e+01	1.09e+08	22	22	22	22	21
	KNITRO	optimal	13	5.99e+01	1.09e+08	15	16	14	15	13
	SNOPT	optimal	25	1.83e+01	1.09e+08	28	1	27	1	0
	WORHP IP	optimal	321	6.60e+02	1.09e+08	404	404	323	1	321
	WORHP IPm	optimal	70	4.81e+01	1.09e+08	77	77	76	1	70
	WORHP SQP	optimal	6	4.70e+01	1.09e+08	7	7	8	3	6
CVXQP2	IPOPT	optimal	24	1.11e+01	8.18e+07	25	25	25	25	24
	KNITRO	optimal	17	2.85e+01	8.18e+07	19	20	18	19	17
	SNOPT	sbasics	93	3.79e+01	8.18e+07	96	1	95	1	0
	WORHP IP	optimal	22	2.25e+01	8.18e+07	168	168	24	1	22
	WORHP IPm	optimal	18	7.12e+00	8.18e+07	88	88	23	1	18
	WORHP SQP	optimal	7	3.62e+01	8.18e+07	8	8	9	3	7
CVXQP3	IPOPT	optimal	19	3.77e+01	1.16e+08	20	20	20	20	19
	KNITRO	maxtime	57	1.82e+03	1.16e+08	59	60	58	59	57
	SNOPT	optimal	11	9.85e+00	1.16e+08	14	1	13	1	0
	WORHP IP	optimal	13	8.06e+01	1.16e+08	45	45	15	1	13
	WORHP IPm	optimal	14	9.14e+01	1.16e+08	21	21	20	1	14
	WORHP SQP	optimal	5	7.34e+01	1.16e+08	6	6	7	3	5
CYCLIC3	IPOPT	optimal	22	7.43e+00	0.00e+00	23	23	23	23	22
	KNITRO	optimal	22	7.68e+00	0.00e+00	24	25	23	24	22
	SNOPT	optimal	24	4.75e+00	0.00e+00	1	27	1	26	0
	WORHP IP	optimal	22	7.35e+00	0.00e+00	24	24	24	24	22
	WORHP IPm	optimal	22	4.98e+00	0.00e+00	24	24	23	23	22
	WORHP SQP	optimal	25	4.17e+02	0.00e+00	26	26	27	27	25
DALE	IPOPT	optimal	27	9.50e-01	3.70e+03	28	28	28	28	27
	KNITRO	optimal	19	1.32e+00	3.70e+03	22	23	21	22	19
	SNOPT	sbasics	422	1.11e+02	3.85e+03	464	1	463	1	0
	WORHP IP	optimal	11	5.60e-01	3.70e+03	13	13	12	1	11
	WORHP IPm	optimal	10	7.70e-01	3.70e+03	15	15	14	1	10
	WORHP SQP	optimal	555	2.96e+01	3.70e+03	556	556	557	3	555
DALLASL	IPOPT	optimal	53	2.00e-01	-2.03e+05	54	54	54	54	53
	KNITRO	optimal	18	8.00e-02	-2.03e+05	21	22	20	21	18
	SNOPT	optimal	216	5.20e-01	-2.03e+05	233	1	232	1	0
	WORHP IP	optimal	33	1.40e-01	-2.03e+05	35	35	35	1	33
	WORHP IPm	optimal	38	1.80e-01	-2.03e+05	42	42	41	1	38
	WORHP SQP	maxiter	10000	5.22e+01	8.37e+07	10001	10001	10002	3	10001
DALLASM	IPOPT	optimal	30	4.00e-02	-4.82e+04	31	31	31	31	30
	KNITRO	optimal	22	3.00e-02	-4.82e+04	25	26	24	25	22
	SNOPT	optimal	102	3.00e-02	-4.82e+04	107	1	106	1	0
	WORHP IP	optimal	26	2.00e-02	-4.82e+04	28	28	28	1	26
	WORHP IPm	maxiter	10000	8.43e+00	-4.46e+04	10002	10002	10001	1	10000
	WORHP SQP	optimal	21	6.00e-02	-4.82e+04	22	22	23	3	21
DALLASS	IPOPT	optimal	30	2.00e-02	-3.24e+04	31	31	31	31	30
	KNITRO	optimal	18	1.00e-02	-3.24e+04	21	22	20	21	18
	SNOPT	optimal	80	1.00e-02	-3.24e+04	86	1	85	1	0
	WORHP IP	optimal	27	1.00e-02	-3.24e+04	29	29	29	1	27
	WORHP IPm	maxiter	10000	1.56e+00	-3.20e+04	10002	10002	10001	1	10000
	WORHP SQP	minalpha	1034	5.60e-01	-3.24e+04	9287	9307	1036	3	1035
DANWOOD	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	19	1.00e-02	0.00e+00	27	28	19	20	18
	SNOPT	infeas	48	1.00e-02	0.00e+00	1	95	1	94	0
	WORHP IP	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP SQP	minalpha	88	2.00e-02	0.00e+00	2864	2870	80	90	79
DANWOODLS	IPOPT	optimal	11	1.00e-02	4.32e-03	17	0	12	0	11
	KNITRO	optimal	10	1.00e-02	4.32e-03	13	0	11	0	10
	SNOPT	optimal	16	1.00e-02	4.32e-03	25	0	24	0	0
	WORHP IP	optimal	11	1.00e-02	4.32e-03	15	0	13	0	11
	WORHP IPm	optimal	11	1.00e-02	4.32e-03	15	0	12	0	11
	WORHP SQP	optimal	15	1.00e-02	4.32e-03	16	0	16	0	15
DECONVB	IPOPT	optimal	1861	1.32e+00	4.32e-08	6448	0	1862	0	1861
	KNITRO	optimal	42	2.00e-02	3.73e-03	67	0	44	0	42
	SNOPT	optimal	389	4.00e-02	2.57e-03	425	0	424	0	0
	WORHP IP	maxiter	10000	5.93e+00	4.27e-03	10232	0	10001	0	10000
	WORHP IPm	maxiter	10000	5.52e+00	1.99e-03	10166	0	10003	0	10000
	WORHP SQP	optimal	18	4.00e-02	7.13e-09	19	0	19	0	18

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DECONVC	IPOPT	optimal	81	3.00e-02	2.57e-03	95	95	82	82	81
	KNITRO	optimal	33	2.00e-02	2.57e-03	37	38	35	36	33
	SNOPT	optimal	662	1.40e-01	2.57e-03	708	708	707	707	0
	WORHP IP	optimal	60	5.00e-02	2.57e-03	64	64	61	61	60
	WORHP IPm	optimal	55	3.00e-02	2.57e-03	143	143	59	59	55
	WORHP SQP	optimal	18	4.00e-02	1.24e-08	19	19	20	20	18
DECONVNE	IPOPT	resto	2	1.00e-02	0.00e+00	3	3	3	3	3
	KNITRO	optimal	1	1.00e-02	0.00e+00	3	4	2	3	1
	SNOPT	optimal	5	1.00e-02	0.00e+00	1	11	1	10	0
	WORHP IP	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	10	4.00e-02	0.00e+00	22	22	12	12	10
DECONVU	IPOPT	optimal	41	3.00e-02	1.06e-10	48	0	42	0	41
	KNITRO	optimal	22	2.00e-02	4.54e-10	26	0	23	0	22
	SNOPT	optimal	711	7.00e-02	1.66e-07	790	0	789	0	0
	WORHP IP	optimal	41	2.00e-02	1.06e-10	46	0	42	0	41
	WORHP IPm	optimal	41	2.00e-02	1.06e-10	46	0	42	0	41
	WORHP SQP	optimal	45	3.00e-02	9.74e-11	135	0	46	0	45
DEGDIAG	IPOPT	optimal	11	1.19e+01	1.67e+04	12	0	12	0	11
	KNITRO	optimal	8	1.70e+00	1.67e+04	10	0	9	0	8
	SNOPT	optimal	203	4.38e+02	1.67e+04	208	0	207	0	0
	WORHP IP	optimal	9	1.34e+01	1.67e+04	11	0	10	0	9
	WORHP IPm	optimal	9	1.38e+01	1.67e+04	12	0	11	0	9
	WORHP SQP	optimal	7	9.19e+01	1.67e+04	8	0	8	0	7
DEGENLPA	IPOPT	optimal	24	1.00e-02	3.05e+00	30	30	25	25	24
	KNITRO	optimal	26	1.00e-02	3.00e+00	29	30	28	29	26
	SNOPT	optimal	0	1.00e-02	3.06e+00	1	1	1	1	0
	WORHP IP	optimal	17	1.00e-02	3.06e+00	19	19	19	1	17
	WORHP IPm	optimal	39	1.00e-02	3.06e+00	77	77	45	1	39
	WORHP SQP	optimal	3	1.00e-02	3.06e+00	3	3	3	2	3
DEGENLPB	IPOPT	optimal	28	1.00e-02	-3.08e+01	42	42	29	29	28
	KNITRO	optimal	258	3.00e-02	-3.07e+01	344	345	260	261	259
	SNOPT	optimal	0	1.00e-02	-3.07e+01	1	1	1	1	0
	WORHP IP	optimal	32	1.00e-02	-3.07e+01	34	34	33	1	32
	WORHP IPm	optimal	57	1.00e-02	-3.07e+01	64	64	63	1	57
	WORHP SQP	optimal	3	1.00e-02	-3.07e+01	4	4	4	2	3
DEGENQP	IPOPT	optimal	11	1.14e+01	1.28e-02	12	24	12	24	11
	KNITRO	optimal	4	9.60e-01	2.89e-08	7	8	6	7	4
	SNOPT	optimal	2	1.95e+00	-1.28e-05	7	1	6	1	0
	WORHP IP	optimal	8	1.27e+01	2.75e-02	10	10	9	1	8
	WORHP IPm	optimal	9	1.36e+01	2.49e-06	11	11	10	1	9
	WORHP SQP	optimal	2	1.59e+02	1.80e-08	3	3	4	3	2
DEGENQPC	IPOPT	optimal	10	5.80e-01	1.83e-03	11	22	11	22	10
	KNITRO	optimal	6	4.00e-01	1.50e-09	9	10	8	9	6
	SNOPT	optimal	2	2.50e-01	-8.45e-06	7	1	6	1	0
	WORHP IP	optimal	8	5.50e-01	4.39e-03	10	10	9	1	8
	WORHP IPm	optimal	8	6.40e-01	2.07e-08	10	10	9	1	8
	WORHP SQP	optimal	2	1.47e+00	2.15e-11	3	3	4	3	2
DEGTRID	IPOPT	optimal	11	1.38e+00	-1.00e+05	12	0	12	0	11
	KNITRO	optimal	8	2.47e+00	-1.00e+05	10	0	9	0	8
	SNOPT	sbasics	29	3.55e+01	-2.92e+03	36	0	35	0	0
	WORHP IP	optimal	9	2.03e+00	-1.00e+05	11	0	10	0	9
	WORHP IPm	optimal	9	2.43e+00	-1.00e+05	11	0	10	0	9
	WORHP SQP	optimal	3	1.72e+00	-1.00e+05	4	0	4	0	3
DEGTRID2	IPOPT	optimal	11	1.78e+00	-1.00e+05	12	0	12	0	11
	KNITRO	optimal	8	2.09e+00	-1.00e+05	10	0	9	0	8
	SNOPT	optimal	205	4.41e+02	-1.00e+05	209	0	208	0	0
	WORHP IP	optimal	9	2.03e+00	-1.00e+05	11	0	10	0	9
	WORHP IPm	optimal	9	2.25e+00	-1.00e+05	11	0	10	0	9
	WORHP SQP	optimal	2	1.08e+00	-1.00e+05	3	0	3	0	2
DEGTRIDL	IPOPT	optimal	11	3.46e+00	5.00e-01	12	12	12	12	11
	KNITRO	optimal	8	8.72e+00	5.00e-01	10	11	9	10	8
	SNOPT	sbasics	56	1.28e+02	9.78e+04	71	1	70	1	0
	WORHP IP	optimal	9	5.67e+00	5.00e-01	11	11	10	1	9
	WORHP IPm	optimal	9	4.34e+00	5.00e-01	11	11	10	1	9
	WORHP SQP	optimal	3	1.48e+01	5.00e-01	4	4	5	3	3
DEMBO7	IPOPT	optimal	42	2.00e-02	1.75e+02	54	54	43	43	42
	KNITRO	optimal	16	1.00e-02	1.75e+02	19	20	18	19	16
	SNOPT	optimal	23	1.00e-02	1.75e+02	30	30	29	29	0
	WORHP IP	optimal	21	1.00e-02	1.75e+02	25	25	22	22	21
	WORHP IPm	optimal	54	1.00e-02	1.75e+02	101	101	60	60	54
	WORHP SQP	optimal	19	2.00e-02	1.75e+02	20	20	21	21	19
DEMYMALO	IPOPT	optimal	11	1.00e-02	-3.00e+00	12	12	12	12	11
	KNITRO	optimal	15	1.00e-02	-3.00e+00	17	18	16	17	15
	SNOPT	optimal	16	1.00e-02	-3.00e+00	1	30	1	29	0
	WORHP IP	optimal	17	1.00e-02	-3.00e+00	19	19	18	18	17
	WORHP IPm	optimal	14	1.00e-02	-3.00e+00	16	16	15	15	14
	WORHP SQP	optimal	14	1.00e-02	-3.00e+00	26	26	16	16	14

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DENSCHNA	IPOPT	optimal	6	1.00e-02	1.10e-23	7	0	7	0	6
	KNITRO	optimal	6	1.00e-02	1.10e-23	8	0	7	0	6
	SNOPT	optimal	9	1.00e-02	8.51e-13	13	0	12	0	0
	WORHP IP	optimal	6	1.00e-02	1.10e-23	8	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	1.10e-23	8	0	7	0	6
	WORHP SQP	optimal	6	1.00e-02	7.50e-21	7	0	7	0	6
DENSCHNB	IPOPT	optimal	6	1.00e-02	9.99e-16	24	0	7	0	6
	KNITRO	optimal	7	1.00e-02	4.63e-20	13	0	8	0	7
	SNOPT	optimal	8	1.00e-02	3.59e-16	11	0	10	0	0
	WORHP IP	optimal	6	1.00e-02	9.99e-16	22	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	9.99e-16	22	0	7	0	6
	WORHP SQP	optimal	7	1.00e-02	1.60e-24	69	0	8	0	7
DENSCHNC	IPOPT	optimal	10	1.00e-02	2.18e-20	11	0	11	0	10
	KNITRO	optimal	10	1.00e-02	2.18e-20	12	0	11	0	10
	SNOPT	optimal	18	1.00e-02	1.59e-14	22	0	21	0	0
	WORHP IP	optimal	10	1.00e-02	2.18e-20	12	0	11	0	10
	WORHP IPm	optimal	10	1.00e-02	2.18e-20	12	0	11	0	10
	WORHP SQP	optimal	10	1.00e-02	4.50e-20	11	0	11	0	10
DENSCHND	IPOPT	optimal	39	1.00e-02	3.18e-10	40	0	40	0	39
	KNITRO	optimal	30	1.00e-02	3.08e-10	39	0	31	0	30
	SNOPT	optimal	67	1.00e-02	2.55e-10	78	0	77	0	0
	WORHP IP	optimal	39	1.00e-02	3.18e-10	41	0	41	0	39
	WORHP IPm	optimal	39	1.00e-02	3.18e-10	41	0	40	0	39
	WORHP SQP	optimal	35	1.00e-02	2.09e-10	36	0	36	0	35
DENSCHNE	IPOPT	optimal	14	1.00e-02	1.86e-17	25	0	15	0	14
	KNITRO	optimal	13	1.00e-02	2.02e-14	15	0	14	0	13
	SNOPT	optimal	31	1.00e-02	1.88e-13	45	0	44	0	0
	WORHP IP	optimal	14	1.00e-02	1.86e-17	20	0	15	0	14
	WORHP IPm	optimal	14	1.00e-02	1.86e-17	20	0	15	0	14
	WORHP SQP	optimal	10	1.00e-02	2.71e-21	30	0	11	0	10
DENSCHNF	IPOPT	optimal	6	1.00e-02	6.51e-22	7	0	7	0	6
	KNITRO	optimal	6	1.00e-02	6.51e-22	8	0	7	0	6
	SNOPT	optimal	9	1.00e-02	1.03e-19	13	0	12	0	0
	WORHP IP	optimal	6	1.00e-02	6.51e-22	8	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	6.51e-22	8	0	7	0	6
	WORHP SQP	optimal	6	1.00e-02	6.51e-22	7	0	7	0	6
DIPIGRI	IPOPT	optimal	11	1.00e-02	6.81e+02	22	22	12	12	11
	KNITRO	optimal	7	1.00e-02	6.81e+02	13	14	8	9	7
	SNOPT	optimal	12	1.00e-02	6.81e+02	17	17	16	16	0
	WORHP IP	optimal	8	1.00e-02	6.81e+02	17	17	9	9	8
	WORHP IPm	optimal	8	1.00e-02	6.81e+02	17	17	9	9	8
	WORHP SQP	optimal	9	1.00e-02	6.81e+02	40	40	11	11	9
DISC2	IPOPT	optimal	30	1.00e-02	1.56e+00	90	180	31	62	30
	KNITRO	optimal	15	1.00e-02	1.56e+00	19	20	17	18	15
	SNOPT	optimal	26	1.00e-02	1.56e+00	1	55	1	54	0
	WORHP IP	optimal	37	1.00e-02	1.56e+00	78	78	38	38	37
	WORHP IPm	optimal	60	1.00e-02	1.56e+00	217	217	62	62	60
	WORHP SQP	optimal	20	3.00e-02	1.56e+00	24	24	22	22	20
DISCS	IPOPT	optimal	175	1.30e-01	1.20e+01	188	376	121	358	175
	KNITRO	optimal	38	2.00e-02	1.20e+01	47	48	40	41	38
	SNOPT	optimal	52	2.00e-02	1.44e+01	1	146	1	145	0
	WORHP IP	infeas	78	4.00e-02	1.38e+01	84	84	79	79	79
	WORHP IPm	infeas	108	5.00e-02	1.60e+01	143	143	109	109	109
	WORHP SQP	optimal	65	7.70e-01	1.53e+01	67	67	67	67	65
DITTERT	IPOPT	optimal	37	7.53e+00	-2.00e+00	38	38	38	38	37
	KNITRO	optimal	132	1.59e+01	-2.00e+00	154	155	134	135	133
	SNOPT	optimal	39	6.40e-01	-2.00e+00	42	42	41	41	0
	WORHP IP	optimal	29	5.88e+00	-2.00e+00	32	32	31	31	29
	WORHP IPm	minalpha	128	1.97e+01	1.00e+00	1198	1198	170	170	129
	WORHP SQP	optimal	9	4.05e+00	-2.00e+00	10	10	11	11	9
DIXCHLNG	IPOPT	optimal	10	1.00e-02	2.47e+03	11	11	11	11	10
	KNITRO	optimal	10	1.00e-02	2.47e+03	12	13	11	12	10
	SNOPT	optimal	29	1.00e-02	2.47e+03	32	32	31	31	0
	WORHP IP	optimal	11	1.00e-02	2.47e+03	55	55	13	13	11
	WORHP IPm	optimal	10	1.00e-02	2.47e+03	12	12	11	11	10
	WORHP SQP	optimal	11	1.00e-02	2.47e+03	12	12	13	13	11
DIXCHLNV	IPOPT	optimal	41	9.01e+00	0.00e+00	53	53	42	42	41
	KNITRO	optimal	21	4.40e+00	3.55e-21	34	35	22	23	21
	SNOPT	optimal	57	2.66e+01	1.59e-15	61	61	60	60	0
	WORHP IP	optimal	95	2.65e+01	2.35e-24	99	99	97	97	95
	WORHP IPm	smallstep	48	7.65e+00	0.00e+00	110	110	50	50	49
	WORHP SQP	optimal	28	9.62e+00	3.28e-15	29	29	30	30	28
DIXMAANA	IPOPT	optimal	7	6.00e-02	1.00e+00	8	0	8	0	7
	KNITRO	optimal	6	6.00e-02	1.00e+00	8	0	7	0	6
	SNOPT	toobig	39	1.24e+01	4.03e+03	43	0	42	0	0
	WORHP IP	optimal	7	5.00e-02	1.00e+00	9	0	8	0	7
	WORHP IPm	optimal	7	7.00e-02	1.00e+00	9	0	8	0	7
	WORHP SQP	optimal	6	5.00e-02	1.00e+00	7	0	7	0	6

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DIXMAANB	IPOPT	optimal	11	8.00e-02	1.00e+00	12	0	12	0	11
	KNITRO	optimal	7	6.00e-02	1.00e+00	9	0	8	0	7
	SNOPT	toobig	53	1.27e+01	1.20e+04	56	0	55	0	0
	WORHP IP	optimal	11	7.00e-02	1.00e+00	13	0	12	0	11
	WORHP IPm	optimal	11	8.00e-02	1.00e+00	13	0	12	0	11
	WORHP SQP	optimal	7	7.00e-02	1.00e+00	8	0	8	0	7
DIXMAANC	IPOPT	optimal	8	6.00e-02	1.00e+00	9	0	9	0	8
	KNITRO	optimal	8	6.00e-02	1.00e+00	10	0	9	0	8
	SNOPT	toobig	63	1.10e+01	2.01e+04	68	0	67	0	0
	WORHP IP	optimal	8	8.00e-02	1.00e+00	10	0	9	0	8
	WORHP IPm	optimal	8	6.00e-02	1.00e+00	10	0	9	0	8
	WORHP SQP	optimal	8	7.00e-02	1.00e+00	9	0	9	0	8
DIXMAAND	IPOPT	optimal	9	7.00e-02	1.00e+00	10	0	10	0	9
	KNITRO	optimal	9	6.00e-02	1.00e+00	11	0	10	0	9
	SNOPT	toobig	77	1.21e+01	3.76e+04	85	0	84	0	0
	WORHP IP	optimal	9	6.00e-02	1.00e+00	11	0	10	0	9
	WORHP IPm	optimal	9	7.00e-02	1.00e+00	11	0	10	0	9
	WORHP SQP	optimal	9	8.00e-02	1.00e+00	10	0	10	0	9
DIXMAANE	IPOPT	optimal	10	8.00e-02	1.00e+00	11	0	11	0	10
	KNITRO	optimal	11	7.00e-02	1.00e+00	17	0	12	0	11
	SNOPT	toobig	230	1.39e+01	6.95e+02	275	0	274	0	0
	WORHP IP	optimal	10	7.00e-02	1.00e+00	12	0	11	0	10
	WORHP IPm	optimal	10	7.00e-02	1.00e+00	12	0	11	0	10
	WORHP SQP	optimal	11	8.00e-02	1.00e+00	62	0	12	0	11
DIXMAANF	IPOPT	optimal	19	1.20e-01	1.00e+00	20	0	20	0	19
	KNITRO	optimal	21	1.30e-01	1.00e+00	27	0	22	0	21
	SNOPT	toobig	72	1.44e+01	9.26e+03	75	0	74	0	0
	WORHP IP	optimal	19	1.10e-01	1.00e+00	21	0	20	0	19
	WORHP IPm	optimal	19	1.20e-01	1.00e+00	21	0	20	0	19
	WORHP SQP	optimal	34	2.60e-01	1.00e+00	140	0	35	0	34
DIXMAANG	IPOPT	optimal	16	1.10e-01	1.00e+00	17	0	17	0	16
	KNITRO	optimal	17	1.00e-01	1.00e+00	19	0	18	0	17
	SNOPT	toobig	89	1.13e+01	1.77e+04	95	0	94	0	0
	WORHP IP	optimal	16	1.10e-01	1.00e+00	18	0	17	0	16
	WORHP IPm	optimal	16	1.00e-01	1.00e+00	18	0	17	0	16
	WORHP SQP	optimal	36	2.30e-01	1.00e+00	76	0	37	0	36
DIXMAANH	IPOPT	optimal	18	1.30e-01	1.00e+00	19	0	19	0	18
	KNITRO	optimal	19	1.20e-01	1.00e+00	21	0	20	0	19
	SNOPT	toobig	111	1.40e+01	3.60e+04	119	0	118	0	0
	WORHP IP	optimal	18	1.10e-01	1.00e+00	20	0	19	0	18
	WORHP IPm	optimal	18	1.10e-01	1.00e+00	20	0	19	0	18
	WORHP SQP	optimal	28	2.00e-01	1.00e+00	98	0	29	0	28
DIXMAANI	IPOPT	optimal	18	1.00e-01	1.00e+00	25	0	19	0	18
	KNITRO	optimal	16	8.00e-02	1.00e+00	18	0	17	0	16
	SNOPT	toobig	539	1.71e+01	1.67e+02	567	0	566	0	0
	WORHP IP	optimal	18	1.00e-01	1.00e+00	23	0	19	0	18
	WORHP IPm	optimal	18	1.20e-01	1.00e+00	23	0	19	0	18
	WORHP SQP	optimal	30	1.80e-01	1.00e+00	79	0	31	0	30
DIXMAANJ	IPOPT	optimal	20	1.40e-01	1.00e+00	21	0	21	0	20
	KNITRO	optimal	20	1.50e-01	1.00e+00	25	0	21	0	20
	SNOPT	toobig	100	1.55e+01	8.76e+03	103	0	102	0	0
	WORHP IP	optimal	20	1.30e-01	1.00e+00	22	0	21	0	20
	WORHP IPm	optimal	20	1.30e-01	1.00e+00	22	0	21	0	20
	WORHP SQP	optimal	31	2.10e-01	1.00e+00	39	0	32	0	31
DIXMAANK	IPOPT	optimal	24	1.60e-01	1.00e+00	37	0	25	0	24
	KNITRO	optimal	23	1.20e-01	1.00e+00	25	0	24	0	23
	SNOPT	toobig	133	1.54e+01	1.72e+04	142	0	141	0	0
	WORHP IP	optimal	24	1.70e-01	1.00e+00	32	0	25	0	24
	WORHP IPm	optimal	24	1.20e-01	1.00e+00	32	0	25	0	24
	WORHP SQP	optimal	50	3.00e-01	1.00e+00	80	0	51	0	50
DIXMAANL	IPOPT	optimal	26	1.80e-01	1.00e+00	27	0	27	0	26
	KNITRO	optimal	23	1.30e-01	1.00e+00	25	0	24	0	23
	SNOPT	toobig	159	1.23e+01	3.56e+04	165	0	164	0	0
	WORHP IP	optimal	26	1.60e-01	1.00e+00	28	0	27	0	26
	WORHP IPm	optimal	26	1.40e-01	1.00e+00	28	0	27	0	26
	WORHP SQP	optimal	55	3.60e-01	1.00e+00	224	0	56	0	55
DIXMAANM	IPOPT	optimal	11	9.00e-02	1.00e+00	12	0	12	0	11
	KNITRO	optimal	10	8.00e-02	1.00e+00	12	0	11	0	10
	SNOPT	toobig	67	1.01e+01	1.49e+02	82	0	81	0	0
	WORHP IP	optimal	11	9.00e-02	1.00e+00	13	0	12	0	11
	WORHP IPm	optimal	11	8.00e-02	1.00e+00	13	0	12	0	11
	WORHP SQP	optimal	26	1.50e-01	1.00e+00	60	0	27	0	26
DIXMAANN	IPOPT	optimal	24	1.60e-01	1.00e+00	33	0	25	0	24
	KNITRO	optimal	20	1.10e-01	1.00e+00	22	0	21	0	20
	SNOPT	toobig	92	1.06e+01	1.57e+03	97	0	96	0	0
	WORHP IP	optimal	24	1.50e-01	1.00e+00	31	0	25	0	24
	WORHP IPm	optimal	24	1.30e-01	1.00e+00	31	0	25	0	24
	WORHP SQP	optimal	42	3.20e-01	1.00e+00	253	0	43	0	42

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DIXMAANO	IPOPT	optimal	24	1.30e-01	1.00e+00	25	0	25	0	24
	KNITRO	optimal	26	1.60e-01	1.00e+00	30	0	27	0	26
	SNOPT	toobig	107	1.37e+01	2.95e+03	110	0	109	0	0
	WORHP IP	optimal	24	1.30e-01	1.00e+00	26	0	25	0	24
	WORHP IPm	optimal	24	1.30e-01	1.00e+00	26	0	25	0	24
	WORHP SQP	optimal	32	1.90e-01	1.00e+00	93	0	33	0	32
DIXMAANP	IPOPT	optimal	27	1.30e-01	1.00e+00	28	0	28	0	27
	KNITRO	optimal	25	1.60e-01	1.00e+00	32	0	26	0	25
	SNOPT	toobig	141	1.55e+01	5.93e+03	145	0	144	0	0
	WORHP IP	optimal	27	1.60e-01	1.00e+00	29	0	29	0	27
	WORHP IPm	optimal	27	1.50e-01	1.00e+00	29	0	28	0	27
	WORHP SQP	optimal	35	2.80e-01	1.00e+00	152	0	36	0	35
DIXON3DQ	IPOPT	optimal	1	6.00e-02	0.00e+00	2	0	2	0	1
	KNITRO	optimal	1	1.00e-01	8.03e-25	3	0	2	0	1
	SNOPT	sbasics	10000	1.80e+02	5.66e-03	11097	0	11096	0	0
	WORHP IP	optimal	1	8.00e-02	8.14e-25	3	0	2	0	1
	WORHP IPm	optimal	1	8.00e-02	8.14e-25	3	0	2	0	1
	WORHP SQP	optimal	11	1.70e-01	1.22e-03	12	0	12	0	11
DJTL	IPOPT	optimal	1524	3.20e-01	-8.95e+03	1651	0	1525	0	1524
	KNITRO	optimal	41	1.00e-02	-8.95e+03	228	0	42	0	41
	SNOPT	toobig	317	1.00e-02	-8.95e+03	1797	0	1796	0	0
	WORHP IP	optimal	1927	8.00e-02	-8.95e+03	5747	0	1929	0	1927
	WORHP IPm	optimal	1927	9.00e-02	-8.95e+03	5747	0	1928	0	1927
	WORHP SQP	optimal	6275	7.70e-01	-8.95e+03	302584	0	6276	0	6275
DMN15102	IPOPT	degree	0	1.35e+01	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	262	8.02e+01	0.00e+00	391	392	261	262	261
	SNOPT	degen	22	1.39e+02	0.00e+00	1	108	1	107	0
	WORHP IP	regular	1517	3.59e+02	0.00e+00	13431	13431	1518	1518	1518
	WORHP IPm	regular	1517	3.36e+02	0.00e+00	13431	13431	1518	1518	1518
	WORHP SQP	minalpha	46	1.75e+03	0.00e+00	3953	3959	30	48	29
DMN15102LS	IPOPT	maxtime	9513	1.81e+03	6.68e+02	11005	0	9514	0	9513
	KNITRO	maxtime	9193	1.81e+03	6.91e+03	9380	0	9194	0	9193
	SNOPT	maxiter	10000	1.67e+02	5.64e+03	12241	0	12240	0	0
	WORHP IP	maxtime	9721	1.82e+03	6.54e+02	10135	0	9723	0	9722
	WORHP IPm	maxtime	9756	1.81e+03	6.58e+02	10132	0	9757	0	9757
	WORHP SQP	maxtime	8732	1.81e+03	2.78e+02	26119	0	8734	0	8733
DMN15103	IPOPT	degree	0	1.82e+01	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	468	7.09e+02	0.00e+00	1111	1112	469	470	466
	SNOPT	noimpr	22	3.66e+02	0.00e+00	1	192	1	191	0
	WORHP IP	regular	2456	1.19e+03	0.00e+00	25720	25720	2457	2457	2457
	WORHP IPm	regular	2456	1.11e+03	0.00e+00	25720	25720	2457	2457	2457
	WORHP SQP	minalpha	43	1.61e+03	0.00e+00	4302	4308	29	45	28
DMN15103LS	IPOPT	maxtime	3633	1.82e+03	1.65e+02	12914	0	3634	0	3633
	KNITRO	noimpr	2684	1.25e+03	8.01e+01	4215	0	2685	0	2685
	SNOPT	maxiter	10000	1.36e+02	9.94e+01	11257	0	11256	0	0
	WORHP IP	maxtime	4042	1.82e+03	1.65e+02	10273	0	4044	0	4043
	WORHP IPm	maxtime	3979	1.82e+03	1.65e+02	5142	0	3980	0	3980
	WORHP SQP	maxtime	3835	1.82e+03	7.82e+01	9783	0	3836	0	3836
DMN15332	IPOPT	degree	0	1.32e+01	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	5806	1.81e+03	0.00e+00	9282	9283	5806	5807	5806
	SNOPT	noimpr	74	1.70e+02	0.00e+00	1	264	1	263	0
	WORHP IP	infeas	3122	6.37e+02	0.00e+00	24674	24674	3123	3123	3123
	WORHP IPm	infeas	3122	7.62e+02	0.00e+00	24674	24674	3123	3123	3123
	WORHP SQP	minalpha	36	1.64e+03	0.00e+00	3988	3994	30	38	29
DMN15332LS	IPOPT	maxtime	8122	1.81e+03	1.59e+02	8460	0	8123	0	8122
	KNITRO	maxtime	6736	1.81e+03	1.22e+02	7413	0	6737	0	6736
	SNOPT	maxiter	10000	1.09e+02	1.20e+03	12250	0	12249	0	0
	WORHP IP	maxtime	8774	1.82e+03	1.58e+02	9119	0	8776	0	8775
	WORHP IPm	maxtime	9409	1.82e+03	1.58e+02	9751	0	9410	0	9410
	WORHP SQP	maxtime	8852	1.81e+03	4.55e+02	21288	0	8854	0	8853
DMN15333	IPOPT	degree	0	2.26e+01	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	1055	7.29e+02	0.00e+00	1545	1546	1056	1057	1055
	SNOPT	noimpr	41	5.32e+02	0.00e+00	1	147	1	146	0
	WORHP IP	regular	929	5.11e+02	0.00e+00	6285	6285	930	930	930
	WORHP IPm	regular	929	3.73e+02	0.00e+00	6285	6285	930	930	930
	WORHP SQP	infeas	22	1.22e+03	0.00e+00	1134	1136	21	23	20
DMN15333LS	IPOPT	maxtime	3740	1.83e+03	8.88e+01	12293	0	3741	0	3740
	KNITRO	maxtime	3618	1.83e+03	9.62e+01	4584	0	3619	0	3619
	SNOPT	maxiter	10000	1.56e+02	7.84e+01	11213	0	11212	0	0
	WORHP IP	maxtime	3659	1.82e+03	8.97e+01	12196	0	3661	0	3660
	WORHP IPm	maxtime	2996	1.82e+03	8.97e+01	5722	0	2997	0	2997
	WORHP SQP	maxtime	4101	1.82e+03	1.01e+02	19051	0	4102	0	4102
DMN37142	IPOPT	degree	0	1.45e+01	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	5411	1.81e+03	0.00e+00	11848	11849	5412	5413	5411
	SNOPT	maxtime	299	1.81e+03	0.00e+00	1	1762	1	1761	0
	WORHP IP	infeas	2775	6.24e+02	0.00e+00	25076	25076	2776	2776	2776
	WORHP IPm	infeas	2775	6.27e+02	0.00e+00	25076	25076	2776	2776	2776
	WORHP SQP	infeas	6	1.02e+03	0.00e+00	1401	1404	4	7	3

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DMN37142LS	IPOPT	maxtime	7894	1.82e+03	1.75e+02	8662	0	7895	0	7894
	KNITRO	maxtime	9659	1.82e+03	1.84e+02	10091	0	9660	0	9659
	SNOPT	maxiter	10000	1.27e+02	2.58e+02	12518	0	12517	0	0
	WORHP IP	maxtime	7388	1.82e+03	1.75e+02	7712	0	7390	0	7389
	WORHP IPm	maxtime	8222	1.82e+03	1.75e+02	8626	0	8223	0	8223
	WORHP SQP	maxtime	7558	1.81e+03	1.17e+02	23331	0	7559	0	7559
DMN37143	IPOPT	degree	0	2.03e+01	0.00e+00	0	0	0	0	0
	KNITRO	maxtime	2626	1.82e+03	0.00e+00	5041	5042	2627	2628	2626
	SNOPT	maxtime	9817	1.83e+03	0.00e+00	1	30234	1	30233	0
	WORHP IP	maxtime	2228	1.05e+03	0.00e+00	9220	9220	2229	2229	2229
	WORHP IPm	maxtime	2068	1.07e+03	0.00e+00	8282	8282	2069	2069	2069
	WORHP SQP	minalpha	12	1.17e+03	0.00e+00	3449	3455	7	14	6
DMN37143LS	IPOPT	maxtime	3870	1.82e+03	9.79e+01	6917	0	3871	0	3870
	KNITRO	maxtime	4006	1.82e+03	9.71e+01	5396	0	4007	0	4006
	SNOPT	maxiter	10000	1.40e+02	1.03e+02	11440	0	11439	0	0
	WORHP IP	maxtime	4271	1.82e+03	9.55e+01	6043	0	4273	0	4272
	WORHP IPm	maxtime	3791	1.82e+03	9.87e+01	5134	0	3792	0	3792
	WORHP SQP	maxtime	4185	1.80e+03	8.58e+01	14064	0	4186	0	4186
DNIEPER	IPOPT	optimal	30	1.00e-02	1.87e+04	31	31	31	31	30
	KNITRO	optimal	29	1.00e-02	1.87e+04	31	32	30	31	29
	SNOPT	optimal	9	1.00e-02	1.87e+04	16	16	15	15	0
	WORHP IP	optimal	38	1.00e-02	1.87e+04	40	40	39	39	38
	WORHP IPm	optimal	40	1.00e-02	1.87e+04	46	46	45	45	40
	WORHP SQP	optimal	30	5.00e-02	1.87e+04	45	45	32	32	30
DQDRTIC	IPOPT	optimal	1	8.00e-02	5.92e-29	2	0	2	0	1
	KNITRO	optimal	1	8.00e-02	0.00e+00	3	0	2	0	1
	SNOPT	toobig	21	1.02e+01	5.42e+06	24	0	23	0	0
	WORHP IP	optimal	1	7.00e-02	5.92e-29	3	0	2	0	1
	WORHP IPm	optimal	1	8.00e-02	5.92e-29	3	0	2	0	1
	WORHP SQP	optimal	5	1.00e-01	1.20e-28	6	0	6	0	5
DQRTIC	IPOPT	optimal	34	1.30e-01	7.03e-07	35	0	35	0	34
	KNITRO	optimal	34	1.60e-01	7.03e-07	36	0	35	0	34
	SNOPT	toobig	104	6.99e+01	4.85e+16	120	0	119	0	0
	WORHP IP	optimal	34	1.70e-01	7.03e-07	36	0	36	0	34
	WORHP IPm	optimal	34	1.70e-01	7.24e-07	36	0	35	0	34
	WORHP SQP	optimal	54	1.50e-01	2.42e-06	55	0	55	0	54
DRCAV1LQ	IPOPT	optimal	49	8.06e+00	5.09e-08	103	0	50	0	49
	KNITRO	optimal	51	6.82e+00	4.13e-08	82	0	53	0	51
	SNOPT	sbasics	10000	2.19e+02	8.88e-05	11129	0	11128	0	0
	WORHP IP	optimal	49	8.23e+00	5.09e-08	85	0	50	0	49
	WORHP IPm	optimal	49	9.43e+00	5.09e-08	85	0	50	0	49
	WORHP SQP	optimal	37	6.98e+00	6.08e-08	44	0	38	0	37
DRCAV2LQ	IPOPT	optimal	102	1.50e+01	6.20e-08	252	0	103	0	102
	KNITRO	optimal	91	1.59e+01	6.68e-08	151	0	93	0	91
	SNOPT	sbasics	10000	2.11e+02	8.85e-05	11230	0	11229	0	0
	WORHP IP	optimal	102	1.73e+01	6.30e-08	183	0	103	0	102
	WORHP IPm	optimal	102	1.72e+01	6.30e-08	183	0	103	0	102
	WORHP SQP	optimal	83	1.59e+01	7.04e-08	360	0	84	0	83
DRCAV3LQ	IPOPT	optimal	483	7.61e+01	1.37e-06	1685	0	484	0	483
	KNITRO	optimal	449	8.06e+01	1.56e-06	839	0	451	0	449
	SNOPT	sbasics	10000	1.99e+02	3.31e-04	11198	0	11197	0	0
	WORHP IP	optimal	481	5.39e+01	1.37e-06	1043	0	482	0	481
	WORHP IPm	optimal	481	8.58e+01	1.37e-06	1043	0	482	0	481
	WORHP SQP	optimal	544	7.64e+01	4.16e-06	3134	0	545	0	544
DRCAVTY1	IPOPT	optimal	7	2.10e+00	0.00e+00	8	8	8	8	7
	KNITRO	optimal	7	1.52e+00	0.00e+00	14	15	9	10	7
	SNOPT	optimal	9	5.82e+00	0.00e+00	1	14	1	13	0
	WORHP IP	optimal	7	1.51e+00	0.00e+00	9	9	8	8	7
	WORHP IPm	optimal	7	1.58e+00	0.00e+00	9	9	8	8	7
	WORHP SQP	optimal	8	1.55e+01	0.00e+00	9	9	10	10	8
DRCAVTY2	IPOPT	optimal	11	3.91e+00	0.00e+00	22	22	12	12	11
	KNITRO	optimal	11	2.23e+00	0.00e+00	26	27	13	14	11
	SNOPT	optimal	35	1.95e+01	0.00e+00	1	58	1	57	0
	WORHP IP	optimal	11	2.31e+00	0.00e+00	23	23	12	12	11
	WORHP IPm	optimal	11	2.37e+00	0.00e+00	23	23	12	12	11
	WORHP SQP	minalpha	179	1.09e+02	0.00e+00	7899	7905	73	181	72
DRCAVTY3	IPOPT	infeas	264	1.32e+02	0.00e+00	373	384	11	267	265
	KNITRO	maxtime	6536	1.80e+03	0.00e+00	36054	36055	6538	6539	6536
	SNOPT	optimal	60	5.93e+01	0.00e+00	1	161	1	160	0
	WORHP IP	infeas	229	5.91e+01	0.00e+00	884	884	230	230	230
	WORHP IPm	infeas	229	5.32e+01	0.00e+00	884	884	230	230	230
	WORHP SQP	minalpha	125	1.17e+02	0.00e+00	7628	7634	36	127	35
DRUGDIS	IPOPT	optimal	108	1.72e+00	4.28e+00	112	112	109	109	108
	KNITRO	optimal	24	9.50e-01	4.28e+00	27	28	26	27	24
	SNOPT	optimal	33	5.27e+01	4.28e+00	1	47	1	46	0
	WORHP IP	optimal	39	1.23e+00	4.28e+00	84	84	41	41	39
	WORHP IPm	optimal	33	1.07e+00	4.28e+00	57	57	35	35	33
	WORHP SQP	optimal	79	1.55e+01	4.28e+00	1878	1880	78	80	77

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DRUGDISE	IPOPT	smallstep	127	2.60e-01	4.04e+02	285	285	128	129	128
	KNITRO	smallstep	297	1.88e+00	4.04e+02	909	910	299	300	296
	SNOPT	optimal	42	2.70e-01	4.36e+02	1	80	1	79	0
	WORHP IP	infeas	1144	3.49e+00	1.02e+02	6581	6581	1145	1145	1145
	WORHP IPm	infeas	1180	3.71e+00	9.72e+01	7305	7305	1183	1183	1181
	WORHP SQP	minalpha	35	7.70e-01	3.45e+01	4400	4406	11	37	10
DTOC1L	IPOPT	optimal	8	1.40e-01	3.94e+00	9	9	9	9	8
	KNITRO	optimal	8	1.50e-01	3.94e+00	10	11	9	10	8
	SNOPT	optimal	32	3.02e+01	3.94e+00	36	1	35	1	0
	WORHP IP	optimal	8	1.30e-01	3.94e+00	10	10	9	1	8
	WORHP IPm	optimal	8	1.40e-01	3.94e+00	10	10	9	1	8
	WORHP SQP	optimal	8	2.30e-01	3.94e+00	9	9	10	3	8
DTOC1NA	IPOPT	optimal	8	2.20e-01	4.14e+00	9	9	9	9	8
	KNITRO	optimal	8	2.30e-01	4.14e+00	10	11	9	10	8
	SNOPT	optimal	31	8.00e+01	4.14e+00	35	35	34	34	0
	WORHP IP	optimal	8	2.40e-01	4.14e+00	10	10	9	9	8
	WORHP IPm	optimal	8	2.40e-01	4.14e+00	10	10	9	9	8
	WORHP SQP	optimal	8	4.20e-01	4.14e+00	12	12	10	10	8
DTOC1NB	IPOPT	optimal	9	2.30e-01	7.14e+00	10	10	10	10	9
	KNITRO	optimal	9	2.40e-01	7.14e+00	11	12	10	11	9
	SNOPT	optimal	37	8.03e+01	7.14e+00	40	40	39	39	0
	WORHP IP	optimal	9	2.60e-01	7.14e+00	11	11	10	10	9
	WORHP IPm	optimal	9	2.70e-01	7.14e+00	11	11	10	10	9
	WORHP SQP	optimal	9	4.40e-01	7.14e+00	10	10	11	11	9
DTOC1NC	IPOPT	optimal	6	1.70e-01	3.52e+01	7	7	7	7	6
	KNITRO	optimal	6	2.10e-01	3.52e+01	8	9	7	8	6
	SNOPT	infeas	2	3.70e-01	1.39e+02	5	5	4	4	0
	WORHP IP	optimal	5	1.90e-01	3.52e+01	7	7	6	6	5
	WORHP IPm	optimal	5	2.00e-01	3.52e+01	7	7	6	6	5
	WORHP SQP	optimal	6	4.10e-01	3.52e+01	7	7	8	8	6
DTOC1ND	IPOPT	optimal	11	2.90e-01	4.76e+01	12	12	12	12	11
	KNITRO	optimal	12	3.50e-01	4.76e+01	15	16	13	14	12
	SNOPT	degen	14	8.76e+00	1.06e+02	44	44	43	43	0
	WORHP IP	optimal	5	2.00e-01	4.76e+01	7	7	6	6	5
	WORHP IPm	optimal	5	1.80e-01	4.76e+01	7	7	6	6	5
	WORHP SQP	optimal	11	5.40e-01	4.76e+01	12	12	13	13	11
DTOC2	IPOPT	optimal	10	2.00e-01	5.09e-01	13	13	11	11	10
	KNITRO	optimal	7	2.30e-01	5.09e-01	10	11	8	9	7
	SNOPT	optimal	101	1.57e+02	5.09e-01	126	126	125	125	0
	WORHP IP	optimal	18	3.70e-01	4.99e-01	20	20	19	19	18
	WORHP IPm	optimal	18	3.90e-01	4.99e-01	20	20	19	19	18
	WORHP SQP	optimal	46	1.17e+00	5.09e-01	47	47	48	48	46
DTOC3	IPOPT	optimal	1	6.00e-02	2.35e+02	2	2	2	2	1
	KNITRO	optimal	1	4.00e-02	2.35e+02	4	5	3	4	1
	SNOPT	optimal	15	4.55e+00	2.35e+02	20	1	19	1	0
	WORHP IP	optimal	5	6.00e-02	2.35e+02	7	7	6	1	5
	WORHP IPm	optimal	5	7.00e-02	2.35e+02	7	7	6	1	5
	WORHP SQP	optimal	6	1.40e-01	2.35e+02	7	7	8	3	6
DTOC4	IPOPT	optimal	3	6.00e-02	2.87e+00	4	4	4	4	3
	KNITRO	optimal	3	9.00e-02	2.87e+00	6	7	5	6	3
	SNOPT	optimal	15	1.11e+01	2.87e+00	18	18	17	17	0
	WORHP IP	optimal	4	9.00e-02	2.87e+00	6	6	5	5	4
	WORHP IPm	optimal	4	7.00e-02	2.87e+00	6	6	5	5	4
	WORHP SQP	optimal	4	1.50e-01	2.87e+00	5	5	6	6	4
DTOC5	IPOPT	optimal	3	1.10e-01	1.54e+00	4	4	4	4	3
	KNITRO	optimal	3	1.30e-01	1.54e+00	5	6	4	5	3
	SNOPT	toobig	26	4.96e+01	1.68e+00	29	29	28	28	0
	WORHP IP	optimal	4	1.20e-01	1.53e+00	6	6	5	5	4
	WORHP IPm	optimal	4	1.20e-01	1.53e+00	6	6	5	5	4
	WORHP SQP	optimal	13	4.50e-01	1.54e+00	14	14	15	15	13
DTOC6	IPOPT	optimal	11	2.40e-01	1.35e+05	12	12	12	12	11
	KNITRO	optimal	11	2.50e-01	1.35e+05	13	14	12	13	11
	SNOPT	maxtime	672	1.80e+03	4.38e+05	3038	3038	3037	3037	0
	WORHP IP	optimal	14	3.90e-01	1.35e+05	16	16	15	15	14
	WORHP IPm	optimal	14	3.80e-01	1.35e+05	16	16	15	15	14
	WORHP SQP	optimal	12	7.60e-01	1.35e+05	13	13	14	14	12
DUAL1	IPOPT	optimal	13	3.00e-02	3.50e-02	14	14	14	14	13
	KNITRO	optimal	15	2.00e-02	3.50e-02	18	19	17	18	15
	SNOPT	optimal	187	3.00e-02	3.50e-02	208	1	207	1	0
	WORHP IP	optimal	11	2.00e-02	3.50e-02	13	13	12	1	11
	WORHP IPm	optimal	11	3.00e-02	3.50e-02	14	14	13	1	11
	WORHP SQP	optimal	5	3.00e-02	3.50e-02	6	6	7	3	5
DUAL2	IPOPT	optimal	11	2.00e-02	3.37e-02	12	12	12	12	11
	KNITRO	optimal	13	2.00e-02	3.37e-02	16	17	15	16	13
	SNOPT	optimal	98	3.00e-02	3.37e-02	105	1	104	1	0
	WORHP IP	optimal	9	3.00e-02	3.37e-02	11	11	10	1	9
	WORHP IPm	optimal	9	3.00e-02	3.37e-02	12	12	11	1	9
	WORHP SQP	optimal	4	3.00e-02	3.37e-02	5	5	6	3	4

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
DUAL3	IPOPT	optimal	10	3.00e-02	1.36e-01	11	11	11	11	10
	KNITRO	optimal	14	4.00e-02	1.36e-01	17	18	16	17	14
	SNOPT	optimal	85	5.00e-02	1.36e-01	99	1	98	1	0
	WORHP IP	optimal	11	4.00e-02	1.36e-01	13	13	12	1	11
	WORHP IPm	optimal	10	4.00e-02	1.36e-01	13	13	12	1	10
	WORHP SQP	optimal	4	4.00e-02	1.36e-01	5	5	6	3	4
DUAL4	IPOPT	optimal	10	2.00e-02	7.46e-01	11	11	11	11	10
	KNITRO	optimal	12	2.00e-02	7.46e-01	15	16	14	15	12
	SNOPT	optimal	28	1.00e-02	7.46e-01	32	1	31	1	0
	WORHP IP	optimal	8	2.00e-02	7.46e-01	10	10	9	1	8
	WORHP IPm	optimal	8	1.00e-02	7.46e-01	11	11	10	1	8
	WORHP SQP	optimal	3	2.00e-02	7.46e-01	4	4	5	3	3
DUALC1	IPOPT	optimal	26	2.00e-02	6.16e+03	33	66	27	54	26
	KNITRO	optimal	17	1.00e-02	6.16e+03	21	22	19	20	17
	SNOPT	optimal	5	1.00e-02	6.16e+03	9	1	8	1	0
	WORHP IP	optimal	15	1.00e-02	6.16e+03	17	17	17	1	15
	WORHP IPm	optimal	23	2.00e-02	6.16e+03	29	29	28	1	23
	WORHP SQP	optimal	10	5.00e-02	6.16e+03	11	11	11	2	10
DUALC2	IPOPT	optimal	22	2.00e-02	3.55e+03	23	46	23	46	22
	KNITRO	optimal	11	1.00e-02	3.55e+03	14	15	13	14	11
	SNOPT	optimal	6	1.00e-02	3.55e+03	11	1	10	1	0
	WORHP IP	optimal	13	1.00e-02	3.55e+03	15	15	15	1	13
	WORHP IPm	optimal	21	1.00e-02	3.55e+03	27	27	26	1	21
	WORHP SQP	optimal	3	3.00e-02	3.55e+03	4	4	4	2	3
DUALC5	IPOPT	optimal	14	1.00e-02	4.27e+02	15	30	15	30	14
	KNITRO	optimal	8	1.00e-02	4.27e+02	11	12	10	11	8
	SNOPT	optimal	17	1.00e-02	4.27e+02	21	1	20	1	0
	WORHP IP	optimal	12	1.00e-02	4.27e+02	14	14	14	1	12
	WORHP IPm	optimal	17	1.00e-02	4.27e+02	22	22	21	1	17
	WORHP SQP	optimal	3	3.00e-02	4.27e+02	4	4	4	2	3
DUALC8	IPOPT	optimal	20	2.00e-02	1.83e+04	22	44	21	42	20
	KNITRO	optimal	10	1.00e-02	1.83e+04	13	14	12	13	10
	SNOPT	optimal	10	1.00e-02	1.83e+04	17	1	16	1	0
	WORHP IP	optimal	15	2.00e-02	1.83e+04	17	17	17	1	15
	WORHP IPm	optimal	23	3.00e-02	1.83e+04	29	29	28	1	23
	WORHP SQP	optimal	3	6.00e-02	1.83e+04	4	4	4	2	3
ECKERLE4	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	121	2.00e-02	0.00e+00	123	124	122	123	121
	SNOPT	infeas	32	1.00e-02	0.00e+00	1	44	1	43	0
	WORHP IP	infeas	20	1.00e-02	0.00e+00	27	27	21	21	21
	WORHP IPm	infeas	20	1.00e-02	0.00e+00	27	27	21	21	21
	WORHP SQP	minalpha	7	3.00e-02	0.00e+00	3782	3788	6	9	5
ECKERLE4LS	IPOPT	optimal	4	1.00e-02	7.00e-01	5	0	5	0	4
	KNITRO	optimal	4	1.00e-02	7.00e-01	6	0	5	0	4
	SNOPT	optimal	4	1.00e-02	7.00e-01	8	0	7	0	0
	WORHP IP	optimal	4	1.00e-02	7.00e-01	6	0	5	0	4
	WORHP IPm	optimal	4	1.00e-02	7.00e-01	6	0	5	0	4
	WORHP SQP	optimal	5	1.00e-02	7.00e-01	7	0	6	0	5
EDENSCH	IPOPT	optimal	12	7.00e-02	1.20e+04	13	0	13	0	12
	KNITRO	optimal	12	6.00e-02	1.20e+04	14	0	13	0	12
	SNOPT	optimal	51	7.05e+01	1.20e+04	57	0	56	0	0
	WORHP IP	optimal	12	6.00e-02	1.20e+04	14	0	13	0	12
	WORHP IPm	optimal	12	5.00e-02	1.20e+04	14	0	13	0	12
	WORHP SQP	optimal	14	8.00e-02	1.20e+04	15	0	15	0	14
EG1	IPOPT	optimal	7	1.00e-02	-1.43e+00	8	0	8	0	7
	KNITRO	optimal	5	1.00e-02	-1.43e+00	10	0	7	0	5
	SNOPT	optimal	9	1.00e-02	-1.13e+00	14	0	13	0	0
	WORHP IP	optimal	7	1.00e-02	-1.43e+00	9	0	8	0	7
	WORHP IPm	optimal	6	1.00e-02	-1.43e+00	9	0	7	0	6
	WORHP SQP	optimal	5	1.00e-02	-1.43e+00	10	0	6	0	5
EG2	IPOPT	optimal	4	2.00e-02	-9.99e+02	5	0	5	0	4
	KNITRO	optimal	3	1.00e-02	-9.99e+02	5	0	4	0	3
	SNOPT	optimal	4	1.00e-02	-9.99e+02	8	0	7	0	0
	WORHP IP	optimal	4	1.00e-02	-9.99e+02	6	0	5	0	4
	WORHP IPm	optimal	4	2.00e-02	-9.99e+02	6	0	5	0	4
	WORHP SQP	optimal	3	1.00e-02	-9.99e+02	4	0	4	0	3
EG3	IPOPT	optimal	173	1.20e+01	1.56e-05	299	598	151	350	173
	KNITRO	infeas	187	2.24e+01	9.52e+13	190	191	188	189	188
	SNOPT	optimal	19	3.74e+01	2.37e-01	23	23	22	22	0
	WORHP IP	optimal	36	3.68e+00	1.28e-01	39	39	37	37	36
	WORHP IPm	optimal	53	6.45e+00	1.28e-01	73	73	54	54	53
	WORHP SQP	maxtime	4133	1.76e+03	2.16e-01	68466	67317	4078	5305	4077
EIGENA	IPOPT	optimal	29	1.12e+02	0.00e+00	34	34	5	31	29
	KNITRO	maxtime	670	1.80e+03	0.00e+00	916	917	672	673	671
	SNOPT	optimal	5	1.11e+00	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	20	1.04e+02	0.00e+00	71	71	22	22	20
	WORHP IPm	optimal	17	9.04e+01	0.00e+00	27	27	20	20	17
	WORHP SQP	optimal	7	4.70e+02	0.00e+00	8	8	9	9	7

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
EIGENA2	IPOPT	optimal	3	2.03e+00	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.17e+01	5.07e-29	5	6	4	5	3
	SNOPT	optimal	5	2.80e-01	4.99e-25	8	8	7	7	0
	WORHP IP	optimal	2	1.98e+01	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	2.31e+01	0.00e+00	4	4	3	3	2
	WORHP SQP	maxtime	139	1.75e+03	3.21e-10	2327	2328	141	141	140
EIGENACO	IPOPT	optimal	3	7.13e+00	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.34e+01	4.75e-28	5	6	4	5	3
	SNOPT	optimal	5	1.07e+00	3.93e-25	8	8	7	7	0
	WORHP IP	optimal	3	6.09e+01	7.21e-28	5	5	4	4	3
	WORHP IPm	optimal	3	5.70e+01	7.21e-28	5	5	4	4	3
	WORHP SQP	optimal	13	3.00e+02	3.34e-14	28	28	15	15	13
EIGENALS	IPOPT	optimal	94	7.22e+01	2.52e-28	151	0	95	0	94
	KNITRO	optimal	110	1.04e+02	6.02e-28	127	0	111	0	110
	SNOPT	optimal	2278	2.69e+01	2.11e-12	2396	0	2395	0	0
	WORHP IP	optimal	94	6.05e+02	2.52e-28	119	0	96	0	94
	WORHP IPm	optimal	94	6.00e+02	2.52e-28	119	0	95	0	94
	WORHP SQP	maxtime	2462	1.74e+03	1.12e+03	2468	0	2463	0	2463
EIGENAU	IPOPT	optimal	1	4.98e+00	0.00e+00	2	2	2	2	1
	KNITRO	optimal	1	2.28e+01	0.00e+00	3	4	2	3	1
	SNOPT	optimal	5	1.05e+00	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	4	2.70e+01	0.00e+00	6	6	5	5	4
	WORHP IPm	optimal	4	3.18e+01	0.00e+00	6	6	5	5	4
	WORHP SQP	optimal	5	2.35e+02	0.00e+00	6	6	7	7	5
EIGENB	IPOPT	infeas	103	6.25e+02	0.00e+00	137	137	12	106	104
	KNITRO	optimal	239	6.54e+02	0.00e+00	1275	1276	240	241	239
	SNOPT	infeas	8	2.59e+00	0.00e+00	1	12	1	11	0
	WORHP IP	maxtime	210	1.72e+03	0.00e+00	814	814	211	211	211
	WORHP IPm	maxtime	261	1.71e+03	0.00e+00	1185	1185	262	262	262
	WORHP SQP	maxtime	31	1.69e+03	0.00e+00	1394	1395	32	33	31
EIGENB2	IPOPT	optimal	117	5.76e+02	2.28e-15	122	123	118	118	117
	KNITRO	optimal	1	1.02e+01	9.80e+01	3	4	2	3	1
	SNOPT	optimal	3	2.20e-01	9.80e+01	6	6	5	5	0
	WORHP IP	optimal	128	1.67e+03	1.76e-17	148	148	129	129	128
	WORHP IPm	maxtime	92	1.78e+03	2.45e+00	94	94	93	93	93
	WORHP SQP	optimal	3	3.54e+01	9.80e+01	4	4	5	5	3
EIGENBCO	IPOPT	maxtime	165	1.80e+03	4.30e-05	246	248	166	166	165
	KNITRO	optimal	1	9.82e+00	4.90e+01	3	4	2	3	1
	SNOPT	optimal	3	1.01e+00	4.90e+01	6	6	5	5	0
	WORHP IP	maxtime	153	1.78e+03	4.26e-05	226	226	154	154	154
	WORHP IPm	maxtime	150	1.78e+03	5.44e-05	221	221	151	151	151
	WORHP SQP	optimal	3	4.45e+01	4.90e+01	4	4	5	5	3
EIGENBLS	IPOPT	optimal	562	1.65e+03	1.20e-12	1538	0	563	0	562
	KNITRO	optimal	608	1.74e+03	6.27e-12	880	0	609	0	608
	SNOPT	sbasics	10000	3.27e+02	4.69e-02	11191	0	11190	0	0
	WORHP IP	maxtime	448	1.78e+03	4.93e-07	807	0	449	0	449
	WORHP IPm	maxtime	497	1.78e+03	1.00e-07	888	0	498	0	498
	WORHP SQP	maxtime	410	1.66e+03	3.61e-07	1971	0	411	0	411
EIGENC	IPOPT	optimal	13	1.94e+02	0.00e+00	36	38	14	14	13
	KNITRO	optimal	8	5.52e+01	0.00e+00	17	18	9	10	8
	SNOPT	infeas	95	1.42e+03	0.00e+00	1	194	1	193	0
	WORHP IP	optimal	240	1.02e+03	0.00e+00	1908	1908	241	241	240
	WORHP IPm	optimal	240	1.02e+03	0.00e+00	1908	1908	241	241	240
	WORHP SQP	maxtime	1	1.70e+03	0.00e+00	668	668	3	3	2
EIGENC2	IPOPT	optimal	14	5.95e+01	1.94e-18	15	15	15	15	14
	KNITRO	optimal	15	6.25e+01	4.71e-20	17	18	16	17	15
	SNOPT	maxtime	1571	1.80e+03	3.51e-06	1835	1835	1834	1834	0
	WORHP IP	optimal	15	1.76e+02	1.62e-15	19	19	17	17	15
	WORHP IPm	optimal	15	1.65e+02	1.64e-22	19	19	16	16	15
	WORHP SQP	optimal	24	4.10e+02	1.39e-15	108	108	26	26	24
EIGENCCO	IPOPT	optimal	47	3.50e+02	8.72e-25	99	100	48	48	47
	KNITRO	optimal	31	2.81e+02	1.14e-15	40	41	32	33	31
	SNOPT	optimal	1604	1.80e+03	3.14e-12	1857	1857	1856	1856	0
	WORHP IP	optimal	35	3.20e+02	1.09e-18	51	51	36	36	35
	WORHP IPm	optimal	35	2.37e+02	1.09e-18	51	51	36	36	35
	WORHP SQP	optimal	51	1.57e+03	2.57e-19	285	285	53	53	51
EIGENCLS	IPOPT	maxtime	219	1.81e+03	2.87e+03	253	0	220	0	219
	KNITRO	optimal	429	1.60e+03	4.81e-18	619	0	430	0	429
	SNOPT	sbasics	10000	2.29e+02	5.31e+00	11196	0	11195	0	0
	WORHP IP	maxtime	222	1.79e+03	2.87e+03	239	0	223	0	223
	WORHP IPm	maxtime	213	1.79e+03	2.87e+03	230	0	214	0	214
	WORHP SQP	maxtime	339	1.66e+03	3.32e+02	947	0	340	0	340
EIGMAXA	IPOPT	optimal	21	1.00e-02	-1.00e+00	26	26	9	23	21
	KNITRO	optimal	18	1.00e-02	-1.00e+00	21	22	20	21	18
	SNOPT	optimal	1	1.00e-02	-1.00e+00	1	4	1	3	0
	WORHP IP	optimal	8	1.00e-02	-1.00e+00	10	10	9	9	8
	WORHP IPm	optimal	8	1.00e-02	-1.00e+00	11	11	10	10	8
	WORHP SQP	optimal	6	3.00e-02	-1.00e+00	118	117	5	8	4

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
EIGMAXB	IPOPT	optimal	8	1.00e-02	-9.67e-04	9	9	9	9	8
	KNITRO	optimal	10	1.00e-02	-4.72e-02	30	31	12	13	10
	SNOPT	maxiter	10000	3.09e+00	-3.90e-01	1	40000	1	39999	0
	WORHP IP	optimal	12	1.00e-02	-9.64e-01	14	14	13	13	12
	WORHP IPm	optimal	13	1.00e-02	-9.67e-04	29	29	14	14	13
	WORHP SQP	smallstep	37	1.40e-01	-5.75e-01	738	738	38	38	37
EIGMAXC	IPOPT	optimal	8	1.00e-02	-1.00e+00	9	9	9	9	8
	KNITRO	optimal	13	2.00e-02	1.00e+00	20	21	15	16	13
	SNOPT	optimal	9	1.00e-02	-1.00e+00	1	17	1	16	0
	WORHP IP	optimal	7	1.00e-02	-1.00e+00	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	-1.00e+00	10	10	9	9	7
	WORHP SQP	zerostep	27	9.00e-02	-1.00e+00	584	584	28	28	27
EIGMINA	IPOPT	optimal	21	1.00e-02	1.00e+00	26	26	9	23	21
	KNITRO	optimal	18	1.00e-02	1.00e+00	21	22	20	21	18
	SNOPT	optimal	1	1.00e-02	1.00e+00	1	4	1	3	0
	WORHP IP	optimal	8	1.00e-02	1.00e+00	10	10	9	9	8
	WORHP IPm	optimal	9	1.00e-02	1.00e+00	12	12	11	11	9
	WORHP SQP	optimal	6	3.00e-02	1.00e+00	118	117	5	8	4
EIGMINB	IPOPT	optimal	8	1.00e-02	9.67e-04	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	9.67e-04	10	11	8	9	6
	SNOPT	maxiter	10000	3.08e+00	1.42e-01	1	39998	1	39997	0
	WORHP IP	optimal	16	1.00e-02	9.67e-04	18	18	17	17	16
	WORHP IPm	optimal	8	1.00e-02	9.67e-04	10	10	9	9	8
	WORHP SQP	optimal	15	6.00e-02	3.39e-01	367	368	16	16	15
EIGMINC	IPOPT	optimal	8	1.00e-02	1.00e+00	9	9	9	9	8
	KNITRO	optimal	13	2.00e-02	-1.00e+00	17	18	15	16	13
	SNOPT	optimal	9	1.00e-02	1.00e+00	1	17	1	16	0
	WORHP IP	optimal	7	1.00e-02	1.00e+00	9	9	8	8	7
	WORHP IPm	optimal	9	1.00e-02	1.00e+00	12	12	11	11	9
	WORHP SQP	optimal	9	8.00e-02	1.00e+00	120	121	10	10	9
ELATTAR	IPOPT	optimal	279	2.10e-01	1.05e+00	404	412	153	290	279
	KNITRO	optimal	475	2.10e-01	1.43e-01	1010	1011	476	477	475
	SNOPT	optimal	490	2.20e-01	1.43e-01	1	1928	1	1927	0
	WORHP IP	optimal	33	2.00e-02	7.42e+01	43	43	34	34	33
	WORHP IPm	optimal	826	5.40e-01	1.43e-01	2386	2386	827	827	826
	WORHP SQP	optimal	360	1.01e+00	8.95e-01	14473	14474	143	362	141
ELEC	IPOPT	optimal	315	6.24e+01	1.84e+04	389	389	316	316	315
	KNITRO	optimal	98	2.44e+01	1.84e+04	376	377	99	100	98
	SNOPT	optimal	718	2.25e+01	1.84e+04	820	820	819	819	0
	WORHP IP	optimal	263	6.00e+01	1.84e+04	276	276	265	265	263
	WORHP IPm	optimal	262	5.61e+01	1.84e+04	275	275	263	263	262
	WORHP SQP	optimal	549	1.26e+02	1.84e+04	2527	2492	509	586	507
ENGVAL1	IPOPT	optimal	8	1.00e-01	5.55e+03	9	0	9	0	8
	KNITRO	optimal	8	1.30e-01	5.55e+03	10	0	9	0	8
	SNOPT	toobig	47	4.45e+01	1.79e+05	50	0	49	0	0
	WORHP IP	optimal	8	1.00e-01	5.55e+03	10	0	9	0	8
	WORHP IPm	optimal	8	1.20e-01	5.55e+03	10	0	9	0	8
	WORHP SQP	optimal	7	1.30e-01	5.55e+03	8	0	8	0	7
ENGVAL2	IPOPT	optimal	21	1.00e-02	1.70e-20	33	0	22	0	21
	KNITRO	optimal	16	1.00e-02	6.47e-19	21	0	17	0	16
	SNOPT	optimal	27	1.00e-02	5.22e-18	35	0	34	0	0
	WORHP IP	optimal	21	1.00e-02	1.70e-20	31	0	22	0	21
	WORHP IPm	optimal	21	1.00e-02	1.70e-20	31	0	22	0	21
	WORHP SQP	optimal	22	1.00e-02	4.60e-14	28	0	23	0	22
ENSO	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	250	3.00e-01	0.00e+00	274	275	251	252	250
	SNOPT	infeas	20	3.00e-02	0.00e+00	1	40	1	39	0
	WORHP IP	infeas	13	2.00e-02	0.00e+00	19	19	14	14	14
	WORHP IPm	infeas	13	2.00e-02	0.00e+00	19	19	14	14	14
	WORHP SQP	infeas	6	3.00e-01	0.00e+00	1118	1120	3	7	2
ENSOLS	IPOPT	optimal	7	1.00e-02	7.89e+02	19	0	8	0	7
	KNITRO	optimal	9	1.00e-02	7.89e+02	16	0	10	0	9
	SNOPT	optimal	26	1.00e-02	7.89e+02	31	0	30	0	0
	WORHP IP	optimal	7	1.00e-02	7.89e+02	14	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	7.89e+02	14	0	8	0	7
	WORHP SQP	optimal	11	2.00e-02	7.89e+02	18	0	12	0	11
EQC	IPOPT	resto	17	1.00e-02	-8.63e+02	54	54	19	19	18
	KNITRO	optimal	16	1.00e-02	-1.01e+03	23	24	18	19	16
	SNOPT	optimal	0	1.00e-02	-8.28e+02	3	1	2	1	0
	WORHP IP	minalpha	31	1.00e-02	-9.99e+02	346	346	73	1	32
	WORHP IPm	smallstep	8	1.00e-02	-8.30e+02	12	12	10	1	9
	WORHP SQP	zerostep	196	3.00e-02	-8.30e+02	196	196	197	3	196
ERRINBAR	IPOPT	optimal	45	2.00e-02	2.80e+01	66	132	42	94	45
	KNITRO	optimal	29	1.00e-02	2.80e+01	34	35	31	32	29
	SNOPT	optimal	1752	1.30e-01	2.80e+01	1	6887	1	6886	0
	WORHP IP	optimal	29	1.00e-02	2.80e+01	31	31	30	30	29
	WORHP IPm	optimal	30	1.00e-02	2.80e+01	37	37	35	35	30
	WORHP SQP	optimal	31	2.00e-02	2.80e+01	32	32	33	33	31

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ERRINROS	IPOPT	optimal	29	1.00e-02	4.04e+01	71	0	30	0	29
	KNITRO	optimal	26	1.00e-02	4.04e+01	55	0	27	0	26
	SNOPT	optimal	261	2.00e-02	3.99e+01	268	0	267	0	0
	WORHP IP	optimal	29	1.00e-02	4.04e+01	51	0	31	0	29
	WORHP IPm	optimal	29	1.00e-02	4.04e+01	51	0	30	0	29
	WORHP SQP	optimal	53	1.00e-02	4.04e+01	204	0	54	0	53
ERRINRSM	IPOPT	optimal	40	1.00e-02	3.85e+01	96	0	41	0	40
	KNITRO	optimal	42	1.00e-02	3.85e+01	70	0	43	0	42
	SNOPT	optimal	330	3.00e-02	3.77e+01	350	0	349	0	0
	WORHP IP	optimal	40	1.00e-02	3.85e+01	112	0	42	0	40
	WORHP IPm	optimal	40	1.00e-02	3.85e+01	112	0	41	0	40
	WORHP SQP	optimal	91	1.00e-02	3.77e+01	500	0	92	0	91
EXPFIT	IPOPT	optimal	8	1.00e-02	2.41e-01	9	0	9	0	8
	KNITRO	optimal	7	1.00e-02	2.41e-01	10	0	8	0	7
	SNOPT	optimal	10	1.00e-02	2.41e-01	19	0	18	0	0
	WORHP IP	optimal	8	1.00e-02	2.41e-01	10	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	2.41e-01	10	0	9	0	8
	WORHP SQP	optimal	13	1.00e-02	2.41e-01	28	0	14	0	13
EXPFITA	IPOPT	optimal	28	1.00e-02	1.14e-03	30	30	29	29	28
	KNITRO	optimal	19	1.00e-02	1.14e-03	21	22	20	21	19
	SNOPT	optimal	23	1.00e-02	1.14e-03	27	1	26	1	0
	WORHP IP	optimal	25	1.00e-02	1.14e-03	27	27	26	1	25
	WORHP IPm	optimal	20	1.00e-02	1.14e-03	23	23	21	1	20
	WORHP SQP	optimal	12	1.00e-02	1.14e-03	13	13	14	3	12
EXPFITB	IPOPT	optimal	33	2.00e-02	5.02e-03	34	34	34	34	33
	KNITRO	optimal	18	1.00e-02	5.02e-03	20	21	19	20	18
	SNOPT	optimal	28	1.00e-02	5.02e-03	34	1	33	1	0
	WORHP IP	optimal	43	1.00e-02	5.02e-03	46	46	45	1	43
	WORHP IPm	optimal	29	1.00e-02	5.02e-03	33	33	30	1	29
	WORHP SQP	optimal	14	2.00e-02	5.02e-03	15	15	16	3	14
EXPFITC	IPOPT	optimal	46	5.00e-02	2.33e-02	49	49	47	47	46
	KNITRO	optimal	18	2.00e-02	2.33e-02	20	21	19	20	18
	SNOPT	optimal	31	2.00e-02	2.33e-02	35	1	34	1	0
	WORHP IP	fritzjohn	14	1.40e-01	5.94e+01	16	16	15	1	15
	WORHP IPm	optimal	64	1.00e-01	2.33e-02	78	78	65	1	64
	WORHP SQP	optimal	13	9.00e-02	2.33e-02	14	14	15	3	13
EXPLIN	IPOPT	optimal	57	7.00e-02	-7.19e+07	68	0	58	0	57
	KNITRO	optimal	62	1.40e-01	-7.19e+07	65	0	64	0	62
	SNOPT	optimal	83	5.00e-02	-7.19e+07	112	0	111	0	0
	WORHP IP	optimal	56	1.00e-01	-7.19e+07	60	0	58	0	56
	WORHP IPm	optimal	54	1.20e-01	-7.19e+07	60	0	59	0	54
	WORHP SQP	minalpha	280	1.39e+00	-7.19e+07	20366	0	281	0	281
EXPLIN2	IPOPT	optimal	25	3.00e-02	-7.20e+07	26	0	26	0	25
	KNITRO	optimal	13	3.00e-02	-7.20e+07	16	0	15	0	13
	SNOPT	optimal	172	5.00e-02	-7.20e+07	207	0	206	0	0
	WORHP IP	optimal	16	3.00e-02	-7.20e+07	18	0	18	0	16
	WORHP IPm	optimal	32	6.00e-02	-7.20e+07	38	0	37	0	32
	WORHP SQP	maxiter	10000	6.30e+01	-7.20e+07	1086392	0	10001	0	10001
EXPQUAD	IPOPT	optimal	32	5.00e-02	-3.68e+09	33	0	33	0	32
	KNITRO	optimal	15	3.00e-02	-3.68e+09	22	0	17	0	15
	SNOPT	optimal	78	4.51e+00	-3.68e+09	103	0	102	0	0
	WORHP IP	optimal	24	4.00e-02	-3.68e+09	27	0	25	0	24
	WORHP IPm	optimal	29	6.00e-02	-3.68e+09	35	0	33	0	29
	WORHP SQP	maxiter	10000	1.45e+02	-3.68e+09	2343585	0	10001	0	10001
EXTRASIM	IPOPT	optimal	5	1.00e-02	1.00e+00	6	6	6	6	5
	KNITRO	optimal	3	1.00e-02	1.00e+00	6	7	5	6	3
	SNOPT	optimal	0	1.00e-02	-1.00e+00	1	1	1	1	0
	WORHP IP	optimal	4	1.00e-02	1.00e+00	6	6	5	1	4
	WORHP IPm	optimal	1	1.00e-02	1.00e+00	3	3	2	1	1
	WORHP SQP	optimal	2	1.00e-02	1.00e+00	2	2	3	3	2
EXTROSNB	IPOPT	optimal	3211	3.91e+00	8.56e-10	8426	0	3212	0	3211
	KNITRO	optimal	3088	2.36e+00	9.63e-10	4450	0	3089	0	3088
	SNOPT	maxiter	10000	6.70e+01	1.93e-05	10743	0	10742	0	0
	WORHP IP	optimal	3116	2.40e+00	9.38e-10	5531	0	3118	0	3116
	WORHP IPm	optimal	3056	2.47e+00	9.97e-10	5361	0	3057	0	3056
	WORHP SQP	optimal	3104	3.21e+00	9.06e-10	20094	0	3105	0	3104
FBRAIN	IPOPT	degree	0	1.10e-01	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	25	4.00e-01	0.00e+00	33	34	25	26	23
	SNOPT	infeas	26	4.20e-01	0.00e+00	1	53	1	52	0
	WORHP IP	infeas	5	1.50e-01	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.50e-01	0.00e+00	7	7	6	6	6
	WORHP SQP	minalpha	211	1.39e+01	0.00e+00	14658	14664	208	213	207
FBRAIN2	IPOPT	degree	0	2.20e-01	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	230	6.41e+00	0.00e+00	1052	1053	230	231	229
	SNOPT	infeas	3208	1.00e+02	0.00e+00	1	25896	1	25895	0
	WORHP IP	infeas	29	8.10e-01	0.00e+00	89	89	30	30	30
	WORHP IPm	infeas	29	7.60e-01	0.00e+00	89	89	30	30	30
	WORHP SQP	minalpha	14	5.15e+00	0.00e+00	2667	2673	11	16	10

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
FBRAIN2LS	IPOPT	optimal	14	3.70e-01	3.68e-01	25	0	15	0	14
	KNITRO	optimal	10	3.50e-01	3.68e-01	13	0	11	0	10
	SNOPT	optimal	35	3.90e-01	3.68e-01	41	0	40	0	0
	WORHP IP	optimal	13	3.80e-01	3.68e-01	16	0	14	0	13
	WORHP IPm	optimal	23	4.90e-01	3.68e-01	25	0	24	0	23
	WORHP SQP	optimal	11	3.50e-01	3.68e-01	14	0	12	0	11
FBRAIN3	IPOPT	degree	0	3.70e-01	0.00e+00	0	0	0	0	0
	KNITRO	maxiter	10000	3.40e+02	0.00e+00	22344	22345	10001	10002	10000
	SNOPT	maxiter	10000	3.55e+02	0.00e+00	1	98007	1	98006	0
	WORHP IP	infeas	689	2.81e+01	0.00e+00	5440	5440	690	690	690
	WORHP IPm	infeas	689	2.84e+01	0.00e+00	5440	5440	690	690	690
	WORHP SQP	minalpha	56	8.55e+00	0.00e+00	4384	4391	40	58	39
FBRAIN3LS	IPOPT	maxiter	10000	1.01e+02	2.41e-01	27074	0	10001	0	10000
	KNITRO	noimpr	716	1.05e+01	2.43e-01	1677	0	717	0	717
	SNOPT	optimal	3365	2.23e+01	2.42e-01	3772	0	3771	0	0
	WORHP IP	maxiter	10000	1.43e+02	2.41e-01	19703	0	10001	0	10000
	WORHP IPm	maxiter	10000	1.44e+02	2.41e-01	19703	0	10001	0	10000
	WORHP SQP	maxiter	10000	1.84e+02	2.42e-01	19803	0	10001	0	10001
FBRAINLS	IPOPT	optimal	8	1.40e-01	4.17e-01	9	0	9	0	8
	KNITRO	optimal	6	1.40e-01	4.17e-01	8	0	7	0	6
	SNOPT	optimal	12	1.40e-01	4.17e-01	20	0	19	0	0
	WORHP IP	optimal	8	1.60e-01	4.17e-01	10	0	9	0	8
	WORHP IPm	optimal	7	1.40e-01	4.17e-01	9	0	8	0	7
	WORHP SQP	optimal	7	1.50e-01	4.17e-01	8	0	8	0	7
FCCU	IPOPT	optimal	8	1.00e-02	1.11e+01	9	9	9	9	8
	KNITRO	optimal	7	1.00e-02	1.11e+01	9	10	8	9	7
	SNOPT	optimal	16	1.00e-02	1.11e+01	19	1	18	1	0
	WORHP IP	optimal	7	1.00e-02	1.11e+01	9	9	8	1	7
	WORHP IPm	optimal	6	1.00e-02	1.11e+01	8	8	7	1	6
	WORHP SQP	optimal	3	1.00e-02	1.11e+01	4	4	5	3	3
FEEDLOC	IPOPT	optimal	32	4.00e-02	1.77e-07	45	90	33	66	32
	KNITRO	optimal	16	2.00e-02	2.13e-10	23	24	18	19	16
	SNOPT	optimal	3	1.00e-02	0.00e+00	1	6	1	5	0
	WORHP IP	optimal	20	3.00e-02	4.39e-07	33	33	21	21	20
	WORHP IPm	optimal	30	4.00e-02	1.36e-08	44	44	31	31	30
	WORHP SQP	optimal	8	2.50e-01	0.00e+00	9	9	9	9	8
FERRISDC	IPOPT	optimal	49	5.38e+01	-2.15e-04	50	50	50	50	49
	KNITRO	optimal	15	1.01e+01	-2.31e-04	18	19	17	18	15
	SNOPT	optimal	0	6.90e-01	0.00e+00	3	1	2	1	0
	WORHP IP	optimal	29	4.80e+01	-2.29e-04	31	31	30	1	29
	WORHP IPm	optimal	43	8.76e+01	-2.29e-04	50	50	44	1	43
	WORHP SQP	optimal	3	1.75e+01	-1.68e-07	4	4	5	3	3
FIVE20B	IPOPT	degree	0	4.20e-01	0.00e+00	0	0	0	0	0
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	sbasics	220	1.66e+03	2.93e+04	242	1	241	1	0
	WORHP IP	killed	-	-	-	-	-	-	-	-
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	maxtime	0	1.70e+03	1.22e+04	1	1	2	2	1
FIVE20C	IPOPT	degree	0	5.20e-01	0.00e+00	0	0	0	0	0
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	maxtime	0	1.82e+03	3.45e+04	1	1	1	1	0
	WORHP IP	killed	-	-	-	-	-	-	-	-
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	maxtime	0	1.70e+03	1.36e+04	1	1	2	2	1
FLETBV3M	IPOPT	optimal	234	2.40e+00	-2.25e+05	240	0	235	0	234
	KNITRO	optimal	260	2.10e+00	-2.23e+05	265	0	261	0	260
	SNOPT	toobig	61	1.99e+01	-9.98e+04	80	0	79	0	0
	WORHP IP	optimal	234	3.11e+00	-2.25e+05	238	0	235	0	234
	WORHP IPm	optimal	234	3.03e+00	-2.25e+05	238	0	235	0	234
	WORHP SQP	optimal	444	8.01e+00	-2.10e+05	1248	0	445	0	444
FLETCBV2	IPOPT	optimal	0	1.00e-01	-5.00e-01	1	0	1	0	0
	KNITRO	optimal	0	9.00e-02	-5.00e-01	2	0	1	0	0
	SNOPT	optimal	0	8.00e-02	-5.00e-01	3	0	2	0	0
	WORHP IP	optimal	0	8.00e-02	-5.00e-01	2	0	1	0	0
	WORHP IPm	optimal	0	9.00e-02	-5.00e-01	2	0	1	0	0
	WORHP SQP	optimal	0	8.00e-02	-5.00e-01	1	0	1	0	0
FLETCBV3	IPOPT	maxiter	10000	7.15e+01	-2.77e+07	10001	0	10001	0	10000
	KNITRO	maxiter	10000	4.12e+01	-3.36e+07	10002	0	10001	0	10000
	SNOPT	unbound	18	5.76e+01	-6.70e+09	34	0	33	0	0
	WORHP IP	maxiter	10000	8.85e+01	-2.76e+07	10002	0	10001	0	10000
	WORHP IPm	maxiter	10000	9.28e+01	-2.76e+07	10002	0	10001	0	10000
	WORHP SQP	maxiter	10000	8.54e+01	-8.88e+07	11731	0	10001	0	10001
FLETCHBV	IPOPT	maxiter	10000	5.51e+01	-2.75e+15	10001	0	10001	0	10000
	KNITRO	maxiter	10000	4.14e+01	-3.49e+15	10002	0	10001	0	10000
	SNOPT	unbound	0	1.00e-01	-2.30e+11	4	0	3	0	0
	WORHP IP	maxiter	10000	6.52e+01	-2.75e+15	10002	0	10001	0	10000
	WORHP IPm	maxiter	10000	8.05e+01	-2.75e+15	10002	0	10001	0	10000
	WORHP SQP	maxiter	10000	1.01e+02	-9.08e+18	21709	0	10001	0	10001

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
FLETCHCR	IPOPT	optimal	1475	1.95e+00	3.40e-18	2511	0	1476	0	1475
	KNITRO	optimal	1476	1.38e+00	2.86e-27	1684	0	1477	0	1476
	SNOPT	optimal	6223	4.27e+01	1.45e-13	6816	0	6815	0	0
	WORHP IP	optimal	1473	7.80e-01	5.69e-22	1867	0	1474	0	1473
	WORHP IPm	optimal	1473	1.44e+00	5.69e-22	1867	0	1474	0	1473
	WORHP SQP	optimal	1431	1.72e+00	9.96e-16	3145	0	1432	0	1431
FLETCHER	IPOPT	optimal	24	1.00e-02	1.95e+01	28	56	25	50	24
	KNITRO	optimal	9	1.00e-02	1.95e+01	12	13	11	12	9
	SNOPT	infeas	0	1.00e-02	4.00e+00	3	3	2	2	0
	WORHP IP	optimal	27	1.00e-02	1.17e+01	31	31	28	28	27
	WORHP IPm	optimal	25	1.00e-02	1.95e+01	30	30	27	27	25
	WORHP SQP	optimal	24	1.00e-02	1.95e+01	1137	1138	20	27	18
FLOSP2HH	IPOPT	infeas	650	8.69e+01	0.00e+00	675	675	9	653	651
	KNITRO	noimpr	29	6.16e+00	0.00e+00	119	120	31	32	30
	SNOPT	infeas	0	1.60e-01	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	84	9.13e+00	0.00e+00	156	156	85	85	84
	WORHP IPm	infeas	84	9.39e+00	0.00e+00	156	156	85	85	84
	WORHP SQP	minalpha	7	1.93e+01	0.00e+00	2745	2751	7	9	6
FLOSP2HL	IPOPT	infeas	8	7.30e-01	0.00e+00	33	33	8	11	9
	KNITRO	infeas	7	2.28e+00	0.00e+00	21	22	9	10	8
	SNOPT	infeas	0	1.70e-01	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	8	7.80e-01	0.00e+00	10	10	9	9	9
	WORHP IPm	infeas	8	8.80e-01	0.00e+00	10	10	9	9	9
	WORHP SQP	minalpha	9	9.38e+00	0.00e+00	3118	3124	11	11	10
FLOSP2HM	IPOPT	infeas	14	9.30e-01	0.00e+00	38	38	14	17	15
	KNITRO	noimpr	13	2.61e+00	0.00e+00	22	23	15	16	13
	SNOPT	infeas	0	1.90e-01	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	18	1.74e+00	0.00e+00	29	29	19	19	19
	WORHP IPm	infeas	18	2.01e+00	0.00e+00	29	29	19	19	19
	WORHP SQP	minalpha	21	2.42e+01	0.00e+00	3094	3100	10	23	9
FLOSP2TH	IPOPT	infeas	117	7.48e+00	0.00e+00	156	156	15	120	118
	KNITRO	optimal	4294	8.38e+02	0.00e+00	18421	18422	4296	4297	4294
	SNOPT	noimpr	140	1.10e+02	0.00e+00	1	240	1	239	0
	WORHP IP	infeas	61	4.23e+00	0.00e+00	120	120	62	62	62
	WORHP IPm	infeas	61	6.77e+00	0.00e+00	120	120	62	62	62
	WORHP SQP	minalpha	7	2.68e+01	0.00e+00	3614	3620	7	9	6
FLOSP2TL	IPOPT	optimal	4	2.20e-01	0.00e+00	5	5	5	5	4
	KNITRO	optimal	4	9.40e-01	0.00e+00	11	12	6	7	4
	SNOPT	optimal	5	7.85e+00	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	5	5.90e-01	0.00e+00	7	7	6	6	5
	WORHP IPm	optimal	5	5.80e-01	0.00e+00	7	7	6	6	5
	WORHP SQP	optimal	9	6.20e+00	0.00e+00	10	10	11	11	9
FLOSP2TM	IPOPT	optimal	10	3.80e-01	0.00e+00	11	11	11	11	10
	KNITRO	optimal	10	2.23e+00	0.00e+00	19	20	12	13	10
	SNOPT	optimal	43	3.20e+01	0.00e+00	1	105	1	104	0
	WORHP IP	optimal	12	1.25e+00	0.00e+00	20	20	14	14	12
	WORHP IPm	optimal	12	1.27e+00	0.00e+00	20	20	13	13	12
	WORHP SQP	optimal	16	1.47e+01	0.00e+00	142	141	5	19	3
FLT	IPOPT	optimal	4	1.00e-02	0.00e+00	5	5	5	5	4
	KNITRO	optimal	4	1.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	infeas	1763	6.00e-02	0.00e+00	3522	3522	3521	3521	0
	WORHP IP	fritzjohn	16	1.00e-02	0.00e+00	18	18	18	18	17
	WORHP IPm	fritzjohn	16	1.00e-02	0.00e+00	18	18	17	17	17
	WORHP SQP	optimal	2053	2.10e-01	1.83e-13	2278	2278	2045	2056	2043
FMINSRF2	IPOPT	optimal	70	2.52e+00	1.00e+00	857	0	71	0	70
	KNITRO	optimal	57	1.94e+00	1.00e+00	244	0	58	0	57
	SNOPT	toobig	574	2.50e+01	1.17e+00	664	0	663	0	0
	WORHP IP	optimal	28	1.28e+00	1.00e+00	195	0	29	0	28
	WORHP IPm	optimal	28	1.44e+00	1.00e+00	195	0	29	0	28
	WORHP SQP	optimal	124	6.66e+00	1.00e+00	5545	0	125	0	124
FMINSURF	IPOPT	maxtime	38	1.82e+03	6.78e+09	628	0	39	0	38
	KNITRO	maxtime	64	1.79e+03	3.96e+00	273	0	65	0	65
	SNOPT	optimal	321	1.72e+01	1.00e+00	374	0	373	0	0
	WORHP IP	maxtime	33	1.79e+03	1.20e+00	549	0	34	0	34
	WORHP IPm	maxtime	35	1.79e+03	1.19e+00	573	0	36	0	36
	WORHP SQP	maxtime	30	1.69e+03	2.25e+00	2211	0	31	0	31
FREURONE	IPOPT	infeas	15	1.00e-02	0.00e+00	84	94	9	18	16
	KNITRO	optimal	9	1.00e-02	0.00e+00	23	24	10	11	9
	SNOPT	infeas	27	1.00e-02	0.00e+00	1	59	1	58	0
	WORHP IP	infeas	15	1.00e-02	0.00e+00	70	70	16	16	16
	WORHP IPm	infeas	15	1.00e-02	0.00e+00	70	70	16	16	16
	WORHP SQP	minalpha	18	1.00e-02	0.00e+00	2602	2608	20	20	19
FREUROTH	IPOPT	optimal	7	1.50e-01	6.08e+05	13	0	8	0	7
	KNITRO	optimal	13	1.90e-01	6.08e+05	15	0	14	0	13
	SNOPT	toobig	35	5.25e+01	3.27e+06	52	0	51	0	0
	WORHP IP	optimal	7	1.10e-01	6.08e+05	11	0	8	0	7
	WORHP IPm	optimal	7	1.40e-01	6.08e+05	11	0	8	0	7
	WORHP SQP	optimal	11	1.90e-01	6.08e+05	12	0	12	0	11

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
GASOIL	IPOPT	optimal	13	5.60e-01	5.24e-03	36	36	14	14	13
	KNITRO	optimal	7	3.50e-01	5.24e-03	11	12	9	10	7
	SNOPT	optimal	15	5.38e+00	5.24e-03	18	18	17	17	0
	WORHP IP	optimal	22	1.12e+00	5.24e-03	68	68	23	23	22
	WORHP IPm	optimal	21	1.11e+00	5.24e-03	87	87	23	23	21
	WORHP SQP	optimal	29	8.33e+00	5.24e-03	92	92	31	31	29
GAUSS1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	131	1.90e-01	0.00e+00	194	195	130	131	130
	SNOPT	infeas	6	1.00e-02	0.00e+00	1	11	1	10	0
	WORHP IP	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP SQP	infeas	4	6.00e-01	0.00e+00	1344	1346	3	5	2
GAUSS1LS	IPOPT	optimal	5	1.00e-02	1.32e+03	6	0	6	0	5
	KNITRO	optimal	5	1.00e-02	1.32e+03	7	0	6	0	5
	SNOPT	optimal	162	5.00e-02	1.32e+03	172	0	171	0	0
	WORHP IP	optimal	5	1.00e-02	1.32e+03	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	1.32e+03	7	0	6	0	5
	WORHP SQP	optimal	13	1.00e-02	1.32e+03	14	0	14	0	13
GAUSS2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	162	2.00e-01	0.00e+00	164	165	163	164	162
	SNOPT	infeas	12	3.00e-02	0.00e+00	1	22	1	21	0
	WORHP IP	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.00e-02	0.00e+00	7	7	6	6	6
	WORHP SQP	infeas	24	1.09e+00	0.00e+00	3593	3598	21	25	20
GAUSS2LS	IPOPT	optimal	5	1.00e-02	1.25e+03	6	0	6	0	5
	KNITRO	optimal	5	1.00e-02	1.25e+03	7	0	6	0	5
	SNOPT	optimal	99	3.00e-02	1.25e+03	117	0	116	0	0
	WORHP IP	optimal	5	1.00e-02	1.25e+03	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	1.25e+03	7	0	6	0	5
	WORHP SQP	optimal	13	2.00e-02	1.25e+03	14	0	14	0	13
GAUSS3	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	71	1.00e-01	0.00e+00	76	77	72	73	71
	SNOPT	infeas	12	4.00e-02	0.00e+00	1	23	1	22	0
	WORHP IP	infeas	7	1.00e-02	0.00e+00	9	9	8	8	8
	WORHP IPm	infeas	7	1.00e-02	0.00e+00	9	9	8	8	8
	WORHP SQP	minalpha	11	9.70e-01	0.00e+00	2676	2682	9	13	8
GAUSS3LS	IPOPT	optimal	11	2.00e-02	1.24e+03	18	0	12	0	11
	KNITRO	optimal	10	1.00e-02	1.24e+03	19	0	11	0	10
	SNOPT	optimal	169	5.00e-02	1.24e+03	180	0	179	0	0
	WORHP IP	optimal	15	2.00e-02	1.24e+03	70	0	17	0	15
	WORHP IPm	optimal	15	2.00e-02	1.24e+03	70	0	16	0	15
	WORHP SQP	optimal	16	2.00e-02	1.24e+03	17	0	17	0	16
GAUSSELM	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	maxtime	1961	1.80e+03	-6.58e+01	1964	1965	1963	1964	1961
	SNOPT	infeas	5	3.78e+02	-1.81e-07	1	30	1	29	0
	WORHP IP	maxtime	397	1.73e+03	-1.55e+00	399	399	398	398	398
	WORHP IPm	maxtime	381	1.74e+03	-5.19e+00	383	383	382	382	382
	WORHP SQP	killed	-	-	-	-	-	-	-	-
GAUSSIAN	IPOPT	optimal	2	1.00e-02	1.13e-08	3	0	3	0	2
	KNITRO	optimal	2	1.00e-02	1.13e-08	4	0	3	0	2
	SNOPT	optimal	3	1.00e-02	1.13e-08	9	0	8	0	0
	WORHP IP	optimal	2	1.00e-02	1.13e-08	4	0	3	0	2
	WORHP IPm	optimal	2	1.00e-02	1.13e-08	4	0	3	0	2
	WORHP SQP	optimal	2	1.00e-02	1.13e-08	3	0	3	0	2
GBRAIN	IPOPT	degree	0	1.00e-01	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	20	4.40e-01	0.00e+00	58	59	21	22	20
	SNOPT	infeas	6	3.50e-01	0.00e+00	1	10	1	9	0
	WORHP IP	infeas	5	1.50e-01	0.00e+00	7	7	6	6	6
	WORHP IPm	infeas	5	1.60e-01	0.00e+00	7	7	6	6	6
	WORHP SQP	infeas	8	1.99e+02	0.00e+00	1789	1791	5	9	4
GBRAINLS	IPOPT	optimal	6	1.40e-01	2.85e+01	7	0	7	0	6
	KNITRO	optimal	6	1.50e-01	2.85e+01	8	0	7	0	6
	SNOPT	optimal	11	1.40e-01	2.85e+01	19	0	18	0	0
	WORHP IP	optimal	6	1.40e-01	2.85e+01	8	0	7	0	6
	WORHP IPm	optimal	6	1.50e-01	2.85e+01	8	0	7	0	6
	WORHP SQP	optimal	7	1.40e-01	2.85e+01	8	0	8	0	7
GENHS28	IPOPT	optimal	1	1.00e-02	9.27e-01	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	9.27e-01	3	4	2	3	1
	SNOPT	optimal	9	1.00e-02	9.27e-01	12	1	11	1	0
	WORHP IP	optimal	2	1.00e-02	9.27e-01	4	4	3	1	2
	WORHP IPm	optimal	2	1.00e-02	9.27e-01	4	4	3	1	2
	WORHP SQP	optimal	3	1.00e-02	9.27e-01	4	4	5	3	3
GENHUMPS	IPOPT	maxiter	10000	7.27e+01	3.01e+07	10001	0	10001	0	10000
	KNITRO	maxiter	10000	8.57e+01	3.05e+07	10004	0	10001	0	10000
	SNOPT	toobig	2666	1.72e+02	7.68e+07	3200	0	3199	0	0
	WORHP IP	maxiter	10000	1.45e+02	3.05e+07	10002	0	10001	0	10000
	WORHP IPm	maxiter	10000	1.20e+02	3.05e+07	10002	0	10001	0	10000
	WORHP SQP	maxiter	10000	1.28e+02	8.38e+06	12041	0	10001	0	10001

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
GENROSE	IPOPT	optimal	392	4.00e-01	1.00e+00	1152	0	393	0	392
	KNITRO	optimal	346	2.30e-01	1.00e+00	638	0	347	0	346
	SNOPT	optimal	1553	3.20e+00	1.00e+00	1719	0	1718	0	0
	WORHP IP	optimal	391	2.30e-01	1.00e+00	818	0	392	0	391
	WORHP IPm	optimal	391	2.50e-01	1.00e+00	818	0	392	0	391
	WORHP SQP	optimal	351	2.80e-01	1.00e+00	942	0	352	0	351
GENROSEB	IPOPT	optimal	15	2.00e-02	1.59e+03	16	0	16	0	15
	KNITRO	optimal	7	1.00e-02	1.59e+03	10	0	9	0	7
	SNOPT	optimal	135	5.00e-02	1.59e+03	141	0	140	0	0
	WORHP IP	optimal	11	2.00e-02	1.59e+03	13	0	12	0	11
	WORHP IPm	optimal	154	1.80e-01	1.59e+03	159	0	158	0	154
	WORHP SQP	optimal	79	6.10e-01	1.59e+03	80	0	80	0	79
GIGOMEZ1	IPOPT	optimal	15	1.00e-02	-3.00e+00	19	19	16	16	15
	KNITRO	optimal	17	1.00e-02	-3.00e+00	19	20	18	19	17
	SNOPT	optimal	12	1.00e-02	-3.00e+00	1	22	1	21	0
	WORHP IP	optimal	15	1.00e-02	-3.00e+00	17	17	16	16	15
	WORHP IPm	optimal	18	1.00e-02	-3.00e+00	24	24	19	19	18
	WORHP SQP	optimal	16	1.00e-02	-3.00e+00	17	17	18	18	16
GIGOMEZ2	IPOPT	optimal	8	1.00e-02	1.95e+00	9	9	9	9	8
	KNITRO	optimal	8	1.00e-02	1.95e+00	10	11	9	10	8
	SNOPT	optimal	8	1.00e-02	1.95e+00	1	11	1	10	0
	WORHP IP	optimal	8	1.00e-02	1.95e+00	10	10	9	9	8
	WORHP IPm	optimal	7	1.00e-02	1.95e+00	9	9	8	8	7
	WORHP SQP	optimal	18	1.00e-02	1.95e+00	19	19	20	20	18
GIGOMEZ3	IPOPT	optimal	8	1.00e-02	2.00e+00	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	2.00e+00	8	9	7	8	6
	SNOPT	optimal	9	1.00e-02	2.00e+00	1	16	1	15	0
	WORHP IP	optimal	7	1.00e-02	2.00e+00	9	9	8	8	7
	WORHP IPm	optimal	6	1.00e-02	2.00e+00	8	8	7	7	6
	WORHP SQP	optimal	7	1.00e-02	2.00e+00	8	8	9	9	7
GILBERT	IPOPT	optimal	23	1.80e-01	2.46e+03	24	24	24	24	23
	KNITRO	optimal	21	1.90e-01	2.46e+03	23	24	22	23	21
	SNOPT	sbasics	57	1.29e+02	1.94e+04	115	115	114	114	0
	WORHP IP	optimal	37	3.00e-01	2.46e+03	39	39	38	38	37
	WORHP IPm	optimal	36	2.70e-01	2.46e+03	38	38	37	37	36
	WORHP SQP	optimal	33	5.30e-01	2.46e+03	89	88	25	36	23
GLIDER	IPOPT	maxiter	10000	1.36e+02	-7.65e+02	14662	14663	967	10018	10000
	KNITRO	optimal	145	4.17e+00	-1.25e+03	232	233	146	147	146
	SNOPT	optimal	253	5.34e+01	-1.25e+03	1	918	1	917	0
	WORHP IP	optimal	681	1.29e+01	-1.25e+03	1499	1499	682	682	681
	WORHP IPm	optimal	441	8.74e+00	-1.25e+03	1745	1745	444	444	441
	WORHP SQP	maxtime	9538	1.74e+03	-2.99e+02	1053161	1062503	9387	9543	9386
GMNCASE1	IPOPT	optimal	11	2.40e-01	2.67e-01	12	12	12	12	11
	KNITRO	optimal	6	1.50e-01	2.67e-01	8	9	7	8	6
	SNOPT	optimal	14	8.00e-02	2.67e-01	18	1	17	1	0
	WORHP IP	optimal	10	2.90e-01	2.67e-01	12	12	11	1	10
	WORHP IPm	optimal	8	2.10e-01	2.67e-01	10	10	9	1	8
	WORHP SQP	optimal	14	8.50e-01	2.67e-01	15	15	16	3	14
GMNCASE2	IPOPT	optimal	11	2.80e-01	-9.94e-01	12	12	12	12	11
	KNITRO	optimal	7	1.70e-01	-9.94e-01	9	10	8	9	7
	SNOPT	optimal	48	1.60e-01	-9.94e-01	54	1	53	1	0
	WORHP IP	optimal	8	2.90e-01	-9.94e-01	10	10	9	1	8
	WORHP IPm	optimal	7	2.50e-01	-9.94e-01	9	9	8	1	7
	WORHP SQP	optimal	5	6.00e-01	-9.94e-01	6	6	7	3	5
GMNCASE3	IPOPT	optimal	10	2.30e-01	1.53e+00	11	11	11	11	10
	KNITRO	optimal	6	1.60e-01	1.53e+00	8	9	7	8	6
	SNOPT	optimal	40	1.40e-01	1.53e+00	44	1	43	1	0
	WORHP IP	optimal	7	2.30e-01	1.53e+00	9	9	8	1	7
	WORHP IPm	optimal	7	2.30e-01	1.53e+00	9	9	8	1	7
	WORHP SQP	optimal	3	5.20e-01	1.53e+00	4	4	5	3	3
GMNCASE4	IPOPT	optimal	38	8.00e-01	5.95e+03	49	49	39	39	38
	KNITRO	optimal	10	2.40e-01	5.95e+03	13	14	12	13	10
	SNOPT	optimal	0	1.20e-01	5.95e+03	3	1	2	1	0
	WORHP IP	optimal	20	6.70e-01	5.95e+03	22	22	22	1	20
	WORHP IPm	optimal	19	6.50e-01	5.95e+03	21	21	20	1	19
	WORHP SQP	optimal	2	9.90e-01	5.95e+03	2	2	3	3	2
GOFFIN	IPOPT	optimal	7	1.00e-02	4.54e-06	8	8	8	8	7
	KNITRO	optimal	4	1.00e-02	1.94e-06	6	7	5	6	4
	SNOPT	optimal	0	1.00e-02	-1.17e-13	1	1	1	1	0
	WORHP IP	optimal	7	1.00e-02	1.08e-05	9	9	8	1	7
	WORHP IPm	optimal	6	1.00e-02	-5.59e-08	8	8	7	1	6
	WORHP SQP	optimal	2	2.00e-02	2.10e-09	2	2	3	3	2
GOTTFR	IPOPT	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	KNITRO	optimal	5	1.00e-02	0.00e+00	9	10	6	7	5
	SNOPT	optimal	5	1.00e-02	0.00e+00	1	10	1	9	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	10	10	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	10	10	6	6	5
	WORHP SQP	optimal	5	1.00e-02	0.00e+00	23	23	7	7	5

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
GOULDQP1	IPOPT	optimal	17	1.00e-02	-3.49e+03	18	18	18	18	17
	KNITRO	optimal	15	1.00e-02	-3.49e+03	18	19	17	18	15
	SNOPT	optimal	9	1.00e-02	-3.49e+03	17	1	16	1	0
	WORHP IP	optimal	15	1.00e-02	-3.49e+03	17	17	17	1	15
	WORHP IPm	optimal	17	1.00e-02	-3.49e+03	23	23	22	1	17
	WORHP SQP	optimal	11	2.00e-02	-3.49e+03	11	11	12	3	11
GOULDQP2	IPOPT	optimal	2	2.40e-01	2.65e-11	3	3	3	3	2
	KNITRO	optimal	1	3.10e-01	1.60e-12	4	5	3	4	1
	SNOPT	optimal	0	1.60e-01	1.85e-12	3	1	2	1	0
	WORHP IP	optimal	3	8.00e-01	1.60e-12	5	5	4	1	3
	WORHP IPm	optimal	3	8.50e-01	1.60e-12	6	6	5	1	3
	WORHP SQP	optimal	0	4.80e-01	1.60e-12	1	1	1	1	0
GOULDQP3	IPOPT	optimal	2	3.30e-01	4.62e-05	3	3	3	3	2
	KNITRO	optimal	1	3.60e-01	6.37e-05	4	5	3	4	1
	SNOPT	optimal	0	6.90e-01	4.76e-05	3	1	2	1	0
	WORHP IP	optimal	7	6.10e-01	2.51e-05	9	9	8	1	7
	WORHP IPm	optimal	3	4.40e-01	6.33e-05	6	6	5	1	3
	WORHP SQP	optimal	2	5.60e-01	2.38e-05	3	3	4	3	2
GPP	IPOPT	optimal	21	2.13e+01	2.32e+05	22	22	22	22	21
	KNITRO	optimal	14	2.02e+01	2.32e+05	16	17	15	16	14
	SNOPT	optimal	12	1.25e+01	2.32e+05	15	15	14	14	0
	WORHP IP	optimal	16	2.66e+01	2.32e+05	18	18	17	17	16
	WORHP IPm	optimal	19	3.22e+01	2.32e+05	21	21	20	20	19
	WORHP SQP	optimal	38	1.60e+02	2.32e+05	1480	1480	40	40	38
GRIDGENA	IPOPT	optimal	7	3.60e-01	2.35e+04	8	0	8	0	7
	KNITRO	optimal	4	2.90e-01	2.35e+04	7	0	6	0	4
	SNOPT	unbound	14	7.89e+00	-8.14e+09	146	0	145	0	0
	WORHP IP	optimal	6	4.20e-01	2.35e+04	8	0	7	0	6
	WORHP IPm	optimal	6	4.50e-01	2.35e+04	8	0	7	0	6
	WORHP SQP	optimal	3	5.10e-01	2.35e+04	4	0	4	0	3
GRIDNETA	IPOPT	optimal	3593	2.93e+01	4.78e+02	3998	3998	3592	3602	3593
	KNITRO	optimal	10	1.50e-01	4.78e+02	13	14	12	13	10
	SNOPT	optimal	53	1.15e+00	4.78e+02	66	1	65	1	0
	WORHP IP	optimal	15	2.30e-01	4.78e+02	17	17	17	1	15
	WORHP IPm	optimal	12	2.00e-01	4.78e+02	17	17	16	1	12
	WORHP SQP	optimal	4932	9.38e+01	4.78e+02	4933	4933	4933	2	4932
GRIDNETB	IPOPT	optimal	1	1.00e-01	1.28e+02	2	2	2	2	1
	KNITRO	optimal	1	1.00e-01	1.28e+02	3	4	2	3	1
	SNOPT	sbasics	122	2.65e+01	1.31e+02	134	1	133	1	0
	WORHP IP	optimal	4	1.20e-01	1.28e+02	6	6	5	1	4
	WORHP IPm	optimal	4	1.30e-01	1.28e+02	6	6	5	1	4
	WORHP SQP	optimal	903	2.16e+01	1.28e+02	904	904	905	3	903
GRIDNETC	IPOPT	optimal	68	1.35e+00	1.62e+02	72	72	69	69	68
	KNITRO	optimal	19	4.10e-01	1.62e+02	22	23	21	22	19
	SNOPT	sbasics	113	2.97e+01	1.62e+02	121	1	120	1	0
	WORHP IP	optimal	19	4.70e-01	1.62e+02	21	21	21	1	19
	WORHP IPm	optimal	21	4.80e-01	1.62e+02	27	27	26	1	21
	WORHP SQP	optimal	3	5.70e-01	1.62e+02	4	4	5	3	3
GRIDNETD	IPOPT	optimal	49	1.45e+00	5.71e+02	50	50	50	50	49
	KNITRO	optimal	11	2.20e-01	5.71e+02	14	15	13	14	11
	SNOPT	optimal	48	1.89e+00	5.71e+02	57	1	56	1	0
	WORHP IP	optimal	15	4.90e-01	5.71e+02	17	17	17	1	15
	WORHP IPm	optimal	14	5.20e-01	5.71e+02	20	20	19	1	14
	WORHP SQP	maxiter	10000	4.12e+02	5.71e+02	11828	11842	10001	2	10001
GRIDNETE	IPOPT	optimal	3	2.80e-01	2.06e+02	4	4	4	4	3
	KNITRO	optimal	3	2.20e-01	2.06e+02	5	6	4	5	3
	SNOPT	sbasics	109	2.26e+01	2.11e+02	115	1	114	1	0
	WORHP IP	optimal	4	2.40e-01	2.06e+02	6	6	5	1	4
	WORHP IPm	optimal	4	2.50e-01	2.06e+02	6	6	5	1	4
	WORHP SQP	optimal	8	7.40e-01	2.06e+02	9	9	10	3	8
GRIDNETF	IPOPT	optimal	22	1.07e+00	2.44e+02	23	23	23	23	22
	KNITRO	optimal	17	6.60e-01	2.44e+02	20	21	19	20	17
	SNOPT	sbasics	110	3.08e+01	2.44e+02	118	1	117	1	0
	WORHP IP	optimal	18	1.20e+00	2.44e+02	20	20	20	1	18
	WORHP IPm	optimal	20	1.24e+00	2.44e+02	26	26	25	1	20
	WORHP SQP	optimal	5	1.38e+00	2.44e+02	6	6	7	3	5
GRIDNETG	IPOPT	optimal	78	2.30e+00	6.16e+02	92	92	79	81	78
	KNITRO	optimal	9	1.90e-01	6.16e+02	12	13	11	12	9
	SNOPT	optimal	52	2.00e+00	6.16e+02	56	1	55	1	0
	WORHP IP	optimal	14	4.50e-01	6.16e+02	16	16	15	1	14
	WORHP IPm	optimal	14	4.80e-01	6.16e+02	20	20	19	1	14
	WORHP SQP	maxiter	10000	3.17e+02	6.16e+02	11384	11384	10001	2	10001
GRIDNETH	IPOPT	optimal	5	3.60e-01	2.06e+02	6	6	6	6	5
	KNITRO	optimal	5	2.70e-01	2.06e+02	7	8	6	7	5
	SNOPT	sbasics	105	2.32e+01	2.11e+02	123	1	122	1	0
	WORHP IP	optimal	5	3.20e-01	2.06e+02	7	7	7	1	5
	WORHP IPm	optimal	5	3.50e-01	2.06e+02	7	7	6	1	5
	WORHP SQP	optimal	8	6.50e-01	2.06e+02	9	9	10	3	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
GRIDNETI	IPOPT	optimal	24	1.03e+00	2.44e+02	25	25	25	25	24
	KNITRO	optimal	17	1.03e+00	2.44e+02	20	21	19	20	17
	SNOPT	sbasics	110	2.32e+01	2.44e+02	115	1	114	1	0
	WORHP IP	optimal	18	9.60e-01	2.44e+02	20	20	20	1	18
	WORHP IPm	optimal	20	1.13e+00	2.44e+02	26	26	25	1	20
	WORHP SQP	optimal	5	1.64e+00	2.44e+02	6	6	7	3	5
GROUPING	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	5	1.00e-02	1.39e+01	8	9	7	8	5
	SNOPT	optimal	0	1.00e-02	1.39e+01	3	3	2	2	0
	WORHP IP	optimal	5	1.00e-02	1.39e+01	7	7	6	6	5
	WORHP IPm	optimal	4	1.00e-02	1.39e+01	26	26	8	8	4
	WORHP SQP	optimal	1	1.00e-02	1.39e+01	1	1	2	2	1
GROWTH	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	35	1.00e-02	0.00e+00	79	80	35	36	34
	SNOPT	infeas	22	1.00e-02	0.00e+00	1	55	1	54	0
	WORHP IP	infeas	15	1.00e-02	0.00e+00	48	48	16	16	16
	WORHP IPm	infeas	15	1.00e-02	0.00e+00	48	48	16	16	16
	WORHP SQP	minalpha	170	5.00e-02	0.00e+00	5216	5223	77	172	76
GROWTHLS	IPOPT	optimal	70	2.00e-02	1.00e+00	170	0	71	0	70
	KNITRO	optimal	74	1.00e-02	1.00e+00	112	0	75	0	74
	SNOPT	optimal	126	1.00e-02	1.00e+00	186	0	185	0	0
	WORHP IP	optimal	72	1.00e-02	1.00e+00	129	0	74	0	72
	WORHP IPm	optimal	76	1.00e-02	1.00e+00	224	0	77	0	76
	WORHP SQP	optimal	75	1.00e-02	1.00e+00	393	0	76	0	75
GULF	IPOPT	optimal	28	1.00e-02	3.40e-22	50	0	29	0	28
	KNITRO	optimal	27	1.00e-02	6.74e-23	39	0	28	0	27
	SNOPT	optimal	50	1.00e-02	1.39e-14	67	0	66	0	0
	WORHP IP	optimal	28	1.00e-02	3.40e-22	39	0	29	0	28
	WORHP IPm	optimal	28	1.00e-02	3.40e-22	39	0	29	0	28
	WORHP SQP	optimal	37	1.00e-02	3.01e-14	55	0	38	0	37
GULFNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	23	1.00e-02	0.00e+00	33	34	22	23	21
	SNOPT	optimal	22	2.00e-02	0.00e+00	1	62	1	61	0
	WORHP IP	maxiter	10000	1.08e+01	0.00e+00	10414	10414	10001	10001	10000
	WORHP IPm	maxiter	10000	1.08e+01	0.00e+00	10414	10414	10001	10001	10000
	WORHP SQP	minalpha	73	3.60e-01	0.00e+00	3976	3982	74	75	73
HADAMALS	IPOPT	optimal	215	4.89e+00	1.24e+02	221	0	216	0	215
	KNITRO	optimal	236	6.12e+00	1.37e+02	239	0	238	0	236
	SNOPT	optimal	8	5.00e-02	7.31e+03	18	0	17	0	0
	WORHP IP	optimal	183	8.10e+00	1.88e+02	185	0	184	0	183
	WORHP IPm	optimal	193	8.40e+00	1.92e+02	201	0	198	0	193
	WORHP SQP	optimal	87	2.36e+01	2.00e+02	88	0	88	0	87
HADAMARD	IPOPT	optimal	195	1.14e+01	1.13e+00	250	500	137	396	195
	KNITRO	optimal	695	2.43e+01	1.15e+00	699	700	697	698	695
	SNOPT	optimal	1529	1.60e+01	1.15e+00	1	5819	1	5818	0
	WORHP IP	optimal	857	4.54e+01	1.13e+00	861	861	858	858	857
	WORHP IPm	optimal	616	3.22e+01	1.15e+00	619	619	617	617	616
	WORHP SQP	minalpha	5	7.52e+00	3.48e-01	2670	2676	6	7	5
HAGER1	IPOPT	optimal	1	4.00e-02	8.81e-01	2	2	2	2	1
	KNITRO	optimal	1	6.00e-02	8.81e-01	3	4	2	3	1
	SNOPT	toobig	21	5.02e+00	1.47e+00	25	1	24	1	0
	WORHP IP	optimal	3	8.00e-02	8.81e-01	5	5	4	1	3
	WORHP IPm	optimal	3	8.00e-02	8.81e-01	5	5	4	1	3
	WORHP SQP	optimal	5	1.40e-01	8.81e-01	6	6	7	3	5
HAGER2	IPOPT	optimal	1	6.00e-02	4.32e-01	2	2	2	2	1
	KNITRO	optimal	1	6.00e-02	4.32e-01	3	4	2	3	1
	SNOPT	sbasics	26	1.00e+01	4.33e-01	30	1	29	1	0
	WORHP IP	optimal	3	7.00e-02	4.32e-01	5	5	5	1	3
	WORHP IPm	optimal	3	7.00e-02	4.32e-01	5	5	4	1	3
	WORHP SQP	optimal	5	1.30e-01	4.32e-01	6	6	7	3	5
HAGER3	IPOPT	optimal	1	7.00e-02	1.41e-01	2	2	2	2	1
	KNITRO	optimal	1	8.00e-02	1.41e-01	3	4	2	3	1
	SNOPT	sbasics	25	1.05e+01	1.50e-01	29	1	28	1	0
	WORHP IP	optimal	3	9.00e-02	1.41e-01	5	5	5	1	3
	WORHP IPm	optimal	3	1.10e-01	1.41e-01	5	5	4	1	3
	WORHP SQP	optimal	5	1.70e-01	1.41e-01	6	6	7	3	5
HAGER4	IPOPT	optimal	7	8.00e-02	2.79e+00	8	8	8	8	7
	KNITRO	optimal	5	9.00e-02	2.79e+00	7	8	6	7	5
	SNOPT	optimal	16	2.77e+00	2.79e+00	22	1	21	1	0
	WORHP IP	optimal	6	9.00e-02	2.79e+00	8	8	7	1	6
	WORHP IPm	optimal	6	8.00e-02	2.79e+00	8	8	7	1	6
	WORHP SQP	minalpha	120	2.90e+00	2.79e+00	4661	4689	122	3	121
HAHN1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	107	2.20e-01	0.00e+00	112	113	108	109	107
	SNOPT	toobig	238	3.00e-01	0.00e+00	1	1516	1	1515	0
	WORHP IP	infeas	12	1.00e-02	0.00e+00	25	25	13	13	13
	WORHP IPm	infeas	12	1.00e-02	0.00e+00	25	25	13	13	13
	WORHP SQP	minalpha	15	3.50e-01	0.00e+00	3698	3704	12	17	11

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HAHN1LS	IPOPT	maxiter	10000	8.39e+00	3.34e+01	106208	0	10001	0	10000
	KNITRO	optimal	76	3.00e-02	1.53e+00	109	0	77	0	76
	SNOPT	unbound	48	1.00e-02	6.71e+03	73	0	72	0	0
	WORHP IP	accept	113	6.00e-02	3.35e+01	801	0	160	0	114
	WORHP IPm	minalpha	113	7.00e-02	3.35e+01	801	0	159	0	114
	WORHP SQP	maxiter	10000	4.20e+00	3.09e+01	10001	0	10001	0	10001
HAIFAL	IPOPT	optimal	115	5.17e+00	-1.28e+01	121	122	116	116	115
	KNITRO	optimal	26	2.04e+00	-1.28e+01	36	37	27	28	26
	SNOPT	optimal	30	9.35e+00	-1.28e+01	1	37	1	36	0
	WORHP IP	optimal	161	2.80e+01	-1.28e+01	654	654	162	162	161
	WORHP IPm	optimal	677	1.72e+02	-1.28e+01	9166	9166	678	678	677
	WORHP SQP	optimal	34	7.10e+01	-1.28e+01	36	36	36	36	34
HAIFAM	IPOPT	optimal	32	3.00e-02	-4.50e+01	33	33	33	33	32
	KNITRO	optimal	26	3.00e-02	-4.50e+01	104	105	27	28	26
	SNOPT	optimal	90	7.00e-02	-4.50e+01	1	227	1	226	0
	WORHP IP	optimal	22	3.00e-02	-4.50e+01	26	26	23	23	22
	WORHP IPm	optimal	27	3.00e-02	-4.50e+01	49	49	28	28	27
	WORHP SQP	optimal	40	2.00e-01	-4.50e+01	74	73	42	43	40
HAIFAS	IPOPT	optimal	8	1.00e-02	-4.50e-01	10	10	9	9	8
	KNITRO	optimal	6	1.00e-02	-4.50e-01	10	11	7	8	6
	SNOPT	optimal	15	1.00e-02	-4.50e-01	1	26	1	25	0
	WORHP IP	optimal	17	1.00e-02	-4.50e-01	19	19	18	18	17
	WORHP IPm	optimal	15	1.00e-02	-4.50e-01	18	18	16	16	15
	WORHP SQP	optimal	9	1.00e-02	-4.50e-01	10	10	11	11	9
HAIRY	IPOPT	optimal	61	2.00e-02	2.00e+01	100	0	62	0	61
	KNITRO	optimal	32	1.00e-02	2.00e+01	54	0	33	0	32
	SNOPT	optimal	17	1.00e-02	2.00e+01	36	0	35	0	0
	WORHP IP	optimal	59	1.00e-02	2.00e+01	78	0	60	0	59
	WORHP IPm	optimal	59	1.00e-02	2.00e+01	78	0	60	0	59
	WORHP SQP	optimal	31	1.00e-02	2.00e+01	42	0	32	0	31
HALDMADS	IPOPT	optimal	75	4.00e-02	1.65e+00	113	113	76	76	75
	KNITRO	optimal	27	1.00e-02	3.41e-02	31	32	28	29	27
	SNOPT	optimal	8	1.00e-02	1.22e-04	1	17	1	16	0
	WORHP IP	optimal	20	1.00e-02	3.29e-02	24	24	21	21	20
	WORHP IPm	optimal	75	1.00e-02	3.47e-02	130	130	76	76	75
	WORHP SQP	optimal	24	3.00e-02	3.22e-02	63	63	26	26	24
HANGING	IPOPT	optimal	30	5.80e-01	-3.15e+04	32	34	31	31	30
	KNITRO	optimal	83	1.68e+00	-3.15e+04	86	87	85	86	83
	SNOPT	maxtime	371	1.80e+03	-3.08e+04	1	1129	1	1128	0
	WORHP IP	optimal	25	8.10e-01	-3.15e+04	27	27	26	26	25
	WORHP IPm	optimal	30	1.14e+00	-3.15e+04	32	32	31	31	30
	WORHP SQP	optimal	39	1.11e+01	-3.15e+04	61	61	41	41	39
HARKERP2	IPOPT	optimal	26	3.46e+02	-4.99e-01	27	0	27	0	26
	KNITRO	optimal	15	2.12e+02	-5.00e-01	17	0	16	0	15
	SNOPT	optimal	43	2.68e+00	-5.00e-01	59	0	58	0	0
	WORHP IP	optimal	23	3.10e+02	-5.00e-01	25	0	25	0	23
	WORHP IPm	smallstep	28	3.59e+02	-4.99e-01	135	0	29	0	29
	WORHP SQP	optimal	42	5.62e+02	-5.00e-01	43	0	43	0	42
HART6	IPOPT	optimal	8	1.00e-02	-3.32e+00	14	0	9	0	8
	KNITRO	optimal	6	1.00e-02	-3.32e+00	9	0	7	0	6
	SNOPT	optimal	11	1.00e-02	-3.32e+00	17	0	16	0	0
	WORHP IP	optimal	7	1.00e-02	-3.32e+00	10	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	-3.32e+00	11	0	8	0	7
	WORHP SQP	optimal	8	1.00e-02	-3.32e+00	22	0	9	0	8
HATFLDA	IPOPT	optimal	9	1.00e-02	9.50e-13	10	0	10	0	9
	KNITRO	optimal	13	1.00e-02	6.29e-19	15	0	14	0	13
	SNOPT	optimal	25	1.00e-02	1.49e-13	31	0	30	0	0
	WORHP IP	optimal	8	1.00e-02	7.26e-15	10	0	9	0	8
	WORHP IPm	optimal	6	1.00e-02	7.42e-15	11	0	7	0	6
	WORHP SQP	optimal	23	1.00e-02	1.66e-13	24	0	24	0	23
HATFLDB	IPOPT	optimal	10	1.00e-02	5.57e-03	11	0	11	0	10
	KNITRO	optimal	11	1.00e-02	5.57e-03	13	0	12	0	11
	SNOPT	optimal	16	1.00e-02	5.57e-03	22	0	21	0	0
	WORHP IP	optimal	9	1.00e-02	5.57e-03	11	0	10	0	9
	WORHP IPm	optimal	8	1.00e-02	5.57e-03	11	0	10	0	8
	WORHP SQP	optimal	20	1.00e-02	5.57e-03	21	0	21	0	20
HATFLDC	IPOPT	optimal	5	1.00e-02	3.81e-14	6	0	6	0	5
	KNITRO	optimal	4	1.00e-02	7.02e-14	6	0	5	0	4
	SNOPT	optimal	21	1.00e-02	1.67e-12	26	0	25	0	0
	WORHP IP	optimal	5	1.00e-02	4.14e-19	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	7.17e-23	7	0	6	0	5
	WORHP SQP	optimal	4	1.00e-02	1.11e-13	5	0	5	0	4
HATFLDD	IPOPT	optimal	20	1.00e-02	6.62e-08	26	0	21	0	20
	KNITRO	optimal	12	1.00e-02	6.62e-08	16	0	13	0	12
	SNOPT	optimal	26	1.00e-02	6.62e-08	30	0	29	0	0
	WORHP IP	optimal	20	1.00e-02	6.62e-08	24	0	21	0	20
	WORHP IPm	optimal	20	1.00e-02	6.62e-08	24	0	21	0	20
	WORHP SQP	optimal	33	1.00e-02	9.80e-06	38	0	34	0	33

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HATFLDE	IPOPT	optimal	20	1.00e-02	5.12e-07	21	0	21	0	20
	KNITRO	optimal	17	1.00e-02	5.12e-07	19	0	18	0	17
	SNOPT	optimal	36	1.00e-02	5.12e-07	45	0	44	0	0
	WORHP IP	optimal	20	1.00e-02	5.12e-07	22	0	21	0	20
	WORHP IPm	optimal	20	1.00e-02	5.12e-07	22	0	21	0	20
	WORHP SQP	optimal	35	1.00e-02	6.53e-05	36	0	36	0	35
HATFLDF	IPOPT	optimal	137	5.00e-02	0.00e+00	1334	1593	130	139	137
	KNITRO	infeas	880	3.00e-02	0.00e+00	4675	4676	881	882	880
	SNOPT	maxiter	10000	4.60e-01	0.00e+00	1	40157	1	40156	0
	WORHP IP	regular	1151	8.00e-02	0.00e+00	10343	10343	1152	1152	1152
	WORHP IPm	regular	1151	7.00e-02	0.00e+00	10343	10343	1152	1152	1152
	WORHP SQP	minalpha	47	3.00e-02	0.00e+00	8614	8631	30	49	29
HATFLDFL	IPOPT	optimal	1222	2.50e-01	6.02e-05	3219	0	1223	0	1222
	KNITRO	optimal	1177	1.00e-02	6.02e-05	1696	0	1178	0	1177
	SNOPT	optimal	361	1.00e-02	6.03e-05	498	0	497	0	0
	WORHP IP	optimal	470	2.00e-02	6.02e-05	813	0	471	0	470
	WORHP IPm	optimal	470	1.00e-02	6.02e-05	813	0	471	0	470
	WORHP SQP	maxiter	10000	5.20e-01	6.02e-05	11984	0	10001	0	10001
HATFLDG	IPOPT	optimal	7	1.00e-02	0.00e+00	20	22	8	8	7
	KNITRO	optimal	8	1.00e-02	0.00e+00	18	19	9	10	8
	SNOPT	optimal	9	1.00e-02	0.00e+00	1	20	1	19	0
	WORHP IP	optimal	6	1.00e-02	0.00e+00	24	24	7	7	6
	WORHP IPm	optimal	6	1.00e-02	0.00e+00	24	24	7	7	6
	WORHP SQP	optimal	8	2.00e-02	0.00e+00	1268	1268	6	11	4
HATFLDH	IPOPT	optimal	14	1.00e-02	-2.45e+01	15	15	15	15	14
	KNITRO	optimal	10	1.00e-02	-2.45e+01	13	14	12	13	10
	SNOPT	optimal	3	1.00e-02	-2.45e+01	8	1	7	1	0
	WORHP IP	optimal	12	1.00e-02	-2.45e+01	14	14	13	1	12
	WORHP IPm	optimal	13	1.00e-02	-2.45e+01	15	15	14	1	13
	WORHP SQP	optimal	3	1.00e-02	-2.45e+01	4	4	5	3	3
HEART6	IPOPT	optimal	100	4.00e-02	0.00e+00	217	217	13	102	100
	KNITRO	optimal	100	1.00e-02	0.00e+00	495	496	101	102	100
	SNOPT	optimal	68	1.00e-02	0.00e+00	1	171	1	170	0
	WORHP IP	optimal	175	1.00e-02	0.00e+00	1089	1089	176	176	175
	WORHP IPm	optimal	175	1.00e-02	0.00e+00	1089	1089	176	176	175
	WORHP SQP	optimal	34	1.00e-02	0.00e+00	681	680	4	37	2
HEART6LS	IPOPT	optimal	889	2.50e-01	8.34e-24	1448	0	890	0	889
	KNITRO	optimal	279	1.00e-02	4.31e-26	327	0	280	0	279
	SNOPT	toobig	2602	8.00e-02	8.83e-02	3335	0	3334	0	0
	WORHP IP	optimal	880	4.00e-02	8.45e-24	1195	0	881	0	880
	WORHP IPm	optimal	880	3.00e-02	8.45e-24	1195	0	881	0	880
	WORHP SQP	optimal	760	6.00e-02	3.95e-17	1541	0	761	0	760
HEART8	IPOPT	optimal	11	1.00e-02	0.00e+00	39	43	12	12	11
	KNITRO	optimal	16	1.00e-02	0.00e+00	59	60	17	18	16
	SNOPT	optimal	41	1.00e-02	0.00e+00	1	85	1	84	0
	WORHP IP	optimal	25	1.00e-02	0.00e+00	92	92	26	26	25
	WORHP IPm	optimal	25	1.00e-02	0.00e+00	92	92	26	26	25
	WORHP SQP	optimal	36	2.00e-02	0.00e+00	1333	1332	8	39	6
HEART8LS	IPOPT	optimal	105	3.00e-02	3.77e-16	188	0	106	0	105
	KNITRO	optimal	108	1.00e-02	2.73e-17	137	0	109	0	108
	SNOPT	optimal	3513	1.20e-01	6.47e-17	4639	0	4638	0	0
	WORHP IP	optimal	105	1.00e-02	3.76e-16	147	0	106	0	105
	WORHP IPm	optimal	105	1.00e-02	3.76e-16	147	0	106	0	105
	WORHP SQP	optimal	77	1.00e-02	5.07e-15	223	0	78	0	77
HELIX	IPOPT	optimal	13	1.00e-02	6.06e-25	25	0	14	0	13
	KNITRO	optimal	15	1.00e-02	1.32e-20	21	0	16	0	15
	SNOPT	optimal	25	1.00e-02	4.07e-16	29	0	28	0	0
	WORHP IP	optimal	13	1.00e-02	6.06e-25	20	0	14	0	13
	WORHP IPm	optimal	13	1.00e-02	6.06e-25	20	0	14	0	13
	WORHP SQP	optimal	17	1.00e-02	3.06e-15	411	0	18	0	17
HELIXNE	IPOPT	optimal	7	1.00e-02	0.00e+00	12	12	8	8	7
	KNITRO	optimal	9	1.00e-02	0.00e+00	13	14	10	11	9
	SNOPT	optimal	8	1.00e-02	0.00e+00	1	12	1	11	0
	WORHP IP	optimal	9	1.00e-02	0.00e+00	13	13	10	10	9
	WORHP IPm	optimal	9	1.00e-02	0.00e+00	13	13	10	10	9
	WORHP SQP	optimal	9	1.00e-02	0.00e+00	10	10	11	11	9
HELSBY	IPOPT	optimal	30	1.60e-01	3.15e+01	31	31	31	31	30
	KNITRO	optimal	56	2.80e-01	3.19e+01	59	60	58	59	56
	SNOPT	optimal	17	1.20e-01	3.19e+01	1	34	1	33	0
	WORHP IP	optimal	60	3.40e-01	3.19e+01	75	75	62	62	60
	WORHP IPm	optimal	44	2.90e-01	3.19e+01	61	61	45	45	44
	WORHP SQP	optimal	40	1.16e+00	3.19e+01	94	94	42	42	40
HET-Z	IPOPT	optimal	12	4.00e-02	1.00e+00	13	13	13	13	12
	KNITRO	optimal	7	3.00e-02	1.00e+00	9	10	8	9	7
	SNOPT	optimal	1	1.00e-02	1.00e+00	1	4	1	3	0
	WORHP IP	optimal	10	3.00e-02	1.00e+00	12	12	11	11	10
	WORHP IPm	optimal	11	2.00e-02	1.00e+00	13	13	12	12	11
	WORHP SQP	optimal	4	4.00e-02	1.00e+00	5	5	6	6	4

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HIE1327D	IPOPT	degree	0	2.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	14	1.54e+00	5.19e+02	17	18	16	17	14
	SNOPT	optimal	290	1.77e+00	5.19e+02	295	1	294	1	0
	WORHP IP	optimal	13	2.89e+00	5.19e+02	15	15	14	1	13
	WORHP IPm	optimal	12	9.00e-01	5.19e+02	16	16	15	1	12
	WORHP SQP	optimal	33	2.39e+01	5.19e+02	34	34	35	3	33
HIE1372D	IPOPT	optimal	17	8.00e-02	2.78e+02	18	18	18	18	17
	KNITRO	optimal	15	2.00e-01	2.78e+02	18	19	17	18	15
	SNOPT	optimal	236	7.00e-01	2.78e+02	242	1	241	1	0
	WORHP IP	optimal	13	2.10e-01	2.78e+02	15	15	14	1	13
	WORHP IPm	optimal	12	1.50e-01	2.78e+02	16	16	15	1	12
	WORHP SQP	optimal	16	9.40e-01	2.78e+02	17	17	18	3	16
HIELOW	IPOPT	optimal	8	1.40e-01	8.74e+02	9	0	9	0	8
	KNITRO	optimal	7	1.80e-01	8.74e+02	12	0	8	0	7
	SNOPT	optimal	23	1.30e-01	8.74e+02	32	0	31	0	0
	WORHP IP	optimal	11	2.90e-01	8.74e+02	69	0	13	0	11
	WORHP IPm	optimal	11	2.80e-01	8.74e+02	69	0	12	0	11
	WORHP SQP	optimal	13	2.90e-01	9.31e+02	22	0	14	0	13
HIER13	IPOPT	degree	0	2.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	10	4.68e+00	1.03e+03	13	14	12	13	10
	SNOPT	optimal	33	5.40e-01	1.03e+03	39	1	38	1	0
	WORHP IP	optimal	11	2.09e+01	1.03e+03	13	13	12	1	11
	WORHP IPm	optimal	9	1.54e+01	1.03e+03	12	12	11	1	9
	WORHP SQP	optimal	21	3.47e+01	1.03e+03	22	22	23	3	21
HIER133A	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	20	5.01e+01	1.03e+03	23	24	22	23	20
	SNOPT	optimal	162	3.37e+00	1.03e+03	181	1	180	1	0
	WORHP IP	optimal	11	4.75e+01	1.03e+03	13	13	12	1	11
	WORHP IPm	optimal	11	1.57e+01	1.03e+03	15	15	14	1	11
	WORHP SQP	optimal	73	1.76e+02	1.03e+03	74	74	75	3	73
HIER133B	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	20	4.83e+01	1.03e+03	23	24	22	23	20
	SNOPT	optimal	162	3.40e+00	1.03e+03	181	1	180	1	0
	WORHP IP	optimal	11	4.83e+01	1.03e+03	13	13	12	1	11
	WORHP IPm	optimal	11	1.61e+01	1.03e+03	15	15	14	1	11
	WORHP SQP	optimal	73	1.21e+02	1.03e+03	74	74	75	3	73
HIER133C	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	20	3.83e+01	1.03e+03	23	24	22	23	20
	SNOPT	optimal	162	3.36e+00	1.03e+03	181	1	180	1	0
	WORHP IP	optimal	11	4.74e+01	1.03e+03	13	13	12	1	11
	WORHP IPm	optimal	11	2.15e+01	1.03e+03	15	15	14	1	11
	WORHP SQP	optimal	73	1.24e+02	1.03e+03	74	74	75	3	73
HIER133D	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	22	4.52e+01	4.11e+03	25	26	24	25	22
	SNOPT	optimal	247	3.15e+00	4.11e+03	267	1	266	1	0
	WORHP IP	optimal	13	5.50e+01	4.11e+03	15	15	14	1	13
	WORHP IPm	optimal	10	1.53e+01	4.11e+03	15	15	14	1	10
	WORHP SQP	optimal	512	3.41e+02	4.11e+03	1337	1341	514	3	512
HIER133E	IPOPT	degree	0	2.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	17	6.69e+01	4.11e+03	20	21	19	20	17
	SNOPT	optimal	174	3.25e+00	4.11e+03	189	1	188	1	0
	WORHP IP	optimal	16	5.33e+01	4.11e+03	18	18	17	1	16
	WORHP IPm	optimal	14	1.31e+01	4.11e+03	19	19	18	1	14
	WORHP SQP	optimal	190	1.21e+02	4.11e+03	191	191	192	3	190
HIER16	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	42	4.70e+02	1.54e+03	47	48	44	45	42
	SNOPT	optimal	43	3.12e+00	1.54e+03	49	1	48	1	0
	WORHP IP	optimal	10	6.50e+01	1.54e+03	12	12	11	1	10
	WORHP IPm	optimal	9	5.94e+01	1.54e+03	14	14	13	1	9
	WORHP SQP	optimal	22	1.12e+02	1.54e+03	23	23	24	3	22
HIER163A	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	16	9.42e+01	1.53e+03	19	20	18	19	16
	SNOPT	optimal	446	5.55e+01	1.53e+03	463	1	462	1	0
	WORHP IP	optimal	18	2.35e+01	1.53e+03	20	20	19	1	18
	WORHP IPm	optimal	14	5.82e+00	1.53e+03	19	19	18	1	14
	WORHP SQP	optimal	51	1.93e+02	1.53e+03	52	52	53	3	51
HIER163B	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	16	7.44e+01	1.53e+03	19	20	18	19	16
	SNOPT	optimal	446	5.59e+01	1.53e+03	463	1	462	1	0
	WORHP IP	optimal	18	3.27e+01	1.53e+03	20	20	19	1	18
	WORHP IPm	optimal	14	7.52e+00	1.53e+03	19	19	18	1	14
	WORHP SQP	optimal	51	2.19e+02	1.53e+03	52	52	53	3	51
HIER163C	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	16	9.11e+01	1.53e+03	19	20	18	19	16
	SNOPT	optimal	446	4.01e+01	1.53e+03	463	1	462	1	0
	WORHP IP	optimal	18	1.87e+01	1.53e+03	20	20	19	1	18
	WORHP IPm	optimal	14	6.70e+00	1.53e+03	19	19	18	1	14
	WORHP SQP	optimal	51	2.79e+02	1.53e+03	52	52	53	3	51

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HIER163D	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	28	8.50e+02	6.14e+03	30	31	29	30	27
	SNOPT	optimal	393	4.73e+01	6.14e+03	402	1	401	1	0
	WORHP IP	optimal	16	1.62e+01	6.14e+03	18	18	17	1	16
	WORHP IPm	optimal	13	7.43e+00	6.14e+03	18	18	17	1	13
	WORHP SQP	optimal	230	6.46e+02	6.14e+03	231	231	232	3	230
HIER163E	IPOPT	degree	0	5.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	28	7.89e+02	6.14e+03	30	31	29	30	27
	SNOPT	optimal	393	4.05e+01	6.14e+03	402	1	401	1	0
	WORHP IP	optimal	16	1.76e+01	6.14e+03	18	18	17	1	16
	WORHP IPm	optimal	13	9.63e+00	6.14e+03	18	18	17	1	13
	WORHP SQP	optimal	230	7.55e+02	6.14e+03	231	231	232	3	230
HILBERTA	IPOPT	optimal	1	1.00e-02	2.96e-31	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.31e-31	3	0	2	0	1
	SNOPT	optimal	7	1.00e-02	9.35e-19	11	0	10	0	0
	WORHP IP	optimal	1	1.00e-02	2.96e-31	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	2.96e-31	3	0	2	0	1
	WORHP SQP	optimal	3	1.00e-02	2.66e-14	4	0	4	0	3
HILBERTB	IPOPT	optimal	1	1.00e-02	4.31e-30	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	2.68e-29	3	0	2	0	1
	SNOPT	optimal	7	1.00e-02	1.87e-15	10	0	9	0	0
	WORHP IP	optimal	1	1.00e-02	3.26e-29	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	3.26e-29	3	0	2	0	1
	WORHP SQP	optimal	2	1.00e-02	4.90e-29	3	0	3	0	2
HIMMELBA	IPOPT	optimal	1	1.00e-02	0.00e+00	2	2	2	2	1
	KNITRO	optimal	0	1.00e-02	0.00e+00	3	4	2	3	0
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP SQP	optimal	1	1.00e-02	0.00e+00	2	2	3	3	1
HIMMELBB	IPOPT	optimal	18	1.00e-02	1.40e-17	26	0	19	0	18
	KNITRO	optimal	8	1.00e-02	3.79e-18	11	0	9	0	8
	SNOPT	optimal	11	1.00e-02	1.70e-11	15	0	14	0	0
	WORHP IP	optimal	18	1.00e-02	1.40e-17	24	0	20	0	18
	WORHP IPm	optimal	18	1.00e-02	1.40e-17	24	0	19	0	18
	WORHP SQP	optimal	20	1.00e-02	1.81e-16	41	0	21	0	20
HIMMELBC	IPOPT	optimal	5	1.00e-02	0.00e+00	8	8	6	6	5
	KNITRO	optimal	5	1.00e-02	0.00e+00	8	9	6	7	5
	SNOPT	optimal	5	1.00e-02	0.00e+00	1	9	1	8	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	WORHP SQP	optimal	5	1.00e-02	0.00e+00	6	6	7	7	5
HIMMELBD	IPOPT	infeas	18	1.00e-02	0.00e+00	77	81	9	21	19
	KNITRO	infeas	33	1.00e-02	0.00e+00	176	177	34	35	34
	SNOPT	infeas	15	1.00e-02	0.00e+00	1	29	1	28	0
	WORHP IP	infeas	14	1.00e-02	0.00e+00	63	63	15	15	15
	WORHP IPm	infeas	14	1.00e-02	0.00e+00	63	63	15	15	15
	WORHP SQP	minalpha	54	1.00e-02	0.00e+00	6856	6862	56	56	55
HIMMELBE	IPOPT	optimal	2	1.00e-02	0.00e+00	3	3	3	3	2
	KNITRO	optimal	1	1.00e-02	0.00e+00	4	5	3	4	1
	SNOPT	optimal	1	1.00e-02	0.00e+00	1	4	1	3	0
	WORHP IP	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	3	1.00e-02	0.00e+00	4	4	5	5	3
HIMMELBF	IPOPT	optimal	75	2.00e-02	3.19e+02	89	0	76	0	75
	KNITRO	optimal	56	1.00e-02	3.19e+02	66	0	57	0	56
	SNOPT	optimal	51	1.00e-02	3.19e+02	54	0	53	0	0
	WORHP IP	optimal	75	1.00e-02	3.19e+02	87	0	77	0	75
	WORHP IPm	optimal	75	1.00e-02	3.19e+02	87	0	76	0	75
	WORHP SQP	optimal	78	1.00e-02	3.19e+02	80	0	79	0	78
HIMMELBG	IPOPT	optimal	6	1.00e-02	3.63e-22	14	0	7	0	6
	KNITRO	optimal	6	1.00e-02	3.63e-22	11	0	7	0	6
	SNOPT	optimal	7	1.00e-02	2.56e-14	11	0	10	0	0
	WORHP IP	optimal	6	1.00e-02	3.63e-22	12	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	3.63e-22	12	0	7	0	6
	WORHP SQP	optimal	4	1.00e-02	7.40e-14	5	0	5	0	4
HIMMELBH	IPOPT	optimal	4	1.00e-02	-1.00e+00	24	0	5	0	4
	KNITRO	optimal	4	1.00e-02	-1.00e+00	10	0	5	0	4
	SNOPT	optimal	6	1.00e-02	-1.00e+00	10	0	9	0	0
	WORHP IP	optimal	4	1.00e-02	-1.00e+00	22	0	5	0	4
	WORHP IPm	optimal	4	1.00e-02	-1.00e+00	22	0	5	0	4
	WORHP SQP	optimal	5	1.00e-02	-1.00e+00	74	0	6	0	5
HIMMELBI	IPOPT	optimal	25	1.00e-02	-1.74e+03	26	26	26	26	25
	KNITRO	optimal	14	1.00e-02	-1.74e+03	17	18	16	17	14
	SNOPT	optimal	61	1.00e-02	-1.74e+03	72	1	71	1	0
	WORHP IP	optimal	23	1.00e-02	-1.74e+03	25	25	24	1	23
	WORHP IPm	optimal	20	1.00e-02	-1.74e+03	22	22	21	1	20
	WORHP SQP	optimal	10	2.00e-02	-1.74e+03	11	11	12	3	10

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HIMMELBJ	IPOPT	maxiter	10000	3.19e+00	0.00e+00	10003	10003	23	10001	10000
	KNITRO	optimal	19	1.00e-02	-1.91e+03	37	38	21	22	19
	SNOPT	optimal	11	1.00e-02	-1.78e+03	141	1	140	1	0
	WORHP IP	optimal	41	2.00e-02	-1.91e+03	111	111	43	1	41
	WORHP IPm	minalpha	78	6.00e-02	1.00e+00	767	767	132	1	79
	WORHP SQP	optimal	44	9.00e-02	-1.91e+03	45	45	46	3	44
HIMMELBK	IPOPT	optimal	17	1.00e-02	5.18e-02	18	18	18	18	17
	KNITRO	optimal	8	1.00e-02	5.18e-02	10	11	9	10	8
	SNOPT	optimal	8	1.00e-02	5.18e-02	1	14	1	13	0
	WORHP IP	optimal	14	1.00e-02	5.18e-02	16	16	15	15	14
	WORHP IPm	optimal	15	1.00e-02	5.18e-02	19	19	18	18	15
	WORHP SQP	optimal	5	1.00e-02	5.18e-02	6	6	7	7	5
HIMMELP1	IPOPT	optimal	11	1.00e-02	-6.21e+01	12	0	12	0	11
	KNITRO	optimal	8	1.00e-02	-6.21e+01	11	0	10	0	8
	SNOPT	optimal	9	1.00e-02	-6.21e+01	20	0	19	0	0
	WORHP IP	optimal	10	1.00e-02	-6.21e+01	12	0	11	0	10
	WORHP IPm	optimal	11	1.00e-02	-6.21e+01	14	0	12	0	11
	WORHP SQP	optimal	11	1.00e-02	-6.21e+01	12	0	12	0	11
HIMMELP2	IPOPT	optimal	18	1.00e-02	-8.20e+00	19	19	19	19	18
	KNITRO	optimal	10	1.00e-02	-6.21e+01	13	14	12	13	10
	SNOPT	optimal	19	1.00e-02	-6.21e+01	36	36	35	35	0
	WORHP IP	optimal	24	1.00e-02	-6.21e+01	26	26	25	25	24
	WORHP IPm	optimal	20	1.00e-02	-8.20e+00	23	23	21	21	20
	WORHP SQP	optimal	13	1.00e-02	-6.21e+01	14	14	15	15	13
HIMMELP3	IPOPT	optimal	12	1.00e-02	-5.90e+01	13	13	13	13	12
	KNITRO	optimal	11	1.00e-02	-5.90e+01	13	14	12	13	11
	SNOPT	optimal	5	1.00e-02	-5.90e+01	9	9	8	8	0
	WORHP IP	optimal	7	1.00e-02	-5.90e+01	9	9	8	8	7
	WORHP IPm	optimal	11	1.00e-02	-5.90e+01	15	15	14	14	11
	WORHP SQP	optimal	9	1.00e-02	-5.90e+01	9	9	10	10	9
HIMMELP4	IPOPT	optimal	24	1.00e-02	-5.90e+01	25	25	25	25	24
	KNITRO	optimal	11	1.00e-02	-5.90e+01	13	14	12	13	11
	SNOPT	optimal	5	1.00e-02	-5.90e+01	10	10	9	9	0
	WORHP IP	optimal	10	1.00e-02	-5.90e+01	12	12	11	11	10
	WORHP IPm	optimal	23	1.00e-02	-5.90e+01	27	27	26	26	23
	WORHP SQP	optimal	9	1.00e-02	-5.90e+01	9	9	10	10	9
HIMMELP5	IPOPT	optimal	59	1.00e-02	-5.90e+01	195	195	56	61	59
	KNITRO	optimal	13	1.00e-02	-5.90e+01	15	16	14	15	13
	SNOPT	optimal	17	1.00e-02	-5.90e+01	44	44	43	43	0
	WORHP IP	optimal	17	1.00e-02	-5.90e+01	19	19	18	18	17
	WORHP IPm	optimal	18	1.00e-02	-5.90e+01	20	20	19	19	18
	WORHP SQP	optimal	11	1.00e-02	-5.90e+01	12	12	13	13	11
HIMMELP6	IPOPT	optimal	23	1.00e-02	-5.90e+01	31	31	24	24	23
	KNITRO	optimal	11	1.00e-02	-5.90e+01	13	14	12	13	11
	SNOPT	optimal	17	1.00e-02	-5.90e+01	50	50	49	49	0
	WORHP IP	optimal	20	1.00e-02	-5.90e+01	22	22	21	21	20
	WORHP IPm	optimal	26	1.00e-02	-5.90e+01	29	29	28	28	26
	WORHP SQP	optimal	15	1.00e-02	-5.90e+01	22	22	17	17	15
HOLMES	IPOPT	optimal	12	7.87e+00	1.25e+03	13	0	13	0	12
	KNITRO	optimal	10	9.93e+00	1.25e+03	13	0	12	0	10
	SNOPT	optimal	26	2.90e-01	1.25e+03	38	0	37	0	0
	WORHP IP	optimal	10	7.87e+00	1.25e+03	12	0	11	0	10
	WORHP IPm	optimal	21	1.30e+01	1.25e+03	28	0	27	0	21
	WORHP SQP	optimal	6	6.46e+00	1.25e+03	7	0	7	0	6
HONG	IPOPT	optimal	8	1.00e-02	2.26e+01	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	2.26e+01	8	9	7	8	6
	SNOPT	optimal	9	1.00e-02	2.26e+01	13	1	12	1	0
	WORHP IP	optimal	6	1.00e-02	2.26e+01	8	8	7	1	6
	WORHP IPm	optimal	8	1.00e-02	2.26e+01	10	10	9	1	8
	WORHP SQP	optimal	6	1.00e-02	2.26e+01	7	7	8	3	6
HS1	IPOPT	optimal	25	1.00e-02	1.33e-15	53	0	26	0	25
	KNITRO	optimal	25	1.00e-02	1.17e-18	35	0	26	0	25
	SNOPT	optimal	37	1.00e-02	3.08e-15	49	0	48	0	0
	WORHP IP	optimal	26	1.00e-02	9.81e-18	34	0	27	0	26
	WORHP IPm	optimal	25	1.00e-02	2.57e-16	38	0	26	0	25
	WORHP SQP	optimal	23	1.00e-02	5.53e-19	80	0	24	0	23
HS10	IPOPT	optimal	12	1.00e-02	-1.00e+00	13	13	13	13	12
	KNITRO	optimal	9	1.00e-02	-1.00e+00	11	12	10	11	9
	SNOPT	optimal	19	1.00e-02	-1.00e+00	1	33	1	32	0
	WORHP IP	optimal	9	1.00e-02	-1.00e+00	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	-1.00e+00	11	11	10	10	9
	WORHP SQP	optimal	12	1.00e-02	-1.00e+00	13	13	14	14	12
HS100	IPOPT	optimal	11	1.00e-02	6.81e+02	22	22	12	12	11
	KNITRO	optimal	7	1.00e-02	6.81e+02	13	14	8	9	7
	SNOPT	optimal	12	1.00e-02	6.81e+02	17	17	16	16	0
	WORHP IP	optimal	8	1.00e-02	6.81e+02	17	17	9	9	8
	WORHP IPm	optimal	8	1.00e-02	6.81e+02	17	17	9	9	8
	WORHP SQP	optimal	9	1.00e-02	6.81e+02	40	40	11	11	9

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS100LNP	IPOPT	optimal	20	1.00e-02	6.81e+02	21	21	21	21	20
	KNITRO	optimal	6	1.00e-02	6.81e+02	8	9	7	8	6
	SNOPT	optimal	15	1.00e-02	6.81e+02	25	25	24	24	0
	WORHP IP	optimal	16	1.00e-02	6.81e+02	24	24	17	17	16
	WORHP IPm	optimal	16	1.00e-02	6.81e+02	24	24	17	17	16
	WORHP SQP	optimal	24	1.00e-02	6.81e+02	37	37	26	26	24
HS100MOD	IPOPT	optimal	9	1.00e-02	6.79e+02	27	27	10	10	9
	KNITRO	optimal	10	1.00e-02	6.79e+02	16	17	11	12	10
	SNOPT	optimal	13	1.00e-02	6.79e+02	18	18	17	17	0
	WORHP IP	optimal	6	1.00e-02	6.79e+02	13	13	7	7	6
	WORHP IPm	optimal	6	1.00e-02	6.79e+02	13	13	7	7	6
	WORHP SQP	optimal	7	1.00e-02	6.79e+02	35	35	9	9	7
HS101	IPOPT	optimal	34	1.00e-02	1.81e+03	76	76	35	35	34
	KNITRO	optimal	37	1.00e-02	1.81e+03	46	47	38	39	37
	SNOPT	optimal	140	1.00e-02	1.81e+03	551	551	550	550	0
	WORHP IP	optimal	83	2.00e-02	1.81e+03	369	369	84	84	83
	WORHP IPm	optimal	208	3.00e-02	1.81e+03	1066	1066	209	209	208
	WORHP SQP	maxiter	10000	2.16e+00	1.81e+03	12528	12532	10002	10002	10001
HS102	IPOPT	optimal	20	1.00e-02	9.12e+02	37	37	21	21	20
	KNITRO	optimal	19	1.00e-02	9.12e+02	23	24	20	21	19
	SNOPT	optimal	62	1.00e-02	9.12e+02	239	239	238	238	0
	WORHP IP	optimal	36	1.00e-02	9.12e+02	40	40	38	38	36
	WORHP IPm	minalpha	22	1.00e-02	1.00e+00	436	436	61	61	23
	WORHP SQP	optimal	20	1.00e-02	9.12e+02	21	21	22	22	20
HS103	IPOPT	optimal	39	2.00e-02	5.44e+02	67	67	40	40	39
	KNITRO	optimal	22	1.00e-02	5.44e+02	24	25	23	24	22
	SNOPT	optimal	49	1.00e-02	5.44e+02	175	175	174	174	0
	WORHP IP	optimal	50	1.00e-02	5.44e+02	188	188	52	52	50
	WORHP IPm	optimal	26	1.00e-02	5.44e+02	30	30	27	27	26
	WORHP SQP	zerostep	1612	3.10e-01	5.44e+02	1855	1855	1613	1613	1612
HS104	IPOPT	optimal	8	1.00e-02	3.95e+00	9	9	9	9	8
	KNITRO	optimal	7	1.00e-02	3.95e+00	9	10	8	9	7
	SNOPT	optimal	22	1.00e-02	3.95e+00	30	30	29	29	0
	WORHP IP	optimal	7	1.00e-02	3.95e+00	9	9	8	8	7
	WORHP IPm	optimal	8	1.00e-02	3.95e+00	11	11	9	9	8
	WORHP SQP	optimal	13	1.00e-02	3.95e+00	17	17	15	15	13
HS105	IPOPT	optimal	17	2.00e-02	1.04e+03	23	23	18	18	17
	KNITRO	optimal	20	3.00e-02	1.04e+03	24	25	22	23	20
	SNOPT	optimal	50	2.00e-02	1.04e+03	64	1	63	1	0
	WORHP IP	optimal	16	4.00e-02	1.04e+03	19	19	17	1	16
	WORHP IPm	optimal	34	1.00e-01	1.04e+03	298	298	40	1	34
	WORHP SQP	optimal	15	3.00e-02	1.04e+03	18	18	17	3	15
HS106	IPOPT	optimal	15	1.00e-02	7.05e+03	16	16	16	16	15
	KNITRO	optimal	22	1.00e-02	7.05e+03	24	25	23	24	22
	SNOPT	optimal	11	1.00e-02	7.05e+03	1	15	1	14	0
	WORHP IP	optimal	1240	2.10e-01	7.05e+03	11690	11690	1241	1241	1240
	WORHP IPm	optimal	1558	2.50e-01	7.05e+03	15174	15174	1559	1559	1558
	WORHP SQP	optimal	58	2.00e-02	7.05e+03	211	207	59	64	57
HS107	IPOPT	optimal	10	1.00e-02	5.06e+03	11	11	11	11	10
	KNITRO	optimal	9	1.00e-02	5.06e+03	11	12	10	11	9
	SNOPT	optimal	8	1.00e-02	5.06e+03	15	15	14	14	0
	WORHP IP	optimal	10	1.00e-02	5.06e+03	14	14	12	12	10
	WORHP IPm	optimal	11	1.00e-02	5.06e+03	16	16	15	15	11
	WORHP SQP	optimal	8	1.00e-02	5.06e+03	9	9	10	10	8
HS108	IPOPT	optimal	16	1.00e-02	-6.75e-01	17	17	17	17	16
	KNITRO	optimal	9	1.00e-02	-8.66e-01	11	12	10	11	9
	SNOPT	optimal	12	1.00e-02	-8.66e-01	20	20	19	19	0
	WORHP IP	optimal	44	1.00e-02	-8.66e-01	76	76	46	46	44
	WORHP IPm	optimal	11	1.00e-02	-8.66e-01	16	16	15	15	11
	WORHP SQP	optimal	17	1.00e-02	-6.75e-01	18	18	19	19	17
HS109	IPOPT	optimal	21	1.00e-02	5.36e+03	52	104	22	44	21
	KNITRO	optimal	12	1.00e-02	5.36e+03	15	16	14	15	12
	SNOPT	optimal	52	1.00e-02	5.36e+03	168	168	167	167	0
	WORHP IP	optimal	45	1.00e-02	5.36e+03	142	142	46	46	45
	WORHP IPm	optimal	27	1.00e-02	5.36e+03	57	57	31	31	27
	WORHP SQP	optimal	45	1.00e-02	5.36e+03	147	144	45	50	43
HS11	IPOPT	optimal	8	1.00e-02	-8.50e+00	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	-8.50e+00	8	9	7	8	6
	SNOPT	optimal	9	1.00e-02	-8.50e+00	16	16	15	15	0
	WORHP IP	optimal	6	1.00e-02	-8.50e+00	8	8	7	7	6
	WORHP IPm	optimal	6	1.00e-02	-8.50e+00	8	8	7	7	6
	WORHP SQP	optimal	9	1.00e-02	-8.50e+00	10	10	11	11	9
HS110	IPOPT	optimal	6	1.00e-02	-4.58e+01	7	0	7	0	6
	KNITRO	optimal	5	1.00e-02	-4.58e+01	7	0	6	0	5
	SNOPT	optimal	6	1.00e-02	-4.58e+01	12	0	11	0	0
	WORHP IP	optimal	5	1.00e-02	-4.58e+01	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	-4.58e+01	7	0	6	0	5
	WORHP SQP	optimal	6	1.00e-02	-4.58e+01	9	0	7	0	6

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS111	IPOPT	optimal	14	1.00e-02	-4.78e+01	15	15	15	15	14
	KNITRO	optimal	14	1.00e-02	-4.78e+01	16	17	15	16	14
	SNOPT	optimal	54	1.00e-02	-4.78e+01	63	63	62	62	0
	WORHP IP	optimal	20	1.00e-02	-4.78e+01	22	22	21	21	20
	WORHP IPm	optimal	20	1.00e-02	-4.78e+01	22	22	21	21	20
	WORHP SQP	optimal	26	1.00e-02	-4.31e+01	27	27	28	28	26
HS111LNP	IPOPT	optimal	14	1.00e-02	-4.78e+01	15	15	15	15	14
	KNITRO	optimal	14	1.00e-02	-4.78e+01	16	17	15	16	14
	SNOPT	optimal	52	1.00e-02	-4.78e+01	62	62	61	61	0
	WORHP IP	optimal	18	1.00e-02	-4.78e+01	20	20	19	19	18
	WORHP IPm	optimal	18	1.00e-02	-4.78e+01	20	20	19	19	18
	WORHP SQP	optimal	176	6.00e-02	-4.29e+01	1427	1420	156	187	154
HS112	IPOPT	optimal	17	1.00e-02	-4.78e+01	18	18	18	18	17
	KNITRO	optimal	6	1.00e-02	-4.78e+01	8	9	7	8	6
	SNOPT	optimal	18	1.00e-02	-4.78e+01	29	1	28	1	0
	WORHP IP	optimal	14	1.00e-02	-4.78e+01	16	16	15	1	14
	WORHP IPm	optimal	20	1.00e-02	-4.78e+01	66	66	22	1	20
	WORHP SQP	optimal	124	3.00e-02	-4.78e+01	1355	1356	126	3	124
HS113	IPOPT	optimal	11	1.00e-02	2.43e+01	12	12	12	12	11
	KNITRO	optimal	7	1.00e-02	2.43e+01	9	10	8	9	7
	SNOPT	optimal	16	1.00e-02	2.43e+01	19	19	18	18	0
	WORHP IP	optimal	9	1.00e-02	2.43e+01	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	2.43e+01	11	11	10	10	9
	WORHP SQP	optimal	9	1.00e-02	2.43e+01	10	10	11	11	9
HS114	IPOPT	optimal	19	1.00e-02	-1.77e+03	20	40	20	40	19
	KNITRO	optimal	8	1.00e-02	-1.77e+03	11	12	10	11	8
	SNOPT	optimal	13	1.00e-02	-1.77e+03	19	19	18	18	0
	WORHP IP	optimal	13	1.00e-02	-1.77e+03	15	15	14	14	13
	WORHP IPm	optimal	12	1.00e-02	-1.77e+03	16	16	15	15	12
	WORHP SQP	optimal	22	1.00e-02	-1.77e+03	23	23	24	24	22
HS116	IPOPT	optimal	25	1.00e-02	9.76e+01	26	26	26	26	25
	KNITRO	optimal	16	1.00e-02	9.76e+01	19	20	18	19	16
	SNOPT	optimal	16	1.00e-02	9.76e+01	1	20	1	19	0
	WORHP IP	optimal	144	3.00e-02	9.76e+01	384	384	145	145	144
	WORHP IPm	optimal	419	9.00e-02	9.76e+01	2154	2154	423	423	419
	WORHP SQP	optimal	24	1.00e-02	9.76e+01	25	25	26	26	24
HS117	IPOPT	optimal	22	1.00e-02	3.23e+01	23	23	23	23	22
	KNITRO	optimal	12	1.00e-02	3.23e+01	14	15	13	14	12
	SNOPT	optimal	18	1.00e-02	3.23e+01	25	25	24	24	0
	WORHP IP	optimal	28	1.00e-02	3.23e+01	30	30	29	29	28
	WORHP IPm	optimal	34	1.00e-02	3.23e+01	42	42	38	38	34
	WORHP SQP	optimal	24	1.00e-02	3.23e+01	25	25	26	26	24
HS118	IPOPT	optimal	11	1.00e-02	6.65e+02	12	12	12	12	11
	KNITRO	optimal	10	1.00e-02	6.65e+02	12	13	11	12	10
	SNOPT	optimal	3	1.00e-02	6.65e+02	8	1	7	1	0
	WORHP IP	optimal	9	1.00e-02	6.65e+02	11	11	10	1	9
	WORHP IPm	optimal	9	1.00e-02	6.65e+02	14	14	13	1	9
	WORHP SQP	optimal	7	1.00e-02	6.65e+02	8	8	9	3	7
HS119	IPOPT	optimal	14	1.00e-02	2.45e+02	15	15	15	15	14
	KNITRO	optimal	10	1.00e-02	2.45e+02	13	14	12	13	10
	SNOPT	optimal	18	1.00e-02	2.45e+02	21	1	20	1	0
	WORHP IP	optimal	12	1.00e-02	2.45e+02	14	14	14	1	12
	WORHP IPm	optimal	12	1.00e-02	2.45e+02	18	18	17	1	12
	WORHP SQP	optimal	10	1.00e-02	2.45e+02	11	11	11	2	10
HS12	IPOPT	optimal	8	1.00e-02	-3.00e+01	9	9	9	9	8
	KNITRO	optimal	7	1.00e-02	-3.00e+01	11	12	8	9	7
	SNOPT	optimal	8	1.00e-02	-3.00e+01	12	12	11	11	0
	WORHP IP	optimal	8	1.00e-02	-3.00e+01	10	10	9	9	8
	WORHP IPm	optimal	7	1.00e-02	-3.00e+01	9	9	8	8	7
	WORHP SQP	optimal	12	1.00e-02	-3.00e+01	13	13	14	14	12
HS13	IPOPT	optimal	54	1.00e-02	9.95e-01	78	78	55	56	54
	KNITRO	optimal	24	1.00e-02	9.99e-01	27	28	26	27	24
	SNOPT	optimal	15	1.00e-02	1.00e+00	18	18	17	17	0
	WORHP IP	optimal	32	1.00e-02	9.99e-01	278	278	36	36	32
	WORHP IPm	optimal	76	1.00e-02	1.00e+00	1468	1468	169	169	76
	WORHP SQP	maxiter	10000	7.20e-01	1.00e+00	10456	10467	10002	10002	10001
HS14	IPOPT	optimal	7	1.00e-02	1.39e+00	8	16	8	16	7
	KNITRO	optimal	5	1.00e-02	1.39e+00	7	8	6	7	5
	SNOPT	optimal	6	1.00e-02	1.39e+00	11	11	10	10	0
	WORHP IP	optimal	6	1.00e-02	1.39e+00	8	8	7	7	6
	WORHP IPm	optimal	6	1.00e-02	1.39e+00	8	8	7	7	6
	WORHP SQP	optimal	10	1.00e-02	1.39e+00	11	11	12	12	10
HS15	IPOPT	optimal	16	1.00e-02	3.06e+02	21	21	17	17	16
	KNITRO	optimal	8	1.00e-02	3.07e+02	10	11	9	10	8
	SNOPT	optimal	7	1.00e-02	3.06e+02	12	12	11	11	0
	WORHP IP	optimal	22	1.00e-02	3.07e+02	66	66	24	24	22
	WORHP IPm	optimal	15	1.00e-02	3.06e+02	21	21	19	19	15
	WORHP SQP	optimal	12	1.00e-02	3.07e+02	12	12	13	13	12

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS16	IPOPT	optimal	18	1.00e-02	2.50e-01	22	22	19	19	18
	KNITRO	optimal	7	1.00e-02	2.50e-01	10	11	9	10	7
	SNOPT	optimal	3	1.00e-02	2.31e+01	6	6	5	5	0
	WORHP IP	optimal	7	1.00e-02	2.31e+01	9	9	8	8	7
	WORHP IPm	optimal	11	1.00e-02	2.31e+01	15	15	14	14	11
	WORHP SQP	optimal	6	1.00e-02	2.31e+01	561	561	7	7	6
HS17	IPOPT	optimal	16	1.00e-02	1.00e+00	17	17	17	17	16
	KNITRO	optimal	6	1.00e-02	1.00e+00	9	10	8	9	6
	SNOPT	optimal	10	1.00e-02	1.00e+00	20	20	19	19	0
	WORHP IP	optimal	13	1.00e-02	1.00e+00	15	15	14	14	13
	WORHP IPm	optimal	12	1.00e-02	1.00e+00	14	14	13	13	12
	WORHP SQP	optimal	9	1.00e-02	1.00e+00	44	44	11	11	9
HS18	IPOPT	optimal	15	1.00e-02	5.00e+00	19	19	16	16	15
	KNITRO	optimal	8	1.00e-02	5.00e+00	11	12	10	11	8
	SNOPT	optimal	17	1.00e-02	5.00e+00	31	31	30	30	0
	WORHP IP	optimal	10	1.00e-02	5.00e+00	12	12	11	11	10
	WORHP IPm	optimal	17	1.00e-02	5.00e+00	19	19	18	18	17
	WORHP SQP	optimal	10	1.00e-02	5.00e+00	11	11	12	12	10
HS19	IPOPT	optimal	14	1.00e-02	-6.96e+03	15	15	15	15	14
	KNITRO	optimal	13	1.00e-02	-6.96e+03	17	18	14	15	13
	SNOPT	optimal	7	1.00e-02	-6.96e+03	10	10	9	9	0
	WORHP IP	optimal	13	1.00e-02	-6.96e+03	15	15	14	14	13
	WORHP IPm	optimal	15	1.00e-02	-6.96e+03	18	18	17	17	15
	WORHP SQP	optimal	10	1.00e-02	-6.96e+03	11	11	12	12	10
HS2	IPOPT	optimal	11	1.00e-02	4.94e+00	17	0	12	0	11
	KNITRO	optimal	8	1.00e-02	4.94e+00	12	0	10	0	8
	SNOPT	optimal	11	1.00e-02	4.94e+00	16	0	15	0	0
	WORHP IP	optimal	10	1.00e-02	4.94e+00	12	0	12	0	10
	WORHP IPm	optimal	9	1.00e-02	4.94e+00	14	0	12	0	9
	WORHP SQP	zerostep	4306	4.70e-01	4.94e+00	60034	0	4306	0	4306
HS20	IPOPT	optimal	6	1.00e-02	4.02e+01	7	7	7	7	6
	KNITRO	optimal	5	1.00e-02	4.02e+01	8	9	7	8	5
	SNOPT	optimal	3	1.00e-02	4.02e+01	6	6	5	5	0
	WORHP IP	optimal	6	1.00e-02	4.02e+01	8	8	7	7	6
	WORHP IPm	optimal	11	1.00e-02	4.02e+01	18	18	15	15	11
	WORHP SQP	optimal	3	1.00e-02	4.02e+01	4	4	5	5	3
HS21	IPOPT	optimal	8	1.00e-02	-1.00e+02	9	9	9	9	8
	KNITRO	optimal	4	1.00e-02	-1.00e+02	7	8	6	7	4
	SNOPT	optimal	1	1.00e-02	-1.00e+02	4	1	3	1	0
	WORHP IP	optimal	6	1.00e-02	-1.00e+02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	-1.00e+02	8	8	7	1	6
	WORHP SQP	optimal	2	1.00e-02	-1.00e+02	3	3	4	3	2
HS21MOD	IPOPT	optimal	12	1.00e-02	-9.60e+01	13	13	13	13	12
	KNITRO	optimal	8	1.00e-02	-9.60e+01	11	12	10	11	8
	SNOPT	optimal	3	1.00e-02	-9.60e+01	6	1	5	1	0
	WORHP IP	optimal	9	1.00e-02	-9.60e+01	11	11	10	1	9
	WORHP IPm	optimal	9	1.00e-02	-9.60e+01	14	14	13	1	9
	WORHP SQP	zerostep	33	1.00e-02	-9.60e+01	33	33	34	3	33
HS22	IPOPT	optimal	6	1.00e-02	1.00e+00	7	7	7	7	6
	KNITRO	optimal	4	1.00e-02	1.00e+00	6	7	5	6	4
	SNOPT	optimal	4	1.00e-02	1.00e+00	7	7	6	6	0
	WORHP IP	optimal	5	1.00e-02	1.00e+00	7	7	6	6	5
	WORHP IPm	optimal	4	1.00e-02	1.00e+00	6	6	5	5	4
	WORHP SQP	optimal	6	1.00e-02	1.00e+00	7	7	8	8	6
HS23	IPOPT	optimal	10	1.00e-02	2.00e+00	12	12	11	11	10
	KNITRO	optimal	7	1.00e-02	2.00e+00	9	10	8	9	7
	SNOPT	optimal	5	1.00e-02	2.00e+00	8	8	7	7	0
	WORHP IP	optimal	9	1.00e-02	2.00e+00	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	2.00e+00	11	11	10	10	9
	WORHP SQP	optimal	10	1.00e-02	2.00e+00	11	11	12	12	10
HS24	IPOPT	optimal	12	1.00e-02	-1.00e+00	14	14	13	13	12
	KNITRO	optimal	6	1.00e-02	-1.00e+00	8	9	7	8	6
	SNOPT	optimal	3	1.00e-02	-1.00e+00	9	1	8	1	0
	WORHP IP	optimal	10	1.00e-02	-1.00e+00	12	12	11	1	10
	WORHP IPm	optimal	8	1.00e-02	-1.00e+00	10	10	9	1	8
	WORHP SQP	optimal	10	1.00e-02	-1.00e+00	10	10	11	3	10
HS25	IPOPT	optimal	35	2.00e-02	1.14e-12	43	0	36	0	35
	KNITRO	optimal	0	1.00e-02	3.28e+01	3	0	2	0	0
	SNOPT	optimal	0	1.00e-02	3.28e+01	3	0	2	0	0
	WORHP IP	optimal	4	1.00e-02	3.28e+01	6	0	5	0	4
	WORHP IPm	optimal	4	1.00e-02	3.28e+01	6	0	5	0	4
	WORHP SQP	optimal	0	1.00e-02	3.28e+01	1	0	1	0	0
HS26	IPOPT	optimal	19	1.00e-02	2.17e-12	20	20	20	20	19
	KNITRO	optimal	19	1.00e-02	2.17e-12	21	22	20	21	19
	SNOPT	optimal	25	1.00e-02	8.32e-12	28	28	27	27	0
	WORHP IP	optimal	18	1.00e-02	1.65e-12	20	20	19	19	18
	WORHP IPm	optimal	18	1.00e-02	1.65e-12	20	20	19	19	18
	WORHP SQP	optimal	19	1.00e-02	2.20e-12	20	20	21	21	19

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS268	IPOPT	optimal	17	1.00e-02	1.70e-07	18	18	18	18	17
	KNITRO	optimal	12	1.00e-02	1.95e-07	15	16	13	14	12
	SNOPT	optimal	45	1.00e-02	-2.55e-11	49	1	48	1	0
	WORHP IP	optimal	13	1.00e-02	1.06e-07	15	15	15	1	13
	WORHP IPm	optimal	12	1.00e-02	2.26e-07	14	14	13	1	12
	WORHP SQP	optimal	15	1.00e-02	7.28e-12	16	16	17	3	15
HS27	IPOPT	optimal	54	1.00e-02	4.00e-02	135	140	55	56	54
	KNITRO	optimal	21	1.00e-02	4.00e-02	44	45	22	23	21
	SNOPT	optimal	11	1.00e-02	4.00e-02	16	16	15	15	0
	WORHP IP	optimal	19	1.00e-02	4.00e-02	58	58	20	20	19
	WORHP IPm	optimal	19	1.00e-02	4.00e-02	58	58	20	20	19
	WORHP SQP	optimal	10	1.00e-02	4.00e-02	20	20	12	12	10
HS28	IPOPT	optimal	1	1.00e-02	3.08e-31	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	3.08e-31	3	4	2	3	1
	SNOPT	optimal	10	1.00e-02	4.14e-14	13	1	12	1	0
	WORHP IP	optimal	1	1.00e-02	4.93e-32	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	4.93e-32	3	3	2	1	1
	WORHP SQP	optimal	3	1.00e-02	3.56e-20	4	4	5	3	3
HS29	IPOPT	optimal	8	1.00e-02	-2.26e+01	9	9	9	9	8
	KNITRO	optimal	23	1.00e-02	-2.26e+01	29	30	24	25	23
	SNOPT	optimal	10	1.00e-02	-2.26e+01	17	17	16	16	0
	WORHP IP	optimal	11	1.00e-02	-2.26e+01	13	13	12	12	11
	WORHP IPm	optimal	11	1.00e-02	-2.26e+01	13	13	12	12	11
	WORHP SQP	optimal	9	1.00e-02	-2.26e+01	10	10	11	11	9
HS3	IPOPT	optimal	4	1.00e-02	8.09e-08	5	0	5	0	4
	KNITRO	optimal	3	1.00e-02	6.19e-10	5	0	4	0	3
	SNOPT	optimal	3	1.00e-02	3.16e-35	10	0	9	0	0
	WORHP IP	optimal	3	1.00e-02	2.75e-10	5	0	4	0	3
	WORHP IPm	optimal	3	1.00e-02	3.00e-12	5	0	4	0	3
	WORHP SQP	optimal	10	1.00e-02	2.42e-08	11	0	11	0	10
HS30	IPOPT	optimal	8	1.00e-02	1.00e+00	12	12	9	9	8
	KNITRO	optimal	5	1.00e-02	1.00e+00	8	9	7	8	5
	SNOPT	optimal	13	1.00e-02	1.00e+00	16	16	15	15	0
	WORHP IP	optimal	9	1.00e-02	1.00e+00	14	14	10	10	9
	WORHP IPm	optimal	5	1.00e-02	1.00e+00	7	7	6	6	5
	WORHP SQP	optimal	10	1.00e-02	1.00e+00	11	11	12	12	10
HS31	IPOPT	optimal	7	1.00e-02	6.00e+00	8	8	8	8	7
	KNITRO	optimal	4	1.00e-02	6.00e+00	7	8	6	7	4
	SNOPT	optimal	7	1.00e-02	6.00e+00	12	12	11	11	0
	WORHP IP	optimal	7	1.00e-02	6.00e+00	10	10	8	8	7
	WORHP IPm	optimal	6	1.00e-02	6.00e+00	9	9	7	7	6
	WORHP SQP	optimal	5	1.00e-02	6.00e+00	6	6	7	7	5
HS32	IPOPT	optimal	12	1.00e-02	1.00e+00	16	32	13	26	12
	KNITRO	optimal	6	1.00e-02	1.00e+00	8	9	7	8	6
	SNOPT	optimal	3	1.00e-02	1.00e+00	6	6	5	5	0
	WORHP IP	optimal	9	1.00e-02	1.00e+00	11	11	10	10	9
	WORHP IPm	optimal	8	1.00e-02	1.00e+00	13	13	12	12	8
	WORHP SQP	optimal	3	1.00e-02	1.00e+00	4	4	5	5	3
HS33	IPOPT	optimal	11	1.00e-02	-4.59e+00	16	16	12	12	11
	KNITRO	optimal	6	1.00e-02	-4.59e+00	9	10	8	9	6
	SNOPT	optimal	5	1.00e-02	-4.00e+00	10	10	9	9	0
	WORHP IP	optimal	14	1.00e-02	-4.59e+00	16	16	15	15	14
	WORHP IPm	optimal	16	1.00e-02	-4.59e+00	20	20	19	19	16
	WORHP SQP	optimal	10	1.00e-02	-4.59e+00	11	11	12	12	10
HS34	IPOPT	optimal	9	1.00e-02	-8.34e-01	10	10	10	10	9
	KNITRO	optimal	6	1.00e-02	-8.34e-01	9	10	8	9	6
	SNOPT	optimal	5	1.00e-02	-8.34e-01	1	9	1	8	0
	WORHP IP	optimal	7	1.00e-02	-8.34e-01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	-8.34e-01	10	10	9	9	7
	WORHP SQP	optimal	6	1.00e-02	-8.34e-01	7	7	8	8	6
HS35	IPOPT	optimal	7	1.00e-02	1.11e-01	8	8	8	8	7
	KNITRO	optimal	5	1.00e-02	1.11e-01	7	8	6	7	5
	SNOPT	optimal	4	1.00e-02	1.11e-01	9	1	8	1	0
	WORHP IP	optimal	5	1.00e-02	1.11e-01	7	7	6	1	5
	WORHP IPm	optimal	5	1.00e-02	1.11e-01	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	1.11e-01	3	3	4	3	2
HS35I	IPOPT	optimal	7	1.00e-02	1.11e-01	8	8	8	8	7
	KNITRO	optimal	5	1.00e-02	1.11e-01	7	8	6	7	5
	SNOPT	optimal	4	1.00e-02	1.11e-01	9	1	8	1	0
	WORHP IP	optimal	6	1.00e-02	1.11e-01	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	1.11e-01	8	8	7	1	6
	WORHP SQP	optimal	2	1.00e-02	1.11e-01	3	3	4	3	2
HS35MOD	IPOPT	optimal	11	1.00e-02	2.50e-01	12	12	12	12	11
	KNITRO	optimal	8	1.00e-02	2.50e-01	10	11	9	10	8
	SNOPT	optimal	3	1.00e-02	2.50e-01	6	1	5	1	0
	WORHP IP	optimal	9	1.00e-02	2.50e-01	11	11	10	1	9
	WORHP IPm	optimal	9	1.00e-02	2.50e-01	11	11	10	1	9
	WORHP SQP	optimal	2	1.00e-02	2.50e-01	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS36	IPOPT	optimal	12	1.00e-02	-3.30e+03	13	13	13	13	12
	KNITRO	optimal	7	1.00e-02	-3.30e+03	9	10	8	9	7
	SNOPT	optimal	4	1.00e-02	-3.30e+03	9	1	8	1	0
	WORHP IP	optimal	7	1.00e-02	-3.30e+03	9	9	8	1	7
	WORHP IPm	optimal	7	1.00e-02	-3.30e+03	12	12	11	1	7
	WORHP SQP	optimal	4	1.00e-02	-3.30e+03	4	4	5	3	4
HS37	IPOPT	optimal	11	1.00e-02	-3.46e+03	12	12	12	12	11
	KNITRO	optimal	6	1.00e-02	-3.46e+03	8	9	7	8	6
	SNOPT	optimal	8	1.00e-02	-3.46e+03	12	1	11	1	0
	WORHP IP	optimal	10	1.00e-02	-3.46e+03	12	12	11	1	10
	WORHP IPm	optimal	10	1.00e-02	-3.46e+03	20	20	11	1	10
	WORHP SQP	optimal	4	1.00e-02	-3.46e+03	5	5	6	3	4
HS38	IPOPT	optimal	40	1.00e-02	1.91e-19	78	0	41	0	40
	KNITRO	optimal	40	1.00e-02	4.41e-20	53	0	41	0	40
	SNOPT	optimal	80	1.00e-02	1.16e-14	102	0	101	0	0
	WORHP IP	optimal	40	1.00e-02	1.12e-18	54	0	42	0	40
	WORHP IPm	optimal	39	1.00e-02	1.15e-20	52	0	40	0	39
	WORHP SQP	optimal	38	1.00e-02	1.90e-20	229	0	39	0	38
HS39	IPOPT	optimal	13	1.00e-02	-1.00e+00	14	14	14	14	13
	KNITRO	optimal	13	1.00e-02	-1.00e+00	15	16	14	15	13
	SNOPT	optimal	17	1.00e-02	-1.00e+00	1	30	1	29	0
	WORHP IP	optimal	22	1.00e-02	-1.00e+00	38	38	23	23	22
	WORHP IPm	optimal	22	1.00e-02	-1.00e+00	38	38	23	23	22
	WORHP SQP	optimal	12	1.00e-02	-1.00e+00	13	13	14	14	12
HS3MOD	IPOPT	optimal	5	1.00e-02	8.09e-08	6	0	6	0	5
	KNITRO	optimal	3	1.00e-02	1.12e-08	5	0	4	0	3
	SNOPT	optimal	6	1.00e-02	9.63e-28	11	0	10	0	0
	WORHP IP	optimal	5	1.00e-02	1.91e-10	7	0	6	0	5
	WORHP IPm	optimal	2	1.00e-02	1.00e-08	4	0	3	0	2
	WORHP SQP	optimal	3	1.00e-02	5.14e-17	4	0	4	0	3
HS4	IPOPT	optimal	5	1.00e-02	2.67e+00	6	0	6	0	5
	KNITRO	optimal	3	1.00e-02	2.67e+00	5	0	4	0	3
	SNOPT	optimal	1	1.00e-02	2.67e+00	5	0	4	0	0
	WORHP IP	optimal	4	1.00e-02	2.67e+00	6	0	5	0	4
	WORHP IPm	optimal	3	1.00e-02	2.67e+00	6	0	5	0	3
	WORHP SQP	optimal	2	1.00e-02	2.67e+00	3	0	3	0	2
HS40	IPOPT	optimal	3	1.00e-02	-2.50e-01	4	4	4	4	3
	KNITRO	optimal	3	1.00e-02	-2.50e-01	5	6	4	5	3
	SNOPT	optimal	5	1.00e-02	-2.50e-01	10	10	9	9	0
	WORHP IP	optimal	4	1.00e-02	-2.50e-01	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	-2.50e-01	6	6	5	5	4
	WORHP SQP	optimal	4	1.00e-02	-2.50e-01	5	5	6	6	4
HS41	IPOPT	optimal	9	1.00e-02	1.93e+00	11	11	10	10	9
	KNITRO	optimal	4	1.00e-02	1.93e+00	7	8	6	7	4
	SNOPT	optimal	7	1.00e-02	1.93e+00	11	1	10	1	0
	WORHP IP	optimal	11	1.00e-02	1.93e+00	13	13	12	1	11
	WORHP IPm	optimal	6	1.00e-02	1.93e+00	9	9	8	1	6
	WORHP SQP	optimal	9	1.00e-02	1.93e+00	10	10	11	3	9
HS42	IPOPT	optimal	3	1.00e-02	1.39e+01	4	4	4	4	3
	KNITRO	optimal	3	1.00e-02	1.39e+01	6	7	5	6	3
	SNOPT	optimal	4	1.00e-02	1.39e+01	7	7	6	6	0
	WORHP IP	optimal	5	1.00e-02	1.39e+01	7	7	6	6	5
	WORHP IPm	optimal	5	1.00e-02	1.39e+01	7	7	6	6	5
	WORHP SQP	optimal	4	1.00e-02	1.39e+01	5	5	6	6	4
HS43	IPOPT	optimal	9	1.00e-02	-4.40e+01	10	10	10	10	9
	KNITRO	optimal	7	1.00e-02	-4.40e+01	9	10	8	9	7
	SNOPT	optimal	9	1.00e-02	-4.40e+01	12	12	11	11	0
	WORHP IP	optimal	7	1.00e-02	-4.40e+01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	-4.40e+01	9	9	8	8	7
	WORHP SQP	optimal	8	1.00e-02	-4.40e+01	9	9	10	10	8
HS44	IPOPT	optimal	18	1.00e-02	-1.30e+01	22	22	19	19	18
	KNITRO	optimal	10	1.00e-02	-1.50e+01	13	14	12	13	10
	SNOPT	optimal	4	1.00e-02	-1.50e+01	9	1	8	1	0
	WORHP IP	optimal	13	1.00e-02	-1.30e+01	19	19	14	1	13
	WORHP IPm	optimal	13	1.00e-02	-1.30e+01	18	18	17	1	13
	WORHP SQP	optimal	3	1.00e-02	-3.00e+00	4	4	5	3	3
HS44NEW	IPOPT	optimal	14	1.00e-02	-1.50e+01	15	15	15	15	14
	KNITRO	optimal	10	1.00e-02	-1.50e+01	12	13	11	12	10
	SNOPT	optimal	4	1.00e-02	-1.50e+01	9	1	8	1	0
	WORHP IP	optimal	16	1.00e-02	-1.50e+01	18	18	17	1	16
	WORHP IPm	optimal	16	1.00e-02	-1.50e+01	20	20	19	1	16
	WORHP SQP	optimal	3	1.00e-02	-3.00e+00	4	4	5	3	3
HS45	IPOPT	optimal	7	1.00e-02	1.00e+00	8	0	8	0	7
	KNITRO	optimal	9	1.00e-02	1.00e+00	15	0	11	0	9
	SNOPT	optimal	3	1.00e-02	1.00e+00	10	0	9	0	0
	WORHP IP	optimal	6	1.00e-02	1.00e+00	8	0	7	0	6
	WORHP IPm	optimal	12	1.00e-02	1.00e+00	17	0	15	0	12
	WORHP SQP	optimal	2	1.00e-02	1.00e+00	2	0	2	0	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS46	IPOPT	optimal	13	1.00e-02	1.44e-11	14	14	14	14	13
	KNITRO	optimal	18	1.00e-02	7.65e-12	22	23	19	20	18
	SNOPT	optimal	27	1.00e-02	1.57e-11	31	31	30	30	0
	WORHP IP	optimal	18	1.00e-02	4.08e-12	20	20	19	19	18
	WORHP IPm	optimal	18	1.00e-02	4.08e-12	20	20	19	19	18
	WORHP SQP	optimal	19	1.00e-02	4.19e-12	20	20	21	21	19
HS47	IPOPT	optimal	16	1.00e-02	3.36e-11	18	18	17	17	16
	KNITRO	optimal	15	1.00e-02	1.46e-10	19	20	16	17	15
	SNOPT	optimal	23	1.00e-02	7.56e-12	28	28	27	27	0
	WORHP IP	optimal	16	1.00e-02	1.07e-10	20	20	17	17	16
	WORHP IPm	optimal	16	1.00e-02	1.07e-10	20	20	17	17	16
	WORHP SQP	optimal	19	1.00e-02	1.78e-10	24	24	21	21	19
HS48	IPOPT	optimal	1	1.00e-02	3.35e-30	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	9.86e-31	3	4	2	3	1
	SNOPT	optimal	7	1.00e-02	3.91e-14	10	1	9	1	0
	WORHP IP	optimal	1	1.00e-02	7.10e-30	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	7.10e-30	3	3	2	1	1
	WORHP SQP	optimal	3	1.00e-02	4.54e-18	4	4	5	3	3
HS49	IPOPT	optimal	16	1.00e-02	1.38e-09	17	17	17	17	16
	KNITRO	optimal	16	1.00e-02	1.38e-09	18	19	17	18	16
	SNOPT	optimal	34	1.00e-02	7.69e-11	37	1	36	1	0
	WORHP IP	optimal	16	1.00e-02	1.38e-09	18	18	17	1	16
	WORHP IPm	optimal	16	1.00e-02	1.38e-09	18	18	17	1	16
	WORHP SQP	optimal	16	1.00e-02	1.38e-09	17	17	18	3	16
HS5	IPOPT	optimal	8	1.00e-02	-1.91e+00	9	0	9	0	8
	KNITRO	optimal	9	1.00e-02	-1.91e+00	11	0	10	0	9
	SNOPT	optimal	8	1.00e-02	-1.91e+00	11	0	10	0	0
	WORHP IP	optimal	6	1.00e-02	-1.91e+00	8	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	-1.91e+00	8	0	7	0	6
	WORHP SQP	optimal	11	1.00e-02	-1.91e+00	12	0	12	0	11
HS50	IPOPT	optimal	9	1.00e-02	0.00e+00	10	10	10	10	9
	KNITRO	optimal	9	1.00e-02	1.23e-32	11	12	10	11	9
	SNOPT	optimal	19	1.00e-02	1.04e-13	23	1	22	1	0
	WORHP IP	optimal	9	1.00e-02	0.00e+00	11	11	11	1	9
	WORHP IPm	optimal	9	1.00e-02	0.00e+00	11	11	10	1	9
	WORHP SQP	optimal	9	1.00e-02	9.15e-15	10	10	11	3	9
HS51	IPOPT	optimal	1	1.00e-02	0.00e+00	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	4.44e-31	3	4	2	3	1
	SNOPT	optimal	6	1.00e-02	1.20e-14	9	1	8	1	0
	WORHP IP	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP SQP	optimal	2	1.00e-02	9.49e-15	3	3	4	3	2
HS52	IPOPT	optimal	1	1.00e-02	5.33e+00	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	5.33e+00	3	4	2	3	1
	SNOPT	optimal	6	1.00e-02	5.33e+00	10	1	9	1	0
	WORHP IP	optimal	3	1.00e-02	5.33e+00	5	5	4	1	3
	WORHP IPm	optimal	3	1.00e-02	5.33e+00	5	5	4	1	3
	WORHP SQP	optimal	3	1.00e-02	5.33e+00	4	4	5	3	3
HS53	IPOPT	optimal	6	1.00e-02	4.09e+00	7	7	7	7	6
	KNITRO	optimal	3	1.00e-02	4.09e+00	5	6	4	5	3
	SNOPT	optimal	6	1.00e-02	4.09e+00	9	1	8	1	0
	WORHP IP	optimal	5	1.00e-02	4.09e+00	7	7	6	1	5
	WORHP IPm	optimal	5	1.00e-02	4.09e+00	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	4.09e+00	3	3	4	3	2
HS54	IPOPT	optimal	15	1.00e-02	-9.08e-01	16	16	16	16	15
	KNITRO	optimal	16	1.00e-02	-9.08e-01	19	20	17	18	16
	SNOPT	optimal	45	1.00e-02	-8.67e-01	56	1	55	1	0
	WORHP IP	optimal	15	1.00e-02	-8.67e-01	17	17	16	1	15
	WORHP IPm	optimal	10	1.00e-02	-8.67e-01	12	12	11	1	10
	WORHP SQP	optimal	20	1.00e-02	-8.67e-01	21	21	22	3	20
HS55	IPOPT	optimal	2	1.00e-02	6.71e+00	3	3	3	3	2
	KNITRO	optimal	4	1.00e-02	6.67e+00	7	8	6	7	4
	SNOPT	optimal	0	1.00e-02	6.67e+00	3	1	2	1	0
	WORHP IP	optimal	9	1.00e-02	6.67e+00	11	11	10	1	9
	WORHP IPm	optimal	5	1.00e-02	6.67e+00	8	8	7	1	5
	WORHP SQP	optimal	2	1.00e-02	6.67e+00	2	2	3	3	2
HS56	IPOPT	optimal	10	1.00e-02	-3.46e+00	11	11	11	11	10
	KNITRO	optimal	7	1.00e-02	-3.46e+00	13	14	8	9	7
	SNOPT	optimal	20	1.00e-02	-3.46e+00	38	38	37	37	0
	WORHP IP	optimal	10	1.00e-02	-3.46e+00	12	12	11	11	10
	WORHP IPm	optimal	10	1.00e-02	-3.46e+00	12	12	11	11	10
	WORHP SQP	optimal	68	1.00e-02	-3.46e+00	1367	1359	68	78	66
HS57	IPOPT	optimal	22	1.00e-02	3.06e-02	28	28	23	23	22
	KNITRO	optimal	10	1.00e-02	2.85e-02	12	13	11	12	10
	SNOPT	noimpr	3	1.00e-02	3.06e-02	58	58	57	57	0
	WORHP IP	optimal	20	1.00e-02	3.06e-02	102	102	21	21	20
	WORHP IPm	optimal	4	1.00e-02	3.06e-02	6	6	5	5	4
	WORHP SQP	optimal	16	1.00e-02	2.85e-02	17	17	18	18	16

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS59	IPOPT	optimal	43	1.00e-02	-7.80e+00	72	72	44	44	43
	KNITRO	optimal	8	1.00e-02	-7.80e+00	11	12	10	11	8
	SNOPT	optimal	12	1.00e-02	-7.80e+00	19	19	18	18	0
	WORHP IP	optimal	13	1.00e-02	-7.80e+00	15	15	14	14	13
	WORHP IPm	optimal	16	1.00e-02	-7.80e+00	18	18	17	17	16
	WORHP SQP	optimal	14	1.00e-02	-6.75e+00	15	15	16	16	14
HS6	IPOPT	optimal	5	1.00e-02	0.00e+00	7	7	6	6	5
	KNITRO	optimal	5	1.00e-02	0.00e+00	7	8	6	7	5
	SNOPT	optimal	4	1.00e-02	0.00e+00	7	7	6	6	0
	WORHP IP	optimal	2	1.00e-02	7.89e-31	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	7.89e-31	4	4	3	3	2
	WORHP SQP	optimal	5	1.00e-02	2.56e-16	41	41	7	7	5
HS60	IPOPT	optimal	7	1.00e-02	3.26e-02	8	8	8	8	7
	KNITRO	optimal	7	1.00e-02	3.26e-02	9	10	8	9	7
	SNOPT	optimal	9	1.00e-02	3.26e-02	14	14	13	13	0
	WORHP IP	optimal	7	1.00e-02	3.26e-02	9	9	8	8	7
	WORHP IPm	optimal	6	1.00e-02	3.26e-02	8	8	7	7	6
	WORHP SQP	optimal	6	1.00e-02	3.26e-02	7	7	8	8	6
HS61	IPOPT	optimal	9	1.00e-02	-1.44e+02	10	10	10	10	9
	KNITRO	optimal	7	1.00e-02	-1.44e+02	10	11	8	9	7
	SNOPT	optimal	35	1.00e-02	-1.44e+02	81	81	80	80	0
	WORHP IP	optimal	6	1.00e-02	-1.44e+02	8	8	7	7	6
	WORHP IPm	optimal	6	1.00e-02	-1.44e+02	8	8	7	7	6
	WORHP SQP	optimal	210	2.00e-02	-8.19e+01	12202	12250	207	213	205
HS62	IPOPT	optimal	7	1.00e-02	-2.63e+04	10	10	8	8	7
	KNITRO	optimal	7	1.00e-02	-2.63e+04	9	10	8	9	7
	SNOPT	optimal	10	1.00e-02	-2.63e+04	17	1	16	1	0
	WORHP IP	optimal	8	1.00e-02	-2.63e+04	61	61	10	1	8
	WORHP IPm	optimal	7	1.00e-02	-2.63e+04	13	13	8	1	7
	WORHP SQP	zerostep	183	3.00e-02	-2.63e+04	6128	6134	184	3	183
HS63	IPOPT	optimal	7	1.00e-02	9.62e+02	8	8	8	8	7
	KNITRO	optimal	5	1.00e-02	9.62e+02	7	8	6	7	5
	SNOPT	optimal	10	1.00e-02	9.62e+02	19	19	18	18	0
	WORHP IP	optimal	6	1.00e-02	9.62e+02	8	8	7	7	6
	WORHP IPm	optimal	6	1.00e-02	9.62e+02	8	8	7	7	6
	WORHP SQP	optimal	13	1.00e-02	9.62e+02	14	14	15	15	13
HS64	IPOPT	optimal	18	1.00e-02	6.30e+03	19	19	19	19	18
	KNITRO	optimal	17	1.00e-02	6.30e+03	19	20	18	19	17
	SNOPT	optimal	22	1.00e-02	6.30e+03	27	27	26	26	0
	WORHP IP	optimal	16	1.00e-02	6.30e+03	18	18	17	17	16
	WORHP IPm	optimal	19	1.00e-02	6.30e+03	25	25	20	20	19
	WORHP SQP	optimal	20	1.00e-02	6.30e+03	21	21	22	22	20
HS65	IPOPT	optimal	28	1.00e-02	9.54e-01	91	91	29	29	28
	KNITRO	optimal	7	1.00e-02	9.54e-01	10	11	9	10	7
	SNOPT	optimal	9	1.00e-02	9.54e-01	12	12	11	11	0
	WORHP IP	optimal	9	1.00e-02	9.54e-01	11	11	10	10	9
	WORHP IPm	optimal	7	1.00e-02	9.54e-01	9	9	8	8	7
	WORHP SQP	optimal	8	1.00e-02	9.54e-01	9	9	10	10	8
HS66	IPOPT	optimal	7	1.00e-02	5.18e-01	8	8	8	8	7
	KNITRO	optimal	4	1.00e-02	5.18e-01	7	8	6	7	4
	SNOPT	optimal	5	1.00e-02	5.18e-01	1	8	1	7	0
	WORHP IP	optimal	7	1.00e-02	5.18e-01	14	14	8	8	7
	WORHP IPm	optimal	6	1.00e-02	5.18e-01	12	12	7	7	6
	WORHP SQP	optimal	3	1.00e-02	5.18e-01	4	4	5	5	3
HS67	IPOPT	optimal	10	1.00e-02	-1.16e+03	11	11	11	11	10
	KNITRO	optimal	10	1.00e-02	-1.16e+03	12	13	11	12	10
	SNOPT	optimal	22	1.00e-02	-1.16e+03	25	25	24	24	0
	WORHP IP	optimal	9	1.00e-02	-1.16e+03	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	-1.16e+03	13	13	12	12	9
	WORHP SQP	optimal	8	1.00e-02	-1.16e+03	9	9	10	10	8
HS68	IPOPT	optimal	17	1.00e-02	-9.20e-01	27	27	18	18	17
	KNITRO	optimal	13	1.00e-02	-9.20e-01	23	24	14	15	13
	SNOPT	optimal	32	1.00e-02	-9.20e-01	44	44	43	43	0
	WORHP IP	optimal	15	1.00e-02	-9.20e-01	19	19	16	16	15
	WORHP IPm	optimal	17	1.00e-02	-9.20e-01	19	19	18	18	17
	WORHP SQP	optimal	25	1.00e-02	-9.20e-01	26	26	27	27	25
HS69	IPOPT	optimal	12	1.00e-02	-9.57e+02	13	13	13	13	12
	KNITRO	optimal	12	1.00e-02	-9.57e+02	14	15	13	14	12
	SNOPT	optimal	14	1.00e-02	-9.57e+02	25	25	24	24	0
	WORHP IP	optimal	12	1.00e-02	-9.57e+02	15	15	14	14	12
	WORHP IPm	diverge	106	1.00e-02	-2.21e+09	167	167	107	107	106
	WORHP SQP	minalpha	394	9.00e-02	-9.57e+02	17922	17983	396	396	395
HS7	IPOPT	optimal	27	1.00e-02	-1.73e+00	28	58	28	28	27
	KNITRO	optimal	22	1.00e-02	-1.73e+00	32	33	23	24	22
	SNOPT	optimal	16	1.00e-02	-1.73e+00	31	31	30	30	0
	WORHP IP	optimal	17	1.00e-02	-1.73e+00	24	24	18	18	17
	WORHP IPm	optimal	17	1.00e-02	-1.73e+00	24	24	18	18	17
	WORHP SQP	optimal	11	1.00e-02	-1.73e+00	124	123	8	14	6

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS70	IPOPT	optimal	28	1.00e-02	7.50e-03	46	46	29	29	28
	KNITRO	optimal	20	1.00e-02	7.50e-03	24	25	21	22	20
	SNOPT	optimal	31	1.00e-02	7.50e-03	43	43	42	42	0
	WORHP IP	optimal	22	1.00e-02	7.50e-03	24	24	23	23	22
	WORHP IPm	optimal	16	1.00e-02	1.86e-01	19	19	18	18	16
	WORHP SQP	optimal	26	1.00e-02	7.50e-03	27	27	28	28	26
HS71	IPOPT	optimal	8	1.00e-02	1.70e+01	9	18	9	18	8
	KNITRO	optimal	6	1.00e-02	1.70e+01	9	10	8	9	6
	SNOPT	optimal	6	1.00e-02	1.70e+01	9	9	8	8	0
	WORHP IP	optimal	7	1.00e-02	1.70e+01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	1.70e+01	11	11	10	10	7
	WORHP SQP	optimal	7	1.00e-02	1.70e+01	8	8	9	9	7
HS72	IPOPT	optimal	16	1.00e-02	7.28e+02	17	17	17	17	16
	KNITRO	optimal	19	1.00e-02	7.28e+02	21	22	20	21	19
	SNOPT	optimal	20	1.00e-02	7.24e+02	1	37	1	36	0
	WORHP IP	optimal	17	1.00e-02	7.28e+02	19	19	18	18	17
	WORHP IPm	optimal	23	1.00e-02	7.28e+02	25	25	24	24	23
	WORHP SQP	optimal	21	1.00e-02	7.28e+02	22	22	23	23	21
HS73	IPOPT	optimal	8	1.00e-02	2.99e+01	9	18	9	18	8
	KNITRO	optimal	7	1.00e-02	2.99e+01	9	10	8	9	7
	SNOPT	optimal	5	1.00e-02	2.99e+01	1	9	1	8	0
	WORHP IP	optimal	7	1.00e-02	2.99e+01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	2.99e+01	9	9	8	8	7
	WORHP SQP	optimal	3	1.00e-02	2.99e+01	4	4	5	5	3
HS74	IPOPT	optimal	9	1.00e-02	5.13e+03	10	20	10	20	9
	KNITRO	optimal	7	1.00e-02	5.13e+03	10	11	9	10	7
	SNOPT	optimal	13	1.00e-02	5.13e+03	16	16	15	15	0
	WORHP IP	optimal	13	1.00e-02	5.13e+03	15	15	14	14	13
	WORHP IPm	optimal	13	1.00e-02	5.13e+03	15	15	14	14	13
	WORHP SQP	optimal	12	1.00e-02	5.13e+03	13	13	14	14	12
HS75	IPOPT	optimal	9	1.00e-02	5.17e+03	10	20	10	20	9
	KNITRO	optimal	8	1.00e-02	5.17e+03	11	12	10	11	8
	SNOPT	optimal	10	1.00e-02	5.17e+03	13	13	12	12	0
	WORHP IP	optimal	185	2.00e-02	5.17e+03	1345	1345	187	187	185
	WORHP IPm	optimal	195	2.00e-02	5.17e+03	1282	1282	196	196	195
	WORHP SQP	optimal	5	1.00e-02	5.17e+03	6	6	7	7	5
HS76	IPOPT	optimal	7	1.00e-02	-4.68e+00	8	8	8	8	7
	KNITRO	optimal	4	1.00e-02	-4.68e+00	6	7	5	6	4
	SNOPT	optimal	7	1.00e-02	-4.68e+00	10	1	9	1	0
	WORHP IP	optimal	6	1.00e-02	-4.68e+00	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	-4.68e+00	11	11	10	1	6
	WORHP SQP	optimal	6	1.00e-02	-4.68e+00	7	7	8	3	6
HS76I	IPOPT	optimal	7	1.00e-02	-4.68e+00	8	8	8	8	7
	KNITRO	optimal	5	1.00e-02	-4.68e+00	7	8	6	7	5
	SNOPT	optimal	7	1.00e-02	-4.68e+00	10	1	9	1	0
	WORHP IP	optimal	5	1.00e-02	-4.68e+00	7	7	6	1	5
	WORHP IPm	optimal	5	1.00e-02	-4.68e+00	9	9	8	1	5
	WORHP SQP	optimal	6	1.00e-02	-4.68e+00	7	7	8	3	6
HS77	IPOPT	optimal	11	1.00e-02	2.42e-01	13	13	12	12	11
	KNITRO	optimal	8	1.00e-02	2.42e-01	11	12	9	10	8
	SNOPT	optimal	12	1.00e-02	2.42e-01	17	17	16	16	0
	WORHP IP	optimal	11	1.00e-02	2.42e-01	13	13	12	12	11
	WORHP IPm	optimal	11	1.00e-02	2.42e-01	13	13	12	12	11
	WORHP SQP	optimal	10	1.00e-02	2.42e-01	14	14	12	12	10
HS78	IPOPT	optimal	4	1.00e-02	-2.92e+00	5	5	5	5	4
	KNITRO	optimal	4	1.00e-02	-2.92e+00	6	7	5	6	4
	SNOPT	optimal	4	1.00e-02	-2.92e+00	8	8	7	7	0
	WORHP IP	optimal	4	1.00e-02	-2.92e+00	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	-2.92e+00	6	6	5	5	4
	WORHP SQP	optimal	6	1.00e-02	-2.92e+00	7	7	8	8	6
HS79	IPOPT	optimal	4	1.00e-02	7.88e-02	5	5	5	5	4
	KNITRO	optimal	4	1.00e-02	7.88e-02	6	7	5	6	4
	SNOPT	optimal	11	1.00e-02	7.88e-02	15	15	14	14	0
	WORHP IP	optimal	4	1.00e-02	7.88e-02	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	7.88e-02	6	6	5	5	4
	WORHP SQP	optimal	4	1.00e-02	7.88e-02	5	5	6	6	4
HS8	IPOPT	optimal	5	1.00e-02	-1.00e+00	6	6	6	6	5
	KNITRO	optimal	6	1.00e-02	-1.00e+00	10	11	7	8	6
	SNOPT	optimal	5	1.00e-02	1.00e+00	1	8	1	7	0
	WORHP IP	optimal	4	1.00e-02	-1.00e+00	8	8	5	5	4
	WORHP IPm	optimal	4	1.00e-02	-1.00e+00	8	8	5	5	4
	WORHP SQP	optimal	5	1.00e-02	-1.00e+00	7	7	7	7	5
HS80	IPOPT	optimal	6	1.00e-02	5.39e-02	7	7	7	7	6
	KNITRO	optimal	6	1.00e-02	5.39e-02	8	9	7	8	6
	SNOPT	optimal	5	1.00e-02	5.39e-02	9	9	8	8	0
	WORHP IP	optimal	5	1.00e-02	5.39e-02	7	7	6	6	5
	WORHP IPm	optimal	5	1.00e-02	5.39e-02	7	7	6	6	5
	WORHP SQP	optimal	5	1.00e-02	5.39e-02	6	6	7	7	5

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS81	IPOPT	optimal	7	1.00e-02	5.39e-02	8	8	8	8	7
	KNITRO	optimal	10	1.00e-02	5.39e-02	12	13	11	12	10
	SNOPT	optimal	8	1.00e-02	5.39e-02	12	12	11	11	0
	WORHP IP	optimal	9	1.00e-02	5.39e-02	11	11	10	10	9
	WORHP IPm	optimal	7	1.00e-02	5.39e-02	9	9	8	8	7
	WORHP SQP	optimal	9	1.00e-02	5.39e-02	20	20	11	11	9
HS83	IPOPT	optimal	16	1.00e-02	-3.07e+04	17	17	17	17	16
	KNITRO	optimal	7	1.00e-02	-3.07e+04	10	11	9	10	7
	SNOPT	optimal	5	1.00e-02	-3.07e+04	9	9	8	8	0
	WORHP IP	optimal	10	1.00e-02	-3.07e+04	12	12	11	11	10
	WORHP IPm	optimal	11	1.00e-02	-3.07e+04	16	16	15	15	11
	WORHP SQP	optimal	8	1.00e-02	-3.07e+04	9	9	10	10	8
HS84	IPOPT	optimal	12	1.00e-02	-5.28e+06	13	13	13	13	12
	KNITRO	optimal	8	1.00e-02	-5.28e+06	10	11	9	10	8
	SNOPT	optimal	27	1.00e-02	-5.28e+06	60	60	59	59	0
	WORHP IP	optimal	13	1.00e-02	-5.28e+06	15	15	15	15	13
	WORHP IPm	optimal	14	1.00e-02	-5.28e+06	19	19	18	18	14
	WORHP SQP	maxiter	10000	1.61e+00	-5.28e+06	10001	10001	10002	10002	10001
HS85	IPOPT	optimal	19	2.00e-02	-2.22e+00	20	20	20	20	19
	KNITRO	optimal	11	1.00e-02	-2.22e+00	16	17	12	13	11
	SNOPT	optimal	10	1.00e-02	-2.22e+00	19	19	18	18	0
	WORHP IP	optimal	13	1.00e-02	-2.22e+00	15	15	14	14	13
	WORHP IPm	optimal	20	1.00e-02	-2.22e+00	29	29	22	22	20
	WORHP SQP	optimal	9	1.00e-02	-2.22e+00	10	10	11	11	9
HS86	IPOPT	optimal	10	1.00e-02	-3.23e+01	11	11	11	11	10
	KNITRO	optimal	8	1.00e-02	-3.23e+01	11	12	10	11	8
	SNOPT	optimal	10	1.00e-02	-3.23e+01	14	1	13	1	0
	WORHP IP	optimal	10	1.00e-02	-3.23e+01	12	12	11	1	10
	WORHP IPm	optimal	10	1.00e-02	-3.23e+01	12	12	11	1	10
	WORHP SQP	optimal	4	1.00e-02	-3.23e+01	5	5	6	3	4
HS87	IPOPT	maxiter	10000	4.21e+00	8.93e+03	76571	76571	10001	10001	10000
	KNITRO	smallstep	8701	4.00e-01	9.00e+03	23799	23800	8703	8704	8701
	SNOPT	noimpr	8	1.00e-02	9.00e+03	60	60	59	59	0
	WORHP IP	maxiter	10000	5.80e-01	9.17e+03	14973	14973	10001	10001	10000
	WORHP IPm	maxiter	10000	6.60e-01	9.17e+03	15866	15866	10001	10001	10000
	WORHP SQP	maxiter	10000	2.00e+00	9.00e+03	90333	95624	10000	10011	9999
HS88	IPOPT	optimal	16	1.00e-02	1.36e+00	18	18	17	17	16
	KNITRO	optimal	13	2.00e-02	1.36e+00	15	16	14	15	13
	SNOPT	optimal	22	1.00e-02	1.36e+00	37	37	36	36	0
	WORHP IP	optimal	24	3.00e-02	1.36e+00	26	26	25	25	24
	WORHP IPm	optimal	21	2.00e-02	1.36e+00	23	23	22	22	21
	WORHP SQP	optimal	16	1.00e-02	1.36e+00	17	17	18	18	16
HS89	IPOPT	optimal	20	3.00e-02	1.36e+00	38	38	21	21	20
	KNITRO	optimal	27	4.00e-02	1.36e+00	37	38	28	29	27
	SNOPT	optimal	42	5.00e-02	1.36e+00	92	92	91	91	0
	WORHP IP	optimal	21	3.00e-02	1.36e+00	33	33	23	23	21
	WORHP IPm	optimal	21	2.00e-02	1.36e+00	23	23	22	22	21
	WORHP SQP	optimal	19	3.00e-02	1.36e+00	20	20	21	21	19
HS9	IPOPT	optimal	3	1.00e-02	-5.00e-01	6	6	4	4	3
	KNITRO	optimal	6	1.00e-02	-5.00e-01	12	13	7	8	6
	SNOPT	optimal	6	1.00e-02	-5.00e-01	11	1	10	1	0
	WORHP IP	optimal	3	1.00e-02	-5.00e-01	7	7	4	1	3
	WORHP IPm	optimal	3	1.00e-02	-5.00e-01	7	7	4	1	3
	WORHP SQP	optimal	6	1.00e-02	-5.00e-01	9	9	8	3	6
HS90	IPOPT	optimal	21	4.00e-02	1.36e+00	28	28	22	22	21
	KNITRO	optimal	65	1.00e-01	1.36e+00	77	78	66	67	66
	SNOPT	optimal	26	4.00e-02	1.36e+00	55	55	54	54	0
	WORHP IP	optimal	19	4.00e-02	1.36e+00	21	21	20	20	19
	WORHP IPm	optimal	20	3.00e-02	1.36e+00	22	22	21	21	20
	WORHP SQP	optimal	27	7.00e-02	1.36e+00	52	52	29	29	27
HS91	IPOPT	optimal	14	3.00e-02	1.36e+00	15	15	15	15	14
	KNITRO	optimal	19	5.00e-02	1.36e+00	26	27	20	21	19
	SNOPT	optimal	37	7.00e-02	1.36e+00	69	69	68	68	0
	WORHP IP	optimal	14	3.00e-02	1.36e+00	16	16	15	15	14
	WORHP IPm	optimal	22	6.00e-02	1.36e+00	25	25	23	23	22
	WORHP SQP	optimal	18	7.00e-02	1.36e+00	47	47	20	20	18
HS92	IPOPT	optimal	19	7.00e-02	1.36e+00	25	25	20	20	19
	KNITRO	optimal	34	1.20e-01	1.36e+00	45	46	35	36	34
	SNOPT	optimal	25	6.00e-02	1.36e+00	52	52	51	51	0
	WORHP IP	optimal	18	5.00e-02	1.36e+00	20	20	19	19	18
	WORHP IPm	optimal	22	6.00e-02	1.36e+00	24	24	23	23	22
	WORHP SQP	optimal	17	9.00e-02	1.36e+00	64	64	19	19	17
HS93	IPOPT	optimal	8	1.00e-02	1.35e+02	9	9	9	9	8
	KNITRO	optimal	5	1.00e-02	1.35e+02	7	8	6	7	5
	SNOPT	optimal	30	1.00e-02	1.35e+02	34	34	33	33	0
	WORHP IP	optimal	7	1.00e-02	1.35e+02	9	9	8	8	7
	WORHP IPm	optimal	12	1.00e-02	1.35e+02	15	15	13	13	12
	WORHP SQP	optimal	8	1.00e-02	1.35e+02	9	9	10	10	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HS95	IPOPT	optimal	14	1.00e-02	1.56e-02	18	18	15	15	14
	KNITRO	optimal	8	1.00e-02	1.56e-02	11	12	10	11	8
	SNOPT	optimal	1	1.00e-02	1.56e-02	1	4	1	3	0
	WORHP IP	optimal	15	1.00e-02	1.56e-02	17	17	16	16	15
	WORHP IPm	optimal	48	1.00e-02	1.56e-02	54	54	52	52	48
	WORHP SQP	optimal	3	1.00e-02	1.56e-02	4	4	5	5	3
HS96	IPOPT	optimal	19	1.00e-02	1.56e-02	24	24	20	20	19
	KNITRO	optimal	9	1.00e-02	1.56e-02	13	14	11	12	9
	SNOPT	optimal	1	1.00e-02	1.56e-02	1	4	1	3	0
	WORHP IP	optimal	16	1.00e-02	1.56e-02	18	18	17	17	16
	WORHP IPm	optimal	48	1.00e-02	1.56e-02	66	66	53	53	48
	WORHP SQP	optimal	3	1.00e-02	1.56e-02	4	4	5	5	3
HS97	IPOPT	optimal	22	1.00e-02	3.14e+00	25	25	23	23	22
	KNITRO	optimal	19	1.00e-02	4.07e+00	23	24	21	22	19
	SNOPT	optimal	14	1.00e-02	3.14e+00	1	38	1	37	0
	WORHP IP	optimal	19	1.00e-02	4.07e+00	21	21	20	20	19
	WORHP IPm	optimal	111	1.00e-02	4.07e+00	120	120	116	116	111
	WORHP SQP	optimal	7	1.00e-02	3.14e+00	8	8	9	9	7
HS98	IPOPT	optimal	22	1.00e-02	4.07e+00	23	23	23	23	22
	KNITRO	optimal	18	1.00e-02	4.07e+00	21	22	20	21	18
	SNOPT	optimal	14	1.00e-02	3.14e+00	1	38	1	37	0
	WORHP IP	optimal	19	1.00e-02	4.07e+00	21	21	20	20	19
	WORHP IPm	optimal	49	1.00e-02	4.07e+00	61	61	55	55	49
	WORHP SQP	optimal	7	1.00e-02	3.14e+00	8	8	9	9	7
HS99	IPOPT	optimal	6	1.00e-02	-8.31e+08	7	7	7	7	6
	KNITRO	optimal	4	1.00e-02	-8.31e+08	6	7	5	6	4
	SNOPT	optimal	10	1.00e-02	-8.31e+08	14	14	13	13	0
	WORHP IP	optimal	6	1.00e-02	-8.31e+08	8	8	8	8	6
	WORHP IPm	smallstep	6	1.00e-02	-8.31e+08	9	9	7	7	7
	WORHP SQP	zerostep	7	1.00e-02	-8.31e+08	7	7	7	7	7
HS99EXP	IPOPT	optimal	24	1.00e-02	-1.26e+12	59	59	25	25	24
	KNITRO	optimal	8	1.00e-02	-1.26e+12	10	11	9	10	8
	SNOPT	unbound	24	1.00e-02	-1.97e+09	45	45	44	44	0
	WORHP IP	optimal	2314	3.20e-01	-1.07e+08	2424	2424	2316	2316	2314
	WORHP IPm	maxiter	10000	1.52e+00	-1.25e+12	10003	10003	10001	10001	10000
	WORHP SQP	maxiter	10000	2.92e+01	-1.76e+12	10113	10112	9999	10003	9998
HUBFIT	IPOPT	optimal	8	1.00e-02	1.69e-02	9	9	9	9	8
	KNITRO	optimal	3	1.00e-02	1.69e-02	6	7	5	6	3
	SNOPT	optimal	4	1.00e-02	1.69e-02	9	1	8	1	0
	WORHP IP	optimal	6	1.00e-02	1.69e-02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	1.69e-02	9	9	8	1	6
	WORHP SQP	optimal	2	1.00e-02	1.69e-02	3	3	4	3	2
HUES-MOD	IPOPT	optimal	25	2.00e-01	3.48e+07	27	27	26	26	25
	KNITRO	optimal	23	1.70e-01	3.48e+07	25	26	24	25	23
	SNOPT	sbasics	22	8.63e+00	4.23e+07	30	1	29	1	0
	WORHP IP	optimal	78	7.70e-01	3.48e+07	80	80	80	1	78
	WORHP IPm	optimal	79	7.10e-01	3.48e+07	85	85	84	1	79
	WORHP SQP	optimal	5	7.60e-01	3.48e+07	6	6	6	2	5
HUESTIS	IPOPT	optimal	25	1.60e-01	1.74e+11	26	26	26	26	25
	KNITRO	optimal	23	2.20e-01	1.74e+11	25	26	24	25	23
	SNOPT	toobig	20	8.74e+00	2.17e+11	23	1	22	1	0
	WORHP IP	optimal	272	2.35e+00	1.74e+11	461	461	273	1	272
	WORHP IPm	optimal	258	2.45e+00	1.74e+11	610	610	263	1	258
	WORHP SQP	optimal	5	1.28e+00	1.74e+11	6	6	6	2	5
HUMPS	IPOPT	optimal	632	1.50e-01	8.39e-14	700	0	633	0	632
	KNITRO	optimal	352	1.00e-02	4.63e-14	417	0	353	0	352
	SNOPT	optimal	58	1.00e-02	7.70e-16	140	0	139	0	0
	WORHP IP	optimal	317	1.00e-02	1.94e-12	333	0	318	0	317
	WORHP IPm	optimal	317	1.00e-02	1.94e-12	333	0	318	0	317
	WORHP SQP	optimal	4134	2.60e-01	9.81e-14	4853	0	4135	0	4134
HVYCRASH	IPOPT	optimal	290	3.33e+00	-1.54e-01	372	377	263	302	290
	KNITRO	optimal	17	2.00e-01	-2.19e-01	20	21	19	20	17
	SNOPT	optimal	52	1.47e+01	-1.11e-02	1	118	1	117	0
	WORHP IP	optimal	11	1.90e-01	-2.18e-01	13	13	12	12	11
	WORHP IPm	optimal	10	2.00e-01	-2.18e-01	12	12	11	11	10
	WORHP SQP	optimal	18	1.78e+00	-2.18e-01	66	65	8	21	6
HYDC20LS	IPOPT	maxiter	10000	8.36e+00	1.69e-06	38488	0	10001	0	10000
	KNITRO	maxiter	10000	3.13e+00	2.20e-06	20397	0	10001	0	10000
	SNOPT	maxiter	10000	2.51e+00	2.43e+02	11105	0	11104	0	0
	WORHP IP	maxiter	10000	4.86e+00	1.89e-06	25301	0	10002	0	10000
	WORHP IPm	maxiter	10000	5.31e+00	1.91e-06	24374	0	10001	0	10000
	WORHP SQP	maxiter	10000	9.24e+00	9.35e-06	44660	0	10001	0	10001
HYDCAR20	IPOPT	optimal	8	1.00e-02	0.00e+00	10	11	9	9	8
	KNITRO	optimal	8	1.00e-02	0.00e+00	11	12	9	10	8
	SNOPT	optimal	15	1.00e-02	0.00e+00	1	37	1	36	0
	WORHP IP	optimal	8	1.00e-02	0.00e+00	12	12	9	9	8
	WORHP IPm	optimal	8	1.00e-02	0.00e+00	12	12	9	9	8
	WORHP SQP	optimal	10	2.00e-02	0.00e+00	12	12	12	12	10

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
HYDCAR6	IPOPT	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	KNITRO	optimal	5	1.00e-02	0.00e+00	8	9	6	7	5
	SNOPT	optimal	7	1.00e-02	0.00e+00	1	13	1	12	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	9	9	6	6	5
	WORHP SQP	optimal	6	1.00e-02	0.00e+00	7	7	8	8	6
HYDROELL	IPOPT	optimal	235	5.50e-01	-3.59e+06	236	236	236	236	235
	KNITRO	optimal	20	1.00e-01	-3.59e+06	23	24	22	23	20
	SNOPT	optimal	5	6.00e-02	-3.59e+06	13	1	12	1	0
	WORHP IP	optimal	231	1.00e+00	-3.59e+06	233	233	232	1	231
	WORHP IPm	optimal	732	2.64e+00	-3.59e+06	736	736	735	1	732
	WORHP SQP	optimal	41	2.14e+00	-3.59e+06	42	42	43	3	41
HYDROELM	IPOPT	optimal	209	2.60e-01	-3.58e+06	210	210	210	210	209
	KNITRO	optimal	18	6.00e-02	-3.58e+06	21	22	20	21	18
	SNOPT	optimal	4	2.00e-02	-3.58e+06	11	1	10	1	0
	WORHP IP	optimal	196	4.10e-01	-3.58e+06	198	198	197	1	196
	WORHP IPm	optimal	436	8.80e-01	-3.58e+06	446	446	439	1	436
	WORHP SQP	optimal	29	6.20e-01	-3.58e+06	30	30	31	3	29
HYDROELS	IPOPT	optimal	120	7.00e-02	-3.58e+06	121	121	121	121	120
	KNITRO	optimal	16	2.00e-02	-3.58e+06	19	20	18	19	16
	SNOPT	optimal	3	1.00e-02	-3.58e+06	9	1	8	1	0
	WORHP IP	optimal	119	7.00e-02	-3.58e+06	121	121	121	1	119
	WORHP IPm	optimal	169	1.00e-01	-3.58e+06	268	268	172	1	169
	WORHP SQP	optimal	29	1.60e-01	-3.58e+06	30	30	31	3	29
HYPCIR	IPOPT	optimal	5	1.00e-02	0.00e+00	8	8	6	6	5
	KNITRO	optimal	4	1.00e-02	0.00e+00	8	9	5	6	4
	SNOPT	optimal	4	1.00e-02	0.00e+00	1	7	1	6	0
	WORHP IP	optimal	4	1.00e-02	0.00e+00	8	8	5	5	4
	WORHP IPm	optimal	4	1.00e-02	0.00e+00	8	8	5	5	4
	WORHP SQP	optimal	4	1.00e-02	0.00e+00	5	5	6	6	4
INDEF	IPOPT	smallstep	108	1.22e+00	1.31e+17	109	0	109	0	109
	KNITRO	noimpr	264	2.99e+00	-1.92e+18	2777	0	265	0	265
	SNOPT	unbound	41	7.53e+00	-7.26e+09	71	0	70	0	0
	WORHP IP	maxiter	10000	7.93e+01	-1.12e+19	10002	0	10001	0	10000
	WORHP IPm	maxiter	10000	1.04e+02	-1.12e+19	10002	0	10001	0	10000
	WORHP SQP	maxiter	10000	8.35e+02	-7.24e+13	10004	0	10001	0	10001
INDEFM	IPOPT	optimal	52	1.05e+01	-1.00e+07	58	0	53	0	52
	KNITRO	optimal	33	8.65e+00	-1.00e+07	39	0	34	0	33
	SNOPT	sbasics	10000	1.16e+03	-1.80e+05	10469	0	10468	0	0
	WORHP IP	optimal	52	1.27e+01	-1.00e+07	56	0	53	0	52
	WORHP IPm	optimal	52	1.72e+01	-1.00e+07	56	0	53	0	52
	WORHP SQP	optimal	67	1.56e+01	-1.00e+07	72	0	68	0	67
INTEGREQ	IPOPT	optimal	2	1.32e+00	0.00e+00	3	3	3	3	2
	KNITRO	optimal	2	1.91e+00	0.00e+00	4	5	3	4	2
	SNOPT	optimal	2	1.62e+00	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	2	1.61e+00	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.45e+00	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	2	7.92e+00	0.00e+00	3	3	4	4	2
INTEQNE	IPOPT	optimal	2	1.00e-02	0.00e+00	3	3	3	3	2
	KNITRO	optimal	2	1.00e-02	0.00e+00	4	5	3	4	2
	SNOPT	optimal	2	1.00e-02	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	2	1.00e-02	0.00e+00	3	3	4	4	2
INTEQNELS	IPOPT	optimal	3	1.00e-02	3.99e-22	4	0	4	0	3
	KNITRO	optimal	3	1.00e-02	3.99e-22	5	0	4	0	3
	SNOPT	optimal	6	1.00e-02	4.85e-15	11	0	10	0	0
	WORHP IP	optimal	3	1.00e-02	3.99e-22	5	0	4	0	3
	WORHP IPm	optimal	3	1.00e-02	3.99e-22	5	0	4	0	3
	WORHP SQP	optimal	3	1.00e-02	6.40e-21	4	0	4	0	3
JANNSON3	IPOPT	optimal	14	6.70e-01	2.00e+04	15	30	15	30	14
	KNITRO	optimal	8	7.80e-01	2.00e+04	10	11	9	10	8
	SNOPT	optimal	19	2.90e+01	2.00e+04	23	23	22	22	0
	WORHP IP	optimal	10	5.50e-01	2.00e+04	12	12	11	11	10
	WORHP IPm	optimal	9	7.40e-01	2.00e+04	11	11	10	10	9
	WORHP SQP	optimal	9	8.60e-01	2.00e+04	10	10	11	11	9
JANNSON4	IPOPT	optimal	13	2.60e-01	9.80e+03	14	14	14	14	13
	KNITRO	optimal	13	6.00e-01	9.80e+03	15	16	14	15	13
	SNOPT	sbasics	26	4.10e+01	9.91e+03	55	55	54	54	0
	WORHP IP	optimal	11	3.40e-01	9.80e+03	13	13	12	12	11
	WORHP IPm	optimal	12	3.80e-01	9.80e+03	14	14	13	13	12
	WORHP SQP	optimal	12	8.60e-01	9.80e+03	13	13	14	14	12
JENSMP	IPOPT	optimal	10	1.00e-02	1.24e+02	11	0	11	0	10
	KNITRO	optimal	10	1.00e-02	1.24e+02	12	0	11	0	10
	SNOPT	optimal	20	1.00e-02	1.24e+02	37	0	36	0	0
	WORHP IP	optimal	10	1.00e-02	1.24e+02	12	0	12	0	10
	WORHP IPm	optimal	10	1.00e-02	1.24e+02	12	0	11	0	10
	WORHP SQP	optimal	9	1.00e-02	1.24e+02	10	0	10	0	9

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
JENSMPNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	34	1.00e-02	0.00e+00	80	81	35	36	33
	SNOPT	maxiter	10000	6.10e-01	0.00e+00	1	30003	1	30002	0
	WORHP IP	infeas	10	1.00e-02	0.00e+00	31	31	11	11	11
	WORHP IPm	infeas	10	1.00e-02	0.00e+00	31	31	11	11	11
	WORHP SQP	infeas	8	1.00e-02	0.00e+00	732	733	8	9	7
JIMACK	IPOPT	optimal	18	1.18e+01	8.67e-01	19	0	19	0	18
	KNITRO	optimal	17	9.29e+00	8.67e-01	19	0	18	0	17
	SNOPT	maxtime	9384	1.80e+03	1.01e+00	10485	0	10484	0	0
	WORHP IP	optimal	18	1.30e+01	8.67e-01	20	0	19	0	18
	WORHP IPm	optimal	18	1.66e+01	8.67e-01	20	0	19	0	18
	WORHP SQP	optimal	23	1.95e+01	8.67e-01	24	0	24	0	23
JJTABEL3	IPOPT	optimal	48	3.90e-01	5.25e+07	49	49	49	49	48
	KNITRO	optimal	26	3.70e-01	5.25e+07	29	30	28	29	26
	SNOPT	optimal	2543	4.16e+02	5.25e+07	2599	1	2598	1	0
	WORHP IP	optimal	117	2.48e+00	5.25e+07	680	680	121	1	117
	WORHP IPm	optimal	76	7.90e-01	5.25e+07	87	87	82	1	76
	WORHP SQP	maxiter	10000	3.33e+02	1.76e+12	25559	24455	8511	2	8511
JNLBRNG1	IPOPT	optimal	12	7.60e-01	-1.80e-01	13	0	13	0	12
	KNITRO	optimal	10	7.50e-01	-1.81e-01	13	0	12	0	10
	SNOPT	toobig	265	1.82e+01	1.15e+01	304	0	303	0	0
	WORHP IP	optimal	12	8.20e-01	-1.81e-01	14	0	13	0	12
	WORHP IPm	optimal	14	1.02e+00	-1.81e-01	17	0	16	0	14
	WORHP SQP	optimal	41	1.00e+01	-1.81e-01	2646	0	42	0	41
JNLBRNG2	IPOPT	optimal	12	7.40e-01	-4.15e+00	13	0	13	0	12
	KNITRO	optimal	11	7.90e-01	-4.15e+00	14	0	13	0	11
	SNOPT	toobig	301	1.84e+01	5.75e+00	352	0	351	0	0
	WORHP IP	optimal	10	7.40e-01	-4.15e+00	12	0	11	0	10
	WORHP IPm	optimal	12	9.40e-01	-4.15e+00	16	0	15	0	12
	WORHP SQP	optimal	3	9.80e-01	-4.15e+00	4	0	4	0	3
JNLBRNGA	IPOPT	optimal	9	5.20e-01	-2.71e-01	10	0	10	0	9
	KNITRO	optimal	9	6.10e-01	-2.71e-01	12	0	11	0	9
	SNOPT	toobig	250	1.74e+01	-2.70e-02	265	0	264	0	0
	WORHP IP	optimal	9	6.00e-01	-2.71e-01	11	0	10	0	9
	WORHP IPm	optimal	23	1.27e+00	-2.71e-01	26	0	25	0	23
	WORHP SQP	optimal	34	6.31e+00	-2.71e-01	704	0	35	0	34
JNLBRNGB	IPOPT	optimal	12	6.50e-01	-6.30e+00	13	0	13	0	12
	KNITRO	optimal	11	3.80e-01	-6.30e+00	14	0	13	0	11
	SNOPT	toobig	409	2.45e+01	-3.51e-01	451	0	450	0	0
	WORHP IP	optimal	18	1.07e+00	-6.30e+00	20	0	19	0	18
	WORHP IPm	optimal	35	1.96e+00	-6.30e+00	38	0	37	0	35
	WORHP SQP	optimal	5	1.18e+00	-6.30e+00	6	0	6	0	5
JUNKTURN	IPOPT	unbound	51	1.87e+02	3.58e+36	127	127	52	52	51
	KNITRO	noimpr	1880	7.60e+01	1.78e-03	17384	17385	1882	1883	1881
	SNOPT	sbasics	405	1.15e+03	5.63e+01	1750	1750	1749	1749	0
	WORHP IP	optimal	174	5.96e+00	1.78e-03	296	296	176	176	174
	WORHP IPm	optimal	172	6.29e+00	1.78e-03	288	288	173	173	172
	WORHP SQP	minalpha	1456	6.60e+01	1.59e-03	37722	37740	305	1458	304
KIRBY2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	22	2.00e-02	0.00e+00	31	32	23	24	22
	SNOPT	infeas	453	1.10e-01	0.00e+00	1	908	1	907	0
	WORHP IP	infeas	8	1.00e-02	0.00e+00	13	13	9	9	9
	WORHP IPm	infeas	8	1.00e-02	0.00e+00	13	13	9	9	9
	WORHP SQP	minalpha	15	1.10e-01	0.00e+00	2471	2477	7	17	6
KIRBY2LS	IPOPT	optimal	12	1.00e-02	3.91e+00	19	0	13	0	12
	KNITRO	optimal	19	1.00e-02	3.91e+00	34	0	20	0	19
	SNOPT	toobig	76	1.00e-02	3.91e+00	103	0	102	0	0
	WORHP IP	optimal	12	1.00e-02	3.91e+00	17	0	14	0	12
	WORHP IPm	optimal	12	1.00e-02	3.91e+00	17	0	13	0	12
	WORHP SQP	zerostep	56	2.00e-02	3.91e+00	105	0	56	0	56
KISSING	IPOPT	optimal	347	2.27e+00	8.46e-01	367	734	330	700	347
	KNITRO	optimal	12	8.00e-02	1.00e+00	15	16	13	14	12
	SNOPT	optimal	26	7.60e-01	1.00e+00	1	72	1	71	0
	WORHP IP	optimal	305	2.08e+00	8.43e-01	363	363	306	306	305
	WORHP IPm	optimal	29	2.10e-01	1.00e+00	39	39	30	30	29
	WORHP SQP	optimal	8	1.40e+00	1.00e+00	349	350	10	10	8
KISSING2	IPOPT	optimal	166	8.00e-01	6.63e+00	181	181	93	168	166
	KNITRO	optimal	145	4.90e-01	5.27e+00	148	149	147	148	145
	SNOPT	optimal	59	2.80e-01	5.27e+00	204	204	203	203	0
	WORHP IP	optimal	210	8.20e-01	6.35e+00	221	221	211	211	210
	WORHP IPm	optimal	151	9.20e-01	6.36e+00	154	154	152	152	151
	WORHP SQP	optimal	81	4.41e+00	5.27e+00	86	86	83	83	81
KIWCRESC	IPOPT	optimal	9	1.00e-02	1.72e-07	11	11	10	10	9
	KNITRO	optimal	9	1.00e-02	-2.54e-07	11	12	10	11	9
	SNOPT	optimal	11	1.00e-02	-1.43e-10	1	19	1	18	0
	WORHP IP	optimal	17	1.00e-02	4.30e-07	24	24	18	18	17
	WORHP IPm	optimal	17	1.00e-02	4.33e-12	25	25	18	18	17
	WORHP SQP	optimal	8	1.00e-02	2.00e+00	37	37	10	10	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
KOEBHELB	IPOPT	optimal	344	1.70e-01	7.75e+01	889	0	345	0	344
	KNITRO	optimal	96	2.00e-02	7.75e+01	149	0	97	0	96
	SNOPT	optimal	151	2.00e-02	7.75e+01	229	0	228	0	0
	WORHP IP	optimal	90	2.00e-02	1.12e+02	168	0	91	0	90
	WORHP IPm	optimal	105	2.00e-02	7.75e+01	171	0	106	0	105
	WORHP SQP	optimal	99	3.00e-02	7.75e+01	493	0	100	0	99
KOWOSB	IPOPT	optimal	8	1.00e-02	3.08e-04	23	0	9	0	8
	KNITRO	optimal	7	1.00e-02	3.08e-04	16	0	8	0	7
	SNOPT	optimal	24	1.00e-02	3.08e-04	34	0	33	0	0
	WORHP IP	optimal	8	1.00e-02	3.08e-04	18	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	3.08e-04	18	0	9	0	8
	WORHP SQP	optimal	10	1.00e-02	3.08e-04	16	0	11	0	10
KOWOSBNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	27	1.00e-02	0.00e+00	30	31	28	29	25
	SNOPT	infeas	5	1.00e-02	0.00e+00	1	16	1	15	0
	WORHP IP	infeas	11	1.00e-02	0.00e+00	24	24	12	12	12
	WORHP IPm	infeas	11	1.00e-02	0.00e+00	24	24	12	12	12
	WORHP SQP	minalpha	44	2.00e-02	0.00e+00	2407	2413	35	46	34
KSIP	IPOPT	optimal	22	1.30e-01	5.76e-01	28	28	23	23	22
	KNITRO	optimal	18	1.10e-01	5.76e-01	20	21	19	20	18
	SNOPT	optimal	15	3.50e-01	5.76e-01	18	1	17	1	0
	WORHP IP	optimal	25	1.90e-01	5.76e-01	30	30	26	1	25
	WORHP IPm	optimal	25	1.70e-01	5.76e-01	27	27	26	1	25
	WORHP SQP	optimal	3	1.40e-01	5.76e-01	4	4	5	3	3
KSS	IPOPT	optimal	6	3.38e+00	0.00e+00	7	7	7	7	6
	KNITRO	optimal	6	1.32e+01	0.00e+00	8	9	7	8	6
	SNOPT	optimal	8	5.49e+00	0.00e+00	1	11	1	10	0
	WORHP IP	optimal	6	1.03e+01	0.00e+00	8	8	7	7	6
	WORHP IPm	optimal	6	1.03e+01	0.00e+00	8	8	7	7	6
	WORHP SQP	optimal	8	1.66e+02	0.00e+00	9	9	10	10	8
KTMODEL	IPOPT	maxiter	10000	1.25e+01	0.00e+00	10042	10042	15	10002	10000
	KNITRO	infeas	0	1.00e-02	0.00e+00	2	3	1	2	0
	SNOPT	infeas	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	regular	87	2.10e-01	0.00e+00	189	189	88	88	88
	WORHP IPm	minalpha	275	1.06e+00	0.00e+00	2143	2143	322	322	276
	WORHP SQP	minalpha	18	1.16e+00	0.00e+00	2536	2545	13	19	13
LAKES	IPOPT	optimal	18	1.00e-02	3.51e+05	52	52	19	19	18
	KNITRO	optimal	11	1.00e-02	3.51e+05	14	15	12	13	11
	SNOPT	optimal	19	1.00e-02	3.51e+05	32	32	31	31	0
	WORHP IP	optimal	112	3.00e-02	3.51e+05	116	116	114	114	112
	WORHP IPm	optimal	102	4.00e-02	3.51e+05	128	128	103	103	102
	WORHP SQP	optimal	51	1.60e-01	3.51e+05	157	156	46	53	45
LANCZOS1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	7	1.00e-02	0.00e+00	9	10	8	9	7
	SNOPT	optimal	10	1.00e-02	0.00e+00	1	17	1	16	0
	WORHP IP	optimal	8	1.00e-02	0.00e+00	12	12	9	9	8
	WORHP IPm	optimal	8	1.00e-02	0.00e+00	12	12	9	9	8
	WORHP SQP	optimal	10	1.00e-02	0.00e+00	208	207	6	13	4
LANCZOS1LS	IPOPT	optimal	162	5.00e-02	2.17e-12	432	0	163	0	162
	KNITRO	optimal	53	1.00e-02	4.29e-06	62	0	54	0	53
	SNOPT	optimal	83	1.00e-02	4.29e-06	95	0	94	0	0
	WORHP IP	optimal	165	1.00e-02	4.98e-13	288	0	166	0	165
	WORHP IPm	optimal	165	1.00e-02	4.98e-13	288	0	166	0	165
	WORHP SQP	optimal	71	1.00e-02	4.29e-06	191	0	72	0	71
LANCZOS2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	21	1.00e-02	0.00e+00	23	24	22	23	21
	SNOPT	optimal	9	1.00e-02	0.00e+00	1	16	1	15	0
	WORHP IP	infeas	9	1.00e-02	0.00e+00	13	13	10	10	10
	WORHP IPm	infeas	9	1.00e-02	0.00e+00	13	13	10	10	10
	WORHP SQP	minalpha	39	8.00e-02	0.00e+00	4537	4545	31	41	30
LANCZOS2LS	IPOPT	optimal	95	4.00e-02	2.39e-11	256	0	96	0	95
	KNITRO	optimal	53	1.00e-02	4.30e-06	62	0	54	0	53
	SNOPT	optimal	85	1.00e-02	4.30e-06	96	0	95	0	0
	WORHP IP	optimal	52	1.00e-02	1.75e-09	100	0	53	0	52
	WORHP IPm	optimal	52	1.00e-02	1.75e-09	100	0	53	0	52
	WORHP SQP	optimal	70	1.00e-02	4.30e-06	182	0	71	0	70
LANCZOS3	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	19	1.00e-02	0.00e+00	21	22	20	21	19
	SNOPT	infeas	11	1.00e-02	0.00e+00	1	26	1	25	0
	WORHP IP	infeas	9	1.00e-02	0.00e+00	13	13	10	10	10
	WORHP IPm	infeas	9	1.00e-02	0.00e+00	13	13	10	10	10
	WORHP SQP	infeas	114	8.00e-02	0.00e+00	5155	5159	15	115	14
LANCZOS3LS	IPOPT	optimal	160	4.00e-02	1.61e-08	428	0	161	0	160
	KNITRO	optimal	55	1.00e-02	4.35e-06	62	0	56	0	55
	SNOPT	optimal	85	1.00e-02	4.35e-06	97	0	96	0	0
	WORHP IP	optimal	159	1.00e-02	1.61e-08	293	0	160	0	159
	WORHP IPm	optimal	159	1.00e-02	1.61e-08	293	0	160	0	159
	WORHP SQP	optimal	60	1.00e-02	4.35e-06	155	0	61	0	60

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LAUNCH	IPOPT	optimal	19	1.00e-02	9.00e+00	25	50	20	40	19
	KNITRO	optimal	8	1.00e-02	9.00e+00	11	12	10	11	8
	SNOPT	optimal	11	1.00e-02	9.00e+00	19	19	18	18	0
	WORHP IP	maxiter	10000	7.47e+00	9.01e+00	164630	164630	10001	10001	10000
	WORHP IPm	optimal	585	3.60e-01	9.00e+00	4898	4898	591	591	585
	WORHP SQP	optimal	15	1.00e-02	9.00e+00	16	16	17	17	15
LCH	IPOPT	optimal	61	4.90e-01	-4.34e+00	62	62	62	62	61
	KNITRO	optimal	37	7.10e-01	-4.34e+00	174	175	38	39	37
	SNOPT	sbasics	651	1.56e+03	1.52e+05	724	724	723	723	0
	WORHP IP	optimal	62	3.80e-01	-4.34e+00	64	64	64	64	62
	WORHP IPm	optimal	63	7.30e-01	-4.34e+00	65	65	64	64	63
	WORHP SQP	optimal	46	6.20e-01	-4.34e+00	114	113	43	49	41
LEAKNET	IPOPT	optimal	17	1.00e-02	8.05e+00	19	19	18	18	17
	KNITRO	optimal	22	1.00e-02	8.00e+00	25	26	24	25	22
	SNOPT	optimal	6	1.00e-02	8.00e+00	1	12	1	11	0
	WORHP IP	optimal	21	1.00e-02	8.00e+00	23	23	22	22	21
	WORHP IPm	optimal	18	1.00e-02	8.00e+00	23	23	22	22	18
	WORHP SQP	optimal	16	3.00e-02	8.05e+00	17	17	18	18	16
LEUVEN1	IPOPT	optimal	94	1.47e+00	-1.52e+07	95	190	95	190	94
	KNITRO	optimal	50	7.10e-01	-1.52e+07	53	54	52	53	50
	SNOPT	optimal	12	4.90e-01	-1.52e+07	15	1	14	1	0
	WORHP IP	optimal	36	4.50e-01	-1.52e+07	38	38	38	1	36
	WORHP IPm	optimal	42	9.20e-01	-1.52e+07	49	49	48	1	42
	WORHP SQP	optimal	7	3.52e+00	-1.52e+07	8	8	8	2	7
LEUVEN2	IPOPT	optimal	167	2.73e+00	-1.41e+07	168	336	168	336	167
	KNITRO	smallstep	1020	2.96e+02	-1.41e+07	10121	10122	1022	1023	1020
	SNOPT	optimal	5	6.40e-01	-1.41e+07	8	1	7	1	0
	WORHP IP	fritzjohn	126	2.52e+00	-1.41e+07	490	490	128	1	127
	WORHP IPm	optimal	352	6.15e+00	-1.41e+07	360	360	359	1	352
	WORHP SQP	optimal	329	4.52e+01	-1.41e+07	38532	38533	322	3	320
LEUVEN3	IPOPT	optimal	815	2.49e+02	-1.99e+09	849	1698	816	1632	815
	KNITRO	maxtime	4859	1.80e+03	-1.94e+09	6350	6351	4861	4862	4860
	SNOPT	optimal	74	7.07e+00	-1.56e+09	93	1	92	1	0
	WORHP IP	infeas	1015	6.77e+02	-1.97e+09	1017	1017	1017	1	1016
	WORHP IPm	regular	602	2.57e+02	-2.00e+09	609	609	607	1	603
	WORHP SQP	maxtime	217	1.05e+03	-5.80e+08	218	218	218	2	218
LEUVEN4	IPOPT	optimal	2212	8.96e+02	-1.99e+09	2355	4710	2183	4428	2212
	KNITRO	maxtime	4278	1.80e+03	3.85e+10	4289	4290	4280	4281	4280
	SNOPT	optimal	293	1.02e+01	-1.26e+09	618	1	617	1	0
	WORHP IP	fritzjohn	1097	3.94e+02	-5.98e+08	1122	1122	1099	1	1098
	WORHP IPm	minalpha	1944	8.73e+02	-1.25e+09	1973	1973	1952	1	1945
	WORHP SQP	maxtime	2091	9.89e+02	-6.20e+08	2219	2218	178	2	178
LEUVEN5	IPOPT	optimal	815	3.00e+02	-1.99e+09	849	1698	816	1632	815
	KNITRO	maxtime	5815	1.80e+03	-1.72e+09	7477	7478	5817	5818	5818
	SNOPT	optimal	74	6.73e+00	-1.56e+09	93	1	92	1	0
	WORHP IP	infeas	1015	8.21e+02	-1.97e+09	1017	1017	1017	1	1016
	WORHP IPm	regular	602	3.83e+02	-2.00e+09	609	609	607	1	603
	WORHP SQP	maxtime	278	9.98e+02	-7.33e+08	279	279	279	2	279
LEUVEN6	IPOPT	optimal	334	9.12e+01	-1.17e+09	348	696	335	670	334
	KNITRO	maxtime	6215	1.80e+03	-1.19e+09	7866	7867	6217	6218	6217
	SNOPT	optimal	33	6.23e+00	-9.20e+08	48	1	47	1	0
	WORHP IP	fritzjohn	235	2.04e+02	-1.15e+09	270	270	237	1	236
	WORHP IPm	optimal	170	8.49e+01	-1.15e+09	340	340	178	1	170
	WORHP SQP	maxtime	278	1.04e+03	-1.87e+08	631	631	279	3	278
LEUVEN7	IPOPT	optimal	29	5.10e-01	6.95e+02	30	30	30	30	29
	KNITRO	optimal	27	4.30e-01	6.95e+02	30	31	29	30	27
	SNOPT	optimal	83	2.80e-01	6.95e+02	87	1	86	1	0
	WORHP IP	optimal	22	5.70e-01	6.95e+02	24	24	23	1	22
	WORHP IPm	optimal	22	5.70e-01	6.95e+02	26	26	25	1	22
	WORHP SQP	optimal	19	1.44e+00	6.95e+02	36	36	21	3	19
LEWISPOL	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	36	1.00e-02	1.16e+00	75	76	37	38	34
	SNOPT	infeas	4	1.00e-02	3.14e+00	8	8	7	7	0
	WORHP IP	infeas	15	1.00e-02	1.21e+00	20	20	16	16	16
	WORHP IPm	infeas	15	1.00e-02	1.21e+00	20	20	16	16	16
	WORHP SQP	infeas	29	2.00e-02	1.20e+00	34	34	30	30	29
LHAIFAM	IPOPT	maxiter	10000	1.14e+01	6.93e-01	10001	10001	2	10001	10000
	KNITRO	nan	0	1.00e-02	6.93e-01	2	3	1	2	0
	SNOPT	infeas	0	1.00e-02	6.93e-01	3	3	2	2	0
	WORHP IP	fritzjohn	103	3.30e-01	-3.47e+01	235	235	104	104	104
	WORHP IPm	minalpha	134	4.20e-01	-2.66e+01	796	796	178	178	135
	WORHP SQP	zerostep	1	1.00e-02	6.93e-01	1	1	1	1	1
LIARWHD	IPOPT	optimal	12	1.60e-01	6.38e-22	13	0	13	0	12
	KNITRO	optimal	12	1.60e-01	6.38e-22	14	0	13	0	12
	SNOPT	toobig	48	5.30e+00	2.22e+05	51	0	50	0	0
	WORHP IP	optimal	12	1.40e-01	6.38e-22	14	0	14	0	12
	WORHP IPm	optimal	12	1.40e-01	6.38e-22	14	0	13	0	12
	WORHP SQP	optimal	15	1.90e-01	1.46e-13	16	0	16	0	15

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LIN	IPOPT	optimal	23	1.00e-02	-1.96e-02	28	28	24	24	23
	KNITRO	optimal	3	1.00e-02	-1.76e-02	6	7	5	6	3
	SNOPT	optimal	3	1.00e-02	-1.76e-02	8	1	7	1	0
	WORHP IP	optimal	11	1.00e-02	-1.96e-02	13	13	12	1	11
	WORHP IPm	optimal	24	1.00e-02	-1.96e-02	30	30	25	1	24
	WORHP SQP	maxiter	10000	1.22e+00	-2.02e-02	83645	83702	10002	3	10001
LINCONT	IPOPT	infeas	118	1.46e+00	5.49e-07	130	130	5	122	119
	KNITRO	smallstep	19	1.90e-01	0.00e+00	24	25	20	21	19
	SNOPT	infeas	0	7.00e-02	1.26e+03	1	1	1	1	0
	WORHP IP	infeas	2	5.00e-02	0.00e+00	4	4	3	1	3
	WORHP IPm	infeas	2	6.00e-02	0.00e+00	4	4	3	1	3
	WORHP SQP	minalpha	6	2.60e+00	0.00e+00	2487	2493	7	3	6
LINSPANH	IPOPT	optimal	13	1.00e-02	-7.70e+01	19	19	14	14	13
	KNITRO	optimal	1	1.00e-02	-7.70e+01	4	5	3	4	1
	SNOPT	optimal	0	1.00e-02	-7.70e+01	1	1	1	1	0
	WORHP IP	optimal	12	1.00e-02	-7.70e+01	14	14	13	1	12
	WORHP IPm	optimal	12	1.00e-02	-7.70e+01	17	17	16	1	12
	WORHP SQP	optimal	1	1.00e-02	-7.70e+01	2	2	3	3	1
LINVERSE	IPOPT	optimal	204	1.49e+00	6.81e+02	964	0	205	0	204
	KNITRO	optimal	36	2.40e-01	6.81e+02	43	0	38	0	36
	SNOPT	optimal	195	6.90e+01	6.81e+02	225	0	224	0	0
	WORHP IP	optimal	294	2.17e+00	6.81e+02	923	0	295	0	294
	WORHP IPm	optimal	31	1.90e-01	6.81e+02	35	0	33	0	31
	WORHP SQP	optimal	90	1.45e+00	6.81e+02	118	0	91	0	90
LIPPERT1	IPOPT	infeas	159	3.67e+01	-1.00e-02	845	1692	136	324	160
	KNITRO	infeas	281	6.35e+01	-1.00e-02	480	481	283	284	282
	SNOPT	noimpr	33	5.76e+02	-1.00e-02	1	87	1	86	0
	WORHP IP	infeas	54	1.57e+01	-1.00e-02	70	70	55	55	54
	WORHP IPm	infeas	49	1.30e+01	-9.00e-03	65	65	51	51	50
	WORHP SQP	minalpha	6	2.27e+02	-1.00e-02	2940	2948	8	8	7
LIPPERT2	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	maxtime	9565	1.80e+03	2.81e+03	36203	36204	9566	9567	9565
	SNOPT	infeas	3	2.17e+02	2.72e+04	1	7	1	6	0
	WORHP IP	maxtime	4533	1.69e+03	2.17e+02	15709	15709	4534	4534	4534
	WORHP IPm	maxtime	4619	1.69e+03	2.23e+02	19100	19100	4620	4620	4620
	WORHP SQP	minalpha	4	6.77e+01	1.00e+00	2447	2453	6	6	5
LISWET1	IPOPT	optimal	18	9.00e-02	7.10e+00	19	19	19	19	18
	KNITRO	optimal	22	1.40e-01	7.22e+00	24	25	23	24	22
	SNOPT	optimal	3	1.00e-02	7.22e+00	7	1	6	1	0
	WORHP IP	optimal	15	9.00e-02	7.22e+00	17	17	16	1	15
	WORHP IPm	optimal	163	7.40e-01	7.22e+00	165	165	164	1	163
	WORHP SQP	optimal	2	9.00e-02	7.22e+00	3	3	4	3	2
LISWET10	IPOPT	optimal	27	1.40e-01	9.80e+00	28	28	28	28	27
	KNITRO	optimal	66	2.80e-01	9.90e+00	68	69	67	68	66
	SNOPT	optimal	2	3.00e-02	9.94e+00	6	1	5	1	0
	WORHP IP	optimal	30	1.70e-01	9.90e+00	32	32	31	1	30
	WORHP IPm	optimal	290	1.46e+00	9.90e+00	340	340	291	1	290
	WORHP SQP	optimal	3	1.30e-01	9.90e+00	4	4	5	3	3
LISWET11	IPOPT	optimal	46	2.10e-01	9.88e+00	47	47	47	47	46
	KNITRO	optimal	31	1.70e-01	9.91e+00	33	34	32	33	31
	SNOPT	optimal	3	4.00e-02	9.91e+00	7	1	6	1	0
	WORHP IP	optimal	27	1.60e-01	9.91e+00	29	29	28	1	27
	WORHP IPm	optimal	300	1.52e+00	9.91e+00	379	379	301	1	300
	WORHP SQP	optimal	2	1.10e-01	9.91e+00	3	3	4	3	2
LISWET12	IPOPT	optimal	32	1.60e-01	3.47e+02	35	35	33	33	32
	KNITRO	optimal	616	2.26e+00	3.48e+02	618	619	617	618	616
	SNOPT	optimal	2	3.00e-02	3.48e+02	6	1	5	1	0
	WORHP IP	optimal	170	8.20e-01	3.48e+02	306	306	172	1	170
	WORHP IPm	optimal	795	3.52e+00	3.48e+02	797	797	796	1	795
	WORHP SQP	optimal	3	1.40e-01	3.48e+02	4	4	5	3	3
LISWET2	IPOPT	optimal	22	1.10e-01	5.00e+00	23	23	23	23	22
	KNITRO	optimal	15	9.00e-02	5.00e+00	17	18	16	17	15
	SNOPT	optimal	3	5.00e-02	5.00e+00	7	1	6	1	0
	WORHP IP	optimal	11	8.00e-02	5.00e+00	13	13	12	1	11
	WORHP IPm	optimal	11	8.00e-02	5.00e+00	13	13	12	1	11
	WORHP SQP	optimal	2	8.00e-02	5.00e+00	3	3	4	3	2
LISWET3	IPOPT	optimal	25	1.20e-01	5.00e+00	26	26	26	26	25
	KNITRO	optimal	12	8.00e-02	5.00e+00	14	15	13	14	12
	SNOPT	optimal	3	8.00e-02	5.00e+00	7	1	6	1	0
	WORHP IP	optimal	21	1.20e-01	5.00e+00	23	23	22	1	21
	WORHP IPm	optimal	11	8.00e-02	5.00e+00	13	13	12	1	11
	WORHP SQP	optimal	2	8.00e-02	5.00e+00	3	3	4	3	2
LISWET4	IPOPT	optimal	26	1.20e-01	5.00e+00	27	27	27	27	26
	KNITRO	optimal	14	9.00e-02	5.00e+00	16	17	15	16	14
	SNOPT	optimal	3	9.00e-02	5.00e+00	7	1	6	1	0
	WORHP IP	optimal	23	1.40e-01	5.00e+00	25	25	24	1	23
	WORHP IPm	optimal	12	8.00e-02	5.00e+00	14	14	13	1	12
	WORHP SQP	optimal	2	1.00e-01	5.00e+00	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LISWET5	IPOPT	optimal	22	1.10e-01	5.00e+00	23	23	23	23	22
	KNITRO	optimal	13	8.00e-02	5.00e+00	15	16	14	15	13
	SNOPT	optimal	3	1.00e-01	5.00e+00	7	1	6	1	0
	WORHP IP	optimal	16	9.00e-02	5.00e+00	18	18	17	1	16
	WORHP IPm	optimal	11	7.00e-02	5.00e+00	13	13	12	1	11
	WORHP SQP	maxiter	10000	5.74e+01	5.00e+00	131211	131211	10002	3	10001
LISWET6	IPOPT	optimal	22	1.10e-01	5.00e+00	25	25	23	23	22
	KNITRO	optimal	8	6.00e-02	5.00e+00	10	11	9	10	8
	SNOPT	optimal	3	6.00e-02	5.00e+00	7	1	6	1	0
	WORHP IP	optimal	18	1.10e-01	5.00e+00	20	20	19	1	18
	WORHP IPm	optimal	10	7.00e-02	5.00e+00	12	12	11	1	10
	WORHP SQP	optimal	2	7.00e-02	5.00e+00	3	3	4	3	2
LISWET7	IPOPT	optimal	16	9.00e-02	9.90e+01	17	17	17	17	16
	KNITRO	optimal	14	9.00e-02	9.99e+01	16	17	15	16	14
	SNOPT	optimal	3	3.00e-02	9.99e+01	7	1	6	1	0
	WORHP IP	optimal	16	9.00e-02	9.96e+01	18	18	17	1	16
	WORHP IPm	optimal	21	1.20e-01	9.99e+01	23	23	22	1	21
	WORHP SQP	optimal	3	7.00e-02	9.99e+01	4	4	5	3	3
LISWET8	IPOPT	optimal	32	1.50e-01	1.43e+02	35	35	33	33	32
	KNITRO	optimal	209	8.10e-01	1.43e+02	211	212	210	211	209
	SNOPT	optimal	2	3.00e-02	1.46e+02	6	1	5	1	0
	WORHP IP	optimal	183	1.14e+00	1.43e+02	370	370	186	1	183
	WORHP IPm	optimal	524	2.51e+00	1.43e+02	568	568	525	1	524
	WORHP SQP	maxiter	10000	4.35e+01	1.43e+02	11300	11300	10002	3	10001
LISWET9	IPOPT	optimal	28	1.50e-01	3.92e+02	29	29	29	29	28
	KNITRO	optimal	418	1.58e+00	3.93e+02	420	421	419	420	418
	SNOPT	optimal	3	3.00e-02	3.93e+02	7	1	6	1	0
	WORHP IP	optimal	218	1.09e+00	3.93e+02	220	220	220	1	218
	WORHP IPm	optimal	788	3.50e+00	3.93e+02	790	790	789	1	788
	WORHP SQP	optimal	3	1.40e-01	3.93e+02	4	4	5	3	3
LMINSURF	IPOPT	optimal	48	1.46e+00	9.00e+00	338	0	49	0	48
	KNITRO	optimal	95	2.83e+00	9.00e+00	379	0	96	0	95
	SNOPT	toobig	2390	5.94e+01	2.28e+01	2640	0	2639	0	0
	WORHP IP	optimal	45	1.29e+00	9.00e+00	247	0	46	0	45
	WORHP IPm	optimal	45	1.51e+00	9.00e+00	247	0	46	0	45
	WORHP SQP	optimal	1087	4.98e+01	9.00e+00	58501	0	1088	0	1087
LOADBAL	IPOPT	optimal	13	1.00e-02	4.53e-01	19	38	14	28	13
	KNITRO	optimal	9	1.00e-02	4.53e-01	12	13	11	12	9
	SNOPT	optimal	52	1.00e-02	4.53e-01	57	1	56	1	0
	WORHP IP	optimal	12	1.00e-02	4.53e-01	15	15	13	1	12
	WORHP IPm	optimal	11	1.00e-02	4.53e-01	14	14	12	1	11
	WORHP SQP	optimal	8	1.00e-02	4.53e-01	9	9	10	3	8
LOBSTERZ	IPOPT	optimal	65	5.01e+01	2.77e+03	202	202	62	67	65
	KNITRO	optimal	80	2.35e+01	2.77e+03	82	83	81	82	80
	SNOPT	optimal	80	2.02e+02	2.77e+03	131	131	130	130	0
	WORHP IP	optimal	23	7.19e+00	2.77e+03	27	27	25	25	23
	WORHP IPm	optimal	91	2.63e+01	2.77e+03	341	341	93	93	91
	WORHP SQP	maxtime	2523	1.70e+03	2.77e+03	11003	10387	2525	3143	2524
LOGHAIRY	IPOPT	optimal	2760	7.10e-01	1.82e-01	3815	0	2761	0	2760
	KNITRO	optimal	2608	4.00e-02	1.82e-01	2893	0	2609	0	2608
	SNOPT	optimal	46	1.00e-02	1.82e-01	124	0	123	0	0
	WORHP IP	optimal	3515	1.10e-01	1.82e-01	4229	0	3516	0	3515
	WORHP IPm	optimal	3515	1.10e-01	1.82e-01	4229	0	3516	0	3515
	WORHP SQP	optimal	4161	2.50e-01	1.82e-01	5493	0	4162	0	4161
LOGROS	IPOPT	optimal	54	1.00e-02	1.87e-14	137	0	55	0	54
	KNITRO	optimal	23	1.00e-02	0.00e+00	34	0	25	0	23
	SNOPT	optimal	78	1.00e-02	3.55e-15	113	0	112	0	0
	WORHP IP	optimal	54	1.00e-02	3.89e-14	75	0	55	0	54
	WORHP IPm	optimal	23	1.00e-02	0.00e+00	37	0	24	0	23
	WORHP SQP	optimal	68	1.00e-02	1.33e-15	484	0	69	0	68
LOOTSMA	IPOPT	optimal	12	1.00e-02	1.41e+00	17	17	13	13	12
	KNITRO	optimal	5	1.00e-02	1.41e+00	8	9	7	8	5
	SNOPT	infeas	12	1.00e-02	4.26e-01	58	58	57	57	0
	WORHP IP	optimal	25	1.00e-02	1.41e+00	27	27	26	26	25
	WORHP IPm	infeas	23	1.00e-02	5.89e+00	28	28	24	24	24
	WORHP SQP	maxiter	10000	2.27e+00	5.01e+00	10557	10558	9999	10002	9998
LOTSCHD	IPOPT	optimal	14	1.00e-02	2.40e+03	15	15	15	15	14
	KNITRO	optimal	10	1.00e-02	2.40e+03	13	14	12	13	10
	SNOPT	optimal	3	1.00e-02	2.40e+03	7	1	6	1	0
	WORHP IP	optimal	12	1.00e-02	2.40e+03	14	14	13	1	12
	WORHP IPm	optimal	12	1.00e-02	2.40e+03	17	17	16	1	12
	WORHP SQP	optimal	2	1.00e-02	2.40e+03	2	2	2	2	2
LSC1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	28	1.00e-02	0.00e+00	40	41	27	28	26
	SNOPT	infeas	7	1.00e-02	0.00e+00	1	25	1	24	0
	WORHP IP	infeas	7	1.00e-02	0.00e+00	13	13	8	8	8
	WORHP IPm	infeas	7	1.00e-02	0.00e+00	13	13	8	8	8
	WORHP SQP	minalpha	313	6.00e-02	0.00e+00	3929	3935	307	315	306

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LSC1LS	IPOPT	optimal	15	1.00e-02	7.71e+00	26	0	16	0	15
	KNITRO	optimal	19	1.00e-02	7.71e+00	32	0	20	0	19
	SNOPT	optimal	57	1.00e-02	7.71e+00	89	0	88	0	0
	WORHP IP	optimal	15	1.00e-02	7.71e+00	21	0	16	0	15
	WORHP IPm	optimal	15	1.00e-02	7.71e+00	21	0	16	0	15
	WORHP SQP	optimal	23	1.00e-02	7.71e+00	44	0	24	0	23
LSC2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	59	1.00e-02	0.00e+00	90	91	60	61	59
	SNOPT	optimal	62	1.00e-02	0.00e+00	1	99	1	98	0
	WORHP IP	infeas	579	5.00e-02	0.00e+00	5292	5292	580	580	579
	WORHP IPm	infeas	579	3.00e-02	0.00e+00	5292	5292	580	580	579
	WORHP SQP	minalpha	55	2.00e-02	0.00e+00	4053	4059	9	57	8
LSC2LS	IPOPT	optimal	49	2.00e-02	1.33e+01	66	0	50	0	49
	KNITRO	optimal	41	1.00e-02	1.33e+01	44	0	42	0	41
	SNOPT	noimpr	158	1.00e-02	1.33e+01	206	0	205	0	0
	WORHP IP	optimal	34	1.00e-02	1.33e+01	40	0	36	0	34
	WORHP IPm	optimal	40	1.00e-02	1.34e+01	46	0	41	0	40
	WORHP SQP	optimal	60	1.00e-02	1.33e+01	74	0	61	0	60
LSNNODOC	IPOPT	optimal	11	1.00e-02	1.23e+02	15	15	12	12	11
	KNITRO	optimal	7	1.00e-02	1.23e+02	10	11	9	10	7
	SNOPT	optimal	6	1.00e-02	1.23e+02	9	1	8	1	0
	WORHP IP	optimal	10	1.00e-02	1.23e+02	12	12	12	1	10
	WORHP IPm	optimal	9	1.00e-02	1.23e+02	14	14	13	1	9
	WORHP SQP	optimal	6	1.00e-02	1.23e+02	6	6	7	3	6
LSQFIT	IPOPT	optimal	7	1.00e-02	3.38e-02	8	8	8	8	7
	KNITRO	optimal	3	1.00e-02	3.38e-02	6	7	5	6	3
	SNOPT	optimal	4	1.00e-02	3.38e-02	9	1	8	1	0
	WORHP IP	optimal	5	1.00e-02	3.38e-02	7	7	6	1	5
	WORHP IPm	optimal	5	1.00e-02	3.38e-02	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	3.38e-02	3	3	4	3	2
LUBRIF	IPOPT	maxtime	173	1.81e+03	5.66e+01	373	373	71	179	173
	KNITRO	maxtime	317	1.80e+03	6.06e+02	410	411	319	320	320
	SNOPT	infeas	174	3.11e+02	0.00e+00	864	864	863	863	0
	WORHP IP	maxtime	139	1.78e+03	6.58e+01	302	302	140	140	140
	WORHP IPm	maxtime	128	1.79e+03	1.11e+01	267	267	129	129	129
	WORHP SQP	maxtime	0	1.69e+03	0.00e+00	1	1	1	1	1
LUBRIFC	IPOPT	infeas	198	1.12e+03	4.16e+00	359	359	137	212	199
	KNITRO	maxtime	323	1.80e+03	2.40e+01	327	328	325	326	324
	SNOPT	infeas	12	1.36e+02	0.00e+00	21	21	20	20	0
	WORHP IP	maxtime	122	1.79e+03	5.96e+00	127	127	123	123	123
	WORHP IPm	maxtime	126	1.79e+03	2.29e+01	155	155	128	128	127
	WORHP SQP	maxtime	2	1.56e+03	2.54e-07	3	3	4	4	3
LUKSAN11	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	1	1.00e-02	0.00e+00	4	5	3	4	1
	SNOPT	optimal	2	1.00e-02	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	385	1.30e-01	0.00e+00	1391	1391	386	386	385
	WORHP IPm	optimal	385	1.30e-01	0.00e+00	1391	1391	386	386	385
	WORHP SQP	minalpha	39	4.20e-01	0.00e+00	3117	3124	32	40	32
LUKSAN11LS	IPOPT	optimal	334	1.30e-01	2.86e-18	621	0	335	0	334
	KNITRO	optimal	334	4.00e-02	1.22e-23	392	0	335	0	334
	SNOPT	optimal	1309	1.90e-01	1.91e-14	1476	0	1475	0	0
	WORHP IP	optimal	334	4.00e-02	1.24e-15	460	0	335	0	334
	WORHP IPm	optimal	334	5.00e-02	1.24e-15	460	0	335	0	334
	WORHP SQP	optimal	330	7.00e-02	6.69e-20	864	0	331	0	330
LUKSAN12	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	164	2.10e-01	0.00e+00	188	189	166	167	164
	SNOPT	infeas	51	3.00e-02	0.00e+00	1	92	1	91	0
	WORHP IP	infeas	14	1.00e-02	0.00e+00	19	19	15	15	15
	WORHP IPm	infeas	14	1.00e-02	0.00e+00	19	19	15	15	15
	WORHP SQP	minalpha	51	5.90e-01	0.00e+00	5466	5481	26	52	26
LUKSAN12LS	IPOPT	optimal	25	2.00e-02	4.29e+03	36	0	26	0	25
	KNITRO	optimal	18	1.00e-02	4.29e+03	28	0	19	0	18
	SNOPT	optimal	399	6.00e-02	2.00e+03	455	0	454	0	0
	WORHP IP	optimal	25	1.00e-02	4.29e+03	31	0	26	0	25
	WORHP IPm	optimal	25	1.00e-02	4.29e+03	31	0	26	0	25
	WORHP SQP	optimal	66	2.00e-02	4.04e+03	143	0	67	0	66
LUKSAN13	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	333	8.70e-01	0.00e+00	825	826	334	335	333
	SNOPT	infeas	102	7.00e-02	0.00e+00	1	235	1	234	0
	WORHP IP	infeas	20	1.00e-02	0.00e+00	35	35	21	21	21
	WORHP IPm	infeas	20	1.00e-02	0.00e+00	35	35	21	21	21
	WORHP SQP	minalpha	9	2.00e-01	0.00e+00	3405	3411	6	11	5
LUKSAN13LS	IPOPT	optimal	19	1.00e-02	2.52e+04	36	0	20	0	19
	KNITRO	optimal	14	1.00e-02	2.52e+04	20	0	15	0	14
	SNOPT	optimal	80	1.00e-02	2.52e+04	93	0	92	0	0
	WORHP IP	optimal	19	1.00e-02	2.52e+04	28	0	20	0	19
	WORHP IPm	optimal	19	1.00e-02	2.52e+04	28	0	20	0	19
	WORHP SQP	optimal	16	1.00e-02	2.52e+04	46	0	17	0	16

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LUKSAN14	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	0	1.00e-02	0.00e+00	2	3	1	2	0
	SNOPT	infeas	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	15	1.00e-02	0.00e+00	22	22	16	16	16
	WORHP IPm	infeas	15	1.00e-02	0.00e+00	22	22	16	16	16
	WORHP SQP	minalpha	35	5.30e-01	0.00e+00	2794	2801	34	36	34
LUKSAN14LS	IPOPT	optimal	11	1.00e-02	1.24e+02	19	0	12	0	11
	KNITRO	optimal	11	1.00e-02	1.24e+02	14	0	12	0	11
	SNOPT	optimal	111	2.00e-02	1.24e+02	122	0	121	0	0
	WORHP IP	optimal	11	1.00e-02	1.24e+02	17	0	12	0	11
	WORHP IPm	optimal	11	1.00e-02	1.24e+02	17	0	12	0	11
	WORHP SQP	optimal	21	1.00e-02	1.24e+02	29	0	22	0	21
LUKSAN15	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	36	1.30e-01	0.00e+00	84	85	37	38	36
	SNOPT	infeas	10	3.00e-02	0.00e+00	1	48	1	47	0
	WORHP IP	infeas	5	2.00e-02	0.00e+00	9	9	6	6	6
	WORHP IPm	infeas	5	2.00e-02	0.00e+00	9	9	6	6	6
	WORHP SQP	minalpha	367	2.72e+00	0.00e+00	4686	4692	355	369	354
LUKSAN15LS	IPOPT	optimal	9	3.00e-02	3.57e+00	27	0	10	0	9
	KNITRO	optimal	7	2.00e-02	3.57e+00	13	0	8	0	7
	SNOPT	optimal	26	3.00e-02	3.57e+00	34	0	33	0	0
	WORHP IP	optimal	9	2.00e-02	3.57e+00	22	0	11	0	9
	WORHP IPm	optimal	9	2.00e-02	3.57e+00	22	0	10	0	9
	WORHP SQP	optimal	10	2.00e-02	3.57e+00	15	0	11	0	10
LUKSAN16	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	30	7.00e-02	0.00e+00	62	63	31	32	30
	SNOPT	infeas	4	1.00e-02	0.00e+00	1	7	1	6	0
	WORHP IP	infeas	21	4.00e-02	0.00e+00	23	23	22	22	21
	WORHP IPm	infeas	21	3.00e-02	0.00e+00	23	23	22	22	21
	WORHP SQP	minalpha	1294	6.34e+00	0.00e+00	6714	6720	1291	1296	1290
LUKSAN16LS	IPOPT	optimal	6	1.00e-02	3.57e+00	12	0	7	0	6
	KNITRO	optimal	6	1.00e-02	3.57e+00	9	0	7	0	6
	SNOPT	optimal	36	1.00e-02	3.57e+00	58	0	57	0	0
	WORHP IP	optimal	6	1.00e-02	3.57e+00	10	0	7	0	6
	WORHP IPm	optimal	6	1.00e-02	3.57e+00	10	0	7	0	6
	WORHP SQP	optimal	7	1.00e-02	3.57e+00	12	0	8	0	7
LUKSAN17	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	42	1.00e-01	0.00e+00	85	86	43	44	42
	SNOPT	maxiter	10000	1.44e+01	0.00e+00	1	20311	1	20310	0
	WORHP IP	infeas	18	3.00e-02	0.00e+00	31	31	19	19	19
	WORHP IPm	infeas	18	2.00e-02	0.00e+00	31	31	19	19	19
	WORHP SQP	infeas	28	5.00e-01	0.00e+00	2259	2263	7	29	6
LUKSAN17LS	IPOPT	optimal	16	2.00e-02	4.93e-01	17	0	17	0	16
	KNITRO	optimal	17	2.00e-02	4.93e-01	22	0	18	0	17
	SNOPT	optimal	417	3.40e-01	4.93e-01	492	0	491	0	0
	WORHP IP	optimal	16	1.00e-02	4.93e-01	18	0	18	0	16
	WORHP IPm	optimal	16	2.00e-02	4.93e-01	18	0	17	0	16
	WORHP SQP	optimal	32	5.00e-02	4.93e-01	46	0	33	0	32
LUKSAN21	IPOPT	optimal	5	1.00e-02	0.00e+00	12	14	6	6	5
	KNITRO	optimal	7	1.00e-02	0.00e+00	17	18	8	9	7
	SNOPT	optimal	9	1.00e-02	0.00e+00	1	24	1	23	0
	WORHP IP	optimal	5	1.00e-02	0.00e+00	15	15	6	6	5
	WORHP IPm	optimal	5	1.00e-02	0.00e+00	15	15	6	6	5
	WORHP SQP	optimal	6	1.00e-02	0.00e+00	18	18	8	8	6
LUKSAN21LS	IPOPT	optimal	11	1.00e-02	2.61e-17	31	0	12	0	11
	KNITRO	optimal	12	1.00e-02	1.65e-10	17	0	13	0	12
	SNOPT	optimal	2529	3.30e-01	2.59e-10	2779	0	2778	0	0
	WORHP IP	optimal	11	1.00e-02	2.61e-17	23	0	12	0	11
	WORHP IPm	optimal	11	1.00e-02	2.61e-17	23	0	12	0	11
	WORHP SQP	optimal	17	1.00e-02	4.74e-13	22	0	18	0	17
LUKSAN22	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	481	8.10e-01	0.00e+00	1773	1774	483	484	481
	SNOPT	infeas	123	4.00e-02	0.00e+00	1	267	1	266	0
	WORHP IP	infeas	26	1.00e-02	0.00e+00	41	41	27	27	27
	WORHP IPm	infeas	26	1.00e-02	0.00e+00	41	41	27	27	27
	WORHP SQP	minalpha	181	6.20e-01	0.00e+00	4349	4355	127	183	126
LUKSAN22LS	IPOPT	optimal	16	1.00e-02	8.69e+02	23	0	17	0	16
	KNITRO	optimal	16	1.00e-02	8.69e+02	20	0	17	0	16
	SNOPT	optimal	267	5.00e-02	8.73e+02	306	0	305	0	0
	WORHP IP	optimal	16	1.00e-02	8.69e+02	21	0	17	0	16
	WORHP IPm	optimal	16	1.00e-02	8.69e+02	21	0	17	0	16
	WORHP SQP	optimal	20	1.00e-02	8.69e+02	51	0	21	0	20
LUKVLE1	IPOPT	optimal	6	5.30e-01	6.23e+00	7	7	7	7	6
	KNITRO	optimal	6	5.60e-01	6.23e+00	8	9	7	8	6
	SNOPT	optimal	21	5.54e+00	6.23e+00	28	28	27	27	0
	WORHP IP	optimal	6	6.20e-01	6.23e+00	8	8	7	7	6
	WORHP IPm	optimal	6	6.40e-01	6.23e+00	8	8	7	7	6
	WORHP SQP	optimal	8	1.39e+00	6.23e+00	9	9	10	10	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LUKVLE10	IPOPT	optimal	13	6.70e-01	3.54e+03	21	21	14	14	13
	KNITRO	optimal	39	2.31e+00	3.54e+03	109	110	40	41	39
	SNOPT	degen	1	3.80e-01	1.30e+04	13	13	12	12	0
	WORHP IP	optimal	14	7.80e-01	3.54e+03	16	16	16	16	14
	WORHP IPm	optimal	13	6.70e-01	3.54e+03	15	15	14	14	13
	WORHP SQP	optimal	55	6.28e+00	3.54e+03	519	517	54	59	52
LUKVLE11	IPOPT	maxiter	10000	1.84e+02	4.20e+08	10205	10352	31	10005	10000
	KNITRO	maxiter	10000	3.77e+02	2.32e+06	96216	96217	10001	10002	10000
	SNOPT	infeas	7449	7.22e+02	6.85e+08	20217	20217	20216	20216	0
	WORHP IP	optimal	13	4.70e-01	1.14e-06	15	15	14	14	13
	WORHP IPm	optimal	13	4.70e-01	1.14e-06	15	15	14	14	13
	WORHP SQP	infeas	8	3.96e+00	1.37e+05	1708	1709	6	9	5
LUKVLE12	IPOPT	optimal	26	6.93e+00	1.93e+05	92	96	27	27	26
	KNITRO	optimal	23	1.11e+00	1.93e+05	38	39	24	25	23
	SNOPT	optimal	136	2.92e+02	1.71e+05	229	229	228	228	0
	WORHP IP	optimal	44	1.14e+00	1.93e+05	59	59	45	45	44
	WORHP IPm	optimal	44	1.19e+00	1.93e+05	59	59	45	45	44
	WORHP SQP	optimal	38	2.34e+01	1.71e+05	87	87	40	40	38
LUKVLE13	IPOPT	optimal	15	3.50e-01	9.16e+04	16	16	16	16	15
	KNITRO	optimal	23	4.00e-01	9.16e+04	31	32	24	25	23
	SNOPT	sbasics	45	1.05e+02	1.03e+05	89	89	88	88	0
	WORHP IP	optimal	44	1.11e+00	9.14e+04	122	122	46	46	44
	WORHP IPm	optimal	32	8.30e-01	9.14e+04	37	37	33	33	32
	WORHP SQP	optimal	48	3.04e+00	9.14e+04	916	912	47	54	45
LUKVLE14	IPOPT	optimal	23	4.70e-01	3.14e+08	24	24	24	24	23
	KNITRO	optimal	25	8.30e-01	3.14e+08	40	41	26	27	25
	SNOPT	sbasics	109	2.59e+02	3.15e+08	156	156	155	155	0
	WORHP IP	optimal	56	1.31e+00	3.14e+08	120	120	58	58	56
	WORHP IPm	optimal	51	1.11e+00	3.14e+08	92	92	52	52	51
	WORHP SQP	optimal	23	1.21e+00	3.14e+08	24	24	25	25	23
LUKVLE15	IPOPT	optimal	280	7.88e+00	3.58e-10	1923	2335	204	328	280
	KNITRO	optimal	99	2.86e+00	2.99e-08	351	352	100	101	99
	SNOPT	maxtime	1108	1.80e+03	1.60e+01	1419	1419	1418	1418	0
	WORHP IP	optimal	54	1.32e+00	8.10e+00	74	74	56	56	54
	WORHP IPm	optimal	60	1.33e+00	8.10e+00	80	80	61	61	60
	WORHP SQP	minalpha	25	8.50e+00	2.15e+07	5722	5729	15	27	14
LUKVLE16	IPOPT	infeas	161	4.52e+00	6.67e+03	606	700	54	177	162
	KNITRO	optimal	129	3.73e+00	3.44e+03	498	499	130	131	129
	SNOPT	sbasics	337	7.51e+02	1.18e+04	619	619	618	618	0
	WORHP IP	optimal	19	5.40e-01	1.78e+04	26	26	21	21	19
	WORHP IPm	optimal	18	4.60e-01	1.78e+04	25	25	19	19	18
	WORHP SQP	minalpha	31	7.56e+00	4.47e+04	4535	4541	22	33	21
LUKVLE17	IPOPT	accept	226	6.85e+00	3.37e+04	1534	1696	210	244	227
	KNITRO	maxiter	10000	1.17e+02	3.25e+04	16247	16248	10001	10002	10000
	SNOPT	sbasics	106	2.62e+02	6.91e+04	280	280	279	279	0
	WORHP IP	optimal	903	1.71e+01	3.32e+04	6543	6543	905	905	903
	WORHP IPm	maxiter	10000	4.12e+02	3.36e+04	114355	114355	10001	10001	10000
	WORHP SQP	minalpha	81	1.11e+01	3.53e+04	7849	7856	24	83	23
LUKVLE18	IPOPT	maxiter	10000	2.19e+02	1.12e+04	125906	125938	10001	10003	10000
	KNITRO	maxiter	10000	2.02e+02	1.11e+04	21921	21922	10001	10002	10000
	SNOPT	infeas	107	4.03e+02	1.49e+04	244	244	243	243	0
	WORHP IP	optimal	549	1.64e+01	1.11e+04	3250	3250	551	551	549
	WORHP IPm	maxiter	10000	2.42e+02	1.12e+04	112227	112227	10001	10001	10000
	WORHP SQP	minalpha	168	1.45e+01	5.61e+04	12050	12059	27	170	26
LUKVLE2	IPOPT	resto	78	4.49e+00	-5.86e+57	79	79	79	79	79
	KNITRO	unbound	31	1.95e+00	-1.73e+22	36	37	32	33	31
	SNOPT	unbound	21	1.28e+01	-4.60e+20	36	36	35	35	0
	WORHP IP	unbound	28	1.77e+00	-1.51e+20	30	30	29	29	28
	WORHP IPm	unbound	28	2.10e+00	-1.51e+20	30	30	29	29	28
	WORHP SQP	minalpha	9095	1.25e+03	-1.00e+20	483278	482260	4413	10122	4412
LUKVLE3	IPOPT	optimal	9	2.60e-01	2.76e+01	10	10	10	10	9
	KNITRO	optimal	9	2.70e-01	2.76e+01	11	12	10	11	9
	SNOPT	toobig	36	1.12e+02	9.48e+05	39	39	38	38	0
	WORHP IP	optimal	9	2.60e-01	2.76e+01	11	11	10	10	9
	WORHP IPm	optimal	9	2.50e-01	2.76e+01	11	11	10	10	9
	WORHP SQP	optimal	13	5.00e-01	2.76e+01	14	14	15	15	13
LUKVLE4	IPOPT	resto	66	3.25e+00	-1.51e+07	211	217	67	68	67
	KNITRO	smallstep	38	3.25e+00	-3.53e+15	205	206	39	40	38
	SNOPT	maxtime	811	1.80e+03	1.83e+05	858	858	857	857	0
	WORHP IP	fritzjohn	75	4.26e+00	-1.04e+08	146	146	76	76	76
	WORHP IPm	fritzjohn	75	4.86e+00	-1.04e+08	146	146	76	76	76
	WORHP SQP	toobig	153	1.11e+01	-5.44e+15	3003	3003	155	155	153
LUKVLE5	IPOPT	optimal	18	1.25e+00	2.64e+00	22	22	19	19	18
	KNITRO	optimal	14	1.01e+00	2.64e+00	17	18	15	16	14
	SNOPT	optimal	113	4.41e+00	4.32e-01	286	286	285	285	0
	WORHP IP	optimal	20	1.30e+00	2.64e+00	25	25	21	21	20
	WORHP IPm	optimal	20	1.22e+00	2.64e+00	25	25	21	21	20
	WORHP SQP	optimal	45	9.81e+00	2.64e+00	170	170	47	47	45

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LUKVLE6	IPOPT	optimal	16	1.07e+00	6.29e+05	17	17	17	17	16
	KNITRO	optimal	15	9.70e-01	6.29e+05	17	18	16	17	15
	SNOPT	maxtime	708	1.80e+03	2.71e+11	847	847	846	846	0
	WORHP IP	optimal	16	1.42e+00	6.29e+05	18	18	18	18	16
	WORHP IPm	optimal	15	1.59e+00	6.29e+05	17	17	16	16	15
	WORHP SQP	optimal	14	1.87e+00	6.29e+05	15	15	16	16	14
LUKVLE7	IPOPT	optimal	11	2.90e-01	-2.17e+03	20	20	12	12	11
	KNITRO	optimal	8	2.00e-01	-2.17e+03	10	11	9	10	8
	SNOPT	toobig	32	6.14e+01	1.47e+07	35	35	34	34	0
	WORHP IP	optimal	15	5.40e-01	-2.17e+03	27	27	16	16	15
	WORHP IPm	optimal	15	6.00e-01	-2.17e+03	27	27	16	16	15
	WORHP SQP	optimal	16	5.40e-01	-2.16e+03	32	32	18	18	16
LUKVLE8	IPOPT	optimal	13	6.80e-01	8.26e+05	19	19	14	14	13
	KNITRO	optimal	14	6.90e-01	8.26e+05	16	17	15	16	14
	SNOPT	optimal	4	2.70e-01	1.60e+06	10	10	9	9	0
	WORHP IP	fritzjohn	26	1.34e+00	1.06e+06	55	55	28	28	26
	WORHP IPm	optimal	21	1.29e+00	1.06e+06	23	23	22	22	21
	WORHP SQP	optimal	15	2.85e+00	8.26e+05	16	16	17	17	15
LUKVLE9	IPOPT	optimal	17	2.10e-01	1.00e+03	26	27	18	18	17
	KNITRO	optimal	14	2.10e-01	1.00e+03	20	21	15	16	14
	SNOPT	sbasics	128	3.11e+02	3.41e+03	154	154	153	153	0
	WORHP IP	optimal	19	2.60e-01	9.99e+02	21	21	20	20	19
	WORHP IPm	optimal	19	2.90e-01	9.99e+02	21	21	20	20	19
	WORHP SQP	optimal	34	6.30e-01	9.99e+02	150	149	34	37	32
LUKVLI1	IPOPT	maxiter	10000	2.36e+02	2.62e+03	13629	13631	10001	10001	10000
	KNITRO	maxiter	10000	2.40e+02	2.34e+03	10003	10004	10001	10002	10000
	SNOPT	sbasics	45	1.73e+02	2.26e+06	54	54	53	53	0
	WORHP IP	maxiter	10000	3.29e+02	3.07e+03	11010	11010	10001	10001	10000
	WORHP IPm	maxiter	10000	1.24e+03	6.65e+03	74172	74172	10001	10001	10000
	WORHP SQP	maxiter	10000	4.28e+02	2.95e+03	22036	22036	10002	10002	10001
LUKVLI10	IPOPT	optimal	78	3.44e+00	3.54e+03	104	104	79	79	78
	KNITRO	optimal	23	1.14e+00	3.54e+03	27	28	24	25	23
	SNOPT	infeas	0	5.64e+00	1.00e+04	3	3	2	2	0
	WORHP IP	optimal	24	1.63e+00	3.54e+03	26	26	25	25	24
	WORHP IPm	optimal	20	1.24e+00	3.54e+03	22	22	21	21	20
	WORHP SQP	optimal	85	1.29e+01	3.53e+03	205	204	78	87	77
LUKVLI11	IPOPT	optimal	22	7.90e-01	2.05e-04	23	23	23	23	22
	KNITRO	optimal	18	6.40e-01	2.12e-07	26	27	19	20	18
	SNOPT	maxtime	594	1.80e+03	9.00e+00	715	715	714	714	0
	WORHP IP	optimal	22	1.02e+00	1.11e-05	24	24	23	23	22
	WORHP IPm	optimal	21	8.70e-01	5.94e-07	23	23	22	22	21
	WORHP SQP	optimal	22	2.19e+00	8.86e-08	23	23	24	24	22
LUKVLI12	IPOPT	optimal	60	1.43e+00	1.84e-06	66	66	61	61	60
	KNITRO	optimal	65	1.73e+00	1.78e-07	78	79	66	67	65
	SNOPT	maxtime	394	1.80e+03	1.50e+04	2652	2652	2651	2651	0
	WORHP IP	optimal	107	4.10e+00	1.71e-07	152	152	108	108	107
	WORHP IPm	optimal	67	2.39e+00	1.26e-07	114	114	68	68	67
	WORHP SQP	optimal	48	2.43e+00	9.06e-10	186	186	50	50	48
LUKVLI13	IPOPT	optimal	35	9.60e-01	1.32e+02	36	36	36	36	35
	KNITRO	optimal	16	5.50e-01	1.32e+02	18	19	17	18	16
	SNOPT	sbasics	35	1.13e+02	6.53e+04	78	78	77	77	0
	WORHP IP	optimal	23	9.80e-01	1.32e+02	25	25	24	24	23
	WORHP IPm	optimal	21	7.90e-01	1.32e+02	23	23	22	22	21
	WORHP SQP	optimal	13	1.34e+00	1.32e+02	14	14	15	15	13
LUKVLI14	IPOPT	optimal	43	1.02e+00	1.56e+04	44	44	44	44	43
	KNITRO	optimal	30	1.00e+00	1.56e+04	34	35	31	32	30
	SNOPT	sbasics	56	2.87e+02	4.17e+07	158	158	157	157	0
	WORHP IP	optimal	26	1.04e+00	1.56e+04	28	28	28	28	26
	WORHP IPm	optimal	26	1.02e+00	1.56e+04	28	28	27	27	26
	WORHP SQP	optimal	16	1.09e+00	1.56e+04	17	17	18	18	16
LUKVLI15	IPOPT	optimal	292	8.39e+00	1.84e-04	325	325	292	296	292
	KNITRO	optimal	68	2.28e+00	6.14e-08	70	71	69	70	68
	SNOPT	maxtime	404	1.80e+03	5.87e+10	2055	2055	2054	2054	0
	WORHP IP	optimal	190	7.03e+00	9.29e-06	194	194	192	192	190
	WORHP IPm	optimal	217	7.28e+00	4.12e+01	220	220	218	218	217
	WORHP SQP	optimal	72	8.79e+00	1.37e-08	369	369	74	74	72
LUKVLI16	IPOPT	optimal	37	1.17e+00	2.97e+03	38	38	38	38	37
	KNITRO	optimal	26	7.30e-01	2.97e+03	32	33	27	28	26
	SNOPT	sbasics	41	1.12e+02	1.41e+04	66	66	65	65	0
	WORHP IP	optimal	51	1.93e+00	2.97e+03	53	53	53	53	51
	WORHP IPm	optimal	39	1.36e+00	2.97e+03	42	42	40	40	39
	WORHP SQP	optimal	17	2.68e+00	2.97e+03	18	18	19	19	17
LUKVLI17	IPOPT	optimal	42	1.29e+00	7.81e+02	43	43	43	43	42
	KNITRO	optimal	17	5.00e-01	7.81e+02	19	20	18	19	17
	SNOPT	maxtime	728	1.80e+03	2.90e+04	775	775	774	774	0
	WORHP IP	optimal	20	8.10e-01	7.81e+02	33	33	21	21	20
	WORHP IPm	optimal	25	8.40e-01	7.81e+02	33	33	26	26	25
	WORHP SQP	optimal	12	7.50e-01	7.81e+02	13	13	14	14	12

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
LUKVLI18	IPOPT	optimal	24	6.70e-01	2.14e-04	25	25	25	25	24
	KNITRO	optimal	16	4.80e-01	7.72e-09	18	19	17	18	16
	SNOPT	sbasics	30	1.18e+02	6.18e+03	53	53	52	52	0
	WORHP IP	optimal	25	9.20e-01	3.08e-04	27	27	26	26	25
	WORHP IPm	optimal	20	6.50e-01	1.27e-06	22	22	21	21	20
	WORHP SQP	optimal	15	9.80e-01	4.36e-07	23	23	17	17	15
LUKVLI2	IPOPT	resto	77	5.50e+00	-9.73e+56	78	78	78	78	78
	KNITRO	unbound	19	1.33e+00	-4.84e+20	21	22	20	21	19
	SNOPT	unbound	123	4.76e+02	-2.93e+19	505	505	504	504	0
	WORHP IP	unbound	33	2.55e+00	-4.24e+21	35	35	34	34	33
	WORHP IPm	diverge	32	2.14e+00	-1.69e+20	34	34	33	33	32
	WORHP SQP	minalpha	99	2.08e+01	-9.97e+19	3243	3241	99	108	98
LUKVLI3	IPOPT	optimal	17	3.10e-01	1.16e+01	18	18	18	18	17
	KNITRO	optimal	17	3.90e-01	1.16e+01	19	20	18	19	17
	SNOPT	toobig	36	8.73e+01	9.48e+05	39	39	38	38	0
	WORHP IP	optimal	17	3.60e-01	1.16e+01	19	19	19	19	17
	WORHP IPm	optimal	17	3.50e-01	1.16e+01	19	19	18	18	17
	WORHP SQP	optimal	16	5.40e-01	1.16e+01	17	17	18	18	16
LUKVLI4	IPOPT	resto	56	2.89e+00	-1.24e+07	86	86	57	57	57
	KNITRO	smallstep	45	3.27e+00	-1.91e+13	222	223	46	47	45
	SNOPT	sbasics	104	2.47e+02	5.17e+05	128	128	127	127	0
	WORHP IP	fritzjohn	193	1.16e+01	-7.48e+07	211	211	194	194	194
	WORHP IPm	fritzjohn	69	4.45e+00	-2.47e+07	104	104	70	70	70
	WORHP SQP	minalpha	73	1.29e+01	-3.58e+05	1610	1615	75	75	74
LUKVLI5	IPOPT	optimal	34	2.04e+00	4.89e-01	35	35	35	35	34
	KNITRO	optimal	22	1.34e+00	4.89e-01	24	25	23	24	22
	SNOPT	optimal	45	8.68e+01	3.72e-01	94	94	93	93	0
	WORHP IP	optimal	43	3.01e+00	5.28e-01	48	48	45	45	43
	WORHP IPm	optimal	43	3.28e+00	5.27e-01	60	60	44	44	43
	WORHP SQP	optimal	28	1.19e+01	5.27e-01	30	30	30	30	28
LUKVLI6	IPOPT	optimal	14	9.30e-01	6.29e+05	15	15	15	15	14
	KNITRO	optimal	17	1.04e+00	6.29e+05	19	20	18	19	17
	SNOPT	sbasics	222	6.65e+02	1.78e+09	273	273	272	272	0
	WORHP IP	optimal	21	2.51e+00	6.29e+05	135	135	23	23	21
	WORHP IPm	optimal	13	1.17e+00	6.29e+05	15	15	14	14	13
	WORHP SQP	optimal	14	1.72e+00	6.29e+05	15	15	16	16	14
LUKVLI7	IPOPT	optimal	22	3.90e-01	-2.17e+03	23	23	23	23	22
	KNITRO	optimal	16	2.90e-01	-2.17e+03	18	19	17	18	16
	SNOPT	sbasics	30	5.95e+01	1.47e+07	74	74	73	73	0
	WORHP IP	optimal	13	5.00e-01	-2.17e+03	15	15	15	15	13
	WORHP IPm	optimal	23	7.90e-01	-2.17e+03	32	32	24	24	23
	WORHP SQP	optimal	21	7.70e-01	-2.16e+03	22	22	23	23	21
LUKVLI8	IPOPT	optimal	42	1.90e+00	1.03e+06	43	43	43	43	42
	KNITRO	optimal	41	1.99e+00	1.06e+06	50	51	42	43	41
	SNOPT	optimal	66	1.61e+01	8.27e+05	109	109	108	108	0
	WORHP IP	fritzjohn	52	3.22e+00	1.06e+06	134	134	54	54	52
	WORHP IPm	optimal	148	8.69e+00	1.06e+06	163	163	149	149	148
	WORHP SQP	optimal	12	7.19e+00	1.06e+06	13	13	14	14	12
LUKVLI9	IPOPT	optimal	30	3.30e-01	9.99e+02	46	46	31	31	30
	KNITRO	optimal	10	1.70e-01	9.99e+02	12	13	11	12	10
	SNOPT	toobig	48	1.37e+02	3.40e+03	83	83	82	82	0
	WORHP IP	optimal	25	3.50e-01	9.99e+02	27	27	26	26	25
	WORHP IPm	optimal	29	4.20e-01	9.99e+02	31	31	30	30	29
	WORHP SQP	optimal	11	2.50e-01	9.99e+02	12	12	13	13	11
MADSEN	IPOPT	optimal	20	1.00e-02	6.16e-01	25	25	21	21	20
	KNITRO	optimal	13	1.00e-02	6.16e-01	15	16	14	15	13
	SNOPT	optimal	10	1.00e-02	6.16e-01	1	14	1	13	0
	WORHP IP	optimal	9	1.00e-02	6.16e-01	11	11	10	10	9
	WORHP IPm	optimal	12	1.00e-02	6.16e-01	14	14	13	13	12
	WORHP SQP	optimal	15	1.00e-02	6.16e-01	25	25	17	17	15
MADSSCHJ	IPOPT	optimal	37	4.41e+00	-4.99e+03	60	60	38	38	37
	KNITRO	optimal	20	6.50e-01	-4.99e+03	22	23	21	22	20
	SNOPT	optimal	29	1.55e+00	-4.99e+03	1	76	1	75	0
	WORHP IP	optimal	22	2.86e+00	-4.99e+03	26	26	23	23	22
	WORHP IPm	optimal	32	5.83e+00	-4.99e+03	41	41	33	33	32
	WORHP SQP	optimal	13	1.86e+01	-4.99e+03	14	14	15	15	13
MAKELA1	IPOPT	optimal	18	1.00e-02	-1.41e+00	19	19	19	19	18
	KNITRO	optimal	12	1.00e-02	-1.41e+00	14	15	13	14	12
	SNOPT	optimal	6	1.00e-02	-1.41e+00	1	10	1	9	0
	WORHP IP	optimal	17	1.00e-02	-1.41e+00	19	19	18	18	17
	WORHP IPm	optimal	15	1.00e-02	-1.41e+00	17	17	16	16	15
	WORHP SQP	optimal	14	1.00e-02	-1.41e+00	16	16	16	16	14
MAKELA2	IPOPT	optimal	7	1.00e-02	7.20e+00	8	8	8	8	7
	KNITRO	optimal	6	1.00e-02	7.20e+00	8	9	7	8	6
	SNOPT	optimal	15	1.00e-02	7.20e+00	1	28	1	27	0
	WORHP IP	optimal	7	1.00e-02	7.20e+00	9	9	8	8	7
	WORHP IPm	optimal	6	1.00e-02	7.20e+00	8	8	7	7	6
	WORHP SQP	optimal	13	1.00e-02	7.20e+00	14	14	15	15	13

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
MAKELA3	IPOPT	optimal	16	1.00e-02	1.81e-06	17	17	17	17	16
	KNITRO	optimal	15	1.00e-02	-9.46e-11	17	18	16	17	15
	SNOPT	optimal	31	1.00e-02	-2.19e-12	1	52	1	51	0
	WORHP IP	optimal	18	1.00e-02	4.39e-06	20	20	19	19	18
	WORHP IPm	optimal	13	1.00e-02	-1.13e-08	15	15	14	14	13
	WORHP SQP	optimal	21	1.00e-02	2.05e-14	22	22	23	23	21
MAKELA4	IPOPT	optimal	7	1.00e-02	3.63e-06	8	8	8	8	7
	KNITRO	optimal	4	1.00e-02	3.73e-06	6	7	5	6	4
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	6	1.00e-02	1.40e-08	8	8	7	1	6
	WORHP IPm	optimal	5	1.00e-02	3.72e-06	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	2.77e-13	2	2	3	3	2
MANCINO	IPOPT	optimal	18	5.60e-01	1.27e-21	19	0	19	0	18
	KNITRO	optimal	8	3.00e-01	1.41e-21	14	0	9	0	8
	SNOPT	optimal	12	1.40e-01	1.74e-20	15	0	14	0	0
	WORHP IP	optimal	16	4.70e-01	1.62e-21	18	0	18	0	16
	WORHP IPm	optimal	16	4.60e-01	1.62e-21	18	0	17	0	16
	WORHP SQP	optimal	12	5.00e-01	2.36e-21	13	0	13	0	12
MANNE	IPOPT	maxiter	10000	1.57e+02	4.45e+01	10002	131251	122	10002	10000
	KNITRO	smallstep	59	3.99e+00	-7.59e-01	356	357	61	62	60
	SNOPT	optimal	0	5.00e-02	-9.75e-01	3	3	2	2	0
	WORHP IP	optimal	601	1.73e+01	-9.72e-01	1330	1330	602	602	601
	WORHP IPm	optimal	621	2.69e+01	-9.74e-01	4299	4299	625	625	621
	WORHP SQP	optimal	90	1.46e+01	-9.75e-01	91	91	91	91	90
MARATOS	IPOPT	optimal	3	1.00e-02	-1.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.00e-02	-1.00e+00	5	6	4	5	3
	SNOPT	optimal	4	1.00e-02	-1.00e+00	9	9	8	8	0
	WORHP IP	optimal	13	1.00e-02	-1.00e+00	25	25	14	14	13
	WORHP IPm	optimal	13	1.00e-02	-1.00e+00	25	25	14	14	13
	WORHP SQP	optimal	4	1.00e-02	-1.00e+00	5	5	6	6	4
MARATOSB	IPOPT	optimal	671	1.50e-01	-1.00e+00	1752	0	672	0	671
	KNITRO	optimal	674	1.00e-02	-1.00e+00	992	0	675	0	674
	SNOPT	optimal	1016	2.00e-02	-1.00e+00	1433	0	1432	0	0
	WORHP IP	optimal	670	2.00e-02	-1.00e+00	1165	0	672	0	670
	WORHP IPm	optimal	670	1.00e-02	-1.00e+00	1165	0	671	0	670
	WORHP SQP	optimal	661	4.00e-02	-1.00e+00	4208	0	662	0	661
MARINE	IPOPT	optimal	38	2.01e+00	1.97e+07	46	46	35	41	38
	KNITRO	optimal	433	6.05e+01	1.97e+07	441	442	435	436	434
	SNOPT	noimpr	168	2.05e+01	1.97e+07	287	287	286	286	0
	WORHP IP	optimal	64	3.34e+00	1.97e+07	120	120	66	66	64
	WORHP IPm	optimal	43	2.19e+00	1.97e+07	49	49	48	48	43
	WORHP SQP	optimal	84	6.26e+02	1.97e+07	215	214	52	87	50
MATRIX2	IPOPT	optimal	12	1.00e-02	3.88e-06	13	13	13	13	12
	KNITRO	optimal	11	1.00e-02	9.53e-08	14	15	13	14	11
	SNOPT	optimal	13	1.00e-02	4.11e-10	16	16	15	15	0
	WORHP IP	optimal	13	1.00e-02	1.00e-06	15	15	14	14	13
	WORHP IPm	optimal	10	1.00e-02	1.80e-06	12	12	11	11	10
	WORHP SQP	optimal	11	1.00e-02	2.39e-07	12	12	13	13	11
MAXLIKA	IPOPT	optimal	28	4.00e-02	1.14e+03	45	0	29	0	28
	KNITRO	optimal	25	3.00e-02	1.14e+03	31	0	27	0	25
	SNOPT	optimal	94	4.00e-02	1.14e+03	110	0	109	0	0
	WORHP IP	optimal	24	4.00e-02	1.14e+03	32	0	25	0	24
	WORHP IPm	optimal	131	4.70e-01	1.14e+03	2423	0	136	0	131
	WORHP SQP	optimal	28	5.00e-02	1.14e+03	56	0	29	0	28
MCCORMCK	IPOPT	optimal	7	1.60e-01	-4.57e+03	8	0	8	0	7
	KNITRO	optimal	7	1.70e-01	-4.57e+03	9	0	8	0	7
	SNOPT	toobig	42	6.13e+01	1.10e+03	47	0	46	0	0
	WORHP IP	optimal	7	1.70e-01	-4.57e+03	9	0	8	0	7
	WORHP IPm	optimal	6	1.70e-01	-4.57e+03	10	0	8	0	6
	WORHP SQP	optimal	5	1.80e-01	-4.57e+03	13	0	6	0	5
MCONCON	IPOPT	optimal	9	1.00e-02	-6.23e+03	10	10	10	10	9
	KNITRO	optimal	6	1.00e-02	-6.23e+03	9	10	8	9	6
	SNOPT	optimal	7	1.00e-02	-6.23e+03	1	10	1	9	0
	WORHP IP	optimal	9	1.00e-02	-6.23e+03	11	11	10	10	9
	WORHP IPm	optimal	10	1.00e-02	-6.23e+03	15	15	14	14	10
	WORHP SQP	optimal	6	1.00e-02	-6.23e+03	7	7	7	7	6
MDHOLE	IPOPT	optimal	43	2.00e-02	8.09e-08	117	0	44	0	43
	KNITRO	optimal	39	1.00e-02	1.70e-07	63	0	40	0	39
	SNOPT	optimal	60	1.00e-02	1.99e-34	90	0	89	0	0
	WORHP IP	optimal	41	1.00e-02	2.45e-08	63	0	42	0	41
	WORHP IPm	optimal	40	1.00e-02	2.96e-12	62	0	42	0	40
	WORHP SQP	optimal	34	1.00e-02	2.04e-12	200	0	35	0	34
MESH	IPOPT	maxiter	10000	1.14e+01	-7.26e+36	10001	840144	10001	20002	10000
	KNITRO	unbound	60	2.00e-02	-2.49e+20	62	63	61	62	60
	SNOPT	unbound	18	1.00e-02	-4.75e+09	39	39	38	38	0
	WORHP IP	diverge	71	2.00e-02	-1.09e+20	73	73	72	72	71
	WORHP IPm	diverge	67	2.00e-02	-1.39e+20	69	69	68	68	67
	WORHP SQP	unbound	229	5.70e-01	-1.26e+20	54623	54623	231	231	229

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
METHANB8	IPOPT	optimal	2	1.00e-02	0.00e+00	3	3	3	3	2
	KNITRO	optimal	2	1.00e-02	0.00e+00	4	5	3	4	2
	SNOPT	optimal	2	1.00e-02	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	2	1.00e-02	0.00e+00	3	3	4	4	2
METHANL8	IPOPT	optimal	4	1.00e-02	0.00e+00	5	5	5	5	4
	KNITRO	optimal	4	1.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	optimal	6	1.00e-02	0.00e+00	1	9	1	8	0
	WORHP IP	optimal	4	1.00e-02	0.00e+00	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	0.00e+00	6	6	5	5	4
	WORHP SQP	optimal	4	1.00e-02	0.00e+00	5	5	6	6	4
METHANOL	IPOPT	optimal	14	1.44e+00	9.02e-03	15	15	15	15	14
	KNITRO	optimal	6	8.00e-01	9.02e-03	8	9	7	8	6
	SNOPT	optimal	444	4.69e+01	9.02e-03	1684	1684	1683	1683	0
	WORHP IP	optimal	15	1.45e+00	9.02e-03	36	36	16	16	15
	WORHP IPm	optimal	8	1.03e+00	9.02e-03	11	11	10	10	8
	WORHP SQP	optimal	35	2.52e+01	9.02e-03	36	36	37	37	35
MEXHAT	IPOPT	optimal	28	1.00e-02	-4.00e-02	44	0	29	0	28
	KNITRO	optimal	28	1.00e-02	-4.00e-02	33	0	29	0	28
	SNOPT	optimal	31	1.00e-02	-4.00e-02	51	0	50	0	0
	WORHP IP	optimal	28	1.00e-02	-4.00e-02	36	0	30	0	28
	WORHP IPm	optimal	28	1.00e-02	-4.00e-02	36	0	29	0	28
	WORHP SQP	optimal	26	1.00e-02	-4.00e-02	41	0	27	0	26
MEYER3	IPOPT	maxiter	10000	5.21e+00	8.79e+01	142992	0	10001	0	10000
	KNITRO	noimpr	202	1.00e-02	8.79e+01	312	0	203	0	202
	SNOPT	maxiter	10000	8.90e-01	8.79e+01	96914	0	96913	0	0
	WORHP IP	accept	213	1.00e-02	8.79e+01	715	0	257	0	214
	WORHP IPm	minalpha	213	1.00e-02	8.79e+01	715	0	257	0	214
	WORHP SQP	zerostep	228	2.00e-02	8.79e+01	1232	0	228	0	228
MEYER3NE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	53	1.00e-02	0.00e+00	157	158	53	54	52
	SNOPT	infeas	12	1.00e-02	0.00e+00	1	30	1	29	0
	WORHP IP	infeas	8	1.00e-02	0.00e+00	16	16	9	9	9
	WORHP IPm	infeas	8	1.00e-02	0.00e+00	16	16	9	9	9
	WORHP SQP	minalpha	22	1.00e-02	0.00e+00	2766	2772	24	24	23
MGH09	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	42	1.00e-02	0.00e+00	48	49	43	44	42
	SNOPT	infeas	28	1.00e-02	0.00e+00	1	53	1	52	0
	WORHP IP	infeas	115	1.00e-02	0.00e+00	488	488	116	116	116
	WORHP IPm	infeas	115	1.00e-02	0.00e+00	488	488	116	116	116
	WORHP SQP	minalpha	12	1.00e-02	0.00e+00	2467	2473	14	14	13
MGH09LS	IPOPT	optimal	71	2.00e-02	3.08e-04	182	0	72	0	71
	KNITRO	optimal	84	1.00e-02	3.08e-04	119	0	85	0	84
	SNOPT	optimal	248	1.00e-02	3.08e-04	360	0	359	0	0
	WORHP IP	optimal	71	1.00e-02	3.08e-04	127	0	72	0	71
	WORHP IPm	optimal	71	1.00e-02	3.08e-04	127	0	72	0	71
	WORHP SQP	maxiter	10000	6.80e-01	9.45e-04	12717	0	10001	0	10001
MGH10	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	46	1.00e-02	0.00e+00	78	79	46	47	45
	SNOPT	infeas	47	1.00e-02	0.00e+00	1	473	1	472	0
	WORHP IP	infeas	2	1.00e-02	0.00e+00	9	9	3	3	3
	WORHP IPm	infeas	2	1.00e-02	0.00e+00	9	9	3	3	3
	WORHP SQP	minalpha	20	3.00e-02	0.00e+00	4518	4524	14	22	13
MGH10LS	IPOPT	maxiter	10000	4.05e+00	8.79e+01	101921	0	10001	0	10000
	KNITRO	noimpr	132	1.00e-02	2.14e+08	357	0	133	0	132
	SNOPT	toobig	53	1.00e-02	1.34e+09	92	0	91	0	0
	WORHP IP	accept	1651	9.00e-02	8.79e+01	3244	0	1700	0	1652
	WORHP IPm	minalpha	1651	8.00e-02	8.79e+01	3244	0	1699	0	1652
	WORHP SQP	minalpha	117	2.00e-02	1.23e+09	2433	0	118	0	118
MGH10S	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	811	1.90e-01	0.00e+00	3151	3152	812	813	812
	SNOPT	degen	89	1.00e-02	0.00e+00	1	297	1	296	0
	WORHP IP	infeas	2	1.00e-02	0.00e+00	9	9	3	3	3
	WORHP IPm	infeas	2	1.00e-02	0.00e+00	9	9	3	3	3
	WORHP SQP	minalpha	5	2.00e-02	0.00e+00	1696	1702	6	7	5
MGH17	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	1194	5.20e-01	0.00e+00	4269	4270	1193	1194	1193
	SNOPT	infeas	18	1.00e-02	0.00e+00	1	57	1	56	0
	WORHP IP	infeas	98	1.00e-02	0.00e+00	227	227	99	99	99
	WORHP IPm	infeas	98	1.00e-02	0.00e+00	227	227	99	99	99
	WORHP SQP	minalpha	357	1.50e-01	0.00e+00	3480	3487	351	359	350
MGH17LS	IPOPT	optimal	48	2.00e-02	7.89e-05	109	0	49	0	48
	KNITRO	optimal	165	2.00e-02	5.46e-05	355	0	166	0	165
	SNOPT	optimal	18	1.00e-02	1.02e+00	32	0	31	0	0
	WORHP IP	optimal	282	2.00e-02	5.46e-05	508	0	284	0	282
	WORHP IPm	optimal	47	1.00e-02	7.91e-05	79	0	48	0	47
	WORHP SQP	optimal	103	1.00e-02	5.46e-05	199	0	104	0	103

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
MGH17S	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	187	4.00e-02	0.00e+00	216	217	188	189	187
	SNOPT	infeas	10	1.00e-02	0.00e+00	1	27	1	26	0
	WORHP IP	infeas	76	2.00e-02	0.00e+00	183	183	77	77	76
	WORHP IPm	infeas	76	1.00e-02	0.00e+00	183	183	77	77	76
	WORHP SQP	minalpha	10	6.00e-02	0.00e+00	4024	4030	7	12	6
MIFFLIN1	IPOPT	optimal	6	1.00e-02	-1.00e+00	7	7	7	7	6
	KNITRO	optimal	5	1.00e-02	-1.00e+00	7	8	6	7	5
	SNOPT	optimal	6	1.00e-02	-1.00e+00	1	10	1	9	0
	WORHP IP	optimal	5	1.00e-02	-1.00e+00	7	7	6	6	5
	WORHP IPm	optimal	5	1.00e-02	-1.00e+00	7	7	6	6	5
	WORHP SQP	optimal	15	1.00e-02	-1.00e+00	17	17	17	17	15
MIFFLIN2	IPOPT	optimal	15	1.00e-02	-1.00e+00	16	16	16	16	15
	KNITRO	optimal	10	1.00e-02	-1.00e+00	12	13	11	12	10
	SNOPT	optimal	10	1.00e-02	-1.00e+00	1	19	1	18	0
	WORHP IP	optimal	16	1.00e-02	-1.00e+00	29	29	17	17	16
	WORHP IPm	optimal	14	1.00e-02	-1.00e+00	17	17	15	15	14
	WORHP SQP	optimal	8	1.00e-02	-1.00e+00	9	9	10	10	8
MINC44	IPOPT	optimal	40	8.87e+00	3.83e-04	44	44	41	41	40
	KNITRO	optimal	71	1.18e+01	3.83e-04	74	75	73	74	71
	SNOPT	optimal	23	6.00e-01	3.83e-04	1	30	1	29	0
	WORHP IP	optimal	2482	6.19e+02	3.83e-04	25654	25654	2483	2483	2482
	WORHP IPm	optimal	4940	6.50e+02	3.83e-04	52051	52051	4942	4942	4940
	WORHP SQP	optimal	22	5.60e+01	3.83e-04	24	24	24	24	22
MINMAXBD	IPOPT	optimal	32	1.00e-02	1.16e+02	37	44	33	33	32
	KNITRO	optimal	10	1.00e-02	1.16e+02	14	15	11	12	10
	SNOPT	optimal	23	1.00e-02	1.16e+02	1	47	1	46	0
	WORHP IP	optimal	62	1.00e-02	1.16e+02	64	64	63	63	62
	WORHP IPm	optimal	26	1.00e-02	1.16e+02	28	28	27	27	26
	WORHP SQP	optimal	26	2.00e-02	1.16e+02	41	41	28	28	26
MINMAXRB	IPOPT	optimal	9	1.00e-02	3.54e-07	11	11	10	10	9
	KNITRO	optimal	5	1.00e-02	-9.42e-11	7	8	6	7	5
	SNOPT	optimal	4	1.00e-02	8.88e-16	1	9	1	8	0
	WORHP IP	optimal	6	1.00e-02	8.79e-07	10	10	7	7	6
	WORHP IPm	optimal	5	1.00e-02	4.17e-12	9	9	6	6	5
	WORHP SQP	optimal	4	1.00e-02	8.27e-16	5	5	6	6	4
MINPERM	IPOPT	smallstep	14	8.28e+00	3.63e-04	38	38	13	16	15
	KNITRO	optimal	99	1.57e+01	3.63e-04	174	175	101	102	99
	SNOPT	optimal	28	7.90e-01	3.63e-04	1	46	1	45	0
	WORHP IP	optimal	5	1.13e+00	3.63e-04	7	7	6	6	5
	WORHP IPm	optimal	8	1.60e+00	3.63e-04	15	15	10	10	8
	WORHP SQP	optimal	4	4.68e+00	3.63e-04	5	5	6	6	4
MINSURF	IPOPT	optimal	4	1.00e-02	1.00e+00	21	0	5	0	4
	KNITRO	optimal	8	1.00e-02	1.00e+00	22	0	10	0	8
	SNOPT	optimal	14	1.00e-02	1.00e+00	26	0	25	0	0
	WORHP IP	optimal	4	1.00e-02	1.00e+00	16	0	5	0	4
	WORHP IPm	optimal	4	1.00e-02	1.00e+00	16	0	5	0	4
	WORHP SQP	optimal	24	1.00e-02	1.00e+00	271	0	25	0	24
MINSURFO	IPOPT	optimal	71	2.58e+00	2.51e+00	374	0	72	0	71
	KNITRO	optimal	16	7.20e-01	2.51e+00	23	0	18	0	16
	SNOPT	toobig	1024	6.01e+01	2.51e+00	1161	0	1160	0	0
	WORHP IP	optimal	72	3.50e+00	2.51e+00	237	0	73	0	72
	WORHP IPm	optimal	57	2.32e+00	2.51e+00	188	0	59	0	57
	WORHP SQP	optimal	40	7.43e+00	2.51e+00	532	0	41	0	40
MISRA1A	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	23	1.00e-02	0.00e+00	30	31	24	25	23
	SNOPT	infeas	23	1.00e-02	0.00e+00	1	59	1	58	0
	WORHP IP	infeas	15	1.00e-02	0.00e+00	53	53	16	16	16
	WORHP IPm	infeas	15	1.00e-02	0.00e+00	53	53	16	16	16
	WORHP SQP	infeas	33	1.00e-02	0.00e+00	1047	1049	21	34	20
MISRA1ALS	IPOPT	optimal	51	1.00e-02	1.25e-01	294	0	52	0	51
	KNITRO	optimal	31	1.00e-02	1.25e-01	42	0	32	0	31
	SNOPT	toobig	7	1.00e-02	1.95e+01	18	0	17	0	0
	WORHP IP	optimal	34	1.00e-02	1.25e-01	114	0	36	0	34
	WORHP IPm	optimal	42	1.00e-02	1.25e-01	66	0	43	0	42
	WORHP SQP	zerostep	39	1.00e-02	1.25e-01	49	0	39	0	39
MISRA1B	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	20	1.00e-02	0.00e+00	27	28	21	22	20
	SNOPT	infeas	8	1.00e-02	0.00e+00	1	50	1	49	0
	WORHP IP	infeas	11	1.00e-02	0.00e+00	30	30	12	12	12
	WORHP IPm	infeas	11	1.00e-02	0.00e+00	30	30	12	12	12
	WORHP SQP	infeas	20	1.00e-02	0.00e+00	815	817	15	21	14
MISRA1BLS	IPOPT	optimal	45	2.00e-02	7.55e-02	325	0	46	0	45
	KNITRO	optimal	26	1.00e-02	7.55e-02	36	0	27	0	26
	SNOPT	toobig	7	1.00e-02	7.32e+00	18	0	17	0	0
	WORHP IP	optimal	31	1.00e-02	7.55e-02	134	0	33	0	31
	WORHP IPm	optimal	37	1.00e-02	7.55e-02	60	0	38	0	37
	WORHP SQP	optimal	34	1.00e-02	7.55e-02	35	0	35	0	34

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
MISRA1C	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	23	1.00e-02	0.00e+00	34	35	24	25	23
	SNOPT	infeas	7	1.00e-02	0.00e+00	1	13	1	12	0
	WORHP IP	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP IPm	infeas	6	1.00e-02	0.00e+00	10	10	7	7	7
	WORHP SQP	infeas	16	1.00e-02	0.00e+00	904	906	15	17	14
MISRA1CLS	IPOPT	optimal	15	1.00e-02	4.10e-02	39	0	16	0	15
	KNITRO	optimal	14	1.00e-02	4.10e-02	21	0	15	0	14
	SNOPT	toobig	8	1.00e-02	4.75e+00	19	0	18	0	0
	WORHP IP	optimal	19	1.00e-02	4.10e-02	107	0	21	0	19
	WORHP IPm	optimal	19	1.00e-02	4.10e-02	107	0	20	0	19
	WORHP SQP	optimal	47	1.00e-02	4.10e-02	1165	0	48	0	47
MISRA1D	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	27	1.00e-02	0.00e+00	40	41	27	28	26
	SNOPT	infeas	7	1.00e-02	0.00e+00	1	15	1	14	0
	WORHP IP	infeas	7	1.00e-02	0.00e+00	17	17	8	8	8
	WORHP IPm	infeas	7	1.00e-02	0.00e+00	17	17	8	8	8
	WORHP SQP	infeas	12	1.00e-02	0.00e+00	724	726	10	13	9
MISRA1DLS	IPOPT	optimal	31	1.00e-02	5.64e-02	62	0	32	0	31
	KNITRO	optimal	26	1.00e-02	5.64e-02	44	0	27	0	26
	SNOPT	toobig	7	1.00e-02	1.11e+00	18	0	17	0	0
	WORHP IP	optimal	24	1.00e-02	5.64e-02	101	0	26	0	24
	WORHP IPm	optimal	31	1.00e-02	5.64e-02	44	0	32	0	31
	WORHP SQP	optimal	31	1.00e-02	5.64e-02	32	0	32	0	31
MISTAKE	IPOPT	optimal	14	1.00e-02	-1.00e+00	15	15	15	15	14
	KNITRO	optimal	9	1.00e-02	-1.00e+00	11	12	10	11	9
	SNOPT	optimal	13	1.00e-02	-1.00e+00	21	21	20	20	0
	WORHP IP	optimal	29	1.00e-02	-1.00e+00	31	31	30	30	29
	WORHP IPm	optimal	14	1.00e-02	-1.00e+00	16	16	15	15	14
	WORHP SQP	optimal	13	1.00e-02	-1.00e+00	14	14	15	15	13
MODBEALE	IPOPT	optimal	9	7.60e-01	1.46e-22	10	0	10	0	9
	KNITRO	optimal	14	9.20e-01	3.86e-27	23	0	15	0	14
	SNOPT	toobig	23	2.57e+01	1.01e+07	26	0	25	0	0
	WORHP IP	optimal	9	6.10e-01	1.46e-22	11	0	11	0	9
	WORHP IPm	optimal	9	5.80e-01	1.46e-22	11	0	10	0	9
	WORHP SQP	optimal	12	8.50e-01	4.67e-16	13	0	13	0	12
MODEL	IPOPT	infeas	30	2.00e-02	0.00e+00	83	166	22	66	31
	KNITRO	infeas	31	1.00e-02	0.00e+00	64	65	33	34	32
	SNOPT	infeas	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	44	2.00e-02	0.00e+00	46	46	45	1	44
	WORHP IPm	infeas	33	1.00e-02	0.00e+00	35	35	34	1	34
	WORHP SQP	minalpha	13	4.80e-01	0.00e+00	2801	2808	11	3	10
MOREBV	IPOPT	optimal	1	6.00e-02	5.83e-15	2	0	2	0	1
	KNITRO	optimal	1	6.00e-02	5.84e-15	3	0	2	0	1
	SNOPT	optimal	77	1.80e-01	5.38e-10	82	0	81	0	0
	WORHP IP	optimal	1	6.00e-02	5.85e-15	3	0	2	0	1
	WORHP IPm	optimal	1	6.00e-02	5.85e-15	3	0	2	0	1
	WORHP SQP	optimal	2	7.00e-02	1.65e-10	3	0	3	0	2
MOREBVNE	IPOPT	optimal	2	1.00e-02	0.00e+00	3	3	3	3	2
	KNITRO	optimal	2	1.00e-02	0.00e+00	4	5	3	4	2
	SNOPT	optimal	2	1.00e-02	0.00e+00	1	5	1	4	0
	WORHP IP	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	0.00e+00	4	4	3	3	2
	WORHP SQP	optimal	2	1.00e-02	0.00e+00	3	3	4	4	2
MOSARQP1	IPOPT	optimal	15	8.00e-02	-3.82e+03	16	16	16	16	15
	KNITRO	optimal	12	1.00e-01	-3.82e+03	14	15	13	14	12
	SNOPT	optimal	35	1.09e+01	-3.82e+03	46	1	45	1	0
	WORHP IP	optimal	12	9.00e-02	-3.82e+03	14	14	13	1	12
	WORHP IPm	optimal	11	9.00e-02	-3.82e+03	16	16	15	1	11
	WORHP SQP	optimal	3	1.20e-01	-3.82e+03	4	4	5	3	3
MOSARQP2	IPOPT	optimal	12	8.00e-02	-5.05e+03	13	13	13	13	12
	KNITRO	optimal	8	7.00e-02	-5.05e+03	10	11	9	10	8
	SNOPT	optimal	41	6.43e+00	-5.05e+03	51	1	50	1	0
	WORHP IP	optimal	9	8.00e-02	-5.05e+03	11	11	10	1	9
	WORHP IPm	optimal	9	8.00e-02	-5.05e+03	14	14	13	1	9
	WORHP SQP	optimal	11	7.50e-01	-5.05e+03	12	12	13	3	11
MPC1	IPOPT	optimal	306	8.75e+00	-2.33e+07	307	614	307	614	306
	KNITRO	optimal	54	1.12e+00	-2.33e+07	57	58	56	57	54
	SNOPT	optimal	4	9.40e-01	-2.33e+07	7	1	6	1	0
	WORHP IP	fritzjohn	344	9.06e+00	-2.33e+07	347	347	346	1	345
	WORHP IPm	optimal	519	1.79e+01	-2.33e+07	527	527	526	1	519
	WORHP SQP	maxtime	1432	1.76e+03	-2.33e+07	1653	1653	1433	2	1433
MPC10	IPOPT	optimal	73	1.20e+00	-1.50e+07	77	154	74	148	73
	KNITRO	maxiter	10000	9.64e+01	-1.50e+07	10011	10012	10002	10003	10000
	SNOPT	optimal	34	1.97e+00	-1.50e+07	51	1	50	1	0
	WORHP IP	optimal	35	7.90e-01	-1.50e+07	37	37	37	1	35
	WORHP IPm	optimal	42	8.40e-01	-1.50e+07	48	48	47	1	42
	WORHP SQP	optimal	138	5.31e+01	-1.50e+07	2651	2651	139	2	138

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
MPC11	IPOPT	optimal	75	1.25e+00	-1.50e+07	80	160	76	152	75
	KNITRO	optimal	47	9.70e-01	-1.50e+07	51	52	49	50	47
	SNOPT	optimal	31	1.85e+00	-1.50e+07	56	1	55	1	0
	WORHP IP	optimal	54	6.60e-01	-1.50e+07	56	56	56	1	54
	WORHP IPm	optimal	77	1.61e+00	-1.50e+07	95	95	84	1	77
	WORHP SQP	optimal	482	7.27e+01	-1.50e+07	10111	10112	479	2	478
MPC12	IPOPT	optimal	75	1.14e+00	-1.50e+07	79	158	76	152	75
	KNITRO	optimal	42	8.10e-01	-1.50e+07	45	46	44	45	42
	SNOPT	optimal	30	1.99e+00	-1.50e+07	45	1	44	1	0
	WORHP IP	optimal	33	6.80e-01	-1.50e+07	35	35	35	1	33
	WORHP IPm	optimal	42	8.30e-01	-1.50e+07	48	48	47	1	42
	WORHP SQP	maxiter	10000	4.77e+02	-1.50e+07	135247	133268	8762	2	8762
MPC13	IPOPT	optimal	80	1.35e+00	-1.50e+07	88	176	81	162	80
	KNITRO	optimal	40	7.80e-01	-1.50e+07	43	44	42	43	40
	SNOPT	optimal	46	2.01e+00	-1.50e+07	60	1	59	1	0
	WORHP IP	optimal	35	7.40e-01	-1.50e+07	37	37	36	1	35
	WORHP IPm	optimal	44	9.00e-01	-1.50e+07	50	50	49	1	44
	WORHP SQP	optimal	635	6.85e+01	-1.50e+07	55923	55923	635	2	634
MPC14	IPOPT	optimal	77	1.29e+00	-1.50e+07	92	184	78	156	77
	KNITRO	optimal	44	8.70e-01	-1.50e+07	47	48	46	47	44
	SNOPT	optimal	49	2.41e+00	-1.50e+07	62	1	61	1	0
	WORHP IP	optimal	35	5.50e-01	-1.50e+07	37	37	36	1	35
	WORHP IPm	optimal	40	6.70e-01	-1.50e+07	46	46	45	1	40
	WORHP SQP	minalpha	3726	1.54e+02	-1.50e+07	305749	305758	3693	2	3693
MPC15	IPOPT	optimal	82	1.36e+00	-1.50e+07	84	168	83	166	82
	KNITRO	smallstep	738	1.08e+01	-1.50e+07	750	751	740	741	738
	SNOPT	optimal	68	1.86e+00	-1.50e+07	73	1	72	1	0
	WORHP IP	optimal	33	7.10e-01	-1.50e+07	35	35	35	1	33
	WORHP IPm	optimal	37	6.20e-01	-1.50e+07	43	43	42	1	37
	WORHP SQP	optimal	620	1.78e+02	-1.50e+07	13297	13306	600	2	599
MPC16	IPOPT	optimal	78	1.31e+00	-1.50e+07	83	166	79	158	78
	KNITRO	optimal	46	9.00e-01	-1.50e+07	49	50	48	49	46
	SNOPT	optimal	53	2.15e+00	-1.50e+07	69	1	68	1	0
	WORHP IP	optimal	42	9.60e-01	-1.50e+07	73	73	44	1	42
	WORHP IPm	optimal	43	8.50e-01	-1.50e+07	49	49	48	1	43
	WORHP SQP	optimal	600	5.06e+01	-1.50e+07	27268	27271	601	2	600
MPC2	IPOPT	optimal	62	1.06e+00	-1.50e+07	65	130	63	126	62
	KNITRO	optimal	1298	1.88e+01	-1.50e+07	1324	1325	1300	1301	1298
	SNOPT	optimal	27	1.49e+00	-1.50e+07	45	1	44	1	0
	WORHP IP	optimal	42	8.90e-01	-1.50e+07	44	44	44	1	42
	WORHP IPm	optimal	53	7.10e-01	-1.50e+07	60	60	59	1	53
	WORHP SQP	optimal	42	3.05e+01	-1.50e+07	518	518	43	2	42
MPC3	IPOPT	optimal	78	1.27e+00	-1.50e+07	81	162	79	158	78
	KNITRO	optimal	42	7.70e-01	-1.50e+07	45	46	44	45	42
	SNOPT	optimal	26	1.65e+00	-1.50e+07	39	1	38	1	0
	WORHP IP	optimal	64	1.33e+00	-1.50e+07	66	66	66	1	64
	WORHP IPm	optimal	83	1.29e+00	-1.50e+07	90	90	89	1	83
	WORHP SQP	optimal	76	2.90e+01	-1.50e+07	1600	1601	77	2	76
MPC4	IPOPT	optimal	66	1.07e+00	-1.50e+07	69	138	67	134	66
	KNITRO	optimal	833	1.28e+01	-1.50e+07	907	908	835	836	833
	SNOPT	optimal	77	1.80e+00	-1.50e+07	81	1	80	1	0
	WORHP IP	optimal	37	8.00e-01	-1.50e+07	39	39	39	1	37
	WORHP IPm	optimal	37	5.70e-01	-1.50e+07	44	44	43	1	37
	WORHP SQP	optimal	39	4.29e+01	-1.50e+07	40	40	40	2	39
MPC5	IPOPT	optimal	66	1.06e+00	-1.50e+07	69	138	67	134	66
	KNITRO	optimal	44	8.70e-01	-1.50e+07	47	48	46	47	44
	SNOPT	optimal	80	1.72e+00	-1.50e+07	87	1	86	1	0
	WORHP IP	optimal	32	6.20e-01	-1.50e+07	34	34	34	1	32
	WORHP IPm	optimal	37	7.50e-01	-1.50e+07	42	42	41	1	37
	WORHP SQP	optimal	39	3.71e+01	-1.50e+07	40	40	40	2	39
MPC6	IPOPT	optimal	73	9.40e-01	-1.50e+07	79	158	74	148	73
	KNITRO	optimal	46	9.10e-01	-1.50e+07	49	50	48	49	46
	SNOPT	optimal	61	2.27e+00	-1.50e+07	71	1	70	1	0
	WORHP IP	optimal	38	8.10e-01	-1.50e+07	40	40	39	1	38
	WORHP IPm	optimal	41	5.30e-01	-1.50e+07	47	47	46	1	41
	WORHP SQP	optimal	493	2.96e+01	-1.50e+07	44846	44844	454	2	453
MPC7	IPOPT	optimal	76	9.60e-01	-1.50e+07	84	168	77	154	76
	KNITRO	optimal	44	8.80e-01	-1.50e+07	47	48	46	47	44
	SNOPT	optimal	56	2.14e+00	-1.50e+07	63	1	62	1	0
	WORHP IP	optimal	37	7.90e-01	-1.50e+07	40	40	39	1	37
	WORHP IPm	optimal	39	6.90e-01	-1.50e+07	45	45	44	1	39
	WORHP SQP	optimal	265	3.41e+01	-1.50e+07	16468	16469	266	2	265
MPC8	IPOPT	optimal	75	1.05e+00	-1.50e+07	80	160	76	152	75
	KNITRO	optimal	42	8.40e-01	-1.50e+07	45	46	44	45	42
	SNOPT	optimal	42	1.97e+00	-1.50e+07	56	1	55	1	0
	WORHP IP	optimal	38	8.10e-01	-1.50e+07	47	47	40	1	38
	WORHP IPm	optimal	38	7.30e-01	-1.50e+07	44	44	43	1	38
	WORHP SQP	optimal	1429	6.46e+01	-1.50e+07	16284	16285	1429	2	1428

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
MPC9	IPOPT	optimal	72	9.70e-01	-1.50e+07	75	150	73	146	72
	KNITRO	optimal	2985	2.90e+01	-1.50e+07	3019	3020	2987	2988	2985
	SNOPT	optimal	64	1.51e+00	-1.50e+07	71	1	70	1	0
	WORHP IP	optimal	36	7.70e-01	-1.50e+07	40	40	38	1	36
	WORHP IPm	optimal	38	6.50e-01	-1.50e+07	43	43	42	1	38
	WORHP SQP	optimal	45	3.31e+01	-1.50e+07	46	46	46	2	45
MRIBASIS	IPOPT	optimal	18	1.00e-02	1.82e+01	23	46	19	38	18
	KNITRO	optimal	10	1.00e-02	1.82e+01	13	14	12	13	10
	SNOPT	optimal	5	1.00e-02	1.82e+01	1	13	1	12	0
	WORHP IP	optimal	24	1.00e-02	1.82e+01	76	76	25	25	24
	WORHP IPm	optimal	1028	7.20e-01	1.82e+01	10858	10858	1031	1031	1028
	WORHP SQP	optimal	6	2.00e-02	1.82e+01	22	22	8	8	6
MSQRTA	IPOPT	optimal	4	5.02e+00	0.00e+00	8	8	5	5	4
	KNITRO	optimal	5	9.43e+00	0.00e+00	8	9	6	7	5
	SNOPT	optimal	4	8.56e+00	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	4	1.36e+00	0.00e+00	9	9	5	5	4
	WORHP IPm	optimal	4	9.70e-01	0.00e+00	9	9	5	5	4
	WORHP SQP	optimal	5	1.31e+02	0.00e+00	6	6	7	7	5
MSQRTALS	IPOPT	optimal	24	1.06e+01	4.22e-16	43	0	25	0	24
	KNITRO	optimal	23	8.96e+00	1.66e-15	32	0	24	0	23
	SNOPT	maxiter	10000	1.04e+02	2.25e-04	11126	0	11125	0	0
	WORHP IP	optimal	24	1.80e+01	4.22e-16	35	0	25	0	24
	WORHP IPm	optimal	24	1.68e+01	4.22e-16	35	0	25	0	24
	WORHP SQP	optimal	33	1.38e+01	2.18e-12	44	0	34	0	33
MSQRTB	IPOPT	optimal	5	7.25e+00	0.00e+00	9	9	6	6	5
	KNITRO	optimal	5	8.92e+00	0.00e+00	9	10	6	7	5
	SNOPT	optimal	5	1.06e+01	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	5	1.71e+00	0.00e+00	10	10	6	6	5
	WORHP IPm	optimal	5	1.69e+00	0.00e+00	10	10	6	6	5
	WORHP SQP	optimal	4	9.59e+01	0.00e+00	5	5	6	6	4
MSQRTBLS	IPOPT	optimal	23	8.70e+00	1.31e-13	49	0	24	0	23
	KNITRO	optimal	21	1.11e+01	1.49e-18	27	0	22	0	21
	SNOPT	maxiter	10000	1.02e+02	6.88e-05	11087	0	11086	0	0
	WORHP IP	optimal	23	1.67e+01	1.31e-13	38	0	24	0	23
	WORHP IPm	optimal	23	1.59e+01	1.31e-13	38	0	24	0	23
	WORHP SQP	optimal	29	1.60e+01	8.32e-12	116	0	30	0	29
MSS1	IPOPT	optimal	496	6.30e-01	-1.40e+01	568	568	462	499	496
	KNITRO	smallstep	7315	3.75e+00	-1.35e+01	11177	11178	7316	7317	7315
	SNOPT	optimal	77	3.00e-02	-1.05e+01	189	189	188	188	0
	WORHP IP	optimal	266	7.50e-01	-1.30e+01	1073	1073	268	268	266
	WORHP IPm	optimal	91	1.30e-01	-1.30e+01	93	93	92	92	91
	WORHP SQP	minalpha	2574	6.54e+00	-1.52e+01	16122	16112	2495	2603	2494
MSS2	IPOPT	optimal	41	2.11e+01	-2.70e+01	43	46	42	42	41
	KNITRO	smallstep	58	1.38e+01	-7.10e+01	105	106	59	60	58
	SNOPT	optimal	80	9.59e+00	-1.81e+00	174	174	173	173	0
	WORHP IP	fritzjohn	99	5.83e+01	-2.70e+01	233	233	101	101	100
	WORHP IPm	optimal	71	1.03e+01	-2.70e+01	143	143	72	72	71
	WORHP SQP	infeas	56	5.31e+01	-1.48e+01	953	949	32	61	31
MSS3	IPOPT	optimal	117	2.63e+02	-3.30e+02	155	155	52	119	117
	KNITRO	maxtime	1238	1.80e+03	-6.37e+01	1508	1509	1239	1240	1238
	SNOPT	optimal	2936	7.60e+01	-8.12e-01	16384	16384	16383	16383	0
	WORHP IP	infeas	16	4.66e+01	-9.89e+02	18	18	17	17	17
	WORHP IPm	infeas	16	6.88e+01	-9.89e+02	18	18	17	17	17
	WORHP SQP	maxtime	190	1.79e+03	-3.29e+02	3884	3868	118	209	117
MUONSINE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	453	8.60e-01	0.00e+00	472	473	452	453	452
	SNOPT	infeas	9	1.00e-02	0.00e+00	1	19	1	18	0
	WORHP IP	infeas	15	2.00e-02	0.00e+00	20	20	16	16	16
	WORHP IPm	infeas	15	2.00e-02	0.00e+00	20	20	16	16	16
	WORHP SQP	infeas	13	5.50e-01	0.00e+00	1598	1601	5	14	4
MWRIGHT	IPOPT	optimal	9	1.00e-02	2.50e+01	10	10	10	10	9
	KNITRO	optimal	8	1.00e-02	2.50e+01	10	11	9	10	8
	SNOPT	optimal	10	1.00e-02	2.50e+01	13	13	12	12	0
	WORHP IP	optimal	7	1.00e-02	2.50e+01	11	11	8	8	7
	WORHP IPm	optimal	7	1.00e-02	2.50e+01	11	11	8	8	7
	WORHP SQP	optimal	6	1.00e-02	4.20e+01	7	7	8	8	6
NASH	IPOPT	infeas	32	2.00e-02	4.91e-05	77	77	13	38	33
	KNITRO	smallstep	20	1.00e-02	0.00e+00	25	26	20	21	18
	SNOPT	infeas	0	1.00e-02	7.20e+01	1	1	1	1	0
	WORHP IP	infeas	18	1.00e-02	0.00e+00	20	20	19	1	19
	WORHP IPm	infeas	9	1.00e-02	0.00e+00	11	11	10	1	10
	WORHP SQP	minalpha	5	5.00e-02	0.00e+00	3780	3786	7	3	6
NCB20	IPOPT	optimal	146	7.74e+00	-1.46e+03	192	0	147	0	146
	KNITRO	optimal	64	3.73e+00	-1.45e+03	82	0	65	0	64
	SNOPT	toobig	354	6.59e+01	6.29e+03	408	0	407	0	0
	WORHP IP	optimal	150	8.90e+00	-1.46e+03	167	0	151	0	150
	WORHP IPm	optimal	150	7.36e+00	-1.46e+03	167	0	151	0	150
	WORHP SQP	optimal	97	7.53e+00	-1.47e+03	394	0	98	0	97

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
NCB20B	IPOPT	optimal	15	1.20e+00	7.35e+03	28	0	16	0	15
	KNITRO	optimal	21	1.45e+00	7.35e+03	35	0	22	0	21
	SNOPT	toobig	3775	1.36e+02	9.08e+03	4213	0	4212	0	0
	WORHP IP	optimal	15	1.14e+00	7.35e+03	23	0	16	0	15
	WORHP IPm	optimal	15	1.09e+00	7.35e+03	23	0	16	0	15
	WORHP SQP	optimal	6	6.20e-01	7.35e+03	16	0	7	0	6
NCVXBQP1	IPOPT	optimal	144	1.31e+01	-1.99e+10	145	0	145	0	144
	KNITRO	optimal	34	2.11e+00	-1.99e+10	36	0	35	0	34
	SNOPT	unbound	7	6.20e-01	-9.86e+09	15	0	14	0	0
	WORHP IP	optimal	143	1.54e+01	-1.99e+10	145	0	145	0	143
	WORHP IPm	optimal	143	2.01e+01	-1.99e+10	149	0	148	0	143
	WORHP SQP	optimal	71	1.12e+02	-1.99e+10	546	0	72	0	71
NCVXBQP2	IPOPT	optimal	455	4.77e+01	-1.33e+10	461	0	456	0	455
	KNITRO	optimal	113	6.73e+00	-1.33e+10	115	0	114	0	113
	SNOPT	unbound	13	1.91e+00	-9.98e+09	27	0	26	0	0
	WORHP IP	optimal	447	4.66e+01	-1.33e+10	457	0	449	0	447
	WORHP IPm	optimal	450	6.30e+01	-1.33e+10	465	0	455	0	450
	WORHP SQP	maxiter	10000	1.13e+03	-1.33e+10	18217	0	10001	0	10001
NCVXBQP3	IPOPT	optimal	741	7.05e+01	-6.44e+09	742	0	742	0	741
	KNITRO	optimal	163	7.75e+00	-6.48e+09	165	0	164	0	163
	SNOPT	optimal	74	7.45e+00	-6.46e+09	93	0	92	0	0
	WORHP IP	optimal	739	7.36e+01	-6.44e+09	741	0	740	0	739
	WORHP IPm	optimal	666	5.91e+01	-6.44e+09	731	0	672	0	666
	WORHP SQP	optimal	102	3.11e+02	-6.45e+09	235	0	103	0	102
NCVXQP1	IPOPT	optimal	338	8.45e+02	-7.51e+09	339	339	339	339	338
	KNITRO	optimal	51	3.24e+02	-7.51e+09	53	54	52	53	51
	SNOPT	optimal	29	1.28e+01	-7.51e+09	59	1	58	1	0
	WORHP IP	accept	342	8.07e+02	-7.51e+09	465	465	385	1	343
	WORHP IPm	optimal	347	7.31e+02	-7.51e+09	354	354	353	1	347
	WORHP SQP	maxtime	43	1.74e+03	-7.51e+09	177	176	45	3	44
NCVXQP2	IPOPT	optimal	529	1.16e+03	-5.84e+09	532	532	530	530	529
	KNITRO	maxtime	353	1.80e+03	-5.49e+09	383	384	354	355	356
	SNOPT	optimal	60	3.49e+01	-5.83e+09	121	1	120	1	0
	WORHP IP	maxtime	652	1.74e+03	-5.84e+09	1430	1430	657	1	653
	WORHP IPm	optimal	853	1.28e+03	-5.84e+09	860	860	859	1	853
	WORHP SQP	infeas	47	1.68e+03	-5.84e+09	47	47	47	2	47
NCVXQP3	IPOPT	optimal	715	1.60e+03	-3.08e+09	716	716	716	716	715
	KNITRO	maxtime	231	1.81e+03	-3.10e+09	233	234	232	233	231
	SNOPT	optimal	84	2.07e+01	-3.00e+09	158	1	157	1	0
	WORHP IP	maxtime	928	1.73e+03	-3.08e+09	1055	1055	930	1	929
	WORHP IPm	maxtime	873	1.74e+03	-3.08e+09	877	877	876	1	874
	WORHP SQP	maxtime	309	1.74e+03	-3.13e+09	887	888	288	3	287
NCVXQP4	IPOPT	optimal	280	2.69e+02	-9.38e+09	286	286	281	281	280
	KNITRO	optimal	171	7.89e+02	-9.38e+09	177	178	172	173	171
	SNOPT	optimal	18	3.34e+00	-9.38e+09	37	1	36	1	0
	WORHP IP	optimal	283	4.76e+02	-9.38e+09	285	285	285	1	283
	WORHP IPm	optimal	293	3.22e+02	-9.38e+09	306	306	299	1	293
	WORHP SQP	optimal	97	7.49e+02	-9.38e+09	3093	3104	98	3	97
NCVXQP5	IPOPT	optimal	420	4.01e+02	-6.63e+09	424	424	421	421	420
	KNITRO	optimal	158	7.40e+02	-6.63e+09	160	161	159	160	158
	SNOPT	optimal	30	3.67e+00	-6.64e+09	52	1	51	1	0
	WORHP IP	optimal	424	4.45e+02	-6.63e+09	426	426	426	1	424
	WORHP IPm	optimal	423	3.87e+02	-6.63e+09	527	527	429	1	423
	WORHP SQP	maxtime	228	1.73e+03	-6.63e+09	2751	2757	230	3	229
NCVXQP6	IPOPT	optimal	794	6.63e+02	-3.42e+09	797	797	795	795	794
	KNITRO	optimal	224	9.95e+02	-3.46e+09	226	227	225	226	224
	SNOPT	optimal	64	9.92e+00	-3.41e+09	113	1	112	1	0
	WORHP IP	optimal	793	8.92e+02	-3.42e+09	795	795	795	1	793
	WORHP IPm	optimal	779	7.55e+02	-3.42e+09	786	786	785	1	779
	WORHP SQP	maxtime	831	1.73e+03	-3.54e+09	11345	11376	826	3	825
NCVXQP7	IPOPT	accept	148	5.13e+02	-5.22e+09	199	199	150	150	149
	KNITRO	optimal	27	5.66e+02	-5.22e+09	29	30	28	29	27
	SNOPT	optimal	8	6.09e+00	-5.22e+09	17	1	16	1	0
	WORHP IP	infeas	161	4.52e+02	-5.22e+09	193	193	163	1	162
	WORHP IPm	optimal	277	6.94e+02	-5.22e+09	284	284	283	1	277
	WORHP SQP	infeas	21	1.15e+03	-5.22e+09	21	21	22	3	21
NCVXQP8	IPOPT	optimal	322	1.20e+03	-3.58e+09	323	323	323	323	322
	KNITRO	maxtime	206	1.80e+03	-3.70e+09	305	306	207	208	206
	SNOPT	optimal	9	6.55e+00	-3.57e+09	19	1	18	1	0
	WORHP IP	fritzjohn	392	1.29e+03	-3.58e+09	477	477	394	1	393
	WORHP IPm	optimal	431	1.20e+03	-3.58e+09	438	438	437	1	431
	WORHP SQP	infeas	24	1.72e+03	-3.58e+09	61	61	25	3	24
NCVXQP9	IPOPT	optimal	416	1.45e+03	-2.12e+09	507	507	417	417	416
	KNITRO	maxtime	92	1.81e+03	-2.12e+09	94	95	93	94	92
	SNOPT	optimal	52	3.01e+01	-2.10e+09	94	1	93	1	0
	WORHP IP	infeas	533	1.73e+03	-2.12e+09	644	644	535	1	534
	WORHP IPm	optimal	531	1.38e+03	-2.12e+09	538	538	537	1	531
	WORHP SQP	maxtime	24	1.72e+03	-2.12e+09	25	25	26	3	25

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
NELSON	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	167	1.00e-01	0.00e+00	269	270	168	169	167
	SNOPT	noimpr	6	1.00e-02	0.00e+00	1	73	1	72	0
	WORHP IP	regular	30	1.00e-02	0.00e+00	61	61	31	31	31
	WORHP IPm	regular	30	1.00e-02	0.00e+00	61	61	31	31	31
	WORHP SQP	toobig	45	1.30e-01	0.00e+00	2544	2546	8	47	6
NELSONLS	IPOPT	smallstep	68	3.00e-02	3.80e+00	194	0	69	0	69
	KNITRO	optimal	73	1.00e-02	3.80e+00	141	0	74	0	73
	SNOPT	toobig	54	1.00e-02	3.88e+00	131	0	130	0	0
	WORHP IP	smallstep	69	1.00e-02	3.80e+00	215	0	70	0	70
	WORHP IPm	smallstep	69	1.00e-02	3.80e+00	215	0	70	0	70
	WORHP SQP	optimal	63	2.00e-02	5.44e+01	532	0	64	0	63
NET1	IPOPT	optimal	43	3.00e-02	9.41e+05	49	98	44	88	43
	KNITRO	optimal	19	1.00e-02	9.41e+05	22	23	21	22	19
	SNOPT	optimal	7	1.00e-02	9.41e+05	15	15	14	14	0
	WORHP IP	optimal	48	1.00e-02	9.41e+05	66	66	50	50	48
	WORHP IPm	optimal	48	1.00e-02	9.41e+05	77	77	53	53	48
	WORHP SQP	optimal	28	8.00e-02	9.41e+05	29	29	29	29	28
NET2	IPOPT	resto	66	6.00e-02	1.19e+06	76	152	66	136	67
	KNITRO	optimal	46	3.00e-02	1.19e+06	53	54	48	49	46
	SNOPT	optimal	14	1.00e-02	1.19e+06	17	17	16	16	0
	WORHP IP	optimal	55	3.00e-02	1.19e+06	57	57	57	57	55
	WORHP IPm	optimal	59	3.00e-02	1.19e+06	80	80	64	64	59
	WORHP SQP	optimal	77	1.20e-01	1.19e+06	104	103	67	79	66
NET3	IPOPT	resto	148	2.70e-01	5.86e+06	169	338	146	304	149
	KNITRO	optimal	49	1.00e-01	5.86e+06	52	53	51	52	49
	SNOPT	optimal	8	3.00e-02	5.86e+06	12	12	11	11	0
	WORHP IP	optimal	883	1.71e+00	5.86e+06	1117	1117	885	885	883
	WORHP IPm	optimal	135	2.50e-01	5.86e+06	247	247	141	141	135
	WORHP SQP	minalpha	38	1.11e+00	5.44e+06	3180	3190	9	39	9
NET4	IPOPT	infeas	594	5.83e+02	2.14e+08	638	1278	233	1214	595
	KNITRO	maxtime	1087	1.80e+03	7.62e+07	1091	1092	1089	1090	1088
	SNOPT	maxtime	226	1.80e+03	4.31e+08	252	252	251	251	0
	WORHP IP	regular	679	7.64e+02	1.44e+08	684	684	680	680	680
	WORHP IPm	regular	674	1.01e+03	1.44e+08	679	679	675	675	675
	WORHP SQP	minalpha	14	1.06e+02	1.78e+08	2147	2153	5	15	5
NGONE	IPOPT	optimal	41	8.90e-01	-6.37e-01	43	43	42	42	41
	KNITRO	optimal	226	5.01e+00	-6.41e-01	237	238	228	229	226
	SNOPT	optimal	15	1.30e-01	-6.09e-01	22	22	21	21	0
	WORHP IP	optimal	90	2.90e+00	-6.37e-01	208	208	91	91	90
	WORHP IPm	optimal	55	1.71e+00	-6.33e-01	66	66	56	56	55
	WORHP SQP	optimal	14	2.84e+00	-6.09e-01	15	15	16	16	14
NINE12	IPOPT	degree	0	1.20e-01	0.00e+00	0	0	0	0	0
	KNITRO	optimal	10	1.40e+01	7.87e+03	13	14	12	13	10
	SNOPT	optimal	123	8.23e+01	7.87e+03	130	1	129	1	0
	WORHP IP	optimal	14	2.31e+02	7.87e+03	16	16	15	1	14
	WORHP IPm	optimal	15	1.47e+02	7.87e+03	19	19	18	1	15
	WORHP SQP	maxtime	710	1.78e+03	7.87e+03	711	711	712	3	711
NINE5D	IPOPT	degree	0	1.50e-01	0.00e+00	0	0	0	0	0
	KNITRO	optimal	14	2.39e+02	1.01e+04	17	18	16	17	14
	SNOPT	optimal	180	3.34e+01	1.01e+04	204	1	203	1	0
	WORHP IP	optimal	11	6.14e+02	1.01e+04	13	13	12	1	11
	WORHP IPm	optimal	12	6.91e+02	1.01e+04	16	16	15	1	12
	WORHP SQP	maxtime	1019	1.76e+03	1.01e+04	1020	1020	1021	3	1020
NINENEW	IPOPT	degree	0	8.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	20	4.27e+01	5.91e+03	23	24	22	23	20
	SNOPT	optimal	135	2.53e+01	5.91e+03	150	1	149	1	0
	WORHP IP	optimal	20	9.58e+01	5.91e+03	22	22	21	1	20
	WORHP IPm	optimal	17	9.14e+01	5.91e+03	21	21	20	1	17
	WORHP SQP	maxtime	1021	1.75e+03	5.94e+03	1022	1022	1023	3	1022
NLMSURF	IPOPT	optimal	40	1.27e+00	3.89e+01	315	0	41	0	40
	KNITRO	optimal	166	4.55e+00	3.89e+01	678	0	167	0	166
	SNOPT	toobig	5820	1.34e+02	7.28e+01	6392	0	6391	0	0
	WORHP IP	optimal	61	1.19e+00	3.89e+01	429	0	62	0	61
	WORHP IPm	optimal	61	1.86e+00	3.89e+01	429	0	62	0	61
	WORHP SQP	optimal	2009	7.30e+01	3.89e+01	113593	0	2010	0	2009
NOBNDTOR	IPOPT	optimal	12	3.40e-01	-4.50e-01	13	0	13	0	12
	KNITRO	optimal	8	3.00e-01	-4.50e-01	11	0	10	0	8
	SNOPT	toobig	211	1.25e+01	-4.16e-01	250	0	249	0	0
	WORHP IP	optimal	12	3.80e-01	-4.50e-01	14	0	13	0	12
	WORHP IPm	optimal	9	3.00e-01	-4.50e-01	11	0	10	0	9
	WORHP SQP	optimal	3	3.60e-01	-4.50e-01	4	0	4	0	3
NONCVXU2	IPOPT	maxtime	670	1.80e+03	2.04e+05	697	0	671	0	670
	KNITRO	maxtime	1058	1.80e+03	7.80e+04	1111	0	1059	0	1058
	SNOPT	sbasics	97	1.13e+01	1.75e+10	132	0	131	0	0
	WORHP IP	maxtime	547	1.75e+03	3.00e+05	559	0	548	0	548
	WORHP IPm	maxtime	548	1.75e+03	2.99e+05	560	0	549	0	549
	WORHP SQP	maxtime	1066	1.75e+03	8.39e+04	10283	0	1067	0	1067

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
NONCVXUN	IPOPT	optimal	2492	6.70e+01	1.16e+04	2974	0	2493	0	2492
	KNITRO	optimal	2596	5.90e+01	1.16e+04	2795	0	2597	0	2596
	SNOPT	sbasics	310	1.74e+01	2.31e+10	361	0	360	0	0
	WORHP IP	optimal	2477	1.03e+02	1.16e+04	2765	0	2479	0	2477
	WORHP IPm	optimal	2478	6.43e+01	1.16e+04	2766	0	2479	0	2478
	WORHP SQP	optimal	2242	6.18e+01	1.16e+04	13663	0	2243	0	2242
NONDIA	IPOPT	optimal	8	1.00e-01	1.66e-17	9	0	9	0	8
	KNITRO	optimal	5	2.10e-01	1.05e-21	7	0	6	0	5
	SNOPT	optimal	41	8.29e+00	7.11e-16	48	0	47	0	0
	WORHP IP	optimal	5	9.00e-02	3.93e-21	7	0	7	0	5
	WORHP IPm	optimal	8	1.00e-01	1.66e-17	10	0	9	0	8
	WORHP SQP	optimal	20	1.90e-01	8.80e-13	21	0	21	0	20
NONDQUAR	IPOPT	optimal	20	1.20e-01	4.09e-11	21	0	21	0	20
	KNITRO	optimal	20	1.40e-01	4.09e-11	22	0	21	0	20
	SNOPT	sbasics	10000	2.27e+02	1.89e-03	11952	0	11951	0	0
	WORHP IP	optimal	20	1.20e-01	4.09e-11	22	0	22	0	20
	WORHP IPm	optimal	20	1.50e-01	4.09e-11	22	0	21	0	20
	WORHP SQP	optimal	81	6.90e-01	9.08e-10	599	0	82	0	81
NONMSQRT	IPOPT	maxtime	5743	1.80e+03	1.96e+03	56539	0	5744	0	5743
	KNITRO	noimpr	230	6.48e+01	7.09e+02	525	0	231	0	230
	SNOPT	sbasics	10000	4.05e+02	4.86e+04	10967	0	10966	0	0
	WORHP IP	minalpha	6381	1.72e+03	7.09e+02	27530	0	6930	0	6382
	WORHP IPm	maxtime	6142	1.80e+03	7.09e+02	24833	0	6545	0	6143
	WORHP SQP	maxtime	5979	1.78e+03	7.09e+02	7693	0	5980	0	5980
NONSCOMP	IPOPT	optimal	17	1.70e-01	5.17e-04	32	0	18	0	17
	KNITRO	optimal	13	1.90e-01	3.64e-05	16	0	14	0	13
	SNOPT	toobig	67	1.70e+02	1.44e+05	75	0	74	0	0
	WORHP IP	optimal	15	1.90e-01	5.19e-04	27	0	16	0	15
	WORHP IPm	optimal	15	1.90e-01	2.94e-05	24	0	16	0	15
	WORHP SQP	optimal	8	1.30e-01	4.63e-11	9	0	9	0	8
NUFFIELD	IPOPT	resto	3302	1.94e+02	-2.30e+00	6893	6912	1189	3338	3303
	KNITRO	maxiter	10000	5.55e+02	-2.38e+00	20408	20409	10002	10003	10001
	SNOPT	maxiter	151	3.55e+02	-1.36e-01	1	608	1	607	0
	WORHP IP	optimal	6806	3.83e+02	-2.51e+00	58591	58591	6807	6807	6806
	WORHP IPm	maxiter	10000	5.78e+02	-2.47e+00	115701	115701	10001	10001	10000
	WORHP SQP	infeas	222	2.69e+01	-7.29e-01	3690	3705	223	223	222
NYSTROM5	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	20	1.00e-02	0.00e+00	22	23	21	22	20
	SNOPT	optimal	110	1.00e-02	0.00e+00	1	244	1	243	0
	WORHP IP	regular	1066	2.20e-01	0.00e+00	9592	9592	1067	1067	1067
	WORHP IPm	regular	1066	2.00e-01	0.00e+00	9592	9592	1067	1067	1067
	WORHP SQP	minalpha	49	6.00e-02	0.00e+00	6437	6443	43	51	42
OBSTCLAE	IPOPT	optimal	14	8.00e-01	1.89e+00	15	0	15	0	14
	KNITRO	optimal	12	8.40e-01	1.89e+00	15	0	14	0	12
	SNOPT	toobig	134	1.38e+01	2.33e+01	162	0	161	0	0
	WORHP IP	optimal	12	8.00e-01	1.89e+00	14	0	13	0	12
	WORHP IPm	optimal	16	1.08e+00	1.89e+00	18	0	17	0	16
	WORHP SQP	optimal	6	2.87e+00	1.89e+00	7	0	7	0	6
OBSTCLAL	IPOPT	optimal	14	5.10e-01	1.89e+00	20	0	15	0	14
	KNITRO	optimal	9	6.50e-01	1.89e+00	12	0	11	0	9
	SNOPT	toobig	195	1.21e+01	1.92e+00	218	0	217	0	0
	WORHP IP	optimal	11	7.90e-01	1.89e+00	13	0	12	0	11
	WORHP IPm	optimal	14	9.50e-01	1.89e+00	16	0	15	0	14
	WORHP SQP	optimal	3	8.40e-01	1.89e+00	4	0	4	0	3
OBSTCLBL	IPOPT	optimal	14	7.90e-01	7.27e+00	15	0	15	0	14
	KNITRO	optimal	7	5.70e-01	7.27e+00	10	0	9	0	7
	SNOPT	toobig	243	1.41e+01	8.78e+00	293	0	292	0	0
	WORHP IP	optimal	13	8.80e-01	7.27e+00	15	0	14	0	13
	WORHP IPm	optimal	8	6.10e-01	7.27e+00	10	0	9	0	8
	WORHP SQP	optimal	3	8.40e-01	7.27e+00	4	0	4	0	3
OBSTCLBM	IPOPT	optimal	10	6.00e-01	7.27e+00	11	0	11	0	10
	KNITRO	optimal	7	5.60e-01	7.27e+00	9	0	8	0	7
	SNOPT	toobig	118	1.70e+01	7.69e+00	125	0	124	0	0
	WORHP IP	optimal	10	7.10e-01	7.27e+00	12	0	11	0	10
	WORHP IPm	optimal	10	7.00e-01	7.27e+00	13	0	12	0	10
	WORHP SQP	optimal	3	4.50e-01	7.27e+00	4	0	4	0	3
OBSTCLBU	IPOPT	optimal	15	8.40e-01	7.27e+00	16	0	16	0	15
	KNITRO	optimal	7	6.00e-01	7.27e+00	10	0	9	0	7
	SNOPT	toobig	211	2.09e+01	1.10e+01	254	0	253	0	0
	WORHP IP	optimal	13	9.00e-01	7.27e+00	15	0	14	0	13
	WORHP IPm	optimal	9	6.80e-01	7.27e+00	11	0	10	0	9
	WORHP SQP	optimal	3	7.50e-01	7.27e+00	4	0	4	0	3
ODC	IPOPT	optimal	69	1.71e+00	-1.14e-02	70	0	70	0	69
	KNITRO	optimal	69	1.74e+00	-1.14e-02	71	0	70	0	69
	SNOPT	toobig	661	2.81e+01	-1.12e-03	716	0	715	0	0
	WORHP IP	optimal	69	1.72e+00	-1.14e-02	71	0	70	0	69
	WORHP IPm	optimal	69	1.71e+00	-1.14e-02	71	0	70	0	69
	WORHP SQP	optimal	70	2.02e+00	-1.14e-02	71	0	71	0	70

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ODFITS	IPOPT	optimal	10	1.00e-02	−2.38e+03	11	11	11	11	10
	KNITRO	optimal	6	1.00e-02	−2.38e+03	8	9	7	8	6
	SNOPT	optimal	11	1.00e-02	−2.38e+03	18	1	17	1	0
	WORHP IP	optimal	7	1.00e-02	−2.38e+03	9	9	8	1	7
	WORHP IPm	optimal	7	1.00e-02	−2.38e+03	9	9	8	1	7
	WORHP SQP	optimal	7	1.00e-02	−2.38e+03	8	8	9	3	7
ODNAMUR	IPOPT	optimal	39	1.31e+03	9.24e+03	40	0	40	0	39
	KNITRO	optimal	25	7.24e+02	9.24e+03	28	0	27	0	25
	SNOPT	sbasics	793	8.00e+01	1.02e+04	865	0	864	0	0
	WORHP IP	maxtime	33	1.79e+03	9.24e+03	35	0	35	0	34
	WORHP IPm	optimal	36	1.51e+03	9.24e+03	41	0	40	0	36
	WORHP SQP	maxtime	1	1.81e+03	2.13e+05	2	0	2	0	2
OET1	IPOPT	optimal	44	8.00e-02	5.38e-01	55	55	45	45	44
	KNITRO	optimal	13	3.00e-02	5.38e-01	15	16	14	15	13
	SNOPT	optimal	0	1.00e-02	5.38e-01	1	1	1	1	0
	WORHP IP	optimal	24	6.00e-02	5.38e-01	34	34	25	1	24
	WORHP IPm	optimal	72	1.60e-01	5.38e-01	80	80	73	1	72
	WORHP SQP	optimal	2	6.00e-02	5.38e-01	113	114	3	3	2
OET2	IPOPT	optimal	76	1.80e-01	8.72e-02	113	121	77	78	76
	KNITRO	optimal	21	6.00e-02	8.72e-02	23	24	22	23	21
	SNOPT	optimal	5	4.00e-02	8.72e-02	1	10	1	9	0
	WORHP IP	optimal	38	1.80e-01	8.72e-02	119	119	39	39	38
	WORHP IPm	optimal	64	4.20e-01	8.72e-02	364	364	65	65	64
	WORHP SQP	optimal	5	1.00e-01	8.72e-02	6	6	7	7	5
OET3	IPOPT	optimal	14	4.00e-02	4.51e-03	17	17	15	15	14
	KNITRO	optimal	9	2.00e-02	4.51e-03	12	13	10	11	9
	SNOPT	optimal	0	2.00e-02	4.50e-03	1	1	1	1	0
	WORHP IP	optimal	10	3.00e-02	4.51e-03	12	12	11	1	10
	WORHP IPm	optimal	17	5.00e-02	4.51e-03	21	21	18	1	17
	WORHP SQP	optimal	5	8.00e-02	4.51e-03	325	326	7	3	5
OET4	IPOPT	optimal	54	1.40e-01	4.30e-03	55	55	55	55	54
	KNITRO	optimal	14	4.00e-02	8.57e-01	16	17	15	16	14
	SNOPT	optimal	5	8.00e-02	4.30e-03	1	9	1	8	0
	WORHP IP	optimal	19	8.00e-02	4.30e-03	42	42	20	20	19
	WORHP IPm	optimal	41	1.60e-01	4.30e-03	75	75	42	42	41
	WORHP SQP	zerostep	10	1.90e-01	4.30e-03	12	12	11	11	10
OET5	IPOPT	optimal	109	2.60e-01	2.65e-03	115	115	110	110	109
	KNITRO	optimal	50	1.10e-01	2.65e-03	53	54	51	52	50
	SNOPT	optimal	20	1.50e-01	2.65e-03	1	38	1	37	0
	WORHP IP	optimal	48	1.70e-01	2.65e-03	78	78	49	49	48
	WORHP IPm	optimal	59	2.70e-01	2.65e-03	161	161	60	60	59
	WORHP SQP	optimal	7	1.70e-01	2.65e-03	8	8	9	9	7
OET6	IPOPT	optimal	93	6.20e-01	2.07e-03	102	102	94	94	93
	KNITRO	optimal	10	5.00e-02	8.72e-02	12	13	11	12	10
	SNOPT	optimal	22	2.10e-01	2.07e-03	1	57	1	56	0
	WORHP IP	optimal	171	6.40e-01	2.08e-03	174	174	172	172	171
	WORHP IPm	optimal	162	7.00e-01	2.08e-03	203	203	163	163	162
	WORHP SQP	optimal	5	2.60e-01	8.72e-02	6	6	7	7	5
OET7	IPOPT	optimal	124	6.90e-01	9.98e-05	134	134	125	125	124
	KNITRO	optimal	10	7.00e-02	8.72e-02	12	13	11	12	10
	SNOPT	optimal	637	4.02e+00	4.43e-05	1	3071	1	3070	0
	WORHP IP	optimal	265	1.51e+00	2.24e-04	335	335	266	266	265
	WORHP IPm	optimal	120	6.80e-01	2.23e-04	134	134	121	121	120
	WORHP SQP	optimal	4	1.40e-01	8.72e-02	5	5	6	6	4
OPTCDEG2	IPOPT	optimal	30	1.90e-01	2.28e+02	31	31	31	31	30
	KNITRO	optimal	25	3.20e-01	2.28e+02	28	29	27	28	25
	SNOPT	optimal	11	2.29e+00	2.28e+02	16	16	15	15	0
	WORHP IP	optimal	29	2.50e-01	2.28e+02	31	31	31	31	29
	WORHP IPm	optimal	30	2.90e-01	2.28e+02	35	35	34	34	30
	WORHP SQP	optimal	16	2.14e+00	2.28e+02	17	17	17	17	16
OPTCDEG3	IPOPT	optimal	28	1.70e-01	4.58e+01	29	29	29	29	28
	KNITRO	optimal	20	2.70e-01	4.58e+01	23	24	22	23	20
	SNOPT	infeas	24	3.30e+00	1.06e+03	44	44	43	43	0
	WORHP IP	optimal	25	2.20e-01	4.58e+01	27	27	26	26	25
	WORHP IPm	optimal	24	2.20e-01	4.58e+01	26	26	25	25	24
	WORHP SQP	optimal	12	1.56e+00	4.58e+01	13	13	13	13	12
OPTCNTRL	IPOPT	optimal	30	1.00e-02	5.50e+02	41	41	31	31	30
	KNITRO	optimal	11	1.00e-02	5.50e+02	14	15	13	14	11
	SNOPT	optimal	3	1.00e-02	5.50e+02	6	6	5	5	0
	WORHP IP	optimal	12	1.00e-02	5.50e+02	14	14	13	13	12
	WORHP IPm	optimal	15	1.00e-02	5.50e+02	21	21	20	20	15
	WORHP SQP	optimal	4	1.00e-02	5.50e+02	5	5	5	5	4
OPTCTRL3	IPOPT	optimal	12	1.20e-01	7.45e+04	14	14	13	13	12
	KNITRO	optimal	15	4.30e-01	7.45e+04	24	25	17	18	15
	SNOPT	maxtime	1840	1.80e+03	7.49e+04	2053	2053	2052	2052	0
	WORHP IP	optimal	12	1.20e-01	7.45e+04	33	33	14	14	12
	WORHP IPm	optimal	12	1.20e-01	7.45e+04	33	33	13	13	12
	WORHP SQP	optimal	13	3.30e-01	7.45e+04	15	15	15	15	13

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
OPTCTRL6	IPOPT	optimal	12	1.20e-01	7.45e+04	14	14	13	13	12
	KNITRO	optimal	15	4.20e-01	7.45e+04	24	25	17	18	15
	SNOPT	maxtime	1815	1.80e+03	7.49e+04	2026	2026	2025	2025	0
	WORHP IP	optimal	12	1.20e-01	7.45e+04	33	33	14	14	12
	WORHP IPm	optimal	12	1.30e-01	7.45e+04	33	33	13	13	12
	WORHP SQP	optimal	13	3.20e-01	7.45e+04	15	15	15	15	13
OPTMASS	IPOPT	optimal	20	1.67e+00	-1.20e-01	21	42	21	42	20
	KNITRO	optimal	26	1.98e+00	-1.20e-01	53	54	27	28	26
	SNOPT	optimal	37	2.29e+01	-1.20e-01	58	58	57	57	0
	WORHP IP	maxiter	10000	9.90e+02	-1.21e-01	127832	127832	10001	10001	10000
	WORHP IPm	optimal	354	3.13e+01	-1.20e-01	2226	2226	355	355	354
	WORHP SQP	optimal	31	6.61e+00	-1.09e-01	60	59	32	34	30
OPTPRLOC	IPOPT	optimal	19	1.00e-02	-1.64e+01	20	20	20	20	19
	KNITRO	optimal	11	1.00e-02	-1.64e+01	14	15	13	14	11
	SNOPT	optimal	7	1.00e-02	-1.64e+01	11	11	10	10	0
	WORHP IP	optimal	18	1.00e-02	-1.64e+01	20	20	19	19	18
	WORHP IPm	optimal	18	1.00e-02	-1.64e+01	23	23	22	22	18
	WORHP SQP	optimal	9	1.00e-02	-1.64e+01	10	10	11	11	9
ORBIT2	IPOPT	optimal	156	4.11e+00	3.12e+02	376	752	130	320	156
	KNITRO	optimal	67	2.14e+00	3.12e+02	83	84	69	70	67
	SNOPT	optimal	60	2.72e+01	3.12e+02	1	75	1	74	0
	WORHP IP	optimal	60	2.34e+00	3.12e+02	162	162	61	61	60
	WORHP IPm	optimal	70	2.49e+00	3.12e+02	148	148	74	74	70
	WORHP SQP	optimal	91	1.08e+01	3.12e+02	4699	4703	40	93	38
ORTHRDM2	IPOPT	optimal	5	5.80e-01	3.11e+02	7	7	6	6	5
	KNITRO	optimal	5	5.70e-01	3.11e+02	8	9	6	7	5
	SNOPT	sbasics	26	5.23e+01	3.30e+02	38	38	37	37	0
	WORHP IP	optimal	5	6.20e-01	3.11e+02	9	9	6	6	5
	WORHP IPm	optimal	5	6.20e-01	3.11e+02	9	9	6	6	5
	WORHP SQP	optimal	8	1.00e+00	3.11e+02	9	9	10	10	8
ORTHRDS2	IPOPT	resto	35	7.10e-01	7.62e+02	108	108	37	37	36
	KNITRO	noimpr	40	8.00e-01	7.62e+02	67	68	41	42	40
	SNOPT	sbasics	94	1.42e+02	7.64e+02	171	171	170	170	0
	WORHP IP	optimal	33	1.03e+00	7.62e+02	430	430	46	46	33
	WORHP IPm	optimal	31	9.90e-01	7.62e+02	372	372	43	43	31
	WORHP SQP	optimal	107	5.00e+00	7.62e+02	5117	5118	108	108	107
ORTHREGA	IPOPT	optimal	82	2.24e+00	2.26e+04	159	160	83	83	82
	KNITRO	optimal	82	2.37e+00	2.26e+04	140	141	83	84	82
	SNOPT	sbasics	36	5.92e+01	2.30e+04	91	91	90	90	0
	WORHP IP	optimal	119	3.81e+00	2.26e+04	255	255	120	120	119
	WORHP IPm	optimal	119	3.61e+00	2.26e+04	255	255	120	120	119
	WORHP SQP	optimal	113	4.71e+00	2.26e+04	1447	1445	113	117	111
ORTHREGB	IPOPT	optimal	2	1.00e-02	4.52e-20	3	3	3	3	2
	KNITRO	optimal	1	1.00e-02	1.26e-29	3	4	2	3	1
	SNOPT	optimal	6	1.00e-02	1.79e-17	10	10	9	9	0
	WORHP IP	optimal	2	1.00e-02	4.51e-20	4	4	3	3	2
	WORHP IPm	optimal	2	1.00e-02	4.51e-20	4	4	3	3	2
	WORHP SQP	optimal	3	1.00e-02	3.76e-18	4	4	5	5	3
ORTHREGC	IPOPT	optimal	12	3.10e-01	9.48e+01	19	19	13	13	12
	KNITRO	optimal	13	3.40e-01	9.48e+01	24	25	14	15	13
	SNOPT	sbasics	46	9.61e+01	6.07e+02	97	97	96	96	0
	WORHP IP	optimal	11	3.00e-01	9.48e+01	13	13	12	12	11
	WORHP IPm	optimal	11	3.00e-01	9.48e+01	13	13	12	12	11
	WORHP SQP	optimal	28	1.02e+00	1.64e+02	106	106	30	30	28
ORTHREGD	IPOPT	optimal	6	2.80e-01	7.62e+02	10	10	7	7	6
	KNITRO	optimal	6	2.80e-01	7.62e+02	9	10	7	8	6
	SNOPT	sbasics	28	4.54e+01	7.63e+02	72	72	71	71	0
	WORHP IP	optimal	6	3.40e-01	7.62e+02	10	10	7	7	6
	WORHP IPm	optimal	6	3.10e-01	7.62e+02	10	10	7	7	6
	WORHP SQP	optimal	7	5.40e-01	7.62e+02	8	8	9	9	7
ORTHREGE	IPOPT	optimal	88	2.47e+00	1.20e+03	119	119	89	89	88
	KNITRO	optimal	112	3.40e+00	3.30e+03	230	231	113	114	112
	SNOPT	sbasics	32	5.39e+01	1.11e+03	50	50	49	49	0
	WORHP IP	optimal	89	2.87e+00	1.05e+03	116	116	90	90	89
	WORHP IPm	optimal	86	3.29e+00	1.05e+03	105	105	87	87	86
	WORHP SQP	optimal	460	3.41e+01	1.06e+03	7178	7157	459	494	457
ORTHREGF	IPOPT	optimal	71	1.08e+00	7.10e+01	126	126	72	72	71
	KNITRO	optimal	35	6.20e-01	6.45e+01	42	43	36	37	35
	SNOPT	sbasics	68	1.19e+02	1.50e+02	134	134	133	133	0
	WORHP IP	optimal	41	8.10e-01	6.63e+01	61	61	42	42	41
	WORHP IPm	optimal	44	7.90e-01	6.44e+01	58	58	45	45	44
	WORHP SQP	optimal	61	1.93e+00	8.09e+01	609	608	62	64	60
ORTHRGDM	IPOPT	optimal	6	8.30e-01	1.51e+03	9	9	7	7	6
	KNITRO	optimal	14	1.17e+00	1.51e+03	24	25	15	16	14
	SNOPT	sbasics	28	4.67e+01	1.79e+03	43	43	42	42	0
	WORHP IP	optimal	7	9.60e-01	1.51e+03	11	11	8	8	7
	WORHP IPm	optimal	7	9.50e-01	1.51e+03	11	11	8	8	7
	WORHP SQP	optimal	8	1.40e+00	1.51e+03	9	9	10	10	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ORTHRGDS	IPOPT	optimal	27	6.00e-01	7.62e+02	30	30	28	28	27
	KNITRO	optimal	33	7.50e-01	2.91e+03	52	53	34	35	33
	SNOPT	optimal	102	1.54e+02	9.14e+02	189	189	188	188	0
	WORHP IP	optimal	31	8.50e-01	8.68e+02	84	84	39	39	31
	WORHP IPm	optimal	48	1.42e+00	8.68e+02	534	534	80	80	48
	WORHP SQP	optimal	13	7.30e-01	7.62e+02	15	15	15	15	13
OSBORNE1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	17	1.00e-02	0.00e+00	28	29	18	19	17
	SNOPT	infeas	1888	1.80e-01	0.00e+00	1	3773	1	3772	0
	WORHP IP	infeas	7	1.00e-02	0.00e+00	13	13	8	8	8
	WORHP IPm	infeas	7	1.00e-02	0.00e+00	13	13	8	8	8
	WORHP SQP	minalpha	89	9.00e-02	0.00e+00	4266	4274	39	91	38
OSBORNE2	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	26	1.00e-02	0.00e+00	29	30	27	28	24
	SNOPT	infeas	11	1.00e-02	0.00e+00	1	29	1	28	0
	WORHP IP	infeas	16	1.00e-02	0.00e+00	37	37	17	17	17
	WORHP IPm	infeas	16	1.00e-02	0.00e+00	37	37	17	17	17
	WORHP SQP	minalpha	42	2.20e-01	0.00e+00	3608	3614	43	44	42
OSBORNEA	IPOPT	optimal	64	2.00e-02	5.46e-05	152	0	65	0	64
	KNITRO	optimal	45	1.00e-02	5.46e-05	71	0	46	0	45
	SNOPT	optimal	101	1.00e-02	5.46e-05	119	0	118	0	0
	WORHP IP	optimal	63	1.00e-02	5.46e-05	106	0	65	0	63
	WORHP IPm	optimal	64	1.00e-02	5.46e-05	108	0	65	0	64
	WORHP SQP	optimal	37	1.00e-02	5.46e-05	39	0	38	0	37
OSBORNEB	IPOPT	optimal	18	1.00e-02	4.01e-02	24	0	19	0	18
	KNITRO	optimal	14	1.00e-02	4.01e-02	22	0	15	0	14
	SNOPT	optimal	81	1.00e-02	4.01e-02	98	0	97	0	0
	WORHP IP	optimal	18	1.00e-02	4.01e-02	22	0	19	0	18
	WORHP IPm	optimal	18	1.00e-02	4.01e-02	22	0	19	0	18
	WORHP SQP	optimal	28	1.00e-02	8.76e-02	53	0	29	0	28
OSCIGRAD	IPOPT	optimal	12	3.71e+00	2.02e-20	13	0	13	0	12
	KNITRO	optimal	12	4.20e+00	6.28e-24	26	0	13	0	12
	SNOPT	toobig	79	5.54e+00	7.28e-17	87	0	86	0	0
	WORHP IP	optimal	13	4.00e+00	6.13e-24	15	0	15	0	13
	WORHP IPm	optimal	14	4.19e+00	5.93e-24	16	0	15	0	14
	WORHP SQP	optimal	22	8.59e+00	7.73e+04	32	0	23	0	22
OSCIGRNE	IPOPT	optimal	6	3.18e+00	0.00e+00	7	7	7	7	6
	KNITRO	optimal	6	3.40e+00	0.00e+00	16	17	7	8	6
	SNOPT	optimal	6	2.89e+00	0.00e+00	1	9	1	8	0
	WORHP IP	optimal	6	3.36e+00	0.00e+00	8	8	7	7	6
	WORHP IPm	optimal	6	3.14e+00	0.00e+00	8	8	7	7	6
	WORHP SQP	optimal	7	8.02e+00	0.00e+00	8	8	9	9	7
OSCIPANE	IPOPT	optimal	7066	4.45e+00	0.00e+00	33483	33824	3	7232	7066
	KNITRO	optimal	0	1.00e-02	0.00e+00	3	4	2	3	0
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	maxiter	10000	1.04e+00	0.00e+00	116139	116139	10001	10001	10000
	WORHP IPm	maxiter	10000	8.10e-01	0.00e+00	116139	116139	10001	10001	10000
	WORHP SQP	minalpha	8	2.00e-02	0.00e+00	3799	3805	10	10	9
OSCIPATH	IPOPT	maxiter	10000	3.05e+00	9.86e-01	26922	0	10001	0	10000
	KNITRO	maxiter	10000	1.70e-01	9.86e-01	14588	0	10001	0	10000
	SNOPT	maxiter	10000	3.50e-01	9.97e-01	13478	0	13477	0	0
	WORHP IP	maxiter	10000	4.50e-01	9.86e-01	17746	0	10001	0	10000
	WORHP IPm	maxiter	10000	3.70e-01	9.86e-01	17746	0	10001	0	10000
	WORHP SQP	maxiter	10000	8.60e-01	9.85e-01	65536	0	10001	0	10001
OSLBQP	IPOPT	optimal	11	1.00e-02	6.25e+00	12	0	12	0	11
	KNITRO	optimal	7	1.00e-02	6.25e+00	10	0	9	0	7
	SNOPT	optimal	3	1.00e-02	6.25e+00	6	0	5	0	0
	WORHP IP	optimal	8	1.00e-02	6.25e+00	10	0	9	0	8
	WORHP IPm	optimal	8	1.00e-02	6.25e+00	13	0	12	0	8
	WORHP SQP	optimal	1	1.00e-02	6.25e+00	2	0	2	0	1
OSORIO	IPOPT	optimal	26	8.80e-01	2.04e+00	27	27	27	27	26
	KNITRO	optimal	15	6.30e-01	2.04e+00	18	19	17	18	15
	SNOPT	sbasics	145	3.22e+01	2.05e+00	167	1	166	1	0
	WORHP IP	optimal	14	7.90e-01	2.04e+00	17	17	15	1	14
	WORHP IPm	optimal	10	5.30e-01	2.04e+00	15	15	14	1	10
	WORHP SQP	optimal	6923	3.13e+02	2.04e+00	33177	33263	6925	3	6923
PALMER1	IPOPT	optimal	691	2.10e-01	1.18e+04	1898	0	692	0	691
	KNITRO	optimal	30	1.00e-02	1.18e+04	39	0	31	0	30
	SNOPT	optimal	19	1.00e-02	1.18e+04	31	0	30	0	0
	WORHP IP	optimal	685	4.00e-02	1.18e+04	998	0	687	0	685
	WORHP IPm	optimal	53	1.00e-02	1.18e+04	188	0	63	0	53
	WORHP SQP	maxiter	10000	3.60e+00	1.18e+04	1054600	0	10001	0	10001
PALMER1A	IPOPT	optimal	45	2.00e-02	8.99e-02	92	0	46	0	45
	KNITRO	optimal	44	1.00e-02	8.99e-02	56	0	45	0	44
	SNOPT	optimal	155	1.00e-02	8.99e-02	197	0	196	0	0
	WORHP IP	optimal	70	1.00e-02	8.99e-02	85	0	72	0	70
	WORHP IPm	optimal	59	1.00e-02	8.99e-02	68	0	61	0	59
	WORHP SQP	optimal	48	1.00e-02	8.99e-02	88	0	49	0	48

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PALMER1B	IPOPT	optimal	20	1.00e-02	3.45e+00	26	0	21	0	20
	KNITRO	optimal	24	1.00e-02	3.45e+00	30	0	25	0	24
	SNOPT	optimal	41	1.00e-02	3.45e+00	58	0	57	0	0
	WORHP IP	optimal	17	1.00e-02	3.45e+00	19	0	19	0	17
	WORHP IPm	optimal	16	1.00e-02	3.45e+00	18	0	17	0	16
	WORHP SQP	optimal	27	1.00e-02	3.45e+00	34	0	28	0	27
PALMER1C	IPOPT	optimal	1	1.00e-02	9.76e-02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	9.76e-02	3	0	2	0	1
	SNOPT	optimal	163	1.00e-02	9.76e-02	170	0	169	0	0
	WORHP IP	optimal	1	1.00e-02	9.76e-02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	9.76e-02	3	0	2	0	1
	WORHP SQP	optimal	39	1.00e-02	9.76e-02	40	0	40	0	39
PALMER1D	IPOPT	optimal	1	1.00e-02	6.53e-01	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	6.53e-01	3	0	2	0	1
	SNOPT	optimal	121	1.00e-02	6.53e-01	128	0	127	0	0
	WORHP IP	optimal	1	1.00e-02	6.53e-01	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	6.53e-01	3	0	2	0	1
	WORHP SQP	optimal	32	1.00e-02	6.53e-01	33	0	33	0	32
PALMER1E	IPOPT	optimal	45	1.00e-02	8.35e-04	97	0	46	0	45
	KNITRO	optimal	68	1.00e-02	8.35e-04	100	0	69	0	68
	SNOPT	optimal	147	1.00e-02	8.35e-04	176	0	175	0	0
	WORHP IP	optimal	38	1.00e-02	8.35e-04	45	0	40	0	38
	WORHP IPm	optimal	55	1.00e-02	8.35e-04	74	0	56	0	55
	WORHP SQP	optimal	60	1.00e-02	8.35e-04	159	0	61	0	60
PALMER2	IPOPT	optimal	898	2.40e-01	3.65e+03	2240	0	899	0	898
	KNITRO	optimal	23	1.00e-02	3.65e+03	35	0	24	0	23
	SNOPT	optimal	25	1.00e-02	3.65e+03	43	0	42	0	0
	WORHP IP	optimal	2628	1.40e-01	3.65e+03	3836	0	2630	0	2628
	WORHP IPm	optimal	18	1.00e-02	3.65e+03	23	0	19	0	18
	WORHP SQP	optimal	17	1.00e-02	3.65e+03	19	0	18	0	17
PALMER2A	IPOPT	optimal	87	2.00e-02	1.71e-02	205	0	88	0	87
	KNITRO	optimal	63	1.00e-02	1.71e-02	92	0	64	0	63
	SNOPT	optimal	97	1.00e-02	1.71e-02	118	0	117	0	0
	WORHP IP	optimal	85	1.00e-02	1.71e-02	117	0	86	0	85
	WORHP IPm	optimal	78	1.00e-02	1.71e-02	105	0	79	0	78
	WORHP SQP	optimal	69	1.00e-02	1.71e-02	143	0	70	0	69
PALMER2B	IPOPT	optimal	18	1.00e-02	6.23e-01	34	0	19	0	18
	KNITRO	optimal	13	1.00e-02	6.23e-01	17	0	14	0	13
	SNOPT	optimal	34	1.00e-02	6.23e-01	46	0	45	0	0
	WORHP IP	optimal	19	1.00e-02	6.23e-01	24	0	21	0	19
	WORHP IPm	optimal	15	1.00e-02	6.23e-01	21	0	16	0	15
	WORHP SQP	optimal	22	1.00e-02	6.23e-01	28	0	23	0	22
PALMER2C	IPOPT	optimal	1	1.00e-02	1.44e-02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.44e-02	3	0	2	0	1
	SNOPT	optimal	154	1.00e-02	1.44e-02	161	0	160	0	0
	WORHP IP	optimal	1	1.00e-02	1.44e-02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	1.44e-02	3	0	2	0	1
	WORHP SQP	optimal	37	1.00e-02	1.44e-02	38	0	38	0	37
PALMER2E	IPOPT	optimal	18	1.00e-02	2.07e-04	24	0	19	0	18
	KNITRO	optimal	45	1.00e-02	2.07e-04	57	0	46	0	45
	SNOPT	optimal	160	1.00e-02	2.07e-04	193	0	192	0	0
	WORHP IP	optimal	24	1.00e-02	2.07e-04	32	0	26	0	24
	WORHP IPm	optimal	25	1.00e-02	2.07e-04	33	0	26	0	25
	WORHP SQP	optimal	71	1.00e-02	2.07e-04	250	0	72	0	71
PALMER3	IPOPT	optimal	166	5.00e-02	2.27e+03	432	0	167	0	166
	KNITRO	optimal	14	1.00e-02	2.27e+03	19	0	15	0	14
	SNOPT	optimal	44	1.00e-02	2.42e+03	53	0	52	0	0
	WORHP IP	optimal	140	1.00e-02	2.27e+03	318	0	141	0	140
	WORHP IPm	optimal	158	1.00e-02	2.42e+03	206	0	161	0	158
	WORHP SQP	optimal	19	1.00e-02	2.27e+03	26	0	20	0	19
PALMER3A	IPOPT	optimal	73	2.00e-02	2.04e-02	182	0	74	0	73
	KNITRO	optimal	73	1.00e-02	2.04e-02	93	0	74	0	73
	SNOPT	optimal	107	1.00e-02	2.04e-02	132	0	131	0	0
	WORHP IP	optimal	70	1.00e-02	2.04e-02	99	0	72	0	70
	WORHP IPm	optimal	74	1.00e-02	2.04e-02	96	0	75	0	74
	WORHP SQP	optimal	33	1.00e-02	2.04e-02	40	0	34	0	33
PALMER3B	IPOPT	optimal	14	1.00e-02	4.23e+00	15	0	15	0	14
	KNITRO	optimal	13	1.00e-02	4.23e+00	19	0	14	0	13
	SNOPT	optimal	32	1.00e-02	4.23e+00	48	0	47	0	0
	WORHP IP	optimal	12	1.00e-02	4.23e+00	14	0	13	0	12
	WORHP IPm	optimal	14	1.00e-02	4.23e+00	17	0	15	0	14
	WORHP SQP	optimal	19	1.00e-02	4.23e+00	23	0	20	0	19
PALMER3C	IPOPT	optimal	1	1.00e-02	1.95e-02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.95e-02	3	0	2	0	1
	SNOPT	optimal	146	1.00e-02	1.95e-02	153	0	152	0	0
	WORHP IP	optimal	1	1.00e-02	1.95e-02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	1.95e-02	3	0	2	0	1
	WORHP SQP	optimal	35	1.00e-02	1.95e-02	36	0	36	0	35

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PALMER3E	IPOPT	optimal	30	1.00e-02	5.07e-05	58	0	31	0	30
	KNITRO	optimal	20	1.00e-02	5.07e-05	25	0	21	0	20
	SNOPT	optimal	114	1.00e-02	5.07e-05	142	0	141	0	0
	WORHP IP	optimal	30	1.00e-02	5.07e-05	44	0	32	0	30
	WORHP IPm	optimal	31	1.00e-02	5.07e-05	45	0	32	0	31
	WORHP SQP	optimal	57	1.00e-02	5.07e-05	184	0	58	0	57
PALMER4	IPOPT	optimal	328	1.10e-01	2.29e+03	856	0	329	0	328
	KNITRO	optimal	11	1.00e-02	2.29e+03	14	0	12	0	11
	SNOPT	optimal	22	1.00e-02	2.42e+03	32	0	31	0	0
	WORHP IP	optimal	306	1.00e-02	2.29e+03	488	0	308	0	306
	WORHP IPm	optimal	318	2.00e-02	2.42e+03	386	0	321	0	318
	WORHP SQP	optimal	20	1.00e-02	2.29e+03	30	0	21	0	20
PALMER4A	IPOPT	optimal	56	2.00e-02	4.06e-02	119	0	57	0	56
	KNITRO	optimal	46	1.00e-02	4.06e-02	62	0	47	0	46
	SNOPT	optimal	84	1.00e-02	4.06e-02	102	0	101	0	0
	WORHP IP	optimal	61	1.00e-02	4.06e-02	79	0	62	0	61
	WORHP IPm	optimal	43	1.00e-02	4.06e-02	58	0	44	0	43
	WORHP SQP	optimal	150	2.00e-02	4.06e-02	1493	0	151	0	150
PALMER4B	IPOPT	optimal	15	1.00e-02	6.84e+00	31	0	16	0	15
	KNITRO	optimal	17	1.00e-02	6.84e+00	22	0	18	0	17
	SNOPT	optimal	27	1.00e-02	6.84e+00	39	0	38	0	0
	WORHP IP	optimal	17	1.00e-02	6.84e+00	23	0	18	0	17
	WORHP IPm	optimal	20	1.00e-02	6.84e+00	27	0	21	0	20
	WORHP SQP	optimal	20	1.00e-02	6.84e+00	24	0	21	0	20
PALMER4C	IPOPT	optimal	1	1.00e-02	5.03e-02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	5.03e-02	3	0	2	0	1
	SNOPT	optimal	144	1.00e-02	5.03e-02	152	0	151	0	0
	WORHP IP	optimal	1	1.00e-02	5.03e-02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	5.03e-02	3	0	2	0	1
	WORHP SQP	optimal	35	1.00e-02	5.03e-02	36	0	36	0	35
PALMER4E	IPOPT	optimal	30	1.00e-02	1.48e-04	48	0	31	0	30
	KNITRO	optimal	28	1.00e-02	1.48e-04	46	0	29	0	28
	SNOPT	optimal	147	1.00e-02	1.48e-04	170	0	169	0	0
	WORHP IP	optimal	25	1.00e-02	1.48e-04	34	0	27	0	25
	WORHP IPm	optimal	24	1.00e-02	1.48e-04	33	0	25	0	24
	WORHP SQP	optimal	44	1.00e-02	1.48e-04	73	0	45	0	44
PALMER5A	IPOPT	maxiter	10000	3.66e+00	2.82e-02	49192	0	10001	0	10000
	KNITRO	smallstep	8398	6.60e-01	5.95e-02	31637	0	8399	0	8398
	SNOPT	maxiter	10000	3.60e-01	3.05e-02	13868	0	13867	0	0
	WORHP IP	maxiter	10000	5.70e-01	2.82e-02	31479	0	10002	0	10000
	WORHP IPm	optimal	6567	3.60e-01	3.11e-02	18651	0	6568	0	6567
	WORHP SQP	maxiter	10000	8.20e-01	2.67e-02	26913	0	10001	0	10001
PALMER5B	IPOPT	optimal	80	3.00e-02	9.75e-03	180	0	81	0	80
	KNITRO	optimal	76	1.00e-02	9.75e-03	110	0	77	0	76
	SNOPT	optimal	1491	5.00e-02	9.75e-03	2028	0	2027	0	0
	WORHP IP	optimal	97	1.00e-02	9.75e-03	152	0	99	0	97
	WORHP IPm	optimal	45	1.00e-02	9.75e-03	65	0	46	0	45
	WORHP SQP	optimal	437	4.00e-02	9.75e-03	1891	0	438	0	437
PALMER5C	IPOPT	optimal	1	1.00e-02	2.13e+00	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	2.13e+00	3	0	2	0	1
	SNOPT	optimal	19	1.00e-02	2.13e+00	22	0	21	0	0
	WORHP IP	optimal	1	1.00e-02	2.13e+00	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	2.13e+00	3	0	2	0	1
	WORHP SQP	optimal	4	1.00e-02	2.13e+00	5	0	5	0	4
PALMER5D	IPOPT	optimal	1	1.00e-02	8.73e+01	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	8.73e+01	3	0	2	0	1
	SNOPT	optimal	31	1.00e-02	8.73e+01	34	0	33	0	0
	WORHP IP	optimal	1	1.00e-02	8.73e+01	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	8.73e+01	3	0	2	0	1
	WORHP SQP	optimal	9	1.00e-02	8.73e+01	10	0	10	0	9
PALMER5E	IPOPT	maxiter	10000	4.30e+00	2.07e-02	87637	0	10001	0	10000
	KNITRO	optimal	7043	3.80e-01	2.07e-02	10254	0	7044	0	7043
	SNOPT	maxiter	10000	4.10e-01	2.07e-02	13284	0	13283	0	0
	WORHP IP	optimal	7530	4.10e-01	2.07e-02	11390	0	7535	0	7530
	WORHP IPm	optimal	7040	3.60e-01	2.07e-02	10229	0	7041	0	7040
	WORHP SQP	optimal	6942	6.70e-01	2.07e-02	42530	0	6943	0	6942
PALMER6A	IPOPT	optimal	125	3.00e-02	5.59e-02	283	0	126	0	125
	KNITRO	optimal	99	1.00e-02	5.59e-02	131	0	100	0	99
	SNOPT	optimal	156	1.00e-02	5.59e-02	196	0	195	0	0
	WORHP IP	optimal	123	1.00e-02	5.59e-02	171	0	124	0	123
	WORHP IPm	optimal	92	1.00e-02	5.59e-02	125	0	93	0	92
	WORHP SQP	optimal	33	1.00e-02	5.59e-02	40	0	34	0	33
PALMER6C	IPOPT	optimal	1	1.00e-02	1.64e-02	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.64e-02	3	0	2	0	1
	SNOPT	optimal	145	1.00e-02	1.64e-02	150	0	149	0	0
	WORHP IP	optimal	1	1.00e-02	1.64e-02	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	1.64e-02	3	0	2	0	1
	WORHP SQP	optimal	35	1.00e-02	1.64e-02	36	0	36	0	35

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PALMER6E	IPOPT	optimal	30	1.00e-02	2.24e-04	59	0	31	0	30
	KNITRO	optimal	29	1.00e-02	2.24e-04	38	0	30	0	29
	SNOPT	optimal	170	1.00e-02	2.24e-04	193	0	192	0	0
	WORHP IP	optimal	32	1.00e-02	2.24e-04	41	0	34	0	32
	WORHP IPm	optimal	30	1.00e-02	2.24e-04	38	0	31	0	30
	WORHP SQP	optimal	49	1.00e-02	2.24e-04	94	0	50	0	49
PALMER7A	IPOPT	optimal	3561	1.00e+00	1.03e+01	9681	0	3562	0	3561
	KNITRO	optimal	3466	1.40e-01	1.03e+01	5050	0	3467	0	3466
	SNOPT	optimal	7081	2.30e-01	1.03e+01	8209	0	8208	0	0
	WORHP IP	optimal	3483	1.60e-01	1.03e+01	5071	0	3484	0	3483
	WORHP IPm	optimal	3439	1.50e-01	1.03e+01	5037	0	3440	0	3439
	WORHP SQP	optimal	3570	2.90e-01	1.03e+01	23394	0	3571	0	3570
PALMER7C	IPOPT	optimal	1	1.00e-02	6.02e-01	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	6.02e-01	3	0	2	0	1
	SNOPT	optimal	153	1.00e-02	6.02e-01	157	0	156	0	0
	WORHP IP	optimal	1	1.00e-02	6.02e-01	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	6.02e-01	3	0	2	0	1
	WORHP SQP	optimal	37	1.00e-02	6.02e-01	38	0	38	0	37
PALMER7E	IPOPT	maxiter	10000	3.55e+00	6.46e+00	40991	0	10001	0	10000
	KNITRO	noimpr	3314	2.00e-01	6.78e+00	13030	0	3313	0	3313
	SNOPT	optimal	1766	6.00e-02	1.02e+01	2352	0	2351	0	0
	WORHP IP	optimal	393	2.00e-02	1.02e+01	649	0	395	0	393
	WORHP IPm	maxiter	10000	5.80e-01	6.50e+00	16495	0	10001	0	10000
	WORHP SQP	optimal	278	3.00e-02	1.02e+01	1697	0	279	0	278
PALMER8A	IPOPT	optimal	45	1.00e-02	7.40e-02	102	0	46	0	45
	KNITRO	optimal	33	1.00e-02	7.40e-02	40	0	34	0	33
	SNOPT	optimal	96	1.00e-02	7.40e-02	125	0	124	0	0
	WORHP IP	optimal	45	1.00e-02	7.40e-02	53	0	46	0	45
	WORHP IPm	optimal	43	1.00e-02	7.40e-02	58	0	44	0	43
	WORHP SQP	optimal	47	1.00e-02	7.40e-02	79	0	48	0	47
PALMER8C	IPOPT	optimal	1	1.00e-02	1.60e-01	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.60e-01	3	0	2	0	1
	SNOPT	optimal	141	1.00e-02	1.60e-01	147	0	146	0	0
	WORHP IP	optimal	1	1.00e-02	1.60e-01	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	1.60e-01	3	0	2	0	1
	WORHP SQP	optimal	34	1.00e-02	1.60e-01	35	0	35	0	34
PALMER8E	IPOPT	optimal	23	1.00e-02	6.34e-03	31	0	24	0	23
	KNITRO	optimal	20	1.00e-02	6.34e-03	26	0	21	0	20
	SNOPT	optimal	84	1.00e-02	6.34e-03	93	0	92	0	0
	WORHP IP	optimal	18	1.00e-02	6.34e-03	29	0	20	0	18
	WORHP IPm	optimal	18	1.00e-02	6.34e-03	25	0	19	0	18
	WORHP SQP	optimal	29	1.00e-02	6.34e-03	30	0	30	0	29
PARKCH	IPOPT	optimal	17	6.08e+01	1.62e+03	24	0	18	0	17
	KNITRO	optimal	16	5.98e+01	1.62e+03	35	0	17	0	16
	SNOPT	optimal	216	1.56e+01	1.62e+03	226	0	225	0	0
	WORHP IP	optimal	17	6.36e+01	1.62e+03	22	0	19	0	17
	WORHP IPm	optimal	17	4.49e+01	1.62e+03	22	0	18	0	17
	WORHP SQP	optimal	25	9.24e+01	1.62e+03	133	0	26	0	25
PDE1	IPOPT	optimal	34	9.94e+02	2.84e-02	39	78	35	70	34
	KNITRO	optimal	23	1.72e+02	1.50e-02	26	27	25	26	23
	SNOPT	memory	0	1.19e+02	0.00e+00	1	1	1	1	0
	WORHP IP	maxtime	115	1.76e+03	3.50e+00	117	117	116	1	116
	WORHP IPm	maxtime	103	1.76e+03	2.49e+00	106	106	105	1	104
	WORHP SQP	optimal	4	5.67e+02	1.49e-02	5	5	6	3	4
PDE2	IPOPT	optimal	28	6.24e+02	1.10e+03	29	58	29	58	28
	KNITRO	optimal	12	6.36e+01	1.10e+03	15	16	14	15	12
	SNOPT	memory	0	1.30e+02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	17	2.78e+02	1.10e+03	19	19	19	1	17
	WORHP IPm	optimal	17	1.51e+02	1.10e+03	21	21	20	1	17
	WORHP SQP	optimal	3	3.97e+02	1.10e+03	4	4	5	3	3
PENALTY1	IPOPT	optimal	39	7.85e+00	9.69e-03	50	0	40	0	39
	KNITRO	optimal	39	9.39e+00	9.69e-03	43	0	40	0	39
	SNOPT	optimal	90	2.24e+00	9.69e-03	112	0	111	0	0
	WORHP IP	optimal	39	1.07e+01	9.69e-03	45	0	41	0	39
	WORHP IPm	optimal	39	1.04e+01	9.69e-03	45	0	40	0	39
	WORHP SQP	optimal	51	1.86e+01	9.69e-03	66	0	52	0	51
PENALTY2	IPOPT	optimal	10	5.00e-02	4.71e+13	11	0	11	0	10
	KNITRO	optimal	10	4.00e-02	4.71e+13	12	0	11	0	10
	SNOPT	optimal	96	5.00e-02	4.71e+13	105	0	104	0	0
	WORHP IP	optimal	11	6.00e-02	4.71e+13	57	0	13	0	11
	WORHP IPm	optimal	11	5.00e-02	4.71e+13	57	0	12	0	11
	WORHP SQP	optimal	0	1.00e-02	4.71e+13	1	0	1	0	0
PENALTY3	IPOPT	optimal	18	3.28e+00	1.00e-03	37	0	19	0	18
	KNITRO	noimpr	21	3.56e+00	9.97e-04	58	0	22	0	22
	SNOPT	maxiter	10000	3.51e+02	9.98e-04	69716	0	69715	0	0
	WORHP IP	optimal	62	7.87e+00	9.97e-04	892	0	75	0	62
	WORHP IPm	optimal	39	5.17e+00	1.00e-03	559	0	40	0	39
	WORHP SQP	optimal	38	8.65e+00	9.97e-04	2275	0	39	0	38

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PENLT1NE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	15	1.00e-02	0.00e+00	30	31	15	16	13
	SNOPT	infeas	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	infeas	20	1.00e-02	0.00e+00	31	31	21	21	21
	WORHP IPm	infeas	20	1.00e-02	0.00e+00	31	31	21	21	21
	WORHP SQP	minalpha	32	2.00e-02	0.00e+00	3640	3646	34	34	33
PENLT2NE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	noimpr	40	1.00e-02	0.00e+00	62	63	42	43	38
	SNOPT	infeas	1912	9.00e-02	0.00e+00	1	5049	1	5048	0
	WORHP IP	infeas	42	1.00e-02	0.00e+00	209	209	43	43	43
	WORHP IPm	infeas	42	1.00e-02	0.00e+00	209	209	43	43	43
	WORHP SQP	minalpha	229	4.00e-02	0.00e+00	4536	4543	63	231	62
PENTAGON	IPOPT	optimal	15	1.00e-02	1.37e-04	18	18	16	16	15
	KNITRO	optimal	8	1.00e-02	1.37e-04	11	12	10	11	8
	SNOPT	optimal	17	1.00e-02	1.37e-04	21	1	20	1	0
	WORHP IP	optimal	14	1.00e-02	1.37e-04	16	16	15	1	14
	WORHP IPm	optimal	19	1.00e-02	1.37e-04	21	21	20	1	19
	WORHP SQP	optimal	15	1.00e-02	1.46e-04	26	26	17	3	15
PENTDI	IPOPT	optimal	11	1.20e-01	-7.49e-01	12	0	12	0	11
	KNITRO	optimal	9	1.20e-01	-7.50e-01	12	0	11	0	9
	SNOPT	optimal	3	3.00e-02	-7.50e-01	7	0	6	0	0
	WORHP IP	optimal	8	1.20e-01	-7.49e-01	10	0	9	0	8
	WORHP IPm	optimal	8	1.00e-01	-7.50e-01	13	0	12	0	8
	WORHP SQP	optimal	1	7.00e-02	-7.50e-01	2	0	2	0	1
PFIT1	IPOPT	infeas	100	5.00e-02	0.00e+00	314	320	17	103	101
	KNITRO	optimal	30	1.00e-02	0.00e+00	81	82	31	32	32
	SNOPT	optimal	22	1.00e-02	0.00e+00	1	71	1	70	0
	WORHP IP	optimal	56	1.00e-02	0.00e+00	321	321	57	57	56
	WORHP IPm	optimal	101	1.00e-02	0.00e+00	723	723	102	102	101
	WORHP SQP	minalpha	91	4.00e-02	0.00e+00	13844	13861	20	93	19
PFIT1LS	IPOPT	optimal	301	1.00e-01	6.77e-13	760	0	302	0	301
	KNITRO	optimal	210	1.00e-02	3.18e-15	307	0	211	0	210
	SNOPT	optimal	353	1.00e-02	1.60e-15	479	0	478	0	0
	WORHP IP	optimal	310	1.00e-02	9.51e-16	449	0	311	0	310
	WORHP IPm	optimal	260	1.00e-02	7.89e-17	360	0	261	0	260
	WORHP SQP	optimal	247	2.00e-02	3.45e-16	1383	0	248	0	247
PFIT2	IPOPT	infeas	152	6.00e-02	0.00e+00	292	302	18	157	153
	KNITRO	infeas	20	1.00e-02	0.00e+00	60	61	21	22	21
	SNOPT	optimal	12	1.00e-02	0.00e+00	1	33	1	32	0
	WORHP IP	optimal	924	8.00e-02	0.00e+00	8665	8665	926	926	924
	WORHP IPm	optimal	872	6.00e-02	0.00e+00	8215	8215	873	873	872
	WORHP SQP	minalpha	87	2.00e-02	0.00e+00	6722	6728	60	89	59
PFIT2LS	IPOPT	optimal	104	3.00e-02	1.07e-13	233	0	105	0	104
	KNITRO	optimal	79	1.00e-02	5.15e-18	115	0	80	0	79
	SNOPT	optimal	132	1.00e-02	2.49e-17	174	0	173	0	0
	WORHP IP	optimal	68	1.00e-02	9.48e-12	93	0	69	0	68
	WORHP IPm	optimal	74	1.00e-02	6.94e-20	98	0	75	0	74
	WORHP SQP	optimal	90	1.00e-02	5.83e-22	363	0	91	0	90
PFIT3	IPOPT	resto	304	1.60e-01	0.00e+00	615	626	10	312	305
	KNITRO	infeas	31	1.00e-02	0.00e+00	70	71	32	33	32
	SNOPT	toobig	6	1.00e-02	0.00e+00	1	22	1	21	0
	WORHP IP	optimal	1420	1.20e-01	0.00e+00	13192	13192	1421	1421	1420
	WORHP IPm	optimal	1326	1.10e-01	0.00e+00	11928	11928	1327	1327	1326
	WORHP SQP	minalpha	68	2.00e-02	0.00e+00	7414	7420	40	70	39
PFIT3LS	IPOPT	optimal	146	4.00e-02	6.21e-14	365	0	147	0	146
	KNITRO	optimal	107	1.00e-02	7.00e-23	148	0	108	0	107
	SNOPT	optimal	222	1.00e-02	1.45e-17	305	0	304	0	0
	WORHP IP	optimal	148	1.00e-02	9.43e-15	216	0	149	0	148
	WORHP IPm	optimal	127	1.00e-02	1.35e-20	180	0	128	0	127
	WORHP SQP	optimal	135	1.00e-02	1.14e-16	658	0	136	0	135
PFIT4	IPOPT	resto	493	2.20e-01	0.00e+00	1071	1139	19	514	494
	KNITRO	infeas	76	1.00e-02	0.00e+00	317	318	77	78	78
	SNOPT	maxiter	10000	5.00e-01	0.00e+00	1	52031	1	52030	0
	WORHP IP	optimal	1992	1.70e-01	0.00e+00	18983	18983	1993	1993	1992
	WORHP IPm	optimal	2000	1.60e-01	0.00e+00	19041	19041	2001	2001	2000
	WORHP SQP	optimal	164	4.00e-02	0.00e+00	7629	7632	166	166	164
PFIT4LS	IPOPT	optimal	233	8.00e-02	2.61e-14	592	0	234	0	233
	KNITRO	optimal	190	1.00e-02	7.86e-19	272	0	191	0	190
	SNOPT	optimal	362	1.00e-02	3.71e-18	493	0	492	0	0
	WORHP IP	optimal	232	1.00e-02	5.14e-14	329	0	233	0	232
	WORHP IPm	optimal	216	1.00e-02	3.22e-19	316	0	217	0	216
	WORHP SQP	optimal	212	2.00e-02	4.86e-20	1127	0	213	0	212
PINENE	IPOPT	optimal	14	4.90e-01	1.99e+01	15	15	15	15	14
	KNITRO	optimal	9	4.20e-01	1.99e+01	12	13	11	12	9
	SNOPT	optimal	40	5.84e+00	1.99e+01	46	46	45	45	0
	WORHP IP	optimal	25	9.50e-01	1.99e+01	38	38	27	27	25
	WORHP IPm	optimal	12	5.10e-01	1.99e+01	14	14	13	13	12
	WORHP SQP	optimal	2572	1.16e+02	1.99e+01	36418	36422	2573	2573	2572

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
POLAK1	IPOPT	optimal	6	1.00e-02	2.72e+00	7	7	7	7	6
	KNITRO	optimal	6	1.00e-02	2.72e+00	8	9	7	8	6
	SNOPT	optimal	12	1.00e-02	2.72e+00	1	19	1	18	0
	WORHP IP	optimal	8	1.00e-02	2.72e+00	10	10	9	9	8
	WORHP IPm	optimal	8	1.00e-02	2.72e+00	10	10	9	9	8
	WORHP SQP	optimal	8	1.00e-02	2.72e+00	9	9	10	10	8
POLAK2	IPOPT	optimal	14	1.00e-02	5.46e+01	16	29	15	15	14
	KNITRO	optimal	5	1.00e-02	5.46e+01	7	8	6	7	5
	SNOPT	optimal	101	1.00e-02	5.46e+01	1	246	1	245	0
	WORHP IP	optimal	6	1.00e-02	5.46e+01	8	8	7	7	6
	WORHP IPm	optimal	4	1.00e-02	5.46e+01	6	6	5	5	4
	WORHP SQP	optimal	45	1.00e-02	5.46e+01	745	745	27	47	25
POLAK3	IPOPT	resto	1842	9.10e-01	-1.24e+02	2979	3800	276	1875	1843
	KNITRO	optimal	17	1.00e-02	5.93e+00	19	20	18	19	17
	SNOPT	optimal	44	1.00e-02	5.93e+00	1	116	1	115	0
	WORHP IP	optimal	30	1.00e-02	5.93e+00	34	34	31	31	30
	WORHP IPm	optimal	28	1.00e-02	5.93e+00	30	30	29	29	28
	WORHP SQP	optimal	16	2.00e-02	5.93e+00	25	25	18	18	16
POLAK4	IPOPT	optimal	6	1.00e-02	2.63e-07	7	7	7	7	6
	KNITRO	optimal	8	1.00e-02	-1.27e-10	10	11	9	10	8
	SNOPT	optimal	5	1.00e-02	1.90e-18	1	8	1	7	0
	WORHP IP	optimal	8	1.00e-02	6.37e-07	10	10	9	9	8
	WORHP IPm	optimal	7	1.00e-02	-2.05e-09	9	9	8	8	7
	WORHP SQP	optimal	3	1.00e-02	-1.07e-11	4	4	5	5	3
POLAK5	IPOPT	optimal	21	1.00e-02	5.00e+01	22	22	22	22	21
	KNITRO	optimal	23	1.00e-02	5.00e+01	32	33	24	25	23
	SNOPT	optimal	41	1.00e-02	5.00e+01	1	60	1	59	0
	WORHP IP	optimal	30	1.00e-02	5.00e+01	32	32	31	31	30
	WORHP IPm	optimal	67	1.00e-02	5.00e+01	241	241	68	68	67
	WORHP SQP	optimal	6	1.00e-02	5.00e+01	36	36	8	8	6
POLAK6	IPOPT	optimal	99	3.00e-02	-4.40e+01	249	437	85	103	99
	KNITRO	optimal	11	1.00e-02	-4.40e+01	13	14	12	13	11
	SNOPT	optimal	30	1.00e-02	-4.40e+01	1	83	1	82	0
	WORHP IP	optimal	21	1.00e-02	-4.40e+01	26	26	22	22	21
	WORHP IPm	optimal	13	1.00e-02	-4.40e+01	16	16	14	14	13
	WORHP SQP	optimal	30	1.00e-02	-4.40e+01	1680	1681	26	32	24
POLYGON	IPOPT	optimal	43	9.70e-01	-7.27e-01	44	44	44	44	43
	KNITRO	optimal	109	3.64e+00	-7.84e-01	116	117	111	112	109
	SNOPT	optimal	309	6.31e+00	-7.85e-01	325	325	324	324	0
	WORHP IP	optimal	164	5.58e+00	-7.83e-01	365	365	165	165	164
	WORHP IPm	optimal	159	6.51e+00	-7.85e-01	1018	1018	160	160	159
	WORHP SQP	optimal	42	6.16e+00	-7.85e-01	43	43	44	44	42
POROUS1	IPOPT	optimal	13	1.54e+00	0.00e+00	20	21	14	14	13
	KNITRO	optimal	12	1.08e+00	0.00e+00	21	22	14	15	12
	SNOPT	optimal	13	1.88e+00	0.00e+00	1	16	1	15	0
	WORHP IP	optimal	13	7.50e-01	0.00e+00	22	22	15	15	13
	WORHP IPm	optimal	13	7.30e-01	0.00e+00	22	22	14	14	13
	WORHP SQP	optimal	14	8.71e+00	0.00e+00	21	21	16	16	14
POROUS2	IPOPT	optimal	8	6.80e-01	0.00e+00	15	16	9	9	8
	KNITRO	optimal	7	8.30e-01	0.00e+00	16	17	9	10	7
	SNOPT	optimal	8	1.61e+00	0.00e+00	1	11	1	10	0
	WORHP IP	optimal	8	4.80e-01	0.00e+00	17	17	9	9	8
	WORHP IPm	optimal	8	4.90e-01	0.00e+00	17	17	9	9	8
	WORHP SQP	optimal	8	6.71e+00	0.00e+00	9	9	10	10	8
PORTFL1	IPOPT	optimal	8	1.00e-02	2.05e-02	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	2.05e-02	8	9	7	8	6
	SNOPT	optimal	17	1.00e-02	2.05e-02	21	1	20	1	0
	WORHP IP	optimal	6	1.00e-02	2.05e-02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	2.05e-02	8	8	7	1	6
	WORHP SQP	optimal	3	1.00e-02	2.05e-02	4	4	5	3	3
PORTFL2	IPOPT	optimal	7	1.00e-02	2.97e-02	8	8	8	8	7
	KNITRO	optimal	6	1.00e-02	2.97e-02	8	9	7	8	6
	SNOPT	optimal	12	1.00e-02	2.97e-02	15	1	14	1	0
	WORHP IP	optimal	6	1.00e-02	2.97e-02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	2.97e-02	8	8	7	1	6
	WORHP SQP	optimal	3	1.00e-02	2.97e-02	4	4	5	3	3
PORTFL3	IPOPT	optimal	8	1.00e-02	3.28e-02	9	9	9	9	8
	KNITRO	optimal	6	1.00e-02	3.27e-02	8	9	7	8	6
	SNOPT	optimal	10	1.00e-02	3.27e-02	13	1	12	1	0
	WORHP IP	optimal	7	1.00e-02	3.27e-02	9	9	8	1	7
	WORHP IPm	optimal	7	1.00e-02	3.27e-02	9	9	8	1	7
	WORHP SQP	optimal	3	1.00e-02	3.27e-02	4	4	5	3	3
PORTFL4	IPOPT	optimal	7	1.00e-02	2.63e-02	8	8	8	8	7
	KNITRO	optimal	6	1.00e-02	2.63e-02	8	9	7	8	6
	SNOPT	optimal	13	1.00e-02	2.63e-02	17	1	16	1	0
	WORHP IP	optimal	6	1.00e-02	2.63e-02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	2.63e-02	8	8	7	1	6
	WORHP SQP	optimal	3	1.00e-02	2.63e-02	4	4	5	3	3

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PORTFL6	IPOPT	optimal	7	1.00e-02	2.58e-02	8	8	8	8	7
	KNITRO	optimal	6	1.00e-02	2.58e-02	8	9	7	8	6
	SNOPT	optimal	11	1.00e-02	2.58e-02	15	1	14	1	0
	WORHP IP	optimal	6	1.00e-02	2.58e-02	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	2.58e-02	8	8	7	1	6
	WORHP SQP	optimal	3	1.00e-02	2.58e-02	4	4	5	3	3
PORTSNQP	IPOPT	optimal	9	2.04e+01	3.33e+04	10	10	10	10	9
	KNITRO	optimal	30	2.09e+01	3.33e+04	32	33	31	32	30
	SNOPT	optimal	10	2.19e+02	3.33e+04	17	1	16	1	0
	WORHP IP	optimal	13	1.23e+01	3.33e+04	15	15	14	1	13
	WORHP IPm	optimal	10	1.75e+01	3.33e+04	14	14	13	1	10
	WORHP SQP	optimal	11	1.54e+02	3.33e+04	18	18	13	3	11
PORTSQP	IPOPT	optimal	20	1.57e+01	3.33e+04	24	24	21	21	20
	KNITRO	optimal	9	1.42e+01	3.33e+04	11	12	10	11	9
	SNOPT	optimal	4	1.51e+02	3.33e+04	8	1	7	1	0
	WORHP IP	optimal	19	1.46e+01	3.33e+04	21	21	20	1	19
	WORHP IPm	optimal	11	1.42e+01	3.33e+04	16	16	15	1	11
	WORHP SQP	optimal	2	3.66e+01	3.33e+04	3	3	4	3	2
POWELL20	IPOPT	optimal	1153	7.73e+00	6.51e+09	1156	1156	1154	1154	1153
	KNITRO	optimal	43	4.80e-01	6.51e+09	45	46	44	45	43
	SNOPT	optimal	3	5.20e-01	6.51e+09	6	1	5	1	0
	WORHP IP	optimal	1229	1.43e+01	6.51e+09	1231	1231	1230	1	1229
	WORHP IPm	optimal	1322	1.63e+01	6.51e+09	1324	1324	1323	1	1322
	WORHP SQP	optimal	804	1.93e+01	6.51e+09	38396	38395	8	2	7
POWELLBC	IPOPT	regular	0	2.27e+00	0.00e+00	1	0	1	0	1
	KNITRO	nan	0	1.44e+00	0.00e+00	3	0	1	0	0
	SNOPT	optimal	1158	2.04e+01	3.11e+05	1492	0	1491	0	0
	WORHP IP	regular	2	8.13e+00	1.00e+20	4	0	3	0	3
	WORHP IPm	minalpha	442	1.69e+02	3.11e+05	725	0	486	0	443
	WORHP SQP	optimal	417	4.88e+02	3.10e+05	455	0	418	0	417
POWELLBS	IPOPT	optimal	11	1.00e-02	0.00e+00	12	12	12	12	11
	KNITRO	optimal	59	1.00e-02	0.00e+00	294	295	60	61	59
	SNOPT	optimal	12	1.00e-02	0.00e+00	1	17	1	16	0
	WORHP IP	optimal	11	1.00e-02	0.00e+00	13	13	12	12	11
	WORHP IPm	optimal	11	1.00e-02	0.00e+00	13	13	12	12	11
	WORHP SQP	optimal	12	1.00e-02	0.00e+00	15	15	14	14	12
POWELLBSLS	IPOPT	optimal	90	3.00e-02	6.52e-25	190	0	91	0	90
	KNITRO	optimal	92	1.00e-02	7.28e-23	116	0	93	0	92
	SNOPT	toobig	104	1.00e-02	8.86e-09	148	0	147	0	0
	WORHP IP	optimal	90	1.00e-02	2.76e-25	132	0	91	0	90
	WORHP IPm	optimal	90	1.00e-02	2.76e-25	132	0	91	0	90
	WORHP SQP	optimal	97	1.00e-02	5.06e-20	247	0	98	0	97
POWELLSE	IPOPT	optimal	13	1.00e-02	0.00e+00	14	14	14	14	13
	KNITRO	optimal	13	1.00e-02	0.00e+00	15	16	14	15	13
	SNOPT	optimal	14	1.00e-02	0.00e+00	1	17	1	16	0
	WORHP IP	optimal	13	1.00e-02	0.00e+00	15	15	14	14	13
	WORHP IPm	optimal	13	1.00e-02	0.00e+00	15	15	14	14	13
	WORHP SQP	optimal	14	1.00e-02	0.00e+00	15	15	16	16	14
POWELLSG	IPOPT	optimal	17	8.00e-02	2.14e-07	18	0	18	0	17
	KNITRO	optimal	17	9.00e-02	2.14e-07	19	0	18	0	17
	SNOPT	sbasics	10000	2.35e+02	6.12e+04	10430	0	10429	0	0
	WORHP IP	optimal	17	7.00e-02	2.14e-07	19	0	19	0	17
	WORHP IPm	optimal	17	8.00e-02	2.14e-07	19	0	18	0	17
	WORHP SQP	optimal	17	1.20e-01	2.63e-07	18	0	18	0	17
POWELLSQ	IPOPT	infeas	24	1.00e-02	0.00e+00	107	114	12	27	25
	KNITRO	optimal	151	1.00e-02	0.00e+00	743	744	152	153	151
	SNOPT	infeas	74	1.00e-02	0.00e+00	1	214	1	213	0
	WORHP IP	optimal	21	1.00e-02	0.00e+00	93	93	22	22	21
	WORHP IPm	optimal	21	1.00e-02	0.00e+00	93	93	22	22	21
	WORHP SQP	optimal	60	1.00e-02	0.00e+00	5027	5029	42	64	40
POWER	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	killed	-	-	-	-	-	-	-	-
	SNOPT	toobig	31	1.03e+01	1.02e+15	34	0	33	0	0
	WORHP IP	maxtime	7	1.84e+03	2.93e+10	9	0	8	0	8
	WORHP IPm	killed	-	-	-	-	-	-	-	-
	WORHP SQP	killed	-	-	-	-	-	-	-	-
PRIMAL1	IPOPT	optimal	16	5.00e-02	-3.50e-02	17	17	17	17	16
	KNITRO	optimal	10	3.00e-02	-3.50e-02	13	14	12	13	10
	SNOPT	optimal	3	2.00e-02	-3.50e-02	7	1	6	1	0
	WORHP IP	optimal	12	4.00e-02	-3.50e-02	14	14	13	1	12
	WORHP IPm	optimal	17	4.00e-02	-3.50e-02	20	20	19	1	17
	WORHP SQP	optimal	4	7.00e-02	-3.50e-02	285	287	6	3	4
PRIMAL2	IPOPT	optimal	15	4.00e-02	-3.37e-02	16	16	16	16	15
	KNITRO	optimal	8	5.00e-02	-3.37e-02	11	12	10	11	8
	SNOPT	optimal	3	6.00e-02	-3.37e-02	7	1	6	1	0
	WORHP IP	optimal	10	4.00e-02	-3.37e-02	12	12	11	1	10
	WORHP IPm	optimal	11	4.00e-02	-3.37e-02	14	14	13	1	11
	WORHP SQP	optimal	2	6.00e-02	-3.37e-02	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
PRIMAL3	IPOPT	optimal	11	1.30e-01	-1.36e-01	12	12	12	12	11
	KNITRO	optimal	8	2.10e-01	-1.36e-01	11	12	10	11	8
	SNOPT	optimal	6	3.00e-01	-1.36e-01	10	1	9	1	0
	WORHP IP	optimal	11	1.00e-01	-1.36e-01	13	13	12	1	11
	WORHP IPm	optimal	8	7.00e-02	-1.36e-01	11	11	10	1	8
	WORHP SQP	optimal	3	2.30e-01	-1.36e-01	4	4	5	3	3
PRIMAL4	IPOPT	optimal	10	9.00e-02	-7.46e-01	11	11	11	11	10
	KNITRO	optimal	7	1.40e-01	-7.46e-01	10	11	9	10	7
	SNOPT	optimal	12	2.15e+00	-7.46e-01	16	1	15	1	0
	WORHP IP	optimal	8	5.00e-02	-7.46e-01	10	10	9	1	8
	WORHP IPm	optimal	7	3.00e-02	-7.46e-01	9	9	8	1	7
	WORHP SQP	zerostep	29	3.60e-01	-7.46e-01	29	29	30	3	29
PRIMALC1	IPOPT	optimal	15	2.00e-02	-6.16e+03	16	16	16	16	15
	KNITRO	optimal	10	1.00e-02	-6.16e+03	13	14	12	13	10
	SNOPT	optimal	10	1.00e-02	-6.16e+03	13	1	12	1	0
	WORHP IP	optimal	11	1.00e-02	-6.16e+03	13	13	12	1	11
	WORHP IPm	optimal	9	1.00e-02	-6.16e+03	14	14	13	1	9
	WORHP SQP	optimal	9	4.00e-02	-6.16e+03	10	10	11	3	9
PRIMALC2	IPOPT	optimal	15	2.00e-02	-3.55e+03	16	16	16	16	15
	KNITRO	optimal	14	1.00e-02	-3.55e+03	17	18	16	17	14
	SNOPT	optimal	9	1.00e-02	-3.55e+03	12	1	11	1	0
	WORHP IP	optimal	10	1.00e-02	-3.55e+03	12	12	11	1	10
	WORHP IPm	optimal	10	1.00e-02	-3.55e+03	15	15	14	1	10
	WORHP SQP	optimal	7	3.00e-02	-3.55e+03	8	8	9	3	7
PRIMALC5	IPOPT	optimal	14	2.00e-02	-4.27e+02	15	15	15	15	14
	KNITRO	optimal	9	1.00e-02	-4.27e+02	12	13	11	12	9
	SNOPT	optimal	7	1.00e-02	-4.27e+02	11	1	10	1	0
	WORHP IP	optimal	8	1.00e-02	-4.27e+02	10	10	9	1	8
	WORHP IPm	optimal	9	1.00e-02	-4.27e+02	14	14	13	1	9
	WORHP SQP	optimal	4	2.00e-02	-4.27e+02	5	5	6	3	4
PRIMALC8	IPOPT	optimal	18	2.00e-02	-1.83e+04	19	19	19	19	18
	KNITRO	optimal	11	2.00e-02	-1.83e+04	14	15	13	14	11
	SNOPT	optimal	11	1.00e-02	-1.83e+04	14	1	13	1	0
	WORHP IP	optimal	15	1.00e-02	-1.83e+04	17	17	16	1	15
	WORHP IPm	optimal	14	2.00e-02	-1.83e+04	19	19	18	1	14
	WORHP SQP	optimal	12	9.00e-02	-1.83e+04	13	13	14	3	12
PROBPENL	IPOPT	optimal	6	2.00e-01	-4.85e-07	7	0	7	0	6
	KNITRO	optimal	5	1.70e-01	3.83e-07	7	0	6	0	5
	SNOPT	optimal	3	5.00e-02	3.99e-07	8	0	7	0	0
	WORHP IP	optimal	9	5.40e-01	-3.42e-06	11	0	11	0	9
	WORHP IPm	optimal	6	3.10e-01	-3.58e-06	8	0	7	0	6
	WORHP SQP	optimal	4	5.80e-01	3.99e-07	5	0	5	0	4
PRODPL0	IPOPT	optimal	18	1.00e-02	5.88e+01	19	38	19	38	18
	KNITRO	optimal	12	1.00e-02	5.88e+01	15	16	14	15	12
	SNOPT	optimal	7	1.00e-02	5.88e+01	1	12	1	11	0
	WORHP IP	optimal	18	1.00e-02	5.88e+01	24	24	19	19	18
	WORHP IPm	optimal	20	1.00e-02	5.88e+01	25	25	24	24	20
	WORHP SQP	optimal	19	4.00e-02	5.88e+01	19	19	20	20	19
PRODPL1	IPOPT	optimal	29	1.00e-02	3.57e+01	30	60	30	60	29
	KNITRO	optimal	10	1.00e-02	3.57e+01	13	14	12	13	10
	SNOPT	optimal	11	1.00e-02	3.57e+01	1	20	1	19	0
	WORHP IP	optimal	21	1.00e-02	3.57e+01	23	23	22	22	21
	WORHP IPm	optimal	22	1.00e-02	3.57e+01	28	28	26	26	22
	WORHP SQP	optimal	11	2.00e-02	3.57e+01	12	12	13	13	11
PSPDOC	IPOPT	optimal	7	1.00e-02	2.41e+00	15	0	8	0	7
	KNITRO	optimal	5	1.00e-02	2.41e+00	11	0	7	0	5
	SNOPT	optimal	12	1.00e-02	2.41e+00	15	0	14	0	0
	WORHP IP	optimal	6	1.00e-02	2.41e+00	12	0	7	0	6
	WORHP IPm	optimal	5	1.00e-02	2.41e+00	12	0	7	0	5
	WORHP SQP	optimal	10	1.00e-02	2.41e+00	61	0	11	0	10
PT	IPOPT	optimal	19	2.00e-02	1.78e-01	20	20	20	20	19
	KNITRO	optimal	12	1.00e-02	1.78e-01	14	15	13	14	12
	SNOPT	optimal	0	1.00e-02	1.78e-01	1	1	1	1	0
	WORHP IP	optimal	10	1.00e-02	1.78e-01	14	14	11	1	10
	WORHP IPm	optimal	25	2.00e-02	1.78e-01	27	27	26	1	25
	WORHP SQP	optimal	2	2.00e-02	1.78e-01	2	2	3	3	2
QC	IPOPT	optimal	28	1.00e-02	-9.57e+02	30	30	29	29	28
	KNITRO	optimal	7	1.00e-02	-9.57e+02	9	10	8	9	7
	SNOPT	optimal	3	1.00e-02	-9.57e+02	9	1	8	1	0
	WORHP IP	optimal	12	1.00e-02	-9.57e+02	14	14	14	1	12
	WORHP IPm	minalpha	13	1.00e-02	-1.14e+03	219	219	56	1	14
	WORHP SQP	optimal	24	1.00e-02	-9.57e+02	24	24	25	3	24
QCNEW	IPOPT	resto	20	1.00e-02	-9.95e+02	32	32	21	21	21
	KNITRO	nan	0	1.00e-02	0.00e+00	3	4	1	2	0
	SNOPT	optimal	1	1.00e-02	-8.07e+02	5	1	4	1	0
	WORHP IP	optimal	5	1.00e-02	-8.07e+02	7	7	6	1	5
	WORHP IPm	regular	26	1.00e-02	-1.99e+03	47	47	28	1	27
	WORHP SQP	optimal	21	1.00e-02	-8.07e+02	3477	3478	22	3	21

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
QPBAND	IPOPT	optimal	18	2.40e+00	-5.00e+04	19	19	19	19	18
	KNITRO	optimal	9	2.28e+00	-5.00e+04	12	13	11	12	9
	SNOPT	optimal	717	9.54e+02	-5.00e+04	835	1	834	1	0
	WORHP IP	optimal	17	2.48e+00	-5.00e+04	19	19	18	1	17
	WORHP IPm	optimal	19	2.73e+00	-5.00e+04	23	23	22	1	19
	WORHP SQP	optimal	6	9.36e+00	-5.00e+04	7	7	8	3	6
QPCBLEND	IPOPT	optimal	19	1.00e-02	-7.84e-03	20	40	20	40	19
	KNITRO	optimal	17	1.00e-02	-7.84e-03	20	21	19	20	17
	SNOPT	optimal	6	1.00e-02	-7.84e-03	11	1	10	1	0
	WORHP IP	optimal	15	1.00e-02	-7.83e-03	17	17	16	1	15
	WORHP IPm	optimal	12	1.00e-02	-7.84e-03	17	17	16	1	12
	WORHP SQP	optimal	1	1.00e-02	-7.84e-03	2	2	3	3	1
QPCBOEI1	IPOPT	optimal	120	2.20e-01	1.15e+07	132	264	121	242	120
	KNITRO	optimal	30	7.00e-02	1.15e+07	33	34	32	33	30
	SNOPT	optimal	13	1.00e-01	1.15e+07	16	1	15	1	0
	WORHP IP	optimal	207	4.30e-01	1.15e+07	229	229	209	1	207
	WORHP IPm	optimal	327	6.40e-01	1.15e+07	336	336	334	1	327
	WORHP SQP	optimal	79	1.30e+00	1.15e+07	950	951	32	2	31
QPCBOEI2	IPOPT	optimal	120	1.00e-01	8.17e+06	121	242	121	242	120
	KNITRO	optimal	76	6.00e-02	8.17e+06	79	80	78	79	76
	SNOPT	optimal	11	1.00e-02	8.17e+06	14	1	13	1	0
	WORHP IP	optimal	156	1.10e-01	8.17e+06	159	159	158	1	156
	WORHP IPm	optimal	205	1.40e-01	8.17e+06	213	213	212	1	205
	WORHP SQP	optimal	49	3.50e-01	8.17e+06	530	531	30	2	29
QPCSTAIR	IPOPT	optimal	212	3.70e-01	6.20e+06	215	430	186	428	212
	KNITRO	optimal	104	2.20e-01	6.20e+06	107	108	106	107	104
	SNOPT	optimal	10	8.00e-02	6.20e+06	13	1	12	1	0
	WORHP IP	optimal	125	2.90e-01	6.20e+06	175	175	128	1	125
	WORHP IPm	optimal	143	3.10e-01	6.20e+06	150	150	149	1	143
	WORHP SQP	optimal	256	2.93e+00	6.20e+06	4441	4482	246	2	245
QPNBAND	IPOPT	optimal	21	2.92e+00	-2.50e+05	22	22	22	22	21
	KNITRO	optimal	6	1.90e+00	-2.50e+05	9	10	8	9	6
	SNOPT	optimal	107	1.06e+02	-2.50e+05	203	1	202	1	0
	WORHP IP	optimal	14	2.95e+00	-2.50e+05	16	16	15	1	14
	WORHP IPm	optimal	17	3.46e+00	-2.50e+05	23	23	22	1	17
	WORHP SQP	optimal	20	2.88e+01	-2.50e+05	22	22	22	3	20
QPNBLEND	IPOPT	optimal	20	1.00e-02	-9.13e-03	21	42	21	42	20
	KNITRO	optimal	16	1.00e-02	-9.14e-03	19	20	18	19	16
	SNOPT	optimal	8	1.00e-02	-9.14e-03	16	1	15	1	0
	WORHP IP	optimal	14	1.00e-02	-9.12e-03	16	16	15	1	14
	WORHP IPm	optimal	15	1.00e-02	-9.13e-03	20	20	19	1	15
	WORHP SQP	optimal	13	6.00e-02	-9.14e-03	14	14	15	3	13
QPNBOEI1	IPOPT	optimal	263	5.80e-01	6.75e+06	269	538	254	530	263
	KNITRO	optimal	75	2.70e-01	6.76e+06	79	80	77	78	76
	SNOPT	optimal	56	1.60e-01	6.74e+06	62	1	61	1	0
	WORHP IP	optimal	1330	5.41e+00	6.75e+06	1332	1332	1332	1	1330
	WORHP IPm	optimal	552	1.08e+00	6.75e+06	559	559	558	1	552
	WORHP SQP	optimal	81	1.54e+00	6.76e+06	952	953	34	2	33
QPNBOEI2	IPOPT	optimal	239	2.80e-01	1.37e+06	250	500	236	482	239
	KNITRO	optimal	191	1.50e-01	1.37e+06	195	196	193	194	192
	SNOPT	optimal	22	1.00e-02	1.37e+06	25	1	24	1	0
	WORHP IP	optimal	220	2.10e-01	1.37e+06	222	222	221	1	220
	WORHP IPm	optimal	327	2.90e-01	1.37e+06	334	334	333	1	327
	WORHP SQP	optimal	51	3.70e-01	1.38e+06	532	533	32	2	31
QPNSTAIR	IPOPT	optimal	241	4.70e-01	5.15e+06	244	488	215	486	241
	KNITRO	optimal	131	3.20e-01	5.15e+06	134	135	133	134	131
	SNOPT	optimal	17	1.00e-01	5.15e+06	20	1	19	1	0
	WORHP IP	optimal	184	4.50e-01	5.15e+06	186	186	186	1	184
	WORHP IPm	optimal	176	4.20e-01	5.15e+06	183	183	182	1	176
	WORHP SQP	optimal	38	1.82e+00	5.15e+06	724	725	28	2	27
QR3D	IPOPT	optimal	10	9.90e-01	0.00e+00	54	54	11	11	10
	KNITRO	optimal	12	5.10e-01	0.00e+00	29	30	13	14	12
	SNOPT	optimal	8	6.00e-02	0.00e+00	1	11	1	10	0
	WORHP IP	optimal	13	5.00e-01	0.00e+00	36	36	14	14	13
	WORHP IPm	optimal	13	7.70e-01	0.00e+00	33	33	14	14	13
	WORHP SQP	optimal	13	2.57e+00	0.00e+00	134	134	15	15	13
QR3DBD	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	11	3.40e-01	0.00e+00	13	14	12	13	11
	SNOPT	optimal	10	1.10e+00	0.00e+00	1	16	1	15	0
	WORHP IP	optimal	13	1.35e+00	0.00e+00	36	36	14	14	13
	WORHP IPm	optimal	13	6.50e-01	0.00e+00	33	33	14	14	13
	WORHP SQP	optimal	14	8.76e+00	0.00e+00	241	240	6	16	5
QR3DLS	IPOPT	optimal	199	1.62e+01	7.28e-13	490	0	200	0	199
	KNITRO	optimal	190	1.41e+01	5.78e-16	297	0	191	0	190
	SNOPT	maxiter	10000	2.94e+01	3.44e-04	11191	0	11190	0	0
	WORHP IP	optimal	206	1.82e+01	2.61e-15	307	0	207	0	206
	WORHP IPm	optimal	204	2.09e+01	9.21e-20	341	0	205	0	204
	WORHP SQP	optimal	302	1.74e+01	4.07e-16	1697	0	303	0	302

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
QRTQUAD	IPOPT	optimal	377	2.84e+00	-2.65e+11	384	0	378	0	377
	KNITRO	optimal	559	4.07e+00	-2.65e+11	565	0	561	0	559
	SNOPT	unbound	6	7.10e-01	-3.34e+09	10	0	9	0	0
	WORHP IP	optimal	379	2.91e+00	-2.65e+11	443	0	381	0	379
	WORHP IPm	optimal	365	2.63e+00	-2.65e+11	376	0	370	0	365
	WORHP SQP	optimal	95	6.22e+00	-2.65e+11	112	0	96	0	95
QUARTC	IPOPT	optimal	34	1.20e-01	7.03e-07	35	0	35	0	34
	KNITRO	optimal	34	1.50e-01	7.03e-07	36	0	35	0	34
	SNOPT	toobig	104	7.41e+01	4.85e+16	120	0	119	0	0
	WORHP IP	optimal	34	1.60e-01	7.03e-07	36	0	36	0	34
	WORHP IPm	optimal	34	1.60e-01	7.24e-07	36	0	35	0	34
	WORHP SQP	optimal	54	2.60e-01	2.42e-06	55	0	55	0	54
QUDLIN	IPOPT	optimal	28	1.50e-01	-1.25e+09	29	0	29	0	28
	KNITRO	optimal	9	1.10e-01	-1.25e+09	12	0	11	0	9
	SNOPT	optimal	10	5.20e-01	-1.25e+09	20	0	19	0	0
	WORHP IP	optimal	18	1.60e-01	-1.25e+09	20	0	20	0	18
	WORHP IPm	optimal	14	1.30e-01	-1.25e+09	19	0	18	0	14
	WORHP SQP	maxiter	10000	1.00e+02	-1.25e+09	759956	0	10001	0	10001
RAT42	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	28	1.00e-02	0.00e+00	35	36	29	30	28
	SNOPT	infeas	10	1.00e-02	0.00e+00	1	22	1	21	0
	WORHP IP	infeas	7	1.00e-02	0.00e+00	11	11	8	8	8
	WORHP IPm	infeas	7	1.00e-02	0.00e+00	11	11	8	8	8
	WORHP SQP	minalpha	86	3.00e-02	0.00e+00	6636	6642	88	88	87
RAT42LS	IPOPT	optimal	28	1.00e-02	8.06e+00	39	0	29	0	28
	KNITRO	optimal	23	1.00e-02	8.06e+00	39	0	24	0	23
	SNOPT	optimal	22	1.00e-02	8.06e+00	45	0	44	0	0
	WORHP IP	optimal	28	1.00e-02	8.06e+00	34	0	29	0	28
	WORHP IPm	optimal	28	1.00e-02	8.06e+00	34	0	29	0	28
	WORHP SQP	optimal	1	1.00e-02	1.82e+04	9	0	2	0	1
RAT43	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	60	1.00e-02	0.00e+00	89	90	60	61	58
	SNOPT	degen	12	1.00e-02	0.00e+00	1	38	1	37	0
	WORHP IP	infeas	17	1.00e-02	0.00e+00	30	30	18	18	18
	WORHP IPm	infeas	17	1.00e-02	0.00e+00	30	30	18	18	18
	WORHP SQP	minalpha	5	4.00e-02	0.00e+00	4002	4008	6	7	5
RAT43LS	IPOPT	optimal	35	1.00e-02	8.79e+03	55	0	36	0	35
	KNITRO	optimal	15	1.00e-02	8.79e+03	29	0	16	0	15
	SNOPT	optimal	7	1.00e-02	1.08e+06	12	0	11	0	0
	WORHP IP	optimal	35	1.00e-02	8.79e+03	47	0	37	0	35
	WORHP IPm	optimal	35	1.00e-02	8.79e+03	47	0	36	0	35
	WORHP SQP	optimal	12	1.00e-02	1.08e+06	13	0	13	0	12
RAYBENDL	IPOPT	regular	32	1.90e-01	-7.35e+09	260	0	33	0	33
	KNITRO	smallstep	85	4.10e-01	-6.04e+17	475	0	86	0	85
	SNOPT	sbasics	10000	2.29e+02	9.66e+01	11296	0	11295	0	0
	WORHP IP	regular	39	1.90e-01	-2.11e+09	382	0	40	0	40
	WORHP IPm	regular	39	1.90e-01	-2.11e+09	382	0	40	0	40
	WORHP SQP	minalpha	93	1.53e+00	-4.30e+08	8397	0	94	0	94
RAYBENDS	IPOPT	regular	18	3.19e+00	-8.63e+08	39	0	19	0	19
	KNITRO	smallstep	23	4.25e+00	-1.96e+13	124	0	24	0	23
	SNOPT	sbasics	10000	3.44e+02	9.63e+01	11258	0	11257	0	0
	WORHP IP	regular	19	3.46e+00	-3.67e+09	25	0	20	0	20
	WORHP IPm	regular	19	3.41e+00	-3.67e+09	25	0	20	0	20
	WORHP SQP	minalpha	67	1.59e+01	-1.71e+10	2709	0	68	0	68
RDW2D51F	IPOPT	optimal	5	4.83e+01	1.74e-03	6	6	6	6	5
	KNITRO	optimal	2	1.74e+01	3.03e-03	5	6	4	5	2
	SNOPT	memory	0	4.18e+02	4.84e-03	3	3	2	2	0
	WORHP IP	optimal	4	2.88e+01	1.99e-03	6	6	5	5	4
	WORHP IPm	optimal	4	4.12e+01	1.72e-03	6	6	5	5	4
	WORHP SQP	optimal	4	1.39e+02	1.15e-03	5	5	6	6	4
RDW2D51U	IPOPT	optimal	1	1.93e+01	8.36e-04	2	2	2	2	1
	KNITRO	optimal	1	1.12e+01	8.36e-04	4	5	3	4	1
	SNOPT	memory	0	4.46e+02	4.75e-03	3	3	2	2	0
	WORHP IP	optimal	2	2.39e+01	8.31e-04	4	4	3	3	2
	WORHP IPm	optimal	2	2.43e+01	8.31e-04	4	4	3	3	2
	WORHP SQP	optimal	4	1.26e+02	8.93e-04	5	5	6	6	4
RDW2D52B	IPOPT	optimal	5	4.75e+01	1.20e-02	6	6	6	6	5
	KNITRO	optimal	5	4.13e+01	1.20e-02	8	9	7	8	5
	SNOPT	memory	0	2.66e+02	1.23e-02	3	3	2	2	0
	WORHP IP	optimal	9	8.73e+01	1.19e-02	11	11	11	11	9
	WORHP IPm	optimal	3	2.96e+01	1.20e-02	6	6	5	5	3
	WORHP SQP	optimal	3	1.97e+02	1.20e-02	4	4	5	5	3
RDW2D52F	IPOPT	optimal	5	4.87e+01	1.18e-02	6	6	6	6	5
	KNITRO	optimal	1	7.45e+00	1.18e-02	4	5	3	4	1
	SNOPT	memory	0	2.92e+02	1.23e-02	3	3	2	2	0
	WORHP IP	optimal	3	2.98e+01	1.20e-02	5	5	4	4	3
	WORHP IPm	optimal	3	2.04e+01	1.20e-02	6	6	5	5	3
	WORHP SQP	optimal	2	5.11e+01	1.20e-02	3	3	4	4	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
RDW2D52U	IPOPT	optimal	0	1.16e+01	1.23e-02	1	1	1	1	0
	KNITRO	optimal	0	4.30e+00	1.23e-02	3	4	2	3	0
	SNOPT	memory	0	3.84e+02	1.23e-02	3	3	2	2	0
	WORHP IP	optimal	1	1.54e+01	1.14e-02	3	3	2	2	1
	WORHP IPm	optimal	1	1.14e+01	1.14e-02	3	3	2	2	1
	WORHP SQP	optimal	1	2.34e+01	1.23e-02	2	2	3	3	1
READING1	IPOPT	optimal	19	1.60e-01	-1.60e-01	20	20	20	20	19
	KNITRO	optimal	17	3.00e-01	-1.60e-01	20	21	19	20	17
	SNOPT	optimal	57	2.15e+01	-1.60e-01	163	163	162	162	0
	WORHP IP	optimal	18	2.50e-01	-1.60e-01	20	20	19	19	18
	WORHP IPm	optimal	34	5.70e-01	-1.60e-01	62	62	36	36	34
	WORHP SQP	optimal	22	1.55e+00	-1.60e-01	23	23	24	24	22
READING2	IPOPT	optimal	9	1.20e-01	-1.24e-02	10	10	10	10	9
	KNITRO	optimal	4	1.40e-01	-1.23e-02	6	7	5	6	4
	SNOPT	optimal	0	4.10e-01	-1.25e-02	1	1	1	1	0
	WORHP IP	optimal	28	4.50e-01	-1.26e-02	30	30	29	1	28
	WORHP IPm	optimal	18	3.30e-01	-1.26e-02	20	20	19	1	18
	WORHP SQP	optimal	14	1.03e+00	-1.26e-02	197	198	16	3	14
READING3	IPOPT	optimal	17	1.80e-01	-1.52e-01	18	18	18	18	17
	KNITRO	optimal	16	2.90e-01	-1.53e-01	19	20	18	19	16
	SNOPT	optimal	21	9.50e-01	-1.53e-01	35	35	34	34	0
	WORHP IP	optimal	15	2.40e-01	-1.53e-01	17	17	17	17	15
	WORHP IPm	optimal	25	3.70e-01	-1.53e-01	27	27	26	26	25
	WORHP SQP	optimal	15	8.70e-01	-1.53e-01	16	16	17	17	15
READING4	IPOPT	optimal	313	5.89e+00	-2.91e-01	668	668	227	317	313
	KNITRO	optimal	84	2.75e+00	-2.91e-01	92	93	86	87	84
	SNOPT	optimal	34	4.64e+00	-2.91e-01	52	52	51	51	0
	WORHP IP	optimal	243	8.50e+00	-2.91e-01	294	294	244	244	243
	WORHP IPm	maxiter	10000	1.13e+03	-1.45e-01	105954	105954	10001	10001	10000
	WORHP SQP	minalpha	71	3.09e+01	6.15e-02	3363	3371	59	73	58
READING5	IPOPT	optimal	5	1.80e-01	-2.25e-17	6	6	6	6	5
	KNITRO	optimal	6	2.20e-01	-4.65e-15	9	10	8	9	6
	SNOPT	optimal	5	9.00e-02	-2.25e-17	8	8	7	7	0
	WORHP IP	infeas	211	4.75e+00	5.08e-02	284	284	212	212	212
	WORHP IPm	optimal	248	5.48e+00	0.00e+00	393	393	249	249	248
	WORHP SQP	optimal	11	1.07e+00	-7.62e-16	265	265	8	13	7
READING6	IPOPT	optimal	18	1.00e-02	-1.45e+02	19	19	19	19	18
	KNITRO	optimal	23	1.00e-02	-1.45e+02	26	27	25	26	23
	SNOPT	optimal	45	1.00e-02	-1.45e+02	59	59	58	58	0
	WORHP IP	optimal	15	1.00e-02	-1.45e+02	17	17	16	16	15
	WORHP IPm	optimal	17	1.00e-02	-1.45e+02	23	23	22	22	17
	WORHP SQP	optimal	18	1.20e-01	-1.45e+02	19	19	20	20	18
READING7	IPOPT	optimal	125	2.27e+01	-1.18e+03	127	127	126	126	125
	KNITRO	optimal	228	2.12e+01	-1.26e+03	231	232	230	231	228
	SNOPT	optimal	12	1.48e+00	-1.25e+03	25	25	24	24	0
	WORHP IP	optimal	154	2.35e+01	-1.11e+03	156	156	155	155	154
	WORHP IPm	optimal	129	3.23e+01	-1.17e+03	141	141	134	134	129
	WORHP SQP	optimal	118	2.69e+02	-1.10e+03	119	119	120	120	118
READING8	IPOPT	optimal	194	2.98e+02	-2.19e+03	195	195	195	195	194
	KNITRO	optimal	227	1.15e+02	-2.63e+03	230	231	229	230	227
	SNOPT	optimal	18	1.10e+01	-2.29e+03	38	38	37	37	0
	WORHP IP	optimal	190	2.33e+02	-2.02e+03	192	192	191	191	190
	WORHP IPm	optimal	174	2.16e+02	-2.15e+03	187	187	180	180	174
	WORHP SQP	maxtime	78	1.74e+03	-2.02e+03	79	79	80	80	79
READING9	IPOPT	optimal	40	8.00e-01	-4.40e-02	41	41	41	41	40
	KNITRO	optimal	78	1.59e+00	-4.43e-02	81	82	80	81	78
	SNOPT	optimal	29	1.70e+01	-4.22e-02	60	60	59	59	0
	WORHP IP	optimal	256	5.82e+00	-4.43e-02	258	258	257	257	256
	WORHP IPm	optimal	446	1.10e+01	-4.43e-02	448	448	447	447	446
	WORHP SQP	optimal	31	4.45e+00	-4.43e-02	32	32	33	33	31
RECIPE	IPOPT	optimal	13	1.00e-02	0.00e+00	14	14	14	14	13
	KNITRO	nan	0	1.00e-02	0.00e+00	3	4	1	2	0
	SNOPT	infeas	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	optimal	13	1.00e-02	0.00e+00	15	15	14	14	13
	WORHP IPm	optimal	13	1.00e-02	0.00e+00	15	15	14	14	13
	WORHP SQP	optimal	13	1.00e-02	0.00e+00	14	14	15	15	13
RES	IPOPT	optimal	11	1.00e-02	0.00e+00	20	40	12	24	11
	KNITRO	optimal	3	1.00e-02	0.00e+00	6	7	5	6	3
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	8	1.00e-02	0.00e+00	10	10	9	1	8
	WORHP IPm	optimal	6	1.00e-02	0.00e+00	10	10	9	1	6
	WORHP SQP	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
RK23	IPOPT	optimal	10	1.00e-02	8.33e-02	11	11	11	11	10
	KNITRO	optimal	6	1.00e-02	8.33e-02	9	10	8	9	6
	SNOPT	optimal	6	1.00e-02	8.33e-02	1	13	1	12	0
	WORHP IP	optimal	8	1.00e-02	8.33e-02	10	10	9	9	8
	WORHP IPm	optimal	16	1.00e-02	8.33e-02	21	21	20	20	16
	WORHP SQP	optimal	7	1.00e-02	8.33e-02	8	8	9	9	7

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ROBOT	IPOPT	optimal	20	1.00e-02	6.59e+00	26	26	18	23	20
	KNITRO	optimal	39	1.00e-02	5.84e+00	43	44	40	41	39
	SNOPT	optimal	18	1.00e-02	5.46e+00	30	30	29	29	0
	WORHP IP	optimal	22	1.00e-02	6.59e+00	24	24	23	23	22
	WORHP IPm	optimal	22	1.00e-02	6.59e+00	24	24	23	23	22
	WORHP SQP	optimal	94	2.00e-02	5.73e+00	1486	1484	83	98	81
ROBOTARM	IPOPT	optimal	554	9.01e+00	9.14e+00	713	713	553	559	554
	KNITRO	optimal	50	1.05e+00	9.14e+00	54	55	52	53	50
	SNOPT	maxiter	591	6.39e+02	3.05e+01	1	2902	1	2901	0
	WORHP IP	optimal	77	1.48e+00	9.14e+00	88	88	79	79	77
	WORHP IPm	optimal	1970	4.03e+01	9.14e+00	6175	6175	1974	1974	1970
	WORHP SQP	zerostep	242	3.05e+01	4.31e+01	5909	5912	66	245	65
ROCKET	IPOPT	optimal	45	3.80e-01	-1.01e+00	52	52	46	46	45
	KNITRO	optimal	26	3.30e-01	-1.01e+00	35	36	28	29	26
	SNOPT	optimal	2345	5.48e+01	-1.01e+00	1	10741	1	10740	0
	WORHP IP	optimal	543	8.18e+00	-1.01e+00	3485	3485	545	545	543
	WORHP IPm	optimal	1347	2.04e+01	-1.01e+00	9295	9295	1348	1348	1347
	WORHP SQP	minalpha	48	2.08e+01	-1.01e+00	2588	2594	50	50	49
ROSENBR	IPOPT	optimal	21	1.00e-02	3.74e-21	45	0	22	0	21
	KNITRO	optimal	21	1.00e-02	3.74e-21	30	0	22	0	21
	SNOPT	optimal	34	1.00e-02	1.17e-15	46	0	45	0	0
	WORHP IP	optimal	21	1.00e-02	3.74e-21	34	0	22	0	21
	WORHP IPm	optimal	21	1.00e-02	3.74e-21	34	0	22	0	21
	WORHP SQP	optimal	19	1.00e-02	6.25e-14	69	0	20	0	19
ROSENMMX	IPOPT	optimal	18	1.00e-02	-4.40e+01	22	47	19	19	18
	KNITRO	optimal	8	1.00e-02	-4.40e+01	15	16	9	10	8
	SNOPT	optimal	23	1.00e-02	-4.40e+01	1	43	1	42	0
	WORHP IP	optimal	9	1.00e-02	-4.40e+01	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	-4.40e+01	11	11	10	10	9
	WORHP SQP	optimal	14	1.00e-02	-4.40e+01	20	20	16	16	14
ROSEPETAL	IPOPT	optimal	27	2.56e+02	-3.07e+04	28	28	28	28	27
	KNITRO	optimal	35	1.41e+02	-3.07e+04	37	38	36	37	35
	SNOPT	optimal	78	6.22e+02	-3.07e+04	1	172	1	171	0
	WORHP IP	optimal	22	2.43e+02	-3.07e+04	24	24	23	23	22
	WORHP IPm	optimal	23	2.60e+02	-3.07e+04	25	25	24	24	23
	WORHP SQP	maxtime	1	9.62e+02	-6.42e+05	2	2	3	3	2
ROSEPETAL2	IPOPT	optimal	35	1.30e+00	-9.91e+05	56	112	36	72	35
	KNITRO	optimal	47	2.14e+00	-9.91e+05	49	50	48	49	47
	SNOPT	sbasics	21	1.46e+02	-1.56e+06	1	53	1	52	0
	WORHP IP	optimal	37	1.87e+00	-9.91e+05	41	41	39	39	37
	WORHP IPm	optimal	32	1.65e+00	-9.91e+05	34	34	33	33	32
	WORHP SQP	maxtime	17	1.88e+03	-3.98e+10	129	129	19	19	18
ROSZMAN1	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	57	2.00e-02	0.00e+00	217	218	58	59	57
	SNOPT	infeas	40	1.00e-02	0.00e+00	1	131	1	130	0
	WORHP IP	infeas	11	1.00e-02	0.00e+00	31	31	12	12	12
	WORHP IPm	infeas	11	1.00e-02	0.00e+00	31	31	12	12	12
	WORHP SQP	minalpha	27	6.00e-02	0.00e+00	3371	3378	26	29	25
ROSZMAN1LS	IPOPT	optimal	28	1.00e-02	4.95e-04	53	0	29	0	28
	KNITRO	optimal	28	1.00e-02	4.95e-04	33	0	29	0	28
	SNOPT	optimal	110	1.00e-02	3.97e-02	130	0	129	0	0
	WORHP IP	optimal	17	1.00e-02	4.95e-04	21	0	19	0	17
	WORHP IPm	optimal	28	1.00e-02	4.95e-04	42	0	29	0	28
	WORHP SQP	optimal	48	1.00e-02	4.95e-04	49	0	49	0	48
ROTDISC	IPOPT	optimal	76	2.60e-01	7.87e+00	103	206	77	154	76
	KNITRO	optimal	15	8.00e-02	7.87e+00	18	19	17	18	15
	SNOPT	optimal	7	1.20e-01	7.87e+00	1	12	1	11	0
	WORHP IP	optimal	38	1.60e-01	7.87e+00	88	88	39	39	38
	WORHP IPm	optimal	48	2.00e-01	7.87e+00	80	80	50	50	48
	WORHP SQP	optimal	438	1.09e+01	7.87e+00	11142	11137	233	445	231
RSNBRNE	IPOPT	optimal	1	1.00e-02	0.00e+00	3	3	2	2	1
	KNITRO	optimal	0	1.00e-02	0.00e+00	3	4	2	3	0
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	optimal	10	1.00e-02	0.00e+00	42	42	11	11	10
	WORHP IPm	optimal	10	1.00e-02	0.00e+00	42	42	11	11	10
	WORHP SQP	optimal	10	1.00e-02	0.00e+00	117	117	12	12	10
S268	IPOPT	optimal	17	1.00e-02	1.70e-07	18	18	18	18	17
	KNITRO	optimal	12	1.00e-02	1.95e-07	15	16	13	14	12
	SNOPT	optimal	45	1.00e-02	-2.55e-11	49	1	48	1	0
	WORHP IP	optimal	13	1.00e-02	1.06e-07	15	15	15	1	13
	WORHP IPm	optimal	12	1.00e-02	2.26e-07	14	14	13	1	12
	WORHP SQP	optimal	15	1.00e-02	7.28e-12	16	16	17	3	15
S277-280	IPOPT	optimal	12	1.00e-02	5.08e+00	13	13	13	13	12
	KNITRO	optimal	3	1.00e-02	5.08e+00	6	7	5	6	3
	SNOPT	optimal	0	1.00e-02	5.08e+00	1	1	1	1	0
	WORHP IP	optimal	9	1.00e-02	5.08e+00	11	11	10	1	9
	WORHP IPm	optimal	6	1.00e-02	5.08e+00	8	8	7	1	6
	WORHP SQP	optimal	2	1.00e-02	5.08e+00	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
S308	IPOPT	optimal	9	1.00e-02	7.73e-01	15	0	10	0	9
	KNITRO	optimal	9	1.00e-02	7.73e-01	12	0	10	0	9
	SNOPT	optimal	12	1.00e-02	7.73e-01	15	0	14	0	0
	WORHP IP	optimal	9	1.00e-02	7.73e-01	13	0	10	0	9
	WORHP IPm	optimal	9	1.00e-02	7.73e-01	13	0	10	0	9
	WORHP SQP	optimal	9	1.00e-02	7.73e-01	12	0	10	0	9
S316-322	IPOPT	optimal	7	1.00e-02	3.34e+02	8	8	8	8	7
	KNITRO	optimal	6	1.00e-02	3.34e+02	8	9	7	8	6
	SNOPT	optimal	6	1.00e-02	3.34e+02	13	13	12	12	0
	WORHP IP	regular	1	1.00e-02	0.00e+00	3	3	2	2	2
	WORHP IPm	regular	1	1.00e-02	0.00e+00	3	3	2	2	2
	WORHP SQP	optimal	7	1.00e-02	3.34e+02	8	8	9	9	7
S365	IPOPT	resto	1	1.00e-02	6.00e+00	2	2	2	2	2
	KNITRO	nan	0	1.00e-02	6.00e+00	3	4	2	3	0
	SNOPT	infeas	0	1.00e-02	6.00e+00	3	3	2	2	0
	WORHP IP	fritzjohn	279	4.00e-02	3.92e-18	1367	1367	280	280	280
	WORHP IPm	minalpha	41	1.00e-02	2.01e-06	507	507	78	78	42
	WORHP SQP	optimal	6	1.00e-02	0.00e+00	7	7	8	8	6
S365MOD	IPOPT	resto	1	1.00e-02	6.00e+00	2	2	2	2	2
	KNITRO	nan	0	1.00e-02	6.00e+00	3	4	2	3	0
	SNOPT	infeas	0	1.00e-02	6.00e+00	3	3	2	2	0
	WORHP IP	infeas	65	1.00e-02	2.50e-01	117	117	66	66	66
	WORHP IPm	minalpha	73	1.00e-02	1.60e-01	840	840	115	115	74
	WORHP SQP	minalpha	35	3.00e-02	2.50e-01	4334	4340	36	37	35
S368	IPOPT	optimal	11	1.00e-02	-7.50e-01	12	0	12	0	11
	KNITRO	optimal	7	1.00e-02	-7.50e-01	11	0	8	0	7
	SNOPT	optimal	10	1.00e-02	-7.50e-01	14	0	13	0	0
	WORHP IP	optimal	8	1.00e-02	-7.50e-01	11	0	9	0	8
	WORHP IPm	optimal	7	1.00e-02	-7.50e-01	12	0	11	0	7
	WORHP SQP	optimal	11	1.00e-02	-9.37e-01	12	0	12	0	11
SANTA	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	53	2.00e-02	0.00e+00	88	89	54	55	53
	SNOPT	infeas	43	1.00e-02	0.00e+00	1	97	1	96	0
	WORHP IP	infeas	16	1.00e-02	0.00e+00	30	30	17	17	17
	WORHP IPm	infeas	16	1.00e-02	0.00e+00	30	30	17	17	17
	WORHP SQP	minalpha	52	9.00e-02	0.00e+00	3358	3364	41	54	40
SANTALS	IPOPT	optimal	32	1.00e-02	1.22e-05	87	0	33	0	32
	KNITRO	optimal	27	1.00e-02	1.22e-05	42	0	28	0	27
	SNOPT	optimal	102	1.00e-02	1.22e-05	113	0	112	0	0
	WORHP IP	optimal	29	1.00e-02	1.22e-05	40	0	30	0	29
	WORHP IPm	optimal	31	1.00e-02	1.22e-05	46	0	32	0	31
	WORHP SQP	optimal	43	1.00e-02	1.22e-05	152	0	44	0	43
SARO	IPOPT	maxtime	7169	1.80e+03	2.51e+02	13374	13374	7170	7170	7169
	KNITRO	infeas	3528	1.04e+03	2.79e+02	12488	12489	3530	3531	3529
	SNOPT	optimal	170	8.24e+01	2.52e+02	1	581	1	580	0
	WORHP IP	maxtime	6448	1.80e+03	2.52e+02	7672	7672	6449	6449	6449
	WORHP IPm	maxtime	6839	1.80e+03	2.52e+02	7973	7973	6843	6843	6839
	WORHP SQP	minalpha	695	2.80e+02	2.52e+02	4139	4125	621	733	621
SAROMM	IPOPT	optimal	136	5.53e+01	8.83e+01	138	276	137	274	136
	KNITRO	optimal	63	2.78e+01	5.74e+01	66	67	65	66	63
	SNOPT	optimal	125	9.32e+01	5.74e+01	1	576	1	575	0
	WORHP IP	optimal	42	1.90e+01	5.74e+01	59	59	44	44	42
	WORHP IPm	optimal	48	1.87e+01	5.74e+01	63	63	52	52	48
	WORHP SQP	optimal	23	2.20e+01	5.74e+01	27	27	24	24	23
SAWPATH	IPOPT	resto	257	1.34e+00	6.36e+02	1771	3992	78	594	258
	KNITRO	optimal	53	2.80e-01	7.50e+02	75	76	55	56	54
	SNOPT	infeas	0	6.00e-02	1.96e+02	1	1	1	1	0
	WORHP IP	infeas	5486	1.02e+02	3.18e+02	47270	47270	5487	5487	5487
	WORHP IPm	infeas	3450	2.30e+01	3.17e+02	22497	22497	3451	3451	3451
	WORHP SQP	minalpha	223	2.15e+01	1.14e+03	4606	4612	189	225	188
SBRYBND	IPOPT	smallstep	16	3.30e-01	2.41e-26	22	0	17	0	17
	KNITRO	optimal	12	2.90e-01	8.90e-22	34	0	13	0	12
	SNOPT	toobig	1797	1.16e+02	7.50e+04	2017	0	2016	0	0
	WORHP IP	smallstep	27	6.20e-01	2.15e-26	32	0	29	0	28
	WORHP IPm	smallstep	13	3.10e-01	1.17e-20	18	0	14	0	14
	WORHP SQP	optimal	48	9.70e-01	7.72e-17	49	0	49	0	48
SCHMVETT	IPOPT	optimal	3	1.40e-01	-1.50e+04	4	0	4	0	3
	KNITRO	optimal	3	1.30e-01	-1.50e+04	5	0	4	0	3
	SNOPT	toobig	107	3.31e+01	-1.46e+04	121	0	120	0	0
	WORHP IP	optimal	3	1.40e-01	-1.50e+04	5	0	4	0	3
	WORHP IPm	optimal	3	1.30e-01	-1.50e+04	5	0	4	0	3
	WORHP SQP	optimal	3	1.30e-01	-1.50e+04	4	0	4	0	3
SCOND1LS	IPOPT	optimal	721	7.94e+00	4.98e-04	1961	0	722	0	721
	KNITRO	optimal	643	9.89e+00	1.89e-08	808	0	644	0	643
	SNOPT	sbasics	10000	2.16e+02	2.55e+01	12217	0	12216	0	0
	WORHP IP	optimal	840	1.11e+01	1.63e-08	1713	0	842	0	840
	WORHP IPm	optimal	673	1.09e+01	1.62e-08	986	0	674	0	673
	WORHP SQP	optimal	620	9.73e+00	6.58e-09	848	0	621	0	620

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
SCOSINE	IPOPT	optimal	130	1.21e+00	-5.00e+03	131	0	131	0	130
	KNITRO	noimpr	189	4.78e+00	-5.00e+03	205	0	190	0	190
	SNOPT	toobig	15	2.36e+01	4.20e+03	41	0	40	0	0
	WORHP IP	optimal	129	1.40e+00	-5.00e+03	131	0	131	0	129
	WORHP IPm	optimal	130	1.27e+00	-5.00e+03	132	0	131	0	130
	WORHP SQP	maxiter	10000	1.49e+02	-4.96e+03	64032	0	10001	0	10001
SCURLY10	IPOPT	optimal	94	3.46e+00	-1.00e+06	122	0	95	0	94
	KNITRO	optimal	76	2.55e+00	-1.00e+06	78	0	77	0	76
	SNOPT	unbound	0	1.40e-01	7.01e+31	3	0	2	0	0
	WORHP IP	optimal	94	4.00e+00	-1.00e+06	114	0	96	0	94
	WORHP IPm	optimal	94	3.34e+00	-1.00e+06	114	0	95	0	94
	WORHP SQP	minalpha	4	1.33e+00	6.51e+31	1259	0	5	0	5
SCURLY20	IPOPT	optimal	87	7.86e+00	-1.00e+06	95	0	88	0	87
	KNITRO	optimal	71	5.44e+00	-1.00e+06	73	0	72	0	71
	SNOPT	unbound	0	1.90e-01	9.03e+32	3	0	2	0	0
	WORHP IP	optimal	87	5.25e+00	-1.00e+06	93	0	89	0	87
	WORHP IPm	optimal	87	6.25e+00	-1.00e+06	93	0	88	0	87
	WORHP SQP	toobig	0	3.10e-01	9.03e+32	1	0	1	0	0
SCURLY30	IPOPT	optimal	74	9.41e+00	-1.00e+06	88	0	75	0	74
	KNITRO	optimal	67	6.51e+00	-1.00e+06	70	0	68	0	67
	SNOPT	unbound	0	2.30e-01	4.16e+33	3	0	2	0	0
	WORHP IP	optimal	74	1.33e+01	-1.00e+06	125	0	76	0	74
	WORHP IPm	optimal	74	7.64e+00	-1.00e+06	125	0	75	0	74
	WORHP SQP	toobig	0	5.30e-01	4.16e+33	1	0	1	0	0
SEMICN2U	IPOPT	optimal	23	3.10e-01	0.00e+00	52	79	24	24	23
	KNITRO	optimal	24	3.60e-01	0.00e+00	68	69	25	26	24
	SNOPT	degen	9	3.40e-01	0.00e+00	1	41	1	40	0
	WORHP IP	optimal	25	3.30e-01	0.00e+00	90	90	26	26	25
	WORHP IPm	optimal	25	5.20e-01	0.00e+00	90	90	26	26	25
	WORHP SQP	optimal	21	7.00e-01	0.00e+00	36	36	23	23	21
SEMICON1	IPOPT	optimal	58	7.00e-01	0.00e+00	59	126	59	59	58
	KNITRO	optimal	57	1.26e+00	0.00e+00	107	108	58	59	57
	SNOPT	optimal	111	3.85e+00	0.00e+00	1	704	1	703	0
	WORHP IP	optimal	876	1.75e+01	0.00e+00	1368	1368	878	878	876
	WORHP IPm	optimal	796	1.63e+01	0.00e+00	1199	1199	797	797	796
	WORHP SQP	optimal	60	2.13e+00	0.00e+00	101	101	62	62	60
SEMICON2	IPOPT	optimal	19	2.90e-01	0.00e+00	20	27	20	20	19
	KNITRO	optimal	19	4.70e-01	0.00e+00	24	25	20	21	19
	SNOPT	optimal	44	1.83e+00	0.00e+00	1	90	1	89	0
	WORHP IP	optimal	124	2.72e+00	0.00e+00	154	154	126	126	124
	WORHP IPm	optimal	136	3.09e+00	0.00e+00	204	204	137	137	136
	WORHP SQP	optimal	20	1.18e+00	0.00e+00	30	30	22	22	20
SENSORS	IPOPT	optimal	35	5.90e-01	-1.99e+03	41	0	36	0	35
	KNITRO	optimal	24	4.70e-01	-2.02e+03	28	0	25	0	24
	SNOPT	optimal	28	6.00e-01	-2.08e+03	44	0	43	0	0
	WORHP IP	optimal	35	5.90e-01	-1.99e+03	39	0	36	0	35
	WORHP IPm	optimal	35	6.30e-01	-1.99e+03	39	0	36	0	35
	WORHP SQP	optimal	23	8.60e-01	-2.09e+03	24	0	24	0	23
SIM2BQP	IPOPT	optimal	7	1.00e-02	8.09e-08	8	0	8	0	7
	KNITRO	optimal	4	1.00e-02	3.01e-09	7	0	6	0	4
	SNOPT	optimal	1	1.00e-02	0.00e+00	4	0	3	0	0
	WORHP IP	optimal	5	1.00e-02	4.59e-09	7	0	6	0	5
	WORHP IPm	optimal	5	1.00e-02	1.00e-12	7	0	6	0	5
	WORHP SQP	optimal	1	1.00e-02	6.52e-09	2	0	2	0	1
SIMBQP	IPOPT	optimal	7	1.00e-02	8.09e-08	8	0	8	0	7
	KNITRO	optimal	5	1.00e-02	3.94e-11	8	0	7	0	5
	SNOPT	optimal	5	1.00e-02	1.29e-40	8	0	7	0	0
	WORHP IP	optimal	6	1.00e-02	1.92e-10	8	0	7	0	6
	WORHP IPm	optimal	5	1.00e-02	1.04e-07	7	0	6	0	5
	WORHP SQP	optimal	2	1.00e-02	1.62e-12	3	0	3	0	2
SIMPLLPA	IPOPT	optimal	8	1.00e-02	1.00e+00	9	9	9	9	8
	KNITRO	optimal	4	1.00e-02	1.00e+00	6	7	5	6	4
	SNOPT	optimal	0	1.00e-02	1.00e+00	1	1	1	1	0
	WORHP IP	optimal	6	1.00e-02	1.00e+00	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	1.00e+00	10	10	9	1	6
	WORHP SQP	optimal	2	1.00e-02	1.00e+00	2	2	3	3	2
SIMPLLPB	IPOPT	optimal	10	1.00e-02	1.10e+00	11	11	11	11	10
	KNITRO	optimal	6	1.00e-02	1.10e+00	12	13	7	8	6
	SNOPT	optimal	0	1.00e-02	1.10e+00	1	1	1	1	0
	WORHP IP	optimal	7	1.00e-02	1.10e+00	9	9	8	1	7
	WORHP IPm	optimal	5	1.00e-02	1.10e+00	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	1.10e+00	2	2	3	3	2
SINEALI	IPOPT	optimal	25	6.00e-02	-9.99e+04	43	0	26	0	25
	KNITRO	optimal	6	1.00e-02	-9.99e+04	11	0	7	0	6
	SNOPT	optimal	123	7.17e+00	-9.99e+04	164	0	163	0	0
	WORHP IP	optimal	21	6.00e-02	-9.99e+04	27	0	22	0	21
	WORHP IPm	optimal	21	6.00e-02	-9.99e+04	27	0	22	0	21
	WORHP SQP	maxiter	10000	3.39e+01	-9.97e+04	10002	0	10001	0	10001

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
SINEVAL	IPOPT	optimal	42	1.00e-02	2.79e-42	110	0	43	0	42
	KNITRO	optimal	41	1.00e-02	7.09e-20	69	0	42	0	41
	SNOPT	optimal	60	1.00e-02	1.91e-17	90	0	89	0	0
	WORHP IP	optimal	42	1.00e-02	5.57e-42	78	0	43	0	42
	WORHP IPm	optimal	42	1.00e-02	5.57e-42	78	0	43	0	42
	WORHP SQP	optimal	40	1.00e-02	6.20e-24	227	0	41	0	40
SINQUAD	IPOPT	optimal	34	6.00e-01	-6.76e+06	198	0	35	0	34
	KNITRO	noimpr	12	2.20e-01	-6.76e+06	41	0	13	0	12
	SNOPT	toobig	56	7.63e+00	-6.76e+06	76	0	75	0	0
	WORHP IP	optimal	31	6.50e-01	-6.76e+06	147	0	33	0	31
	WORHP IPm	optimal	31	6.40e-01	-6.76e+06	147	0	32	0	31
	WORHP SQP	optimal	30	5.60e-01	-5.76e+06	93	0	31	0	30
SINROSNB	IPOPT	optimal	305	1.43e+00	1.88e+02	972	972	305	307	305
	KNITRO	optimal	101	3.30e-01	0.00e+00	117	118	102	103	101
	SNOPT	maxtime	9426	1.80e+03	6.92e-06	10200	10200	10199	10199	0
	WORHP IP	optimal	74	4.00e-01	1.41e+00	76	76	76	76	74
	WORHP IPm	optimal	85	4.40e-01	1.41e+00	87	87	86	86	85
	WORHP SQP	optimal	91	3.80e+00	8.01e+02	92	92	93	93	91
SINVALNE	IPOPT	optimal	1	1.00e-02	0.00e+00	3	3	2	2	1
	KNITRO	optimal	1	1.00e-02	0.00e+00	4	5	3	4	1
	SNOPT	optimal	1	1.00e-02	0.00e+00	1	4	1	3	0
	WORHP IP	optimal	15	1.00e-02	0.00e+00	71	71	16	16	15
	WORHP IPm	optimal	15	1.00e-02	0.00e+00	71	71	16	16	15
	WORHP SQP	optimal	7	1.00e-02	0.00e+00	61	61	9	9	7
SIPOW1	IPOPT	optimal	40	1.20e-01	-1.00e+00	45	45	41	41	40
	KNITRO	optimal	21	7.00e-02	-1.00e+00	23	24	22	23	21
	SNOPT	optimal	0	4.00e-02	-1.00e+00	1	1	1	1	0
	WORHP IP	optimal	115	4.30e-01	-1.00e+00	121	121	116	1	115
	WORHP IPm	optimal	128	4.90e-01	-1.00e+00	135	135	129	1	128
	WORHP SQP	smallstep	13	1.10e-01	-1.00e+00	13	13	14	3	13
SIPOW1M	IPOPT	optimal	96	2.40e-01	-1.00e+00	100	100	97	97	96
	KNITRO	optimal	22	7.00e-02	-1.00e+00	24	25	23	24	22
	SNOPT	optimal	0	4.00e-02	-1.00e+00	1	1	1	1	0
	WORHP IP	optimal	115	4.30e-01	-1.00e+00	118	118	116	1	115
	WORHP IPm	optimal	126	4.80e-01	-1.00e+00	129	129	127	1	126
	WORHP SQP	optimal	3	9.00e-02	-1.00e+00	237	238	4	3	3
SIPOW2	IPOPT	optimal	31	8.00e-02	-1.00e+00	34	34	32	32	31
	KNITRO	optimal	16	4.00e-02	-1.00e+00	18	19	17	18	16
	SNOPT	optimal	0	2.00e-02	-1.00e+00	1	1	1	1	0
	WORHP IP	optimal	95	3.50e-01	-1.00e+00	100	100	96	1	95
	WORHP IPm	optimal	125	4.90e-01	-1.00e+00	163	163	126	1	125
	WORHP SQP	optimal	3	8.00e-02	-1.00e+00	4	4	5	3	3
SIPOW2M	IPOPT	optimal	29	8.00e-02	-1.00e+00	37	37	30	30	29
	KNITRO	optimal	18	4.00e-02	-1.00e+00	20	21	19	20	18
	SNOPT	optimal	0	2.00e-02	-1.00e+00	1	1	1	1	0
	WORHP IP	optimal	103	3.70e-01	-1.00e+00	106	106	104	1	103
	WORHP IPm	optimal	115	4.10e-01	-1.00e+00	118	118	116	1	115
	WORHP SQP	optimal	2	7.00e-02	-1.00e+00	3	3	4	3	2
SIPOW3	IPOPT	optimal	15	5.00e-02	5.35e-01	20	20	16	16	15
	KNITRO	optimal	8	4.00e-02	5.35e-01	10	11	9	10	8
	SNOPT	optimal	0	2.00e-02	5.35e-01	1	1	1	1	0
	WORHP IP	optimal	11	6.00e-02	5.35e-01	13	13	12	1	11
	WORHP IPm	optimal	10	5.00e-02	5.35e-01	12	12	11	1	10
	WORHP SQP	optimal	2	1.10e-01	5.35e-01	3	3	4	3	2
SIPOW4	IPOPT	optimal	13	7.00e-02	2.72e-01	17	17	14	14	13
	KNITRO	optimal	9	3.00e-02	2.72e-01	11	12	10	11	9
	SNOPT	optimal	0	3.00e-02	2.72e-01	1	1	1	1	0
	WORHP IP	optimal	12	6.00e-02	2.72e-01	15	15	13	1	12
	WORHP IPm	optimal	14	7.00e-02	2.72e-01	16	16	15	1	14
	WORHP SQP	smallstep	4	1.10e-01	2.72e-01	4	4	5	3	4
SISSER	IPOPT	optimal	14	1.00e-02	4.16e-10	15	0	15	0	14
	KNITRO	optimal	14	1.00e-02	4.16e-10	16	0	15	0	14
	SNOPT	optimal	10	1.00e-02	2.26e-09	16	0	15	0	0
	WORHP IP	optimal	14	1.00e-02	4.16e-10	16	0	15	0	14
	WORHP IPm	optimal	14	1.00e-02	4.16e-10	16	0	15	0	14
	WORHP SQP	optimal	14	1.00e-02	4.17e-10	15	0	15	0	14
SMBANK	IPOPT	optimal	23	2.00e-02	-7.13e+06	24	24	24	24	23
	KNITRO	optimal	17	1.00e-02	-7.13e+06	20	21	19	20	17
	SNOPT	optimal	92	1.00e-02	-7.13e+06	109	1	108	1	0
	WORHP IP	optimal	15	1.00e-02	-7.13e+06	17	17	16	1	15
	WORHP IPm	optimal	16	1.00e-02	-7.13e+06	18	18	17	1	16
	WORHP SQP	maxiter	10000	1.89e+01	-2.06e+06	10001	10001	10002	3	10001
SMMPSF	IPOPT	optimal	526	3.08e+00	1.03e+06	530	1060	395	1056	526
	KNITRO	optimal	29	1.20e-01	1.03e+06	32	33	31	32	29
	SNOPT	optimal	10	5.00e-02	1.03e+06	1	22	1	21	0
	WORHP IP	optimal	1204	1.09e+01	1.03e+06	1208	1208	1206	1206	1204
	WORHP IPm	regular	753	6.38e+00	2.14e+06	759	759	755	755	754
	WORHP SQP	optimal	64	1.38e+00	1.03e+06	231	228	46	68	45

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
SNAIL	IPOPT	optimal	62	1.00e-02	8.24e-17	148	0	63	0	62
	KNITRO	optimal	66	1.00e-02	3.33e-22	91	0	67	0	66
	SNOPT	optimal	100	1.00e-02	3.16e-14	140	0	139	0	0
	WORHP IP	optimal	62	1.00e-02	3.02e-14	104	0	63	0	62
	WORHP IPm	optimal	62	1.00e-02	3.02e-14	104	0	63	0	62
	WORHP SQP	optimal	64	1.00e-02	9.75e-19	282	0	65	0	64
SNAKE	IPOPT	optimal	12	1.00e-02	-2.00e-04	14	14	13	13	12
	KNITRO	optimal	6	1.00e-02	-2.00e-06	8	9	7	8	6
	SNOPT	optimal	4	1.00e-02	-8.73e-14	1	8	1	7	0
	WORHP IP	maxiter	10000	1.74e+00	-7.18e+05	334425	334425	10001	10001	10000
	WORHP IPm	maxiter	10000	1.44e+00	-9.03e+02	239336	239336	10001	10001	10000
	WORHP SQP	minalpha	126	3.00e-02	-2.12e+03	5601	5609	105	129	104
SOSQP1	IPOPT	optimal	6	8.00e-02	-1.73e-11	8	8	7	7	6
	KNITRO	optimal	4	1.10e-01	5.69e-11	6	7	5	6	4
	SNOPT	optimal	0	3.00e-02	0.00e+00	3	1	2	1	0
	WORHP IP	optimal	5	7.00e-02	3.40e-08	7	7	6	1	5
	WORHP IPm	optimal	5	8.00e-02	-3.03e-11	9	9	8	1	5
	WORHP SQP	optimal	2	1.00e-01	1.78e-11	2	2	3	3	2
SOSQP2	IPOPT	optimal	13	9.00e-02	-1.25e+03	14	14	14	14	13
	KNITRO	optimal	12	2.20e-01	-1.25e+03	14	15	13	14	12
	SNOPT	optimal	23	1.65e+01	-1.25e+03	29	1	28	1	0
	WORHP IP	optimal	13	1.60e-01	-1.25e+03	15	15	14	1	13
	WORHP IPm	optimal	14	1.80e-01	-1.25e+03	16	16	15	1	14
	WORHP SQP	optimal	23	1.86e+00	-1.25e+03	24	24	25	3	23
SPANHYD	IPOPT	accept	27	1.00e-02	2.40e+02	29	29	28	28	28
	KNITRO	optimal	7	1.00e-02	2.40e+02	14	15	9	10	7
	SNOPT	optimal	11	1.00e-02	2.40e+02	16	1	15	1	0
	WORHP IP	optimal	43	2.00e-02	2.40e+02	390	390	45	1	43
	WORHP IPm	optimal	18	1.00e-02	2.40e+02	65	65	22	1	18
	WORHP SQP	optimal	10	5.00e-02	2.40e+02	11	11	12	3	10
SPARSINE	IPOPT	optimal	17	8.52e+01	1.53e-08	18	0	18	0	17
	KNITRO	optimal	48	3.64e+02	5.58e-14	53	0	49	0	48
	SNOPT	toobig	540	2.54e+01	2.12e+06	600	0	599	0	0
	WORHP IP	optimal	15	7.23e+01	2.06e-12	17	0	17	0	15
	WORHP IPm	optimal	17	1.14e+02	1.53e-08	19	0	18	0	17
	WORHP SQP	optimal	155	3.81e+02	8.14e-14	359	0	156	0	155
SPARSQUR	IPOPT	optimal	20	2.37e+02	1.15e-07	21	0	21	0	20
	KNITRO	optimal	20	5.49e+02	1.15e-07	22	0	21	0	20
	SNOPT	toobig	42	1.07e+01	6.99e+06	45	0	44	0	0
	WORHP IP	optimal	20	4.52e+02	1.15e-07	22	0	22	0	20
	WORHP IPm	optimal	20	3.61e+02	1.15e-07	22	0	21	0	20
	WORHP SQP	optimal	20	3.29e+02	1.21e-07	21	0	21	0	20
SPECAN	IPOPT	optimal	11	7.90e-01	1.65e-13	12	0	12	0	11
	KNITRO	optimal	9	7.40e-01	1.65e-13	11	0	10	0	9
	SNOPT	optimal	33	7.30e-01	1.65e-13	37	0	36	0	0
	WORHP IP	optimal	11	8.00e-01	1.66e-13	14	0	13	0	11
	WORHP IPm	optimal	10	7.70e-01	1.66e-13	14	0	11	0	10
	WORHP SQP	minalpha	206	3.96e+01	1.13e-12	33201	0	207	0	207
SPIN	IPOPT	optimal	6	1.50e-01	0.00e+00	7	7	7	7	6
	KNITRO	optimal	7	1.50e-01	0.00e+00	9	10	8	9	7
	SNOPT	optimal	9	2.20e-01	0.00e+00	1	13	1	12	0
	WORHP IP	optimal	7	1.60e-01	0.00e+00	9	9	8	8	7
	WORHP IPm	optimal	7	1.60e-01	0.00e+00	9	9	8	8	7
	WORHP SQP	optimal	8	5.10e-01	0.00e+00	9	9	10	10	8
SPIN2	IPOPT	optimal	4	6.00e-02	0.00e+00	5	5	5	5	4
	KNITRO	optimal	4	7.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	optimal	5	3.00e-02	0.00e+00	1	9	1	8	0
	WORHP IP	optimal	4	8.00e-02	0.00e+00	6	6	5	5	4
	WORHP IPm	optimal	4	8.00e-02	0.00e+00	6	6	5	5	4
	WORHP SQP	optimal	4	1.50e-01	0.00e+00	5	5	6	6	4
SPIN2OP	IPOPT	optimal	138	1.59e+00	3.27e-13	505	514	96	148	138
	KNITRO	optimal	36	5.00e-01	6.08e-18	89	90	37	38	36
	SNOPT	optimal	32	1.30e-01	6.32e-15	77	77	76	76	0
	WORHP IP	optimal	152	2.54e+00	2.19e-17	202	202	153	153	152
	WORHP IPm	optimal	152	1.34e+00	2.19e-17	202	202	153	153	152
	WORHP SQP	optimal	11	3.70e-01	9.66e-17	13	13	13	13	11
SPINOP	IPOPT	resto	211	6.52e+00	8.07e-05	1066	1140	118	216	212
	KNITRO	maxiter	10000	1.27e+02	1.16e-01	45420	45421	10001	10002	10000
	SNOPT	maxiter	10000	9.78e+02	4.79e-01	57494	57494	57493	57493	0
	WORHP IP	regular	21	6.50e-01	8.93e-07	23	23	22	22	22
	WORHP IPm	regular	21	6.40e-01	8.93e-07	23	23	22	22	22
	WORHP SQP	minalpha	602	2.71e+01	4.27e+00	12355	12352	107	613	106
SPIRAL	IPOPT	optimal	63	1.00e-02	1.72e-07	64	64	64	64	63
	KNITRO	optimal	116	1.00e-02	-8.06e-11	150	151	117	118	116
	SNOPT	optimal	77	1.00e-02	-1.09e-12	1	106	1	105	0
	WORHP IP	optimal	54	1.00e-02	3.45e-07	56	56	55	55	54
	WORHP IPm	optimal	55	1.00e-02	-2.05e-10	57	57	56	56	55
	WORHP SQP	maxiter	10000	1.08e+00	5.73e+01	90398	90365	9918	10035	9917

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
SPMSQRT	IPOPT	degree	0	8.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	6	3.60e-01	0.00e+00	8	9	7	8	6
	SNOPT	optimal	17	7.58e+00	0.00e+00	1	34	1	33	0
	WORHP IP	optimal	4	1.50e-01	0.00e+00	9	9	5	5	4
	WORHP IPm	optimal	4	1.50e-01	0.00e+00	9	9	5	5	4
	WORHP SQP	optimal	4	4.70e+01	0.00e+00	5	5	6	6	4
SPMSRTLS	IPOPT	optimal	22	3.20e-01	1.86e-15	41	0	23	0	22
	KNITRO	optimal	15	2.40e-01	1.86e-15	28	0	16	0	15
	SNOPT	toobig	1552	5.12e+01	1.09e+03	1704	0	1703	0	0
	WORHP IP	optimal	22	3.60e-01	1.86e-15	33	0	23	0	22
	WORHP IPm	optimal	22	2.90e-01	1.86e-15	33	0	23	0	22
	WORHP SQP	maxiter	10000	1.07e+02	9.24e+00	70309	0	10001	0	10001
SREADIN3	IPOPT	optimal	18	1.90e-01	-1.52e-01	19	19	19	19	18
	KNITRO	optimal	18	3.00e-01	-1.53e-01	20	21	19	20	18
	SNOPT	optimal	35	3.30e+00	-1.53e-01	124	124	123	123	0
	WORHP IP	optimal	37	6.00e-01	-1.53e-01	39	39	39	39	37
	WORHP IPm	optimal	28	4.30e-01	-1.53e-01	30	30	29	29	28
	WORHP SQP	optimal	15	1.06e+00	-1.53e-01	25	25	17	17	15
SROSENBR	IPOPT	optimal	8	7.00e-02	3.30e-22	14	0	9	0	8
	KNITRO	optimal	8	7.00e-02	3.30e-22	11	0	9	0	8
	SNOPT	toobig	26	4.62e+01	9.70e+03	31	0	30	0	0
	WORHP IP	optimal	8	6.00e-02	3.30e-22	12	0	10	0	8
	WORHP IPm	optimal	8	5.00e-02	3.30e-22	12	0	9	0	8
	WORHP SQP	optimal	8	8.00e-02	5.26e-12	9	0	9	0	8
SSBRYBND	IPOPT	optimal	22	4.00e-01	2.32e-21	31	0	23	0	22
	KNITRO	optimal	20	5.70e-01	1.33e-25	65	0	21	0	20
	SNOPT	toobig	398	6.59e+01	7.50e+04	473	0	472	0	0
	WORHP IP	optimal	27	4.80e-01	1.03e-18	34	0	29	0	27
	WORHP IPm	optimal	27	4.90e-01	1.60e-13	34	0	28	0	27
	WORHP SQP	optimal	27	6.90e-01	6.90e-14	28	0	28	0	27
SSC	IPOPT	optimal	2	2.00e-01	-2.08e+00	3	0	3	0	2
	KNITRO	optimal	2	1.90e-01	-2.08e+00	4	0	3	0	2
	SNOPT	toobig	474	2.86e+01	-2.01e+00	534	0	533	0	0
	WORHP IP	optimal	2	2.00e-01	-2.08e+00	4	0	3	0	2
	WORHP IPm	optimal	2	2.10e-01	-2.08e+00	4	0	3	0	2
	WORHP SQP	optimal	4	2.30e-01	-2.08e+00	5	0	5	0	4
SSCOSINE	IPOPT	optimal	71	5.70e-01	-5.00e+03	72	0	72	0	71
	KNITRO	noimpr	267	2.57e+00	-5.00e+03	477	0	268	0	267
	SNOPT	toobig	2663	9.25e+01	6.34e+02	2801	0	2800	0	0
	WORHP IP	optimal	71	7.20e-01	-5.00e+03	73	0	73	0	71
	WORHP IPm	optimal	71	7.10e-01	-5.00e+03	73	0	72	0	71
	WORHP SQP	maxiter	10000	1.14e+02	-4.82e+03	10033	0	10001	0	10001
SSEBLIN	IPOPT	optimal	64	2.00e-02	1.62e+07	65	130	65	130	64
	KNITRO	optimal	14	1.00e-02	1.62e+07	17	18	16	17	14
	SNOPT	optimal	0	1.00e-02	1.62e+07	1	1	1	1	0
	WORHP IP	optimal	61	2.00e-02	1.62e+07	63	63	63	1	61
	WORHP IPm	optimal	47	1.00e-02	1.62e+07	52	52	51	1	47
	WORHP SQP	optimal	51	1.60e-01	1.62e+07	279	280	53	3	51
SSEBNLN	IPOPT	resto	88	4.00e-02	1.62e+07	165	330	90	180	89
	KNITRO	optimal	63	1.10e-01	1.62e+07	68	69	65	66	63
	SNOPT	optimal	7	1.00e-02	1.62e+07	1	15	1	14	0
	WORHP IP	accept	232	2.60e-01	1.62e+07	2302	2302	235	235	233
	WORHP IPm	regular	282	1.30e-01	1.62e+07	307	307	286	286	283
	WORHP SQP	minalpha	1336	4.69e+00	1.72e+07	13333	13352	1334	1339	1333
SSI	IPOPT	maxiter	10000	2.63e+00	3.73e-11	26614	0	10001	0	10000
	KNITRO	maxiter	10000	1.00e-01	3.73e-11	14636	0	10001	0	10000
	SNOPT	optimal	3930	1.10e-01	2.57e-09	5355	0	5354	0	0
	WORHP IP	optimal	8883	3.30e-01	5.33e-11	15657	0	8884	0	8883
	WORHP IPm	optimal	8883	2.90e-01	5.33e-11	15657	0	8884	0	8883
	WORHP SQP	zerostep	5804	3.40e-01	1.85e-10	35992	0	5804	0	5804
SSINE	IPOPT	infeas	180	7.00e-02	0.00e+00	733	738	107	187	181
	KNITRO	maxiter	10000	2.50e-01	0.00e+00	40850	40851	10001	10002	10000
	SNOPT	optimal	1296	5.00e-02	0.00e+00	1	5086	1	5085	0
	WORHP IP	maxiter	10000	8.20e-01	0.00e+00	139338	139338	10001	10001	10000
	WORHP IPm	maxiter	10000	7.90e-01	0.00e+00	139338	139338	10001	10001	10000
	WORHP SQP	optimal	503	5.00e-02	0.00e+00	14006	14006	505	505	503
SSNLBEAM	IPOPT	optimal	163	9.60e-01	3.40e+02	185	185	164	164	163
	KNITRO	optimal	331	3.39e+00	3.40e+02	335	336	333	334	331
	SNOPT	optimal	1	1.00e-01	3.50e+02	4	4	3	3	0
	WORHP IP	optimal	236	2.26e+00	3.40e+02	239	239	237	237	236
	WORHP IPm	optimal	707	6.42e+00	3.40e+02	712	712	709	709	707
	WORHP SQP	optimal	34	2.23e+00	3.40e+02	35	35	36	36	34
STANCMIN	IPOPT	optimal	10	1.00e-02	4.25e+00	11	11	11	11	10
	KNITRO	optimal	6	1.00e-02	4.25e+00	8	9	7	8	6
	SNOPT	optimal	0	1.00e-02	4.25e+00	3	1	2	1	0
	WORHP IP	optimal	8	1.00e-02	4.25e+00	10	10	9	1	8
	WORHP IPm	optimal	8	1.00e-02	4.25e+00	11	11	10	1	8
	WORHP SQP	optimal	2	1.00e-02	4.25e+00	3	3	4	3	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
STATIC3	IPOPT	unbound	162	2.00e-01	-6.18e+44	163	163	163	163	162
	KNITRO	unbound	21	3.00e-02	-3.91e+20	24	25	23	24	21
	SNOPT	unbound	11	5.00e-02	-5.51e+09	19	1	18	1	0
	WORHP IP	unbound	35	5.00e-02	-1.07e+21	37	37	36	1	35
	WORHP IPm	diverge	35	5.00e-02	-1.07e+21	37	37	36	1	35
	WORHP SQP	unbound	324	4.85e+00	-1.14e+21	89594	89594	326	3	324
STCQP1	IPOPT	optimal	13	1.54e+01	3.67e+05	14	14	14	14	13
	KNITRO	optimal	12	9.70e-01	3.67e+05	15	16	14	15	12
	SNOPT	toobig	55	4.77e+01	3.68e+05	71	1	70	1	0
	WORHP IP	optimal	9	3.86e+01	3.67e+05	11	11	10	1	9
	WORHP IPm	optimal	10	4.95e+01	3.67e+05	16	16	15	1	10
	WORHP SQP	optimal	46	2.86e+02	3.67e+05	153	153	48	3	46
STCQP2	IPOPT	optimal	14	4.30e+00	3.72e+04	15	15	15	15	14
	KNITRO	optimal	11	3.66e+00	3.72e+04	14	15	13	14	11
	SNOPT	toobig	284	4.05e+01	3.77e+04	315	1	314	1	0
	WORHP IP	optimal	11	4.19e+00	3.72e+04	13	13	13	1	11
	WORHP IPm	optimal	10	3.66e+00	3.72e+04	16	16	15	1	10
	WORHP SQP	optimal	10	1.43e+01	3.72e+04	11	11	12	3	10
STEENBRA	IPOPT	optimal	2498	1.56e+01	1.70e+04	2506	2506	2495	2500	2498
	KNITRO	optimal	19	1.50e-01	1.70e+04	22	23	21	22	19
	SNOPT	optimal	19	1.00e-02	1.70e+04	25	1	24	1	0
	WORHP IP	optimal	29	1.00e-01	1.70e+04	33	33	31	1	29
	WORHP IPm	optimal	25	8.00e-02	1.70e+04	30	30	29	1	25
	WORHP SQP	optimal	65	1.32e+00	1.70e+04	66	66	67	3	65
STEENBRB	IPOPT	accept	9531	6.92e+01	1.31e+04	9644	9644	9525	9535	9532
	KNITRO	optimal	120	1.15e+00	1.11e+04	141	142	121	122	121
	SNOPT	optimal	153	5.90e-01	9.08e+03	210	1	209	1	0
	WORHP IP	optimal	83	7.80e-01	9.08e+03	91	91	85	1	83
	WORHP IPm	optimal	84	9.80e-01	9.08e+03	93	93	85	1	84
	WORHP SQP	maxiter	10000	1.70e+02	3.38e+04	10113	10112	10002	3	10001
STEENBRC	IPOPT	resto	3717	1.98e+01	1.76e+05	3768	3768	3701	3720	3718
	KNITRO	optimal	170	2.78e+00	2.75e+04	199	200	170	171	169
	SNOPT	optimal	503	2.44e+00	2.85e+04	717	1	716	1	0
	WORHP IP	optimal	113	2.41e+00	2.75e+04	167	167	114	1	113
	WORHP IPm	optimal	149	2.72e+00	2.85e+04	264	264	151	1	149
	WORHP SQP	maxiter	10000	3.31e+02	1.93e+06	10001	10001	10002	3	10001
STEENBRD	IPOPT	resto	506	3.26e+00	5.17e+05	527	527	500	509	507
	KNITRO	optimal	140	1.78e+00	9.03e+03	146	147	142	143	140
	SNOPT	optimal	221	5.80e-01	9.03e+03	326	1	325	1	0
	WORHP IP	optimal	152	1.78e+00	9.25e+03	255	255	154	1	152
	WORHP IPm	minalpha	141	1.10e+00	1.00e+00	2373	2373	226	1	142
	WORHP SQP	maxiter	10000	1.57e+02	1.00e+04	10641	10637	10002	3	10001
STEENBRE	IPOPT	accept	2883	4.09e+01	2.83e+04	3018	3018	2867	2886	2884
	KNITRO	optimal	177	3.25e+00	2.75e+04	254	255	179	180	177
	SNOPT	optimal	350	1.48e+00	2.85e+04	517	1	516	1	0
	WORHP IP	optimal	160	3.04e+00	2.75e+04	434	434	162	1	160
	WORHP IPm	optimal	166	3.25e+00	2.75e+04	272	272	169	1	166
	WORHP SQP	maxiter	10000	1.97e+02	3.04e+04	10001	10001	10002	3	10001
STEENBRF	IPOPT	resto	1302	8.20e+00	4.55e+04	1344	1344	1297	1307	1303
	KNITRO	optimal	153	1.72e+00	1.11e+04	177	178	154	155	154
	SNOPT	optimal	174	5.80e-01	8.99e+03	243	1	242	1	0
	WORHP IP	optimal	46	3.50e-01	8.99e+03	48	48	48	1	46
	WORHP IPm	optimal	83	7.70e-01	8.99e+03	100	100	86	1	83
	WORHP SQP	maxiter	10000	2.34e+02	1.11e+05	10001	10001	10002	3	10001
STEENBRG	IPOPT	accept	2612	2.60e+01	2.90e+04	2859	2859	2596	2615	2613
	KNITRO	optimal	121	2.44e+00	2.74e+04	139	140	121	122	120
	SNOPT	optimal	562	2.56e+00	2.83e+04	853	1	852	1	0
	WORHP IP	optimal	237	5.12e+00	2.74e+04	599	599	243	1	237
	WORHP IPm	optimal	156	3.24e+00	2.74e+04	258	258	159	1	156
	WORHP SQP	maxiter	10000	2.66e+02	2.91e+04	11106	11107	10002	3	10001
STEERING	IPOPT	optimal	16	1.20e-01	5.55e-01	22	22	17	17	16
	KNITRO	optimal	6	9.00e-02	5.55e-01	9	10	7	8	6
	SNOPT	optimal	51	3.55e+00	5.55e-01	1	87	1	86	0
	WORHP IP	optimal	17	1.50e-01	5.55e-01	20	20	19	19	17
	WORHP IPm	optimal	11	1.10e-01	5.55e-01	15	15	12	12	11
	WORHP SQP	optimal	48	1.11e+00	5.55e-01	52	52	50	50	48
STNQP1	IPOPT	optimal	18	9.71e+00	-3.12e+05	19	19	19	19	18
	KNITRO	optimal	6	7.00e-01	-3.12e+05	9	10	8	9	6
	SNOPT	toobig	25	5.55e+01	-3.11e+05	46	1	45	1	0
	WORHP IP	optimal	10	2.05e+00	-3.12e+05	12	12	11	1	10
	WORHP IPm	optimal	11	2.46e+01	-3.12e+05	16	16	15	1	11
	WORHP SQP	optimal	8	4.07e+01	-3.12e+05	9	9	10	3	8
STNQP2	IPOPT	optimal	20	1.04e+01	-5.75e+05	21	21	21	21	20
	KNITRO	optimal	7	3.35e+00	-5.75e+05	10	11	9	10	7
	SNOPT	toobig	24	3.67e+01	-5.75e+05	44	1	43	1	0
	WORHP IP	optimal	12	3.51e+00	-5.75e+05	14	14	13	1	12
	WORHP IPm	optimal	16	5.01e+00	-5.75e+05	21	21	20	1	16
	WORHP SQP	optimal	8	1.05e+01	-5.75e+05	9	9	10	3	8

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
STRATEC	IPOPT	optimal	24	3.26e+01	2.21e+03	56	0	25	0	24
	KNITRO	optimal	19	2.81e+01	2.21e+03	26	0	20	0	19
	SNOPT	optimal	204	9.68e+00	2.21e+03	225	0	224	0	0
	WORHP IP	optimal	24	3.61e+01	2.21e+03	42	0	26	0	24
	WORHP IPm	optimal	24	3.62e+01	2.21e+03	42	0	25	0	24
	WORHP SQP	optimal	37	3.28e+01	2.21e+03	52	0	38	0	37
SUPERSIM	IPOPT	optimal	1	1.00e-02	6.67e-01	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	6.67e-01	4	5	3	4	1
	SNOPT	optimal	0	1.00e-02	6.67e-01	1	1	1	1	0
	WORHP IP	optimal	5	1.00e-02	6.67e-01	7	7	6	1	5
	WORHP IPm	optimal	5	1.00e-02	6.67e-01	7	7	6	1	5
	WORHP SQP	optimal	2	1.00e-02	6.67e-01	2	2	3	3	2
SVANBERG	IPOPT	optimal	27	9.10e-01	8.36e+03	29	30	28	28	27
	KNITRO	optimal	16	7.80e-01	8.36e+03	19	20	17	18	16
	SNOPT	optimal	61	3.26e+01	8.36e+03	77	77	76	76	0
	WORHP IP	optimal	13	7.70e-01	8.36e+03	15	15	14	14	13
	WORHP IPm	optimal	15	9.00e-01	8.36e+03	17	17	16	16	15
	WORHP SQP	optimal	36	1.18e+01	8.36e+03	651	652	20	38	18
SWOPF	IPOPT	optimal	15	1.00e-02	6.79e-02	16	32	16	32	15
	KNITRO	optimal	9	1.00e-02	6.79e-02	11	12	10	11	9
	SNOPT	optimal	20	1.00e-02	6.79e-02	1	36	1	35	0
	WORHP IP	optimal	15	1.00e-02	6.79e-02	33	33	16	16	15
	WORHP IPm	optimal	13	1.00e-02	6.79e-02	19	19	14	14	13
	WORHP SQP	optimal	7	1.00e-02	6.79e-02	8	8	9	9	7
SYNPOP24	IPOPT	degree	0	9.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	0	9.00e-02	0.00e+00	2	3	1	2	0
	SNOPT	infeas	0	2.02e+02	0.00e+00	1	1	1	1	0
	WORHP IP	regular	102	1.77e+03	0.00e+00	318	318	103	1	103
	WORHP IPm	infeas	38	1.13e+02	0.00e+00	41	41	39	1	39
	WORHP SQP	minalpha	20	3.01e+02	0.00e+00	1807	1813	21	2	21
SYNTHES1	IPOPT	optimal	9	1.00e-02	7.59e-01	10	10	10	10	9
	KNITRO	optimal	6	1.00e-02	7.59e-01	9	10	8	9	6
	SNOPT	optimal	7	1.00e-02	7.59e-01	11	11	10	10	0
	WORHP IP	optimal	7	1.00e-02	7.59e-01	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	7.59e-01	12	12	11	11	7
	WORHP SQP	optimal	5	1.00e-02	7.59e-01	6	6	7	7	5
SYNTHES2	IPOPT	optimal	13	1.00e-02	-5.54e-01	14	28	14	28	13
	KNITRO	optimal	7	1.00e-02	-5.54e-01	10	11	9	10	7
	SNOPT	optimal	7	1.00e-02	-5.54e-01	11	11	10	10	0
	WORHP IP	optimal	18	1.00e-02	-5.54e-01	20	20	19	19	18
	WORHP IPm	optimal	16	1.00e-02	-5.54e-01	29	29	21	21	16
	WORHP SQP	optimal	6	1.00e-02	-5.54e-01	7	7	7	7	6
SYNTHES3	IPOPT	optimal	12	1.00e-02	1.51e+01	13	26	13	26	12
	KNITRO	optimal	8	1.00e-02	1.51e+01	11	12	10	11	8
	SNOPT	optimal	8	1.00e-02	1.51e+01	11	11	10	10	0
	WORHP IP	optimal	22	1.00e-02	1.51e+01	24	24	23	23	22
	WORHP IPm	optimal	18	1.00e-02	1.51e+01	26	26	22	22	18
	WORHP SQP	optimal	6	1.00e-02	1.51e+01	7	7	8	8	6
TABLE1	IPOPT	maxiter	10000	6.66e+01	4.45e+05	10036	10036	9610	10008	10000
	KNITRO	optimal	26	1.80e-01	3.71e+05	29	30	28	29	26
	SNOPT	optimal	788	1.04e+02	3.71e+05	938	1	937	1	0
	WORHP IP	optimal	17	2.10e-01	3.71e+05	19	19	19	1	17
	WORHP IPm	optimal	17	1.70e-01	3.71e+05	21	21	20	1	17
	WORHP SQP	maxiter	10000	4.70e+01	6.11e+10	17802	17844	10001	2	10001
TABLE3	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	30	3.66e+00	4.36e+04	33	34	32	33	30
	SNOPT	optimal	1709	9.69e+02	4.36e+04	1768	1	1767	1	0
	WORHP IP	optimal	18	2.60e+00	4.36e+04	20	20	19	1	18
	WORHP IPm	optimal	20	6.00e+00	4.36e+04	26	26	25	1	20
	WORHP SQP	maxiter	10000	1.26e+03	8.44e+07	10337	10334	9986	2	9986
TABLE4	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	30	3.66e+00	4.36e+04	33	34	32	33	30
	SNOPT	optimal	1709	1.01e+03	4.36e+04	1768	1	1767	1	0
	WORHP IP	optimal	18	4.95e+00	4.36e+04	20	20	19	1	18
	WORHP IPm	optimal	20	5.58e+00	4.36e+04	26	26	25	1	20
	WORHP SQP	maxiter	10000	1.48e+03	8.44e+07	10337	10334	9986	2	9986
TABLE5	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	30	3.68e+00	4.36e+04	33	34	32	33	30
	SNOPT	optimal	1709	9.73e+02	4.36e+04	1768	1	1767	1	0
	WORHP IP	optimal	18	4.89e+00	4.36e+04	20	20	19	1	18
	WORHP IPm	optimal	20	3.11e+00	4.36e+04	26	26	25	1	20
	WORHP SQP	maxiter	10000	1.30e+03	8.44e+07	10337	10334	9986	2	9986
TABLE6	IPOPT	maxiter	10000	5.73e+01	4.30e+05	10041	10041	9565	10009	10000
	KNITRO	optimal	26	1.70e-01	3.71e+05	29	30	28	29	26
	SNOPT	optimal	788	1.13e+02	3.71e+05	938	1	937	1	0
	WORHP IP	optimal	16	1.60e-01	3.71e+05	18	18	17	1	16
	WORHP IPm	optimal	18	2.30e-01	3.71e+05	23	23	22	1	18
	WORHP SQP	maxiter	10000	5.56e+01	6.11e+10	10001	10001	10001	2	10001

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
TABLE7	IPOPT	maxiter	10000	2.29e+01	6.07e+04	10001	10001	10001	10001	10000
	KNITRO	optimal	20	5.00e-02	5.96e+04	23	24	22	23	20
	SNOPT	optimal	530	2.74e+00	5.96e+04	557	1	556	1	0
	WORHP IP	optimal	19	7.00e-02	5.96e+04	21	21	20	1	19
	WORHP IPm	optimal	14	5.00e-02	5.96e+04	20	20	19	1	14
	WORHP SQP	maxiter	10000	1.84e+01	6.92e+06	10001	10001	10002	3	10001
TABLE8	IPOPT	optimal	12	3.00e-02	1.90e+00	13	13	13	13	12
	KNITRO	optimal	8	4.00e-02	1.90e+00	11	12	10	11	8
	SNOPT	optimal	110	6.66e+00	1.90e+00	126	1	125	1	0
	WORHP IP	optimal	10	4.00e-02	1.90e+00	12	12	11	1	10
	WORHP IPm	optimal	8	4.00e-02	1.90e+00	11	11	10	1	8
	WORHP SQP	maxiter	10000	4.03e+01	1.90e+00	10001	10001	10002	3	10001
TAME	IPOPT	optimal	5	1.00e-02	0.00e+00	6	6	6	6	5
	KNITRO	optimal	1	1.00e-02	1.23e-32	4	5	3	4	1
	SNOPT	optimal	3	1.00e-02	4.93e-32	7	1	6	1	0
	WORHP IP	optimal	4	1.00e-02	0.00e+00	6	6	5	1	4
	WORHP IPm	optimal	3	1.00e-02	0.00e+00	6	6	5	1	3
	WORHP SQP	optimal	1	1.00e-02	0.00e+00	2	2	3	3	1
TARGUS	IPOPT	optimal	76	6.00e-02	1.08e+03	77	77	77	77	76
	KNITRO	optimal	20	1.00e-02	1.08e+03	23	24	22	23	20
	SNOPT	optimal	409	9.00e-02	1.08e+03	420	1	419	1	0
	WORHP IP	optimal	11	1.00e-02	1.08e+03	13	13	12	1	11
	WORHP IPm	optimal	10	1.00e-02	1.08e+03	15	15	14	1	10
	WORHP SQP	maxiter	10000	5.33e+00	1.12e+03	10001	10001	10002	3	10001
TENBARS1	IPOPT	optimal	30	1.00e-02	2.30e+03	33	66	31	62	30
	KNITRO	optimal	15	1.00e-02	2.30e+03	18	19	17	18	15
	SNOPT	optimal	572	5.00e-02	2.30e+03	1	2140	1	2139	0
	WORHP IP	optimal	69	1.00e-02	2.30e+03	78	78	70	70	69
	WORHP IPm	optimal	77	1.00e-02	2.30e+03	95	95	81	81	77
	WORHP SQP	optimal	38	3.00e-02	2.30e+03	100	100	40	40	38
TENBARS2	IPOPT	optimal	30	1.00e-02	2.30e+03	34	34	31	31	30
	KNITRO	optimal	29	1.00e-02	2.30e+03	40	41	31	32	29
	SNOPT	optimal	217	1.00e-02	2.28e+03	1	577	1	576	0
	WORHP IP	optimal	83	1.00e-02	2.30e+03	96	96	84	84	83
	WORHP IPm	optimal	89	1.00e-02	2.28e+03	137	137	93	93	89
	WORHP SQP	optimal	33	2.00e-02	2.30e+03	43	43	35	35	33
TENBARS3	IPOPT	optimal	16	1.00e-02	2.25e+03	17	17	17	17	16
	KNITRO	optimal	31	1.00e-02	2.25e+03	40	41	33	34	31
	SNOPT	optimal	130	1.00e-02	2.25e+03	1	340	1	339	0
	WORHP IP	optimal	57	1.00e-02	2.25e+03	64	64	58	58	57
	WORHP IPm	optimal	58	1.00e-02	2.25e+03	76	76	62	62	58
	WORHP SQP	optimal	34	2.00e-02	2.25e+03	220	218	34	38	32
TENBARS4	IPOPT	optimal	17	1.00e-02	3.68e+02	18	36	18	36	17
	KNITRO	optimal	14	1.00e-02	3.68e+02	21	22	16	17	14
	SNOPT	optimal	672	5.00e-02	3.68e+02	1	2871	1	2870	0
	WORHP IP	optimal	43	1.00e-02	3.68e+02	47	47	44	44	43
	WORHP IPm	optimal	71	1.00e-02	3.68e+02	82	82	75	75	71
	WORHP SQP	optimal	50	2.00e-02	3.68e+02	262	261	52	53	50
TESTQUAD	IPOPT	optimal	1	3.00e-02	0.00e+00	2	0	2	0	1
	KNITRO	optimal	1	3.00e-02	3.39e-24	3	0	2	0	1
	SNOPT	toobig	38	2.12e+01	3.59e+08	41	0	40	0	0
	WORHP IP	optimal	1	2.00e-02	2.37e-24	3	0	2	0	1
	WORHP IPm	optimal	1	3.00e-02	2.37e-24	3	0	2	0	1
	WORHP SQP	optimal	4	4.00e-02	1.53e-23	5	0	5	0	4
TFI1	IPOPT	optimal	44	3.00e-02	5.33e+00	80	80	45	45	44
	KNITRO	optimal	7	1.00e-02	5.33e+00	9	10	8	9	7
	SNOPT	optimal	10	1.00e-02	5.33e+00	18	18	17	17	0
	WORHP IP	optimal	14	1.00e-02	5.33e+00	18	18	15	15	14
	WORHP IPm	optimal	66	3.00e-02	5.33e+00	402	402	67	67	66
	WORHP SQP	optimal	11	4.00e-02	5.33e+00	12	12	13	13	11
TFI2	IPOPT	optimal	12	1.00e-02	6.49e-01	15	15	13	13	12
	KNITRO	optimal	10	1.00e-02	6.49e-01	13	14	12	13	10
	SNOPT	optimal	0	1.00e-02	6.49e-01	1	1	1	1	0
	WORHP IP	optimal	9	1.00e-02	6.49e-01	11	11	10	1	9
	WORHP IPm	optimal	9	1.00e-02	6.49e-01	11	11	10	1	9
	WORHP SQP	optimal	2	1.00e-02	6.49e-01	2	2	3	3	2
TFI3	IPOPT	optimal	15	1.00e-02	4.30e+00	16	16	16	16	15
	KNITRO	optimal	11	1.00e-02	4.30e+00	14	15	13	14	11
	SNOPT	optimal	3	1.00e-02	4.30e+00	7	1	6	1	0
	WORHP IP	optimal	12	1.00e-02	4.30e+00	14	14	13	1	12
	WORHP IPm	optimal	21	1.00e-02	4.30e+00	24	24	22	1	21
	WORHP SQP	optimal	3	1.00e-02	4.30e+00	4	4	5	3	3
THURBER	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	infeas	38	1.00e-02	0.00e+00	86	87	38	39	37
	SNOPT	infeas	11	1.00e-02	0.00e+00	1	29	1	28	0
	WORHP IP	regular	28	1.00e-02	0.00e+00	42	42	29	29	29
	WORHP IPm	regular	28	1.00e-02	0.00e+00	42	42	29	29	29
	WORHP SQP	infeas	39	3.00e-02	0.00e+00	939	941	36	40	35

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
THURBERLS	IPOPT	optimal	19	1.00e-02	5.64e+03	37	0	20	0	19
	KNITRO	optimal	24	1.00e-02	5.64e+03	32	0	25	0	24
	SNOPT	optimal	145	1.00e-02	5.64e+03	162	0	161	0	0
	WORHP IP	optimal	19	1.00e-02	5.64e+03	27	0	21	0	19
	WORHP IPm	optimal	19	1.00e-02	5.64e+03	27	0	20	0	19
	WORHP SQP	optimal	31	1.00e-02	5.64e+03	32	0	32	0	31
TOINTGOR	IPOPT	optimal	7	1.00e-02	1.37e+03	8	0	8	0	7
	KNITRO	optimal	7	1.00e-02	1.37e+03	9	0	8	0	7
	SNOPT	optimal	131	1.00e-02	1.37e+03	134	0	133	0	0
	WORHP IP	optimal	7	1.00e-02	1.37e+03	9	0	8	0	7
	WORHP IPm	optimal	7	1.00e-02	1.37e+03	9	0	8	0	7
	WORHP SQP	optimal	5	1.00e-02	1.37e+03	6	0	6	0	5
TOINTGSS	IPOPT	optimal	1	6.00e-02	1.00e+01	2	0	2	0	1
	KNITRO	optimal	1	7.00e-02	1.00e+01	3	0	2	0	1
	SNOPT	toobig	43	7.00e+01	2.70e+04	47	0	46	0	0
	WORHP IP	optimal	1	6.00e-02	1.00e+01	3	0	2	0	1
	WORHP IPm	optimal	1	6.00e-02	1.00e+01	3	0	2	0	1
	WORHP SQP	optimal	15	2.60e-01	1.00e+01	52	0	16	0	15
TOINTPSP	IPOPT	optimal	20	1.00e-02	2.26e+02	83	0	21	0	20
	KNITRO	optimal	13	1.00e-02	2.26e+02	35	0	14	0	13
	SNOPT	optimal	36	1.00e-02	2.26e+02	57	0	56	0	0
	WORHP IP	optimal	17	1.00e-02	2.26e+02	52	0	18	0	17
	WORHP IPm	optimal	17	1.00e-02	2.26e+02	52	0	18	0	17
	WORHP SQP	optimal	21	1.00e-02	2.26e+02	309	0	22	0	21
TOINTQOR	IPOPT	optimal	1	1.00e-02	1.18e+03	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	1.18e+03	3	0	2	0	1
	SNOPT	optimal	43	1.00e-02	1.18e+03	46	0	45	0	0
	WORHP IP	optimal	1	1.00e-02	1.18e+03	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	1.18e+03	3	0	2	0	1
	WORHP SQP	optimal	4	1.00e-02	1.18e+03	5	0	5	0	4
TORSION1	IPOPT	optimal	11	3.40e-01	-4.30e-01	12	0	12	0	11
	KNITRO	optimal	8	3.20e-01	-4.30e-01	11	0	10	0	8
	SNOPT	toobig	244	1.35e+01	-4.17e-01	275	0	274	0	0
	WORHP IP	optimal	10	3.50e-01	-4.30e-01	12	0	11	0	10
	WORHP IPm	optimal	8	3.00e-01	-4.30e-01	10	0	9	0	8
	WORHP SQP	optimal	3	3.70e-01	-4.30e-01	4	0	4	0	3
TORSION2	IPOPT	optimal	9	3.00e-01	-4.30e-01	10	0	10	0	9
	KNITRO	optimal	7	2.80e-01	-4.30e-01	9	0	8	0	7
	SNOPT	toobig	481	2.24e+01	-5.83e-02	563	0	562	0	0
	WORHP IP	optimal	9	3.20e-01	-4.30e-01	11	0	10	0	9
	WORHP IPm	optimal	7	2.80e-01	-4.30e-01	9	0	8	0	7
	WORHP SQP	optimal	3	3.80e-01	-4.30e-01	4	0	4	0	3
TORSION3	IPOPT	optimal	10	3.70e-01	-1.22e+00	11	0	11	0	10
	KNITRO	optimal	8	3.10e-01	-1.22e+00	11	0	10	0	8
	SNOPT	optimal	105	9.06e+00	-1.22e+00	122	0	121	0	0
	WORHP IP	optimal	11	4.00e-01	-1.22e+00	13	0	12	0	11
	WORHP IPm	optimal	8	3.10e-01	-1.22e+00	10	0	9	0	8
	WORHP SQP	optimal	3	3.70e-01	-1.22e+00	4	0	4	0	3
TORSION4	IPOPT	optimal	10	3.10e-01	-1.22e+00	11	0	11	0	10
	KNITRO	optimal	7	2.90e-01	-1.22e+00	9	0	8	0	7
	SNOPT	toobig	1145	7.67e+01	-6.82e-01	1240	0	1239	0	0
	WORHP IP	optimal	9	3.40e-01	-1.22e+00	11	0	10	0	9
	WORHP IPm	optimal	10	3.30e-01	-1.22e+00	13	0	12	0	10
	WORHP SQP	optimal	3	3.90e-01	-1.22e+00	4	0	4	0	3
TORSION5	IPOPT	optimal	10	3.30e-01	-2.86e+00	11	0	11	0	10
	KNITRO	optimal	7	2.90e-01	-2.86e+00	10	0	9	0	7
	SNOPT	optimal	43	1.40e+00	-2.86e+00	48	0	47	0	0
	WORHP IP	optimal	10	3.60e-01	-2.86e+00	12	0	11	0	10
	WORHP IPm	optimal	8	3.20e-01	-2.86e+00	10	0	9	0	8
	WORHP SQP	optimal	2	2.90e-01	-2.86e+00	3	0	3	0	2
TORSION6	IPOPT	optimal	10	3.40e-01	-2.86e+00	11	0	11	0	10
	KNITRO	optimal	7	2.80e-01	-2.86e+00	9	0	8	0	7
	SNOPT	optimal	109	1.47e+02	-2.86e+00	125	0	124	0	0
	WORHP IP	optimal	8	3.20e-01	-2.86e+00	10	0	9	0	8
	WORHP IPm	optimal	8	3.00e-01	-2.86e+00	10	0	9	0	8
	WORHP SQP	optimal	3	3.40e-01	-2.86e+00	4	0	4	0	3
TORSIONA	IPOPT	optimal	11	4.20e-01	-4.18e-01	12	0	12	0	11
	KNITRO	optimal	8	3.40e-01	-4.18e-01	11	0	10	0	8
	SNOPT	toobig	245	1.33e+01	-4.04e-01	276	0	275	0	0
	WORHP IP	optimal	10	4.40e-01	-4.18e-01	12	0	11	0	10
	WORHP IPm	optimal	10	4.40e-01	-4.18e-01	12	0	11	0	10
	WORHP SQP	optimal	806	9.15e+01	-4.18e-01	105658	0	807	0	806
TORSIONB	IPOPT	optimal	9	3.40e-01	-4.18e-01	10	0	10	0	9
	KNITRO	optimal	7	3.30e-01	-4.18e-01	9	0	8	0	7
	SNOPT	toobig	507	2.85e+01	-4.82e-02	595	0	594	0	0
	WORHP IP	optimal	9	3.60e-01	-4.18e-01	11	0	10	0	9
	WORHP IPm	optimal	7	3.30e-01	-4.18e-01	9	0	8	0	7
	WORHP SQP	optimal	3	4.10e-01	-4.18e-01	4	0	4	0	3

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
TORSIONC	IPOPT	optimal	10	4.00e-01	-1.20e+00	11	0	11	0	10
	KNITRO	optimal	8	3.40e-01	-1.20e+00	11	0	10	0	8
	SNOPT	optimal	89	9.13e+00	-1.20e+00	102	0	101	0	0
	WORHP IP	optimal	11	4.40e-01	-1.20e+00	13	0	12	0	11
	WORHP IPm	optimal	7	3.20e-01	-1.20e+00	9	0	8	0	7
	WORHP SQP	optimal	2	5.30e-01	-1.20e+00	3	0	3	0	2
TORSIOND	IPOPT	optimal	9	3.60e-01	-1.20e+00	10	0	10	0	9
	KNITRO	optimal	7	3.10e-01	-1.20e+00	9	0	8	0	7
	SNOPT	toobig	1024	8.36e+01	-6.63e-01	1121	0	1120	0	0
	WORHP IP	optimal	9	4.20e-01	-1.20e+00	11	0	10	0	9
	WORHP IPm	optimal	7	3.10e-01	-1.20e+00	9	0	8	0	7
	WORHP SQP	optimal	3	4.10e-01	-1.20e+00	4	0	4	0	3
TORSIONE	IPOPT	optimal	10	3.90e-01	-2.85e+00	11	0	11	0	10
	KNITRO	optimal	7	3.30e-01	-2.85e+00	10	0	9	0	7
	SNOPT	optimal	43	1.52e+00	-2.85e+00	50	0	49	0	0
	WORHP IP	optimal	10	4.10e-01	-2.85e+00	12	0	11	0	10
	WORHP IPm	optimal	8	3.50e-01	-2.85e+00	10	0	9	0	8
	WORHP SQP	optimal	3	4.50e-01	-2.85e+00	135	0	4	0	3
TORSIONF	IPOPT	optimal	10	3.60e-01	-2.85e+00	11	0	11	0	10
	KNITRO	optimal	7	3.40e-01	-2.85e+00	9	0	8	0	7
	SNOPT	optimal	110	1.74e+02	-2.85e+00	134	0	133	0	0
	WORHP IP	optimal	9	3.80e-01	-2.85e+00	11	0	10	0	9
	WORHP IPm	optimal	8	3.20e-01	-2.85e+00	10	0	9	0	8
	WORHP SQP	optimal	3	3.60e-01	-2.85e+00	4	0	4	0	3
TOYSARAH	IPOPT	resto	1266	1.97e+01	5.14e+12	1277	1277	3	1268	1267
	KNITRO	smallstep	957	1.02e+02	5.74e+09	1376	1377	958	959	958
	SNOPT	infeas	0	8.00e-02	2.89e+03	1	1	1	1	0
	WORHP IP	infeas	171	1.03e+01	5.70e+09	794	794	172	1	172
	WORHP IPm	maxiter	10000	5.24e+02	5.70e+09	125695	125695	10333	1	10000
	WORHP SQP	minalpha	50	8.27e+00	1.69e+19	1730	1731	9	2	9
TQUARTIC	IPOPT	optimal	1	1.00e-01	1.30e-22	2	0	2	0	1
	KNITRO	optimal	1	1.20e-01	5.26e-26	3	0	2	0	1
	SNOPT	toobig	48	6.03e+00	8.04e-01	63	0	62	0	0
	WORHP IP	optimal	1	1.00e-01	6.04e-31	3	0	2	0	1
	WORHP IPm	optimal	1	9.00e-02	6.04e-31	3	0	2	0	1
	WORHP SQP	optimal	8	1.60e-01	4.30e-11	10	0	9	0	8
TRAINF	IPOPT	optimal	33	1.90e-01	3.10e+00	34	34	34	34	33
	KNITRO	optimal	25	3.00e-01	3.10e+00	28	29	27	28	25
	SNOPT	optimal	20	9.40e-01	3.10e+00	27	27	26	26	0
	WORHP IP	optimal	54	4.60e-01	3.10e+00	56	56	56	56	54
	WORHP IPm	optimal	123	8.30e-01	3.10e+00	126	126	125	125	123
	WORHP SQP	optimal	24	1.80e+00	3.10e+00	25	25	26	26	24
TRAINH	IPOPT	optimal	56	4.00e-01	1.23e+01	57	57	57	57	56
	KNITRO	optimal	36	4.30e-01	1.23e+01	44	45	38	39	36
	SNOPT	optimal	61	1.87e+00	1.23e+01	73	73	72	72	0
	WORHP IP	optimal	63	6.30e-01	1.23e+01	68	68	65	65	63
	WORHP IPm	optimal	124	1.16e+00	1.23e+01	148	148	127	127	124
	WORHP SQP	optimal	109	7.63e+00	1.23e+01	2830	2830	39	111	37
TRIDIA	IPOPT	optimal	1	4.00e-02	6.39e-25	2	0	2	0	1
	KNITRO	optimal	1	3.00e-02	9.77e-26	3	0	2	0	1
	SNOPT	toobig	83	6.30e+01	4.50e+06	93	0	92	0	0
	WORHP IP	optimal	1	3.00e-02	8.69e-25	3	0	2	0	1
	WORHP IPm	optimal	1	5.00e-02	8.69e-25	3	0	2	0	1
	WORHP SQP	optimal	5	7.00e-02	3.24e-20	6	0	6	0	5
TRIGGER	IPOPT	optimal	15	1.00e-02	0.00e+00	27	27	16	16	15
	KNITRO	optimal	174	1.00e-02	0.00e+00	978	979	175	176	174
	SNOPT	optimal	15	1.00e-02	0.00e+00	1	30	1	29	0
	WORHP IP	optimal	13	1.00e-02	0.00e+00	20	20	15	15	13
	WORHP IPm	optimal	15	1.00e-02	0.00e+00	22	22	16	16	15
	WORHP SQP	optimal	14	1.00e-02	0.00e+00	15	15	16	16	14
TRIMLOSS	IPOPT	optimal	29	1.00e-02	9.06e+00	30	60	30	60	29
	KNITRO	optimal	18	1.00e-02	9.06e+00	21	22	20	21	18
	SNOPT	optimal	25	1.00e-02	9.06e+00	1	39	1	38	0
	WORHP IP	optimal	22	1.00e-02	9.06e+00	26	26	23	23	22
	WORHP IPm	optimal	49	3.00e-02	9.06e+00	154	154	50	50	49
	WORHP SQP	optimal	12	4.00e-02	9.06e+00	13	13	14	14	12
TRO11X3	IPOPT	resto	1387	1.19e+00	-9.90e+14	1423	2846	1380	2778	1388
	KNITRO	smallstep	152	2.10e-01	-9.03e+07	212	213	152	153	151
	SNOPT	noimpr	10000	2.66e+00	-9.93e+07	1	19988	1	19987	0
	WORHP IP	infeas	111	9.00e-02	-4.77e+04	115	115	112	112	112
	WORHP IPm	minalpha	1441	1.33e+00	-4.70e+12	1592	1592	1482	1482	1442
	WORHP SQP	minalpha	392	6.17e+00	2.17e+01	7832	7846	378	394	377
TRO21X5	IPOPT	unbound	3489	9.75e+00	-1.05e+20	3666	7412	3367	6998	3489
	KNITRO	smallstep	1645	1.12e+01	-3.16e+07	4161	4162	1646	1647	1646
	SNOPT	noimpr	10000	1.36e+01	-9.91e+07	1	20019	1	20018	0
	WORHP IP	infeas	91	4.50e-01	-2.44e+04	130	130	92	92	92
	WORHP IPm	minalpha	726	3.16e+00	-7.52e+09	865	865	773	773	727
	WORHP SQP	minalpha	22	3.52e+00	4.26e+02	3142	3149	14	24	13

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
TRO3X3	IPOPT	optimal	13	1.00e-02	9.00e+00	21	42	14	28	13
	KNITRO	infeas	390	8.00e-02	-1.56e+03	1806	1807	391	392	391
	SNOPT	optimal	21	1.00e-02	9.00e+00	1	27	1	26	0
	WORHP IP	optimal	1407	5.10e-01	9.00e+00	21776	21776	1409	1409	1407
	WORHP IPm	optimal	25	1.00e-02	9.00e+00	29	29	28	28	25
	WORHP SQP	optimal	45	1.00e-01	9.00e+00	50	50	46	46	45
TRO41X9	IPOPT	maxiter	10000	1.60e+02	1.05e+03	12302	24604	9994	20004	10000
	KNITRO	maxiter	10000	2.00e+02	6.61e+02	20194	20195	10000	10001	10000
	SNOPT	maxiter	218	2.61e+02	2.81e+03	1	819	1	818	0
	WORHP IP	infeas	64	1.63e+00	-4.89e+04	80	80	65	65	65
	WORHP IPm	minalpha	464	1.27e+01	-3.35e+09	1819	1819	700	700	465
	WORHP SQP	minalpha	233	7.43e+01	2.53e+02	20369	20424	99	235	98
TRO4X4	IPOPT	optimal	16	1.00e-02	9.00e+00	34	68	17	34	16
	KNITRO	infeas	238	1.40e-01	-6.85e+01	872	873	239	240	240
	SNOPT	optimal	20	1.00e-02	9.00e+00	1	27	1	26	0
	WORHP IP	optimal	20	1.00e-02	9.00e+00	31	31	21	21	20
	WORHP IPm	maxiter	10000	4.31e+00	-2.02e+15	10030	10030	10003	10003	10000
	WORHP SQP	optimal	41	1.50e-01	9.00e+00	186	185	36	44	34
TRO5X5	IPOPT	optimal	18	1.00e-02	9.00e+00	42	84	19	38	18
	KNITRO	infeas	248	2.10e-01	-1.20e+02	500	501	249	250	249
	SNOPT	optimal	32	2.00e-02	9.00e+00	1	41	1	40	0
	WORHP IP	infeas	1224	1.57e+00	9.00e+00	20984	20984	1299	1299	1225
	WORHP IPm	maxiter	10000	9.90e+00	-2.33e+15	10010	10010	10003	10003	10000
	WORHP SQP	minalpha	161	2.79e+00	6.52e+00	6466	6481	133	163	132
TRO6X2	IPOPT	unbound	1797	1.75e+00	-1.38e+21	1833	4186	1767	3610	1797
	KNITRO	maxiter	10000	2.46e+00	-5.88e+14	10051	10052	10001	10002	10000
	SNOPT	optimal	107	2.00e-02	1.22e+03	1	117	1	116	0
	WORHP IP	maxiter	10000	2.91e+00	-8.97e+09	11787	11787	10001	10001	10000
	WORHP IPm	minalpha	450	1.50e-01	-1.86e+11	568	568	490	490	451
	WORHP SQP	minalpha	43	2.20e-01	1.15e+02	4542	4551	20	45	19
TRUSPYR1	IPOPT	optimal	13	1.00e-02	1.12e+01	15	30	14	28	13
	KNITRO	optimal	8	1.00e-02	1.12e+01	11	12	10	11	8
	SNOPT	optimal	21	1.00e-02	1.12e+01	1	32	1	31	0
	WORHP IP	optimal	12	1.00e-02	1.12e+01	20	20	13	13	12
	WORHP IPm	optimal	75	1.00e-02	1.12e+01	406	406	77	77	75
	WORHP SQP	optimal	7	1.00e-02	1.12e+01	8	8	9	9	7
TRUSPYR2	IPOPT	optimal	12	1.00e-02	1.12e+01	13	26	13	26	12
	KNITRO	optimal	8	1.00e-02	1.12e+01	11	12	10	11	8
	SNOPT	optimal	4	1.00e-02	1.12e+01	1	7	1	6	0
	WORHP IP	optimal	13	1.00e-02	1.12e+01	15	15	15	15	13
	WORHP IPm	optimal	48	1.00e-02	1.12e+01	190	190	49	49	48
	WORHP SQP	optimal	7	1.00e-02	1.12e+01	8	8	9	9	7
TRY-B	IPOPT	optimal	19	1.00e-02	2.07e-15	20	20	20	20	19
	KNITRO	optimal	9	1.00e-02	6.77e-21	11	12	10	11	9
	SNOPT	optimal	8	1.00e-02	1.00e+00	11	11	10	10	0
	WORHP IP	optimal	9	1.00e-02	1.36e-14	11	11	10	10	9
	WORHP IPm	optimal	9	1.00e-02	2.18e-17	11	11	10	10	9
	WORHP SQP	optimal	7	1.00e-02	1.00e+00	8	8	9	9	7
TWIRIBG1	IPOPT	resto	245	1.27e+03	-9.76e-01	679	1358	217	522	246
	KNITRO	optimal	83	2.33e+02	-1.04e+00	85	86	84	85	83
	SNOPT	optimal	49	8.51e+00	-1.05e+00	1	95	1	94	0
	WORHP IP	optimal	56	1.94e+02	-1.04e+00	58	58	57	57	56
	WORHP IPm	optimal	141	2.99e+02	-1.04e+00	458	458	144	144	141
	WORHP SQP	maxtime	1046	1.76e+03	-1.04e+00	1047	1047	1048	1048	1047
TWIRIMD1	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	optimal	162	4.11e+01	-1.03e+00	164	165	163	164	162
	SNOPT	optimal	33	1.33e+00	-1.03e+00	1	62	1	61	0
	WORHP IP	optimal	52	1.55e+01	-1.03e+00	54	54	53	53	52
	WORHP IPm	optimal	48	1.03e+01	-1.03e+00	104	104	49	49	48
	WORHP SQP	maxtime	5950	1.74e+03	-1.03e+00	6063	6062	5952	5953	5951
TWIRISM1	IPOPT	optimal	78	1.07e+00	-1.01e+00	147	294	79	160	78
	KNITRO	optimal	118	9.00e-01	-1.00e+00	153	154	120	121	118
	SNOPT	optimal	30	1.10e-01	-1.01e+00	1	60	1	59	0
	WORHP IP	optimal	47	5.40e-01	-1.00e+00	49	49	48	48	47
	WORHP IPm	optimal	48	4.80e-01	-1.01e+00	91	91	50	50	48
	WORHP SQP	optimal	4543	7.97e+01	-1.01e+00	4544	4544	4545	4545	4543
TWO5IN6	IPOPT	degree	0	8.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	11	8.62e+01	4.59e+03	14	15	13	14	11
	SNOPT	optimal	103	6.20e+00	4.59e+03	116	1	115	1	0
	WORHP IP	optimal	11	9.32e+02	4.59e+03	13	13	12	1	11
	WORHP IPm	optimal	10	3.37e+02	4.59e+03	14	14	13	1	10
	WORHP SQP	optimal	761	1.73e+03	4.59e+03	762	762	763	3	761
TWOBARS	IPOPT	optimal	9	1.00e-02	1.51e+00	10	10	10	10	9
	KNITRO	optimal	5	1.00e-02	1.51e+00	7	8	6	7	5
	SNOPT	optimal	8	1.00e-02	1.51e+00	15	15	14	14	0
	WORHP IP	optimal	8	1.00e-02	1.51e+00	10	10	9	9	8
	WORHP IPm	optimal	5	1.00e-02	1.51e+00	7	7	6	6	5
	WORHP SQP	optimal	10	1.00e-02	1.51e+00	11	11	12	12	10

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
TWOD	IPOPT	killed	-	-	-	-	-	-	-	-
	KNITRO	maxtime	9	1.87e+03	3.39e-03	12	13	11	12	9
	SNOPT	memory	0	1.39e+01	3.20e+03	1	1	1	1	0
	WORHP IP	maxtime	17	1.78e+03	8.63e-03	19	19	18	1	18
	WORHP IPm	maxtime	17	1.80e+03	9.50e-03	20	20	19	1	18
	WORHP SQP	maxtime	0	1.78e+03	8.01e-02	1	1	2	2	1
UBH1	IPOPT	optimal	5	1.60e-01	1.12e+00	6	6	6	6	5
	KNITRO	optimal	2	1.30e-01	1.12e+00	5	6	4	5	2
	SNOPT	sbasics	100	6.21e+01	1.96e+00	103	1	102	1	0
	WORHP IP	optimal	4	2.40e-01	1.12e+00	6	6	5	1	4
	WORHP IPm	optimal	4	2.60e-01	1.12e+00	6	6	5	1	4
	WORHP SQP	optimal	33	5.06e+00	1.33e+00	34	34	34	2	33
UBH5	IPOPT	optimal	5	1.00e-01	1.12e+00	6	6	6	6	5
	KNITRO	optimal	3	1.20e-01	1.12e+00	6	7	5	6	3
	SNOPT	degen	9	2.24e+00	-3.37e+01	1	19	1	18	0
	WORHP IP	optimal	5	1.20e-01	1.12e+00	7	7	6	6	5
	WORHP IPm	optimal	5	1.30e-01	1.12e+00	7	7	6	6	5
	WORHP SQP	optimal	41	4.82e+00	1.12e+00	421	422	42	42	41
VANDANIUMS	IPOPT	infeas	13	1.00e-02	0.00e+00	37	37	4	16	14
	KNITRO	infeas	8	1.00e-02	0.00e+00	42	43	9	10	9
	SNOPT	infeas	15	1.00e-02	0.00e+00	1	32	1	31	0
	WORHP IP	infeas	1674	3.10e-01	0.00e+00	1676	1676	1675	1675	1674
	WORHP IPm	infeas	1674	3.00e-01	0.00e+00	1676	1676	1675	1675	1674
	WORHP SQP	minalpha	6	3.00e-02	0.00e+00	2449	2455	6	8	5
VANDERM1	IPOPT	resto	2	2.00e-01	0.00e+00	3	6	3	6	3
	KNITRO	optimal	2232	6.67e+01	0.00e+00	5620	5621	2233	2234	2233
	SNOPT	optimal	3047	7.89e+00	0.00e+00	1	3191	1	3190	0
	WORHP IP	maxiter	10000	2.72e+02	0.00e+00	44699	44699	12793	12793	10000
	WORHP IPm	maxiter	10000	2.60e+02	0.00e+00	14760	14760	10001	10001	10000
	WORHP SQP	optimal	17	1.40e+00	0.00e+00	19	19	19	19	17
VANDERM2	IPOPT	resto	2	2.20e-01	0.00e+00	3	6	3	6	3
	KNITRO	optimal	2232	4.90e+01	0.00e+00	5620	5621	2233	2234	2233
	SNOPT	optimal	3047	8.30e+00	0.00e+00	1	3191	1	3190	0
	WORHP IP	maxiter	10000	3.03e+02	0.00e+00	44699	44699	12793	12793	10000
	WORHP IPm	maxiter	10000	2.51e+02	0.00e+00	14760	14760	10001	10001	10000
	WORHP SQP	optimal	17	1.38e+00	0.00e+00	19	19	19	19	17
VANDERM3	IPOPT	resto	2	2.20e-01	0.00e+00	3	6	3	6	3
	KNITRO	optimal	83	2.29e+00	0.00e+00	161	162	84	85	84
	SNOPT	maxiter	10000	3.18e+01	0.00e+00	1	10397	1	10396	0
	WORHP IP	optimal	8127	3.51e+02	0.00e+00	102673	102673	19104	19104	8127
	WORHP IPm	fritzjohn	2282	9.56e+01	0.00e+00	23918	23918	4092	4092	2283
	WORHP SQP	optimal	25	1.55e+00	0.00e+00	27	27	27	27	25
VANDERM4	IPOPT	resto	1	1.50e-01	0.00e+00	2	4	2	4	1
	KNITRO	noimpr	2	1.50e-01	0.00e+00	23	24	1	2	1
	SNOPT	infeas	0	2.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	toobig	0	6.00e-02	0.00e+00	2	2	1	1	0
	WORHP IPm	toobig	0	6.00e-02	0.00e+00	2	2	1	1	0
	WORHP SQP	toobig	0	6.00e-02	0.00e+00	1	1	1	1	0
VARDIM	IPOPT	optimal	29	1.10e-01	1.42e-24	30	0	30	0	29
	KNITRO	optimal	29	1.10e-01	2.62e-24	31	0	30	0	29
	SNOPT	optimal	150	4.00e-02	1.96e-12	188	0	187	0	0
	WORHP IP	optimal	29	1.10e-01	2.91e-24	31	0	31	0	29
	WORHP IPm	optimal	29	1.20e-01	2.91e-24	31	0	30	0	29
	WORHP SQP	optimal	31	3.10e-01	8.80e-18	32	0	32	0	31
VARDIMNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	0	1.00e-02	0.00e+00	3	4	2	3	0
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	3	1	2	0
	WORHP IP	optimal	7	1.00e-02	0.00e+00	9	9	8	8	7
	WORHP IPm	optimal	7	1.00e-02	0.00e+00	9	9	8	8	7
	WORHP SQP	optimal	7	1.00e-02	0.00e+00	8	8	8	8	7
VAREIGVL	IPOPT	optimal	11	1.00e-02	3.14e-13	12	0	12	0	11
	KNITRO	optimal	7	1.00e-02	8.57e-18	9	0	8	0	7
	SNOPT	optimal	21	1.00e-02	2.46e-12	24	0	23	0	0
	WORHP IP	optimal	11	1.00e-02	3.14e-13	13	0	12	0	11
	WORHP IPm	optimal	11	1.00e-02	3.14e-13	13	0	12	0	11
	WORHP SQP	optimal	6	1.00e-02	3.58e-18	11	0	7	0	6
VESUVIA	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	1085	1.17e+01	0.00e+00	1941	1942	1086	1087	1085
	SNOPT	infeas	19	2.70e-01	0.00e+00	1	67	1	66	0
	WORHP IP	infeas	89	5.90e-01	0.00e+00	199	199	90	90	90
	WORHP IPm	infeas	89	4.90e-01	0.00e+00	199	199	90	90	90
	WORHP SQP	minalpha	9	1.00e+01	0.00e+00	2466	2472	6	11	5
VESUVIALS	IPOPT	optimal	48	1.80e-01	9.91e+02	67	0	49	0	48
	KNITRO	optimal	62	2.30e-01	1.50e+03	97	0	63	0	62
	SNOPT	toobig	123	1.30e-01	2.28e+03	152	0	151	0	0
	WORHP IP	optimal	65	1.90e-01	9.91e+02	69	0	67	0	65
	WORHP IPm	optimal	84	2.50e-01	9.91e+02	88	0	85	0	84
	WORHP SQP	optimal	87	2.90e-01	9.91e+02	100	0	88	0	87

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
VESUVIO	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	280	1.72e+00	0.00e+00	290	291	280	281	279
	SNOPT	infeas	12	1.90e-01	0.00e+00	1	25	1	24	0
	WORHP IP	infeas	18	1.60e-01	0.00e+00	103	103	19	19	19
	WORHP IPm	infeas	18	1.60e-01	0.00e+00	103	103	19	19	19
	WORHP SQP	infeas	17	9.91e+00	0.00e+00	1151	1153	7	18	6
VESUVIOLS	IPOPT	optimal	10	6.00e-02	9.91e+02	23	0	11	0	10
	KNITRO	optimal	9	6.00e-02	9.91e+02	13	0	10	0	9
	SNOPT	optimal	250	2.40e-01	9.91e+02	273	0	272	0	0
	WORHP IP	optimal	10	8.00e-02	9.91e+02	69	0	12	0	10
	WORHP IPm	optimal	10	7.00e-02	9.91e+02	69	0	11	0	10
	WORHP SQP	optimal	37	1.40e-01	9.91e+02	38	0	38	0	37
VESUVIOU	IPOPT	degree	0	3.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	smallstep	45	3.50e-01	0.00e+00	86	87	44	45	43
	SNOPT	infeas	12	1.60e-01	0.00e+00	1	27	1	26	0
	WORHP IP	infeas	23	1.40e-01	0.00e+00	41	41	24	24	24
	WORHP IPm	infeas	23	1.30e-01	0.00e+00	41	41	24	24	24
	WORHP SQP	infeas	13	7.20e+00	0.00e+00	952	954	9	14	8
VESUVIOULS	IPOPT	optimal	8	5.00e-02	4.77e-01	15	0	9	0	8
	KNITRO	optimal	10	6.00e-02	4.77e-01	17	0	11	0	10
	SNOPT	optimal	142	1.50e-01	4.77e-01	162	0	161	0	0
	WORHP IP	optimal	8	5.00e-02	4.77e-01	13	0	10	0	8
	WORHP IPm	optimal	8	4.00e-02	4.77e-01	13	0	9	0	8
	WORHP SQP	optimal	36	1.30e-01	4.77e-01	37	0	37	0	36
VIBRBEAM	IPOPT	optimal	59	3.00e-02	3.32e-01	75	0	60	0	59
	KNITRO	optimal	44	1.00e-02	1.02e+01	84	0	45	0	44
	SNOPT	optimal	255	2.00e-02	1.75e+00	278	0	277	0	0
	WORHP IP	optimal	36	1.00e-02	1.56e-01	48	0	38	0	36
	WORHP IPm	optimal	36	1.00e-02	1.56e-01	48	0	37	0	36
	WORHP SQP	optimal	70	2.00e-02	1.56e-01	95	0	71	0	70
WACHBIEG	IPOPT	infeas	14	1.00e-02	-1.00e+00	24	24	10	18	15
	KNITRO	optimal	29	1.00e-02	1.00e+00	35	36	30	31	29
	SNOPT	optimal	4	1.00e-02	1.00e+00	1	7	1	6	0
	WORHP IP	optimal	17	1.00e-02	1.00e+00	21	21	18	18	17
	WORHP IPm	optimal	17	1.00e-02	1.00e+00	23	23	19	19	17
	WORHP SQP	optimal	13	1.00e-02	1.00e+00	479	480	14	14	13
WALL10	IPOPT	optimal	41	5.50e-01	-4.56e+05	721	0	42	0	41
	KNITRO	optimal	19	1.40e-01	-4.56e+05	23	0	21	0	19
	SNOPT	maxiter	10000	1.69e+02	-6.53e+01	11140	0	11139	0	0
	WORHP IP	optimal	26	2.40e-01	-4.56e+05	203	0	28	0	26
	WORHP IPm	optimal	47	5.10e-01	-4.56e+05	679	0	50	0	47
	WORHP SQP	optimal	45	8.20e-01	-4.56e+05	46	0	46	0	45
WALL100	IPOPT	optimal	15	2.71e+01	-8.95e+03	16	0	16	0	15
	KNITRO	optimal	13	2.19e+01	-8.95e+03	16	0	15	0	13
	SNOPT	toobig	485	1.19e+02	-4.03e-01	548	0	547	0	0
	WORHP IP	optimal	14	3.28e+01	-8.95e+03	16	0	16	0	14
	WORHP IPm	optimal	19	3.79e+01	-8.95e+03	22	0	21	0	19
	WORHP SQP	optimal	19	1.59e+02	-8.95e+03	23	0	20	0	19
WALL20	IPOPT	optimal	59	3.79e+00	-5.22e+06	1107	0	60	0	59
	KNITRO	smallstep	23	5.10e+00	-5.22e+06	95	0	25	0	23
	SNOPT	toobig	2069	6.72e+01	-2.38e+00	2334	0	2333	0	0
	WORHP IP	optimal	51	2.64e+00	-5.22e+06	595	0	58	0	51
	WORHP IPm	optimal	35	1.24e+00	-5.22e+06	160	0	36	0	35
	WORHP SQP	minalpha	183	4.35e+01	-5.22e+06	15003	0	184	0	184
WALL50	IPOPT	optimal	74	2.47e+01	-9.55e+06	322	0	75	0	74
	KNITRO	optimal	40	6.06e+00	-9.55e+06	43	0	42	0	40
	SNOPT	toobig	1739	1.05e+02	-6.41e-01	1976	0	1975	0	0
	WORHP IP	optimal	55	1.50e+01	-9.55e+06	517	0	57	0	55
	WORHP IPm	optimal	58	1.86e+01	-9.55e+06	64	0	63	0	58
	WORHP SQP	optimal	46	7.41e+01	-9.55e+06	1594	0	46	0	46
WATER	IPOPT	optimal	21	1.00e-02	1.05e+04	22	22	22	22	21
	KNITRO	optimal	13	1.00e-02	1.05e+04	16	17	15	16	13
	SNOPT	optimal	17	1.00e-02	1.05e+04	27	1	26	1	0
	WORHP IP	optimal	24	1.00e-02	1.05e+04	26	26	25	1	24
	WORHP IPm	optimal	25	1.00e-02	1.05e+04	30	30	29	1	25
	WORHP SQP	optimal	72	2.00e-02	1.05e+04	73	73	74	3	72
WATSON	IPOPT	optimal	16	1.00e-02	3.68e-07	17	0	17	0	16
	KNITRO	optimal	17	1.00e-02	3.30e-07	19	0	18	0	17
	SNOPT	optimal	79	1.00e-02	1.56e-07	86	0	85	0	0
	WORHP IP	optimal	16	1.00e-02	3.68e-07	18	0	17	0	16
	WORHP IPm	optimal	16	1.00e-02	3.68e-07	18	0	17	0	16
	WORHP SQP	optimal	19	1.00e-02	2.09e-07	20	0	20	0	19
WATSONNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	4	1.00e-02	0.00e+00	6	7	5	6	4
	SNOPT	optimal	4	1.00e-02	0.00e+00	1	8	1	7	0
	WORHP IP	optimal	4	1.00e-02	0.00e+00	6	6	5	5	4
	WORHP IPm	optimal	4	1.00e-02	0.00e+00	6	6	5	5	4
	WORHP SQP	optimal	4	1.00e-02	0.00e+00	5	5	6	6	4

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
WEEDS	IPOPT	optimal	25	1.00e-02	2.59e+00	41	0	26	0	25
	KNITRO	optimal	22	1.00e-02	2.59e+00	35	0	23	0	22
	SNOPT	optimal	39	1.00e-02	2.59e+00	54	0	53	0	0
	WORHP IP	optimal	29	1.00e-02	2.59e+00	56	0	31	0	29
	WORHP IPm	optimal	34	1.00e-02	2.59e+00	43	0	36	0	34
	WORHP SQP	optimal	37	1.00e-02	2.59e+00	41	0	38	0	37
WOMFLET	IPOPT	optimal	11	1.00e-02	6.05e+00	12	12	12	12	11
	KNITRO	optimal	7	1.00e-02	6.05e+00	10	11	8	9	7
	SNOPT	optimal	15	1.00e-02	-2.95e-16	1	29	1	28	0
	WORHP IP	optimal	25	1.00e-02	6.05e+00	27	27	26	26	25
	WORHP IPm	optimal	20	1.00e-02	6.05e+00	30	30	21	21	20
	WORHP SQP	optimal	18	1.00e-02	1.53e-09	20	20	20	20	18
WOODS	IPOPT	optimal	40	2.00e-01	4.84e-24	85	0	41	0	40
	KNITRO	optimal	40	1.40e-01	3.04e-15	52	0	41	0	40
	SNOPT	toobig	35	6.73e+01	3.52e+05	39	0	38	0	0
	WORHP IP	optimal	40	1.60e-01	4.84e-24	62	0	42	0	40
	WORHP IPm	optimal	40	1.50e-01	4.84e-24	62	0	41	0	40
	WORHP SQP	optimal	37	2.50e-01	2.83e-13	201	0	38	0	37
WOODSNE	IPOPT	infeas	9	1.00e-01	-8.93e+03	36	38	7	12	10
	KNITRO	noimpr	5	6.00e-02	-8.93e+03	8	9	7	8	5
	SNOPT	infeas	0	4.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	infeas	16	1.40e-01	-8.93e+03	32	32	17	17	17
	WORHP IPm	infeas	16	1.30e-01	-8.93e+03	32	32	17	17	17
	WORHP SQP	minalpha	13	1.43e+00	-8.93e+03	2456	2462	15	15	14
YAO	IPOPT	optimal	27	1.30e-01	1.96e+02	33	33	28	28	27
	KNITRO	optimal	33	1.80e-01	1.98e+02	36	37	35	36	33
	SNOPT	optimal	2	3.00e-02	1.98e+02	7	1	6	1	0
	WORHP IP	optimal	991	4.42e+00	1.98e+02	993	993	992	1	991
	WORHP IPm	optimal	1462	5.11e+00	1.98e+02	1464	1464	1463	1	1462
	WORHP SQP	optimal	3	1.00e-01	1.98e+02	4	4	5	3	3
YATP1LS	IPOPT	optimal	20	3.48e+01	7.80e-20	31	0	21	0	20
	KNITRO	optimal	18	3.09e+01	8.77e-20	26	0	19	0	18
	SNOPT	toobig	917	3.07e+01	2.11e+02	981	0	980	0	0
	WORHP IP	optimal	19	6.76e+01	4.58e-21	21	0	21	0	19
	WORHP IPm	optimal	19	4.04e+01	3.54e-19	21	0	20	0	19
	WORHP SQP	optimal	15	1.79e+01	5.44e-14	16	0	16	0	15
YATP1NE	IPOPT	optimal	3	1.60e-01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.40e-01	0.00e+00	5	6	4	5	3
	SNOPT	optimal	5	1.40e-01	0.00e+00	1	10	1	9	0
	WORHP IP	optimal	3	1.20e-01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	1.50e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	6	6.60e-01	0.00e+00	7	7	8	8	6
YATP1SQ	IPOPT	optimal	3	1.14e+01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	8.45e+00	0.00e+00	5	6	4	5	3
	SNOPT	optimal	6	3.46e+01	0.00e+00	1	12	1	11	0
	WORHP IP	optimal	3	1.22e+01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	8.74e+00	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	4	2.89e+02	0.00e+00	5	5	6	6	4
YATP1SS	IPOPT	optimal	3	1.40e-01	0.00e+00	4	4	4	4	3
	KNITRO	optimal	3	1.50e-01	0.00e+00	5	6	4	5	3
	SNOPT	optimal	5	1.50e-01	0.00e+00	1	10	1	9	0
	WORHP IP	optimal	3	1.50e-01	0.00e+00	5	5	4	4	3
	WORHP IPm	optimal	3	1.40e-01	0.00e+00	5	5	4	4	3
	WORHP SQP	optimal	6	6.60e-01	0.00e+00	7	7	8	8	6
YATP2LS	IPOPT	optimal	31	1.03e+02	5.75e-16	32	0	32	0	31
	KNITRO	optimal	16	2.50e+01	5.03e-22	18	0	17	0	16
	SNOPT	sbasics	10000	2.08e+02	1.03e+06	10635	0	10634	0	0
	WORHP IP	optimal	28	8.72e+01	2.93e-16	36	0	29	0	28
	WORHP IPm	optimal	28	7.85e+01	2.93e-16	36	0	29	0	28
	WORHP SQP	optimal	27	4.74e+01	3.06e-12	32	0	28	0	27
YATP2SQ	IPOPT	optimal	7	1.75e+01	0.00e+00	29	29	8	8	7
	KNITRO	optimal	5	1.51e+01	0.00e+00	10	11	6	7	5
	SNOPT	infeas	135	4.11e+02	0.00e+00	1	463	1	462	0
	WORHP IP	optimal	8	3.61e+01	0.00e+00	21	21	9	9	8
	WORHP IPm	optimal	8	3.61e+01	0.00e+00	21	21	9	9	8
	WORHP SQP	optimal	7	3.19e+02	0.00e+00	8	8	9	9	7
YFIT	IPOPT	optimal	49	1.00e-02	6.72e-13	100	0	50	0	49
	KNITRO	optimal	36	1.00e-02	6.67e-13	55	0	37	0	36
	SNOPT	optimal	70	1.00e-02	6.67e-13	96	0	95	0	0
	WORHP IP	optimal	51	1.00e-02	6.67e-13	77	0	52	0	51
	WORHP IPm	optimal	33	1.00e-02	6.67e-13	52	0	34	0	33
	WORHP SQP	optimal	39	1.00e-02	6.67e-13	92	0	40	0	39
YFITNE	IPOPT	degree	0	1.00e-02	0.00e+00	0	0	0	0	0
	KNITRO	optimal	11	1.00e-02	0.00e+00	13	14	12	13	11
	SNOPT	noimpr	15	1.00e-02	0.00e+00	1	45	1	44	0
	WORHP IP	optimal	10	1.00e-02	0.00e+00	14	14	11	11	10
	WORHP IPm	optimal	10	1.00e-02	0.00e+00	14	14	11	11	10
	WORHP SQP	optimal	8	1.00e-02	0.00e+00	126	125	3	10	2

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
YFITU	IPOPT	optimal	35	1.00e-02	6.67e-13	92	0	36	0	35
	KNITRO	optimal	35	1.00e-02	6.67e-13	53	0	36	0	35
	SNOPT	optimal	70	1.00e-02	6.67e-13	96	0	95	0	0
	WORHP IP	optimal	35	1.00e-02	6.67e-13	63	0	36	0	35
	WORHP IPm	optimal	35	1.00e-02	6.67e-13	63	0	36	0	35
	WORHP SQP	optimal	39	1.00e-02	6.67e-13	92	0	40	0	39
YORKNET	IPOPT	optimal	55	5.00e-02	1.39e+04	57	57	56	56	55
	KNITRO	optimal	22	3.00e-02	1.39e+04	36	37	24	25	22
	SNOPT	optimal	41	6.00e-02	2.50e+04	82	82	81	81	0
	WORHP IP	infeas	1240	2.84e+00	1.44e+04	17826	17826	1244	1244	1241
	WORHP IPm	optimal	55	5.00e-02	1.39e+04	76	76	61	61	55
	WORHP SQP	minalpha	242	1.20e+00	1.44e+04	5963	5965	226	247	226
ZAMB2	IPOPT	optimal	34	2.60e-01	-1.11e+01	35	35	35	35	34
	KNITRO	optimal	24	3.70e-01	-1.11e+01	27	28	26	27	24
	SNOPT	optimal	156	3.03e+01	-1.11e+01	167	167	166	166	0
	WORHP IP	optimal	32	3.60e-01	-1.11e+01	34	34	33	33	32
	WORHP IPm	optimal	34	3.90e-01	-1.11e+01	37	37	36	36	34
	WORHP SQP	optimal	15	9.90e-01	-1.11e+01	16	16	17	17	15
ZAMB2-10	IPOPT	optimal	26	1.00e-02	-1.58e+00	27	27	27	27	26
	KNITRO	optimal	18	2.00e-02	-1.58e+00	21	22	20	21	18
	SNOPT	optimal	57	4.00e-02	-1.58e+00	76	76	75	75	0
	WORHP IP	optimal	23	2.00e-02	-1.58e+00	25	25	24	24	23
	WORHP IPm	optimal	25	2.00e-02	-1.58e+00	27	27	26	26	25
	WORHP SQP	optimal	15	6.00e-02	-1.58e+00	16	16	17	17	15
ZAMB2-11	IPOPT	optimal	20	2.00e-02	-1.12e+00	21	21	21	21	20
	KNITRO	optimal	13	2.00e-02	-1.12e+00	16	17	15	16	13
	SNOPT	optimal	48	4.00e-02	-1.12e+00	52	52	51	51	0
	WORHP IP	optimal	18	1.00e-02	-1.12e+00	20	20	19	19	18
	WORHP IPm	optimal	23	2.00e-02	-1.12e+00	30	30	24	24	23
	WORHP SQP	optimal	26	1.50e-01	-1.12e+00	27	27	28	28	26
ZAMB2-8	IPOPT	optimal	19	1.00e-02	-1.53e-01	20	20	20	20	19
	KNITRO	optimal	12	1.00e-02	-1.53e-01	15	16	14	15	12
	SNOPT	optimal	32	1.00e-02	-1.53e-01	42	42	41	41	0
	WORHP IP	optimal	17	1.00e-02	-1.53e-01	19	19	18	18	17
	WORHP IPm	optimal	16	1.00e-02	-1.53e-01	19	19	18	18	16
	WORHP SQP	optimal	7	1.00e-02	-1.53e-01	8	8	9	9	7
ZAMB2-9	IPOPT	optimal	20	1.00e-02	-3.55e-01	21	21	21	21	20
	KNITRO	optimal	12	1.00e-02	-3.55e-01	15	16	14	15	12
	SNOPT	optimal	49	1.00e-02	-3.55e-01	53	53	52	52	0
	WORHP IP	optimal	15	1.00e-02	-3.55e-01	17	17	16	16	15
	WORHP IPm	optimal	13	1.00e-02	-3.55e-01	15	15	14	14	13
	WORHP SQP	optimal	12	3.00e-02	-3.55e-01	13	13	14	14	12
ZANGWIL2	IPOPT	optimal	1	1.00e-02	-1.82e+01	2	0	2	0	1
	KNITRO	optimal	1	1.00e-02	-1.82e+01	3	0	2	0	1
	SNOPT	optimal	3	1.00e-02	-1.82e+01	7	0	6	0	0
	WORHP IP	optimal	1	1.00e-02	-1.82e+01	3	0	2	0	1
	WORHP IPm	optimal	1	1.00e-02	-1.82e+01	3	0	2	0	1
	WORHP SQP	optimal	1	1.00e-02	-1.82e+01	2	0	2	0	1
ZANGWIL3	IPOPT	optimal	1	1.00e-02	0.00e+00	2	2	2	2	1
	KNITRO	optimal	1	1.00e-02	0.00e+00	3	4	2	3	1
	SNOPT	optimal	0	1.00e-02	0.00e+00	1	1	1	1	0
	WORHP IP	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP IPm	optimal	1	1.00e-02	0.00e+00	3	3	2	1	1
	WORHP SQP	optimal	1	1.00e-02	0.00e+00	2	2	3	3	1
ZECEVIC2	IPOPT	optimal	8	1.00e-02	-4.12e+00	9	9	9	9	8
	KNITRO	optimal	5	1.00e-02	-4.12e+00	8	9	7	8	5
	SNOPT	optimal	3	1.00e-02	-4.12e+00	7	1	6	1	0
	WORHP IP	optimal	6	1.00e-02	-4.12e+00	8	8	7	1	6
	WORHP IPm	optimal	6	1.00e-02	-4.12e+00	8	8	7	1	6
	WORHP SQP	optimal	2	1.00e-02	-4.12e+00	3	3	4	3	2
ZECEVIC3	IPOPT	optimal	21	1.00e-02	9.73e+01	22	22	22	22	21
	KNITRO	optimal	9	1.00e-02	9.73e+01	12	13	11	12	9
	SNOPT	optimal	7	1.00e-02	9.73e+01	12	12	11	11	0
	WORHP IP	optimal	11	1.00e-02	9.73e+01	13	13	12	12	11
	WORHP IPm	optimal	9	1.00e-02	9.73e+01	11	11	10	10	9
	WORHP SQP	optimal	8	1.00e-02	9.73e+01	9	9	10	10	8
ZECEVIC4	IPOPT	optimal	9	1.00e-02	7.56e+00	10	10	10	10	9
	KNITRO	optimal	12	1.00e-02	7.56e+00	15	16	14	15	12
	SNOPT	optimal	6	1.00e-02	7.56e+00	9	9	8	8	0
	WORHP IP	optimal	9	1.00e-02	7.56e+00	11	11	10	10	9
	WORHP IPm	optimal	10	1.00e-02	7.56e+00	12	12	11	11	10
	WORHP SQP	optimal	15	1.00e-02	7.56e+00	16	16	17	17	15
ZIGZAG	IPOPT	optimal	286	1.47e+00	8.64e+01	320	640	228	580	286
	KNITRO	optimal	221	2.40e+00	8.64e+01	228	229	223	224	222
	SNOPT	optimal	2	2.20e-01	8.64e+01	5	5	4	4	0
	WORHP IP	optimal	72	5.80e-01	8.64e+01	86	86	73	73	72
	WORHP IPm	optimal	215	1.95e+00	8.64e+01	435	435	221	221	215
	WORHP SQP	minalpha	140	2.36e+01	3.56e+01	7090	7100	36	141	36

instance	solver	status	iter	time	obj	nf	ng	ndf	ndg	nhm
ZY2	IPOPT	optimal	9	1.00e-02	2.00e+00	10	10	10	10	9
	KNITRO	optimal	5	1.00e-02	2.00e+00	7	8	6	7	5
	SNOPT	optimal	5	1.00e-02	2.00e+00	10	10	9	9	0
	WORHP IP	optimal	6	1.00e-02	2.00e+00	8	8	7	7	6
	WORHP IPm	optimal	5	1.00e-02	2.00e+00	9	9	8	8	5
	WORHP SQP	optimal	7	1.00e-02	2.00e+00	7	7	8	8	7

Table B.2: Comparison of the nonlinear programming solvers IPOPT, KNITRO, WORHP IP, WORHP IPm and WORHP SQP on the CUTEst test set.

Glossary of Symbols

Roman Symbols

A, B, C	$\in \mathbb{R}^{n\times m}$	Matrices defined temporarily	7
a, b, c, t, v	$\in \mathbb{R}^{n}$	Vectors or scalars defined temporarily	7
D	$\in \mathbb{R}^{n\times m}$	General diagonal scaling matrix	35
d	$\in \mathbb{R}^{n_x}$	General primal step	12
D_f	$\in \mathbb{R}$	Scaling for objective function	154
D_g	$\in \mathbb{R}^{n_g\times n_g}$	Diagonal scaling matrix for equality constraints	154
D_h	$\in \mathbb{R}^{n_h\times n_h}$	Diagonal scaling matrix for inequality constraints	154
d_λ	$\in \mathbb{R}^{n_g}$	General dual step of λ	136
d_ν	$\in \mathbb{R}^{n_h}$	General dual step of ν	136
d_s	$\in \mathbb{R}^{n_g}$	General step of slack variables s	30
E	$:= \mathrm{diag}(e)$	Identity matrix of appropriate size	7
e	$:= (1,\ldots,1)^\top$	Vector of ones of appropriate size	7
$F_j(t)$	$\in \mathbb{R}$	Performance profile for solver run j	153
$f(x;p)$	$\in \mathbb{R}$	Nonlinear parameter dependent objective function	19
$f(x)$	$\in \mathbb{R}$	Nonlinear objective function	8
f^*	$:= f(x^*)$	Value of the objective function at the optimal solution	10
$f^*(p)$	$:= f(x^*;p)$	Value of nonlinear parameter dependent objective function at optimal solution	19
$f_p(x;p)$	$\in \mathbb{R}$	Nonlinear parameter dependent objective function part	19
$g(x;p)$	$\in \mathbb{R}^{n_g}$	Nonlinear parameter dependent inequality constraints	19
$g(x)$	$\in \mathbb{R}^{n_g}$	Nonlinear inequality constraints	8
g^*	$:= g(x^*)$	Value of the inequality constraints at the optimal solution or certificate of infeasibility	10
$g^*(p)$	$:= g(x^*;p)$	Value of nonlinear parameter dependent inequality constraints at optimal solution	19
g^k	$:= g(x^k)$	Value of inequality constraints at iteration k	48

$g_p(x;p)$	$\in \mathbb{R}^{n_g}$	Nonlinear parameter dependent equality constraints part	19
$H(x)$	$:= \operatorname{diag}(h(x))$	Diagonal matrix with $h(x)$ on its diagonal	15
$h(x;p)$	$\in \mathbb{R}^{n_h}$	Nonlinear parameter dependent equality constraints	19
$h(x)$	$\in \mathbb{R}^{n_h}$	Nonlinear equality constraints	8
H_*	$:= \operatorname{diag}(h(x^*))$	Diagonal matrix with $h(x^*)$ on its diagonal	20
h^*	$:= h(x^*)$	Value of the equality constraints at the optimal solution or certificate of infeasibility	10
$h^*(p)$	$:= h(x^*;p)$	Value of nonlinear parameter dependent equality constraints at optimal solution	19
$\widehat{H}_k$	$\in \mathbb{R}^{n_x \times n_x}$	Augmented Hessian in reduced linear equation system at iteration k	75
h^k	$:= h(x^k)$	Value of equality constraints at iteration k	48
h_L	$\in \mathbb{R}^{n_h}$	Lower bound of general constraints	150
$h_p(x;p)$	$\in \mathbb{R}^{n_h}$	Nonlinear parameter dependent inequality constraint part	19
h_U	$\in \mathbb{R}^{n_h}$	Upper bound of general constraints	150
J	$\in \mathbb{R}^{n \times m}$	Jacobian matrix of some vector valued function	45
K	$\in \mathbb{R}^{n+m \times n+m}$	Jacobian matrix of some KKT conditions	45
k	$\in \mathbb{N}_0$	Iteration index for iterative solution algorithms	7
$L(x,\lambda;p)$	$\in \mathbb{R}$	Lagrangian function of parameter dependent equality constrained nonlinear program	124
$L(x,\lambda,\nu;p)$	$\in \mathbb{R}$	Lagrangian function of parameter dependent nonlinear program	19
$L(x,\lambda,\nu)$	$\in \mathbb{R}$	Lagrangian function	15
L^*	$:= L(x^*,\lambda^*,\nu^*)$	Value of Lagrangian function at the optimal solution	20
$L^*(p)$	$:= L(x^*,\nu^*,\lambda^*;p)$	Value of Lagrangian function of parameter dependent nonlinear program at optimal solution	20
ℓ_0	$\in \mathbb{R}$	Lagrangian multiplier for objective function in Fritz-John conditions	16
l_ε	$\in \mathbb{N}_0$	Non-monotonicity level of tolerances	85
l_f	$\in \mathbb{N}_0$	Non-monotonicity level of filter or PLPF	43
L^k	$:= L(x^k,\lambda^k,\nu^k)$	Value of Lagrangian function at iteration k	48
l_m	$\in \mathbb{N}_0$	Non-monotonicity level of merit function	39
l_ρ	$\in \mathbb{N}_0$	Number of adaptive penalty update trials	132
m	$\in \mathbb{N}_0$	Number of dimensions, strongly related to $n_g + n_h$	7
M_k, M_*	$\in \mathbb{R}^{(2n_x+n_g)^2}$	Linear equation system matrix at iteration k (at optimal solution)	74

$\widetilde{M}_k, \widetilde{M}_*$	$\in \mathbb{R}^{(n_x+n_g)^2}$	Reduced linear equation system matrix at iteration k (at optimal solution)	75
N	$\in \mathbb{R}^{n\times m}$	Basis matrix of some null space	46
n	$\in \mathbb{N}_0$	Number of dimensions, strongly related to n_x	7
n_g	$\in \mathbb{N}_0$	Number of nonlinear inequality constraints	9
n_h	$\in \mathbb{N}_0$	Number of nonlinear equality constraints	9
n_{p}	$\in \mathbb{N}$	Number of test set instances in performance profile	151
n_p	$\in \mathbb{N}_0$	Number of nonlinear parameters	19
n_{s}	$\in \mathbb{N}$	Number of solver runs in performance profile	152
n_x	$\in \mathbb{N}$	Number of optimization variables	9
o		Landau symbol for asymptotically dominated	7
P	$\in \mathbb{R}^{n\times m}$	General permutation matrix	141
p	$\in \mathbb{R}^{n_p+n_x+n_g+n_h}$	Parameters in sensitivity analysis	19
p^0	$\in \mathbb{R}^{n_p+n_x+n_g+n_h}$	Reference parameters for p in sensitivity analysis	19
p_c	$\in \mathbb{R}^{n_x}$	Shift parameter in barrier term in sensitivity analysis	136
p_c^0	$\in \mathbb{R}^{n_x}$	Reference parameter for p_c in sensitivity analysis	136
p_f	$\in \mathbb{R}^{n_x}$	Linear parameters in the objective function in sensitivity analysis	19
p_f^0	$\in \mathbb{R}^{n_x}$	Reference parameters for p_f in sensitivity analysis	21
p_g	$\in \mathbb{R}^{n_g}$	Constant parameters in the inequality constraints in sensitivity analysis	11
p_g^0	$\in \mathbb{R}^{n_g}$	Reference parameters for p_g in sensitivity analysis	21
p_h	$\in \mathbb{R}^{n_h}$	Constant parameters in the equality constraints in sensitivity analysis	19
p_h^0	$\in \mathbb{R}^{n_h}$	Reference parameters for p_h in sensitivity analysis	21
p_n	$\in \mathbb{R}^{n_p}$	Nonlinear parameters in sensitivity analysis	19
p_n^0	$\in \mathbb{R}^{n_p}$	Reference parameters for p_n in sensitivity analysis	21
Q, Q_k, Q_*	$\in \mathbb{R}^{n\times n}$	Hessian matrix of some scalar valued function (at iteration k / at the primal-dual optimal solution)	45
$Q_{i,j}$	$\in \mathbb{R}$	Quantity in performance profile for problem i and solver run j	152
q	$\in \mathbb{R}$	Local convergence order	27
$Q_{\mathrm{feas}}(w,d,\rho)$	$\in \mathbb{R}$	Quality function measuring sufficient constraint violation decrease	131
$Q_{\mathrm{opt}}(w,d,\rho)$	$\in \mathbb{R}$	Quality function measuring progress towards optimal solution	131
$R_{i,j}$	$\in \mathbb{R}$	Ratio in performance profile for problem i and solver run j	152

r	$\in \mathbb{R}$	Local convergence rate	27
S, S_k, S_*	$:= \mathrm{diag}(s)$	Diagonal matrix with s (s^k / s^*) on its diagonal	30
$s, s_k, \bar{s}$	$\in \mathbb{R}^{n_g}$	Slack variables mainly for inequality constraints (at iteration k / at a limit point)	30
s^*	$\in \mathbb{R}^{n_g}$	Slack variables at optimal solution	30
$w(\bar{\mu}, \bar{\nu})$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Primal-dual optimal solution of the modified barrier subproblem	105
$\widetilde{w}(p)$	$:= (\widetilde{x}(p), \widetilde{y}(p), \widetilde{z}(p))$	Approximation of primal-dual optimal solution of parameter dependent nonlinear program	139
$w(p)$	$:= (x(p), y(p), z(p))$	Primal-dual optimal solution of parameter dependent nonlinear program	139
$w, w_k, \bar{w}$	$:= (x, y, z)$	Primal-dual variables (at iteration k / at a limit point)	74
w^*	$:= (x^*, y^*, z^*)$	Primal-dual variables at optimal solution	74
$X(\bar{\mu}, \bar{\nu})$	$:= \mathrm{diag}(x(\bar{\mu}, \bar{\nu}))$	Diagonal matrix with $x(\bar{\mu}, \bar{\nu})$ on its diagonal	107
$\widetilde{X}(p)$	$:= \mathrm{diag}(\widetilde{x}(p))$	Diagonal matrix with $\widetilde{x}(p)$ on its diagonal	137
X, X_k, X_*	$:= \mathrm{diag}(x)$	Diagonal matrix with x (x^k / x^*) on its diagonal	49
$x(\bar{\mu}, \bar{\nu})$	$\in \mathbb{R}^{n_x}$	Optimal solution of the modified barrier subproblem	105
$x(\mu)$	$\in \mathbb{R}^{n_x}$	Optimal solution of the barrier subproblem	51
$\widetilde{x}(p)$	$\in \mathbb{R}^{n_x}$	Approximation of optimal solution of perturbed nonlinear program	61
$x(p)$	$\in \mathbb{R}^{n_x}$	Optimal solution of parameter dependent nonlinear program	19
$x, x^k, \bar{x}$	$\in \mathbb{R}^{n_x}$	Optimization variables (at iteration k / at a limit point)	8
x^*	$\in \mathbb{R}^{n_x}$	Optimal solution or certificate of infeasibility	9
x_L	$\in \mathbb{R}^{n_x}$	Lower bound of box constraints	150
x_U	$\in \mathbb{R}^{n_x}$	Upper bound of box constraints	150
$y(\bar{\mu}, \bar{\nu})$	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multipliers of equality constraints of the modified barrier subproblem	105
$\widetilde{y}(p)$	$\in \mathbb{R}^{n_g}$	Approximation of optimal Lagrangian multiplier of the equality constraints of perturbed nonlinear program	137
$y(p)$	$\in \mathbb{R}^{n_h}$	Optimal Lagrangian multipliers of the equality constraints of parameter dependent nonlinear program	58
$y, y_k, \bar{y}$	$\in \mathbb{R}^{n_h}$	Lagrangian multipliers of equality constraints in penalty subproblem (at iteration k / at a limit point)	8
y^*	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multiplier of equality constraints in penalty subproblem	70
$y_\mathrm{G}, y_\mathrm{G}^k$	$\in \mathbb{R}^{n_g}$	Lagrangian multipliers for equality constraints (at iteration k)	151

$y_{\mathrm{H}}, y_{\mathrm{H}}^{k}$	$\in \mathbb{R}^{n_h}$	Lagrangian multipliers for equality constraints (at iteration k)	151
$\widetilde{Z}(p)$	$:= \mathrm{diag}(\widetilde{z}(p))$	Diagonal matrix with $\widetilde{z}(p)$ on its diagonal	142
Z, Z_k, Z_*	$:= \mathrm{diag}(z)$	Diagonal matrix with z (z^k / z^*) on its diagonal	53
$z(\bar{\mu}, \bar{\nu})$	$:= \mathrm{diag}(y(\bar{\mu}, \bar{\nu}))$	Diagonal matrix with $y(\bar{\mu}, \bar{\nu})$ on its diagonal	105
$z(\mu)$	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multipliers of inequality constraints of the barrier subproblem	53
$\widetilde{z}(p)$	$\in \mathbb{R}^{n_x}$	Approximation of optimal Lagrangian multiplier for the inequality constraints of the perturbed nonlinear program	137
$z(p)$	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multipliers of the inequality constraints of parameter dependent nonlinear program	124
$z, z_k, \bar{z}$	$\in \mathbb{R}^{n_g}$	Lagrangian multipliers of inequality constraints in barrier subproblem (at iteration k / at a limit point)	52
$z_{\mathrm{HL}}, z_{\mathrm{HL}}^{k}$	$\in \mathbb{R}^{n_h}$	Lagrangian multipliers of lower inequality general constraints (at iteration k)	150
$z_{\mathrm{HU}}, z_{\mathrm{HU}}^{k}$	$\in \mathbb{R}^{n_h}$	Lagrangian multipliers of upper inequality general constraints (at iteration k)	151
$z_{\mathrm{XL}}, z_{\mathrm{XL}}^{k}$	$\in \mathbb{R}^{n_x}$	Lagrangian multipliers of lower inequality box constraints (at iteration k)	150
$z_{\mathrm{XU}}, z_{\mathrm{XU}}^{k}$	$\in \mathbb{R}^{n_x}$	Lagrangian multipliers of upper inequality box constraints (at iteration k)	150

Greek Symbols

α, α_k	$\in (0,1]$	Step size of optimization variables (at iteration k)	27
$\widehat{\alpha}_k$	$\in (0,1]$	Maximal allowed step size for second-order-correction or refinement step of primal optimization variables in iteration k	82
α_k^z	$\in (0,1]$	Step size of dual variables for inequality constraints	83
$\alpha_{\max,k}$	$\in (0,1]$	Maximal allowed step size of optimization variables in iteration k	53
$\alpha_{\min}$	$\in \mathbb{R}_{0+}$	Minimum step size	43
$\widehat{\alpha}_k^z$	$\in (0,1]$	Maximal allowed step size for second-order-correction or refinement step of dual optimization variables in iteration k	165
β	$\in (0,1)$	Step size reduction	40
$\chi_b^a(c)$	$\in \mathbb{R}_{0+} \cup \{\infty\}$	Indicator function returning 0 if c fulfills condition b and a otherwise	50

δ_d	$\in \mathbb{R}_{0+}$	Dual regularization	46
δ_k	$\in \mathbb{R}_{0+}$	Filter or PLPF envelope	40
$\bar{\delta}_\mathrm{p}$	$\in \mathbb{R}_{0+}$	Previous value of primal regularization	161
δ_p	$\in \mathbb{R}_{0+}$	Primal regularization	46
δ_p^0	$\in \mathbb{R}_{+}$	Initial value of primal regularization	160
$\delta_\mathrm{p}^\mathrm{min}$	$\in \mathbb{R}_{+}$	Minimum value of primal regularization	160
ε	$\in \mathbb{R}_{+}$	A very small positive number, i.e. machine precision for numerical computations	7
$\varepsilon_\mathrm{desc}$	$\mathbb{R}_{+}$	Minimum descent parameter	160
$\varepsilon_\mathrm{frac}$	$\in (0,1)$	Fraction-to-the-boundary parameter	53
$\varepsilon_{\mathrm{frac},k}$	$\in (0,1)$	Fraction-to-the-boundary parameter sequence	78
$\varepsilon_{\lambda,k}$	$\in \mathbb{R}_{+}$	Tolerance for Lagrangian multiplier λ update	84
$\varepsilon_{\mu,k}$	$\in \mathbb{R}_{+}$	Tolerance for barrier parameter update	86
$\varepsilon_{\nu,k}$	$\in \mathbb{R}_{+}$	Tolerance for Lagrangian multiplier ν update	86
$\varepsilon_{\rho,k}$	$\in \mathbb{R}_{+}$	Tolerance for penalty parameter update	83
ε_tol	$\in \mathbb{R}_{+}$	Optimality tolerance	29
η	$\in (\sigma,1)$	Fraction of descent in Wolfe condition	38
γ	$\in \mathbb{R}_{0+}$	Balancing parameter in active set approximation	48
Γ_bar^k	$\in \mathbb{R}_{0+}$	Lowpass filter for barrier subprogram	169
γ_f	$\in (0,1)$	Fraction of envelope in filter or PLPF condition	40
Γ_pen^k	$\in \mathbb{R}_{0+}$	Lowpass filter for penalty subprogram	169
$\kappa_{\delta,\mathrm{inc}}^0$	> 1	First increase factor of primal regularization	160
κ_1	$\in (0,1)$	Proportionality for feasibility descent in adaptive parameter updates	132
κ_2	$\in (0,1)$	Proportionality for optimality quality function in adaptive parameter updates	132
κ_3	$\in (0,1)$	Proportionality for descent in adaptive parameter updates	132
κ_adj	> 1	Dual adjustment parameter	156
$\kappa_{\delta,\mathrm{dec}}$	$\in (0,1)$	Decrease factor of primal regularization	160
$\kappa_{\delta,\mathrm{inc}}$	> 1	Increase factor of primal regularization	160
κ_λ	$\in (0,1)$	Dual trust-region parameter	84
κ_μ	$\in (0,1)$	Barrier decrease parameter	87
κ_π	$\in (0,1)$	Penalty decrease parameter	85
κ_sc	$\in \mathbb{R}_{+}$	Maximum gradient scaling parameter	157
κ_ς	$\in (0,1]$	Exponent in boundary shift update	87
κ_τ	$\in (1,\infty)$	Penalty increase parameter	85

κ_z	$\in \mathbb{R}_+$	Dual projection parameter	83
$\lambda(\mu)$	$\in \mathbb{R}^{n_h}$	Optimal Lagrangian multipliers of the equality constraints of the barrier subproblem	53
$\lambda(p)$	$\in \mathbb{R}^{n_h}$	Optimal Lagrangian multipliers of the equality constraints of parameter dependent nonlinear program	19
λ, λ_k, $\bar{\lambda}$	$\in \mathbb{R}^{n_h}$	Lagrangian multipliers for the equality constraints (at iteration k / at a limit point)	13
λ^*	$\in \mathbb{R}^{n_h}$	Optimal Lagrangian multipliers for the equality constraints	15
λ_+	$\in \mathbb{N}_0$	Number of positive eigenvalues of a matrix	46
λ_-	$\in \mathbb{N}_0$	Number of negative eigenvalues of a matrix	46
λ_0	$\in \mathbb{N}_0$	Number of zero eigenvalues of a matrix	46
λ_G, λ_G^k	$\in \mathbb{R}^{n_g}$	Lagrangian multiplier parameters for equality constraints (at iteration k)	151
λ_H, λ_H^k	$\in \mathbb{R}^{n_h}$	Lagrangian multiplier parameters for equality constraints (at iteration k)	151
μ, μ_k, $\bar{\mu}$	$\in \mathbb{R}_+$	Barrier parameter (at iteration k / at a limit point)	51
$\widetilde{\nu}(p)$	$\in \mathbb{R}^{n_x}$	Approximation of optimal Lagrangian multiplier ν of perturbed nonlinear program	136
$\nu(p)$	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multipliers of the inequality constraints of parameter dependent nonlinear program	19
ν, ν_k, $\bar{\nu}$	$\in \mathbb{R}^{n_g}$	Lagrangian multipliers for the inequality constraints (at iteration k / at a limit point)	13
ν^*	$\in \mathbb{R}^{n_g}$	Optimal Lagrangian multipliers for the inequality constraints	15
ν_HL, ν_HL^k	$\in \mathbb{R}^{n_h}$	Lagrangian multiplier parameters for lower inequality general constraints (at iteration k)	151
ν_HU, ν_HU^k	$\in \mathbb{R}^{n_h}$	Lagrangian multiplier parameters for upper inequality general constraints (at iteration k)	151
ν_XL, ν_XL^k	$\in \mathbb{R}^{n_x}$	Lagrangian multiplier parameters for lower inequality box constraints (at iteration k)	151
ν_XU, ν_XU^k	$\in \mathbb{R}^{n_x}$	Lagrangian multiplier parameters for upper inequality box constraints (at iteration k)	151
Ω		Landau symbol for asymptotically bounded below	7
$\omega(p)$	$:= (x(p), \lambda(p), \nu(p))$	Primal-dual optimal solution of parameter dependent nonlinear program	34
ω, ω_k, $\bar{\omega}$	$:= (x, \lambda, \nu)$	Primal-dual variables (at iteration k / at a limit point)	74
ω^*	$:= (x^*, \lambda^*, \nu^*)$	Primal-dual variables at optimal solution	74

$\Phi(w_k; p_k)$	$\in \mathbb{R}^{2n_x+n_g}$	Vector of KKT conditions of original problem in penalty-interior-point algorithm	86
$\Phi(x, \lambda, z; \mu)$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Vector of KKT conditions of barrier subproblem	53
$\Phi(x, y; \tau)$	$\in \mathbb{R}^{n_x+n_g}$	Vector of KKT conditions of penalty subproblem for an equality constrained nonlinear program	58
$\Phi(x, \lambda)$	$\in \mathbb{R}^{n_x+n_g}$	Vector of equality KKT conditions of equality constrained problem	28
$\Phi(x, \lambda, \nu; p)$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Vector of equality KKT conditions of parameter dependent nonlinear program	20
$\Phi(x, \lambda, \nu)$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Vector of equality KKT conditions	16
$\Phi_{\text{bar}}(w_k, p_k)$	$\in \mathbb{R}^{2n_x+n_g}$	Vector of KKT conditions of barrier subproblem in penalty-interior-point algorithm	84
$\varphi_{\text{bar}}(x; \mu)$	$\in \mathbb{R}$	Barrier function	50
$\Phi_{\text{pen}}(w_k, p_k)$	$\in \mathbb{R}^{2n_x+n_g}$	Vector of KKT conditions of penalty subproblem in penalty-interior-point algorithm	84
$\varphi_{\text{pen}}(x; \tau)$	$\in \mathbb{R}$	Penalty function	56
π, π_k, $\bar{\pi}$	$\in \mathbb{R}_+$	Penalty parameter (at iteration k / at a limit point)	68
$\Psi(x; \tau)$	$\in \mathbb{R}$	Merit function	36
ρ, ρ_k, $\bar{\rho}$	$\in \mathbb{R}^{n_g+2n_x+3}$	Parameters in barrier-penalty function (at iteration k / at a limit point)	68
ϱ, ϱ_k, $\bar{\varrho}$	$:= \lVert g(x) \rVert_2 / \tau$	Fixation of constraint violation factor for step calculation (at iteration k / at a limit point)	71
Σ, Σ_k, $\bar{\Sigma}$	$\in \mathbb{R}_+^{n_x}$	Diagonal matrix with ς (ς^k / $\bar{\varsigma}$) on its diagonal	55
σ	$\in (0, \frac{1}{2})$	Fraction of descent in Armijo condition	38
ς, ς_k, $\bar{\varsigma}$	$\in \mathbb{R}_+$	Primal shift in modified barrier function (at iteration k / at a limit point)	51
ς_{HL}, ς_{HL}^k	$\in \mathbb{R}^{n_h}$	Primal shift in modified barrier for lower inequality general constraints (at iteration k)	151
ς_{HU}, ς_{HU}^k	$\in \mathbb{R}^{n_h}$	Primal shift in modified barrier for upper inequality general constraints (at iteration k)	151
ς_{XL}, ς_{XL}^k	$\in \mathbb{R}^{n_x}$	Primal shift in modified barrier for lower inequality box constraints (at iteration k)	151
ς_{XU}, ς_{XU}^k	$\in \mathbb{R}^{n_x}$	Primal shift in modified barrier for upper inequality box constraints (at iteration k)	151
τ, τ_k, $\bar{\tau}$	$\in \mathbb{R}_+$	Penalty parameter (at iteration k / at a limit point)	36
$\bar{\tau}$	$\in \mathbb{R}_+$	Sufficiently large penalty parameter for exact penalty function	37
τ_{c}	$\in \mathbb{R}_+$	Constant penalty factor for perturbed complementarity condition in primal-dual merit function	79

τ_{f}	$\in \mathbb{R}_+$	Constant penalty factor for perturbed feasibility condition in primal-dual merit function	79
$\tau_{\max}$	$\in \mathbb{R}_+$	Maximum penalty parameter τ	85
Θ		Landau symbol for asymptotically bounded above and below	7
$\theta(x)$	$\in \mathbb{R}_{0+}$	Constraint violation	36
$\theta_{\max}$	$\in \mathbb{R}_+$	Maximal allowed constraint violation	41
$\Upsilon(x;\rho)$	$\in \mathbb{R}$	Combined barrier-penalty function	69
υ_1	$\in (0,1)$	Factor in tolerance for penalty parameter update	85
υ_2	$\in (0,1)$	Factor in tolerance for penalty parameter update	85
υ_3	$\in (0,1)$	Factor in tolerance for penalty parameter update	85
υ_4	$\in (0,1)$	Factor in tolerance for barrier parameter update	88
υ_5	$\in (0,1)$	Factor in tolerance for barrier parameter update	88
υ_6	$\in (0,1)$	Factor in tolerance for barrier parameter update	88
$\xi_{\lambda,k}$	$\in \mathbb{R}_+$	Non-negative sequence for tolerance of Lagrangian multiplier λ update	85
$\xi_{\mu,k}$	$\in \mathbb{R}_+$	Non-negative sequence for tolerance of barrier parameter update	88
$\xi_{\nu,k}$	$\in \mathbb{R}_+$	Non-Negative sequence for tolerance of Lagrangian multiplier ν update	88
$\xi_{\rho,k}$	$\in \mathbb{R}_+$	Non-negative sequence for tolerance of penalty parameter update	85

Delta Symbols

$\Delta\lambda^k$	$\in \mathbb{R}^{n_g}$	Standard step direction of λ	28
$\overline{\Delta\lambda}^{k,j}$	$\in \mathbb{R}^{n_g}$	Inexact step direction of λ	66
$\widetilde{\Delta\lambda}^{k,j}$	$\in \mathbb{R}^{n_g}$	Refinement step direction of λ	64
$\Delta\nu$	$\in \mathbb{R}^{n_x}$	Perturbation of parameter ν	137
$\Delta\nu^k$	$\in \mathbb{R}^{n_x}$	Standard step direction of ν	48
$\widehat{\Delta\omega}^k$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Second-order-correction step direction of ω	119
$\Delta\omega^k$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Standard step direction of ω	119
Δp	$\in \mathbb{R}^{n_p+n_x+n_g+n_h}$	Perturbation of parameter p	19
Δp_c	$\in \mathbb{R}^{n_x}$	Perturbation of parameter p_c	138
Δp_g	$\in \mathbb{R}^{n_g}$	Perturbation of parameter p_g	138
$\Delta\rho$	$\in \mathbb{R}^{n_g+2n_x+3}$	Perturbation of parameter ρ	137
$\widetilde{\Delta s}^k$	$\in \mathbb{R}^{n_h}$	Complementarity refinement step of s^k	165
Δs^k	$\in \mathbb{R}^{n_h}$	Standard step direction of s	31

$\widehat{\Delta w}^k$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Second-order-correction step direction of ω	82
$\widetilde{\Delta w}^k(\rho)$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Approximation of parameter dependent step direction of w	131
$\Delta w^k(\rho)$	$\in \mathbb{R}^{n_x+n_g+n_h}$	Parameter dependent step direction of w	131
Δw^k	$\in \mathbb{R}^{n_x+n_g+n_h}$	Standard step direction of w	74
$\widetilde{\Delta x}^k$	$\in \mathbb{R}^{n_x}$	Complementarity refinement step of x^k	165
$\widehat{\Delta x}^k$	$\in \mathbb{R}^{n_x}$	Second-order-correction step direction of x	82
$\widetilde{\Delta x}^k(\rho)$	$\in \mathbb{R}^{n_x}$	Approximation of parameter dependent step direction of x	130
$\Delta x^k(\rho)$	$\in \mathbb{R}^{n_x}$	Parameter dependent step direction of x	129
Δx^k	$\in \mathbb{R}^{n_x}$	Standard step direction of x	27
$\overline{\Delta x}^{k,j}$	$\in \mathbb{R}^{n_x}$	Inexact step direction of x	66
$\widetilde{\Delta x}^{k,j}$	$\in \mathbb{R}^{n_x}$	Refinement step direction of x	63
$\widehat{\Delta y}^k$	$\in \mathbb{R}^{n_g}$	Second-order-correction step direction of y	82
$\widetilde{\Delta y}^k(\rho)$	$\in \mathbb{R}^{n_g}$	Approximation of parameter dependent step direction of y	130
$\Delta y^k(\rho)$	$\in \mathbb{R}^{n_g}$	Parameter dependent step direction of y	129
Δy^k	$\in \mathbb{R}^{n_g}$	Standard step direction of y	58
$\widetilde{\Delta y}^k_{\mathrm{G}}$	$\in \mathbb{R}^{n_g}$	Complementarity refinement step of y^k_{G}	165
Δy^k_{G}	$\in \mathbb{R}^{n_g}$	Standard step direction of y^k_{G}	159
$\widetilde{\Delta y}^k_{\mathrm{H}}$	$\in \mathbb{R}^{n_h}$	Complementarity refinement step of y^k_{H}	165
Δy^k_{H}	$\in \mathbb{R}^{n_h}$	Standard step direction of y^k_{H}	159
$\widehat{\Delta z}^k$	$\in \mathbb{R}^{n_x}$	Second-order-correction step direction of z	82
$\widetilde{\Delta z}^k(\rho)$	$\in \mathbb{R}^{n_x}$	Approximation of parameter dependent step direction of z	130
$\Delta z^k(\rho)$	$\in \mathbb{R}^{n_x}$	Parameter dependent step direction of z	129
Δz^k	$\in \mathbb{R}^{n_x}$	Standard step direction of z	53
$\widetilde{\Delta z}^k_{\mathrm{HL}}$	$\in \mathbb{R}^{n_h}$	Complementarity refinement step of z^k_{HL}	165
Δz^k_{HL}	$\in \mathbb{R}^{n_h}$	Standard step direction of z^k_{HL}	159
$\widetilde{\Delta z}^k_{\mathrm{HU}}$	$\in \mathbb{R}^{n_h}$	Complementarity refinement step of z^k_{HU}	165
Δz^k_{HU}	$\in \mathbb{R}^{n_h}$	Standard step direction of z^k_{HU}	159
$\widetilde{\Delta z}^k_{\mathrm{XL}}$	$\in \mathbb{R}^{n_x}$	Complementarity refinement step of z^k_{XL}	165
Δz^k_{XL}	$\in \mathbb{R}^{n_x}$	Standard step direction of z^k_{XL}	159
$\widetilde{\Delta z}^k_{\mathrm{XU}}$	$\in \mathbb{R}^{n_x}$	Complementarity refinement step of z^k_{XU}	165
Δz^k_{XU}	$\in \mathbb{R}^{n_x}$	Standard step direction of z^k_{XU}	159

Calligraphic Symbols

$\widetilde{\mathscr{A}}$, $\widetilde{\mathscr{A}}_k$	$\subseteq \{1,\ldots,n_h\}$	Active set approximation (at iteration k)	48
$\mathscr{A}(x;p)$	$\subseteq \{1,\ldots,n_x\}$	Set of active inequality constraints of parameter dependent nonlinear program	19
$\mathscr{A}(x)$	$\subseteq \{1,\ldots,n_h\}$	Set of active inequality constraints	9
$\mathscr{B}_\varepsilon(x)$	$\subseteq \mathbb{R}^{n_x}$	Neighbourhood around x with distance ε	7
$\mathscr{D}\left(\mathscr{F}^k\right)$	$\subseteq \mathbb{R}^2$	Acceptable region of filter or PLPF	41
$\overline{\mathscr{D}}\left(\tau;\mathscr{F}^k\right)$	$\in \mathbb{R}$	Piecewise Linear Penalty Function (PLPF)	44
$\mathscr{D}$	$\subseteq \mathbb{R}^{n_x}$	Set of feasible points	9
$\mathscr{F}_k$	$\subseteq \mathbb{R}^2$	Set of historic data points stored in filter or PLPF	41
$\mathscr{F}_k(l)$	$\subseteq \mathbb{R}^2$	Set of historic data points stored in non-monotone filter or PLPF	43
$\mathscr{F}_{\text{mag},k}(l)$	$\subseteq \mathbb{R}^2$	Set of historic data points stored in non-monotone filter or PLPF for magic step	88
$\mathscr{H}_{\text{L}}$	$\subseteq \mathbb{N}_0$	Set of indices for lower bounded general constraints	150
$\mathscr{H}_{\text{U}}$	$\subseteq \mathbb{N}_0$	Set of indices for upper bounded general constraints	150
$\widetilde{\mathscr{I}}$, $\widetilde{\mathscr{I}}_k$	$\subseteq \{1,\ldots,n_h\}$	Inactive set approximation (at iteration k)	48
$\mathscr{I}(x;p)$	$\subseteq \{1,\ldots,n_x\}$	Set of inactive inequality constraints of parameter dependent nonlinear program	21
$\mathscr{I}(x)$	$\subseteq \{1,\ldots,n_h\}$	Set of inactive inequality constraints	9
$\mathscr{K}$	$\subseteq \mathbb{N}_0$	Index set of iterations	7
$\mathscr{K}_\lambda$	$\subseteq \mathbb{N}_0$	Index set of iterations with Lagrangian multiplier λ^k updates	85
$\mathscr{K}_\mu$	$\subseteq \mathbb{N}_0$	Index set of iterations with barrier updates	88
$\mathscr{K}_\nu$	$\subseteq \mathbb{N}_0$	Index set of iterations with Lagrangian multiplier ν^k updates	88
$\mathscr{K}_\pi$	$\subseteq \mathbb{N}_0$	Index set of iterations with penalty updates	85
$\mathscr{M}$	$\subseteq \mathbb{R}_+$	Admissible set for adaptive barrier parameter updates	132
$\mathscr{N}(x)$	$\subseteq \mathbb{R}^{n_x}$	Neighborhood around x of appropriate size	7
$\mathscr{O}$		Landau symbol for asymptotically bounded above	7
$\mathscr{P}$	$\subseteq \mathbb{R}^{n_p+n_x+n_g+n_h}$	Parameter neighborhood of sensitivity analysis	19
$\mathscr{P}_{\text{lin}}$	$\subseteq \mathbb{R}^{n_p+n_x+n_g+n_h}$	Linear approximation of parameter neighborhood of sensitivity analysis	24
$\mathscr{T}_{\text{crit}}(x,\nu^*)$	$\subseteq \mathbb{R}^{n_x}$	Critical cone	17
$\mathscr{T}_{\mathscr{D}}(x)$	$\subseteq \mathbb{R}^{n_x}$	Tangent cone	14
$\mathscr{T}_{\text{lin}}(x)$	$\subseteq \mathbb{R}^{n_x}$	Linearized tangent cone	14
$\mathscr{X}_{\text{L}}$	$\subseteq \mathbb{N}_0$	Set of indices for lower bounded box constraints	150

$\mathscr{X}_{\mathrm{U}}$	$\subseteq \mathbb{N}_0$	Set of indices for upper bounded box constraints	150

Blackboard Symbols

$\mathbb{N}$	Set of natural numbers excluding zero	7
$\mathbb{N}_0$	Set of natural numbers including zero	7
$\mathbb{R}$	Set of real numbers	7
$\mathbb{R}_+$	Set of positive real numbers	7
$\mathbb{R}_{0+}$	Set of nonnegative real numbers	7

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