A STUDY ON MEASURE-GEOMETRIC LAPLACIANS ON THE REAL LINE

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Ableiten ist ein Handwerk – Integrieren eine Kunst.

In Gedenken an Bernd O. Stratmann, der mir diese Weisheit nahebrachte

Abstract

A Study On Measure-Geometric Laplacians on the Real Line

In this thesis, we consider measure-geometric differential operators on the real line as they were introduced by Freiberg and Zähle in 2002. We define derivatives ∇_μ and Laplace operators Δ_μ with respect to different types of compactly supported finite measures μ.
We first discuss the class of continuous measures ν. We deduce a harmonic calculus for ∇_ν and Δ_ν from the classical (weak) analysis. It is shown that the eigenvalues of Δ_ν do not depend on the chosen measure and the eigenfunctions are given explicitly. The results are illustrated through examples of (fractal) measures.
In the following, the framework of Freiberg and Zähle is extended by dropping the condition that the measures are atomless. We consider distributions δ which have finite support and define analogous operators ∇_δ and Δ_δ. Using that they act on finite-dimensional function spaces, we give matrix representations for both operators and obtain analytic properties for them. General observations on the spectral properties of Δ_δ are discussed and compared to the atomless case. For uniform discrete probability distributions, we determine the eigenvalues and eigenfunctions of the associated Laplacian.
We then study measures η which have a continuous and an atomic part. Again we define the operators ∇_η and Δ_η and obtain properties similar to those for weak first and second order derivatives. We give a systematic way to calculate the eigenvalues and eigenfunctions of Δ_η and determine them for two leading examples. Differences and similarities to the previous cases are discussed and problems for the general solution of the eigenvalue problem are indicated.

Zusammenfassung

A Study On Measure-Geometric Laplacians on the Real Line

In dieser Dissertationsschrift werden maßgeometrische Ableitungsoperatoren untersucht, wie sie Freiberg und Zähle 2002 einführten. Wir definieren Ableitungen ∇_μ und Laplace-Operatoren Δ_μ bezüglich verschiedener endlicher Maße μ mit kompaktem Träger.
Als Erstes werden Maße ν mit stetigen Verteilungsfunktionen betrachtet und aus der schwachen Ableitungstheorie wird eine harmonische Analysis für ∇_ν und Δ_ν entwickelt. Es wird gezeigt, dass die Eigenwerte von Δ_ν unabhängig von dem gewählten Maß sind und die Eigenfunktionen werden explizit angegeben. Durch Beispiele (fraktaler) Maße werden die Ergebnisse veranschaulicht.
Anschließend wird die Theorie von Freiberg und Zähle erweitert, indem auf die Bedingung der Atomfreiheit der Maße verzichtet wird. Wir betrachten Maße δ die endliche, gewichtete Summen von Dirac-Punktmassen sind und führen die Operatoren ∇_δ und Δ_δ ein. Da diese auf endlichdimensionalen Funktionenräumen definiert sind, kann eine Matrixdarstellung für beide Operatoren angegeben werden, mit deren Hilfe verschiedene analytische Eigenschaften bewiesen werden können. Es werden allgemeine Aussagen zu den Spektraleigenschaften von Δ_δ diskutiert und mit denen des vorangegangenen Falls verglichen. Für gleichverteilte, diskrete Wahrscheinlichkeitsmaße werden die Eigenwerte und Eigenfunktionen des zugehörigen Laplace-Operators explizit bestimmt.
Abschließend werden Maße η betrachtet, die sowohl einen stetigen als auch einen diskreten Anteil besitzen. Wir definieren die Operatoren ∇_η und Δ_η und beweisen Eigenschaften welche analog zu denen in der schwachen Ableitungstheorie sind. Wir entwickeln eine Möglichkeit die Eigenwerte und Eigenfunktionen von Δ_η zu ermitteln und bestimmen diese für zwei wichtige Beispiele. Gemeinsamkeiten und Unterschiede zu den vorangegangenen Fällen werden diskutiert und Probleme bei der Lösung des allgemeinen Eigenwertproblems veranschaulicht.

Contents

CHAPTER 1

Introduction

1.1 Background and motivation

It is a classical analytic problem to find the solution of the Dirichlet eigenvalue problem. This question seeks to find which functions u and scalars λ fulfil

$$\begin{cases} -\Delta_n u = \lambda u \text{ on } \Omega, \\ u|_{\partial\Omega} \equiv 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ and Δ_n is the Laplace-operator in $\mathbb{R}^n$. For this problem, Weyl [Wey15] showed that the eigenvalues λ are non-negative, of finite multiplicity and that their only accumulation point is $+\infty$. Furthermore, he proved in this paper that, when assuming some regularity conditions on Ω, for all $n \in \mathbb{N}$ the eigenvalue counting function

$$N_n(x) \coloneqq \#\{n \in \mathbb{N} \colon \lambda_n \leq x \text{ is a solution to (1.1)}\},$$

where the eigenvalues are counted with respect to their multiplicities, satisfies the following asymptotic behaviour as x tends to infinity:

$$N_n(x) = \frac{b_n \Lambda_n(\Omega)}{(2\pi)^n}\, x^{\frac{n}{2}} + o\left(x^{\frac{n}{2}}\right). \tag{1.2}$$

Here b_n is the volume of the n-dimensional unit sphere and Λ_n denotes the n-dimensional Lebesgue measure. Métivier [Mét77] generalised this result to any bounded domain of $\mathbb{R}^n$ and the error term $o(x^{n/2})$ was specified in terms of the Minkowski dimension by for example Lapidus and Pomerance (see [Lap91, LP96]).

Extending observations in the original work of Weyl, Kac approached the inverse spectral problem. He wanted to know if one can reconstruct the geometry of an n-dimensional manifold from the eigenvalues of the associated Laplacian. This led in [Kac66] to his famous question "Can one hear the shape of a drum?" Milnor [Mil64] gave the first counterexample to this when he showed in 1964 the existence of a pair of 16-dimensional tori whose associated Laplacians

have the same eigenvalues but which have different shapes. Twenty years later, Urakawa [Ura82] produced, for $n \geq 4$, the first examples of domains in $\mathbb{R}^n$ which are isospectral but not congruent. The problem in two dimensions remained open until 1992, when Gordon, Webb, and Wolpert [GWW92] constructed a pair of regions in the plane that have different shapes but whose associated Laplacians have identical eigenvalues.

Questions of this type were then also transferred to fractals and to domains with fractal boundaries. For these the physicist Berry [Ber79] conjectured in 1979 that the analogue of (1.2), for $x \to \infty$, is

$$N_d(x) = b_d \mathcal{H}^d(\Omega) x^{\frac{d}{2}} + o\left(x^{\frac{d}{2}}\right), \tag{1.3}$$

where d is the Hausdorff dimension of Ω, b_d is a constant independent of Ω and $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure. This formula is known today as the Weyl-Berry conjecture and it was disproven by Brossard and Camora [BC86] who suggested to replace the Hausdorff dimension with the upper box dimension. This holds true in the plane if the boundary has dimension one, but it again was mostly disproven by Lapidus and Pomerance [LP93, LP96] for higher dimensions. Nevertheless, as observed by Weyl [Wey12], Berry [Ber79, Ber80], Lapidus [Lap91, Lap93], Laidus and Pomerance [LP90, LP93, LP96], Beals and Greiner [BG09] and many others, the spectrum of a Laplacian still tells us a lot about the shape of the underlying geometric structure it is defined on.

Taking one step back, one first needs to think about the definition of a derivative and of a Laplacian on a fractal set, since these structures do not have a tangent space. There are many different approaches to this and we shortly introduce three well-known ones. Kigami constructed in [Kig89] a Laplacian on the Sierpinski gasket and this theory was developed further by Kusuoka [Kus89], Kigami [Kig93], Kigami and Lapidus [KL93, KL01], Strichartz [Str06] and many others to the class of "post critically finite" fractals. This analytic approach, for which the classic book by Kigami [Kig01] provides a full introduction, considers a sequence of discrete Laplacians on finite graphs, where in the limit the graphs approximate the fractal structure and, using an appropriate rescaling, the discrete Laplacians converge to a well-defined operator which resembles a Laplace operator on a fractal.

Motivated by studies in physics (see for example [Liu86, HBA02]), Kusuoka [Kus87] and Goldstein [Gol87] independently developed a probabilistic approach to define a Laplacian on a fractal. They considered a sequence of random walks on graphs approximating the fractal structure and showed, again using an appropriate rescaling, that they converge to a diffusion process on the fractal, called Brownian motion. Here the Laplacian is defined as the infinitesimal generator of the Brownian motion. Further important work on this approach was done by, for example Barlow and Perkins [BP88], Barlow and Bass [BB89] and by Lindstrøm [Lin90], but again certain assumptions on the class of fractals need to be made.

A different probabilistic approach, which is due to Denker and Sato [DS99, DS01] and was also studied by Ju, Lau and Wang [JLW12], defines the Laplacian using Martin kernels and the Martin boundary. One again considers random walks on approximating graphs and looks at the set Σ^* of finite words in the code space. One defines transition probabilities, Green's function and the Martin kernel on these sets and these imply a metric d on Σ^*. The Martin boundary is then the difference between the closure of Σ^* with respect to d and Σ^* itself. One

can show that a specific translation operator can be extended to the whole fractal and this extension then resembles a Laplacian.

All three approaches have a strong reliance on graph approximations to the underlying fractal. To avoid this dependence on the construction of the fractals, Freiberg introduced in her PhD thesis [Fre00] in joint work with her supervisor Zähle a measure-geometric approach to define a Laplace operator on one-dimensional fractals. Motivated by the fundamental theorem of calculus, and based on the works of Feller [Fel57] and Kac and Kreĭn [KK74], they introduced a first order differential operator ∇_ν and a second order differential operator $\Delta_{\mu,\nu} \coloneqq \nabla_\mu \circ \nabla_\nu$, where μ and ν are two atomless Borel probability measures supported on a compact interval such that $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu)$. Their research led to a series of publications [FZ02, Fre03a, Fre03b, Fre05, Zäh05] and these operators are often referred to as generalised Kreĭn-Feller operators, since they extend the classic work of [Fel57, KK74]. Feller, Kac and Kreĭn considered, as well as Itô and McKean [IM65], Kontani [Kot76] and Fujita [Fuj87], the operators $\Delta_{\mu,\Lambda}$, where Λ denotes the Lebesgue measure. More recent work on Kreĭn-Feller operators $\Delta_{\mu,\Lambda}$ for μ being a self-similar measure was done by Bird, Ngai and Teplyaev [BNT03] and by Freiberg and Löbus [FL04a, FL04b]. Arzt [Arz15] studied the asymptotic behaviour of the eigenvalue counting function of $\Delta_{\mu,\nu}$. He also developed generalised trigonometric functions and identities, which allowed him to obtain information on the growth of the supremum of normalised eigenfunctions $\Delta_{\mu,\Lambda}$ for a certain class of measures μ.
In [FZ02], Freiberg and Zähle considered the special class of generalised Kreĭn-Feller operators where $\mu = \nu$, namely $\Delta_\mu \coloneqq \Delta_{\mu,\mu} = \nabla_\mu \circ \nabla_\mu$. Here again μ is an atomless Borel probability measure. They developed a harmonic calculus for the operators ∇_μ and Δ_μ, including an analogue to the Gauss-Green formula and a kernel representation of the resolvent operator of Δ_μ. Further, they obtain scaling properties of the trace of the resolvent operator which they use together with a Tauberian theorem to obtain spectral asymptotics of Δ_μ for the case that μ is a self-similar measure on a Cantor-like set. More precisely, they proved for the eigenvalue counting function N_μ of $-\Delta_\mu$, when assuming either Dirichlet or von Neumann boundary conditions, that there exist constants $C > 0$ and $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ holds

$$C^{-1} \leq \frac{N_\mu(x)}{\sqrt{x}} \leq C.$$

In this thesis we consider measure-geometric differential operators for the case $\mu = \nu$ and have two main goals. The first is to strengthen the results on the eigenvalues and eigenfunctions and to additionally consider the case of periodic boundary conditions. Under all types of boundary conditions we show that the eigenvalues of Δ_μ are independent of the chosen measure, namely that the they coincide with those of the classical Laplacian. Furthermore, we show that the corresponding eigenfunctions are the composition of sine and cosine functions with the distribution function of the measure. The second goal is to generalise the approach from Freiberg and Zähle in order to define first and second order differential operators with respect to distributions with finite support and to measures with a continuous and an atomic part. For both cases we develop a harmonic calculus and obtain analytic properties of the introduced derivatives, the measure-geometric Laplacians and their domains. Also spectral properties of the Laplacians with respect to these measures are studied.

1.2 Outline and statement of main results

The main results of the thesis are Theorem 2.1.19, Theorem 2.1.20, Theorem 2.2.5, Corollary 3.1.13, Theorem 3.1.15, Corollary 3.3.3, Theorem 4.1.22, Theorem 4.2.4, Theorem 4.3.2 and Theorem 4.3.6. In the following we give a general outline of the thesis and discuss the main results in more detail.
The mathematical part of the thesis consists of three main chapters, each concerned with a different class of measures and the corresponding differential operators. We discuss atomless measures in Chapter 2, discrete distributions with finite support in Chapter 3 and a mixture of both these types in Chapter 4. Chapter 5 then concludes the thesis with an outlook on further questions concerning measure-geometric Laplace operators.

Chapter 2: Atomless Measures. Here we discuss measure-geometric derivatives and Laplacians with respect to atomless measures ν. We discuss the case that ν is supported on the unit interval. In Section 2.1 we give analytic properties of the ν-derivative ∇_ν and of the ν-Laplacian Δ_ν. Many results under Dirichlet and von Neumann boundary conditions build up on work done by Freiberg [Fre00] and Freiberg and Zähle [FZ02]. Additionally, we consider periodic boundary conditions and develop the theory for the ν-Laplacian there.
We recall and extend the results of Freiberg and Zähle in Proposition 2.1.14, where we show that the ν-Laplacian is an unbounded linear operator, it fulfils analogues of the Gauss-Green formulas and its domain is closed under multiplication with a product formula holding. In the first main result of this chapter, we show that when assuming different types of boundary conditions, the ν-Laplace operator is still densely defined.

Theorem 2.1.19 *The sets $\mathscr{D}^2_{\nu,\star}$, where $\star \in \{D, N, P\}$, are dense in L^2_ν.*

Here the sets $\mathscr{D}^2_{\nu,\star}$ are subsets of $\mathscr{D}^2_\nu$, which is the domain of Δ_ν, when assuming Dirichlet, von Neumann or periodic boundary conditions.
In the second main result of this chapter, we prove that the ν-Laplacian, when assuming one of the three types of boundary conditions, is non-positive and self-adjoint. These are very important observations, since analogous properties are well-known for the weak Laplacian.

Theorem 2.1.20 *For $\star \in \{D, N, P\}$, the operator Δ_ν restricted to $\mathscr{D}^2_{\nu,\star}$ is non-positive and self-adjoint.*

Section 2.2, the second part of this chapter, discusses the spectral properties of Δ_ν. Parts of these results are published in [KSW16] by Keßeböhmer, Samuel and Weyer. The first important result establishes a relation between the eigenfunctions of Δ_ν under Dirichlet boundary conditions and those in the von Neumann case. When assuming periodic boundary conditions, we have that the ν-derivative of an eigenfunction is again an eigenfunction.

Proposition 2.2.1

(i) If $f \in \mathscr{D}^2_\nu$ is an eigenfunction of Δ_ν fulfilling Dirichlet boundary conditions, then $\nabla_\nu f$ is also an eigenfunction of Δ_ν fulfilling von Neumann boundary conditions and, apart from the constant function, vice versa. Additionally, the corresponding eigenvalues coincide.

(ii) If $f \in \mathscr{D}^2_{\nu,P}$ is an eigenfunction of Δ_ν, then $\nabla_\nu f$ is an element of $\mathscr{D}^2_{\nu,P}$ and also an eigenfunction with the same corresponding eigenvalue.

The eigenvalue problem for Δ_ν is then solved under the three types of boundary conditions. This is done by first solving the special case that ν is the Lebesgue measure and then using a Volterra type equation to obtain the general case. It is shown, for probability measures, that the eigenvalues of Δ_ν are independent of the chosen measure and that the corresponding eigenfunctions are the composition of sine and cosine functions with the distribution function of ν which is denoted by F_ν. This implies that the asymptotic behaviour of the eigenvalue counting function is the same as for the classical Laplace operator.

Theorem 2.2.5 *Let $\lambda_n := -(\pi n)^2$, for $n \in \mathbb{N}_0$.*

(i) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,D}$ are λ_n, for $n \in \mathbb{N}$, with corresponding eigenfunctions

$$f_\nu^{(n)}(x) := \sin(\pi n F_\nu(x)), \quad \text{for } x \in [0,1].$$

(ii) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,N}$ are λ_n, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions

$$g_\nu^{(n)}(x) := \cos(\pi n F_\nu(x)), \quad \text{for } x \in [0,1].$$

(iii) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,P}$ are λ_{2n}, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions $f_\nu^{(2n)}$ for $n \in \mathbb{N}$ and $g_\nu^{(2n)}$ for $n \in \mathbb{N}_0$.

We conclude the chapter with the discussion of three examples in Section 2.3. These are self-similar measures, Salem measures and an absolutely continuous measure (with respect to Lebesgue measure) for which the eigenfunctions under von Neumann boundary conditions are the Chebyshev polynomials of the first kind.

Chapter 3: Purely Atomic Measures. The class of measures which is considered in this chapter are discrete distributions with finite support. Parts of the results are published in [KSW18a]. Unlike in the previous case, the set L^2_δ is now finite-dimensional. To obtain a well-defined operator, one needs to assume periodic boundary conditions on the space of δ-differentiable functions. In Section 3.1, we first introduce the δ-derivative ∇_δ, give a matrix representation for the operator and obtain basic properties. With the help of an introduced Dirichlet form, we define the δ-Laplace operator Δ_δ and also obtain a matrix representation for it. We then discuss connections to Graph Laplacians and discrete Laplacians on non-uniform grids. We see that the operators are independent of the position of the atoms and only the weights play a role. Important properties of the δ-Laplacian are then studied and lead to the first main results of this chapter.

Corollary 3.1.13 *The δ-Laplace operator Δ_δ is defined everywhere on L^2_δ.*

Theorem 3.1.15 *The operator Δ_δ is linear, bounded, self-adjoint and non-positive on L^2_δ.*

The second part of this chapter, Section 3.2, describes general spectral properties of the δ-Laplacian. Using the matrix representation, we obtain bounds on the eigenvalues and prove that zero is always a simple eigenvalue.

We strengthen these results in Section 3.3, where we discuss three leading examples and see that for some of these cases the obtained bounds are sharp. Since the case of a single atom

is trivial, we first solve the case of two weighted Dirac point masses. The second example is the set of uniform discrete probability distributions with N atoms. Since the δ-Laplacian does not depend on the positions of the atoms, uniform means here that all atoms have the same weight. In this case, the operator has a close connection to discrete Fourier transform and with the help of a Fourier ansatz we obtain the eigenvalues and eigenfunctions of Δ_δ.

Corollary 3.3.3 *The eigenvalues of the operator Δ_δ are $\lambda_l = -2N^2 + 2N^2\cos(2\pi l/N)$, for $l \in \{0,\dots,N-1\}$, with corresponding eigenfunctions $f_l \in \mathscr{D}^2_\delta$, where*

1. *f_0 is the constant function with value 1,*

and, for $j \in \{1,\dots,N\}$,

2. $f_l|_{(z_{j-1},z_j]} = \sin\left(2\pi l(j-1)N^{-1}\right)$ *for $0 < l < N/2$, and*
3. $f_l|_{(z_{j-1},z_j]} = \cos\left(2\pi l(j-1)N^{-1}\right)$ *for $N/2 \le l \le N-1$.*

We show that when N tends to infinity, not only does the measures converge weakly to the Lebesgue measure, but the eigenvalues and eigenfunctions of Δ_δ also converge to those of Δ_Λ. The last example which is discussed is a case of six atoms with alternating atom weights. Here we obtain an important difference to case of atomless measures, namely that the corresponding eigenfunctions cannot be written as the composition of sine or cosine functions with the distribution function of δ.

Chapter 4: Mixed Measures. This chapter focuses on measure-geometric differential operators with respect to measures which a continuous part ν and a purely atomic part δ. The results presented are partly published in [KSW18b]. We start in Section 4.1 with the definition of the η-differentiable functions and the η-derivative ∇_η, for which again periodic boundary conditions have to be assumed. We obtain properties of ∇_η and its adjoint ∇^*_η, which allow us, with the help of a Dirichlet form, to define the η-Laplacian Δ_η. This operator is studied and we see that it has many properties in common with the weak Laplacian.

Theorem 4.1.22 *The operator Δ_η is densely defined on the space of square-η-integrable functions, is linear, unbounded, self-adjoint, non-positive, has compact resolvent and its eigenfunctions form a basis of L^2_η.*

In the second part, Section 4.2, we discuss the spectral properties of Δ_η. We obtain the general form of the eigenfunctions of Δ_η and give a system of equations which has to be solved to obtain the eigenvalues and the parameters of the corresponding eigenfunctions. Without loss of generality, the assumption is made that there is an atom in one.

Theorem 4.2.4 *A square-η-integrable function f is an eigenfunction of Δ_η with corresponding eigenvalue λ if and only if it is of the form*

$$f(x) = \begin{cases} a_1 \sin(bF_\nu(x) + \gamma_1) & \text{if } x \in (0, z_1], \\ \vdots & \vdots \\ a_N \sin(bF_\nu(x) + \gamma_N) & \text{if } x \in (z_{N-1}, 1], \end{cases}$$

and $\lambda = -b^2$*, where* $b, a_1, \dots, a_N, \gamma_1, \dots, \gamma_N \in \mathbb{R}$ *satisfy the following system of equations:*

$$\alpha_j b a_{j+1} \cos(bF_\nu(z_j) + \gamma_{j+1}) = a_{j+1} \sin(bF_\nu(z_j) + \gamma_{j+1}) - a_j \sin(bF_\nu(z_j) + \gamma_j),$$
$$\alpha_j b^2 a_j \sin(bF_\nu(z_j) + \gamma_j) = a_j b \cos(bF_\nu(z_j) + \gamma_j) - a_{j+1} b \cos(bF_\nu(z_j) + \gamma_{j+1}),$$

for $j \in \{1, \dots, N-1\}$*, and*

$$\alpha_N b a_1 \cos(\gamma_1) = a_1 \sin(\gamma_1) - a_N \sin(bF_\nu(1) + \gamma_N),$$
$$\alpha_N b^2 a_N \sin(bF_\nu(1) + \gamma_N) = a_N b \cos(bF_\nu(1) + \gamma_N) - a_1 b \cos(\gamma_1).$$

In Section 4.3, we then solve this system of equations for two examples and we indicate where problems arise in the general case. We first consider measures of the form $\eta = \Lambda + \alpha\delta_1$.

Theorem 4.3.2 *The eigenvalues of* Δ_η *are* $\lambda^{(k,1)} = -(b^{(k,1)}(\alpha))^2$ *for* $k \in \mathbb{Z}$*, with corresponding eigenfunctions* $f^{(k,1)}(x) := \sin(b^{(k,1)}(\alpha)x + \gamma^{(k,1)}(\alpha))$*, where* $\gamma^{(k,1)}(\alpha)$ *denotes the unique solution in the interval* $(-\pi/2, \pi/2)$ *to the equation*

$$\tan\left(\gamma^{(k,1)}\right) = -2\gamma^{(k,1)}\alpha + \alpha\frac{\pi}{2} + 2\pi\alpha k + 1,$$

and

$$b^{(k,1)}(\alpha) := -2\gamma^{(k,1)}(\alpha) + \frac{\pi}{2} + 2\pi k.$$

We further obtain that all eigenvalues are simple and that the asymptotic behaviour of the eigenvalue counting function is the same as for the weak Laplacian.
In the second example we consider measures of the form $\eta = \Lambda + \alpha\delta_{1/2} + \alpha\delta_1$ and also solve the system of equations to obtain the eigenvalues and eigenfunctions.

Theorem 4.3.6 *The eigenvalues of* Δ_η *are* $\lambda^{(k,2)} := -(b^{(k,2)}(\alpha))^2$*, for* $k \in \mathbb{Z}$*, with corresponding eigenfunctions* $f^{(k,2)}$*.*

$$f^{(k,2)}(x) = \begin{cases} \sin\left(b^{(k,2)}(\alpha)x + \gamma_1^{(k,2)}(\alpha)\right) & \text{if } x \in \left(0, \frac{1}{2}\right], \\ \sin\left(b^{(k,2)}(\alpha)x + \gamma_2^{(k,2)}(\alpha)\right) & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $\gamma_1^{(k,2)}(\alpha)$ *denotes the unique solution in* $(-\pi/2, \pi/2)$ *to the equation*

$$\tan\left(\gamma_1^{(k,2)}\right) = 1 - 4\alpha\gamma_1^{(k,2)} + \alpha\pi + 2\pi\alpha k$$

and $\gamma_2^{(k,2)}(\alpha)$ *and* $b^{(k,2)}$ *are defined by*

$$\gamma_2^{(k,2)}(\alpha) := 3\gamma_1^{(k,2)}(\alpha) - \frac{\pi}{2},$$
$$b^{(k,2)}(\alpha) := -4\gamma_1^{(k,2)}(\alpha) + \pi + 2\pi k.$$

Further, each eigenvalue has multiplicity one.

We again obtain that the asymptotic behaviour of the eigenvalue counting function is as for atomless probability measures.
We then look at a class of measures with two non-uniformly spaced atoms. Here we show that either not all eigenvalues are simple, or that there are eigenfunctions for which in the general form discussed above, we do not have that $|a_1| = |a_2|$.

1.3 Comparison of the different Laplacians

We give a short comparison of the measure-geometric Laplacians with respect to the different types of measures. Here ν, δ and η denote the same measures as described in Chapters 2 to 4, namely ν is an atomless measure, δ is a discrete distribution with finite support and η is a measure with both a continuous and an atomic part. When writing μ, we refer to all three types of measures simultaneously. Since for δ and η it is necessary to assume periodic boundary conditions in order to get well-defined operators, we restrict the ν-Laplacian also to the set $\mathscr{D}^2_{\nu,P}$ throughout this section.

The sets of μ-differentiable functions are defined as subsets of L^2_μ. Hence, a unique characteristic of the purely atomic case is that L^2_δ is finite-dimensional. This allows us, different to the other two cases, to obtain matrix representations for ∇_δ and Δ_δ. For all three types of measures, the μ-Laplacian is defined to be $-\nabla^*_\mu \circ \nabla_\mu$. Note that in the atomless case it is necessary to assume the periodic boundary conditions to obtain this.
The μ-Laplace operators have various properties which are analogous to the properties of the classical and the weak Laplacian. Namely, we obtain in Theorems 2.1.19, 2.1.20, 3.1.15 and 4.1.22 and Corollary 3.1.13 that in all three cases, Δ_μ is densely-defined on L^2_μ, is linear, self-adjoint and non-positive. Unlike in the other two cases, the domain of the operator Δ_δ is even the whole of L^2_δ.
We see in Theorem 3.1.15, that the Laplacian with respect to a purely atomic measure is a bounded operator. This is different to the classical Laplace operator as well as to the measure-geometric Laplacians with respect to atomless measures or mixed measures. For these types of measures, we observe in Proposition 2.1.14 and Theorem 4.1.20 that the associated Laplace operators are unbounded.
From Corollaries 2.1.17, 3.1.14 and 4.1.19 it follows, for all discussed measures μ, that the μ-harmonic functions are the constants. For the observation in the atomless case, it is used that we assumed periodic boundary conditions. These observations are similar to the classical Laplacian under periodic boundary conditions. Since the kernels of the μ-Laplacians coincide, we have for all types of measures that zero is a simple eigenvalue of Δ_μ.
Apart from this fact, when looking at the spectral properties of Δ_μ, we see big differences between the three classes of measures. In the atomless case, the eigenvalues are independent of the chosen (probability) measure and they always coincide with the those from the classical Laplacian. Further, apart from zero, all eigenvalues have multiplicity two. Neither of these facts hold true for purely atomic and mixed measures in general. The first difference for discrete distributions is that Δ_δ has only finitely many eigenvalues. In Section 3.3, we see that there can be simple eigenvalues which are non-zero. Also, the eigenvalues depend on the chosen measure, namely on the chosen weights of the point masses. In the case of measures

with a continuous and an atomic part, the eigenvalues are also dependent on the measures. Here not only the weights of the atoms play an important role, but also their positions relative to continuous part. In the examples discussed in Sections 4.3.1 and 4.3.2 we see that all eigenvalues can be simple. It is worth mentioning that in both examples we obtain the same asymptotic behaviour of the eigenvalue counting functions as in the case of atomless measures. But it is unclear whether this holds in general.
A last difference we want to point out is about the eigenfunctions of Δ_μ. For atomless measures, we have a strong connection to the classical case. More precisely, we see that the eigenfunctions of Δ_ν are exactly the eigenfunctions of the classical case, composed with the distribution function F_ν. For mixed measures, we can deduce from Theorem 4.2.4 that we have at most one eigenfunction of this form. For purely atomic measures, this property depends on the chosen measure δ. While in the example discussed in Section 4.3.1 the eigenfunctions are compostions of sine and cosine functions with the left-continuous distribution function F_δ^{lc}, we prove that this is not the case for the measure given in Section 4.3.2.

1.4 Notation and basic definitions

In this section we give basic definitions and introduce notation used throughout the thesis.

1. Following conventions, we let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the *sets of natural, rational, real and complex numbers*, respectively, and $\mathbb{Z}$ denote the *set of integers*. Here, the set of natural numbers does not include zero and we write $\mathbb{N}_0$ for the *set of non-negative integers*. Further, we denote the *extended real line* by $\overline{\mathbb{R}}$.
2. For $z \in \mathbb{C}$, we let $\overline{z}$ denote its *complex conjugate*, and for $v = (v_1, \dots, v_N)$ being a complex-valued vector, we denote by $\overline{v}$ the vector $(\overline{v_1}, \dots, \overline{v_N})$. The null vector is denoted by $\underline{0}$.
3. For a set $A \subset \mathbb{R}$, we let $\overline{A}$ denote the *closure* of A, which is defined to be the smallest closed subset of $\mathbb{R}$ containing A. The *interior* of A, namely the largest open subset of $\mathbb{R}$ which is fully contained in A, is denoted by A°. The *convex hull* of A is defined to be the smallest convex subset of $\mathbb{R}$ containing A and we denote it by $\operatorname{conv}(A)$.
4. For $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, we define the sets $A + c := \{a + c \colon a \in A\}$ and $c \cdot A := \{c \cdot a \colon a \in A\}$. For the case that $c = -1$, we write $-A$ instead of $(-1) \cdot A$.
5. If F is a finite subset of $\mathbb{R}$, we let $|F|$ denote its *cardinality*.
6. Let $B \subseteq A \subseteq \mathbb{R}$. We define the *characteristic function of B*, denoted by $\mathbb{1}_B \colon A \to \mathbb{R}$, as

$$\mathbb{1}_B(x) := \begin{cases} 0 & \text{if } x \in A \setminus B, \\ 1 & \text{if } x \in B. \end{cases}$$

   For $\mathbb{1}_A$, we write $\mathbb{1}$.
7. For $X \subset \mathbb{R}$, we denote the *set of continuous functions on X* by $C(X)$ and, for $k \in \mathbb{N}$, the *set of k-times continuously differentiable functions* by $C^k(X)$. An important subset is the *set of smooth functions* $C^\infty(X) := \bigcap_{k=1}^\infty C^k(X)$. The set $C_B(X)$ denotes the *set of bounded continuous functions on X*.

For a function $\phi\colon X \to X$, we write $\operatorname{supp}(\phi)$ for the *support of the function* ϕ, which is the closure of the subset of X where ϕ is non-zero. We define the *set of compactly supported continuous functions* $C_c(X) \coloneqq \{\phi \in C(X)\colon \operatorname{supp}(\phi) \subseteq X^{\mathrm{o}} \text{ compact}\}$ and set $C_c^\infty(X) \coloneqq \{\phi \in C^\infty(X)\colon \operatorname{supp}(\phi) \subseteq X^{\mathrm{o}} \text{ compact}\}$ to be the *set of test functions*.
When X is clear from the context, it is omitted the notation of the function spaces. If $S(X)$ is a set of functions on X, the set of its non-negative elements is denoted by $S^+(X)$.

8. The *Borel σ-algebra* on the set of real numbers is denoted by $\mathfrak{B}$.
9. A map $\mu\colon \mathfrak{B} \to \overline{\mathbb{R}}$ is called a *(Borel) measure* if it satisfies the following properties:
   (i) $\mu(\emptyset) = 0$
   (ii) $\mu(\mathfrak{B}) \subseteq [0, \infty]$
   (iii) For every sequence $(B_n)_{n\in\mathbb{N}}$ with $B_n \in \mathfrak{B}$ and $B_m \cap B_n = \emptyset$, for all $m, n \in \mathbb{N}$ with $m \neq n$, it holds that
   $$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n).$$
   (iv) For every $x \in \mathbb{R}$ exists an open neighbourhood U_x of x such that $\mu(U_x) < \infty$.

   When saying measure, we always refer to a Borel measure. A measure is called *finite*, if $\mu(\mathbb{R}) < \infty$. We say that a measure is *atomless*, if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$.
10. The *support of a measure* μ is the set of points for which every neighbourhood has positive measure. This set is closed and we denote it by $\operatorname{supp}(\mu)$. For $K \subseteq \operatorname{supp}(\mu)$, we let $\mu|_K$ be the *restriction* of μ to the set K, that is, $\mu|_K(A) \coloneqq \mu(A \cap K)$ for all $A \in \mathfrak{B}$. When it is clear from the context, we write μ instead of $\mu|_K$.
11. The *(one-dimensional) Lebesgue measure on* $\mathbb{R}$ is denoted by Λ. For $z \in \mathbb{R}$, let δ_z denote the *Dirac point mass at z*. That is the measure defined, for $B \in \mathfrak{B}$, by
    $$\delta_z(B) \coloneqq \begin{cases} 0 & \text{if } z \in \mathbb{R} \setminus B, \\ 1 & \text{if } z \in B. \end{cases}$$
    For S being a countable subset of $\mathbb{R}$, we introduce the notation $\delta_S \coloneqq \sum_{s\in S} \delta_s$. A measure μ is said to be *purely atomic*, if $\mu = \delta_T$ and T is a finite subset of $\mathbb{R}$. For $d \in \mathbb{R}$ we set $d\mathbb{Z} \coloneqq \{dk\colon k \in \mathbb{Z}\}$ and call $\delta_{d\mathbb{Z}}$ a *Dirac comb*.
12. For a measure μ, the *distribution function of* μ, denoted by F_μ, is defined by
    $$\begin{aligned} F_\mu\colon \operatorname{conv}(\operatorname{supp}(\mu)) &\to \mathbb{R} \\ x &\mapsto \mu((-\infty, x]). \end{aligned}$$
    We write F_μ^{lc} for the *left-continuous distribution function of* μ, which is defined as
    $$\begin{aligned} F_\mu^{lc}\colon \operatorname{conv}(\operatorname{supp}(\mu)) &\to \mathbb{R} \\ x &\mapsto \mu((-\infty, x)). \end{aligned}$$
    The measure μ is atomless if and only if $F_\mu = F_\mu^{lc}$.

13. Fix $X \subset \mathbb{R}$. A function $f\colon X \to \mathbb{R}$ is said to be *measurable* if the pre-image of a Borel set under f is also a Borel set. For a measure μ, the *set of real-valued square-μ-integrable-functions with domain X*, denoted by $\mathfrak{L}^2_\mu(X)$, contains the measurable functions f, such that $\int_X |f|^2 \,\mathrm{d}\nu < \infty$. We set $\mathcal{N}_\mu(X)$ to be the set of $\mathfrak{L}^2_\mu(X)$-functions which are constant zero μ-almost everywhere and define the quotient space $L^2_\mu(X) \coloneqq \mathfrak{L}^2_\mu(X)/\mathcal{N}_\mu(X)$. When we write $f \in L^2_\mu(X)$, we mean that there exists an equivalence class of $L^2_\mu(X)$ to which f belongs. When it is clear from the context we use this notation for a function $f \in \mathfrak{L}^2_\mu(X)$ and for the equivalence class in $L^2_\mu(X)$ to which it belongs. We omit the base space X in the notation of the function spaces when it is clear what the domain of the functions is. We equip L^2_μ with the inner product given by

$$\langle f, g\rangle_\mu \coloneqq \int f g \,\mathrm{d}\mu.$$

The inner product defines a norm on L^2_μ, namely $\|f\|_\mu \coloneqq \sqrt{\langle f, f\rangle_\mu}$. We drop the measure in the index of the inner product and the norm when it is clear from context.

14. For a Hilbert space H, we denote by $\mathrm{id}\colon H \to H$ the *identity operator*.

15. The *domain* of a linear operator L between two vector spaces is denote by $\mathrm{dom}(L)$. By $\mathrm{ran}(L)$ we denote its *range* and by $\ker(L)$ its *kernel*.

CHAPTER 2

Atomless Measures

In this chapter we consider differentiation with respect to atomless measures ν supported on a subset of the closed interval $M := [a, b]$, where $a < b$ are two fixed real numbers. More precisely, we require the following conditions on ν:

- $\{a, b\} \subseteq \operatorname{supp}(\nu) \subseteq M$,
- $0 < P_\nu := \nu(M) < \infty$ and
- $\nu(\{z\}) = 0$ for all $z \in M$.

These properties imply that the distribution function F_ν is a continuous and monotonically increasing function mapping M onto $[0, P_\nu]$. Note that F_ν is not necessarily strictly monotonically increasing, see for example the case where ν is a singular continuous measure, discussed in Section 2.3.1.

In 2002, Freiberg and Zähle introduced the ν-derivative ∇_ν and the ν-Laplace operator Δ_ν for this setting in [FZ02]. They obtain many of the results of Section 2.1 under Dirichlet and von Neumann boundary conditions. Hence, we will give in this section only short proofs and refer the reader for more details to [Fre00, FZ02]. In this thesis we extend these results and additionally develop the theory accordingly for the case of periodic boundary conditions.
Freiberg and Zähle obtained the asymptotic behaviour of the eigenvalue counting function for self-similar measures on Cantor-like sets. In [KSW16], Keßeböhmer, Samuel and Weyer strengthened the results and some of the results from Section 2.2 can be found there. We show that the eigenvalues do not depend on the chosen probability measure and the corresponding eigenfunctions are given explicitly in terms of the distribution function. Again, we discuss in this chapter additionally the case of periodic boundary conditions. Furthermore, extending [KSW16], we discuss the multiplicity of the eigenvalues and give the general asymptotic behaviour of the eigenvalue counting function.

For ease of notation, we assume that $a = 0$ and $b = 1$, hence $M = [0, 1]$, and that ν is a probability measure, namely that $P_\nu = 1$. The results also hold for general measures ν as described above. In [KSW16], some of the results are stated and proven for general intervals M. If ν is not a probability measure similar results can be obtained when rescaling by P_ν in the appropriate steps.

Further important literature on this case is by Freiberg [Fre03a, Fre03b, Fre05], Freiberg and Löbus [FL04a, FL04b], Fujita [Fuj87], Kac and Kreĭn [KK58, KK74] and Feller [Fel57]. They discuss the generalised operators $\nabla_\nu \circ \nabla_\Lambda$ and $\nabla_\mu \circ \nabla_\nu$, where μ is also an atomless probability measure with $\operatorname{supp}\mu \subset \operatorname{supp}\nu$. Arzt discusses in [Arz15] the operator $\nabla_\nu \circ \nabla_\Lambda$ for a special class of fractal measures ν. He gives bounds for the eigenvalue counting function and constructs generalised trigonometric functions in order to obtain further properties of the eigenvalues,

The chapter is divided into three sections. In Section 2.1 we define the ν-derivative ∇_ν and the ν-Laplacian Δ_ν and obtain basic properties of these operators. The second part, Section 2.2, is about the spectral properties of Δ_ν and we prove the main theorem of this section, Theorem 2.2.5, on the eigenvalues and eigenfunctions of Δ_ν. We conclude by considering three examples in Section 2.3, namely inhomogeneous Cantor measures, Salem measures and a measure ν having the Chebyshev polynomials as eigenfunctions of the ν-Laplacian Δ_ν.

2.1 Derivative and Laplacian with respect to atomless measures

The function space on which we shall define the ν-derivative is a subset of $\mathfrak{L}^2_\nu$. It will be shown later that this subset has a natural embedding into L^2_ν and even is dense there.

Definition 2.1.1. The *set of ν-differentiable functions* $\mathscr{D}^1_\nu$ is given

$$\begin{aligned}\mathscr{D}^1_\nu = \mathscr{D}^1_\nu([0,1]) \coloneqq \Big\{ f \in \mathfrak{L}^2_\nu \colon \text{there exists } f' \in L^2_\nu \text{ such that} \\ f(x) = f(0) + \int_0^x f' \,\mathrm{d}\nu \text{ for all } x \in [0,1] \Big\}.\end{aligned} \tag{2.1}$$

This definition is motivated by the fundamental theorem of calculus, which can be seen when choosing $\nu = \Lambda$ and $f \in C^1$ with derivative f'. One can read the integral equation (2.1) as that the integral over the derivative f' equals the difference of the evaluation of the function f at the boundary points.

The integral equation in (2.1) yields that a function $f \in \mathscr{D}^1_\nu$ is piecewise constant outside the support of ν. More precisely, the equality $f(x) = f(y_x)$ holds for every $x \in [0,1] \setminus \operatorname{supp}(\nu)$, and $y_x \coloneqq \sup\{y \in \operatorname{supp}(\nu) \colon y < x\}$.

Moreover, functions in $\mathscr{D}^1_\nu$ have the following properties:

Proposition 2.1.2. *(i) Every function in $\mathscr{D}^1_\nu$ is continuous on* $[0,1]$.

(ii) For $f \in \mathscr{D}^1_\nu$ the function $f' \in L^2_\nu$ in (2.1) *is unique.*

Proof. We use the definitions and results on converging and monotone sequences of sets and on signed measures, which are introduced in Appendix A.

For (i) it is sufficient to show the left- and the right-continuity on $\operatorname{supp}(\nu)$. To this end, we define for every Borel set B

$$\varrho(B) \coloneqq \int_B f' \,\mathrm{d}\nu. \tag{2.2}$$

This defines a signed measure on $\mathfrak{B}$ with $\varrho([0,z]) = f(z) - f(0)$, for $z \in [0,1]$. Fix $x \in \operatorname{supp}(\nu)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{supp}(\nu)$ with $x_n \nearrow x$, for $n \to \infty$. Define $A \coloneqq [0,x]$ and, for $n \in \mathbb{N}$, set $A_n \coloneqq [0,x_n]$. Noting that $A_n \nearrow A$ as $n \to \infty$, by the measure-geometric arguments given in Lemma A.1.4 we have that $\varrho(A_n) \to \varrho(A)$. Hence, $f(x_n) \to f(x)$ which is the left-continuity on $\operatorname{supp}(\nu)$. The right-continuity also follows by Lemma A.1.4, using decreasing sequences of sets $(B_n)_{n\in\mathbb{N}}$ with $B_1 = [0,1]$, due to the fact that $|\varrho(B_1)|^2 \le \|f'\|^2 < \infty$.
The measure ϱ defined in (2.2) is absolutely continuous with respect to ν. The Radon-Nikodym theorem implies that f' is uniquely defined ν-almost everywhere, which yields (ii). □

From these observations, we conclude that if $f, g \in \mathscr{D}_\nu^1$ with $f \neq g$, then $\|f - g\| \neq 0$. Hence, there exists a natural embedding $\pi\colon \mathscr{D}_\nu^1 \to L_\nu^2$. In the following we will not distinguish between $\mathscr{D}_\nu^1$ and $\pi(\mathscr{D}_\nu^1)$ and write $\mathscr{D}_\nu^1$ in both cases.

Definition 2.1.3. For $f \in \mathscr{D}_\nu^1$ let f' be as in (2.1). The operator

$$\begin{aligned} \nabla_\nu\colon \mathscr{D}_\nu^1 &\to L_\nu^2 \\ f &\mapsto f' \end{aligned}$$

is called the *ν-derivative*.

A special case is $\nu = \Lambda$. Here, from the fundamental theorem of calculus, it holds that a function f lies in $\mathscr{D}_\Lambda^1$ if and only if it is absolutely continuous. This implies that the classical derivative of f exists for Λ-almost every $x \in [0,1]$ and that it coincides with the function f' given in (2.1) (in the L_Λ^2-sense). Since a measurable function is absolutely continuous if and only if the weak derivative exists and the weak derivative coincides with the classical derivative Λ-almost everywhere, we have that $\mathscr{D}_\Lambda^1$ coincides with the Sobolev space H_1^2 and the Λ-derivative with the weak derivative. Thus, we have that $C^1 \subseteq \mathscr{D}_\Lambda^1 \subseteq C \subseteq L_\Lambda^2$, where each set is dense, with respect to $\|\cdot\|_\Lambda$, in the succeeding one. For more details on absolutely continuous functions and the proofs of the statements, see [Gor94, p. 57ff].

Analogously to the classical case, we obtain the following properties of the ν-derivative.

Proposition 2.1.4. *Fix $f, g \in \mathscr{D}_\nu^1$ and $a, b \in [0,1]$ with $a \le b$.*

(i) *The operator $\nabla_\nu\colon \mathscr{D}_\nu^1 \to L_\nu^2$ is linear.*

(ii) $\ker(\nabla_\nu) = \{c\mathbb{1} : c \in \mathbb{R}\}$

(iii) $\int_a^b \nabla_\nu f \,\mathrm{d}\nu = f(b) - f(a)$

(iv) *It holds that $f \cdot g \in \mathscr{D}_\nu^1$ with $\nabla_\nu(f \cdot g) = (\nabla_\nu f) \cdot g + f \cdot (\nabla_\nu g)$.*

(v) $\int_a^b (\nabla_\nu f) \cdot g \,\mathrm{d}\nu = [f \cdot g]_a^b - \int_a^b f \cdot (\nabla_\nu g) \,\mathrm{d}\nu$

Proof. The properties (i), (ii) and (iii) directly follow from the definition of ∇_ν and the linearity of the integral. The product rule (iv) follows from the definition of $\mathscr{D}^1_\nu$ and Fubini's theorem and implies together with (iii) the integration by parts formula stated in (v). □

Lemma 2.1.5. *We have that*

(i) $\nu \circ F_\nu^{-1} = \Lambda$ *and*

(ii) $\Lambda \circ F_\nu = \nu$.

Proof. It is sufficient to show the equality of the measures on a generator of $\mathfrak{B}$ which is stable under intersection. To this end, let $[a_1, b_1]$ be an arbitrary closed subinterval of $[0,1]$. For (i) we set $T \coloneqq \{x \in [0,1] \colon a_1 \leq F_\nu(x) \leq b_1\}$ and observe that

$$\nu(F_\nu^{-1}([a_1,b_1])) = \nu(T) = F_\nu(\sup\{x \colon x \in T\}) - F_\nu(\inf\{x \colon x \in T\}) = b_1 - a_1 = \Lambda([a_1,b_1]).$$

This yields the result stated in (i). For (ii) we observe, for a subinterval $[a_2, b_2] \subseteq [0,1]$, that

$$\Lambda(F_\nu([a_2,b_2])) = \Lambda([F_\nu(a_2), F_\nu(b_2)]) = F_\nu(b_2) - F_\nu(a_2) = \nu([a_2,b_2]),$$

concluding the proof. □

Lemma 2.1.6. *A function f lies in $\mathscr{D}^1_\Lambda$ with derivative $\nabla_\Lambda f$ if and only if $f \circ F_\nu \in \mathscr{D}^1_\nu$ with $\nabla_\nu(f \circ F_\nu) = \nabla_\Lambda f \circ F_\nu$.*

Proof. We only show the forward direction explicitly. The backward direction follows in a similar manner from Lemma 2.1.5 by concatenation with F_ν^{-1} when needed and thus is omitted here.

For $f \in \mathscr{D}^1_\Lambda$, there exists by definition $\nabla_\Lambda f$ with

$$f(x) = f(0) + \int_0^x \nabla_\Lambda f \, \mathrm{d}\Lambda,$$

for all $x \in [0,1]$, and so, using Lemma 2.1.5,

$$f(F_\nu(x)) = f(0) + \int_0^{F_\nu(x)} \nabla_\Lambda f \, \mathrm{d}\Lambda = f(0) + \int_0^x (\nabla_\Lambda f) \circ F_\nu \, \mathrm{d}\nu.$$

This, combined with the fact that $F_\nu(0) = 0$, yields the forward direction. □

As already discussed for the special case $\nu = \Lambda$, it holds in general that the set of ν-differentiable functions is dense in L^2_ν.

Proposition 2.1.7. *The set $\mathscr{D}^1_\nu$ is a dense subset of L^2_ν.*

Proof. Let p be a real-valued polynomial on $[0,1]$. Then p is differentiable with derivative p' and it holds for all $x \in [0,1]$ that

$$\int_0^x p' \, \mathrm{d}\Lambda = p(x) - p(0).$$

This shows that $p \in \mathscr{D}^1_\Lambda$ and since the set of all polynomials separates points, the Stone-Weierstrass theorem implies that $\mathscr{D}^1_\Lambda$ is dense in L^2_Λ. The result for arbitrary measures ν then follows from Lemma 2.1.5 and Lemma 2.1.6 by composition of the polynomials with the density function F_ν. □

As we will see in the following, it is not only the set of ν-differentiable functions which is dense in L^2_ν, but also the range of the ν-derivative ∇_ν.

Proposition 2.1.8. *The range of ∇_ν is dense in L^2_ν.*

Proof. The set of bounded continuous functions C_B is dense in L^2_ν. Setting $f(x) \coloneqq \int_0^x g \, \mathrm{d}\nu$, for $g \in C_B$ and $x \in [0,1]$, we observe that f is continuous and bounded, and so $f \in L^2_\nu$. Since $g \in C_B$, we have that $f \in \mathscr{D}^1_\nu$ with $\nabla_\nu f = g$, and the result follows. □

One final important property of the ν-derivative that we show is the unboundedness. This property is again analogous to the theory of classical differentiation and it is proven by giving an explicit family of functions.

Proposition 2.1.9. *The ν-derivative ∇_ν is an unbounded operator on $\mathscr{D}^1_\nu$.*

Proof. Set $\psi_n(x) \coloneqq n\pi F_\nu(x)$ for $x \in [0,1]$ and $n \in \mathbb{N}$. The measure identities in Lemma 2.1.5 can be used to obtain that $\int_0^x n\pi \cos(n\pi F_\nu(y)) \, \mathrm{d}\nu(y) = \sin(n\pi F_\nu(x))$ for $x \in [0,1]$ and $n \in \mathbb{N}$. This yields $(\sin \circ \psi_n) \in \mathscr{D}^1_\nu$ with $\nabla_\nu(\sin \circ \psi_n) = n\pi \cdot (\cos \circ \psi_n)$. The result then follows from the observation that $\|\sin \circ \psi_n\| = \|\cos \circ \psi_n\| = 1/\sqrt{2}$ for all $n \in \mathbb{N}$. □

Before defining the ν-Laplacian, we state a result on the adjoint operator of ∇_ν in a restricted setting. This result will allow us later to give a comparison between Laplace operators with respect to different types of measures. We denote the set of the *periodic ν-differentiable functions* by $\mathscr{D}^1_{\nu,P} \coloneqq \{f \in \mathscr{D}^1_\nu : f(0) = f(1)\}$ and look at the ν-derivative ∇_ν restricted to $\mathscr{D}^1_{\nu,P}$, which we denote by $\nabla_{\nu,P}$ for easier distinction of the cases. Note that it follows from the periodic boundary condition that for all $f \in \mathscr{D}^1_{\nu,P}$ holds

$$\int_0^1 \nabla_{\nu,P} f \, \mathrm{d}\nu = 0. \tag{2.3}$$

Letting ϱ denote the natural quotient map from L^2_ν to $L^2_\nu/\{c\mathbb{1} : c \in \mathbb{R}\}$, an argument similar to the one given in Proposition 2.1.8 implies that $\varrho(\mathrm{ran}(\nabla_{\nu,P}))$, namely the image under ϱ of the range of $\nabla_{\nu,P}$, is dense in the quotient space $L^2_\nu/\{c\mathbb{1} : c \in \mathbb{R}\}$.

We denote the adjoint operator of ∇_ν by ∇^*_ν. In the following we prove that the domains of ∇_ν and ∇^*_ν coincide and that the operators only differ in sign.

Lemma 2.1.10. *The domain of $\nabla^*_{\nu,P}$ is equal to $\mathscr{D}^1_{\nu,P}$ and it holds that $\nabla^*_{\nu,P} = -\nabla_{\nu,P}$.*

Proof. A function f belongs to $\operatorname{dom}(\nabla^*_{\nu,P})$, if there exists $h \in L^2_\nu$ such that $\langle f, \nabla_\nu g\rangle = \langle h, g\rangle$ for all $g \in \mathscr{D}^1_{\nu,P}$; in which case $\nabla^*_{\nu,P} f = h$. Thus, if $f \in \operatorname{dom}(\nabla^*_{\nu,P})$, for all $g \in \mathscr{D}^1_{\nu,P}$, we have

$$\begin{aligned}\int_0^1 f\nabla_{\nu,P} g \,\mathrm{d}\nu &= \int_0^1 \nabla^*_{\nu,P} f g \,\mathrm{d}\nu = \int_0^1 \nabla^*_{\nu,P} f(x)\left(g(0) + \int_0^x \nabla_{\nu,P} g(y)\,\mathrm{d}\nu(y)\right)\mathrm{d}\nu(x)\\ &= g(0)\int_0^1 \nabla^*_{\nu,P} f \,\mathrm{d}\nu + \int_0^1\int_0^x \nabla^*_{\nu,P} f(x)\nabla_{\nu,P} g(y)\,\mathrm{d}\nu(y)\,\mathrm{d}\nu(x).\end{aligned}$$

Using the fact that $\langle \nabla^*_{\nu,P} f, \mathbb{1}\rangle = \langle f, \nabla_{\nu,P}\mathbb{1}\rangle = \langle f, 0\rangle = 0$ and Fubini's theorem, one obtains that

$$\int_0^1 \left(f(y) - \int_y^1 \nabla^*_{\nu,P} f(x)\,\mathrm{d}\nu(x)\right)\nabla_{\nu,P} g(y)\mathrm{d}\nu(y) = 0. \tag{2.4}$$

From the argument given above that $\varrho(\operatorname{ran}(\nabla_{\nu,P}))$ is dense in the quotient space $L^2_\nu/\{c\mathbb{1} : c \in \mathbb{R}\}$, we may conclude from (2.4) that

$$\varrho\left(f - \int \mathbb{1}_{[\cdot,1]}\nabla^*_{\nu,P} f(x)\,\mathrm{d}\nu(x)\right) = 0.$$

Hence, there exists $c \in \mathbb{R}$ such that, for ν-almost every $y \in [0,1]$,

$$f(y) - \int_y^1 \nabla^*_{\nu,P} f \,\mathrm{d}\nu = c.$$

Thus f is a continuous function with $c = f(1) = f(0)$. Combining this with the fact that $\langle \nabla^*_{\nu,P} f, \mathbb{1}\rangle = 0$, we obtain $f(y) = f(0) + \int_0^y -\nabla^*_{\nu,P} f \,\mathrm{d}\nu$ for all $y \in [0,1]$, which implies that $\operatorname{dom}(\nabla^*_{\nu,P}) \subseteq \mathscr{D}^1_{\nu,P}$.

For the backward direction, it is sufficient to show that all $f, g \in \mathscr{D}^1_{\nu,P}$ satisfy the equation $\langle \nabla_{\nu,P} f, g\rangle = -\langle f, \nabla_{\nu,P} g\rangle$, since this also implies that $\mathscr{D}^1_{\nu,P} \subseteq \operatorname{dom}(\nabla^*_{\nu,P})$. This follows directly from Proposition 2.1.4 (v) and the periodic boundary conditions. □

As mentioned earlier, the last lemma was stated merely for completeness and will be used later to obtain an additional result on the ν-Laplace operator. We now return to the set $\mathscr{D}^1_\nu$ and the operator ∇_ν, not assuming any boundary conditions.

We now use the ν-derivative to define a Laplace operator with respect to the measure ν. This operator and its domain will be introduced and different representations and analytic properties will be obtained. In many aspects the operator behaves similarly to the classical theory.

Definition 2.1.11. The *set of twice ν-differentiable functions* is defined by

$$\begin{aligned}\mathscr{D}^2_\nu = \mathscr{D}^2_\nu([0,1]) \coloneqq \Big\{ f \in \mathscr{D}^1_\nu :\ &\text{there exists } f'' \in L^2_\nu \text{ such that}\\ &\nabla_\nu f(x) = \nabla_\nu f(0) + \int_0^x f''(y)\,\mathrm{d}\nu(y) \text{ for all } x \in [0,1]\Big\}.\end{aligned} \tag{2.5}$$

Recall that, by Proposition 2.1.2, the ν-derivative of a function is unique in L^2_ν and that ν-differentiable functions are continuous. These observations lead to a natural embedding of $\mathscr{D}^1_\nu$ in L^2_ν. In this sense, the pointwise evaluation of L^2_ν-functions in (2.5) is to be understood as evaluating the continuous representative of the equivalence class which fulfils the equality $f(x) = f(y_x)$ for every $x \in \operatorname{supp}(\nu)^c$, where $y_x := \sup\{y \in \operatorname{supp}(\nu) \colon y < x\}$.

Proposition 2.1.12. *The set $\mathscr{D}^2_\nu$ has the following properties:*

(i) For $f \in \mathscr{D}^2_\nu$ the function $f'' \in L^2_\nu$ given in (2.5) *is unique.*

(ii) $\mathscr{D}^2_\nu = \{f \in \mathscr{D}^1_\nu \colon \nabla_\nu f \in \mathscr{D}^1_\nu\}$

(iii) The set $\mathscr{D}^2_\nu$ is a dense subset of L^2_ν.

Proof. The statement (i) follows in a similar manner as Proposition 2.1.2 using the uniqueness of the ν-derivative. By definition (ii) holds true. Since the set of polynomials is a subset of $\mathscr{D}^2_\Lambda$, the same arguments as in the proof of Proposition 2.1.7 show (iii). □

On the set $\mathscr{D}^2_\nu$ we now define the measure-geometric analogue to the classical Laplacian.

Definition 2.1.13. For $f \in \mathscr{D}^2_\nu$ let f'' be as in (2.5). The operator

$$\begin{aligned} \Delta_\nu \colon \mathscr{D}^2_\nu &\to L^2_\nu \\ f &\mapsto f'' \end{aligned}$$

is called the *ν-Laplace operator* or the *ν-Laplacian*.

From Proposition 2.1.12 (i) it follows that Δ_ν is well-defined. Furthermore, from its definition and Proposition 2.1.12 (ii) it follows that the ν-Laplace operator is the composition of the ν-derivative with itself, namely

$$\Delta_\nu = \nabla_\nu \circ \nabla_\nu. \tag{2.6}$$

We now obtain analytic properties of Δ_ν which are analogous to the classical case, such as the Gauß-Green formulas (see Proposition 2.1.14 (v) and (vi)).

Proposition 2.1.14. *Fix $f, g \in \mathscr{D}^2_\nu$, $h \in \mathscr{D}^1_\nu$ and $a, b \in [0,1]$ with $a \le b$.*

(i) The operator $\Delta_\nu \colon \mathscr{D}^2_\nu \to L^2_\nu$ is linear.

(ii) The operator $\Delta_\nu \colon \mathscr{D}^2_\nu \to L^2_\nu$ is unbounded.

(iii) It holds that $f \cdot g \in \mathscr{D}^2_\nu$ with $\Delta_\nu(f \cdot g) = (\Delta_\nu f) \cdot g + f \cdot (\Delta_\nu g) + 2(\nabla_\nu f) \cdot (\nabla_\nu g)$.

(iv) $\int_a^b \Delta_\nu f \, \mathrm{d}\nu = \nabla_\nu f(b) - \nabla_\nu f(a)$

(v) $\int_a^b (\Delta_\nu f) \cdot h \, \mathrm{d}\nu = [(\nabla_\nu f) \cdot h]_a^b - \int_a^b (\nabla_\nu f) \cdot (\nabla_\nu h) \, \mathrm{d}\nu$

(vi) $\int_a^b (\Delta_\nu f) \cdot g - f \cdot (\Delta_\nu g) \, \mathrm{d}\nu = [(\nabla_\nu f) \cdot g - f \cdot (\nabla_\nu g)]_a^b$

Proof. Combining (2.6) with the linearity of ∇_ν implies (i). Part (ii) follows analogously to Proposition 2.1.9, obtaining that $\Delta_\nu(\sin(n\pi F_\nu(\cdot))) = -n^2\pi^2 \sin(n\pi F_\nu(\cdot))$ for $n \in \mathbb{N}$. Applying Proposition 2.1.4 (iv) three times yields (iii). Part (iv) is a direct consequence of Proposition 2.1.4 and Proposition 2.1.12 (ii). The integration by parts formula in (v) follows from (iv), Proposition 2.1.4 and Fubini's theorem and it yields (vi). □

Combining the integral equations in (2.1) and (2.5) leads to a different representation of functions in $\mathscr{D}_\nu^2$. Furthermore, this new integral equation provides an alternative defining equation for the set $\mathscr{D}_\nu^2$.

Lemma 2.1.15. *(i) Every function $f \in \mathscr{D}_\nu^2$ has the following representation:*

$$f(x) = f(0) + \nabla_\nu f(0) \cdot F_\nu(x) + \int_0^x (F_\nu(x) - F_\nu(y))\, \Delta_\nu f(y)\, \mathrm{d}\nu(y)$$

(ii) Let $c_1, c_2 \in \mathbb{R}$ and $g \in L_\nu^2$. Then the function

$$f(x) \coloneqq c_1 + c_2 \cdot F_\nu(x) + \int_0^x (F_\nu(x) - F_\nu(y))\, g(y)\, \mathrm{d}\nu(y)$$

lies in $\mathscr{D}_\nu^2$ with continuous version of the ν-derivative given by $\nabla_\nu f(x) = c_2 + \int_0^x g\, \mathrm{d}\nu$ for $x \in [0,1]$ and $\Delta_\nu f = g$.

Proof. The representation (i) follows from substituting (2.5) in (2.1) and applying Fubini's theorem to the resulting double integral.

To obtain (ii), note that it follows from the definition of f that $f(0) = c_1$. We define the function $h(y) \coloneqq c_2 + \int_0^y g\, \mathrm{d}\nu$ which is continuous, since the measure ν is atomless. With Fubini's theorem we can deduce that

$$f(x) - f(0) = \int_0^x h\, \mathrm{d}\nu,$$

which implies that $f \in \mathscr{D}_\nu^1$ with $\nabla_\nu f = h$. Also, from the definition of h, it follows that $h(0) = c_2$ and that $h \in \mathscr{D}_\nu^1$. This concludes the proof, since in Proposition 2.1.12 (ii) we showed that $\mathscr{D}_\nu^2 = \{f \in \mathscr{D}_\nu^1 \colon \nabla_\nu f \in \mathscr{D}_\nu^1\}$. □

The results from Lemma 2.1.15 can be generalised using the finite additivity of the measure ν. Here, if $a \le b$, we use the integral identity $\int_b^a f\, \mathrm{d}\nu = -\int_a^b f\, \mathrm{d}\nu$ for an integrable function f.

Corollary 2.1.16. *Let $x_0 \in [0,1]$.*

(i) If $f \in \mathscr{D}_\nu^2$, then the function has the following representation:

$$f(x) = f(x_0) + \nabla_\nu f(x_0) \cdot (F_\nu(x) - F_\nu(x_0)) + \int_{x_0}^x (F_\nu(x) - F_\nu(y))\, \Delta_\nu f(y)\, \mathrm{d}\nu(y).$$

(ii) Let $c_1, c_2 \in \mathbb{R}$ and $g \in L^2_\nu$. The function

$$f(x) \coloneqq c_1 + c_2 \cdot (F_\nu(x) - F_\nu(x_0)) + \int_{x_0}^{x} (F_\nu(x) - F_\nu(y))\, g(y)\, \mathrm{d}\nu(y)$$

lies in $\mathscr{D}^2_\nu$ with continuous version of the ν-derivative given by $\nabla_\nu f(x) = c_2 + \int_{x_0}^{x} g \,\mathrm{d}\nu$ for $x \in [0,1]$ and $\Delta_\nu f = g$.

Having these integral representations for functions in $\mathscr{D}^2_\nu$, we can now determine the kernel of Δ_ν easily. Functions in the kernel of the classical Laplacian are often referred to as *harmonic functions*. Analogously, we say that a function $f \in \mathscr{D}^2_\nu$ is *ν-harmonic*, if it vanishes under Δ_ν. The *set of the ν-harmonic functions* is denoted by

$$\mathcal{H}_\nu = \left\{ f \in \mathscr{D}^2_\nu \colon \Delta_\nu f \equiv 0 \right\}.$$

It is a direct consequence of Lemma 2.1.15 that $\mathcal{H}_\nu$ is a two-dimensional vector space and that it consists of functions of the following form.

Corollary 2.1.17.

$$\mathcal{H}_\nu = \{ x \mapsto A + BF_\nu(x) \colon A, B \in \mathbb{R} \}.$$

For $\nu = \Lambda$, the set $\mathcal{H}_\Lambda$ is the set of linear functions. This means that a function is Λ-harmonic if and only if it is harmonic.

To study the operator further, we assume boundary conditions from now on. Here we discuss three different types of boundary conditions, namely homogeneous Dirichlet, homogeneous von Neumann and periodic boundary conditions.

Definition 2.1.18. (i) The set $\mathscr{D}^2_{\nu,D} \coloneqq \{ f \in \mathscr{D}^2_\nu \colon f(0) = f(1) = 0 \}$ is the set of twice ν-differentiable functions fulfilling *homogeneous Dirichlet boundary conditions*.

(ii) The set $\mathscr{D}^2_{\nu,N} \coloneqq \{ f \in \mathscr{D}^2_\nu \colon \nabla_\nu f(0) = \nabla_\nu f(1) = 0 \}$ is the set of twice ν-differentiable functions fulfilling *homogeneous von Neumann boundary conditions*.

(iii) The set $\mathscr{D}^2_{\nu,P} \coloneqq \{ f \in \mathscr{D}^2_\nu \colon f(0) = f(1) \text{ and } \nabla_\nu f(0) = \nabla_\nu f(1) \}$ is the set of twice ν-differentiable functions fulfilling *periodic boundary conditions*.

To simplify the notation we will omit the word homogeneous and just write Dirichlet and von Neumann boundary conditions. When talking about more than one of the three sets $\mathscr{D}^2_{\nu,D}$, $\mathscr{D}^2_{\nu,N}$ and $\mathscr{D}^2_{\nu,P}$ simultaneously, we write $\mathscr{D}^2_{\nu,\star}$, where $\star \in M$ and $M \subseteq \{D, N, P\}$. The set M will be specified if it is not clear from the context. If we write $f, g \in \mathscr{D}^2_{\nu,\star}$ for $\star \in \{D, N\}$ we mean that either $f, g \in \mathscr{D}^2_{\nu,D}$ or $f, g \in \mathscr{D}^2_{\nu,N}$ and not $f \in \mathscr{D}^2_{\nu,D}$ and $g \in \mathscr{D}^2_{\nu,N}$ or vice versa. For other $M \subseteq \{D, N, P\}$ the term $f, g \in \mathscr{D}^2_{\nu,\star}$ for $\star \in M$ is meant in a similar way.

In the next theorem, we see that when restricting the ν-Laplacian to one of these sets, it is still densely defined on L^2_ν.

Theorem 2.1.19. *The sets $\mathscr{D}^2_{\nu,\star}$, where $\star \in \{D, N, P\}$, are dense in L^2_ν.*

Proof. The set $C_c^\infty = C_c^\infty([0,1])$ denotes the set of test functions on the interval $[0,1]$. Note that these functions are differentiable and, since they are continuous on $[0,1]$, lie in L^2_Λ. By classical analytic results one can show that the space C_c^∞ is a dense subspace of L^2_Λ. Further, every function in C_c^∞ fulfils Dirichlet, von Neumann and periodic boundary conditions, which implies that $\mathscr{D}^2_{\Lambda,\star}$ is a dense subset of L^2_Λ. Combining this with Lemma 2.1.6 and Proposition 2.1.12 (ii) yields the general result. □

Periodic and Dirichlet boundary conditions imply that the integral of the ν-derivative vanishes, namely that for $f \in \mathscr{D}^2_{\nu,\star}$,where $\star \in \{D, P\}$,

$$\int_0^1 \nabla_\nu f \, \mathrm{d}\nu = 0. \tag{2.7}$$

Hence, in these cases the only constant function which can occur as a ν-derivative is the zero function. By definition, this observation also holds true under von Neumann boundary conditions, even though equation (2.7) does not hold necessarily.

Note that from Lemma 2.1.10 it follows that on $\mathscr{D}^2_{\nu,P}$ the representation of the ν-Laplace operator (2.6) can be rewritten as

$$\Delta_\nu = -\nabla_\nu^* \circ \nabla_\nu. \tag{2.8}$$

Assuming one of the three types of boundary conditions, we can show various properties of the ν-Laplace operator which are similar to the classical case.

Theorem 2.1.20. *For $\star \in \{D, N, P\}$, the operator Δ_ν restricted to $\mathscr{D}^2_{\nu,\star}$ is non-positive and self-adjoint.*

Proof. Let $f, g \in \mathscr{D}^2_{\nu,\star}$. The non-positivity of Δ_ν follows from Proposition 2.1.14 (v) and the respective boundary conditions which yield

$$\langle \Delta_\nu f, f \rangle = -\int_0^1 (\nabla_\nu f)^2 \, \mathrm{d}\nu \leq 0.$$

Proposition 2.1.14 (vi) established that

$$\int_0^1 (\Delta_\nu f) \cdot g - f \cdot (\Delta_\nu g) \, \mathrm{d}\nu = \Big[(\nabla_\nu f) \cdot g - f \cdot (\nabla_\nu g) \Big]_0^1.$$

The evaluation of the right-hand side of this equation gives us zero for all three types of boundary conditions, which implies that $\langle \Delta_\nu f, g \rangle = \langle f, \Delta_\nu g \rangle$ and hence, Δ_ν is symmetric.

To show the self-adjointness of the ν-Laplacian it is left to show that $\mathrm{dom}(\Delta_\nu) = \mathrm{dom}(\Delta_\nu^*)$, when restricting the operator to $\mathscr{D}^2_{\nu,\star}$. The symmetry of the operator implies $\mathrm{dom}(\Delta_\nu) \subseteq \mathrm{dom}(\Delta_\nu^*)$ it only the reverse inclusion is left to show. For proof of the self-adjointness under Dirichlet

and von Neumann boundary conditions we use a result by Freiberg, which is not proven in this thesis. In [Fre00, Theorem 2.3.10], it is shown that Δ_ν restricted to either $\mathscr{D}^2_{\nu,D}$ or $\mathscr{D}^2_{\nu,N}$ is surjective. This implies that for $f \in \mathrm{dom}(\Delta^*_\nu)$ there exists $g \in \mathscr{D}^2_{\nu,\star}$ such that $\Delta^*_\nu f = \Delta_\nu g$. For all $h \in \mathscr{D}^2_{\nu,\star}$ then holds that

$$\langle f, \Delta_\nu h\rangle = \langle \Delta^*_\nu f, h\rangle = \langle \Delta_\nu g, h\rangle = \langle g, \Delta_\nu h\rangle.$$

Hence, the surjectivity implies that $\langle f, k\rangle = \langle g, k\rangle$ for all $k \in L^2_\nu$. This yields $f = g$ and so we have $\mathrm{dom}(\Delta^*_\nu) \subseteq \mathrm{dom}(\Delta_\nu)$.

The case of periodic boundary conditions follows from Lemma 2.1.10. Here we showed that $\mathrm{dom}(\nabla_{\nu,P}) = \mathrm{dom}(\nabla^*_{\nu,P})$ with $\nabla^*_{\nu,P} = -\nabla_{\nu,P}$, where the additional index indicates the assumed boundary conditions. This together with (2.6) yields the result. □

2.2 Eigenvalues and eigenfunctions

In this section we discuss the spectral properties of the operator Δ_ν. It will be shown that the eigenvalues do not depend on the chosen probability measure, but the eigenfunctions do. We will see that the eigenfunctions are sine and cosine functions which are composed with the distribution function of the measure ν.

The definitions of the main objects, such as eigenvalue, eigenfunction and eigenspace, and basic results on them can be found in Appendix C.
We examine the eigenvalue equation for Δ_ν, namely we look for $\lambda \in \mathbb{R}$ and $f \in \mathscr{D}^2_{\nu,\star}$, where $\star \in \{D, N, P\}$, such that

$$\Delta_\nu f = \lambda f. \tag{2.9}$$

Two observations on the eigenfunctions can be made directly from (2.9) using the different types of boundary conditions.

Proposition 2.2.1. *(i) If $f \in \mathscr{D}^2_\nu$ is an eigenfunction of Δ_ν fulfilling Dirichlet boundary conditions, then $\nabla_\nu f$ is also an eigenfunction of Δ_ν fulfilling von Neumann boundary conditions and, apart from the constant function, vice versa. Additionally, the corresponding eigenvalues coincide.*

(ii) If $f \in \mathscr{D}^2_{\nu,P}$ is a non-constant eigenfunction of Δ_ν, then $\nabla_\nu f$ is an element of $\mathscr{D}^2_{\nu,P}$ and also an eigenfunction with the same corresponding eigenvalue.

Proof. Let f be an eigenfunction of Δ_ν with corresponding eigenvalue λ. Equation (2.6) and the linearity of ∇_ν shown in Proposition 2.1.4 (i) yield

$$\Delta_\nu(\nabla_\nu f) = \nabla_\nu(\Delta_\nu f) = \nabla_\nu(\lambda f) = \lambda(\nabla_\nu f).$$

If f fulfils Dirichlet boundary conditions then $\Delta_\nu f$ also fulfils them, since f is an eigenfunction. Therefore, $\nabla_\nu f$ satisfies von Neumann boundary conditions since $\nabla_\nu \circ \nabla_\nu f(0) = \Delta_\nu f(0) =$

$\lambda f(0) = 0$ and also $\nabla_\nu \circ \nabla_\nu f(1) = \Delta_\nu f(1) = \lambda f(1) = 0$. If f fulfils von Neumann boundary conditions, we have that $\nabla_\nu f(0) = \nabla_\nu f(1) = 0$, which means by definition that $\nabla_\nu f$ satisfies Dirichlet boundary conditions. If f fulfils periodic boundary conditions, then $\nabla_\nu f$ also satisfies periodic boundary conditions. □

In the following, we show that the eigenvalues of Δ_ν do not depend on the measure ν and give a full characterisation of the corresponding eigenfunctions. This is done in two steps: In Theorem 2.2.2, we first look at the case $\nu = \Lambda$ and in a second step we obtain the solution for a general probability measures ν using a Volterra type equation given in Lemma 2.2.3.

Theorem 2.2.2. *Let $\lambda_n := -(\pi n)^2$, for $n \in \mathbb{N}_0$.*

(i) The eigenvalues of Δ_Λ on $\mathscr{D}^2_{\Lambda,D}$ are λ_n, for $n \in \mathbb{N}$, with corresponding eigenfunctions

$$f^{(n)}_\Lambda(x) = \sin(\pi n x), \qquad \text{for } x \in [0,1].$$

(ii) The eigenvalues of Δ_Λ on $\mathscr{D}^2_{\Lambda,N}$ are λ_n, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions

$$g^{(n)}_\Lambda(x) = \cos(\pi n x), \qquad \text{for } x \in [0,1].$$

(iii) The eigenvalues of Δ_Λ on $\mathscr{D}^2_{\Lambda,P}$ are λ_{2n}, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions $f^{(2n)}$ for $n \in \mathbb{N}$ and $g^{(2n)}$ for $n \in \mathbb{N}_0$.

Proof. On the set of twice differentiable functions supported on $[0,1]$ the Λ-Laplacian coincides with the classical Laplacian. Hence, $f^{(n)}_\Lambda$ and $g^{(n)}_\Lambda$ fulfil the eigenvalue equation (2.9) for $\nu = \Lambda$. To conclude the proof, it is sufficient to show that the set $B := \{f^{(n)}_\Lambda, g^{(m)}_\Lambda : n \in \mathbb{N}, m \in \mathbb{N}_0\}$ forms an orthogonal system and that $\operatorname{span}(B)$ is dense in L^2_Λ. These are well-known facts which are proven for example in [Alt99, Example 7.9]. Looking at the different types of boundary conditions yields the three cases (i) – (iii).

For completeness, we give a sketch of the proof as done in [Alt99, Example 7.9]. To do this, we look, for $k \in \mathbb{Z}$, at the complex functions

$$e_k(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathbf{i}kx}$$

and show that they form a orthonormal basis of $L^2_{\mathbb{C}}((-\pi,\pi))$, where $L^2_{\mathbb{C}}((-\pi,\pi))$ is the set of (equivalence classes) of complexed-valued and square-Λ-integrable functions on $(-\pi,\pi)$. An appropriate rescaling by 2π and the identities

$$\sin(x) = \frac{\mathrm{e}^{\mathbf{i}x} - \mathrm{e}^{-\mathbf{i}x}}{2\mathbf{i}} \quad \text{and} \quad \cos(x) = \frac{\mathrm{e}^{\mathbf{i}x} + \mathrm{e}^{-\mathbf{i}x}}{2}$$

then imply that B is an orthogonal system and $\operatorname{span}(B)$ is dense in L^2_Λ.

The orthogonality of the functions e_k can be obtained directly from the definition of the inner product on $L^2_{\mathbb{C}}((-\pi,\pi))$, which is given for $f,g \in L^2_{\mathbb{C}}((-\pi,\pi))$ by $\langle f,g\rangle_{L^2_{\mathbb{C}}} = \int_{-\pi}^{\pi} f\,\overline{g}\,\mathrm{d}\Lambda$. We observe

$$\begin{aligned}\langle e_k, e_l\rangle_{L^2_{\mathbb{C}}} &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathbf{i}kx}\mathrm{e}^{-\mathbf{i}lx}\,\mathrm{d}\Lambda(x)\\ &= \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,\mathrm{d}\Lambda(x) = 1 & \text{if } k = l,\\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathbf{i}(k-l)x}\,\mathrm{d}\Lambda(x) = 0 & \text{if } k \neq l,\end{cases}\end{aligned}$$

and hence that $E \coloneqq \{e_k \colon k \in \mathbb{Z}\}$ is a orthonormal system.
The set $C_0^\infty((-\pi,\pi))$ is dense in $L^2_{\mathbb{C}}((-\pi,\pi))$ (see [Alt99, Satz 2.14]) and hence it suffices to show that every $f \in C_0^\infty((-\pi,\pi))$ can be approximated, with respect to the $L^2_{\mathbb{C}}$-norm, by a linear combination of functions in E. To this end define the Fourier sum

$$P_n f \coloneqq \sum_{|k|\le n} \langle f, e_k\rangle_{L^2_{\mathbb{C}}} e_k$$

and note that from Bessel's inequality follows that

$$\sum_{|k|\le n} |\langle f, e_k\rangle_{L^2_{\mathbb{C}}}|^2 \le \|f\|^2_{L^2_{\mathbb{C}}} < \infty.$$

This implies that the inequality also holds in the limit for $n \to \infty$, namely when the sum is over all integers. Hence, $P_n f$ is a Cauchy sequence and therefore it converges and we set $\tilde{f} \coloneqq \lim_{n\to\infty} P_n f$ where $\tilde{f} \in L^2_{\mathbb{C}}((-\pi,\pi))$. The convergence with respect to the $L^2_{\mathbb{C}}$ yields the existence of a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $P_{n_k} f(x) = \tilde{f}(x)$ for Λ-almost every $x \in (-\pi,\pi)$. Now the result follows from the pointwise convergence of the Fourier sum as for example given in [Alt99, Lemma 7.11]. □

From the integral representation given in Lemma 2.1.15 (i) and the eigenvalue equation (2.9) it follows that every eigenfunction f of Δ_ν fulfils the Volterra type equation, for $x \in [0,1]$,

$$f(x) = f(0) + \nabla_\nu f(0)\cdot F_\nu(x) + \lambda \int_0^x (F_\nu(x) - F_\nu(y))\, f(y)\,\mathrm{d}\nu(y), \tag{2.10}$$

where λ is the eigenvalue corresponding to f. More details on Volterra equations can be found in [Wer07, p. 265]. Now we show the reverse, namely that for every choice of $f(0)$ and $\nabla_\nu f(0)$ there exists a unique function which fulfils this Volterra type equation.

Lemma 2.2.3. *Let $\kappa \in \mathbb{R}$ be given. Under the boundary conditions $f(0) = A$ and $\nabla_\nu f(0) = B$, for $A, B \in \mathbb{R}$, there exists a unique solution to the integral equation*

$$f(x) = A + BF_\nu(x) + \kappa \int_0^x (F_\nu(x) - F_\nu(y)) f(y)\,\mathrm{d}\nu(y),$$

for $x \in [0,1]$.

Proof. To prove this lemma, we want to apply Banach's fixed point theorem. To this end, we introduce the norm $\|\cdot\|_{\alpha,\nu}$ for $\alpha > 0$ and show that $C = C([0,1]))$ is complete with respect to this norm. In a second step we show that interpreting the integral equation as an operator, for certain $\alpha > 0$, it is contracting with respect to $\|\cdot\|_{\alpha,\nu}$.

Define, for $\alpha > 0$, on C the norm

$$\|f\|_{\alpha,\nu} \coloneqq \sup\{|f(x)|\mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\}.$$

This norm is equivalent to the maximum norm $\|f\|_\infty \coloneqq \sup\{|f(x)| \colon x \in [0,1]\}$, since, for $f \in C$,

$$\mathrm{e}^{-\alpha}\|f\|_\infty \le \|f\|_{\alpha,\nu} \le \|f\|_\infty.$$

The set C equipped with $\|\cdot\|_\infty$ is complete and thus the equivalence of the norms implies that C equipped with $\|\cdot\|_{\alpha,\nu}$ is also complete.

For $\alpha > |\kappa|$ the operator

$$T(f)(x) = A + BF_\nu(x) + \kappa \int_0^x (F_\nu(x) - F_\nu(y))f(y)\,\mathrm{d}\nu(y)$$

is a contraction on C with respect to $\|\cdot\|_{\alpha,\nu}$. This can be seen by the following chain of inequalities, where $f, g \in C$.

$$\begin{aligned}
\|T(f) - T(g)\|_{\alpha,\nu} &= \sup\{|T(f)(x) - T(g)(x)|\mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\} \\
&= \sup\left\{\left|\kappa \int_0^x (F_\nu(x) - F_\nu(y))(f(y) - g(y))\,\mathrm{d}\nu(y)\right| \mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\right\} \\
&\le \sup\left\{\alpha \int_0^x |(F_\nu(x) - F_\nu(y))| \cdot |f(y) - g(y)|\,\mathrm{d}\nu(y)\,\mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\right\} \\
&\le \sup\left\{\alpha \int_0^x |f(y) - g(y)|\mathrm{e}^{-\alpha F_\nu(y)}\mathrm{e}^{\alpha F_\nu(y)}\,\mathrm{d}\nu(y)\,\mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\right\} \\
&\le \sup\left\{\alpha\|f - g\|_{\alpha,\nu} \int_0^x \mathrm{e}^{\alpha F_\nu(y)}\,\mathrm{d}\nu(y)\,\mathrm{e}^{-\alpha F_\nu(x)} \colon x \in [0,1]\right\} \\
&= \sup\left\{\|f - g\|_{\alpha,\nu}\left(1 - \mathrm{e}^{-\alpha F_\nu(x)}\right) \colon x \in [0,1]\right\} \\
&= \|f - g\|_{\alpha,\nu}\left(1 - \mathrm{e}^{-\alpha}\right)
\end{aligned}$$

Then, the result follows by Banach's fixed point theorem. □

Definition 2.2.4. The *pseudoinverse* $\check{F}_\nu^{-1} \colon [0,1] \to [0,1]$ of the distribution function F_ν is defined by

$$\check{F}_\nu^{-1}(x) \coloneqq \inf\{y \in [0,1] \colon F_\nu(y) \ge x\}.$$

Note that $F_\nu \circ \check{F}_\nu^{-1}(x) = x$, for all $x \in [0,1]$, and $\check{F}_\nu^{-1} \circ F_\nu(y) = y$, for ν-almost all $y \in [0,1]$. Furthermore, $\check{F}_\nu^{-1}(1) = b$ since b belongs to the support of ν.

We are now in a position to prove the main result of this section, namely the characterisation of the eigenvalues and eigenfunctions of Δ_ν under Dirichlet, von Neumann and periodic boundary conditions.

Theorem 2.2.5. *Let $\lambda_n \coloneqq -(\pi n)^2$, for $n \in \mathbb{N}_0$.*

(i) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,D}$ are λ_n, for $n \in \mathbb{N}$, with corresponding eigenfunctions

$$f_\nu^{(n)}(x) \coloneqq \sin(\pi n F_\nu(x)), \quad \text{for } x \in [0,1].$$

(ii) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,N}$ are λ_n, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions

$$g_\nu^{(n)}(x) \coloneqq \cos(\pi n F_\nu(x)), \quad \text{for } x \in [0,1].$$

(iii) The eigenvalues of Δ_ν on $\mathscr{D}^2_{\nu,P}$ are λ_{2n}, for $n \in \mathbb{N}_0$, with corresponding eigenfunctions $f_\nu^{(2n)}$ for $n \in \mathbb{N}$ and $g_\nu^{(2n)}$ for $n \in \mathbb{N}_0$.

Proof. We present the proof under Dirichlet boundary conditions, namely (i). Proposition 2.2.1 then implies (ii) for all $n \neq 0$. The case $n = 0$ follows from the fact that $g_\nu^{(0)} \in \mathcal{H}_\nu$, which means that $\Delta_\nu g_\nu^{(0)} = 0 = 0 \cdot g_\nu^{(0)}$. In the case (iii) of periodic boundary conditions it is sufficient to check for which $n \in \mathbb{N}$ the functions $f_\nu^{(n)} \in \mathscr{D}^2_{\nu,D}$ and $g_\nu^{(n)} \in \mathscr{D}^2_{\nu,N}$ lie in $\mathscr{D}^2_{\nu,P}$.

First we show that the functions $f_\nu^{(n)}$ are eigenfunctions. Second we prove that if l_κ is an eigenfunction of Δ_ν with eigenvalue κ, then $l_\kappa \circ \check{F}_\nu^{-1}$ is an eigenfunction of Δ_Λ. Thus, the functions $f_\nu^{(n)}$ are, up to scalar multiplication, all the eigenfunctions of Δ_ν under Dirichlet boundary conditions.

We start by determining $\nabla_\nu f_\nu^{(n)}$ and to this end, fix $n \in \mathbb{N}$. By definition of the ν-derivative, we have, for $x \in [0,1]$,

$$f_\nu^{(n)}(x) = f_\nu^{(n)}(0) + \int_0^x \nabla_\nu f_\nu^{(n)}(y)\, \mathrm{d}\nu(y).$$

By definition we have Dirichlet boundary conditions, which yield $f_\nu^{(n)}(0) = 0$. Therefore, Lemma 2.1.5 implies, for $x \in [0,1]$, that $\nabla_\nu f_\nu^{(n)}(x) = \pi n \cos(\pi n F_\nu(x))$, since

$$\begin{aligned}\int_0^x \pi n \cos(\pi n F_\nu(y))\, \mathrm{d}\nu(y) &= \int_0^{F_\nu(x)} \pi n \cos(\pi n y)\, \mathrm{d}\Lambda(y) \\ &= \sin(\pi n F_\nu(x)) = f_\nu^{(n)}(x).\end{aligned}$$

Note that $\nabla_\nu f_\nu^{(n)}(0) = \pi n$. Using Lemma 2.1.5 again, we have that, for $x \in [0,1]$,

$$\begin{aligned}\nabla_\nu f_\nu^{(n)}(0) + \int_0^x (-\pi^2 n^2) \sin(\pi n F_\nu(y))\, \mathrm{d}\nu(y) &= \pi n - \pi^2 n^2 \int_0^{F_\nu(x)} \sin(\pi n y)\, \mathrm{d}\Lambda(y) \\ &= \pi n \cos(\pi n F_\nu(x)) = \nabla_\nu f_\nu^{(n)}(x),\end{aligned}$$

and hence

$$\Delta_\nu f_\nu^{(n)}(x) = -\pi^2 n^2 \sin(\pi n F_\nu(x)) = -\pi^2 n^2 f_\nu^{(n)}(x).$$

This implies that $f_\nu^{(n)}$ fulfils the eigenvalue equation (2.9).

We now show by way of contradiction that there are no further eigenfunctions, up to scalar multiplication. Therefore, recall that we showed in Theorem 2.1.20 that Δ_ν is a non-positive operator and hence, all eigenvalues are non-positive. Suppose that l_κ is an eigenfunction of Δ_ν with corresponding eigenvalue $\kappa \leq 0$. If $\nabla_\nu l_\kappa(0) = 0$, equation (2.10) and Lemma 2.2.3 imply that $l_\kappa \equiv 0$, which contradicts the assumption of l_κ being an eigenfunction. Hence, it follows that $\nabla_\nu l_\kappa(0) \neq 0$ and so, without loss of generality, we may assume that $\nabla_\nu l_\kappa(a) = 1$.

Using (2.10) and the properties of the pseudoinverse, we observe that, for $z \in [0,1]$,

$$\begin{aligned} l_\kappa \circ \check{F}_\nu^{-1}(z) &= z + \kappa \int \mathbb{1}_{[0,\check{F}_\nu^{-1}(z)]}(\check{F}_\nu^{-1} \circ F_\nu(y))(z - F_\nu(y)) l_\kappa(y)\, \mathrm{d}\nu(y) \\ &= z + \kappa \int \mathbb{1}_{[0,z]}(F_\nu(y))(z - F_\nu(y)) l_\kappa \circ \check{F}_\nu^{-1} \circ F_\nu(y)\, \mathrm{d}\nu(y) \\ &= z + \kappa \int \mathbb{1}_{[0,z]}(y)(z-y) l_\kappa \circ \check{F}_\nu^{-1}(y)\, \mathrm{d}\nu \circ F_\nu^{-1}(y) \\ &= z + \kappa \int_0^z (z-y) l_\kappa \circ \check{F}_\nu^{-1}(y)\, \mathrm{d}\Lambda(y). \end{aligned} \tag{2.11}$$

For $\alpha > -\kappa$, the operator

$$T(f)(z) \mapsto z + \kappa \int_0^z (z-y) f(y)\, \mathrm{d}\Lambda(y)$$

is contractive with respect to the norm

$$\|f\|_{\alpha,\Lambda} = \sup\left\{|f(x)| \mathrm{e}^{-\alpha F_\Lambda(x)} \colon x \in [0,1]\right\}$$

on the set of continuous functions supported on $[0,1]$. The following chain of inequalities is analog to the one in the proof of Lemma 2.2.3 and hence, we do not give every step in detail and refer the reader to the earlier proof for more information.

$$\begin{aligned} \|T(f) - T(g)\|_{\alpha,\Lambda} &= \sup\left\{\left|\kappa \int_0^x (x-y)(f(y) - g(y)\, \mathrm{d}\Lambda(y)\right| \mathrm{e}^{-\alpha x} \colon x \in [0,1]\right\} \\ &\leq \sup\left\{\alpha \int_0^x |f(y) - g(y)| \mathrm{e}^{-\alpha y} \mathrm{e}^{\alpha y}\, \mathrm{d}\Lambda(y)\, \mathrm{e}^{-\alpha x} \colon x \in [0,1]\right\} \\ &= \sup\left\{\|f-g\|_{\alpha,\Lambda}\left(1 - \mathrm{e}^{-\alpha x}\right) \colon x \in [0,1]\right\} \\ &= \|f-g\|_{\alpha,\Lambda}\left(1 - \mathrm{e}^{-\alpha}\right) \end{aligned}$$

In the proof of Lemma 2.2.3 it was also shown that C is complete with respect to $\|\cdot\|_{\alpha,\Lambda}$ and thus, Banach's fixed point theorem can be applied and it implies that the function $l_\kappa \circ \check{F}_\nu^{-1}$ in

equation (2.11) is unique.
By the integration by parts formula given in Proposition 2.1.4 (v), for $z \in [0,1]$, we have

$$z + \kappa \int_0^z (z-y) \frac{\sin(\sqrt{-\kappa}y)}{\sqrt{-\kappa}} \, \mathrm{d}\Lambda(y) = z + \kappa \left[(z-y) \frac{-\cos(\sqrt{-\kappa}y)}{-\kappa} \right]_0^z - \kappa \int_0^z \frac{\cos(\sqrt{-\kappa}y)}{-\kappa} \, \mathrm{d}\Lambda(y)$$
$$= \int_0^z \cos(\sqrt{-\kappa}y) \, \mathrm{d}\Lambda(y) = \frac{\sin(\sqrt{-\kappa}z)}{\sqrt{-\kappa}},$$

which implies that $l_\kappa \circ \check{F}_\nu^{-1}(z) = \sin(\sqrt{-\kappa}z)/\sqrt{-\kappa}$. From the fact that $l_\kappa(0) = l_\kappa(1) = 0$, it follows that $l_\kappa \circ \check{F}_\nu^{-1}(0) = l_\kappa \circ \check{F}_\nu^{-1}(1) = 0$. Therefore, $l_\kappa \circ \check{F}_\nu^{-1}$ is a Dirichlet eigenfunction of Δ_Λ, and hence of the form $z \mapsto \sin(\pi n z)$. This means that there exists an $n \in \mathbb{N}$ such that $\kappa = -(\pi n)^2$ and $l_\kappa = f_\nu^{(n)}$ and hence, no further eigenfunctions of Δ_ν exist. □

Remark 2.2.6. An alternative proof for the fact that $f_\nu^{(n)}$ is an eigenfunction with eigenvalue $-\pi^2 n^2$ of Δ_ν may be obtained by adapting methods of [Arz15]. Assume homogeneous Dirichlet boundary conditions and define $p_0(x) \coloneqq 1$ and

$$p_n(x) \coloneqq \int_0^x p_{n-1}(y) \, \mathrm{d}\nu(y),$$

for $n \in \mathbb{N}$ and $x \in [0,1]$. By an inductive argument using Lemma 2.1.5 it follows that $p_n(x) = (n!)^{-1}(F_\nu(x))^n$. Since $p_n(0) = 0$, by definition, we have $\nabla_\nu p_{n+1} = p_n$ and by noting that

$$\sin(\pi n F_\nu(x)) = \sum_{k=0}^{\infty} (-1)^k (\pi n)^{2k+1} \frac{F_\nu(x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k (\pi n)^{2k+1} p_{2k+1}(x),$$

we have that $x \mapsto \sin(\pi n F_\nu(x))$ is an eigenfunction of Δ_ν under homogeneous Dirichlet boundary conditions. Similar arguments can be used when assuming von Neumann or periodic boundary conditions.

We conclude this section with an overview over the dimensions of the eigenspaces and the asymptotics of the eigenvalue counting functions under the different boundary conditions, which are a direct consequence of Theorem 2.2.5.

Corollary 2.2.7. *Let λ_n denote the eigenvalues of Δ_ν given in Theorem 2.2.5.*

(i) Under Dirichlet boundary conditions it holds, for all $n \in \mathbb{N}$, that $\dim(\mathrm{Eig}(\lambda_n)) = 1$.

(ii) Under von Neumann boundary conditions it holds, for all $n \in \mathbb{N}_0$, that $\dim(\mathrm{Eig}(\lambda_n)) = 1$.

(iii) Under periodic boundary conditions it holds $\dim(\mathrm{Eig}(\lambda_0)) = 1$ *and, for all $n \in \mathbb{N}$, that* $\dim(\mathrm{Eig}(\lambda_{2n})) = 2$.

Since the eigenvalues of Δ_ν under Dirichlet and von Neumann boundary conditions coincide, and since every second of these eigenvalues is an eigenvalue with multiplicity two under periodic boundary conditions, we have that the asymptotic behaviour of the eigenvalue counting function is always the same.

Corollary 2.2.8. *Let $N_\nu\colon \mathbb{R}^+ \to \mathbb{R}$ denote the eigenvalue counting function of $-\Delta_\nu$. When assuming Dirichlet, von Neumann or periodic boundary conditions, it holds that*

$$\lim_{x\to\infty} \frac{\pi N_\nu(x)}{\sqrt{x}} = 1.$$

2.3 Examples

We discuss three examples for atomless measures, namely ν being an inhomogeneous Cantor measure, a Salem measure and an absolutely continuous measure with respect to Λ such that the eigenfunctions of Δ_ν under von Neumann boundary conditions are the Chebyshev polynomials.

2.3.1 Inhomogeneous Cantor measures

We now give a brief introduction to iterated function systems on the real line. We refer to [Fal97, Fal14, Hut81] for a detailed discussion of iterated function systems in general and for the proofs of the facts stated in this chapter.

Fix a closed interval $[a,b] \subseteq \mathbb{R}$. A function $\phi\colon [a,b] \to [a,b]$ is said to be a *contraction on* $[a,b]$ if there exists a number r with $0 < r < 1$ such that $|\phi(x) - \phi(y)| \le r|x-y|$ for all $x, y \in [a,b]$. If equality holds, ϕ is called a *similarity* with *contraction ratio* r. We call $\Phi = ([a,b];\, \phi_1, \dots, \phi_N)$ an *iterated function system* if $N \in \mathbb{N}$ with $N \ge 2$ and each $\phi_i\colon [a,b] \to [a,b]$ is a contraction. For every iterated function system Φ there exists a unique *attractor*, that is a non-empty compact set $E \subseteq [a,b]$ with

$$E = \bigcup_{i=1}^{N} \phi_i(E). \tag{2.12}$$

If all of the contractions in Φ are similarities, then the set E is called *self-similar*. Moreover, in this case, if all of the contraction ratios of the members of Φ are equal, then E is called a *homogeneous self-similar set*; otherwise E is called an *inhomogeneous self-similar set*. We say that Φ satisfies the *strong separation condition*, if the union in (2.12) is disjoint. If the iterated function system Φ with attractor E satisfies the strong separation condition, then E is homeomorphic to the Cantor set, and, in particular, is totally disconnected.

Further, if $\mathbf{p} = (p_1, \dots, p_N)$ is a probability vector with $p_i \in (0,1)$, for all $i \in \{1, 2, \dots, N\}$, then there exists a unique Borel probability measure $\nu = \nu(\Phi, \mathbf{p})$ supported on E satisfying

$$\nu(A) = \sum_{i=1}^{N} p_i \nu(\phi_i^{-1}(A)), \tag{2.13}$$

for all Borel sets A. If the attractor E is self-similar, the measure ν is called a *self-similar measure*. We now show that if Φ satisfies the strong separation condition and all contractions

are injective, then this measure is atomless and hence, we can define the operators ∇_ν and Δ_ν with respect to such self-similar measures. To this end, note that equation (2.12) implies that for every $x \in E$ there exists a sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2,\dots,N\}^{\mathbb{N}}$ such that for all $j \in \mathbb{N}$

$$x \in \phi_{i_j} \circ \phi_{i_{j-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}([a,b]).$$

It follows from the pairwise disjointness of the sets $\phi_i([a,b])$ that this sequence is unique and it is often referred to as the *coding of x*. Applying the self-similarity equation given in (2.13), we have by monotonicity of the measure, and the injectivity of the contractions for $j \in \mathbb{N}$ and $x \in E$, that

$$\nu(\{x\}) \leq \prod_{k=1}^{j} p_{i_k}.$$

Since $p_i < 1$ for all $i \in \{1,2,\dots N\}$, it follows that $\nu(\{x\}) = 0$ for all $x \in E$. Hence, ν is an atomless measure.

We now consider the example $[a,b] = [0,1]$, $\Phi = ([0,1];\ \phi_1(x) = x/2,\ \phi_2(x) = x/3 + 2/3)$ and $\mathbf{p} = (0.7,\ 0.3)$. Here the attractor E is an inhomogeneous self-similar set. which is often referred to as a *Cantor-like set*. Letting $\nu_{(1)}$ denote the to Φ and $\mathbf{p}$ associated self-similar measure, we give a graphical representation of the distribution function $F_{\nu_{(1)}} : [0,1] \to [0,1]$ in Figure 2.1.

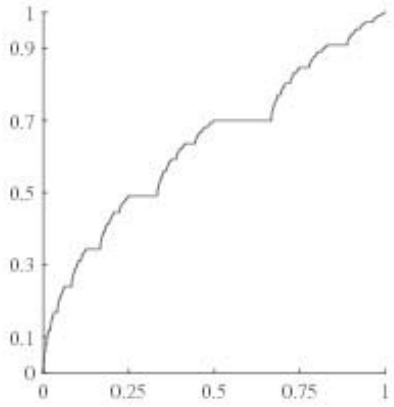
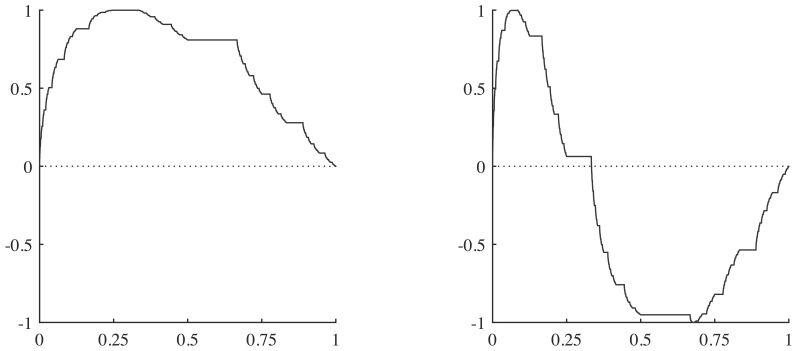
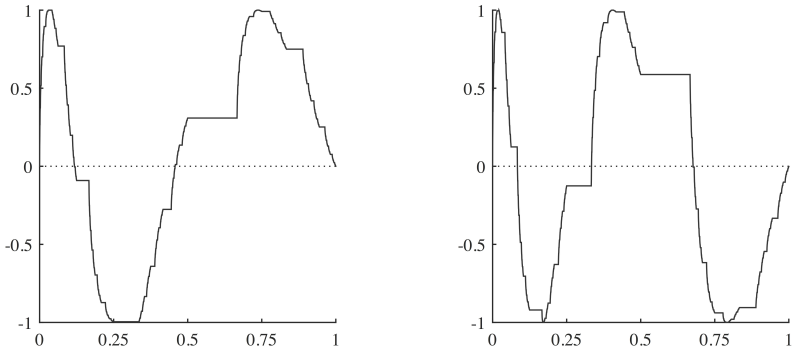


Figure 2.1. Distribution function $F_{\nu_{(1)}}$ for the self-similar measure $\nu_{(1)}$ associated to Φ and $\mathbf{p}$.

When assuming Dirichlet boundary conditions, it follows from Theorem 2.2.5 that the eigenfunctions of $\Delta_{\nu_{(1)}}$ are $f^{(n)}_{\nu_{(1)}}(x) := \sin(\pi n F_{\nu_{(1)}}(x))$ for $n \in \mathbb{N}$ and $x \in [0,1]$. In Figure 2.2, we give graphical representations of the eigenfunctions $f^{(1)}_{\nu_{(1)}}, f^{(2)}_{\nu_{(1)}}, f^{(3)}_{\nu_{(1)}}$ and $f^{(4)}_{\nu_{(1)}}$ which have the corresponding eigenvalues $\lambda_1 = -\pi^2, \lambda_2 = -4\pi^2, \lambda_3 = -9\pi^2$ and $\lambda_4 = -16\pi^2$.



Graph of $f^{(1)}_{\nu_{(1)}}$ Graph of $f^{(2)}_{\nu_{(1)}}$



Graph of $f^{(3)}_{\nu_{(1)}}$ Graph of $f^{(4)}_{\nu_{(1)}}$

Figure 2.2. Eigenfunctions $f^{(k)}_{\nu_{(1)}}$ of $\Delta_{\nu_{(1)}}$, for $k \in \{1,2,3,4\}$, where $\nu_{(1)}$ is the self-similar measure associated to Φ and $\mathbf{p}$.

2.3.2 Salem measures

As a second class of examples, we consider the so-called Salem measures, which go back to the work of Salem [Sal43] and which were popularised by Riesz and Szőkefalvi-Nagy [RSN90]. This is a family of absolutely continuous probability measures $\{\nu_{p,q} \colon p,q \in (0,1)\}$, whose distribution functions $\{F_{\nu_{p,q}} \colon p,q \in (0,1)\}$ arise from the following endomorphisms of $[0,1]$. For $r \in (0,1)$, we define the function

$$S_r(x) \coloneqq \begin{cases} \dfrac{x}{r} & \text{if } x \in [0,r], \\ \dfrac{x-r}{1-r} & \text{if } x \in (r,1] \end{cases}$$

which is a piecewise linear function with two full branches. The maps $F_{\nu_{p,q}} \colon [0,1] \to [0,1]$ are then defined to be the unique non-zero functions fulfilling $S_p \circ F_{\nu_{p,q}} = F_{\nu_{p,q}} \circ S_q$.

In [Sal43], the original work of Salem, a special subclass of these distribution functions is discussed, namely when $p+q=1$ with $p \neq q$. In his article, Salem gives an explicit

construction of the functions $F_{\nu_{p,q}}$. This works similarly for general $p, q \in (0,1)$ and we give a brief introduction to it. We start with the identity on the unit interval, namely $f_0(x) = x$, for $x \in [0,1]$, and fix $p, q \in [0,1]$. The function f_1, defined for $x \in [0,1]$ by

$$f_1(x) := \begin{cases} \dfrac{px}{q} & \text{if } x \in [0,q], \\ \dfrac{(1-p)(x-q)}{1-q} + p & \text{if } x \in (q,1], \end{cases}$$

is a piecewise linear, strictly monotonically increasing, continuous function with $f_1(0) = 0$, $f_1(q) = p$ and $f_1(1) = 1$. In the next step, the transformation of f_0 to f_1 is applied to both branches of f_1. This means, that the branch between 0 and q is split up into two linear functions, where the x-axis is divided in the ratio $q : (1-q)$ and the y-axis in the ratio $p : (1-p)$. The same is done for the branch between q and 1. We denote the resulting function by f_2. In Figure 2.3, one can see a graphical representation of f_1 and f_2 for $p = 0.5$ and $q = 0.7$.

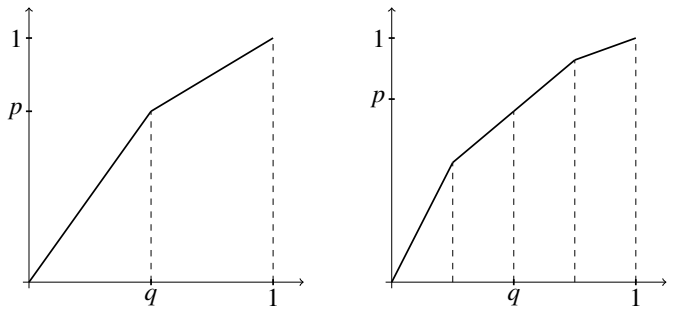


Figure 2.3. Graphs of f_1 and f_2 for $p = 0.7$ and $q = 0.5$.

Continuing this procedu f_n with 2^n branches. If $n \to \infty$, the functions f_n In Figure 2.4 we give a graphical representation alem measure $\nu_{0.7,0.5}$.

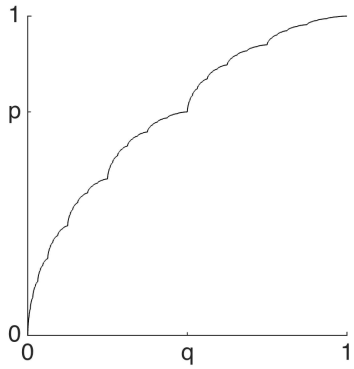
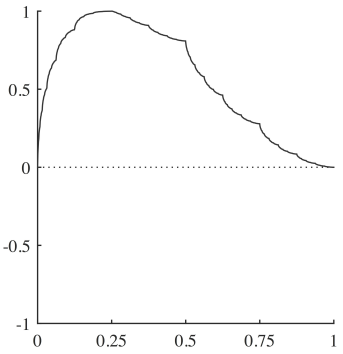
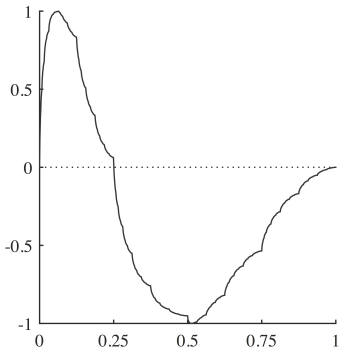


Figure 2.4. Distribution function $F_{\nu_{p,q}}$ of the Salem measure for $p = 0.7$ and $q = 0.5$.

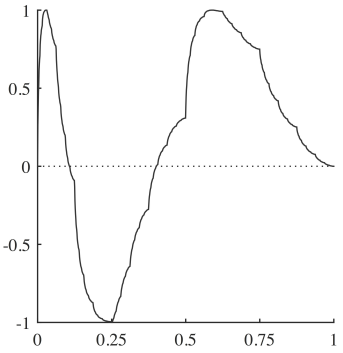
One can verify that $F_{\nu_{p,q}}$ is strictly monotonically increasing, continuous and, for $p \neq q$, differentiable Lebesgue-almost everywhere with the derivative equal to zero, where it exists. For more details and further properties of these functions we refer the reader to [Sal43, RSN90, PVB07, JKPS09]. The continuity of $F_{\nu_{p,q}}$ implies that the measure $\nu_{p,q}$ is atomless and hence, falls in the class of measures discussed in this chapter. We now fix $p = 0.7$ and $q = 0.5$ and set $\nu_{(2)} \coloneqq \nu_{0.7,0.5}$. In Figure 2.5, we give graphical representations of the eigenfunctions $f^{(1)}_{\nu_{(2)}}, f^{(2)}_{\nu_{(2)}}, f^{(3)}_{\nu_{(2)}}$ and $f^{(4)}_{\nu_{(2)}}$ of the $\nu_{(2)}$-Laplacian under Dirichlet boundary conditions.



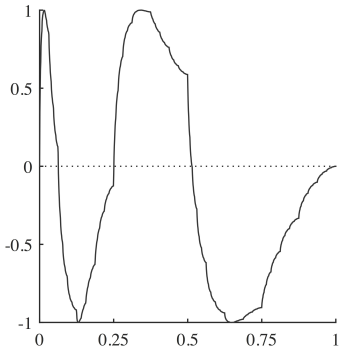
Graph of $f^{(1)}_{\nu_{(2)}}$



Graph of $f^{(2)}_{\nu_{(2)}}$



Graph of $f^{(3)}_{\nu_{(2)}}$



Graph of $f^{(4)}_{\nu_{(2)}}$

Figure 2.5. Eigenfunctions $f^{(k)}_{\nu_{(2)}}$ of $\Delta_{\nu_{(2)}}$, for $k \in \{1,2,3,4\}$, where $\nu_{(2)}$ is the Salem measure for $p = 0.7$ and $q = 0.5$.

2.3.3 Chebyshev polynomials as eigenfunctions

In this section, we construct a measure $\nu_{(3)}$ on the interval $[-1,1]$ such that the von Neumann eigenfunctions of $\Delta_{\nu_{(3)}}$ are the Chebyshev polynomials of the first kind. Since we do not consider the Chebyshev polynomials of the second kind, we only write Chebyshev polynomials when talking about those of the first kind.

The Chebyshev polynomials occur in different areas of mathematics and are a well-studied example class for many different properties, such as being the polynomials with largest possible leading coefficient under the restriction that their sup-norm $\|\cdot\|_\infty$ is bounded by one. They are also used in approximation and filtration theory, since they minimise the maximal error when interpolating polynomials. The Chebyshev polynomials are the solutions to the Sturm-Liouville differential equations of the form

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

For further details on different properties of this function family, see [Det95, Riv90].

The *Chebyshev polynomials* are given, for $n \in \mathbb{N}_0$, by

$$\begin{aligned} T_n \colon [-1,1] &\to [-1,1], \\ x &\mapsto \cos(n \arccos(x)). \end{aligned}$$

They can also be defined inductively by

$$T_0(x) \coloneqq 1, \quad T_1(x) \coloneqq x, \quad \text{and} \quad T_{n+1}(x) \coloneqq 2xT_n(x) - T_{n-1}(x),$$

which directly implies that T_n is a polynomial of degree n. The first six Chebyshev polynomials are

$$\begin{aligned} T_0(x) &= 1, & T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, & T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, & T_5(x) &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

To prove that the two representations lead to the same functions, one needs the trigonometric addition formula

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta), \tag{2.14}$$

where $\alpha, \beta \in \mathbb{R}$. Using the identity $\cos(\arccos(x)) = x$, which holds for $x \in [-1,1]$, this yields for every $n \in \mathbb{N}$

$$\begin{aligned} T_{n+1}(x) &= \cos(n\arccos(x) + \arccos(x)) = \cos(n\arccos(x))x - \sin(n\arccos(x))\sin(\arccos(x)), \\ T_{n-1}(x) &= \cos(n\arccos(x) - \arccos(x)) = \cos(n\arccos(x))x + \sin(n\arccos(x))\sin(\arccos(x)), \end{aligned}$$

and therefore

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\arccos(x))x = 2xT_n(x).$$

Note that T_{2k} is an even function and T_{2k+1} is an odd function, for $k \in \mathbb{N}_0$.

We now construct a measure ν such that $\Delta_\nu T_n = \lambda_n T_n$ holds for $n \in \mathbb{N}_0$, and where $\lambda_n = -(\pi n)^2$. To this end, choose $[a,b] = [-1,1]$ and let Λ denote the Lebesgue measure restricted to the

interval $[-1,1]$. Consider the absolutely continuous probability measure $\nu_{(3)}$ supported on $[-1,1]$ given by

$$\frac{\mathrm{d}\nu_{(3)}}{\mathrm{d}\Lambda}(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

The distribution function $F_{\nu_{(3)}}\colon [-1,1] \to [0,1]$ can be determined explicitly via integration:

$$F_{\nu_{(3)}}(x) = \frac{1}{\pi}\int_{-1}^{x} \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}\Lambda(t) = \frac{1}{\pi}\left(\arcsin(x) + \frac{\pi}{2}\right) = \frac{1}{\pi}\arccos(-x).$$

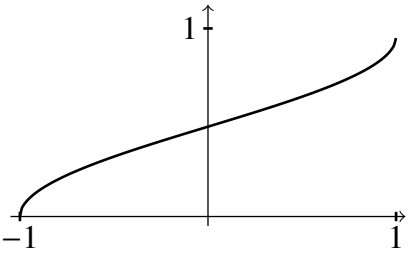


Figure 2.6. Distribution function $F_{\nu_{(3)}}$ of the absolutely continuous measure $\nu_{(3)}$.

The analogue of Theorem 2.2.5 in the setting $[a,b] = [-1,1]$ yields that the eigenfunctions of $\Delta_{\nu_{(3)}}$ under von Neumann boundary conditions are of the form $g^{(n)}_{\nu_{(3)}}(x) = \cos(\pi n F_{\nu_{(3)}}(x))$, where $n \in \mathbb{N}_0$ and $x \in [-1,1]$. The following calculation shows that $g^{(n)}_{\nu_{(3)}} = (-1)^n T_n$ and hence, for all $n \in \mathbb{N}$, the Chebyshev polynomial T_n fulfils the eigenvalue equation of $\Delta_{\nu_{(3)}}$ with corresponding eigenvalue $\lambda_n = -(\pi n)^2$.

$$g^{(n)}_{\nu_3}(x) = \cos(\pi n F_{\nu_3}(x)) = \cos(n\arccos(-x)) = T_n(-x) = (-1)^n T_n(x)$$

The functions $g^{(0)}_{\nu_{(3)}}, g^{(1)}_{\nu_{(3)}}, g^{(2)}_{\nu_{(3)}}$ and $g^{(3)}_{\nu_{(3)}}$ are shown in the left plot of Figure 2.7.

Combining (2.14) with the identity $\sin(\arccos(x)) = \sqrt{1-x^2}$ which holds on $[-1,1]$, we get

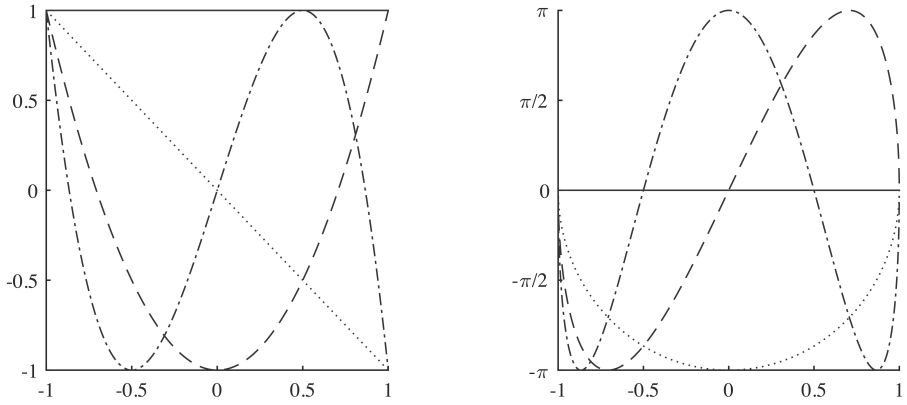
$$\begin{aligned}\frac{dT_n}{dx}(x) &= \frac{n\sin(n\arccos(x))}{\sqrt{1-x^2}} = \frac{n\sin(\arccos(x))\sin(n\arccos(x))}{1-x^2}\\ &= \frac{n(\cos(\arccos(x))\cos(n\arccos(x)) - \cos((n+1)\arccos(x)))}{1-x^2}\\ &= \frac{nxT_n(x) - nT_{n+1}(x)}{1-x^2}\end{aligned}$$

Hence, we observe, for $n \in \mathbb{N}_0$ and $x \in [-1,1]$, that

$$\nabla_{\nu_{(3)}} g^{(n)}_{\nu_3}(x) = (-1)^n \nabla_{\nu_{(3)}} T_n(x) = (-1)^n \pi\sqrt{1-x^2}\frac{\mathrm{d}T_n}{\mathrm{d}x}(x) = \frac{(-1)^n \pi n(xT_n(x) - T_{n+1}(x))}{\sqrt{1-x^2}}.$$

The right plot of Figure 2.7 illustrates the derivatives of $g^{(n)}_{\nu_{(3)}}$ for $n = 0,1,2,3$. Recalling

Proposition 2.2.1, it holds, for $n \in \mathbb{N}$, that $\nabla_{\nu_{(3)}} g^{(n)}_{\nu_{(3)}}$ are eigenfunctions of $\Delta_{\nu_{(3)}}$ when assuming Dirichlet boundary conditions.



Graphs of $g^{(i)}_{\nu_{(3)}}$ for $i \in \{0,1,2,3\}$ Graphs of $\nabla_{\nu_{(3)}} g^{(i)}_{\nu_{(3)}}$ for $i \in \{0,1,2,3\}$

Figure 2.7. Eigenfunctions $g^{(0)}_{\nu_{(3)}}, g^{(1)}_{\nu_{(3)}}, g^{(2)}_{\nu_{(3)}}$ and $g^{(3)}_{\nu_{(3)}}$ of $\Delta_{\nu_{(3)}}$, and their $\nu_{(3)}$-derivatives.

CHAPTER 3

Purely Atomic Measures

In most of the literature on differentiation with respect to measures, a crucial assumption is that the measures do not have point masses. In this chapter, we show that the framework of Freiberg and Zähle can be extended to included purely atomic distributions δ, supported on an interval $M := (a, b]$. Fixing $N \in \mathbb{N}$, the assumptions on δ are the following:

- $a < z_1 < z_2 < \cdots < z_N \leq b$,
- $0 < \alpha_i < \infty$ for $i \in \{1, \ldots, N\}$ and
- $\delta = \sum_{i=1}^{N} \alpha_i \delta_{z_i}$.

This means, that the measure δ is a finite sum of weighted Dirac point masses. We call z_i the *position of the i-th atom* and the corresponding α_i its *weight*. The distribution function F_δ is a right-continuous step function mapping M to $[0, \sum_{i=1}^{N} \alpha_i]$. Note that, different to the continuous setting, we have that $F_\delta \neq F_\delta^{lc}$. More precisely, it holds that F_δ^{lc} is left-continuous and that $F_\delta(x) = F_\delta^{lc}(x)$ if and only if $x \notin \{z_1, \ldots, z_N\}$. Another important difference to the measures discussed in Chapter 2 is that the space L_δ^2 is finite-dimensional. We will see that every equivalence class in L_δ^2 has a δ-differentiable representative.

In order to obtain well-defined measure-geometric differential operator on a subset of L_δ^2, we need to assume periodic boundary conditions. The two main operators defined in this chapter are the δ-derivative ∇_δ and the δ-Laplacian Δ_δ. For both operators, we give matrix representations which allow us to obtain many properties of ∇_δ and Δ_δ. Unlike in the case when one has a measure with a continuous distribution function, we prove that the operator $\nabla_\delta \circ \nabla_\delta$ is no longer symmetric. To circumvent this problem, we consider the operator ∇_δ, its adjoint ∇_δ^* and define the δ-Laplacian to be $\Delta_\delta := -\nabla_\delta^* \circ \nabla_\delta$.
We will see that the eigenvalues of Δ_δ only depend on N and on the weights α_i of the atoms and not on their positions z_i. Unlike in the atomless case, the corresponding eigenfunctions are not necessarily of the form $f_\kappa^\mu(\cdot) := \sin(\pi\kappa F(\cdot))$ or $g_\kappa^\mu(\cdot) := \cos(\pi\kappa F(\cdot))$, for $\kappa \in \mathbb{R}$ and F being either F_δ or F_δ^{lc}.

For ease of notation, we assume that $a = 0$ and $b = 1$ and, hence, that $M = (0, 1]$. Further, we assume that δ is a probability measure, namely $\sum_{i=1}^{N} \alpha_i = 1$. The results presented in this chapter also hold in general for measures δ as described above and they can be obtained in

a similar manner. As we will see later, the operators defined here and their properties do not depend on the distances between the atoms and, therefore, the general case follows from $M = (0, 1]$ by an appropriate rescaling.

Parts of the results presented in this chapter are published by Keßeböhmer, Samuel and Weyer in [KSW18a]. We give more details on the results shown there and extend them by obtaining that ∇_δ and Δ_δ are bounded operators and discussing the δ-harmonic functions. Furthermore, we obtain bounds on the smallest eigenvalue of Δ_ν.

The chapter is divided in three sections. In Section 3.1, we define the δ-derivative ∇_δ and the δ-Laplace operator Δ_ν and obtain matrix representations of these operators and analytic properties. The second part, Section 3.2, is about general results and bounds on the eigenvalues of Δ_ν. We concluded this Chapter with Section 3.3, where we explicitly solve the eigenvalue problem of Δ_ν for three leading examples.

3.1 Derivative and Laplacian with respect to purely atomic measures

The space L^2_δ is a finite-dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_0^1 fg \, \mathrm{d}\delta = \sum_{i=1}^{N} \alpha_i f(z_i) g(z_i).$$

Unlike in the definition of the set of ν-differentiable functions for an atomless measure ν, we assume periodic boundary conditions for functions which are δ-differentiable. The reason for this is discussed right after the definition of δ-differentiable functions.
Before we give this definition, we need to introduce the following notation. For a function $f\colon (0, 1] \to \mathbb{R}$ we let $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}$ be the periodic extension of f, that is $\mathbf{f}(x) = f(x)$ for all $x \in (0, 1]$ and $\mathbf{f}(x) = \mathbf{f}(x + k)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Definition 3.1.1. The set of *δ-differentiable functions on* $(0, 1]$ *with periodic boundary conditions* is defined by

$$\begin{aligned} \mathscr{D}^1_\delta = \mathscr{D}^1_\delta((0,1]) := \Big\{ f \in \mathfrak{L}^2_\delta \colon \ &\text{there exists } f' \in L^2_\delta \text{ with} \\ &\mathbf{f}(x) = \mathbf{f}(y) + \int \mathbb{1}_{[y,x)} \mathbf{f}' \, \mathrm{d}(\delta * \delta_{\mathbb{Z}}) \text{ for all } x, y \in \mathbb{R} \text{ with } y < x \Big\}. \end{aligned} \tag{3.1}$$

Here $\delta * \delta_{\mathbb{Z}}$ denotes the convolution of the measure δ and the Dirac comb $\delta_{\mathbb{Z}}$ as defined in Appendix A. In the following we always assume periodic boundary conditions and when we write δ-differentiable function we mean an element of $\mathscr{D}^1_\delta$. The definition of $\mathscr{D}^1_\delta$ is coherent with the case of a continuous measure introduced in Definition 2.1.1 when assuming periodic boundary conditions. This follows from Proposition 2.1.4 (iii).

In order to obtain a well-defined operator on a subset of L^2_δ, it becomes clear why periodic boundary conditions have to be assume. If there were no boundary conditions and if $z_N = 1$, the set $\mathscr{D}^1_\delta$ is independent of the weight α_N, since we integrate in (3.1) over the interval $[y, x)$.

For $z_N \neq 1$, two functions f_1 and f_2 which only differ on the δ-null set $(z_N, 1]$, could have functions f_1' and f_2' from different L^2_δ-equivalence classes fulfilling the respective integral equation (3.1).

Throughout the chapter we assume that $z_N = 1$, where the general case can be obtained from the following translation argument. Let $0 < z_1 < \cdots < z_N < 1$, fix $\alpha_i > 0$ with $\sum_{i=1}^N \alpha_i = 1$ and set $\tilde{z}_i = z_i + 1 - z_N$, for $i \in \{1, \ldots, N\}$, which implies $\tilde{z}_N = 1$. Comparing the measures $\delta \coloneqq \sum_{i=1}^N \alpha_i \delta_{z_i}$ and $\tilde{\delta} \coloneqq \sum_{i=1}^N \alpha_i \delta_{\tilde{z}_i}$, it holds that $\tilde{f} \in \mathscr{D}^1_{\tilde{\delta}}$ if and only if $f \in \mathscr{D}^1_\delta$, where f is defined by

$$f(x) \coloneqq \begin{cases} \tilde{f}(x - z_N + 1) & \text{if } x \in (0, z_N], \\ \tilde{f}(x - z_N) & \text{if } x \in (z_N, 1]. \end{cases}$$

Hence, the assumption $z_N = 1$ can be made without loss of generality.

For ease of notation, we set $z_0 \coloneqq 0 = z_N - 1$ and $z_{N+1} \coloneqq z_1 + 1$. Since δ is a linear combination of Dirac point masses, we can obtain from the defining equation given in (3.1), for $f \in \mathscr{D}^1_\delta$ and $x \in (z_1, 1]$, that

$$f(x) = f(z_1) + \sum_{\substack{i \in \{1,\ldots,N\} \\ 0 \leq z_i < x}} \alpha_i f'(z_i). \tag{3.2}$$

For other $x, y \in \mathbb{R}$, the integral is the sum of the weighted function values of $\mathbf{f}'$ over all atoms of $\delta * \delta_{\mathbb{Z}}$ which lie in the interval $[x, y)$. We define the sets $A_i \coloneqq (z_{i-1}, z_i]$, for $i \in \{1, \ldots, N+1\}$. These subsets of the real line form a partition of $(0, 1]$ induced by the set of atoms.

Proposition 3.1.2. *(i) A function lies in $\mathscr{D}^1_\delta$ if and only if it is constant on A_i, for all $i \in \{1, \ldots, N\}$. Hence, all functions in $\mathscr{D}^1_\delta$ are piecewise constant and left-continuous.*

(ii) For $f \in \mathscr{D}^1_\delta$ the function $f' \in L^2_\delta$ given in (3.1) *is unique.*

Proof. It follows from the integral equation (3.2) that every function in $\mathscr{D}^1_\delta$ is constant on A_i, for $i \in \{1, \ldots, N\}$, which is the forward direction in (i). The backward direction can also be obtained from (3.2), by setting $f'(z_i) \coloneqq (\mathbf{f}(z_{i+1}) - \mathbf{f}(z_i))/\alpha_i$.
In order to prove (ii), assume there are two functions $g, h \in L^2_\delta$, both fulfilling the defining integral equation (3.1). Equation (3.2) yields, for $i \in \{1, \ldots, N\}$ that $\mathbf{f}(z_{i+1}) = f(z_i) + \alpha_i g(z_i)$ and $\mathbf{f}(z_{i+1}) = f(z_i) + \alpha_i h(z_i)$, which implies $g(z_i) = h(z_i)$ and hence, $g = h$. □

From the functions being piecewise constant, it follows that $f \in \mathscr{D}^1_\delta$ is uniquely determined by the vector $(f(z_1), \ldots, f(z_N))^\top$. Further, if $f, g \in \mathscr{D}^1_\delta$ with $f \neq g$, then $\|f - g\| \neq 0$. Thus we may view $\mathscr{D}^1_\delta$ as a collection of real-valued square-δ-integrable functions, or as a collection of equivalence classes in L^2_δ. This means there exists a natural embedding $\pi \colon \mathscr{D}^1_\delta \to L^2_\delta$ and in the following we do not distinguish between $\mathscr{D}^1_\delta$ and $\pi(\mathscr{D}^1_\delta)$.

Definition 3.1.3. For $f \in \mathscr{D}^1_\delta$ let f' be as in Definition 3.1.1. The operator

$$\begin{aligned} \nabla_\delta \colon \mathscr{D}^1_\delta &\to L^2_\delta \\ f &\mapsto f' \end{aligned}$$

is called the *δ-derivative*.

Since for every $f \in \mathscr{D}^1_\delta$ the function f' is unique, the operator ∇_δ is well-defined. Unlike in the atomless case, the domain of this operator is not infinite-dimensional, since L^2_δ is finite-dimensional. By the linearity of the integral, the linearity of ∇_δ follows.

Corollary 3.1.4. *The operator $\nabla_\delta \colon \mathscr{D}^1_\delta \to L^2_\delta$ is linear and, therefore, continuous.*

Further, since $\mathbf{f}(x) = \mathbf{f}(x+1)$, for $x \in \mathbb{R}$, it follows that

$$\sum_{i \in \{1,\dots,N\}} \alpha_i \nabla_\delta f(z_i) = 0. \tag{3.3}$$

If δ is a Dirac point mass, that is $N = 1$, then (3.3) becomes $\alpha_1 \nabla_\delta f(z_1) = 0$. Hence, from (3.2) and the fact that $\alpha_1 > 0$, it follows that $\mathscr{D}^1_\delta$ is the set of constant functions and that the operator ∇_δ is the null-operator, and so, from here on we assume that $N \geq 2$.

Using Proposition 3.1.2 (i), we obtained that every function $f \in \mathscr{D}^1_\delta$ is uniquely determined by the vector $(f(z_1), \dots, f(z_N))^\top$. Thinking of the operator ∇_δ as acting on L^2_δ as described above, it is sufficient to look at the values of the δ-derivative at the atoms in order to understand how the δ-derivative acts on $\mathscr{D}^1_\delta$. We can use the defining integral equation (3.2) to obtain these values. Evaluating this equation at $x = z_2$ implies that

$$f(z_2) = f(z_1) + \alpha_1 \nabla_\delta f(z_1),$$

and hence,

$$\nabla_\delta f(z_1) = \frac{f(z_2) - f(z_1)}{\alpha_1}.$$

Similarly, evaluating the integral equation at the third atom (if $N \geq 3$) yields

$$f(z_3) = f(z_1) + \alpha_1 \nabla_\delta f(z_1) + \alpha_2 \nabla_\delta f(z_2) = f(z_2) + \alpha_2 \nabla_\delta f(z_2),$$

which can be rearranged to

$$\nabla_\delta f(z_2) = \frac{f(z_3) - f(z_2)}{\alpha_2}.$$

Continuing this inductively for the consecutive atoms we get the general formula

$$\nabla_\delta f(z_i) = \frac{f(z_{i+1}) - f(z_i)}{\alpha_i}, \tag{3.4}$$

for $i \in \{1, \dots, N-1\}$. To obtain an equation for $\nabla_\delta f(z_N)$, one can either consider the periodic extension or deduce directly from (3.2) and (3.3) that

$$\nabla_\delta f(z_N) = \frac{-\sum_{i=1}^{N-1} \alpha_i \nabla_\delta f(z_i)}{\alpha_N} = \frac{f(z_1) - f(z_N)}{\alpha_N}.$$

Therefore, equation (3.4) holds for all $i \in \{1, \dots, N\}$. Namely, the value of the δ-derivative of a $\mathscr{D}^1_\delta$-function at the atom z_i is the difference of the function values on A_{i+1} and on A_i divided by the weight α_i.

Summarising what has been shown so far, every function in $\mathscr{D}^1_\delta$ is a step function which is constant on the parts of the partition $\{A_i \colon i \in \{1, \dots, N\}\}$ induced by the set of atoms. The δ-derivative of such a function reflects a weighted measuring of the height differences between consecutive parts. The δ-derivative depends only on the weights of the atoms of δ and not on the distances between them.

Considering formula (3.4), it is clear that

$$\ker(\nabla_\delta) = \{c\mathbb{1} \colon c \in \mathbb{R}\}. \tag{3.5}$$

Different to the classical theory and also to the case of atomless measures, the δ-derivative is not unbounded.

Proposition 3.1.5. *The δ-derivative ∇_δ is an bounded operator on $\mathscr{D}^1_\delta$.*

Proof. Since $\mathscr{D}^1_\delta$ is finite-dimensional and the δ-derivative is linear, this results follows from a general result in operator theory. One can also see this from the following chain of inequalities, which use (3.4) and where $f \in \mathscr{D}^1_\delta$.

$$\begin{aligned} \|\nabla_\delta f\|^2 &= \sum_{i=1}^{N} \left(\frac{f(z_{i+1}) - f(z_i)}{\alpha_i} \right)^2 = \sum_{i=1}^{N} \frac{1}{\alpha_i^2} \left(f(z_{i+1})^2 - 2 f(z_i) f(z_{i+1}) + f(z_i)^2 \right) \\ &\leq \max_i \left(\frac{2}{\alpha_i^2} \right) \sum_{i=1}^{N} f(z_i)^2 \leq \max_i \left(\frac{2}{\alpha_i^3} \right) \cdot \|f\|^2 \end{aligned}$$

□

Since all differences and sums in the general form of the derivative (3.4) are finite, we have $\pi(\mathscr{D}^1_\delta) = L^2_\delta$, where π is the embedding of $\mathscr{D}^1_\delta$ into L^2_δ discussed earlier. In other words, every equivalence class of L^2_δ has a δ-differentiable representative.
Letting ϱ denote the quotient map from L^2_δ to $L^2_\delta / \{c\mathbb{1} \colon c \in \mathbb{R}\}$, we can show that the orthogonal complement of the range of ∇_δ is equal to the set of constant functions.

Proposition 3.1.6. *The image under ϱ of the range of ∇_δ is equal to the quotient space $L^2_\delta / \{c\mathbb{1} \colon c \in \mathbb{R}\}$, namely $\varrho(\operatorname{ran}(\nabla_\delta)) = L^2_\delta / \{c\mathbb{1} \colon c \in \mathbb{R}\}$.*

Proof. From the observation $\pi(\mathscr{D}^1_\delta) = L^2_\delta$ and from (3.3), it follows that the range of ∇_δ is the set of all functions $f \in L^2_\delta$ with $\sum_{i \in \{1,\dots,N\}} \alpha_i f(z_i) = 0$, which yields the result. □

As discussed previously, both a function $f \in \mathscr{D}^1_\delta$ and its δ-derivative $\nabla_\delta f$ are uniquely determined by their values at the atoms. We now give a matrix representation for the operator ∇_δ, namely an $N \times N$-matrix A_δ with

$$A_\delta (f(z_1), \dots, f(z_N))^\top = (\nabla_\delta f(z_1), \dots, \nabla_\delta f(z_N))^\top.$$

The matrix A_δ can be obtained from the formula given in (3.4).

$$A_\delta \coloneqq \begin{pmatrix} -\alpha_1^{-1} & \alpha_1^{-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\alpha_2^{-1} & \alpha_2^{-1} & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\alpha_3^{-1} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -\alpha_{N-2}^{-1} & \alpha_{N-2}^{-1} & 0 \\ 0 & 0 & 0 & \ddots & 0 & -\alpha_{N-1}^{-1} & \alpha_{N-1}^{-1} \\ \alpha_N^{-1} & 0 & 0 & \cdots & 0 & 0 & -\alpha_N^{-1} \end{pmatrix}$$

In accordance with (3.5), the matrix A_δ has rank $N-1$. This can be obtained by adding $\alpha_N^{-1}\alpha_i$ times the i-th row to the N-th row, for $i \in \{1, \dots, N-1\}$. This leads to an upper triangular matrix with one row of zeros.
Comparing the function values $(A_\delta)_{i,j}$ and $(A_\delta)_{j,i}$ for $i, j \in \{1, \dots, N\}$, one immediately sees that the matrix A_δ is symmetric if and only if $N = 2$ and $\alpha_1 = \alpha_2$.

For an atomless measure ν we defined the ν-Laplacian Δ_ν as the concatenation of the ν-derivative ∇_ν with itself, see equation (2.6). When assuming periodic boundary conditions, we saw in (2.8) that this coincides with taking the negative of the concatenation of ∇_ν with its adjoint operator. We will see that these approaches lead to different operators in the case of purely atomic measures. To see this, we will analyse the matrix A_δ^2 and see that in general it is not symmetric. Furthermore, we show that in the case where it is symmetric, it coincides with $-A_\delta^\top A_\delta$. The operator represented by this matrix corresponds to the δ-Laplacian we define and study later.

Proposition 3.1.7. *The matrix A_δ^2 is symmetric if and only if $N = 2$ and $\alpha_1 = \alpha_2$.*

Proof. In the case that $N = 2$, we obtain

$$A_\delta^2 = \begin{pmatrix} -\alpha_1^{-1} & \alpha_1^{-1} \\ \alpha_2^{-1} & -\alpha_2^{-1} \end{pmatrix} \cdot \begin{pmatrix} -\alpha_1^{-1} & \alpha_1^{-1} \\ \alpha_2^{-1} & -\alpha_2^{-1} \end{pmatrix} = \begin{pmatrix} \alpha_1^{-2} + \alpha_1^{-1}\alpha_2^{-1} & -\alpha_1^{-2} - \alpha_1^{-1}\alpha_2^{-1} \\ -\alpha_2^{-2} - \alpha_1^{-1}\alpha_2^{-1} & \alpha_2^{-2} + \alpha_1^{-1}\alpha_2^{-1} \end{pmatrix},$$

which is a symmetric matrix if and only if $\alpha_1 = \alpha_2$, since $\alpha_1 > 0$ and $\alpha_2 > 0$.

For $N \geq 3$, we have that every row and every column of A_δ^2 has three non-zero entries. The

matrix is of the following form:

$$\begin{pmatrix} \alpha_1^{-2} & -\alpha_1^{-2}-\alpha_1^{-1}\alpha_2^{-1} & \alpha_1^{-1}\alpha_2^{-1} & 0 & \cdots & 0 \\ 0 & \alpha_2^{-2} & -\alpha_2^{-2}-\alpha_2^{-1}\alpha_3^{-1} & \ddots & \ddots & \vdots \\ \vdots & 0 & \alpha_3^{-2} & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \alpha_{N-2}^{-1}\alpha_{N-1}^{-1} \\ \alpha_{N-1}^{-1}\alpha_N^{-1} & \ddots & \ddots & \ddots & \alpha_{N-1}^{-2} & -\alpha_{N-1}^{-2}-\alpha_{N-1}^{-1}\alpha_N^{-1} \\ -\alpha_N^{-2}-\alpha_N^{-1}\alpha_1^{-1} & \alpha_N^{-1}\alpha_1^{-1} & 0 & \cdots & 0 & \alpha_N^{-2} \end{pmatrix}.$$

Note that $(A_\delta^2)_{1,2} = -\alpha_1^{-2}-\alpha_1^{-1}\alpha_2^{-1} < 0$, whereas $(A_\delta^2)_{2,1} = \alpha_2^{-1}\alpha_3^{-1} > 0$, in the case of $N = 3$ and $(A_\delta^2)_{2,1} = 0$ for $N > 3$. Hence, for $N \geq 3$, it holds that $(A_\delta^2)_{1,2} \neq (A_\delta^2)_{2,1}$, which concludes the proof. □

This means, defining the δ-Laplacian as $\nabla_\delta \circ \nabla_\delta$, would lead to a non-symmetric operator. Since self-adjointness is a key property of the classical Laplacian, we expect an operator to be at least symmetric if it is to resemble a generalised Laplace operator. For the difference between the notion of symmetry and self-adjointness, see Appendix Chapter C.

To avoid this problem, we define the δ-Laplacian Δ_δ using a Dirichlet form. This is analogous to the program of Kigami [Kig93, Kig01, Kig03], Kigami and Lapidus [KL01] and others, who follow this approach in order to define a Laplacian on fractals. Note that they consider a different Laplacian to those discussed in this thesis.

Definition 3.1.8. For $f, g \in \mathscr{D}_\delta^1$ we define

$$\mathcal{E}(f,g) = \mathcal{E}_\delta(f,g) \coloneqq \langle \nabla_\delta f, \nabla_\delta g \rangle.$$

and refer to $\mathcal{E}$ as the *δ-energy form*.

By the definition of the integral and the formula for the δ-derivative given in (3.4) it holds for $f, g \in \mathscr{D}_\delta^1$ that

$$\mathcal{E}(f,g) = \sum_{i=1}^{N} \alpha_i \nabla_\delta f(z_i) \nabla_\delta g(z_i) = \sum_{i=1}^{N} \alpha_i^{-1} \Big(\mathbf{f}(z_{i+1}) - \mathbf{f}(z_i)\Big) \cdot \Big(\mathbf{g}(z_{i+1}) - \mathbf{g}(z_i)\Big). \tag{3.6}$$

We now show that the δ-energy form is a Dirichlet form on $\mathscr{D}_\delta^1$. The definition of a Dirichlet form is given in the Appendix B. When referring to the Conditions (i) – (iii) of a Dirichlet form, we mean the condition given in Definition B.1.
Recall that it has already been shown that every equivalence class of L_δ^2 has a δ-differentiable representative and hence $\mathcal{E}_\delta$ is everywhere defined on the Hilbert space L_δ^2. Therefore, it is sufficient to show that properties (i) – (iii) are fulfilled.

Lemma 3.1.9. *The δ-energy form $\mathcal{E}$ is a Dirichlet form on $\mathscr{D}^1_\delta$.*

Proof. The δ-energy form $\mathcal{E}$ is bilinear, since the inner product is bilinear, ∇_δ is linear and every equivalence class of L^2_δ has a δ-differentiable representative. The symmetry and the non-negativity of $\mathcal{E}$ follow directly from the properties of the inner product and hence, Condition (i) of the definition of a Dirichlet form is fulfilled.

Set $\langle f,g\rangle_{\mathcal{E}} \coloneqq \langle f,g\rangle + \mathcal{E}(f,g)$, for $f,g \in \mathscr{D}^1_\delta$. To prove condition (ii) of the definition of a Dirichlet form, it is sufficient to show that $\mathscr{D}^1_\delta$ is complete with respect to $\|\cdot\|_{\mathcal{E}} \coloneqq \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{E}}}$. For every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $(\mathscr{D}^1_\delta, \langle\cdot,\cdot\rangle_{\mathcal{E}})$, we have that both $(f_n)_{n\in\mathbb{N}}$ and $(\nabla_\delta f_n)_{n\in\mathbb{N}}$ are Cauchy-sequences in L^2_δ, since $\langle\cdot,\cdot\rangle$ and $\mathcal{E}(\cdot,\cdot)$ are non-negative. Hence, there exist $\tilde{f}_0, \tilde{f}_1 \in L^2_\delta$ with $\lim_{n\to\infty}\|f_n - \tilde{f}_0\|_\delta = 0$ and $\lim_{n\to\infty}\|\nabla_\delta f_n - \tilde{f}_1\|_\delta = 0$, where $\|\cdot\|_\delta$ is the norm induced by $\langle\cdot,\cdot\rangle$. Since ∇_δ is continuous, see Corollary 3.1.4, we have $\tilde{f}_0 \in \mathscr{D}^1_\delta$ with $\nabla_\delta \tilde{f}_0 = \tilde{f}_1$ which implies that $\lim_{n\to\infty} f_n = \tilde{f}_0$, with respect to $\|\cdot\|_{\mathcal{E}}$.

To conclude the proof it is sufficient to show that $\mathcal{E}$ has the Markov property. To this end, define for $u \in \mathscr{D}^1_\delta$ the function $\hat{u}\colon (0,1] \to \mathbb{R}$ by $\hat{u}(x) \coloneqq \min(\max(u(x),0),1)$. Since this function lies in L^2_δ and and is constant on the sets A_i for $i \in \{1,\dots,N\}$, from Proposition 3.1.2 (i) it follows that $\hat{u}$ belongs to $\mathscr{D}^1_\delta$. By definition of $\hat{u}$ it holds, for $i \in \{1,\dots,N\}$, that

$$|\hat{\mathbf{u}}(z_{i+1}) - \hat{\mathbf{u}}(z_i)| \le \left|\mathbf{u}(z_{i+1}) - \mathbf{u}(z_i)\right|.$$

Therefore, equation (3.6) yields $\mathcal{E}(\hat{u},\hat{u}) \le \mathcal{E}(u,u)$. □

With this at hand, we now use the δ-energy form to define the δ-Laplace operator and its corresponding domain.

Definition 3.1.10. The *set of twice δ-differentiable functions* is defined by

$$\mathscr{D}^2_\delta = \mathscr{D}^2_\delta((0,1]) \coloneqq \left\{f \in \mathscr{D}^1_\delta\colon \text{ there exists } f'' \in L^2_\delta \text{ such that } \mathcal{E}(f,g) = -\langle f'', g\rangle \text{ for all } g \in \mathscr{D}^1_\delta\right\}.$$

Proposition 3.1.11. *For $f \in \mathscr{D}^2_\delta$ the function $f'' \in L^2_\delta$ given in Definition 3.1.10 is unique.*

Proof. Let $f \in \mathscr{D}^2_\delta$ and assume that there exist two functions $h_1, h_2 \in L^2_\delta$ for which the equalities $\mathcal{E}(f,g) = -\langle h_1, g\rangle = -\langle h_2, g\rangle$ hold for all $g \in \mathscr{D}^1_\delta$. Define, for $i \in \{1,\dots,N\}$, the function $g_i \coloneqq \mathbb{1}_{A_i}$ and note that $g_i \in \mathscr{D}^1_\delta$. Hence, $\langle h_1, g_i\rangle = \alpha_i h_1(z_i)$ and $\langle h_2, g_i\rangle = \alpha_i h_2(z_i)$ for every $i \in \{1,\dots,N\}$, which yields $h_1 = h_2$. □

This ensures that the now introduced δ-Laplacian is a well-defined operator.

Definition 3.1.12. Letting $f \in \mathscr{D}^2_\delta$ and f'' be as in Definition 3.1.10, the operator

$$\begin{aligned}\Delta_\delta\colon \mathscr{D}^2_\delta &\to L^2_\delta\\ f &\mapsto f''\end{aligned}$$

is called the *δ-Laplacian* or the *δ-Laplace operator*.

By definition, we can observe for $f \in \mathscr{D}_\delta^2$ and an arbitrary $g \in \mathscr{D}_\delta^1$ that

$$\mathcal{E}(f,g) = \langle \nabla_\delta f, \nabla_\delta g \rangle = -\langle \Delta_\delta f, g \rangle, \tag{3.7}$$

from which we can directly deduce that

$$\Delta_\delta = -\nabla_\delta^* \circ \nabla_\delta. \tag{3.8}$$

As done in the proof of Proposition 3.1.11, define, for $i \in \{1,\dots,N\}$, the functions $g_i \coloneqq \mathbb{1}_{A_i}$ which are in $\mathscr{D}_\delta^1$ with

$$\nabla_\delta g_1(z_j) = \begin{cases} -\alpha_1^{-1} & \text{if } j = 1, \\ \alpha_N^{-1} & \text{if } j = N, \\ 0 & \text{otherwise.} \end{cases}$$

and, for $i \in \{2,\dots,N\}$,

$$\nabla_\delta g_i(z_j) = \begin{cases} \alpha_{i-1}^{-1} & \text{if } j = i-1, \\ -\alpha_i^{-1} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting these functions in equation (3.7) implies, for $i \in \{1,\dots,N\}$, that

$$\Delta_\delta f(z_i) = \frac{\nabla_\delta f(z_i) - \nabla_\delta f(z_{i-1})}{\alpha_i}.$$

We set $\alpha_0 \coloneqq \alpha_N$ and combine this with (3.4) to obtain

$$\Delta_\delta f(z_i) = \frac{f(z_{i+1}) - f(z_i)}{\alpha_i^2} - \frac{f(z_i) - f(z_{i-1})}{\alpha_{i-1}\alpha_i}, \tag{3.9}$$

for $i \in \{1,\dots,N\}$, which allows us to calculate $\Delta_\delta f$ for every $f \in \mathscr{D}_\delta^2$ explicitly. Further, we note that when substituting the formula for the first and the second δ-derivative, namely (3.4) and (3.9), in (3.7) the equation holds true for every $f, g \in L_\delta^2$. As indicated in the proof of Lemma 3.1.9, this means that $\pi(\mathscr{D}_\delta^2) = L_\delta^2$, where π is the embedding of $\mathscr{D}_\delta^2$ into L_δ^2. In other words, every equivalence class of L_δ^2 has a representative lying in $\mathscr{D}_\delta^2$.

Corollary 3.1.13. *The δ-Laplace operator Δ_δ is defined everywhere on L_δ^2.*

To define the δ-Laplacian weakly over the δ-energy form can also be motivated from the case of an atomless measure ν, more precisely by the correspondence between equations (2.8) and (3.8). Analogously to the classical integration by parts formula, we have seen in Proposition 2.1.14 (v) that in the case of an atomless measure ν the equation

$$\int_a^b (\Delta_\nu f) \cdot g \, \mathrm{d}\nu = \Big[(\nabla_\nu f) \cdot g \Big]_a^b - \int_a^b (\nabla_\nu f) \cdot (\nabla_\nu g) \, \mathrm{d}\nu$$

holds for all $a, b \in [0,1]$ with $a \leq b$ and all $f \in \mathscr{D}_\nu^2$ and $g \in \mathscr{D}_\nu^1$. When choosing $a = 0$ and $b = 1$ and assuming that $\nabla_\nu f$ and g fulfil periodic boundary conditions, this formula reduces to

$$\int_0^1 (\Delta_\nu f) \cdot g \, \mathrm{d}\nu = -\int_0^1 (\nabla_\nu f) \cdot (\nabla_\nu g) \, \mathrm{d}\nu,$$

which is analogous to (3.7) in the purely atomic setting.

Using (3.8) together with the matrix representation for the δ-derivative, we can also find a $N \times N$-matrix B_δ such that

$$B_\delta (f(z_1), \ldots, f(z_N))^\top = (\Delta_\delta f(z_1), \ldots, \Delta_\delta f(z_N))^\top.$$

From (3.9) it follows that this matrix is the negative of the Gramian matrix corresponding to A_δ, namely $B_\delta = -A_\delta^\top A_\delta$. For $N = 2$, we have that

$$B_\delta = \begin{pmatrix} -\alpha_1^{-2} - \alpha_2^{-2} & \alpha_1^{-2} + \alpha_2^{-2} \\ \alpha_1^{-2} + \alpha_2^{-2} & -\alpha_1^{-2} - \alpha_2^{-2} \end{pmatrix}, \tag{3.10}$$

and, for $N \geq 3$, the matrix B_δ is

$$\begin{pmatrix} -\alpha_N^{-2} - \alpha_1^{-2} & \alpha_1^{-2} & 0 & \cdots & 0 & 0 & \alpha_N^{-2} \\ \alpha_1^{-2} & -\alpha_1^{-2} - \alpha_2^{-2} & \alpha_2^{-2} & \cdots & 0 & 0 & 0 \\ 0 & \alpha_2^{-2} & -\alpha_2^{-2} - \alpha_3^{-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{N-3}^{-2} - \alpha_{N-2}^{-2} & \alpha_{N-2}^{-2} & 0 \\ 0 & 0 & 0 & \cdots & \alpha_{N-2}^{-2} & -\alpha_{N-2}^{-2} - \alpha_{N-1}^{-2} & \alpha_{N-1}^{-2} \\ \alpha_N^{-2} & 0 & 0 & \cdots & 0 & \alpha_{N-1}^{-2} & -\alpha_{N-1}^{-2} - \alpha_N^{-2} \end{pmatrix}.$$

Note that only in the case that $N = 2$ and $\alpha_1 = \alpha_2$ we have that $B_\delta = -A_\delta^2$. As discussed in Proposition 3.1.7 this is also the only case where A_δ^2 is symmetric. We empasise that the matrix representation of the operator Δ_δ only depends on the weight of the atoms of δ and not on the distances between them.

This matrix representation shows that the δ-Laplacian, for $N \geq 2$, is closely related to two other types of Laplacians, namely the graph Laplacian on a weighted cycle graph and a discrete Laplacian on a non-uniform grid.
Assume that we have a graph $\mathcal{G} = (V, E)$, where V is the set of vertices and E the set of undirected edges. Assume that $V = \{v_1, \ldots, v_N\}$ and that there is an edge $e_{v_i, v_j} \in E$ if and only if $j = i + 1$ or $i = N$ and $j = 1$. The graph $\mathcal{G}$ is connected and the degree of each vertice is two, hence it is a so-called *cycle graph*. The *graph Laplacian* $L_\mathcal{G}$ associated to the graph $\mathcal{G}$ is the difference of the matrix with the degree of the vertices along the diagonal and zero

everywhere else and the adjacency matrix, namely

$$L_{\mathcal{G}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{3.11}$$

This is exactly the negative of B_δ for $\delta = \sum_{i=1}^N \delta_{z_i}$, where z_i are as defined throughout this chapter. A full discussion of the spectral properties of Δ_δ will be given in Section 3.3.2. The general case that not all $\alpha_i = 1$ is obtained by looking at weighted graphs $\mathcal{G}^* = (V, E, w)$, where V and E are as above and $w\colon E \to \mathbb{R}^+$ associates the weights $w(e_{v_i,v_{i+1}}) = \alpha_i^{-2}$ and $w(e_{v_N,v_1}) = \alpha_N^{-2}$ to the edges and is discussed, for example, in [Moh91].
The second type of Laplace operators for which there exists a connection to the δ-Laplacian is the class of discrete Laplacians on a non-uniform grid. Assume we have a twice differentiable function $u\colon [0,1] \to \mathbb{R}$. If we want to approximate $u''(x_2)$ for a fixed point $x_2 \in (0,1)$ based on the values U_1, U_2 and U_3 at three unequally spaced points $x_1, x_2, x_3 \in (a,b)$, one way to do this is by interpolating with a quadratic function and differentiating twice. Let $h_1 \coloneqq x_2 - x_1$ and $h_2 \coloneqq x_3 - x_2$ and denote the interpolating polynomial by $p\colon [0,1] \to \mathbb{R}$. Using techniques of finite difference methods, one obtains that

$$p''(x_2) = \frac{2U_1}{h_1(h_1+h_2)} - \frac{2U_2}{h_1 h_2} + \frac{2U_3}{h_2(h_1+h_2)}.$$

This yields that the difference between the δ-Laplacian and the discrete Laplacian is only the weighting of the values U_i. Namely, we have the correspondences

$$h_1 = \sqrt{\frac{2\alpha_1^2\alpha_2}{\alpha_1+\alpha_2}} \quad \text{and} \quad h_2 = \sqrt{\frac{2\alpha_2^3}{\alpha_1+\alpha_2}}$$

When assuming periodic boundary conditions and looking at the approximations for $u''(x_1)$ and $u''(x_3)$ similar relations between the distances h_1, h_2 and h_3, where $h_3 \coloneqq x_1 - x_3 + b - a$, and the weights α_1, α_2 and α_3 can be obtained. Note that choosing the an uniform grid, namely $h_1 = h_2 = h_3$, corresponds to the case of choosing equally weighted atoms for the δ-Laplacian. For further details about discrete Laplacians on (non-uniform) grids see, for example, [LeV07, Example 1.4 and Section 2.18].

We now continue the study of the δ-Laplacian. From the equality $B_\delta = -A_\delta^\top A_\delta$ it follows that the kernels of B_δ and A_δ coincide and therefore, the rank of B_δ is $N-1$. To see the equality of the kernels, note that if for some vector $v \in \mathbb{R}^N \setminus \underline{0}$ holds that $A_\delta v = \underline{0}$, then also $A_\delta^\top A_\delta v = \underline{0}$ and hence $\ker(B_\delta) \subseteq \ker(A_\delta)$. For the other inclusion let $w \in \mathbb{R}^N \setminus \underline{0}$ be such that $A_\delta^\top A_\delta w = \underline{0}$. It then holds that $w^\top A_\delta^\top A_\delta w = (A_\delta w)^\top (A_\delta w) = \underline{0}$ and hence, that $A_\delta w = \underline{0}$. Let $\mathcal{H}_\delta$ denote the *set of δ-harmonic functions*, namely $\mathcal{H}_\delta \coloneqq \{f \in \mathscr{D}_\delta^2 \colon \Delta_\delta f = 0\}$. We then obtain from (3.5) the following.

Corollary 3.1.14.

$$\mathcal{H}_\delta = \{c\mathbb{1} : c \in \mathbb{R}\}$$

We conclude this section by showing important analytic properties of Δ_δ, which we also know for the classical Laplace operator. This supports that it is reasonable to call the operator developed in this chapter a Laplacian with respect to a purely atomic measure.

Theorem 3.1.15. *The operator Δ_δ is linear, bounded, self-adjoint and non-positive on L^2_δ.*

Proof. In Corollary 3.1.4 we showed that ∇_δ is a linear operator and since $\mathcal{E}$ is a Dirichlet form, see Lemma 3.1.9, it is bilinear. Together with the definition of Δ_δ, this implies the linearity of the operator. Similar to the proof of Proposition 3.1.5, equation (3.9) can be used to show that Δ_δ is bounded. Self-adjointness can easily be obtained, since Δ_δ is a finite-dimensional operator. Therefore, it is sufficient to check that its matrix representation is symmetric, namely $B_\delta = B_\delta^\top$, which is clear by definition. Using (3.7), we have for $f \in \mathscr{D}^2_\delta$ that $\langle \Delta_\delta f, f\rangle = -\langle \nabla_\delta f, \nabla_\delta f\rangle \leq 0$, and hence, Δ_δ is non-positive. □

3.2 Spectral properties

In this section, results about the eigenvalues and eigenfunctions of Δ_δ are discussed for measures $\delta = \sum_1^N \alpha_i \delta_{z_i}$ where $N \geq 2$, $0 < z_1 \cdots < z_N = 1$ and $\alpha_i > 0$, for $i \in \{1, \ldots, N\}$. As mentioned in the previous section, in the case that $N = 1$ we have that ∇_δ is the null-operator and its domain is the set of constant functions. Therefore, an analysis of the spectral properties can be omitted in this case.

By definition of the matrix B_δ, in order to find the eigenvalues and eigenfunctions of Δ_δ, it suffices to compute the eigenvalues and eigenvectors of B_δ. Recall that B_δ is an $N \times N$-matrix, which means that there exist at most N real eigenvalues. We do not give a general solution for the eigenvalue problem of matrices of this form, but we obtain some general spectral properties and bounds of the eigenvalues and then give a complete solution for three leading examples.

Since we know from Corollary 3.1.14 that the kernel of the operator is non-trivial, zero is an eigenvalue of Δ_δ. From the fact that $\mathcal{H}_\delta$ is one-dimensional, it also follows that the eigenvalue zero is simple, namely that $\dim(\mathrm{Eig}(0)) = 1$.

Corollary 3.2.1. *The operator Δ_δ has a simple eigenvalue at $\lambda = 0$ where the corresponding eigenfunctions are the non-zero constants.*

We now use the properties of Δ_δ together with general results in matrix theory to deduce that all eigenvalues are real and to obtain explicit bounds for them.

Proposition 3.2.2. *The operator Δ_δ has N real eigenvalues λ with* $2 \min_{i \in \{1,\ldots,N\}} (B_\delta)_{i,i} \leq \lambda \leq 0$.

Proof. We first prove that all λ are real-valued, before we obtain the bounds in the second step. The first part is a well-known result, but we state the proof for completeness. It follows from the fundamental theorem of algebra that the matrix B_δ has N complex eigenvalues λ with corresponding non-zero eigenvectors $v \in \mathbb{C}^N$. The eigenvalue equation implies that $\overline{v}^\top (B_\delta v) = \overline{v}^\top (\lambda v) = \lambda(\overline{v} \cdot v)$, where $\overline{v} \cdot v$ denotes the dot product of $\overline{v}$ and v. On the other hand, using that the matrix B_δ is real-valued and symmetric, namely $B_\delta = \overline{B_\delta} = B_\delta^\top$, we obtain that $\overline{v}^\top (B_\delta v) = (B_\delta \overline{v})^\top v = (\overline{B_\delta}\overline{v})^\top v = (\overline{\lambda v})^\top v = \overline{\lambda}(\overline{v} \cdot v)$. Since v is not the null vector, this implies that $\lambda = \overline{\lambda}$ and hence that all N eigenvalues of B_δ are real.

We know from Theorem 3.1.15 that Δ_δ is a non-positive operator which implies that all eigenvalues are less than or equal to zero.

A general result in linear algebra is that the spectral radius, namely the maximum of the absolute values of the eigenvalues, is bounded from above by every matrix norm. To see that, let λ be an eigenvalue of B_δ with corresponding eigenvector v and let $\|\cdot\|_M$ denote a norm on the set of $N \times N$-matrices $\mathcal{M}_N$. Define the matrix $V \in \mathcal{M}_N$ as the N-fold concatenation of v, namely $V := [v, \cdots, v]$. Since v is an eigenvector it holds that $B_\delta V = \lambda V$ and since v is not the null vector V has norm strictly greater than zero. From

$$|\lambda| \cdot \|V\|_M = \|\lambda V\|_M = \|B_\delta V\|_M \leq \|B_\delta\|_M \cdot \|V\|_M$$

follows that $|\lambda| \leq \|B_\delta\|_M$, since $\|V\|_M > 0$.

Now choose $\|\cdot\|_M$ to be the row sum norm $\|\cdot\|_\infty$, which, for $A \in \mathcal{M}_N$, is defined as

$$\|A\|_\infty := \max_{i \in \{1,\dots,N\}} \sum_{j=1}^{N} |a_{ij}|.$$

From the general form of B_δ it follows that for every eigenvalue λ the following inequality holds.

$$|\lambda| \leq 2 \max_{i \in \{1,\dots,N\}} |(B_\delta)_{i,i}|$$

Combining this with the already proven non-positivity of the eigenvalues and the fact that $(B_\delta)_{i,i} < 0$, for $i \in \{1, \dots, N\}$, concludes the proof. □

It is clear that the upper bound is always sharp, since B_δ has a non-trivial kernel. As pointed out in the proof, depending on the choice of the weights α_i, one could possibly get a better lower bound on the eigenvalues of B_δ, by studying different matrix norms. In the following section, we will see three examples for which the lower bound stated in Proposition 3.2.2 is sharp and one where that is not the case.

We have seen that zero is always and eigenvalue of B_δ and that all other eigenvalues are negative. This implies that the spread of the matrix, namely the absolute value of the difference between the smallest and the biggest eigenvalue, is equal to the negative of the smallest eigenvalue. Since the matrix B_δ is real-valued and symmetric, we can use the results of Mirsky [Mir56, Mir57] to obtain bounds on the smallest eigenvalue.

Corollary 3.2.3. *Set ω to be the absolute value of the smallest eigenvalue of B_δ. A lower bound for ω is given by*

$$\max\left\{2\max_{i\neq j}(B_\delta)_{i,j},\ \max_{i\neq j}\sqrt{\left((B_\delta)_{i,i}-(B_\delta)_{j,j}\right)^2+4\left((B_\delta)_{i,j}\right)^2}\right\}\leq\omega$$

and an upper bound by

$$\omega\leq\min\left\{\sqrt{2\sum_{i,j=1}^{N}\left((B_\delta)_{i,j}\right)^2},\ 2\max_{i\in\{1,\dots,N\}}(B_\delta)_{i,i}\right\}.$$

Proof. The lower bounds follow from [Mir57, Corollary to Theorem 1, Theorem 2] and the upper bounds are a consequence of [Mir56, Corollary to Theorem 1] and Proposition 3.2.2. □

3.3 Examples for eigenvalues and eigenfunctions

In this section we discuss three examples of discrete distributions δ. These are that δ is the weighted sum of two Dirac-point masses, namely $N=2$; that δ is a uniform discrete probability distribution; and that δ is a probability measure with six atoms and alternating weights. In all three settings we are able to explicitly calculate the eigenvalues and eigenvectors. Further, in the second and third example, we point out connections to the case of atomless measures ν, which were discussed in Chapter 2.

3.3.1 Two atoms

In the case $\delta_{(1)}=\alpha_1\delta_{z_1}+\alpha_2\delta_{z_2}$, we can explicitly calculate all eigenvalues and eigenvectors of $B_{\delta_{(1)}}$ and, hence, know the eigenvalues and eigenfunctions of $\Delta_{\delta_{(1)}}$.

Proposition 3.3.1. *The eigenvalues of $\Delta_{\delta_{(1)}}$ are $\lambda_0=0$ and $\lambda_1=-2(\alpha_1^{-2}+\alpha_2^{-2})$ with corresponding eigenfunctions,*

$$f_0(x)=1,\ \textit{for all } x\in[0,1],\quad \textit{and}\quad f_1(x)=\begin{cases}1 & \textit{for } x\in[0,z_1]\cup(z_2,1],\\ -1 & \textit{otherwise.}\end{cases}$$

Proof. In (3.10) it was shown that the matrix representation of the $\delta_{(1)}$-Laplacian is

$$B_{\delta_{(1)}}=\begin{pmatrix}-\alpha_1^{-2}-\alpha_2^{-2} & \alpha_1^{-2}+\alpha_2^{-2}\\ \alpha_1^{-2}+\alpha_2^{-2} & -\alpha_1^{-2}-\alpha_2^{-2}\end{pmatrix}.$$

For this matrix, a direct calculation shows that $B_{\delta_{(1)}}(1,1)^\top=(0,0)^\top=\lambda_0(1,1)^\top$ and also that $B_{\delta_{(1)}}(1,-1)^\top=(-2(\alpha_1^{-2}+\alpha_2^{-2}),2(\alpha_1^{-2}+\alpha_2^{-2}))^\top=\lambda_1(1,-1)^\top$. □

Note that in this case the eigenvalues of $B_{\delta_{(1)}}$ coincide with the lower and the upper bound stated in Proposition 3.2.2.

3.3.2 Uniform discrete probability distributions

We now consider the case that $\delta = \delta_{(2)}$ is a uniform discrete probability distribution. This means that $\delta_{(2)} = \sum_{i=1}^{N} N^{-1}\delta_{z_i}$ with $N \geq 3$ and $0 < z_1 < \cdots < z_N = 1$. Note that uniform means in this context that all atoms have the same weight, but there is no condition on the position of the atoms, since the operator Δ_δ is independent of the positions.

In this case, the matrix representation of $\Delta_{\delta_{(2)}}$ is of the form

$$B_{\delta_{(2)}} = \begin{pmatrix} -2N^2 & N^2 & 0 & \cdots & 0 & 0 & N^2 \\ N^2 & -2N^2 & N^2 & \cdots & 0 & 0 & 0 \\ 0 & N^2 & -2N^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2N^2 & N^2 & 0 \\ 0 & 0 & 0 & \cdots & N^2 & -2N^2 & N^2 \\ N^2 & 0 & 0 & \cdots & 0 & N^2 & -2N^2 \end{pmatrix}. \tag{3.12}$$

Note that $B_{\delta_{(2)}}$ is closely connected to the matrix of the graph Laplacian given in (3.11). To be precise, we have that $B_{\delta_{(2)}} = -N^2 L_{\mathcal{G}}$. The matrix representation of $\Delta_{\delta_{(2)}}$ is a *circulant matrix*, namely a special type of matrix where each row vector is rotated one element to the right relative to the preceding row vector. That is equivalent to the columns of the matrix being cyclic permutations of the first column with offset equal to the column index. Circulant matrices are well studied, since they resemble a discrete Fourier transform. For further details about the discrete Fourier transform, we refer the reader to [Smi07].

The close relation to discrete Fourier transform will also be seen in the proof of the following theorem on the spectrum of the matrix $B_{\delta_{(2)}}$. The chosen approach leads to complex-valued eigenvectors and, just after the proof, from these we deduce real-valued ones, namely we consider the real and the imaginary parts.

Theorem 3.3.2. *The eigenvalues of the matrix $B_{\delta_{(2)}}$ given in* (3.12) *are of the form*

$$\lambda_l = -2N^2 + 2N^2 \cos\left(\frac{2\pi l}{N}\right)$$

with corresponding complex-valued eigenvectors

$$v^{(l)} = \left(1, \exp\left(\mathbf{i}\frac{2\pi l}{N}\right), \exp\left(\mathbf{i}\frac{2\pi 2l}{N}\right), \ldots, \exp\left(\mathbf{i}\frac{2\pi(N-1)l}{N}\right)\right)^\top$$

for $l \in \{0, \ldots, N-1\}$.

Proof. Set $m_1 := -2N^2, m_2 = m_N := N^2$ and $m_i = 0$, for $i \in \{3, \ldots, N-1\}$. We now look for $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^N$ such that the eigenvalue equation $B_{\delta_{(2)}} v = \lambda v$ is fulfilled. The eigenvalue

equation can be reformulated as a system of N difference equations of the form

$$\sum_{k=1}^{N-j} m_k v_{k+j} + \sum_{k=N-j+1}^{N} m_k v_{k-N+j} = \lambda v_{j+1}, \tag{3.13}$$

where $j \in \{0,\dots,N-1\}$ and, as usual, v_i denotes the i-th component of the vector v. Set $\varphi_l^k \coloneqq \exp(\mathbf{i}2\pi kl/N)$, for $k,l \in \{0,\dots,N-1\}$. We now choose the eigenvector $v^{(l)}$ following the ansatz that $v_k^{(l)} \coloneqq \varphi_l^{k-1}$ and substitute this into (3.13). This yields for $j \in \{0,\dots,N-1\}$

$$\sum_{k=1}^{N-j} m_k \varphi_l^{k+j-1} + \sum_{k=N-j+1}^{N} m_k \varphi_l^{k-N+j-1} = \lambda_l \varphi_l^j.$$

Using the facts that $\varphi_l^{-N} = 1$ and that $\varphi_l^k \neq 0$ for all $k,l \in \{0,\dots,N-1\}$, we can divide both sides by φ_l^j and obtain that $\lambda_l = \sum_{k=1}^{N} m_k \varphi_l^{k-1}$. Hence, for $l \in \{0,\dots,N-1\}$, it holds that

$$\begin{aligned}\lambda_l &= -2N^2 + N^2 \exp\left(\mathbf{i}\frac{2\pi l}{N}\right) + N^2 \exp\left(\mathbf{i}\frac{2\pi l(N-1)}{N}\right)\\ &= -2N^2 + 2N^2 \cos\left(\frac{2\pi l}{N}\right)\end{aligned}$$

and that λ_l and $v^{(l)}$ fulfil the eigenvalue equation $B_{\delta_{(2)}} v^{(l)} = \lambda_l v^{(l)}$. □

If N is even, the result gives a second example for which the bounds in Proposition 3.2.2 are sharp, since we have that $\lambda_0 = 0$ and $\lambda_{N/2} = -2N^{-2} = (B_{\delta_{(2)}})_{i,i}$, for all $i \in \{1,\dots,N\}$. In contrast to that, the lower bound is not sharp if N is odd, since there does not exist an $l \in \{0,\dots,N-1\}$ such that $\cos(2\pi l/N) = 0$.

Further, it can be obtained that all eigenvalues apart from at most two occur in pairs. To be precise, we have that $\lambda_i = \lambda_{N-i}$, for $i \in \{1,\dots,\lfloor N-1 \rfloor/2\}$, and $\lambda_i \neq \lambda_j$, for $j \notin \{i, N-i\}$. In Corollary 3.2.1 we showed that zero is always a simple eigenvalue and hence, if N is odd, all other eigenvalues have multiplicity two. If N is even, $\lambda_{N/2} = -4N^2$ is also simple.

To obtain real-valued eigenfunctions of the operator $\Delta_{\delta_{(2)}}$, one uses the fact that if v is a complex eigenvector of the matrix $B_{\delta_{(2)}}$, then both $\mathrm{Im}(v)$ and $\mathrm{Re}(v)$ are also eigenvectors and additionally they are linearly independent. Using the identities $\mathrm{Re}(\exp(\mathbf{i}x)) = \cos(x)$ and $\mathrm{Im}(\exp(\mathbf{i}x)) = \sin(x)$ we obtain can obtain the eigenfunctions of $\Delta_{\delta_{(2)}}$.

Corollary 3.3.3. *The eigenvalues of the operator $\Delta_{\delta_{(2)}}$ are $\lambda_l = -2N^2 + 2N^2\cos(2\pi l/N)$, for $l \in \{0,\dots,N-1\}$, with corresponding eigenfunctions $f_l \in \mathscr{D}^2_{\delta_{(2)}}$, where*

1. *f_0 is the constant function with value 1,*

and, for $j \in \{1,\dots,N\}$,

2. *$f_l|_{A_j} = \sin\left(2\pi l(j-1)N^{-1}\right)$ for $0 < l < N/2$, and*
3. *$f_l|_{A_j} = \cos\left(2\pi l(j-1)N^{-1}\right)$ for $N/2 \leq l \leq N-1$.*

The eigenfunctions of $\Delta_{\delta_{(2)}}$ are closely related to the eigenfunctions of Δ_ν, where ν is an atomless measure. From Theorem 2.2.5 we know that the eigenfunctions of Δ_ν under periodic boundary conditions are of the form

$$f^{(2n)}(x) = \sin(2\pi n F_\nu(x))$$

for $n \in \mathbb{N}$ and

$$g^{(2m)}(x) = \cos(2\pi m F_\nu(x))$$

for $m \in \mathbb{N}_0$, where F_ν is the distribution function of ν. Recall that from the definition of the distribution function F_ν and its left-continuous analogue F_ν^{lc} it follows that $F_\nu^{lc} = F_\nu$, since ν is atomless.

When now looking at purely atomic measures δ the latter equation no longer holds true for F_δ and F_δ^{lc}. We have that the two functions coincide outside the atoms, but that F_δ is right-continuous and F_δ^{lc} is left-continuous. Corollary 3.3.3 can now also be read as that the eigenfunctions of Δ_δ are of the form $\sin(2\pi n F^{lc}_{\delta_{(2)}}(x))$ and $\cos(2\pi m F^{lc}_{\delta_{(2)}}(x))$, where the possible choices for n and m depend on the total number of atoms.

Further, strengthening the connection between the purely atomic and the atomless case, in the situation of Corollary 3.3.3 we have that with increasing N we approximate the weak Laplacian Δ_Λ, where Λ denotes the Lebesgue measure on $(0,1]$. To clarify this relation we denote by $\delta^N_{(2)}$ the uniform discrete probability distribution with N equally spaced atoms, namely $\delta^N_{(2)} = \sum_{i=1}^N N^{-1}\delta_{i/N}$. It then holds that

$$\operatorname*{w-lim}_{N\to\infty} \delta^N_{(2)} = \Lambda, \tag{3.14}$$

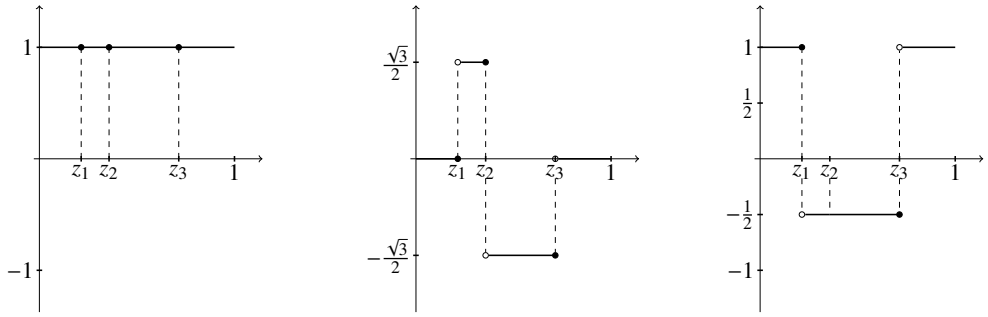
where this notation means that the measures converge weakly. See for the definition of weak convergence Appendix A. Since every bounded, continuous function is Riemann and Lebesgue integrable and the integrals coincide, the weak convergence stated in (3.14) follows from the fact that $\int f \, \mathrm{d}\delta^N_{(2)}$ is equal to the right Riemann sum of f over $(0,1]$ on a uniform partition with N intervals.

For a better readability, we denote the left-continuous distribution function of $\delta^{(N)}_{(2)}$ by $F^{lc}_{(N)}$ and the corresponding $\delta^{(N)}_{(2)}$-Laplacian by $\Delta_{(N)}$. The weak convergence of the measures implies that $F^{lc}_{(N)} \to \mathrm{id}$ uniformly and hence, that the eigenfunctions of $\Delta_{(N)}$ converge uniformly to the eigenfunctions of Δ_Λ. We also obtain that the eigenvalues of $\Delta_{(N)}$ converge to $-(2\pi l)^2$, the corresponding eigenvalues of the classical weak Laplacian. To see that, we take the eigenvalues of the $\delta^{(N)}_{(2)}$-Laplacian which are $\lambda_l = -2N^2 + 2N^2\cos(2\pi l/N)$, for $l \in \{0,\dots,N-1\}$, and look at the Taylor series of the cosine at zero, which is

$$1 - \frac{x^2}{2} + O(x^4).$$

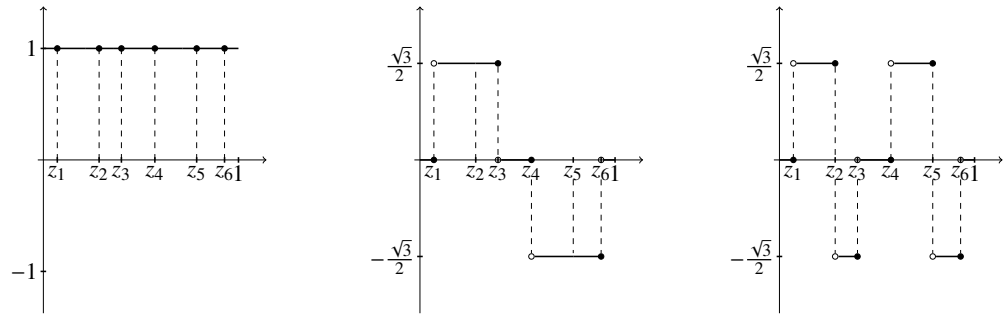
This means we can rewrite the equation for λ_l as $\lambda_l = -(2\pi l)^2 + O((2\pi l/N)^4)$, where for every fixed $l \in \mathbb{N}$ the error term vanishes when N tends to infinity. Note that the weak convergence

of the measure, the uniform convergence of the eigenfunctions and the convergence of the eigenvalues also hold true if the atoms in $\delta_{(2)}^N$ are not equally spaced.

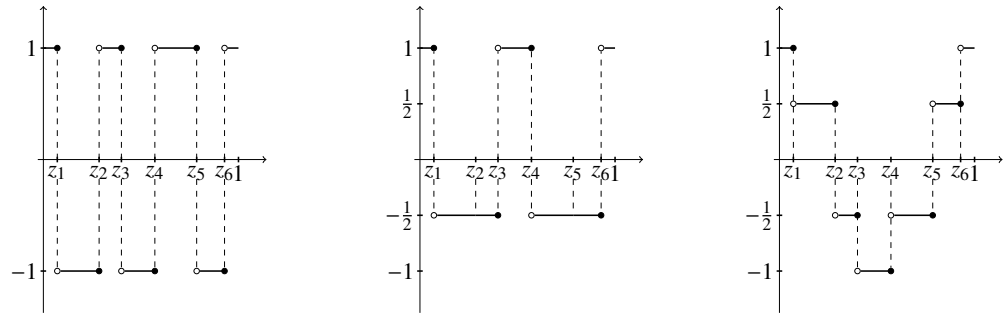


Graph of f_0 Graph of f_1 Graph of f_2

Figure 3.1. Graphs of the eigenfunctions f_k of $\Delta_{\delta_{(2)}}$, for $k \in \{0,1,2\}$, where $N = 3$ and hence, $\delta_{(2)} = \sum_{i=1}^{3} 3^{-1}\delta_{z_i}$.



Graph of f_0 Graph of f_1 Graph of f_2



Graph of f_3 Graph of f_4 Graph of f_5

Figure 3.2. Graphs of the eigenfunctions f_k of $\Delta_{\delta_{(2)}}$, for $k \in \{0,1,2,3,4,5\}$, where $N = 6$ and hence, $\delta_{(2)} = \sum_{i=1}^{6} 6^{-1}\delta_{z_i}$.

For $N = 3$, namely $\delta_{(2)} = \sum_{i=1}^{3} 3^{-1}\delta_{z_i}$, we have that $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = -27$. A graphical representation of the corresponding eigenfunctions f_k, for $k \in \{0,1,2\}$, is given in Figure 3.1. In the case of six atoms with uniform mass, that is $\delta_{(2)} = \sum_{i=1}^{6} 6^{-1}\delta_{z_i}$, the eigenvalues of $\Delta_{\delta_{(2)}}$ are $\lambda_0 = 0$, $\lambda_1 = \lambda_5 = -36$, $\lambda_2 = \lambda_4 = -108$ and $\lambda_3 = -144$. The corresponding eigenfunctions f_k, for $k \in \{0,1,2,3,4,5\}$, can be seen in Figure 3.2.

3.3.3 Alternating atom weights

In this paragraph we show that, unlike in the case of continuous distributions, the eigenfunctions for distributions with finite support are not in general of the form $f_\delta^{(2\kappa)} = \sin(2\pi\kappa F_\delta^{lc}(\cdot))$ or $g_\delta^{(2\kappa)} = \cos(2\pi\kappa F_\delta^{lc}(\cdot))$, for some $\kappa \in \mathbb{R}$. The example discussed here is for the case $N = 6$, but it can easily be generalised to every even $N \in \mathbb{N}$.

We consider a probability measure with six atoms and two different atom weights occurring alternately. To this end, let m_1 and m_2 denote two positive real numbers with $3m_1 + 3m_2 = 1$ and consider the discrete distribution $\delta_{(3)} = m_1\delta_{\{z_1,z_3,z_5\}} + m_2\delta_{\{z_2,z_4,z_5\}}$, where the atoms are ordered as usual. The operator $\Delta_{\delta_{(3)}}$ is represented by the matrix

$$B_{\delta_{(3)}} = \begin{pmatrix} -m_1^{-2}-m_2^{-2} & m_1^{-2} & 0 & 0 & 0 & m_2^{-2} \\ m_1^{-2} & -m_1^{-2}-m_2^{-2} & m_2^{-2} & 0 & 0 & 0 \\ 0 & m_2^{-2} & -m_1^{-2}-m_2^{-2} & m_1^{-2} & 0 & 0 \\ 0 & 0 & m_1^{-2} & -m_1^{-2}-m_2^{-2} & m_2^{-2} & 0 \\ 0 & 0 & 0 & m_2^{-2} & -m_1^{-2}-m_2^{-2} & m_1^{-2} \\ m_2^{-2} & 0 & 0 & 0 & m_1^{-2} & -m_1^{-2}-m_2^{-2} \end{pmatrix}.$$

The characteristic polynomial p of this matrix is

$$p(\lambda) = \lambda\left(2\left(m_1^{-2}+m_2^{-2}\right)+\lambda\right)\left(\lambda^2 + 2\left(m_1^{-2}+m_2^{-2}\right)\lambda + 3m_1^{-2}m_2^{-2}\right)^2$$

and the roots of p are the eigenvalues of $B_{\delta_{(3)}}$.

Theorem 3.3.4. *The eigenvalues of $B_{\delta_{(3)}}$ are*

$$\lambda_0 = 0, \qquad \lambda_1 = \lambda_5 = -(m_1^{-2}+m_2^{-2}) + \sqrt{m_1^{-4}+m_2^{-4}-m_1^{-2}m_2^{-2}},$$
$$\lambda_3 = -2(m_1^{-2}+m_2^{-2}), \qquad \lambda_2 = \lambda_4 = -(m_1^{-2}+m_2^{-2}) - \sqrt{m_1^{-4}+m_2^{-4}-m_1^{-2}m_2^{-2}}.$$

Setting $r := m_2/m_1$, the corresponding eigenvectors, up to scalar multiplication, are

$$\begin{aligned}
v^{(0)} &= (1,1,1,1,1,1)^\top, \\
v^{(1)} &= \left(r^2, \sqrt{1+r^4-r^2}, 1-r^2, -\sqrt{1+r^4-r^2}, -1, 0\right)^\top, \\
v^{(2)} &= \left(r^2, -\sqrt{1+r^4-r^2}, 1-r^2, \sqrt{1+r^4-r^2}, -1, 0\right)^\top, \\
v^{(3)} &= (1,-1,1,-1,1,-1)^\top, \\
v^{(4)} &= \left(\sqrt{1+r^4-r^2}, 1-r^2, -\sqrt{1+r^4-r^2}, r^2, 0, -1\right)^\top, \\
v^{(5)} &= \left(\sqrt{1+r^4-r^2}, r^2-1, -\sqrt{1+r^4-r^2}, -r^2, 0, 1\right)^\top.
\end{aligned}$$

Proof. For $l \in \{0,\dots,5\}$, a substitution shows that $p(\lambda_l) = 0$ and $B_{\delta_{(3)}}v^{(l)} = \lambda_l v^{(l)}$. □

In the previous example of an discrete probability distribution, see Section 3.3.2, we saw that all eigenvalues apart from at most two have multiplicity two. Further, the ansatz to find the eigenvectors of the matrix lead to complex-valued eigenvectors, with entries lying on the unit circle. We now look at the similar pairs of eigenvectors in the example discussed here and proof that they lie on an ellipse rather than on a circle.
Taking the eigenvalues with multiplicity two, namely $\lambda_1 = \lambda_5$ and $\lambda_2 = \lambda_4$, the sets of tuples

$$S_{1,5} := \left\{\left(v_1^{(1)}, v_1^{(5)}\right), \ldots, \left(v_6^{(1)}, v_6^{(5)}\right)\right\} \quad \text{and} \quad S_{2,4} := \left\{\left(v_1^{(2)}, v_1^{(4)}\right), \ldots, \left(v_6^{(2)}, v_6^{(4)}\right)\right\}$$

consist of three pairs of points of the form (x, y) and $(-y, -x)$. Both sets determine the same conic section, namely,

$$\mathfrak{E} := \left\{(x, y) \in \mathbb{R}^2 : \ \sqrt{r^{-4} - r^{-2} + 1}\left(x^2 + y^2 - 1\right) = \left(2 - r^{-2}\right)xy\right\}.$$

This means, if we take any five points of the set $S_{1,5}$ and solve the general form for conic sections

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

for $A, B, C, D, E, F \in \mathbb{R}$, we obtain the equation for $\mathfrak{E}$ and see that the sixth point of $S_{1,5}$ also lies in $\mathfrak{E}$. This can be verified by substituting the points of $S_{1,5}$ in the defining equation of $\mathfrak{E}$. From the fact that $B^2 - 4AC = -3r^{-4} < 0$, it follows that $\mathfrak{E}$ is an ellipse. The same holds true for the set $S_{2,4}$ and, furthermore, both ellipses coincide.
The resulting ellipse is centred at the origin and the symmetry axes are the lines $y = x$ and $y = -x$. This follows from the defining equation of $\mathfrak{E}$, since if a pair (x, y) lies in $\mathfrak{E}$, then also (y, x) and $(-y, -x)$ are elements of $\mathfrak{E}$.

We now look at the example $m_1 = 1/4$ and $m_2 = 1/12$, namely we consider the measure $\delta_{(3)} = 4^{-1}\delta_{\{z_1, z_3, z_5\}} + 12^{-1}\delta_{\{z_2, z_4, z_5\}}$. We have that $r = 3^{-1}$ and using the formulas above, it follows that the eigenvalues are $\lambda_0 = 0$, $\lambda_1 = \lambda_5 = -16(10 - \sqrt{73})$, $\lambda_2 = \lambda_4 = -16(10 + \sqrt{73})$ and $\lambda_3 = -320$ with corresponding eigenvectors

$$\begin{aligned}
v^{(0)} &= (1, 1, 1, 1, 1, 1)^\top, \\
v^{(1)} &= \frac{1}{9}\left(1, \sqrt{73}, 8, -\sqrt{73}, -9, 0\right)^\top, \\
v^{(2)} &= \frac{1}{9}\left(1, -\sqrt{73}, 8, \sqrt{73}, -9, 0\right)^\top, \\
v^{(3)} &= (1, -1, 1, -1, 1, -1)^\top, \\
v^{(4)} &= \frac{1}{9}\left(\sqrt{73}, 8, -\sqrt{73}, 1, 0, -9\right)^\top, \\
v^{(5)} &= \frac{1}{9}\left(\sqrt{73}, -8, -\sqrt{73}, -1, 0, 9\right)^\top,
\end{aligned}$$

A graphical representation of the corresponding eigenfunction f_k, for $k \in \{0, 1, 2, 3, 4, 5\}$, is given in Figure 3.3.

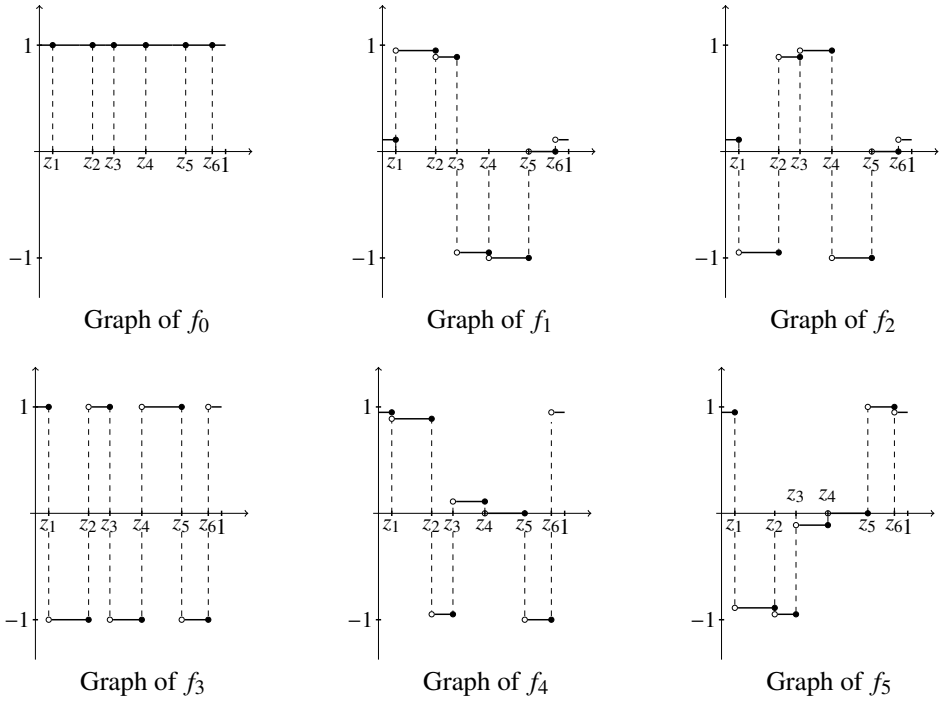


Figure 3.3. Graphs of the eigenfunctions f_k of $\Delta_{\delta_{(3)}}$, for $k \in \{0,1,2,3,4,5\}$, where $m_1 = 1/4$ and $m_2 = 1/12$ and hence, $\delta_{(3)} = 4^{-1}\delta_{\{z_1,z_3,z_5\}} + 12^{-1}\delta_{\{z_2,z_4,z_6\}}$.

The ellipse coming from the pairs of eigenvalues with multiplicity two is for $m_1 = 1/4$ and $m_2 = 1/12$ of the form

$$\mathfrak{E} := \left\{(x,y) \in \mathbb{R}^2 : \ \sqrt{73}\left(x^2 + y^2 - 1\right) = -7xy\right\}.$$

A graphical representation is shown in Figure 3.4 together with the point plots of $S_{1,5}$ and $S_{2,4}$.

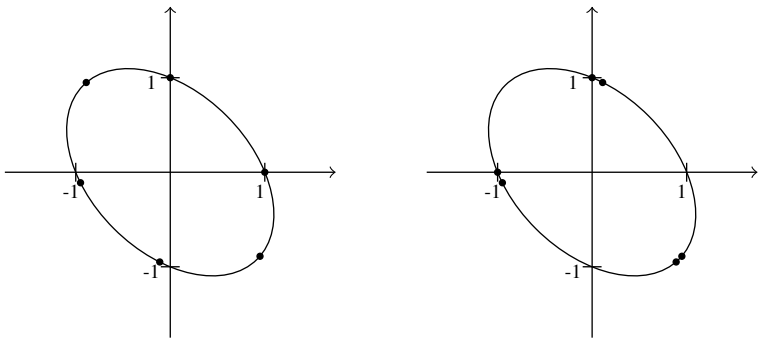


Figure 3.4. Point plot of $S_{1,5}$ (left) and point plot of $S_{2,4}$ (right) together with the ellipse $\mathfrak{E}$ given by $\sqrt{73}(x^2 + y^2 - 1) = -7xy$.

In the uniformly distributed case, namely in Theorem 3.3.2 and Corollary 3.3.3, the analogous set of tuples lies on the unit circle. This is reflected in the ansatz used to determine the

eigenvectors and eigenvalues there. Here we see that the case of a uniform discrete probability distribution is a special case for which the eigenfunctions are a closely connected to the atomless case. In general, the eigenfunctions are, $\kappa \in \mathbb{R}$, no longer of the form

$$w^{(\kappa)} := \left(\sin\left(\pi\kappa F^{lc}_{\delta_{(3)}}(z_1)\right), \ldots, \sin\left(\pi\kappa F^{lc}_{\delta_{(3)}}(z_N)\right)\right)^\top \text{ or}$$

$$u^{(\kappa)} := \left(\cos\left(\pi\kappa F^{lc}_{\delta_{(3)}}(z_1)\right), \ldots, \cos\left(\pi\kappa F^{lc}_{\delta_{(3)}}(z_N)\right)\right)^\top$$

In the example discussed here, where $N = 6$ and $m_1 = 1/4$ and $m_2 = 1/12$, a direct calculation shows that, for all $\lambda \in \mathbb{R} \setminus \{0\}$, the equations $B_{\delta_{(3)}} w^{(\kappa)} = \lambda w^{(\kappa)}$ or $B_{\delta_{(3)}} u^{(\kappa)} = \lambda u^{(\kappa)}$ only hold when κ is such that $w^{(\kappa)} = \underline{0}$ or $u^{(\kappa)} = \underline{0}$, respectively. An analogous result also holds true when replacing $F^{lc}_{\delta_{(3)}}$ by $F_{\delta_{(3)}}$.

In the more general case that $N \in \mathbb{N}$ with $N \geq 8$ we consider, for $m_1, m_2 > 0$ such that $N(m_1 + m_2) = 2$, the probability measure $\delta_{(3)} = m_1\delta_{\{z_1, z_3 \ldots, z_{N-1}\}} + m_2\delta_{\{z_2, z_4, \ldots, z_N\}}$. It is clear that $\lambda_0 = 0$ and $\lambda_{N/2} = -2(m_1^{-2} + m_2^{-2})$ are eigenvalues of $\Delta_{\delta_{(3)}}$ with corresponding eigenvectors

$$v^{(0)} = (1, \quad 1, \ldots, 1, \quad 1)^\top,$$
$$v^{(N/2)} = (1, -1, \ldots, 1, -1)^\top$$

and hence, both bounds given in Proposition 3.2.2 are sharp.

Based on numerical experiments, we conjecture that for the other eigenvalues and their corresponding eigenvectors a result on the ellipses similar to the case $N = 6$ holds true.

Conjecture 3.3.5. *Let $\delta_{(3)} = m_1\delta_{\{z_1, z_3 \ldots, z_{N-1}\}} + m_2\delta_{\{z_2, z_4, \ldots, z_N\}}$, where $N \in \mathbb{N}$ is even with $N \geq 6$ and $m_1, m_2 > 0$ are such that $N(m_1 + m_2) = 2$. Apart from $\lambda_0 = 0$ and $\lambda_{N/2} = -2(m_1^{-2} + m_2^{-2})$ all eigenvalues of $\Delta_{\delta_{(3)}}$ have multiplicity two. Letting $v, w \in \mathbb{R}^N$ be two linearly independent eigenvectors to the same eigenvalue λ_l, the points in the sets $\{(v_i, w_i) \colon i \in \{1, \ldots, N\}\}$ always lie on an ellipse.*

We want to emphasise that the ellipses for different pairs of eigenvalues are not expected to coincide in general.

CHAPTER 4

Mixed Measures

The third class of measures with respect to which we define measure-geometric differential operators are mixtures of the previous two cases. More precisely, we look at measures $\eta = \nu + \delta$ supported on a subset of the interval $M := (a, b]$, where ν is as in Chapter 2 and δ as in Chapter 3. This means we require the following conditions on ν and δ:

- $\operatorname{supp}(\nu) \subseteq \overline{M}$ with $0 < P_\nu := \nu(M) < \infty$,
- $\nu(\{z\}) = 0$ for all $z \in M$ and
- $\delta = \sum_{i=1}^{N} \alpha_i \delta_{z_i}$, where $N \in \mathbb{N}$, $a < z_1 < \ldots < z_N \leq b$ and $0 < \alpha_i < \infty$ for $i \in \{1, \ldots, N\}$.

As in the case of purely atomic measures, we call the z_i the *position of the i-th atom* and α_i its *weight*. The measure η consists of a continuous part ν and a purely atomic part δ and the assumptions yield that both parts are non-trivial. In the following, we refer to this type of measure as a mixed measure.
The distribution function F_η is a right-continuous and monotonically increasing map from M to $[0, P_\nu + \sum_{i=1}^{N} \alpha_i]$. It is also piecewise continuous with discontinuities only occurring at the position of the atoms. We have that $F_\eta(x) = F_\eta^{lc}(x)$ if and only if $z \notin \{z_1, \ldots, z_N\}$.

As in the case of purely atomic measures, we will again assume periodic boundary conditions for the η-differentiable functions in order to obtain well-defined operators. The reason for this assumption is the same as in the previous chapter and we refer to the discussions there for more details. Unlike in the case of purely atomic measures, the function space L^2_η is infinite-dimensional.

We introduce the η-derivative ∇_η and, again using a Dirichlet form, the η-Laplacian Δ_η. We show that Δ_η has many properties in common with the classical weak Laplacian, such as being unbounded, self-adjoint and non-positive. We also prove that the operator has compact resolvent, from which we deduce that the eigenfunctions of the η-Laplacian form a basis of L^2_η. We prove the general form of the eigenvalues and eigenfunctions and give a systematic way to obtain the needed parameters. Some of the results are published by Keßeböhmer, Samuel and Weyer in [KSW18b].
We extend the results shown there, by proving that Δ_η is unbounded and discussing the η-harmonic functions. Further, we strengthen the results on the asymptotic behaviour of

the eigenvalues in the examples considered in Sections 4.3.1 and 4.3.2. The discussion in Section 4.3.3 on general properties of the eigenvalues and eigenfunctions, also extends the results of [KSW18b].

This chapter is structured as follows. In Section 4.1 we define ∇_η and Δ_η and prove analytic properties of these operators. We discuss general spectral properties of Δ_η in Section 4.2. Here we give a system of equations which allows one to obtain the eigenvalues and find a general form of the eigenfunctions. In Section 4.3 we then solve this system of equations for two classes of examples and we indicate the problems arising in the general case.

4.1 Derivative and Laplacian with respect to mixed measures

Let η be a measure of the form described above. The space L^2_η is equipped with the inner product given by

$$\langle f, g\rangle = \int_a^b fg \,\mathrm{d}\eta = \int_a^b fg \,\mathrm{d}\nu + \sum_{i=1}^N \alpha_i f(z_i) g(z_i).$$

We again use the notion of the periodic extension of a function $f\colon M \to \mathbb{R}$. This new function is denoted as $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}$ and defined by $\mathbf{f}(x) = f(x)$ for all $x \in M$ and $\mathbf{f}(x) = \mathbf{f}(x + (b-a)k)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$.

A definition similar to the one in the case of purely atomic measures is used for what it means for a function to be differentiable with respect to η. In particular, this means that we again assume that the functions fulfil periodicity conditions.

Definition 4.1.1. The set of *η-differentiable functions on M* is defined as

$$\begin{aligned}\mathscr{D}^1_\eta = \mathscr{D}^1_\eta(M) \coloneqq \Big\{ f \in \Omega^2_\eta \colon\ &\text{there exists } f' \in L^2_\eta \text{ with} \\ &\mathbf{f}(x) = \mathbf{f}(y) + \int \mathbb{1}_{[y,x)} \mathbf{f}' \,\mathrm{d}\eta * \delta_{(b-a)\mathbb{Z}} \text{ for all } x, y \in \mathbb{R} \text{ with } y < x \Big\}.\end{aligned} \tag{4.1}$$

Here $\eta * \delta_{(b-a)\mathbb{Z}}$ denotes the convolution of the measure η and the Dirac comb $\delta_{(b-a)\mathbb{Z}}$; defined in Appendix A. Further, we assume, without loss of generality, that $a = 0$ and $b = 1$ and hence, that $M = (0, 1]$. In the general case, analogous results to those shown here hold true and they can be obtained by an appropriate rescaling. By the following translation argument, we may assume without loss of generality that $z_N = 1$.

Remark 4.1.2. Fix $0 < z_1 < \ldots < z_N < 1$ and $\alpha_i > 0$, for $i \in \{1, \ldots, N\}$, and define the two measures $\eta = \nu + \sum_{i=1}^N \alpha_i \delta_{z_i}$ and $\tilde{\eta} = \tilde{\nu} + \sum_{i=1}^N \alpha_i \delta_{\tilde{z}_i}$, such that $\tilde{z}_i = z_i + 1 - z_N$, for all $i \in \{1, \ldots, N\}$, and $\nu * \delta_{\mathbb{Z}}(A) = \tilde{\nu} * \delta_{\mathbb{Z}}(A + 1 - z_N)$, for all Borel sets A. This means that $\tilde{\eta}$ is the translation by $1 - z_N$ of the measure η. It then holds that $\tilde{f} \in \mathscr{D}^1_{\tilde{\eta}}$ if and only if $f \in \mathscr{D}^1_\eta$, where f is defined by

$$f(x) \coloneqq \begin{cases} \tilde{f}(x - z_N + 1) & \text{if } x \in (0, z_N], \\ \tilde{f}(x - z_N) & \text{if } x \in (z_N, 1]. \end{cases}$$

Hence, from now on the assumption $z_N = 1$ is made throughout this chapter.

For ease of notation, we set $z_0 := 0 = z_N - 1$ and $z_{N+1} = 1 + z_1$ and define, similar to the purely atomic setting, the sets $A_i := (z_{i-1}, z_i]$, for $i \in \{1, \dots, N+1\}$. If we now look at the restrictions of f to a set A_i, we see that the integral equation only considers the continuous part of the measure. This is, for $f \in \mathscr{D}_\eta^1$ and $x, y \in A_i$ with $i \in \{1, \dots, N\}$ and $y < x$, that

$$f(x) = f(y) + \int \mathbb{1}_{[y,x)} f' \, \mathrm{d}\nu. \tag{4.2}$$

We emphasise that the integration is with respect to the measure ν, since there is no atom in the interval $[y, x)$. If we now have two indices $i_0, i_1 \in \{1, \dots, N\}$ with $i_0 < i_1$ and look at $y \in A_{i_0}$ and $x \in A_{i_1}$, the integral in (4.1) can be split into two parts:

$$f(x) = f(y) + \int \mathbb{1}_{[y,x)} f' \, \mathrm{d}\nu + \sum_{i=i_0}^{i_1 - 1} \alpha_i f'(z_i) \tag{4.3}$$

Analogous equations hold true for $x, y \in \mathbb{R}$, with the difference that one has to use the periodic continuation of the functions. Using these observations, we can characterise the functions lying in $\mathscr{D}_\eta^1$ and prove the uniqueness of the function f' in the defining integral equation.

Proposition 4.1.3. *(i) Every function $f \in \mathscr{D}_\eta^1$ is left-continuous with discontinuities occurring only in a subset of $\{z_1, \dots, z_N\}$.*

(ii) For $f \in \mathscr{D}_\eta^1$ the function $f' \in L_\eta^2$ given in (4.1) *is unique.*

Proof. Both properties follow from the integral representations (4.2) and (4.3) in combination with the analogous results for continuous measure and purely atomic measures, which are given in Proposition 2.1.2 and Proposition 3.1.2. □

The integral representation (4.1) together with the uniqueness of $f' \in L_\eta^2$ implies that if $f, g \in \mathscr{D}_\eta^1$ with $f \neq g$, then $\|f - g\|_\eta \neq 0$. Thus, we may view $\mathscr{D}_\eta^1$ as a collection of real-valued square-η-integrable functions, or as a collection of equivalence classes of L_η^2. This means that there exists a natural embedding $\pi \colon \mathscr{D}_\eta^1 \to L_\eta^2$ and thus we do not distinguish between $\mathscr{D}_\eta^1$ and $\pi(\mathscr{D}_\eta^1)$. Furthermore, the uniqueness of f' ensures that the following operator is well-defined.

Definition 4.1.4. For $f \in \mathscr{D}_\eta^1$ let f' be as in Definition 4.1.1. The operator

$$\begin{aligned} \nabla_\eta \colon \mathscr{D}_\eta^1 &\to L_\eta^2 \\ f &\mapsto f' \end{aligned}$$

is called the *η-derivative*.

By the linearity of the integral equation in (4.1), it follows that the η-derivative is linear.

Corollary 4.1.5. *The operator* $\nabla_\eta \colon \mathscr{D}^1_\eta \to L^2_\eta$ *is linear.*

Further, since $\mathbf{f}(x) = \mathbf{f}(x+1)$, for $x \in \mathbb{R}$, it follows that

$$\int \mathbb{1}_{(0,1]} \nabla_\eta f \, \mathrm{d}\eta = 0, \tag{4.4}$$

or equivalently that

$$\sum_{i=1}^{N} \alpha_i \nabla_\eta f(z_i) = -\int_0^1 \nabla_\eta f \, \mathrm{d}\nu.$$

This implies that the zero function is the only constant function which can occur as an η-derivative. The functions for which this happens lie, by definition, the kernel of ∇_η and it is clear from (4.1) that

$$\ker(\nabla_\eta) = \{c\mathbb{1} : c \in \mathbb{R}\}. \tag{4.5}$$

Remark 4.1.6. To simplify notation and to make the results more easily understandable, we assume that there exists $\underline{c} = (c_1, \dots, c_N) \in \{0,1\}^N$, such that $\eta = \Gamma + \sum_{i=1}^N \alpha_i \delta_{z_i}$, where $\mathrm{d}\Gamma = (\sum_{i=1}^N c_i \mathbb{1}_{A_i}) \mathrm{d}\Lambda$. Lemma 2.1.5 and the integral representation outside the atoms given in (4.2) implicate that appropriately composing the operator with the distribution function F_ν is sufficient to obtain the results of this section for measures with a different continuous part ν. Since the continuous and the atomic parts of the measure η shall not be trivial, we assume that $\underline{c} \neq \underline{0}$.

With Remark 4.1.6 at hand, we will assume throughout this section that $\eta = \Gamma + \sum_{i=1}^N \alpha_i \delta_{z_i}$, with Γ as described there.

As with the classical weak derivative, $\nabla_\eta f$ reflects local properties of f. More precisely, for $i \in \{1, \dots, N\}$, it follows from the integral equation for the continuous part given in (4.2) that

$$\nabla_\eta f(x) = f'_\Gamma(x), \tag{4.6}$$

for $x \in A_i^{\mathrm{o}}$, where $f'_\Gamma \in L^2_\Gamma$ is such that

$$f(x) = \lim_{\varepsilon \searrow 0} f(z_{i-1} + \varepsilon) + \int \mathbb{1}_{[z_{i-1}, x)} f'_\Gamma \, \mathrm{d}\Gamma.$$

If $c_i = 1$, then $f'_\Gamma|_{A_i^{\mathrm{o}}}$ coincides with the weak derivative, and if $c_i = 0$, then $f'_\Gamma|_{A_i^{\mathrm{o}}}$ can be chosen arbitrarily. This means that outside the atoms, only the continuous part of the measure plays a role and applying the η-derivative is similar to weak differentiation.
Equation (4.3) implies the behaviour of the η-derivative at the atoms, namely, for $i \in \{1, \dots, N\}$, it holds that

$$\nabla_\eta f(z_i) = \lim_{\varepsilon \searrow 0} \frac{\mathbf{f}(z_i + \varepsilon) - \mathbf{f}(z_i)}{\alpha_i}. \tag{4.7}$$

Hence, we have at an atom, that the η-derivative of a function is the weighted height of the jump this function makes there. This is similar to the purely atomic case discussed in (3.4). Understanding how the η-derivative ∇_η acts on a $\mathscr{D}^1_\eta$-function allows us to obtain first important results on the operator and its domain.

Proposition 4.1.7. *The η-derivative ∇_η is an unbounded operator on $\mathscr{D}^1_\eta$.*

Proof. Set $K \coloneqq \Gamma((0,1])$ and define, for $m \in \mathbb{N}$ and $x \in (0,1]$, the function

$$g_m(x) \coloneqq \sin\left(\frac{2\pi m}{K} F_\Gamma(x)\right).$$

It then holds that $g_m(x) \in \mathscr{D}^1_\eta$, with

$$\nabla_\eta g_m(x) = \begin{cases} \frac{2\pi m}{K} \cos\left(\frac{2\pi m}{K} F_\Gamma(x)\right) & \text{if } x \in (0,1] \setminus \{z_1, \dots, z_N\}, \\ 0 & \text{otherwise.} \end{cases}$$

A direct calculation show, for all $m \in \mathbb{N}$, that $\|g_m\|_\eta \leq K + \sum_{i=1}^N \alpha_i$ and $\|\nabla_\eta g_m\|_\eta = \sqrt{2}\pi m/K$. Hence, the results follows. □

The η-derivative is also densely defined on L^2_η as we will see in the following important proposition. In the proof we define functions lying in $\mathscr{D}^1_\eta$ which span L^2_η. These functions will also play an important role in the proof of Theorem 4.1.20 and therefore, they are chosen in a way which might not be intuitive on first sight.

Proposition 4.1.8. *The set $\mathscr{D}^1_\eta$ is dense in L^2_η with respect to $\|\cdot\|_\eta$.*

Proof. The proof can be performed in two parts. We first show that for a given $f \in L^2_\Gamma$ and $\varepsilon > 0$, there exists $g \in \mathscr{D}^1_\eta$ with $\|f - g\|_\Gamma < \varepsilon$. In the second step, for $i \in \{1, \dots, N\}$, we construct a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathscr{D}^1_\eta$ with $\lim_{n\to\infty} \|h_n - \mathbb{1}_{\{z_i\}}\|_\eta = 0$.

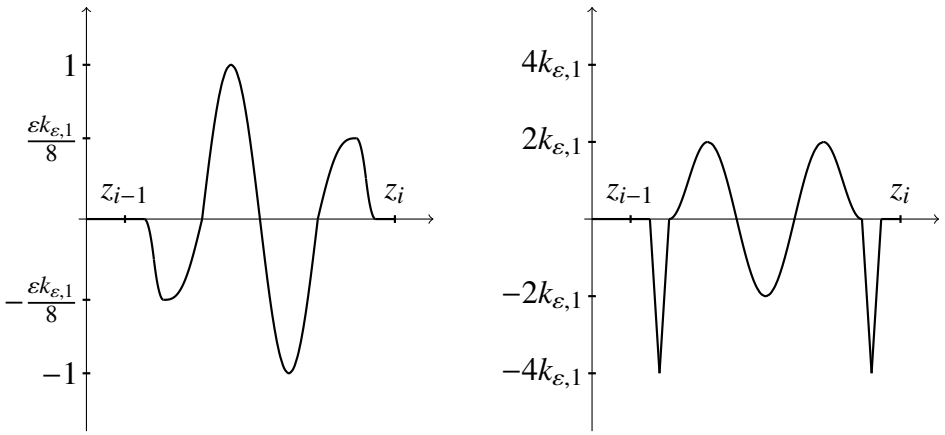
We show the first statement by restricting the problem to the intervals A_i. To this end, fix $i \in \{1, \dots, N\}$. If $c_i = 0$, the norm with respect to $\Gamma|_{A_i}$ of every function equals zero and hence, the approximation on A_i holds trivially.
For the case that $c_i = 1$, we now construct a family of $\mathscr{D}^1_\eta$-functions, namely the set $\{f_{\varepsilon,n}, g_{\varepsilon,n}\}$ with some conditions on ε and n, which allows us to approximate every L^2_Γ-function. We will have that, for at fixed ε, the functions $f_{\varepsilon,n}$ are of the form $\sin(n\cdot)$ and $g_{\varepsilon,n}$ ressembles $\cos(n\cdot)$, both rescaled to the interval $(z_{i-1} + \varepsilon, z_i - \varepsilon)$.

To this end, fix for $0 < \varepsilon < (z_i - z_{i-1})/2$ and $n \in \mathbb{N}_0$ the constant $k_{\varepsilon,n} \coloneqq \pi n/(z_i - z_{i-1} - 2\varepsilon)$.

Further, define for the same ε and n and for $x \in (0,1]$, the function $f_{\varepsilon,n}(x) := \int_0^x f'_{\varepsilon,n}\,\mathrm{d}\Lambda$, where

$$f'_{\varepsilon,n}(x) := \begin{cases} -\frac{32k_{\varepsilon,n}}{\varepsilon}\left(x-\left(z_{i-1}+\frac{\varepsilon}{4}\right)\right) & \text{if } x \in \left(z_{i-1}+\frac{\varepsilon}{4}, z_{i-1}+\frac{3\varepsilon}{8}\right], \\ \frac{32k_{\varepsilon,n}}{\varepsilon}\left(x-\left(z_{i-1}+\frac{\varepsilon}{2}\right)\right) & \text{if } x \in \left(z_{i-1}+\frac{3\varepsilon}{8}, z_{i-1}+\frac{\varepsilon}{2}\right], \\ -k_{\varepsilon,n}\cos\left(\frac{2\pi}{\varepsilon}\left(x-\left(z_{i-1}+\frac{\varepsilon}{2}\right)\right)\right)+k_{\varepsilon,n} & \text{if } x \in \left(z_{i-1}+\frac{\varepsilon}{2}, z_{i-1}+\varepsilon\right], \\ 2k_{\varepsilon,n}\cos\left(2k_{\varepsilon,n}(x-(z_{i-1}+\varepsilon))\right) & \text{if } x \in (z_{i-1}+\varepsilon, z_i-\varepsilon], \\ k_{\varepsilon,n}\cos\left(\frac{2\pi}{\varepsilon}\,(x-(z_i-\varepsilon))\right)+k_{\varepsilon,n} & \text{if } x \in \left(z_i-\varepsilon, z_i-\frac{\varepsilon}{2}\right], \\ -\frac{32k_{\varepsilon,n}}{\varepsilon}\left(x-\left(z_i-\frac{\varepsilon}{2}\right)\right) & \text{if } x \in \left(z_i-\frac{\varepsilon}{2}, z_i-\frac{3\varepsilon}{8}\right], \\ \frac{32k_{\varepsilon,n}}{\varepsilon}\left(x-\left(z_i-\frac{\varepsilon}{4}\right)\right) & \text{if } x \in \left(z_i-\frac{3\varepsilon}{8}, z_i-\frac{\varepsilon}{4}\right], \\ 0 & \text{otherwise.} \end{cases}$$

It holds by definition that $f_{\varepsilon,n}(x) = 0$, for $x \in (0,1] \setminus A_i^o$, and $f_{\varepsilon,n}(x) = \sin\left(2k_{\varepsilon,n}(x-(z_i+\varepsilon))\right)$, for $x \in (z_{i-1}+\varepsilon, z_i-\varepsilon)$. To clarify the definition of $f_{\varepsilon,n}$ and $f'_{\varepsilon,n}$, a graphical representation of both functions, dependant on ε, is given in Figure 4.1, for the case $n = 1$.
The functions $f_{\varepsilon,n}$ are weakly differentiable and since $\Gamma|_{A_i} = \Lambda|_{A_i}$, we have $f_{\varepsilon,n} \in \mathscr{D}^1_{\Gamma|_{A_i}}$. Since the support of $f_{\varepsilon,n}$ is bounded away from the atoms, it holds that $f_{\varepsilon,n} \in \mathscr{D}^1_\eta$ with $\nabla_\eta f_{\varepsilon,n} = f'_{\varepsilon,n}$.



Graph of $f_{\varepsilon,1}$ Graph of $\nabla_\eta f_{\varepsilon,1}$

Figure 4.1. Graphs of the functions $f_{\varepsilon,n}$ and $f'_{\varepsilon,n}$ for $n = 1$.

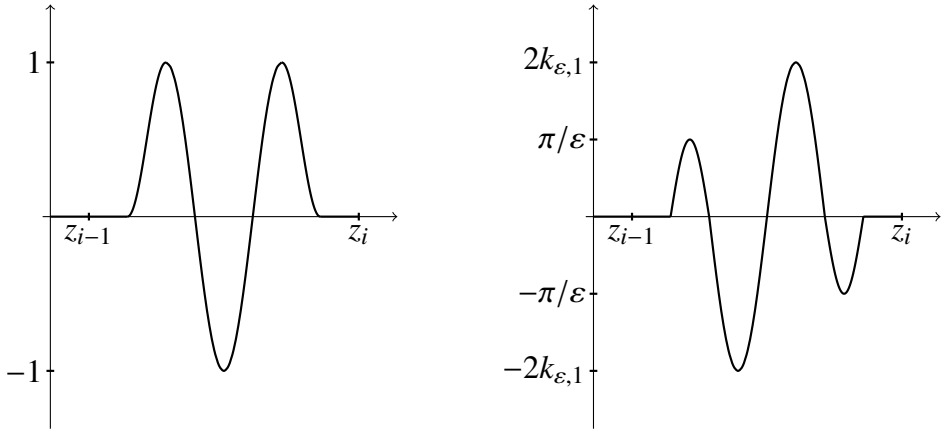
For the same parameters $0 < \varepsilon < (z_i - z_{i-1})/2$ and $n \in \mathbb{N}_0$, we set

$$g_{\varepsilon,n}(x) := \begin{cases} \frac{1}{2}\left(1-\cos\left(\frac{2\pi}{\varepsilon}\left(x-\left(z_{i-1}+\frac{\varepsilon}{2}\right)\right)\right)\right) & \text{if } x \in \left(z_{i-1}+\frac{\varepsilon}{2}, z_{i-1}+\varepsilon\right], \\ \cos\left(2k_{\varepsilon,n}(x-(z_{i-1}+\varepsilon))\right) & \text{if } x \in (z_{i-1}+\varepsilon, z_i-\varepsilon], \\ \frac{1}{2}\left(1+\cos\left(\frac{2\pi}{\varepsilon}\,(x-(z_i-\varepsilon))\right)\right) & \text{if } x \in \left(z_i-\varepsilon, z_i-\frac{\varepsilon}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

We obtain that $g_{\varepsilon,n}(x) = 0$, for $x \in (0,1] \setminus A_i^o$, and that $g_{\varepsilon,n}(x) = \cos(2k_{\varepsilon,n}(x-(z_i+\varepsilon)))$, for $x \in (z_{i-1}+\varepsilon, z_i-\varepsilon)$. The functions $g_{\varepsilon,n}$ are again weakly differentiable and, by the same argument given for the functions $f_{\varepsilon,n}$, we have that $g_{\varepsilon,n} \in \mathscr{D}_\eta^1$ with η-derivative

$$\nabla_\eta g_{\varepsilon,n}(x) = \begin{cases} \frac{\pi}{\varepsilon}\sin\left(\frac{2\pi}{\varepsilon}\left(x-\left(z_{i-1}+\frac{\varepsilon}{2}\right)\right)\right) & \text{if } x \in \left(z_{i-1}+\frac{\varepsilon}{2}, z_{i-1}+\varepsilon\right], \\ -2k_{\varepsilon,n}\sin\left(2k_{\varepsilon,n}(x-(z_{i-1}+\varepsilon))\right) & \text{if } x \in (z_{i-1}+\varepsilon, z_i-\varepsilon], \\ -\frac{\pi}{\varepsilon}\sin\left(\frac{2\pi}{\varepsilon}(x-(z_i-\varepsilon))\right) & \text{if } x \in \left(z_i-\varepsilon, z_i-\frac{\varepsilon}{2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

For a better understanding of the definition of $g_{\varepsilon,n}$ and its η-derivative $\nabla_\eta g_{\varepsilon,n}$, a graphical representation of both functions, dependant on ε, is given in Figure 4.2, for the case $n = 1$.



Graph of $g_{\varepsilon,1}$ Graph of $\nabla_\eta g_{\varepsilon,1}$

Figure 4.2. Graph of the functions $g_{\varepsilon,n}$ and $\nabla_\eta g_{\varepsilon,n}$ for $n = 1$.

We fix the index set $\Im := \{i \in \{1,\dots,N\} \colon c_i = 1\}$, and for $i \in \Im$, we define the set

$$\mathcal{F}_i := \left\{ f_{\varepsilon,n}, g_{\varepsilon,n} \colon 0 < \varepsilon < \frac{z_i - z_{i-1}}{2} \text{ and } n \in \mathbb{N}_0 \right\}.$$

The fact that, for $i \in \Im$, the set $\{x \mapsto \sin(2\pi n x/\Lambda(A_i)) \colon n \in \mathbb{N}\} \cup \{x \mapsto \cos(2\pi n x/\Lambda(A_i)) \colon n \in \mathbb{N}_0\}$ forms a basis of $L^2_{\Lambda|_{A_i}}$ yields that the span of $\bigcup_{i\in\Im} \mathcal{F}_i \subseteq \mathscr{D}_\eta^1$ is a dense subset of L^2_Γ with respect to the norm $\|\cdot\|_\Gamma$. This completes the first part of the proof.

In order to prove the second statement given in the beginning, let $i \in \{1,\dots,N-1\}$ be fixed. The case $i = N$ follows by a similar argument and is only omitted here for ease of notation. Our aim is to approximate the function $\mathbb{1}_{\{z_i\}}$ with respect to $\|\cdot\|_\eta$ by functions $h_n \in \mathscr{D}_\eta^1$. We divide the proof into four cases, namely when c_i and c_{i+1} are zero or one. The different cases are indicated by the superscript index of the approximating functions, namely we introduce the functions $h^{0,0}$, $h_n^{0,1}$, $h_n^{1,1}$ and $h_n^{1,0}$. The first superscript is equal to the value c_i and the second to c_{i+1}. A graphical representation of these functions is given in Figure 4.3.

We start with the case of an isolated atom, that is $c_i = c_{i+1} = 0$. We define the function $h^{0,0} := \mathbb{1}_{A_i}$ which lies in $\mathscr{D}^1_\eta$ with

$$\nabla_\eta h^{0,0}(x) := \begin{cases} \frac{1}{\alpha_{i-1}} & \text{if } x \in [z_{i-1}, z_i), \\ -\frac{1}{\alpha_i} & \text{if } x \in [z_i, z_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

and for which $\|h^{0,0} - \mathbb{1}_{\{z_i\}}\|_\eta = 0$. Hence, no sequence of functions is needed to approximate $\mathbb{1}_{\{z_i\}}$ in this case.

We now consider the two cases when $c_{i+1} = 1$. We let P be the smallest natural number with $1/P < \min\{z_i - z_{i-1} : i \in \{1, \dots, N\}\}$ and define, for $n \in \mathbb{N}$ with $n \geq P$, the functions

$$h^{0,1}_n(x) := \begin{cases} 1 & \text{if } x \in (z_{i-1}, z_i], \\ \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left(z_i, z_i + \frac{1}{n}\right), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h^{1,1}_n(x) := \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left(z_i - \frac{1}{n}, z_i + \frac{1}{n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\lim_{n\to\infty} \|h^{0,1}_n - \mathbb{1}_{\{z_i\}}\|_\eta = \|h^{1,1}_n - \mathbb{1}_{\{z_i\}}\|_\eta = 0$. Moreover, $h^{0,1}_n, h^{1,1}_n \in \mathscr{D}^1_\eta$ with

$$\nabla_\eta h^{0,1}_n(x) := \begin{cases} \frac{1}{\alpha_i} & \text{if } x \in [z_{i-1}, z_i), \\ \frac{n\pi}{2}\sin\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left[z_i, z_i + \frac{1}{n}\right), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nabla_\eta h^{1,1}_n(x) := \begin{cases} \frac{n\pi}{2}\sin\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left(z_i - \frac{1}{n}, z_i + \frac{1}{n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we consider the case $c_i = 1$ and $c_{i+1} = 0$. For this let $j \in \{i+2, \dots, N\}$ be the smallest integer with $c_j = 1$. If no such j exists, then let $j \in \{1, \dots, i-1\}$ be the smallest integer with $c_j = 1$. Observe that in the first case it is sufficient to approximate $\mathbb{1}_{\{z_i, \dots, z_j\}}$ and, in the second case, $\mathbb{1}_{\{z_1, \dots, z_j, z_i, \dots, z_N\}}$, since the previous three cases then allow us to approximate $\mathbb{1}_{z_i}$ for all other choices of c_i and c_{i+1}. Here we prove the former of these two cases as the latter follows

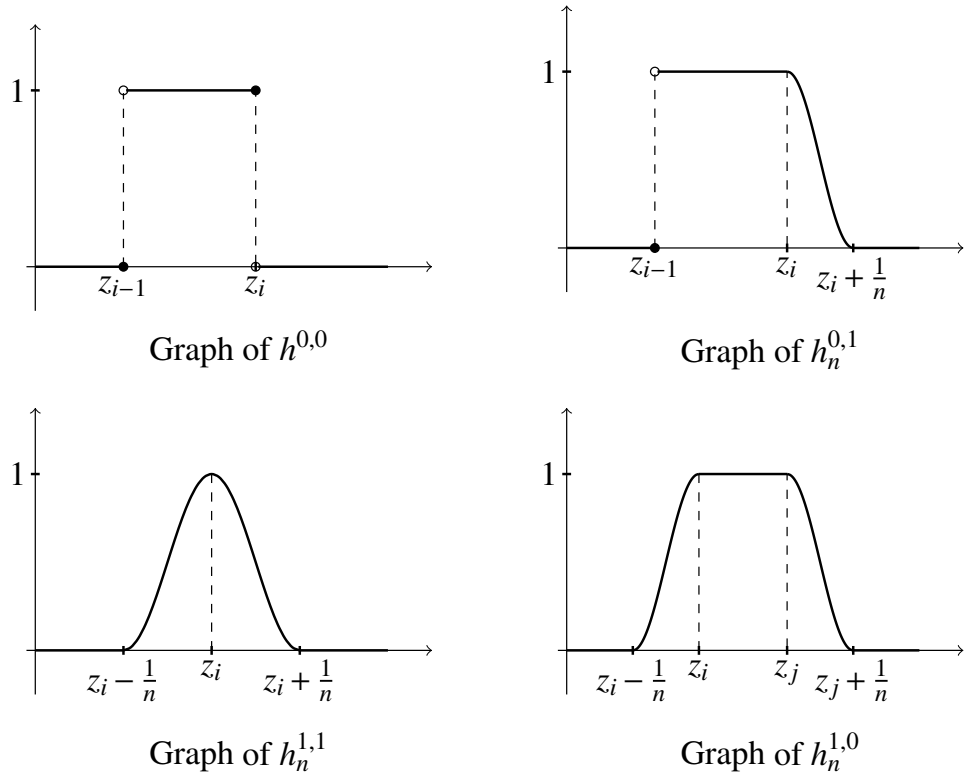


Figure 4.3. Graphs of the functions $h^{0,0}$, $h_n^{0,1}$, $h_n^{1,1}$ and $h_n^{1,0}$.

analogously, using the periodic boundary conditions. For $n \in \mathbb{N}$ with $n \geq P$, we set

$$h_n^{1,0}(x) := \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left(z_i - \frac{1}{n}, z_i\right], \\ 1 & \text{if } x \in (z_i, z_j], \\ \frac{1}{2} - \frac{1}{2}\cos\left(n\pi\left(x - \left(z_j + \frac{1}{n}\right)\right)\right) & \text{if } x \in \left(z_j, z_j + \frac{1}{n}\right], \\ 0 & \text{otherwise.} \end{cases}$$

By definition, we have $h_n^{1,0} \in \mathscr{D}_\eta^1$ with

$$\nabla_\eta h_n^{1,0}(x) := \begin{cases} \frac{n\pi}{2}\sin\left(n\pi\left(x - \left(z_i - \frac{1}{n}\right)\right)\right) & \text{if } x \in \left[z_i - \frac{1}{n}, z_i\right), \\ 0 & \text{if } x \in [z_i, z_j), \\ \frac{n\pi}{2}\sin\left(n\pi\left(x - \left(z_j + \frac{1}{n}\right)\right)\right) & \text{if } x \in \left[z_j, z_j + \frac{1}{n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

and $\lim_{n\to\infty} \|h_n^{1,0} - \mathbb{1}_{\{z_i,\dots,z_j\}}\|_\eta = 0$, which concludes the proof. □

From this proposition follows that the η-derivative is densely defined.

We now study the range of the η-derivative. To be more precise, we will show that the image of the range of ∇_η under ϱ is dense in $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$, where ϱ denotes the (natural) quotient map from L^2_η to $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$. In the purely atomic case, we showed in Proposition 3.1.6 that $\varrho(\mathrm{ran}(\nabla_\delta)) = L^2_\delta/\{c\mathbb{1} : c \in \mathbb{R}\}$. Now, we do not have equality, but still denseness of $\varrho(\mathrm{ran}(\nabla_\eta))$ in $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$. By the continuity of the inner product, this implies that the orthogonal complement of the range of ∇_η is equal to the set of constant functions.

Lemma 4.1.9. *The image under ϱ of the range of ∇_η is dense in the space $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$.*

Proof. The set of bounded continuous functions $C_B = C_B((0,1])$ is dense in L^2_η, and so the image of $E := \{g \in C_B : \langle g, \mathbb{1}\rangle_\eta = 0\}$ under ϱ is dense in $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$. For $g \in E$, setting $f(x) = \langle g, \mathbb{1}_{[0,x)}\rangle_\eta$ for $x \in (0,1]$, observe that f is left-continuous and bounded, and so $f \in L^2_\eta$. Since $g \in E$, by (4.4), we have $f \in \mathscr{D}^1_\eta$, where $\nabla_\eta f = g$. In other words, E is contained in the range of ∇_η and hence the image of the range of ∇_η under ϱ is dense in $L^2_\eta/\{c\mathbb{1} : c \in \mathbb{R}\}$. □

To prepare the definition of the measure-geometric Laplacian, we now study ∇^*_η, the adjoint of the η-derivative. The definition of the adjoint operator can be found in Appendix C. In the following lemma we show that the domain of ∇^*_η is equal to

$$\begin{aligned}\mathscr{D}^{1*}_\eta := \Big\{ f \in \mathfrak{L}^2_\eta : \text{there exists } f^* \in L^2_\eta \text{ with} \\ \mathbf{f}(x) = \mathbf{f}(y) + \int \mathbb{1}_{(x,y]}\mathbf{f}^* \,\mathrm{d}\eta * \delta_{\mathbb{Z}} \text{ for all } x, y \in \mathbb{R} \text{ with } x < y \Big\}.\end{aligned} \tag{4.8}$$

Note that the crucial differences to the definition of $\mathscr{D}^1_\eta$ is that the integral equation holds for all $x < y$ and that we integrate over the interval $(x, y]$.
Following similar arguments to the ones given in the discussion about $\mathscr{D}^1_\eta$, one can obtain that if $f \in \mathscr{D}^{1*}_\eta$, then f is right-continuous with discontinuities occurring only at points in a subset of $\{z_1, \dots, z_N\}$. Moreover, if $f, g \in \mathscr{D}^{1*}_\eta$ with $f \neq g$, then $\|f - g\|_\eta \neq 0$. Thus, as with $\mathscr{D}^1_\eta$, we may view $\mathscr{D}^{1*}_\eta$ as a collection of real-valued square-η-integrable functions, or as a collection of equivalence classes of L^2_η. As with $\mathscr{D}^1_\eta$, this implies an embedding $\pi : \mathscr{D}^{1*}_\eta \to L^2_\eta$ and in the following we will not distinguish between $\mathscr{D}^{1*}_\eta$ and $\pi(\mathscr{D}^{1*}_\eta)$.

Lemma 4.1.10. *The domain of ∇^*_η is equal to $\mathscr{D}^{1*}_\eta$.*

Proof. Let $f \in \mathrm{dom}(\nabla^*_\eta)$. Using the fact that $\langle \nabla^*_\eta f, \mathbb{1}\rangle_\eta = \langle f, \nabla_\eta \mathbb{1}\rangle_\eta = \langle f, 0\rangle_\eta = 0$ and Fubini's theorem, we obtain, for $g \in \mathscr{D}^1_\eta$,

$$\begin{aligned}\int \mathbb{1}_{(0,1]} f \nabla_\eta g \,\mathrm{d}\eta &= \int \mathbb{1}_{(0,1]} \nabla^*_\eta f g \,\mathrm{d}\eta \\ &= \int \mathbb{1}_{(0,1]}(x)(\nabla^*_\eta f)(x)\left(\mathbf{g}(0) + \int \mathbb{1}_{[0,x)}(y)(\nabla_\eta g)(y)\,\mathrm{d}\eta(y)\right)\mathrm{d}\eta(x) \\ &= \int \mathbb{1}_{[0,1)}(y)\nabla_\eta g(y) \int \mathbb{1}_{(y,1]}(x)(\nabla^*_\eta f)(x)\,\mathrm{d}\eta(x)\,\mathrm{d}\eta(y).\end{aligned}$$

Combining this with Lemma 4.1.9 and the fact that $\langle \mathbb{1}, \nabla_\eta g \rangle_\eta = 0$, which was obtained in (4.4), implies $\varrho(f - \int \mathbb{1}_{(\cdot,1]} \nabla_\eta^* f \, \mathrm{d}\eta) = 0$. Hence, there exists a constant c such that, for η-almost all $y \in (0,1]$, it holds that

$$f(y) - \int \mathbb{1}_{(y,1]} \nabla_\eta^* f \, \mathrm{d}\eta = c.$$

This yields that f is a right-continuous function and from $\langle \nabla_\eta^* f, \mathbb{1} \rangle_\eta = 0$ it follows that $c = \mathbf{f}(0)$. When setting $f^* \coloneqq \nabla_\eta^* f$, since $\mathbf{f}(0) = f(1)$ and again $\langle \nabla_\eta^* f, \mathbb{1} \rangle_\eta = 0$, we have that $\mathbf{f}^*$ fulfils the integral equation in (4.8) and therefore, $\mathrm{dom}(\nabla_\eta^*) \subseteq \mathscr{D}_\eta^{1*}$.

For the reverse inclusion, let $g \in \mathscr{D}_\eta^1$, $f \in \mathscr{D}_\eta^{1*}$ and f^* be as in (4.8). By Fubini's theorem and the fact that $\langle f^*, \mathbb{1} \rangle_\eta = \langle \nabla_\eta g, \mathbb{1} \rangle_\eta = 0$, we have the following chain of equalities, which yields the result.

$$\begin{aligned} \int \nabla_\eta g \; f \, \mathrm{d}\eta &= \int \nabla_\eta g(y) \left(f(1) + \int \mathbb{1}_{(y,1]}(x) f^*(x) \, \mathrm{d}\eta(x) \right) \mathrm{d}\eta(y) \\ &= \int f^*(x) \int \nabla_\eta g(y) \mathbb{1}_{[0,x)}(y) \, \mathrm{d}\eta(y) \mathrm{d}\eta(x) = \int f^* g \, \mathrm{d}\eta \end{aligned}$$ □

Corollary 4.1.11. *For $f \in \mathrm{dom}(\nabla_\eta^*)$, we have that $\nabla_\eta^* f = f^*$, where f^* is defined as in* (4.8)*.*

Similar to the η-derivative, we have that the function $\nabla_\eta^* f$, when it exists, reflects local properties of f. Indeed, we have

$$\nabla_\eta^* f(z_i) = \lim_{\varepsilon \searrow 0} \frac{\mathbf{f}(z_i) - \mathbf{f}(z_i - \varepsilon)}{\alpha_i} \quad \text{and} \quad \nabla_\eta^* f(x) = f_\Gamma^*(x), \tag{4.9}$$

for $i \in \{1, \dots, N\}$ and $x \in A_i^\circ$, where $f_\Gamma^* \in L_\Lambda^2$ is equal to the negative of the weak derivative of f on A_i°, if $c_i = 1$, and otherwise can be chosen arbitrarily on A_i°. By a proof similar to that of Proposition 4.1.8, we can concluded that $\mathscr{D}_\eta^{1*}$ is dense in L_η^2 with respect to $\|\cdot\|_\eta$. This denseness is now used to show that the η-derivative is a closed operator.

Proposition 4.1.12. *The operator ∇_η is closed.*

Proof. The general result given in Theorem C.2.1 implies that ∇_η^{**} is closed and hence, it suffices to show that $\mathrm{dom}(\nabla_\eta) = \mathrm{dom}(\nabla_\eta^{**})$. This follows by an argument analogous to the one given in the proof of Lemma 4.1.10, using the denseness of $\mathscr{D}_\eta^{1*}$ in L_η^2. □

Analogous to the purely atomic case, having these properties of ∇_η, its adjoint ∇_η^* and the set $\mathscr{D}_\eta^1$ at hand, the last part needed to define the η-Laplacian Δ_η is a bilinear form $\mathcal{E}$. We first give the definition of $\mathcal{E}$ and in Lemma 4.1.14 we prove that $\mathcal{E}$ is a Dirichlet form.

Definition 4.1.13. For $f, g \in \mathscr{D}_\eta^1$ we define

$$\mathcal{E}(f,g) = \mathcal{E}_\eta(f,g) \coloneqq \langle \nabla_\eta f, \nabla_\eta g \rangle$$

and refer to $\mathcal{E}$ as the *η-energy form*.

By the definition of the integral and the observations regarding the η-derivative in (4.6) and (4.7), it holds, for $f, g \in \mathscr{D}_\eta^1$, that

$$\begin{aligned}\mathcal{E}(f,g) &= \int_0^1 \nabla_\eta f \nabla_\eta g \, \mathrm{d}\nu + \sum_{i=1}^N \alpha_i \nabla_\eta f(z_i) \nabla_\eta g(z_i) \\ &= \int_0^1 \nabla_\eta f \nabla_\eta g \, \mathrm{d}\nu + \sum_{i=1}^N \alpha_i^{-1} \Big(\lim_{\varepsilon \searrow 0} \mathbf{f}(z_i+\varepsilon) - \mathbf{f}(z_i) \Big) \cdot \Big(\lim_{\varepsilon \searrow 0} \mathbf{g}(z_i+\varepsilon) - \mathbf{g}(z_i) \Big).\end{aligned} \tag{4.10}$$

Lemma 4.1.14. *The η-energy form $\mathcal{E}$ is a Dirichlet form on $\mathscr{D}_\eta^1$.*

Proof. Using the properties of the inner product $\langle \cdot, \cdot \rangle$ and the operator ∇_η, it follows that $\mathcal{E}$ is bilinear, symmetric and that $\mathcal{E}(u,u) \geq 0$, for all $u \in \mathscr{D}_\eta^1$. Moreover, this yields that $\mathscr{D}_\eta^1$ equipped with $\langle \cdot, \cdot \rangle_{\mathcal{E}} \coloneqq \langle \cdot, \cdot \rangle + \mathcal{E}(\cdot, \cdot)$ is an inner product space. All that remains to show is the completeness of $\mathscr{D}_\eta^1$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and the Markov property.

If $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathscr{D}_\eta^1, \langle \cdot, \cdot \rangle_{\mathcal{E}})$, then both $(f_n)_{n \in \mathbb{N}}$ and $(\nabla_\eta f_n)_{n \in \mathbb{N}}$ are Cauchy-sequences in L_η^2. Hence, they converge in L_η^2, namely there exist $\widetilde{f_0}, \widetilde{f_1} \in L_\eta^2$ such that $\lim_{n \to \infty} \| f_n - \widetilde{f_0} \|_\eta = 0$ and $\lim_{n \to \infty} \| \nabla_\eta f_n - \widetilde{f_1} \|_\eta = 0$. In Proposition 4.1.12 we proved that ∇_η is closed, which implies that $\nabla_\eta \widetilde{f_0} = \widetilde{f_1}$. Thus, (f_n) converges to $\widetilde{f_0} \in \mathscr{D}_\eta^1$ with respect to the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$.

For the Markov property it suffices to show, for $u \in \mathscr{D}_\eta^1$, that $\hat{u} \in \mathscr{D}_\eta^1$ and that $\mathcal{E}(\hat{u}, \hat{u}) \leq \mathcal{E}(u,u)$, where $\hat{u} \coloneqq \min(\max(u, 0), 1)$. Therefore, define, for $x \in (0,1] \setminus \{z_1, \dots, z_N\}$,

$$\hat{u}'(x) \coloneqq \begin{cases} \nabla_\eta u(x) & \text{if } u(x) \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

and, for $i \in \{1, \dots, N\}$, set

$$\hat{u}'(z_i) \coloneqq \lim_{\varepsilon \searrow 0} \frac{\hat{\mathbf{u}}(z_i + \varepsilon) - \hat{\mathbf{u}}(z_i)}{\alpha_i}.$$

By substitution if follows, for $x, y \in \mathbb{R}$ with $y < x$, that this function fulfils

$$\hat{\mathbf{u}}(x) = \hat{\mathbf{u}}(y) + \int \mathbb{1}_{[y,x)} \hat{\mathbf{u}}' \, \mathrm{d}\eta * \delta_{\mathbb{Z}}.$$

Hence, $\hat{u} \in \mathscr{D}_\eta^1$ with $\nabla_\eta \hat{\mathbf{u}} = \hat{\mathbf{u}}'$. By definition, $|\nabla_\eta \hat{\mathbf{u}}(x)| \leq |\nabla_\eta \mathbf{u}(x)|$ for all $x \in (0,1] \setminus \{z_1, \dots, z_N\}$. Further, for $i \in \{1, \dots, N\}$, we have

$$\lim_{\varepsilon \searrow 0} |\hat{\mathbf{u}}(z_i + \varepsilon) - \hat{\mathbf{u}}(z_i)| \leq \lim_{\varepsilon \searrow 0} |\mathbf{u}(z_i + \varepsilon) - \mathbf{u}(z_i)|,$$

and so, by (4.7), it follows that $|\nabla_\eta \hat{\mathbf{u}}(z_i)| \leq |\nabla_\eta \mathbf{u}(z_i)|$, for $i \in \{1, \dots, N\}$. Combining all of this with (4.10) implies that $\mathcal{E}(u^+, u^+) \leq \mathcal{E}(u,u)$ and concludes the proof. □

We now have everything at hand to define the η-Laplace operator and its domain. Similar to the purely atomic case, this is done using the η-energy form.

Definition 4.1.15. The *set of twice η-differentiable functions* is defined by

$$\mathscr{D}_\eta^2 = \mathscr{D}_\eta^2((0,1]) \coloneqq \left\{ f \in \mathscr{D}_\eta^1 : \text{there exists } f'' \in L_\eta^2 \text{ such that } \mathcal{E}(f,g) = -\langle f'', g\rangle \text{ for all } g \in \mathscr{D}_\eta^1 \right\}.$$

Proposition 4.1.16. *For $f \in \mathscr{D}_\eta^2$, the function $f'' \in L_\eta^2$ in Definition 4.1.15 is unique.*

Proof. Let $f \in \mathscr{D}_\eta^2$ and assume that there exist two functions $h_1, h_2 \in L_\eta^2$ for which the equalities $\mathcal{E}(f,g) = -\langle h_1, g\rangle = -\langle h_2, g\rangle$ hold for all $g \in \mathscr{D}_\eta^1$. Following the argument in the proof of Proposition 2.1.2, we get that $h_1(x) = h_2(x)$ for Γ-almost every $x \in (0,1]$ and it remains to show that $h_1(z_i) = h_2(z_i)$, for $i \in \{1,\dots,N\}$. To this end, define for all such i the function $g_i \coloneqq \mathbb{1}_{A_i}$ and note that $g_i \in \mathscr{D}_\eta^1$. The identity $\langle h_1, g\rangle = \langle h_2, g\rangle$ implies that

$$\alpha_i h_1(z_i) + \int \mathbb{1}_{A_i} h_1 \, \mathrm{d}\Gamma = \alpha_i h_2(z_i) + \int \mathbb{1}_{A_i} h_2 \, \mathrm{d}\Gamma,$$

and since the integrals coincide, we have $h_1 = h_2$, which concludes the proof. □

The uniqueness of the functions fulfilling the integral equation in Definition 4.1.15, ensures that the η-Laplacian we introduce now is a well-defined operator.

Definition 4.1.17. Letting $f \in \mathscr{D}_\eta^2$ and f'' be as in Definition 4.1.15, the operator

$$\begin{aligned} \Delta_\eta \colon \mathscr{D}_\eta^2 &\to L_\eta^2 \\ f &\mapsto f'' \end{aligned}$$

is called the *η-Laplacian* or the *η-Laplace operator*.

By definition, we observe for $f \in \mathscr{D}_\eta^2$ and an arbitrary $g \in \mathscr{D}_\eta^1$ that

$$\mathcal{E}(f,g) = \langle \nabla_\eta f, \nabla_\eta g\rangle = -\langle \Delta_\eta f, g\rangle, \tag{4.11}$$

and thus

$$\Delta_\eta = -\nabla_\eta^* \circ \nabla_\eta. \tag{4.12}$$

Furthermore, this implies another important characterisation of the functions lying in $\mathscr{D}_\eta^2$.

Corollary 4.1.18.

$$\mathscr{D}_\eta^2 = \left\{ f \in \mathscr{D}_\eta^1 : \nabla_\eta f \in \mathscr{D}_\eta^{1*} \right\}$$

With this characterisation at hand, one can now easily obtain the *set of η-harmonic functions*

$$\mathcal{H}_\eta \coloneqq \left\{ f \in \mathscr{D}_\eta^2 : \Delta_\eta f = 0 \right\}.$$

For $f \in \mathcal{H}_\eta$, the integral equation (4.8) implies that $\nabla_\eta f$ is constant. As discussed earlier in (4.4), the periodicity of f implies that the only constant function which can occur as an η-derivative is the zero function. Hence, we have $\nabla_\eta f = 0$ and (4.5) implies that the η-harmonic functions are the constants.

Corollary 4.1.19.

$$\mathcal{H}_\eta = \{c\mathbb{1} : c \in \mathbb{R}\}.$$

In the following theorem, we show further important properties of the η-Laplacian, which are analogous to those of the classical Laplace operator.

Theorem 4.1.20. *The operator Δ_η is densely defined on L^2_η, linear, unbounded, self-adjoint and non-positive.*

Proof. That the operator is densely defined follows from the observation that the functions $f_{\varepsilon,n}, g_{\varepsilon,n}, h^{0,0}, h^{0,1}_n, h^{1,1}_n$ and $h^{1,0}_n$, as defined in Proposition 4.1.8, lie in $\mathscr{D}^2_\eta$. Which functions are defined depends on the measure η, especially on the c_i and on the parameters ε and n discussed there. For $f_{\varepsilon,n}$ and $g_{\varepsilon,n}$ this follows from (4.9), since their η-derivatives are weakly differentiable with support bounded away from the atoms. For the other functions one chooses the right-continuous representative as the η-derivative, which are also given in the proof of Proposition 4.1.8, and varifies that these lie in $\mathscr{D}^{1*}_\eta$.
Linearity follows from the linearity of ∇_η and the bilinearity of the inner product. For all choices of η, there exists an $i \in \{1, \dots, N\}$ such that, for sufficiently large $n \in \mathbb{N}$, either the function $h^{1,1}_n$ or the $h^{1,0}_n$ lies in $\mathscr{D}^2_\eta$. For these functions holds that $\|h^{1,1}_n\| \leq \eta((0,1])$ and that $\|h^{1,0}_n\| \leq \eta((0,1])$. Furthermore, a direct calculation yields $\|\Delta_\eta h^{1,1}_n\| = \|\Delta_\eta h^{1,0}_n\| = n^{3/2}\pi^2/2$, which implies that Δ_η is unbounded.
The fact that Δ_η is self-adjoint is a consequence of Theorem C.2.1 and Proposition 4.1.12. From (4.11), it follows that $\langle \Delta_\eta f, f\rangle_\eta = -\langle \nabla_\eta f, \nabla_\eta f\rangle_\eta \leq 0$ and hence, we have that the operator Δ_η is non-positive. □

We conclude this section by showing that Δ_η has compact resolvent. An introduction to the resolvent formalism for unbounded operators and an overview on compact operators can be found in Appendix C. We denote the closed unit ball in a normed space $(X, \|\cdot\|)$ by $B(X, \|\cdot\|)$ and, for $i \in \{1, \dots, N\}$, let $(W^{1,2}_i, \|\cdot\|_i)$ denote the Sobolev space $(W^{1,2}(A^{\mathrm{o}}_i), \|\cdot\|_{1,2})$.

Theorem 4.1.21. *The operator Δ_η has compact resolvent.*

Proof. Let λ denote a fixed element of the resolvent set $\rho(\Delta_\eta)$. We show that the embedding $\pi\colon \mathscr{D}^1_\eta \to L^2_\eta$ is compact, and that $(\lambda \mathrm{id} - \Delta_\eta)^{-1}\colon L^2_\eta \to \mathscr{D}^1_\eta$ is continuous. This is sufficient for the proof since $R_{\Delta_\eta}(\lambda) = \pi \circ (\lambda \mathrm{id} - \Delta_\eta)^{-1}$ and the composition of a compact operator and a continuous operator is also compact.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $B(\mathscr{D}^1_\eta, \|\cdot\|_\mathcal{E})$, where $\|\cdot\|_\mathcal{E}$ is the by $\langle\cdot,\cdot\rangle_\mathcal{E}$ induced norm. In order to show that the embedding π is compact, it sufficieses to show that $(f_n)_{n\in\mathbb{N}}$ has a convergent subsequence with respect to $\|\cdot\|_\eta$. It is known, for $i \in \{1, \dots, N\}$ with $c_i = 1$, that the unit ball

$B(W_i^{1,2}, \|\cdot\|_i)$ is compact in L^2_Λ. By (4.6) and (4.7), if $f \in B(\mathscr{D}^1_\eta, \|\cdot\|_\mathcal{E})$, then $f|_{A_i^\circ} \in B(W_i^{1,2}, \|\cdot\|_i)$. This yields the existence of a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for $i \in \{1,\dots,N\}$, the sequence $(f_{n_k}|_{A_i^\circ})_{k\in\mathbb{N}}$ converges in $W_i^{1,2}$. Combined with the Bolzano-Weierstrass theorem and the left continuity of elements in $\mathscr{D}^1_\eta$, this yields the existence of a subsequence $(n_l)_{l\in\mathbb{N}}$, such that $(f_{n_l})_{l\in\mathbb{N}}$ converges with respect to $\|\cdot\|_{\eta|_{A_i}}$ for all $i \in \{1,\dots,N\}$, and hence, with respect to $\|\cdot\|_\eta$.

To conclude the proof it is sufficient to find a constant C such that $\|(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\|_\mathcal{E} \le C\|f\|_\eta$ for all $f \in L^2_\eta$. This is done in the following sequence of inequalities, in which we use Hölder's inequality and the fact that, since $(\Delta_\eta - \lambda\,\mathrm{id})^{-1}$ is a bounded linear operator on L^2_η, there exists a $K \in \mathbb{R}$ with $\|(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\|_\eta \le K\|f\|_\eta$.

$$
\begin{aligned}
\|(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\|^2_\mathcal{E} =& \langle (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta + \langle \nabla_\eta(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, \nabla_\eta(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta \\
=& \langle (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta + \langle \Delta_\eta(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta \\
&+ \langle \lambda(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta - \langle \lambda(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta \\
\le& (1+|\lambda|)\,\|(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\|^2_\eta + \langle f, (\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\rangle_\eta \\
\le& (1+|\lambda|)K^2\,\|f\|^2_\eta + \|f\|_\eta \cdot \|(\Delta_\eta - \lambda\,\mathrm{id})^{-1} f\|_\eta \\
\le& \Big((1+|\lambda|)\,K^2 + K\Big)\|f\|^2_\eta
\end{aligned}
$$

□

That the operator Δ_η has compact resolvent together with its self-adjointness implies that the eigenfunctions of Δ_η form a basis of L^2_η and that all eigenvalues of Δ_η have finite multiplicity. Since the resolvent is compact and self-adjoint, this can be obtained from the spectral theorem as stated in Theorem C.1.1 and the fact that, for $\kappa \in \rho(\Delta_\eta)$, a function f is an eigenfunction of Δ_η with corresponding eigenvalue λ if and only if it is an eigenfunction of $R_{\Delta_\eta}(\kappa)$ with eigenvalue $1/(\kappa-\lambda)$. This result is also known as the spectral theorem for symmetric operators with compact resolvent.

We conclude this section by summarising the results on the η-Laplacian.

Theorem 4.1.22. *The operator Δ_η is densely defined on the space of square-η-integrable functions, is linear, unbounded, self-adjoint, non-positive, has compact resolvent and its eigenfunctions form a basis of L^2_η.*

Proof. This is a consequence of Remarks 4.1.2 and 4.1.6 and Theorems 4.1.20 and 4.1.21 and the spectral theorem given in Theorem C.1.1. □

4.2 Spectral properties

We now analyse the spectral properties of Δ_η. As discussed at the end of the last section, Δ_η having compact resolvent implies that its eigenfunctions form a basis of L^2_η. Also we know that all eigenvalues are non-positive and form a countable unbounded monotonic sequence, as well as that they all have finite multiplicity. As in the previous section with Remark 4.1.6, we again restrict our setting in order to simplify the notation and to make the results better readable and easier to understand.

Remark 4.2.1. We assume that $\eta = \Lambda + \sum_{i=1}^{N} \alpha_i \delta_{z_i}$, where $0 < z_1 < \ldots < z_N = 1$ and $\alpha_i > 0$ for $i \in \{1, \ldots, N\}$. From Lemma 2.1.5 and the integral representation outside the atoms given in (4.2), it follows that in order to obtain the results of this section for measures with a different continuous part ν, one needs to appropriately compose the occurring functions with the distribution function F_ν, similar to the case of an atomless measure.

We again assume without loss of generality that $z_N = 1$, which can be justified by the translation argument given in Remark 4.1.2. Note that the assumptions made for this section are different to the ones in Remark 4.1.6. Namely, we assume here that between every two atoms there is Lebesgue measure, while in the previous section, it was important to distinguish wether two consecutive atoms have mass between them or not; see for example Proposition 4.1.8 and Theorem 4.1.21. When now looking at the eigenfunctions of Δ_η, this distinction is no longer important, since the composition with the distribution function F_ν would lead to constant functions if there were no mass between the atoms. We will see that the crucial step in finding the eigenfunctions and eigenvalues is studying the behaviour at the atoms.

When not stated differently, we assume in the following that the measure η is of the form described in Remark 4.2.1. To find the system of equations which a function f has to fulfil to be an eigenfunction of Δ_η, the key properties to use are the observations we made on the function spaces $\mathscr{D}_\eta^1$ and $\mathscr{D}_\eta^{1*}$ and Corollary 4.1.18, where we showed that $\mathscr{D}_\eta^2 = \{f \in \mathscr{D}_\eta^1 : \nabla_\eta f \in \mathscr{D}_\eta^{1*}\}$. Namely, we have that $\nabla_\eta f$ is right-continuous and that $\Delta_\eta f$ is left-continuous, which gives us conditions on the parameters in the general form of the eigenfunctions.

Theorem 4.2.2. *A square-η-integrable function f is an eigenfunction of Δ_η with corresponding eigenvalue λ if and only if it is of the form*

$$f(x) = \begin{cases} a_1 \sin(bx + \gamma_1) & \text{if } x \in (0, z_1], \\ \vdots & \vdots \\ a_N \sin(bx + \gamma_N) & \text{if } x \in (z_{N-1}, 1], \end{cases} \tag{4.13}$$

and $\lambda = -b^2$, where $b, a_1, \ldots, a_N, \gamma_1, \ldots, \gamma_N \in \mathbb{R}$ satisfy the following system of equations:

$$\begin{aligned} \alpha_j b a_{j+1} \cos(bz_j + \gamma_{j+1}) &= a_{j+1} \sin(bz_j + \gamma_{j+1}) - a_j \sin(bz_j + \gamma_j), \\ \alpha_j b^2 a_j \sin(bz_j + \gamma_j) &= a_j b \cos(bz_j + \gamma_j) - a_{j+1} b \cos(bz_j + \gamma_{j+1}), \end{aligned} \tag{4.14}$$

for $j \in \{1, \ldots, N-1\}$, and

$$\begin{aligned} \alpha_N b a_1 \cos(\gamma_1) &= a_1 \sin(\gamma_1) - a_N \sin(b + \gamma_N), \\ \alpha_N b^2 a_N \sin(b + \gamma_N) &= a_N b \cos(b + \gamma_N) - a_1 b \cos(\gamma_1). \end{aligned} \tag{4.15}$$

Proof. Since the continuous part of η is the Lebesgue measure, we conclude from (4.6) and (4.9) that Δ_η behaves like the weak Laplacian outside the atoms. Combining this with (4.7), the fact that elements of $\mathrm{dom}(\nabla_\eta)$ are left-continuous, and Picard-Lindelöf's theorem, we get that if $\Delta_\eta f = \lambda f$, for some $\lambda \in \mathbb{R}$ and $f \in \mathscr{D}_\eta^2$, then there exist $b, a_1, \ldots, a_N, \gamma_1, \ldots, \gamma_N \in \mathbb{R}$ with $\lambda = -b^2$, such that f is of the form given in (4.13).

We recall that in (4.7) and (4.9) we showed that, at the atoms, the operators ∇_η and ∇_η^* act as weighted differences of the function values to the right and to the left. This implies, since $\nabla_\eta \mathbf{f}$ is right-continuous and $\mathbf{\Delta}_\eta \mathbf{f}$ is left-continuous, that f is an eigenfunction of Δ_η if and only if $b, a_1, \dots, a_N, \gamma_1, \dots, \gamma_N$ satisfy (4.14) and (4.15). This concludes the proof. □

Note, without loss of generality, we may assume for the general form of the eigenfunctions (4.13) that $\gamma_1 \in (-\pi/2, \pi/2]$. This can be ensured by adapting b and $a_1, \dots, a_N$ in an appropriate manner. From now on, when talking about (4.13), we always assume that $\gamma_1 \in (-\pi/2, \pi/2]$.

Let f be an eigenfunction of Δ_η with eigenvalue λ, and let $b, a_1, \dots, a_N$ be as in (4.13). If $b = 0$, then f is a step function of the form $f|_{A_i} = a_i \sin(\gamma_i)$. Equations (4.14) and (4.15) then imply that $a_i \sin(\gamma_i) = a_j \sin(\gamma_j)$ for all $i, j \in \{1, \dots, N\}$ and hence, that f is a constant function. By the fact that $\lambda = -b^2$ it follows that f has corresponding eigenvalue zero. In accordance with Corollary 4.1.19, this implies that the set of η-harmonic functions is one-dimensional.

In the case that $b \neq 0$, equations (4.14) yield that if $a_i = 0$ for some $i \in \{1, \dots, N-1\}$, then $a_{j+1} \cos(bz_j + \gamma_{j+1}) = a_{j+1} \sin(bz_j + \gamma_{j+1}) = 0$, and hence, $a_{j+1} = 0$. Using (4.15), an analogous argument yields that $a_N = 0$ results in $a_1 = 0$. These two observations imply that, if $a_i = 0$ for some $i \in \{1, \dots, N\}$, then $f \equiv 0$.

Corollary 4.2.3. *Every constant function is an eigenfunction with corresponding eigenvalue equal to zero. Moreover, the eigenspace* $\mathrm{Eig}(0) := \{f \in \mathscr{D}_\eta^2 \colon \Delta_\eta f = 0\}$ *is one-dimensional. Further, if* $f \notin \mathrm{Eig}(0)$ *is an eigenfunction of* Δ_η*, then* $b \neq 0$ *and* $a_i \neq 0$ *for all* $i \in \{1, \dots, N\}$*, where* $b, a_1, \dots, a_N$ *are as in* (4.13)*.*

For completeness, we state the theorem above for measures with a continuous part different to the one discussed in Remark 4.2.1. To make the distinction between the two cases clear, we write here $\eta' = \nu' + \delta$ for this general case. If $\nu' = \Lambda$, the following theorem coincides with Theorem 4.2.2.

Theorem 4.2.4. *A square-*η'*-integrable function* f *is an eigenfunction of* $\Delta_{\eta'}$ *with corresponding eigenvalue* λ *if and only if it is of the form*

$$f(x) = \begin{cases} a_1 \sin(bF_{\nu'}(x) + \gamma_1) & \text{if } x \in (0, z_1], \\ \vdots & \vdots \\ a_N \sin(bF_{\nu'}(x) + \gamma_N) & \text{if } x \in (z_{N-1}, 1], \end{cases}$$

and $\lambda = -b^2$*, where* $b, a_1, \dots, a_N, \gamma_1, \dots, \gamma_N \in \mathbb{R}$ *satisfy the following system of equations:*

$$\begin{aligned} \alpha_j b a_{j+1} \cos(bF_{\nu'}(z_j) + \gamma_{j+1}) &= a_{j+1} \sin(bF_{\nu'}(z_j) + \gamma_{j+1}) - a_j \sin(bF_{\nu'}(z_j) + \gamma_j), \\ \alpha_j b^2 a_j \sin(bF_{\nu'}(z_j) + \gamma_j) &= a_j b \cos(bF_{\nu'}(z_j) + \gamma_j) - a_{j+1} b \cos(bF_{\nu'}(z_j) + \gamma_{j+1}), \end{aligned}$$

for $j \in \{1, \dots, N-1\}$*, and*

$$\begin{aligned} \alpha_N b a_1 \cos(\gamma_1) &= a_1 \sin(\gamma_1) - a_N \sin(bF_{\nu'}(1) + \gamma_N), \\ \alpha_N b^2 a_N \sin(bF_{\nu'}(1) + \gamma_N) &= a_N b \cos(bF_{\nu'}(1) + \gamma_N) - a_1 b \cos(\gamma_1). \end{aligned}$$

Proof. This follows similarly to the proof of Theorem 4.2.2 when composing the discussed function with the distribution function $F_{\nu'}$ of the continuous part of the measure. □

Coming back to measures of the form described in Remark 4.2.1, if the atoms are equally distributed, namely $z_i = i/N$ for $i \in \{1,\dots,N\}$, two further properties of the eigenfunctions of Δ_η can be obtained. Both properties will play an important role when investigating the example in Section 4.3.2.

Proposition 4.2.5. *If f is an eigenfunction of Δ_η, then, for $r \in \{2,\dots,N\}$,*

$$f_r(x) = \begin{cases} a_{N-r+2}\sin\left(\dfrac{bx+b(N-r+1)}{N}+\gamma_{N-r+2}\right) & \text{if } x \in \left(0,\dfrac{1}{N}\right], \\ \quad\vdots & \quad\vdots \\ a_N \sin\left(\dfrac{bx+b(N-r+1)}{N}+\gamma_N\right) & \text{if } x \in \left(\dfrac{r-2}{N},\dfrac{r-1}{N}\right], \\ a_1 \sin\left(\dfrac{bx-b(r-1)}{N}+\gamma_1\right) & \text{if } x \in \left(\dfrac{r-1}{N},\dfrac{r}{N}\right], \\ \quad\vdots & \quad\vdots \\ a_{N-r+1}\sin\left(\dfrac{bx-b(r-1)}{N}+\gamma_{N-r+1}\right) & \text{if } x \in \left(\dfrac{N-1}{N},1\right], \end{cases}$$

is an eigenfunction of Δ_{η_r}, where $\eta_r := \Lambda + \sum_{i=1}^{r-1} \alpha_{i+N-r+1}\delta_{z_i} + \sum_{i=r}^{N} \alpha_{i-r+1}\delta_{z_i}$.

Proof. Since the measure η_r and the function $\mathbf{f}_r$ are shifted by the same value, the system of equations that f fulfils, namely (4.14) and (4.15), coincides with the system for f_r. □

Note that, if all α_i coincide, then $\eta_r = \eta$ for all $r \in \{2,\dots,N\}$. This means, that in the case of equally weighted and equally distributed atoms, every shift of an eigenfunction by the distance between any two atoms also leads to an eigenfunction.

For the more general class of measures, where the atoms are equally distributed and the atom weights occur in a periodic pattern, we obtain that some eigenfunctions of the η-Laplacians are periodically extended, rescaled solutions of the eigenvalue problem for the measure-geometric Laplacian with respect to a related measure.

Proposition 4.2.6. *Let $N = pk$ with $p,k \in \mathbb{N}$ and fix $\alpha_i > 0$ for $i \in \{1,\dots,p\}$. Set $\alpha_{i+p} := \alpha_i$, for $i \in \{1,\dots,N-p\}$, and define the measures*

$$\eta^{(p)} := \Lambda + \sum_{i=1}^{p} \alpha_i\, \delta_{\frac{i}{p}} \quad \text{and} \quad \eta^{(N)} := \Lambda + \sum_{i=1}^{N} \frac{\alpha_i}{k}\, \delta_{\frac{i}{N}}.$$

If $f^{(p)}$ is an eigenfunction of $\Delta_{\eta^{(p)}}$ with eigenvalue $\lambda^{(p)}$, then $f^{(N)}$ is an eigenfunction of $\Delta_{\eta^{(N)}}$ with eigenvalue $\lambda^{(n)} := k^2\lambda^{(p)}$, where $f^{(N)}(x) := \mathbf{f}^{(p)}(kx)$ for $x \in (0,1]$.

Proof. Since $f^{(p)}$ is an eigenfunction of $\Delta_{\eta^{(p)}}$, it follows that (4.14) and (4.15) are fulfilled. The fact that $f^{(N)}(x) = f^{(N)}(x + j/k)$, for all $j \in \mathbb{Z}$, together with the periodic distribution of the atoms yields that $f^{(N)}$ fulfils the analogous equations for being an eigenfunction of $\Delta_{\eta^{(N)}}$. □

This implies, that in the case of N equally weighted and equally distributed atoms, we have that rescaled solutions to the one atom case occur as eigenfunctions. More precisely, if $f^{(1)}$ is an eigenfunction of $\Delta_{\eta^{(1)}}$, where $\eta^{(1)} = \Lambda + N\alpha\delta_1$, for some $\alpha > 0$, then $f^{(N)}$ is an eigenfunction of $\Delta_{\eta^{(N)}}$, where $\eta^{(N)} = \Lambda + \sum_{i=1}^{N} \alpha\delta_{i/N}$ and $f^{(N)}(x) \coloneqq \mathbf{f}^{(1)}(Nx)$ for $x \in (0, 1]$. Similar results hold for every factor of N.

4.3 Examples for eigenvalues and eigenfunctions

We now look at three possible choices for the measure η. In Section 4.3.1, we solve the system of equations given in (4.14) and (4.15) for the case $N = 1$ and illustrate the results in an example. The second part, Section 4.3.2, deals with the case $N = 2$ under the condition that the Dirac point masses are uniformly distributed and equally weighted. Again the system of equations is solved and an example illustrates the results in this setting. We outline the problems which arise when one attempts to solve the system of equations (4.14) and (4.15) for the general case in Section 4.3.3. This is done by showing that some parameters of the eigenfunctions, which can be simplified in the two previous cases, are not easy to handle in general.

4.3.1 One atom

Here we consider the case $N = 1$, that is when $\eta = \Lambda + \alpha\delta_1$, for some $\alpha > 0$. As discussed in Remark 4.2.1, this implicates the results for a general $z \in (0, 1]$. The main results of this section are Theorem 4.3.2 and Corollary 4.3.3, in which we explicitly compute the eigenvalues and eigenfunctions of Δ_η. For the proofs of these results we require the following preliminaries. For $\beta > 0$ and $k \in \mathbb{Z}$, let $c_k^{\pm} = c_k^{\pm}(\beta)$ denote the unique solution in the interval $(-\pi/2, \pi/2)$ to the equation

$$\tan\left(c_k^{\pm}\right) = -2c_k^{\pm}\beta \pm \beta\frac{\pi}{2} + 2\pi\beta k \pm 1 \tag{4.16}$$

and set

$$\xi_k^{\pm} = \xi_k^{\pm}(\beta) \coloneqq \frac{\tan\left(c_k^{\pm}\right) \mp 1}{\beta}. \tag{4.17}$$

For every $k \in \mathbb{Z}$, equation (4.16) has a unique solution, because the tangent is strictly monotonically increasing on $(-\pi/2, \pi/2)$ with $\lim_{x\to-\pi/2} \tan(x) = -\infty$ and $\lim_{x\to\pi/2} \tan(x) = \infty$, and the right side of the equation is a strictly monotonically decreasing linear function. Substituting (4.16) in (4.17) yields the alternative representation

$$\xi_k^{\pm} = -2c_k^{\pm} \pm \frac{\pi}{2} + 2\pi k.$$

A direct calculation, using that the tangent is odd, yields that $-c^-_{-k} = c^+_k$ for $k \in \mathbb{Z}$. With this at hand it also follows that $-\xi^-_{-k} = \xi^+_k$.

Analogously, for $\beta < 0$ we denote by $C^\pm_k = C^\pm_k(\beta)$ the set of solutions to (4.16). Note that, from the arguments given for the uniqueness of the solution for positive β and the fact that the tangent has a unique inflection point in zero, it follows that the cardinality $|C^\pm_k|$ of $C^\pm_k$ is one, two or three. We denote the elements of $C^\pm_k$ by $c^\pm_{k,i} = c^\pm_{k,i}(\beta)$, where $i \in \{1,\dots,|C^\pm_k|\}$. For every $c^\pm_{k,i}$ we define the values $\xi^\pm_{k,i} = \xi^\pm_{k,i}(\beta)$ similarly to (4.17) and denote by $\Xi^\pm_k = \Xi^\pm_k(\beta)$ the set $\{\xi^\pm_{k,i} : i \in \{1,\dots,|C^\pm_k|\}\}$. The same argument as above yields, for $k \in \mathbb{Z}$ that $-C^-_{-k} = C^+_k$ and $-\Xi^-_{-k} = \Xi^+_k$.

Lemma 4.3.1. *Let $\beta \in \mathbb{R} \setminus \{0\}$. The pair $(\xi, c) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$ is a solution to the system of equations*

$$\begin{aligned} \beta\xi\cos(c) &= \sin(c) - \sin(\xi + c) \\ \beta\xi^2\sin(\xi + c) &= \xi\cos(\xi + c) - \xi\cos(c) \end{aligned} \tag{4.18}$$

if and only if

$$(\xi, c) \in \begin{cases} \left\{\left(\xi^\pm_k, c^\pm_k\right) : k \in \mathbb{Z}\right\} & \text{if } \beta > 0, \\ \left\{\left(\xi^\pm_{k,i}, c^\pm_{k,i}\right) : k \in \mathbb{Z} \text{ and } i \in \{1,\dots,|C^\pm_k|\}\right\} & \text{if } \beta < 0. \end{cases}$$

The system of equations given in (4.18) *is also solved by $(0, c)$ for all $c \in \mathbb{R}$. Further, if $c = \pi/2 + \pi k$, for some $k \in \mathbb{Z}$, then (ξ, c) is a solution to* (4.18) *if and only if $\xi = 0$.*

Proof. The backward implication of the first statement follows by substituting the given values for ξ and c directly into (4.18). We now show the forward implication. Substituting the first equation of (4.18) into the second equation, and using the identity $\cos^2(\arcsin(x)) = 1 - x^2$, for $x \in [-1, 1]$, we obtain

$$\beta\xi\sin(c) - \beta^2\xi^2\cos(c) = \sqrt{1 - (\sin(c) - \beta\xi\cos(c))^2} - \cos(c),$$

which implies that

$$(\beta\xi\sin(c) - \beta^2\xi^2\cos(c) + \cos(c))^2 = 1 - (\sin(c) - \beta\xi\cos(c))^2. \tag{4.19}$$

If $\cos(c) = 0$, this formula reduces to $\beta^2\xi^2\sin^2(c) = 0$, which is a contradiction to $\beta, \xi \neq 0$ and hence, we have $\cos(c) \neq 0$. Expanding equation (4.19) and dividing by $\cos^2(c)$ yields

$$\beta^2\xi^2 - 2\beta\xi\tan(c) + \tan^2(c) - 1 = 0.$$

Thus, we have that either $\xi = 0$ and $c = \pm\pi/4 + \pi k$, for some $k \in \mathbb{Z}$, or that $\beta\xi = \tan(c) \pm 1$. Since the first case again contradicts our assumption that $\xi \neq 0$, the second case must apply

and we continue with substituting $\beta\xi = \tan(c) \pm 1$ into the first equation of (4.18), which yields

$$(\tan(c) \pm 1)\cos(c) = \sin(c) - \sin\left(\frac{\tan(c) \pm 1}{\beta} + c\right),$$

or equivalently,

$$\cos(c) = \sin\left(\frac{\mp\tan(c) - 1}{\beta} \mp c\right).$$

Using the identity $\arcsin(\cos(x)) = \pi/2 - |x|$, for $x \in (-\pi/2, \pi/2)$, leads to the following four cases:

1. $\beta\xi = \tan(c) + 1$ and $\tan(c) = -2c\beta - \beta\dfrac{\pi}{2} + 2\pi\beta k - 1$, for $k \in \mathbb{Z}$,
2. $\beta\xi = \tan(c) + 1$ and $\tan(c) = -\beta\dfrac{\pi}{2} + 2\pi\beta k - 1$, for $k \in \mathbb{Z}$,
3. $\beta\xi = \tan(c) - 1$ and $\tan(c) = -2c\beta + \beta\dfrac{\pi}{2} + 2\pi\beta k + 1$, for $k \in \mathbb{Z}$, or
4. $\beta\xi = \tan(c) - 1$ and $\tan(c) = \beta\dfrac{\pi}{2} + 2\pi\beta k + 1$, for $k \in \mathbb{Z}$.

By substituting these values into (4.18), one sees that Cases 1 and 3 yield solutions to (4.18). Cases 2 and 4 do not yield solutions, except when $c = 0$, but this is the same solution given by Cases 1 and 3 when $c = 0$.

For $\xi = 0$, equations (4.18) are trivially fulfilled for all $c \in \mathbb{R}$. The last statement of the lemma follows by substituting the given values for ξ and c directly into (4.18). The first equation yields $\sin(\xi + \pi/2) = 1$ and hence that $\xi = 2\pi n$ for $n \in \mathbb{Z}$. Substituting this into the second equation gives us $\beta\xi^2 = 0$, which implies that $n = 0$. □

As we will see in the following, this lemma plays the key role in finding the eigenfunctions and eigenvalues of the η-Laplacian. To solve the eigenvalue problem, we start by formulating the general observations from Theorem 4.2.2 in our setting. By (4.13), if $\Delta_\eta f = \lambda f$, for some $\lambda \in \mathbb{R}$, then there exist $a, b \in \mathbb{R}$ and $\gamma \in (-\pi/2, \pi/2]$ such that $\lambda = -b^2$ and

$$f(x) = a\sin(bx + \gamma),$$

for $x \in (0, 1]$. By linearity, we may assume, without loss of generality, that $a = 1$. Further, (4.14) and (4.15) imply that f is an eigenfunction of Δ_η if and only if b and γ satisfy the following system of equations.

$$\begin{aligned} \alpha b\cos(\gamma) &= \sin(\gamma) - \sin(b + \gamma) \\ \alpha b^2\sin(b + \gamma) &= b\cos(b + \gamma) - b\cos(\gamma) \end{aligned} \tag{4.20}$$

Thus, if f is non-constant, then $\gamma \neq \pi/2$. Suppose, by way of contradiction, that $\gamma = \pi/2$. In this case (4.20) implies that $0 = 1 - \sin(b + \pi/2)$ and $\alpha b^2\sin(b + \pi/2) = b\cos(b + \pi/2)$. The

first equation yields $b = 2\pi n$ for some $n \in \mathbb{Z}$ and substituting this into the second equation yields $\alpha(2\pi n)^2 = 0$, and thus $n = 0$. Hence, $b = 0$, which implies $f = \mathbb{1}$.

For $k \in \mathbb{Z}$, let $\gamma^{(k,1)} = \gamma^{(k,1)}(\alpha)$ denote the unique solution in the interval $(-\pi/2, \pi/2)$ to the equation

$$\tan\left(\gamma^{(k,1)}\right) = -2\gamma^{(k,1)}\alpha + \alpha\frac{\pi}{2} + 2\pi\alpha k + 1,$$

and set

$$b^{(k,1)}(\alpha) := -2\gamma^{(k,1)}(\alpha) + \frac{\pi}{2} + 2\pi k.$$

The uniqueness of $\gamma^{(k,1)}$ follows by the same argument given for the uniqueness of $c_k^{\pm}$ in (4.16) in the case that $\beta > 0$. As we will shortly see, $b^{(k,1)}$ and $\gamma^{(k,1)}$ completely determine the eigenfunctions and eigenvalues of Δ_η. We have introduced the extra index 1 to indicate that they give rise to solutions to the eigenvalue problem when η has a single atom. This will become important in the following section where we consider measures with two atoms.

Notice, if $k = 0$, then $\gamma^{(k,1)} = \pi/4$ and $b^{(k,1)} = 0$; if $b^{(k,1)} = 0$, then $\gamma^{(k,1)} = \pi/4$ and $k = 0$; if $\gamma^{(k,1)} = \pi/4$, then $k = 0$ and $b^{(k,1)} = 0$.

Theorem 4.3.2. *The eigenvalues of Δ_η are $\lambda^{(k,1)} = -(b^{(k,1)}(\alpha))^2$ for $k \in \mathbb{Z}$, with corresponding eigenfunctions $f^{(k,1)}(x) := \sin(b^{(k,1)}(\alpha)x + \gamma^{(k,1)}(\alpha))$.*

Proof. This is a direct consequence of Lemma 4.3.1 with $\beta = \alpha$, $c_k^+ = \gamma^{(k,1)}$ and $\xi_k^+ = b^{(k,1)}$ and the observation that $-c_{-k}^- = c_k^+$ and $-\xi_{-k}^- = \xi_k^+$. This means the resulting eigenfunctions are the negative of the ones obtained from the first case and hence, due to linearity, do not have to be considered separately. □

Note that, the only eigenfunction $f^{(k,1)}$ with $\mathbf{f}^{(k,1)}$ continuous is the constant function $f^{(0,1)}$. Indeed, if there exists $k \in \mathbb{Z} \setminus \{0\}$ with $\mathbf{f}^{(k,1)}$ continuous, then $\sin(b^{(k,1)} + \gamma^{(k,1)}) = \sin(\gamma^{(k,1)})$. By (4.20), this implies $\cos(\gamma^{(k,1)}) = 0$ as $b^{(k,1)} \neq 0$, contradicting the fact that $\gamma^{(k,1)} \in (-\pi/2, \pi/2)$ for $k \neq 0$.
Further, for $k_1, k_2 \in \mathbb{Z}$ with $k_1 \neq k_2$, by definition $\gamma^{(k_1,1)} \neq \gamma^{(k_2,1)}$ and, since $\gamma^{(k,1)} \in (-\pi/2, \pi/2)$ for $k \in \mathbb{Z} \setminus \{0\}$, we have $(b^{(k_1,1)})^2 \neq (b^{(k_2,1)})^2$. Hence, all eigenvalues have multiplicity one.

That the eigenvalues of Δ_η do not occur in pairs is in contrast to the case when η is atomless. From the definition of $\gamma_{(k,1)}$ follows that

$$\lim_{k\to-\infty} \gamma^{(k,1)} = -\frac{\pi}{2} \quad \text{and} \quad \lim_{k\to+\infty} \gamma^{(k,1)} = \frac{\pi}{2}.$$

Substituting this into the defining equation for $b^{(k,1)}$ yields

$$\lim_{k\to-\infty} \frac{b^{(k,1)}}{2\pi k} = 1 \quad \text{and} \quad \lim_{k\to+\infty} \frac{b^{(k,1)}}{2\pi k} = 1.$$

Let λ_k denote the k-th largest eigenvalue of Δ_η and recall that $\lambda^{(k,1)} = -(b^{(k,1)})^2$. It follows from the definition of $b^{(k,1)}$ that

$$\lim_{k\to\infty} \sqrt{\lambda_k} - \sqrt{\lambda_{k-1}} = \pi$$

and also that

$$\lim_{k\to\infty} \frac{-\lambda_k}{\left(k\pi + \frac{\pi}{2}\right)^2} = 1.$$

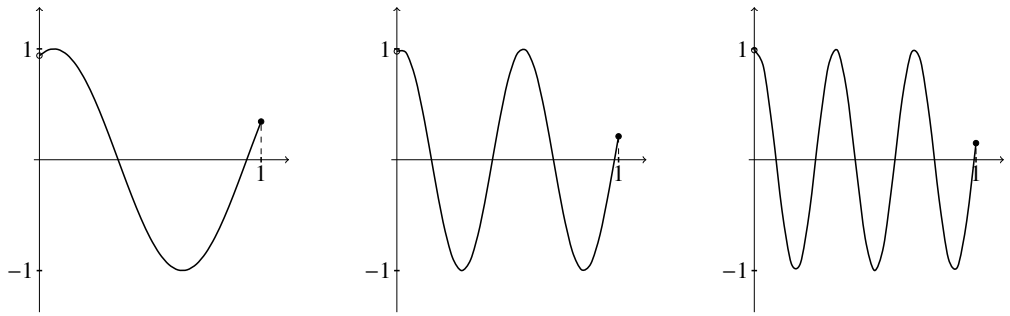
This observation also implies the asymptotic behaviour of the eigenvalue counting function, which coincides with the one for the atomless case.

Corollary 4.3.3. *Letting $N_\eta\colon \mathbb{R}^+ \to \mathbb{R}$ denote the eigenvalue counting function of $-\Delta_\eta$, it holds that*

$$\lim_{x\to\infty} \frac{\pi N_\eta(x)}{\sqrt{x}} = 1.$$

We conclude this section with an example where α is chosen explicitly as π^{-1}.

Example 4.3.4. For the measure $\eta_{(1)} = \Lambda + \pi^{-1}\delta_1$ we have that $\lambda^{(1,1)} \approx -29.3$, $\lambda^{(2,1)} \approx -130.4$ and $\lambda^{(3,1)} \approx -309.1$; see Figure 4.4 for a graphical representation of $f^{(1,1)}$, $f^{(2,2)}$ and $f^{(3,1)}$.



Graph of $f^{(1,1)}$ Graph of $f^{(2,1)}$ Graph of $f^{(3,1)}$

Figure 4.4. Graphs of the eigenfunctions $f^{(k,1)}$ of $\Delta_{\eta_{(1)}}$, for $k \in \{1,2,3\}$, where $\eta_{(1)} = \Lambda + \pi^{-1}\delta_1$.

4.3.2 Two uniformly distributed and equally weighted atoms

Let α denote a positive real number, let $z_1, z_2 \in (0,1]$ be such that $z_2 - z_1 = 1/2$ and let $\eta = \Lambda + \sum_{i=1}^{2} \alpha\delta_{i/2}$. As discussed in Remark 4.2.1, we may assume, without loss of generality, that $z_2 = 1$, and hence that $z_1 = 1/2$. The main results of this section are Theorem 4.3.6 and Corollary 4.3.7, in which we determine the eigenvalues and eigenfunctions of Δ_η.

The general form of the eigenfunctions and the system of equations which has to be fulfilled follow from the general results discussed in Theorem 4.2.2. By (4.13), if $\Delta_\eta f = \lambda f$, for some

$\lambda \in \mathbb{R}$, then there exist $b, a_1, a_2, \gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 \in (-\pi/2, \pi/2]$ such that

$$f(x) = \begin{cases} a_1 \sin(bx + \gamma_1) & \text{if } x \in \left(0,\ \frac{1}{2}\right], \\ a_2 \sin(bx + \gamma_2) & \text{if } x \in \left(\frac{1}{2},\ 1\right]. \end{cases}$$

and $\lambda = -b^2$. By Corollary 4.2.3, the constant function $\mathbb{1}$ is an eigenfunction of Δ_η with corresponding eigenvalue zero and the eigenspace $\mathrm{Eig}(0) = \{f \in \mathscr{D}_\eta^2 \colon \Delta_\eta f \equiv 0\}$ is one-dimensional. Thus the only solutions with $b = 0$ are those with $f \in \mathrm{Eig}(0)$.

With this at hand, we may assume that $b \neq 0$. From the system of equations given in (4.14) and (4.15) it follows that if f is an eigenfunction, then $a_1, a_2, b, \gamma_1, \gamma_2$ fulfil the following equations.

$$\begin{aligned} \alpha b a_1 \cos(\gamma_1) &= a_1 \sin(\gamma_1) - a_2 \sin(b + \gamma_2) \\ \alpha b a_2 \sin(b + \gamma_2) &= a_2 \cos(b + \gamma_2) - a_1 \cos(\gamma_1) \\ \alpha b a_2 \cos(b/2 + \gamma_2) &= a_2 \sin(b/2 + \gamma_2) - a_1 \sin(b/2 + \gamma_1) \\ \alpha b a_1 \sin(b/2 + \gamma_1) &= a_1 \cos(b/2 + \gamma_1) - a_2 \cos(b/2 + \gamma_2) \end{aligned} \tag{4.21}$$

As discussed directly prior to Corollary 4.2.3, we have $a_1, a_2 \neq 0$, since the equations (4.21) yield $f \equiv 0$ otherwise.

By Proposition 4.2.5, we have that

$$f_2(x) = \begin{cases} a_2 \sin\left(bx + \frac{b}{2} + \gamma_2\right) & \text{if } x \in \left(0,\ \frac{1}{2}\right], \\ a_1 \sin\left(bx - \frac{b}{2} + \gamma_1\right) & \text{if } x \in \left(\frac{1}{2},\ 1\right], \end{cases} \tag{4.22}$$

is also an eigenfunction of Δ_η. Hence, without loss of generality, we may assume that $a_1 = 1$ and that $|a_2| \leq 1$.

Now our aim is to find all tuples $(b, a_2, \gamma_1, \gamma_2) \in \mathbb{R}^4$ such that f is a non-constant eigenfunction. This is done in three steps. We start with the special cases $a_2 = 1$ and $b = \pm 1/\alpha$. In the second step we discuss the case $a_2 = 1$ and $b \neq \pm 1/\alpha$. Since $(b, a_2, \gamma_1, \gamma_2)$ leads to an eigenfunction if and only if $(b, -a_2, \gamma_1, \gamma_2 + \pi)$ does, the first two steps also solve the case $a_2 = -1$. We end with showing that f is not an eigenfunction of Δ_η, if $|a_2| < 1$.

Suppose that $a_2 = 1$ and $b = -1/\alpha$. The first two equations of (4.21) in tandem imply that $\sin(\gamma_1) = -\cos(b + \gamma_2)$ and hence, we can conclude that $b + \gamma_2 = \gamma_1 + \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. Substituting this into the first equation yields $\gamma_1 = 0$ and hence, $\gamma_2 = -b + \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. When substituting these values into the third and fourth equations of (4.21), we obtain that $b = \pi + 2\pi m$ for some $m \in \mathbb{Z}$. Since we assumed that $b = -1/\alpha$ we have that f is an eigenfunction if and only if $\alpha = 1/(\pi + 2\pi m)$ for some $m \in \mathbb{Z}$, in which case $\gamma_2 \bmod 2\pi = 3\pi/2$.

In a similar manner we study the case $a_2 = 1$ and $b = 1/\alpha$. Here, the first two equations of (4.21) result in $\sin(\gamma_1) = \cos(b + \gamma_2)$ and hence, $b + \gamma_2 = -\gamma_1 + \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. When substituting this into the first equation, one obtains that $\gamma_1 = \arctan(2)$ and hence,

$\gamma_2 = -b - \arctan(2) + \pi/2 + 2\pi k$ for some $k \in \mathbb{Z}$. Looking again at the third and fourth equations of (4.21), we obtain that $b = 2\arctan(1/2) - 2\arctan(2) + 2\pi m$ for some $m \in \mathbb{Z}$ and hence, f is an eigenfunction if and only if $\alpha = 1/(2\arctan(1/2) - 2\arctan(2) + 2\pi m)$ for some $m \in \mathbb{Z}$ and we get $\gamma_2 \bmod 2\pi = \arctan(2) - 2\arctan(1/2) + \pi/2$.

Note that we can assume in both cases that $m \in \mathbb{N}_0$, since $\alpha > 0$. We combine the obtained results in the following statement.

Corollary 4.3.5. *If and only if there exists $m \in \mathbb{N}_0$ such that either $\alpha = \alpha' := 1/(\pi + 2\pi m)$ or $\alpha = \alpha'' := 1/(2\arctan(1/2) - 2\arctan(2) + 2\pi m)$, then $\lambda = -1/\alpha^2$ is an eigenvalue of Δ_η with multiplicity one. The corresponding eigenfunction is*

$$f(x) = \begin{cases} \sin\left(-\frac{x}{\alpha}\right) & \text{if } x \in \left(0, \frac{1}{2}\right] \text{ and } \alpha = \alpha', \\ \sin\left(-\frac{x}{\alpha} + \frac{3\pi}{2}\right) & \text{if } x \in \left(\frac{1}{2}, 1\right] \text{ and } \alpha = \alpha', \\ \sin\left(\frac{x}{\alpha} + \arctan(2)\right) & \text{if } x \in \left(0, \frac{1}{2}\right] \text{ and } \alpha = \alpha'', \\ \sin\left(\frac{x}{\alpha} + \arctan(2) - 2\arctan\left(\frac{1}{2}\right) + \frac{\pi}{2}\right) & \text{if } x \in \left(\frac{1}{2}, 1\right] \text{ and } \alpha = \alpha''. \end{cases}$$

We now consider the case $a_2 = 1$ and $b \in \mathbb{R} \setminus \{0, \pm 1/\alpha\}$. By way of contradiction, we first show that $\cos(\gamma_1) \neq 0$. To this end, assume that $\cos(\gamma_1) = 0$. The first equation of (4.21) implies that $\sin(\gamma_1) = \sin(b + \gamma_2) = 1$, which also yields $\cos(b + \gamma_2) = 0$. Substituting this into the second equation of (4.21) implies $\alpha b = 0$, which is a contradiction since $\alpha > 0$ and $b \neq 0$. A similar argument for the third and the fourth equations of (4.21) shows that also $\cos(b/2 + \gamma_2) \neq 0$. This implies that f is discontinuous at the atoms and hence that $b + \gamma_2 - \gamma_1 \neq 0$ and $\gamma_1 - \gamma_2 \neq 0$. Thus, we have $\gamma_1 \neq \pi/2$.

Define $g\colon (0,1] \to \mathbb{R}$ by $g(x) := \sin((b + \gamma_2 - \gamma_1)x + \gamma_1)$ and set $\beta_1 := \alpha b/(b + \gamma_2 - \gamma_1)$. Setting $\beta = \beta_1$, $c = \gamma_1$ and $\xi = b + \gamma_2 - \gamma_1$, the equalities in (4.21) imply that g fulfils those of (4.18). So, by Lemma 4.3.1, there exists a $k \in \mathbb{Z}$ with $b + \gamma_2 = -\gamma_1 \pm \pi/2 + 2\pi k$.

If $b + \gamma_2 = -\gamma_1 + \pi/2 + 2\pi k$, then $\sin(b + \gamma_2) = \cos(\gamma_1)$ and $\cos(b/2 + \gamma_2) = \sin(b/2 + \gamma_1)$. Combining this with (4.21) yields

$$\begin{aligned} \tan(\gamma_1) &= \alpha b + 1, & \tan(b + \gamma_2) &= \frac{1}{\alpha b + 1}, \\ \tan\left(\frac{b}{2} + \gamma_2\right) &= \alpha b + 1, & \tan\left(\frac{b}{2} + \gamma_1\right) &= \frac{1}{\alpha b + 1}. \end{aligned} \tag{4.23}$$

Substituting the first equation of (4.23) into the last and using the trigonometric identity $\tan(x) = -\cot(x + \pi/2)$, yields

$$b = -4\gamma_1 + \pi + 2\pi k. \tag{4.24}$$

From the first equation it follows that $\gamma_1 = \gamma_1^{(k,2)}$, for some $k \in \mathbb{Z}$, where $\gamma_1^{(k,2)}$ denotes the unique solution in $(-\pi/2, \pi/2)$ to the equation

$$\tan\left(\gamma_1^{(k,2)}\right) = 1 - 4\alpha\gamma_1^{(k,2)} + \alpha\pi + 2\pi\alpha k. \tag{4.25}$$

That (4.25) has a unique solution is again a result of the tangent being strictly monotonically increasing on $(-\pi/2, \pi/2)$ with $\lim_{x\to-\pi/2}\tan(x) = -\infty$ and $\lim_{x\to\pi/2}\tan(x) = \infty$, and the right side of the equation being a strictly monotonically decreasing linear function, since $\alpha > 0$. Further, equation (4.24) implies that for every $k \in \mathbb{Z}$, there exists a unique b, which we denote by $b^{(k,2)}$, namely

$$b^{(k,2)} = -4\gamma_1^{(k,2)} + \pi + 2\pi k, \tag{4.26}$$

and from the formula $b + \gamma_2 = -\gamma_1 + \pi/2 + 2\pi k$ obtained above, it follows that

$$\gamma_2 = \gamma_2^{(k,2)} = 3\gamma_1^{(k,2)} - \frac{\pi}{2}, \tag{4.27}$$

where an occurring additional multiple of 2π is omitted due to the periodicity of the sine function. Substituting (4.25), (4.26) and (4.27) in (4.23), one sees that all equations hold true for $k \in \mathbb{Z}$. Observe that $\gamma_1^{(0,2)} = \pi/4$, and hence that $b^{(0,2)} = 0$, contradicting our initial assumption that f is a non-constant eigenfunction.

If $b + \gamma_2 = -\gamma_1 - \pi/2 + 2\pi k$, then $\sin(b+\gamma_2) = -\cos(\gamma_1)$ and $\cos(b/2+\gamma_2) = -\sin(b/2+\gamma_1)$ and hence, (4.21) yields

$$\tan(\gamma_1) = \alpha b - 1, \qquad \tan(b+\gamma_2) = \frac{1}{\alpha b - 1},$$

$$\tan\left(\frac{b}{2} + \gamma_2\right) = \alpha b - 1, \qquad \tan\left(\frac{b}{2} + \gamma_1\right) = \frac{1}{\alpha b - 1}.$$

Similar to the case when $N = 1$, analogous calculations yield for this case that $b = -b^{(k,2)}$, $\gamma_1 = -\gamma_1^{(k,2)}$ and $\gamma_2 = -\gamma_2^{(k,2)}$ for some $k \in \mathbb{Z}$. This means that the resulting eigenfunctions are the negative of those obtained from the case $b + \gamma_2 = -\gamma_1 + \pi/2 + 2\pi k$ and hence, due to linearity, do not have to be considered separately.

We set $f^{(0,2)} := \mathbb{1}$ and $\lambda^{(0,2)} := 0$ and obtain from Corollary 4.2.3 that $f^{(0,2)}$ is also an eigenfunction of Δ_η with corresponding eigenvalue $\lambda^{(0,2)}$.

If α is of the form discussed in Corollary 4.3.5, then the eigenfunction f given there belongs to $\{f^{(k,2)} : k \in \mathbb{Z}\}$. Indeed, a direct calculation shows that, if there exists $m \in \mathbb{N}_0$ with $\alpha = 1/(2\arctan(1/2) - 2\arctan(2) + 2\pi m)$, then $f = f^{(m,2)}$; and if $\alpha = 1/(\pi + 2\pi m)$ for some $m \in \mathbb{N}_0$, then $f = f^{(-m-1,2)}$.

Since $b^{(k,2)}/2 + \gamma_2^{(k,2)} = \gamma_1^{(k,2)} + \pi k$, we have, for $x \in (0, 1/2]$, that

$$f^{(k,2)}\left(x + \frac{1}{2}\right) = \sin\left(b^{(k,2)}x + \frac{b^{(k,2)}}{2} + \gamma_2^{(k,2)}\right) = \sin\left(b^{(k,2)}x + \gamma_1^{(k,2)} + \pi k\right).$$

This means that if k is even, the eigenfunctions are periodic with period $1/2$, namely $f = f_2$ where f_2 is defined as in (4.22). As discussed in Proposition 4.2.6, we have that these eigenfunctions are concatenations of rescaled solutions for a measure with one atom. To be precise, if $k = 2m$, then $\gamma_1^{(k,2)}(\alpha) = \gamma^{(m,1)}(2\alpha)$ and $b^{(k,2)}(\alpha) = 2b^{(m,1)}(2\alpha)$. On the other hand, if k is odd, then $f = -f_2$.

Summarising the results proven above, we have, for $k \in \mathbb{Z}$, that

$$f^{(k,2)}(x) = \begin{cases} \sin\left(b^{(k,2)}x + \gamma_1^{(k,2)}\right) & \text{if } x \in \left(0, \frac{1}{2}\right], \\ \sin\left(b^{(k,2)}x + \gamma_2^{(k,2)}\right) & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

is an eigenfunction of Δ_η with corresponding eigenvalue $\lambda^{(k,2)} = -(b^{(k,2)})^2$.

We now show, by way of contradiction, that there does not exist any further eigenfunction of Δ_η. To this end, assume that h is an eigenfunction of Δ_η in the orthogonal complement of $\operatorname{span}\{f^{(k,2)} : k \in \mathbb{Z}\}$. By the discussion following Proposition 4.2.5, we can assume that h is of the form

$$h(x) = \begin{cases} \sin(bx + \gamma_1) & \text{if } x \in \left(0, \frac{1}{2}\right], \\ a_2 \sin(bx + \gamma_2) & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

for some $b \in \mathbb{R}$ and $a_2, \gamma_1, \gamma_2 \in \mathbb{R}$ with $|a_2| < 1$. Defining the function $\psi(x) := x/2$, for $x \in \mathbb{R}$, by Theorem 4.1.21 and Theorem 4.3.2, we can conclude that $\{f^{(2m,2)} \circ \psi : m \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2_{\Lambda + 2\alpha\delta_1}$, and thus, $\{f^{(2m,2)}|_{(0,1/2]} : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2_{\Lambda|_{(0,1/2]} + \alpha\delta_{1/2}}$.

This fact, together with the periodicity of the function $f^{(2m,2)}$, implies that $\langle h, f\rangle_\eta = 0$ for all $f \in \operatorname{span}\{f^{(2m,2)} : m \in \mathbb{Z}\}$. This yields, for Borel sets $A \subset (0, 1/2]$,

$$\int \mathbb{1}_A(x)\left(h(x) + h\left(x + \frac{1}{2}\right)\right) \mathrm{d}\eta(x) = 0.$$

From this integral equation, it follows that $\eta(\{x \in (0, 1/2] : h(x) > -h(x + 1/2)\}) = 0$ and that $\eta(\{x \in (0, 1/2] : h(x) < -h(x + 1/2)\}) = 0$. Hence, we have $h(x) = -h(x + 1/2)$ for η-almost all $x \in (0, 1/2]$. Since $h \in \mathscr{D}^2_\eta$, it is left-continuous, which implies that $h(x) = -h(x + 1/2)$ for all $x \in (0, 1/2]$. Therefore,

$$h(x) = \begin{cases} \sin(bx + \gamma_1) & \text{if } x \in \left(0, \frac{1}{2}\right], \\ -\sin\left(b\left(x - \frac{1}{2}\right) + \gamma_1\right) & \text{if } x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

contradicting our initial assumption that $|a_2| < 1$.

For $k_1, k_2 \in \mathbb{Z}$ with $k_1 \neq k_2$ we observe from (4.25) that $\gamma^{(k_1,1)} \neq \gamma^{(k_2,1)}$. Further, we now show, again by way of contradiction, that $(b^{(k_1,1)})^2 \neq (b^{(k_2,1)})^2$, which implies that all eigenvalues of Δ_η have multiplicity one. To this end, assume that $(b^{(k_1,1)})^2 = (b^{(k_2,1)})^2$, which means, by definition (4.26), that

$$-4\gamma_1^{(k_1,2)} + \pi + 2\pi k_1 = \pm(-4\gamma_1^{(k_2,2)} + \pi + 2\pi k_2). \tag{4.28}$$

We start with the case $-4\gamma_1^{(k_1,2)} + \pi + 2\pi k_1 = -4\gamma_1^{(k_2,2)} + \pi + 2\pi k_2$ for which we obtain that $\gamma_1^{(k_1,2)} - \gamma_1^{(k_2,2)} = \pi/2(k_1 - k_2)$. Since $k_1 \neq k_2$ and since $\gamma_1^{(k_1,2)}, \gamma_1^{(k_2,2)} \in (-\pi/2, \pi/2)$, this implies that $|k_1 - k_2| = 1$. Without loss of generality, we can assume that $k_2 = k_1 + 1$, which implies

$\gamma_1^{(k_2,2)} = \gamma_1^{(k_1,2)} + \pi/2$. Again using $\gamma_1^{(k_2,2)} \in (-\pi/2, \pi/2)$, we have $\gamma_1^{(k_1,2)} \in (-\pi/2, 0)$. Using the defining formula (4.25), we obtain that

$$\begin{aligned}\tan\left(\gamma_1^{(k_2,2)}\right) = \tan\left(\gamma_1^{(k_1,2)} + \pi/2\right) &= -\frac{1}{\tan\left(\gamma_1^{(k_1,2)}\right)} \\ &= 1 - 4\alpha\gamma_1^{(k_2,2)} + \alpha\pi + 2\pi\alpha k_2 \\ &= 1 - 4\alpha\left(\gamma_1^{(k_1,2)} + \frac{\pi}{2}\right) + \alpha\pi + 2\pi\alpha\,(k_1+1) \\ &= 1 - 4\alpha\gamma_1^{(k_1,2)} + \alpha\pi + 2\pi\alpha k_1 = \tan\left(\gamma_1^{(k_1,2)}\right).\end{aligned}$$

This implies that $\gamma_1^{(k_1,2)} = \gamma_1^{(k_2,2)}$ and hence, yields a contradiction. In the second case of (4.28), namely $-4\gamma_1^{(k_1,2)} + \pi + 2\pi k_1 = 4\gamma_1^{(k_2,2)} - \pi - 2\pi k_2$, the same line of reasoning as in the first case yields that $k_1 = -k_2 - p$, where $p \in \{0,1,2\}$. Substituting this in (4.25) and following the same argument as before yields that in this case (4.28) can only be fulfilled for $k_1 = -k_2 = 0$, which contradicts our initial assumption that $k_1 \neq k_2$. Hence, we have shown that all eigenvalues of Δ_η are simple, namely that $\dim(\mathrm{Eig}(\lambda^{(k,2)})) = 1$, for $k \in \mathbb{Z}$.

Combining all of the above we obtain the following.

Theorem 4.3.6. *The eigenvalues of Δ_η are $\lambda^{(k,2)} := -(b^{(k,2)})^2$ for $k \in \mathbb{Z}$, with corresponding eigenfunctions $f^{(k,2)}$. Further, each eigenvalue has multiplicity one.*

Notice, the only eigenfunction $f^{(k,2)}$ with $\mathbf{f}^{(k,2)}$ continuous at an atom is the constant function $f^{(0,2)}$. Indeed, if there existed $k \in \mathbb{Z}\setminus\{0\}$ such that $\mathbf{f}^{(k,2)}$ were continuous at an atom, we would have $\cos(\gamma_1^{(k,2)}) = 0$ or $\cos(b^{(k,2)}/2 + \gamma_2^{(k,2)}) = 0$. Substituting the defining equations for $b^{(k,2)}$ and $\gamma_2^{(k,2)}$ into the latter, we obtain, in both cases, that $\gamma_1^{(k,2)} = \pi/2$, which contradicts the fact that $\gamma_1^{(k,2)} \in (-\pi/2, \pi/2)$.

We also obtain the asymptotic behaviour of the parameters $\gamma_1^{(k,2)}, \gamma_2^{(k,2)}$ and $b^{(k,2)}$. From (4.25) it directly follows that

$$\lim_{k\to-\infty} \gamma_1^{(k,2)} = -\frac{\pi}{2} \quad \text{and} \quad \lim_{k\to\infty} \gamma_1^{(k,2)} = \frac{\pi}{2}.$$

Using (4.27) this implies the asymptotic behaviour of $\gamma_2^{(k,2)}$, namely

$$\lim_{k\to-\infty} \gamma_2^{(k,2)} \bmod 2\pi = 0 \quad \text{and} \quad \lim_{k\to\infty} \gamma_2^{(k,2)} \bmod 2\pi = \pi.$$

Since $\gamma_1^{(k,2)}$ is bounded, it is clear from (4.26) that

$$\lim_{k\to-\infty} \frac{b^{(k,2)}}{2\pi k} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{b^{(k,2)}}{2\pi k} = 1.$$

As in the case of mixed measures with one atom (see Section 4.3.1), we have that the eigenvalues do not occur in pairs. However, in contrast to what we have seen there, the

distances between the pairs $\lambda^{(k,2)}$ and $\lambda^{(-k,2)}$ converge to zero and the distances between $\sqrt{-\lambda^{(k,2)}}$ and $\sqrt{-\lambda^{(k+1,2)}}$ converge to 2π. This means, if λ_k denotes the k-th largest eigenvalue of Δ_η, that

$$\lim_{k\to\infty} \lambda_{2k} - \lambda_{2k-1} = 0$$

and also that

$$\lim_{k\to\infty} \sqrt{-\lambda_{2k+1}} - \sqrt{-\lambda_{2k}} = 2\pi$$

The asymptotic distribution of the eigenvalues is similar to the case of atomless measures.

Corollary 4.3.7. *Letting $N_\eta\colon \mathbb{R}^+ \to \mathbb{R}$ denote the eigenvalue counting function of $-\Delta_\eta$, it holds that*

$$\lim_{x\to\infty} \frac{\pi N_\eta(x)}{\sqrt{x}} = 1.$$

We conclude this section with an example where α is chosen explicitly as π^{-1}.

Example 4.3.8. For the measure $\eta_{(2)} = \Lambda + \pi^{-1}\delta_{\frac{1}{2}} + \pi^{-1}\delta_1$, we have that $\lambda^{(-2,2)} = -4\pi^2$, $\lambda^{(-1,2)} = -\pi^2$, $\lambda^{(1,2)} \approx -21.8$, $\lambda^{(2,2)} \approx -106.9$, $\lambda^{(3,2)} \approx -267.2$ and $\lambda^{(4,2)} \approx -505.3$. We give a graphical representation of $f^{(k,2)}$ for $k \in \{-2,-1,1,2,3,4\}$ in Figure 4.5.

4.3.3 Two non-uniformly distributed atoms

In Theorem 4.1.21 we showed that the operator Δ_η has compact resolvent and hence, as discussed later, the eigenfunctions of the η-Laplacian always form a basis of L^2_η. In the previous example in Section 4.3.2, we saw that the eigenfunctions of the form given in (4.13) fulfil that $|a_1| = |a_2|$ and that all eigenvalues are simple. By linearity, we could restrict ourselves to the case $|a_1| = |a_2| = 1$ and by adapting γ_2 we could even assume that $a_1 = a_2 = 1$. This observations give rise to the question whether it always holds true that the a_i coincide and that the eigenvalues are simple. We will disprove this conjecture in this chapter by giving a class of measures for which this is not true. Namely, we look at measures of the form $\eta = \Lambda + \alpha_1\delta_{z_1} + \alpha_2\delta_1$, where α_1, α_2 and z_1 fulfil the following properties:

(1) $\alpha_1, \alpha_2 \in \mathbb{Q} \cap \mathbb{R}^+$,

(2) $\dfrac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} \notin \mathbb{Z}$,

(3) $\sqrt{\alpha_1^2 + \alpha_2^2 + 6\alpha_1\alpha_2} \in \mathbb{R} \setminus \mathbb{Q}$, and

(4) $z_1 = \dfrac{1}{2} - \dfrac{\pi}{4k}$ for some $k \in \mathbb{N}$ with $k \geq 2$.

An example for possible choices of the weights which fulfil Conditions (1) – (3) is $\alpha_1 = 2$ and $\alpha_2 = 1/2$.

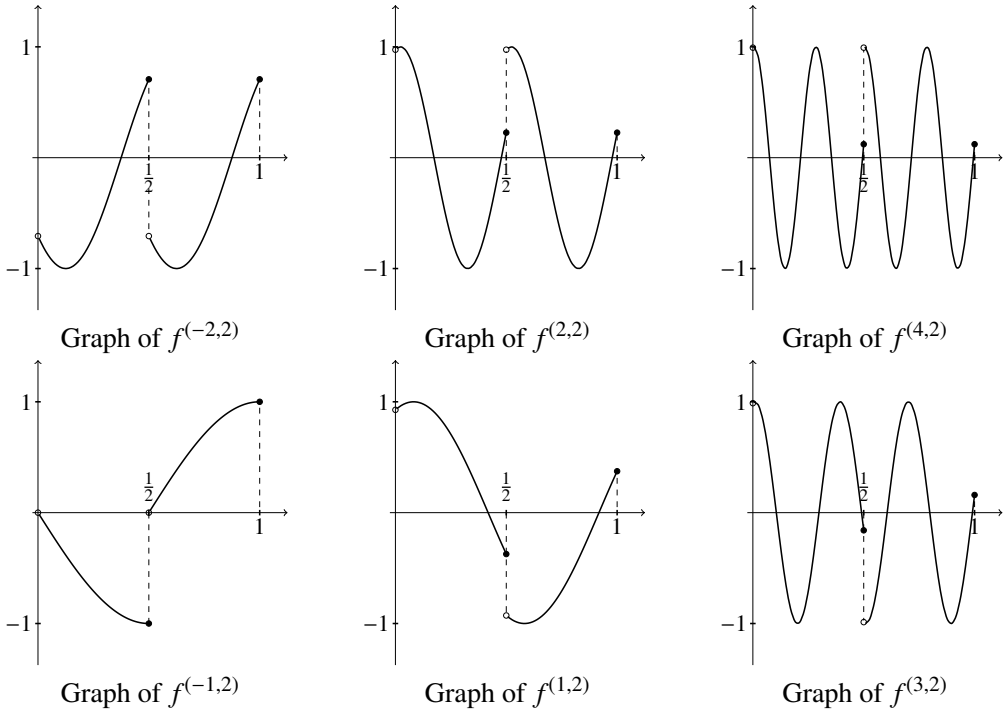


Figure 4.5. Graphs of the eigenfunctions $f^{(k,2)}$ of $\Delta_{\eta_{(2)}}$, for $k \in \{-2,-1,1,2,3,4\}$, where $\eta_{(2)} = \Lambda + \pi^{-1}\delta_{1/2} + \pi^{-1}\delta_1$.

Theorem 4.3.9. *Let $\eta_{(3)} = \Lambda + \alpha_1\delta_{z_1} + \alpha_2\delta_1$ be a measure which fulfils Conditions (1) – (4). The operator $\Delta_{\eta_{(3)}}$ has eigenfunction of the form*

$$f(x) = \begin{cases} \sin(bx+\gamma_1) & \text{if } x \in (0,z_1], \\ a_2\sin(bx+\gamma_2) & \text{if } x \in (z_1,1], \end{cases}$$

where $b, a_2, \gamma_1, \gamma_2 \in \mathbb{R}$ with $|a_2| < 1$. Furthermore, it might have eigenvalues with multiplicity strictly greater than one.

In Theorem 4.2.2, we obtained the general form of the eigenfunctions and the system of equations which have to be fulfilled by the parameters $a_1, a_2, \gamma_1, \gamma_2$ and b. We will prove by contradiction that there exists an eigenvalue for which there is no corresponding eigenfunction with $a_1 = a_2 = 1$. Therefore, we assume $a_1 = a_2 = 1$ and in the following we show that the only possible solutions to the system of equations (4.14) and (4.15) are $b \in \{0, \pm 1/\alpha_1, \pm 1/\alpha_2\}$.

To this end, we assume that all eigenfunctions are of the form

$$f(x) = \begin{cases} \sin(bx+\gamma_1) & \text{if } x \in (0,z_1], \\ \sin(bx+\gamma_2) & \text{if } x \in (z_1,1]. \end{cases} \tag{4.29}$$

Hence, equations (4.14) and (4.15) imply that γ_1, γ_2 and b have to fulfil

$$\begin{aligned}
\alpha_1 b\cos(\gamma_1) &= \sin(\gamma_1) - \sin(b+\gamma_2),\\
\alpha_1 b\sin(b+\gamma_2) &= \cos(b+\gamma_2) - \cos(\gamma_1),\\
\alpha_2 b\cos(bz_1+\gamma_2) &= \sin(bz_1+\gamma_2) - \sin(bz_1+\gamma_1),\\
\alpha_2 b\sin(bz_1+\gamma_1) &= \cos(bz_1+\gamma_1) - \cos(bz_1+\gamma_2).
\end{aligned} \tag{4.30}$$

Let f be an eigenfunction of Δ_η with $b \in \mathbb{R}\setminus\{0, \pm 1/\alpha_1, \pm 1/\alpha_2\}$. Similar to the argument given just after Corollary 4.3.5, we first show, by way of contradiction, that $\cos(\gamma_1) \neq 0$. To this end, assume that $\cos(\gamma_1) = 0$. The first equation of (4.30) implies that $\sin(\gamma_1) = \sin(b+\gamma_2) = 1$, which also yields $\cos(b+\gamma_2) = 0$. Substituting this into the second equation of (4.30), implies that $\alpha_1 b = 0$, which is a contradiction since $\alpha_1 > 0$ and $b \neq 0$. A similar argument for the third and the fourth equations of (4.30) shows that $\cos(bz_1+\gamma_2) \neq 0$. This implies that f is discontinuous at the atoms and hence, that $b+\gamma_2-\gamma_1 \neq 0$ and $\gamma_1-\gamma_2 \neq 0$.
Define the two non-constant functions $g_1, g_2\colon (0,1] \to \mathbb{R}$ by $g_1(x) \coloneqq \sin((b+\gamma_2-\gamma_1)x+\gamma_1)$ and $g_2(x) \coloneqq \sin((\gamma_1-\gamma_2)x+bz_1+\gamma_2)$ and set $\beta_1 \coloneqq \alpha_1 b/(b+\gamma_2-\gamma_1)$ and $\beta_2 \coloneqq \alpha_2 b/(\gamma_1-\gamma_2)$. Setting $\beta = \beta_1$, $c = \gamma_1$ and $\xi = b+\gamma_2-\gamma_1$, equations (4.30) imply that g_1 fulfils those of (4.18). Similarly, we have that (4.30) implies that g_2 fulfils the equations (4.18), when setting $\beta = \beta_2$, $c = bz_1+\gamma_2$ and $\xi = \gamma_1-\gamma_2$. Hence, it follows from Lemma 4.3.1 that there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$\begin{aligned}
b+\gamma_2-\gamma_1 &= -2\gamma_1 + \sigma_1\frac{\pi}{2} + 2\pi k_1 \quad \text{and}\\
\gamma_1-\gamma_2 &= -2bz_1 - 2\gamma_2 + \sigma_2\frac{\pi}{2} + 2\pi k_2,
\end{aligned} \tag{4.31}$$

where $\sigma_1, \sigma_2 \in \{-1, 1\}$ can be chosen independently. By equating both formulas in (4.31), we can conclude that there exists $k \in \mathbb{Z}$ such that

$$b = \frac{\pi}{1-2z_1}\left(\sigma_1\frac{1}{2} + \sigma_2\frac{1}{2} + 2k\right). \tag{4.32}$$

Condition (4) on the position of the atom z_1 yields that $b \in \mathbb{Z}$, for all $k \in \mathbb{Z}$ and all choices for σ_1 and σ_2.
Equations (4.31) imply that $\sin(b+\gamma_2) = \sigma_1\cos(\gamma_1)$ and that $\sin(bz_1+\gamma_1) = \sigma_2\cos(bz_1+\gamma_2)$. Substituting this back into the system of equations given in (4.30) yields

$$\tan(\gamma_1) = \alpha_1 b + \sigma_1, \qquad \tan(b+\gamma_2) = \frac{1}{\alpha_1 b + \sigma_1},$$

$$\tan(bz_1+\gamma_2) = \alpha_2 b + \sigma_2, \qquad \tan(bz_1+\gamma_1) = \frac{1}{\alpha_2 b + \sigma_2}.$$

These equations imply that $\gamma_1 = \arctan(\alpha_1 b + \sigma_1)$ and $bz_1+\gamma_1 = \arctan(1/(\alpha_2 b+\sigma_2)) + \pi k$ for some $k \in \mathbb{Z}$. Using the identity $\tan(x) = -\cot(x+\pi/2)$, we have that there exists $k \in \mathbb{Z}$ such

that

$$bz_1 - \frac{\pi}{2} + \pi k = -\arctan(\alpha_1 b + \sigma_1) - \arctan(\alpha_2 b + \sigma_2).$$

If we now substitute Condition (4), which is on the position of the atom z_1, in this equation, we can restate the formula in a way that it only depends on the weights α_1 and α_2, but not on the position z_1, namely

$$\frac{b}{2} + \sigma_1 \frac{\pi}{4} + \sigma_2 \frac{\pi}{4} - \frac{\pi}{2} + \pi k = -\arctan(\alpha_1 b + \sigma_1) - \arctan(\alpha_2 b + \sigma_2). \tag{4.33}$$

We now want to rewrite this equation, using the addition formula for the inverse tangent. Namely, we want to use that for $x, y \in \mathbb{R}$ with $xy \neq 1$ there exists an $n \in \mathbb{Z}$ such

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right) + \pi n. \tag{4.34}$$

First we treat the cases in (4.33) where the addition formula can not be applied, that is where $(\alpha_1 b + \sigma_1)(\alpha_1 b + \sigma_1) = 1$. Depending on the choices of σ_1 and σ_2 we obtain four cases, namely

1. $\alpha_1\alpha_2 b^2 + \alpha_1 b + \alpha_2 b = 0$,
2. $\alpha_1\alpha_2 b^2 + \alpha_1 b - \alpha_2 b - 2 = 0$,
3. $\alpha_1\alpha_2 b^2 - \alpha_1 b + \alpha_2 b - 2 = 0$, and
4. $\alpha_1\alpha_2 b^2 - \alpha_1 b - \alpha_2 b = 0$.

The set of solutions for b in all cases is

$$\left\{0,\ \pm\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2},\ \pm\frac{\sqrt{\alpha_1^2+\alpha_2^2+6\alpha_1\alpha_2}-\alpha_1+\alpha_2}{2\alpha_1\alpha_2},\ \pm\frac{\sqrt{\alpha_1^2+\alpha_2^2+6\alpha_1\alpha_2}+\alpha_1-\alpha_2}{2\alpha_1\alpha_2}\right\}.$$

By Conditions (2) and (3) on the atom weights, these are non-integer numbers and hence contradict (4.32). For all other possible choices of b we can apply (4.34) to (4.33) and obtain that there exists $k \in \mathbb{Z}$ such that

$$\frac{b}{2} + \sigma_1 \frac{\pi}{4} + \sigma_2 \frac{\pi}{4} - \frac{\pi}{2} + \pi k = -\arctan\left(\frac{\alpha_1 b + \alpha_2 b + \sigma_1 + \sigma_2}{1 - (\alpha_1 b + \sigma_1)(\alpha_2 b + \sigma_2)}\right),$$

or equivalently,

$$-\tan\left(\frac{b}{2} + \sigma_1 \frac{\pi}{4} + \sigma_2 \frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{\alpha_1 b + \alpha_2 b + \sigma_1 + \sigma_2}{1 - (\alpha_1 b + \sigma_1)(\alpha_2 b + \sigma_2)}. \tag{4.35}$$

Depending on the choices of σ_1 and σ_2, the left-hand side is either equal to $-\tan(b/2)$ or to $\cot(b/2)$. To obtain a contradiction, we use a general theorem on trigonometric functions, see [Niv56, Corollary 2.7].

Corollary 4.3.10. *The trigonometric functions are irrational at non-zero rational values of the arguments.*

In our setting, since b is an integer, it follows from Condition (1) that the right-hand side of equation (4.35) is rational for all choices of σ_1 and σ_2. Since the left-hand side is either $-\tan(b/2)$ or $\cot(b/2)$, this is a contradiction to Corollary 4.3.10 for $b \neq 0$. This means there exists no $b \in \mathbb{R} \setminus \{0, \pm 1/\alpha_1, \pm 1/\alpha_2\}$ which fulfils (4.32) and (4.35).

In conclusion, we have that the only candidates for b which lead to a non-constant eigenfunction of the form (4.29) are $b \in \{\pm 1/\alpha_1, \pm 1/\alpha_2\}$. We showed in Theorem 4.1.21 that Δ_η has compact resolvent and hence, the eigenfunctions form a basis for L^2_η. Further, the spectral theorem for compact, self-adjoint operators implied that all eigenvalues have finite multiplicity. This yields that there exist eigenfunctions of the form

$$f(x) = \begin{cases} \sin(bx + \gamma_1) & \text{if } x \in (0, z_1], \\ a_2 \sin(bx + \gamma_2) & \text{if } x \in (z_1, 1], \end{cases}$$

with $|a_2| < 1$. Note, for $\lambda = -1/\alpha_1^2$ or $\lambda = -1/\alpha_2^2$, there might exist eigenfunctions with $|a_1| = |a_2|$ which also might have multiplicity strictly greater than one.

CHAPTER 5

Outlook

There are many different directions in which the theory developed here can be extended. On the one hand, there are open questions concerning the cases discussed here, and on the other hand, one can generalise the measures and the considered operators.

A first direction in which the theory could be extended, is to look at atomless signed measures. In Chapter 2 we developed the theory for atomless measures ν and at the heart of many proofs was that the distribution function F_ν is continuous and monotonically increasing. For signed measures $\tilde{\nu}$ we would still have that $F_{\tilde{\nu}}$ is continuous, but is is not necessarily monotone anymore. Having some parts where the function is increasing and some where it is decreasing immediately results in problems with, for example Lemma 2.1.5, and looking at the pseudo-inverse $\check{F}^{-1}_{\tilde{\nu}}$, both being crucial to many proofs in Chapter 2.

For δ being a purely atomic measure, in (3.10) and the following paragraph, we gave the general form of the matrix representation B_δ of the δ-Laplacian. To find the eigenvalues and eigenvectors of B_δ is equivalent to finding the eigenvalues and eigenfunctions of Δ_δ and in Section 3.2 some general results and bounds were obtained. Of course the long time goal is to solve the eigenvalue problem completely and hence, to give a full picture of the eigenvalues and eigenfunctions in dependence of the measures δ. There are different ways to approach this problem or to at least obtain partial results.
For a fixed measure δ, especially for N sufficiently large, the matrix B_δ is sparse. Using this fact, one way to obtain better bounds on the eigenvalues of B_δ might be the use of numerical algorithms. For example the Lanczos algorithm could be used to tridiagonalise the matrix and then general eigendecomposition algorithms might allow for efficiently finding numerical approximations to the eigenvalues and the eigenvectors. We refer the reader for further details on the Lanczos algorithm to [Lan50, ON70] and for a possible eigendecomposition algorithm, the divide-and-conquer eigenvalue algorithm, to [Dem97, Chapter 5.3.3]. The quality of the results of this approach and the time needed to obtain these highly depend on the measure δ. Even if it could be assured that the numerical approximations are arbitrarily good, which it can not in general, one might not get the general solution from this.
A more theoretical approach to obtain further results on this might be the continuation of the analysis of the case discussed in Section 3.3.3. Proving or disproving Conjecture 3.3.5 is a starting point to obtaining further results. In a second step, one then could extend the

setting to general purely atomic measures with periodic atom weights. By this we mean measures of the following form: Let $N = pk$ with $p, k \in \mathbb{N}$ and fix $\alpha_i > 0$ for $i \in \{1, \dots, p\}$. Set $\alpha_{i+p} \coloneqq \alpha_i$, for $i \in \{1, \dots, N-p\}$, and define the measure $\delta = \sum_{i=1}^{N} \alpha_i \, \delta_{i/N}$. The case discussed in Section 3.3.3 is $p = 2$, $k = 3$ and hence, $N = 6$ and Conjecture 3.3.5 is then a speculation for the case $p = 2$ and general $k \in \mathbb{N}$. Varying p and k might give a deeper insight in the behaviour of the eigenvalues and eigenvectors for these special cases and from these one might deduce further results for general measures δ.
A natural extension to the class of purely atomic measures as they are discussed in Chapter 3, are measures with infinitely many atoms. To this end, for $i \in \mathbb{N}$, fix $\alpha_i > 0$ such that $\sum_{i=1}^{N} \alpha_i < \infty$. Further, choose $z_i \in (0, 1]$ such that $z_i \neq z_j$ for all $i, j \in \mathbb{N}$ with $i \neq j$, and define the measure $\tilde{\delta} \coloneqq \sum_{i=1}^{\infty} \alpha_i \delta_{z_i}$. One could define the operators $\nabla_{\tilde{\delta}}$ and $\Delta_{\tilde{\delta}}$ and their domains analogously to the case discussed here, but many properties are unclear. Unlike in the case discussed here, we expect that the results on these operators are not only dependent on the weights of the atoms, but also on the number and locations of the accumulation points of their positions.

In the case of mixed measures η, we saw that the eigenfunctions of the η-Laplacian form a basis of L^2_η and in Theorem 4.2.2 we obtained the general form of the eigenfunctions and gave a system of equations characterising the eigenvalues and eigenfunctions. Having this explicit system, it is desirable to find the general solution to it and hence, to get a full picture of the eigenvalues and eigenfunctions of Δ_η.
We obtained in Corollary 4.3.3 and Corollary 4.3.7 the asymptotic behaviour of the eigenvalue counting function in the case of δ being a Dirac point mass and δ being the sum of equally weighted and equally distributed Dirac point masses. In both cases, the growth behaviour was similar to the classical case and to the case of atomless measures. So the question arises, if the eigenvalue counting function always behaves like this or if one can find assumptions on the measure η which ensures this. If it holds true that the asymptotic distribution of the eigenvalues is always the same, this would fit very well to the theory developed in Chapter 2, where we saw that the behaviour of the eigenvalue counting function does not depend on the chosen atomless probability measure.
In Section 3.3.2 we solved the eigenvalue problem of the δ-Laplacian for uniform discrete probability distributions δ. Furthermore, it was shown that if the number of atoms tends to infinity, then not only the measures converge weakly to the Lebesgue-measure Λ, but also the eigenvalues and eigenfunctions of Δ_δ converge to the ones of Δ_Λ. Similarly, one can now look at probability measures of the form $\eta^N = 2^{-1}\Lambda + \sum_{i=1}^{N} (2N)^{-1} \delta_{i/N}$, which also converge weakly over N to the Lebesgue-measure. It would be interesting to investigate the connections between the operators Δ_{η^N} and Δ_Λ and to see if there is a relation between the eigenvalues and eigenfunctions. Also a comparison with the uniform discrete probability distributions would be possible and might lead to deeper insight on the dependence of the differential operators on the measures with respect to which they are defined.
If one has developed the theory further in the directions discussed above for atomless and purely atomic measures, one can then try to join these two approaches and look at measures of the form $\tilde{\eta} = \tilde{\nu} + \tilde{\delta}$, where $\tilde{\nu}$ is an atomless signed measure and $\tilde{\delta}$ is a purely atomic measure with infinitely many atoms. Getting a full picture of the measure-geometric differential operators $\nabla_{\tilde{\eta}}$ and $\Delta_{\tilde{\eta}}$ would be the long-term goal in the research which was developed throughout this thesis.

The paper by Freiberg and Zähle [FZ02] was part of a series of publications of both authors, some of them being joint work and some not. The focus in most of the other papers, see [Fre03a, Fre03b, Fre05, Zäh05], and also the PhD thesis of Freiberg [Fre00] lies on operators of the form $\Delta_{\mu,\nu} = \nabla_\mu \circ \nabla_\nu$, where μ and ν are two atomless probability measures with $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu)$. These operators are often referred to as (generalised) Kreĭn-Feller operators as they go back to work of Kac and Kreĭn [KK58, KK74] as well as Feller [Fel57], where the operators $\Delta_{\mu,\Lambda}$ are discussed. More recent results on operators of this form can be found in the work of Arzt [Arz15]. A next big step would be to combine the work done on these operators with the results on atomless measures obtained in Chapter 2. There is a reasonable hope to be able to obtain further results on the spectral properties of the operators $\Delta_{\mu,\Lambda}$ and probably being even able to solve the eigenvalue problem completely or at least give new results on spectral aspects like the asymptotic behaviour of the eigenvalue counting function.

As the extension from atomless to purely atomic and mixed measures was made here for the operators $\Delta_{\nu,\nu}$, one could also try to do so for the operators $\Delta_{\mu,\Lambda}$ or $\Delta_{\mu,\nu}$ in general. As the theory for δ- and η-Laplacians indicates, one needs to analyse if the operator $\Delta_{\mu,\nu}$ should be defined as $\nabla_\mu \circ \nabla_\nu$ or as $-\nabla_\mu^* \circ \nabla_\nu$ when the measures μ and ν have atomic parts. In general, it is unclear if both approaches lead to derivative operators. Further, we expect that in general the operators do not coincide and may behave very differently. One important assumption which needs to be made in order to obtain well-defined operators is the condition on the supports of the measure discussed above, namely $\operatorname{supp}(\mu) \subseteq \operatorname{supp}(\nu)$. The further analysis of these Kreĭn-Feller operators might be an important step in order to use measure-geometric differential operators to the differential equations on fractals or other irregular structures.

Appendices

APPENDIX A

Basics On Measure Theory

We give the definitions for monotonic sequences of sets and signed measures, as well as the convolution of measures, and obtain some basic properties of them.

A.1 Sequences of sets and signed measures

We introduce monotonic sequences of sets and signed measures in order to state a lemma we use in the proof of Proposition 2.1.2. There is a great variety of books on measure theory and we refer the reader to [Els05] for the proofs of the results stated here and for a further discussion of the topic. We start with the definitions on converging and monotone sequences of sets, which can be found in [Els05, Chapter I.2].

Definition A.1.1. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a set X. The set

$$\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is called the *limes superior* and the set

$$\underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

the *limes inferior* of $(A_n)_{n\in\mathbb{N}}$. The sequence $(A_n)_{n\in\mathbb{N}}$ is said to be *convergent*, if

$$\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n,$$

and the common value is called the *limit* of $(A_n)_{n\in\mathbb{N}}$.

Definition A.1.2. A sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of a set X is said to be *monotonically increasing*, if $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and *monotonically decreasing*, if $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. If it is either monotonically increasing or monotonically decreasing, we say that $(A_n)_{n\in\mathbb{N}}$ is *monotone*.

If $(A_n)_{n\in\mathbb{N}}$ is monotonically increasing and converges to a set $A \subseteq X$, we write $A_N \nearrow A$, and if it is monotonically decreasing and converging to A, we write $A_N \searrow A$.

We now look at the definition of signed measure, as stated in [Els05, Definition VII.1.1].

Definition A.1.3. Let $\mathfrak{A}$ be an σ-algebra on $\mathbb{R}$ and $\overline{\mathbb{R}}$. A map $\mu\colon \mathfrak{A} \to \overline{\mathbb{R}}$ is a *signed measure* if it fulfils the following properties:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\mathfrak{A}) \subseteq [-\infty, \infty)$ or $\mu(\mathfrak{A}) \subseteq (-\infty, \infty]$

(iii) For every sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathfrak{A}$ and $A_m \cap A_n = \emptyset$, for all $m, n \in \mathbb{N}$ with $m \neq n$, it holds that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The following lemma on the convergence of the measure of monotone sequences is stated in proven in [Els05, Lemma VII.1.4].

Lemma A.1.4. *Let $\mu\colon \mathfrak{A} \to \overline{R}$ be a signed measure.*

(i) From $A_n \in \mathfrak{A}$ and $A_n \nearrow A$ follows: $\mu(A_n) \to \mu(A)$.

(ii) From $A_n \in \mathfrak{A}$ and $A_n \searrow B$ and $\mu(A_1) < \infty$ follows: $\mu(A_n) \to \mu(B)$.

A.2 Convergence and convolution of measures

Following [Els05, Chapter VIII.4.2], we define the *weak convergence* of positive, finite Borel measures.

Definition A.2.1. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of positive, finite Borel measures on X. We say that $(\mu_n)_{n\in\mathbb{N}}$ *converges weakly* to a positive finite Borel measures μ on X, if, for all $f \in C_B(X)$, it holds that

$$\lim_{n\to\infty} \int_X f \, \mathrm{d}\mu_n = \int_X f \, \mathrm{d}\mu.$$

In this case we write

$$\operatorname*{w\text{-}lim}_{N\to\infty} \mu_n = \mu.$$

There exist several equivalent definitions to the one given here. For example in the Portmanteau theorem (see [Els05, Theorm VIII.4.10]) some other characterisations of weak convergence of measures can be found.

For the definition of the convolution of two measures we follow [BF75, Definition 1.8] and adapt it to our situation. In the book [BF75], further results on the convolution of measures are discussed.

Definition A.2.2. Let μ and ν be positive Radon measures on $\mathbb{R}$. The convolution of μ and ν exists, if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) < \infty,$$

for all $f \in C_c^+(\mathbb{R})$. If the convolutions of μ and ν exists, the mapping

$$f \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y)$$

of $C_c(\mathbb{R})$ into $\mathbb{C}$ defines a positive measure on $\mathbb{R}$ called the *convolution of μ and ν* and denoted by $\mu * \nu$.

If the convolution of μ and ν exists so does convolution of ν and μ and $\mu * \nu = \nu * \mu$.

In Chapters 3 and 4, we discuss convolutions of the form $\mu * \delta_{\mathbb{Z}}$, where μ is a Borel measure with $\mathrm{supp}(\mu) \subseteq [0,1]$ and $\delta_{\mathbb{Z}} = \sum_{k\in\mathbb{Z}} \delta_k$. In this case holds the following equality, for every Borel set B and $l \in \mathbb{Z}$.

$$\begin{aligned}
\mu * \delta_{\mathbb{Z}}(B+l) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{B+l}\,(x+y)\, \mathrm{d}\delta_{\mathbb{Z}}(x)\, \mathrm{d}\mu(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{(B-y)+l}\,(x)\, \mathrm{d}\delta_{\mathbb{Z}}(x)\, \mathrm{d}\mu(y) \\
&= \int_{\mathbb{R}} \delta_{\mathbb{Z}}((B-y)+l)\, \mathrm{d}\mu(y) = \int_{\mathbb{R}} \delta_{\mathbb{Z}}(B-y)\, \mathrm{d}\mu(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(x+y)\, \mathrm{d}\delta_{\mathbb{Z}}(x)\, \mathrm{d}\mu(y) \\
&= \mu * \delta_{\mathbb{Z}}(B)
\end{aligned}$$

Fix $a, b \in \mathbb{R}$ with $a < b$ and let μ be a Borel measure with $\mathrm{supp}(\mu) \subseteq [a,b]$. A similar argument to the above yields, for $B \in \mathfrak{B}$ and $l \in \mathbb{Z}$, that

$$\mu * \delta_{(b-a)\mathbb{Z}}\,(B + (b-a)l) = \mu * \delta_{(b-a)\mathbb{Z}}\,(B).$$

APPENDIX B

Dirichlet Forms

The book [FOT94] provides a full introduction to Dirichlet forms. We mainly follow [FOT94, Chapter 1] and refer the reader to this chapter for further details and the proofs of the statements given.

Definition B.1. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and let $\mathscr{D}$ be a dense subspace of H. Then $\mathrm{E}\colon \mathscr{D} \times \mathscr{D} \to \mathbb{R}$ is a *Dirichlet form on* $\mathscr{D}$ if it fulfils the following properties:

(i) E is a non-negative symmetric bilinear form.

(ii) The set $\mathscr{D}$ equipped with the inner product defined by $\langle u, v \rangle_{\mathrm{E}} \coloneqq \langle u, v \rangle_H + \mathrm{E}(u, v)$ is a real Hilbert space.

(iii) For any $u \in \mathscr{D}$, the function

$$\hat{u}(x) \coloneqq \begin{cases} 1 & \text{if } u(x) > 1, \\ 0 & \text{if } u(x) < 0, \\ u(x) & \text{otherwise,} \end{cases}$$

lies in $\mathscr{D}$ with $\mathrm{E}(\hat{u}, \hat{u}) \leq \mathrm{E}(u, u)$.

The property (iii) is known as the *Markov property* and it can be replaced by equivalent conditions, which are extensively studied in [FOT94, Chapter 1].

By definition, the set $\mathscr{D}$ equipped with $\langle \cdot, \cdot \rangle_{\mathrm{E}}$ is an inner product space and hence, it is sufficient to show that $\mathscr{D}$ is complete with respect to the norm induced by this inner product in order to prove property (ii).

APPENDIX C

Basics On Operator Theory

We give an overview on some basic definitions and results on bounded and unbounded operators. We mainly follow the books [RS80, Rud91, Alt99] and give more detailed references in the text.

Throughout this chapter, we let H denote a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Note that the theory can be developed equivalently for real Hilbert spaces, but following the literature we referred to, we assume that H is a complex Hilbert space.

C.1 Bounded operators

We start with some definitions and fundamental results on bounded operators and refer the reader for the proofs to [RS80, Chapter VI], [Rud91, Part I - Chapter 4] and [Alt99, Chapter 8 and 9].

Let $\mathcal{B}(H)$ denote the set of bounded linear operators $T\colon H \to H$, namely the set of linear mappings T for which there exists a constant K such that for all $h \in H$ holds $\|T(h)\| \leq K\|h\|$.

The *spectrum* of an operator $T \in \mathcal{B}(H)$ is defined as the set

$$\sigma(T) := \{\lambda \in \mathbb{C}\colon\ T - \lambda \mathrm{id}\colon H \to H \text{ is not invertible}\}$$

and the *resolvent set* of T is the complement of the spectrum, namely $\rho(T) = \mathbb{C} \setminus \sigma(T)$. An element $f \in H$ is called an *eigenfunction of* T (or, depending on the context, *eigenvector of* T) if there exists a scalar $\lambda \in \mathbb{C}$ such that

$$T(f) = \lambda f.$$

This equation is called the *eigenvalue equation for* T and the scalar λ an *eigenvalue of* T *with corresponding eigenfunction* f. Every eigenvalue of T is element of $\sigma(T)$. Since T is a linear operator, if f and g are eigenfunctions of T with the same eigenvalue λ, then $af + bg$ fulfils the eigenfunction equation for T for all $a, b \in \mathbb{C}$ and the same scalar λ. Hence, to every

eigenvalue λ there exists a linear subspace of H, the so-called *eigenspace of* λ, such that all functions in that space satisfy the eigenvalue equation for T for λ. We denote the eigenspace of the eigenvalue λ by $\mathrm{Eig}(\lambda)$ and refer to the (topological) dimension of $\mathrm{Eig}(\lambda)$ as *multiplicity of* λ. If an eigenvalue has multiplicity one, we call it *simple*.

An operator $T \in \mathcal{B}(H)$ is said to be *compact* if the closure of $T(U)$ is compact, where U denotes the open unit ball in H. This definition is equivalent to the sequence $(T(h_n))_{n\in\mathbb{N}}$ having a convergent subsequence, for any bounded sequence $(h_n)_{n\in\mathbb{N}}$ in H. This fact will be used in the proof of Theorem 4.1.21 and for a proof we refer to [Rud91, Theorem A4].

The *adjoint* of an operator $T \in \mathcal{B}(H)$ is the operator $T^* \in B(H)$ for which, for all $h_1, h_2 \in H$, holds

$$\langle T(h_1), h_2\rangle = \langle h_1, T^*(h_2)\rangle.$$

The adjoint is well-defined by the virtue of Riesz Representation theorem. The operator $T \in \mathcal{B}(H)$ is said to be *self-adjoint* if $T^* = T$. Note that the spectrum of a self-adjoint operator is fully contained in $\mathbb{R}$.

The following theorem is often referred to as the *Spectral Theorem for Compact Self-Adjoint Operators* and proofs can be found in [Alt99, Satz 10.12] and in [Wer07, Theorem VI.3.2]. The latter is the reference we follow for the statement of the theorem.

Theorem C.1.1. *Let $T \in \mathcal{B}(H)$ be a compact and self-adjoint operator. Then there exists a, possibly finite, orthonormal system $e_1, e_2, \ldots \in H$ and a sequence $\lambda_1, \lambda_2, \ldots \in \mathbb{C}\setminus\{0\}$, either finite or converging to zero, such that*

$$H = \ker T \oplus \overline{\mathrm{span}\{e_n : n \in \mathbb{N}\}}$$

and, for all $h \in H$, holds

$$T(h) = \sum_{n=1}^{\infty} \lambda_n \langle h, e_n\rangle e_n.$$

Furthermore, the scalars λ_n are exactly the eigenvalues of T which are different from zero, and hence they are real-valued. The elements $e_n \in H$ are the normalised eigenvectors corresponding to λ_n.

C.2 Unbounded operators

We now discuss unbounded operators on H and introduce some important objects and discuss properties of them. For this part, we mainly follow [RS80, Chapter VIII], in which one finds the proofs of the stated results. In the following T denotes a densely defined linear operator on H and $\mathrm{dom}(T)$ is its domain.

The *graph* of T is defined to be the set

$$\Gamma(T) := \{(f, T(f)) \in H \times H : f \in \mathrm{dom}(T)\}.$$

If $\Gamma(T)$ is a closed subset of $H \times H$, we say that T is *closed*. If T_1 is a densely defined operator on H and if $\Gamma(T_1) \supseteq \Gamma(T)$, then T_1 is called an *extension* of T. When T has a closed extension, T is said to be *closable*. The smallest closed extension of T, which we denote by $\overline{T}$, is called the *closure* of T.

We let $\mathrm{dom}(T^*)$ be the set of $h \in H$ for which there exists $g \in H$ with

$$\langle T(f), h\rangle = \langle f, g\rangle,$$

for all $f \in \mathrm{dom}(T)$. For each such $h \in \mathrm{dom}(T^*)$, we define $T^*(h) \coloneqq g$. We refer to the linear operator T^* as the *adjoint* of T.

We call T *symmetric* if $\mathrm{dom}(T) \subseteq \mathrm{dom}(T^*)$ and $T(h) = T^*(h)$ for all $h \in \mathrm{dom}(T)$. Equivalently, T is symmetric if and only if $\langle T(h_1), h_2\rangle_\eta = \langle h_1, T(h_2)\rangle_\eta$ for all $h_1, h_2 \in \mathrm{dom}(T)$. If, in addition to T being symmetric, we have that $\mathrm{dom}(T) = \mathrm{dom}(T^*)$, then we say that T is *self-adjoint*.

The following important result on unbounded and densely defined operators is crucial in the proofs of Proposition 4.1.12 and Theorem 4.1.20 and we refer for its proof to [RS80, Theorem VIII.1].

Theorem C.2.1. *If T is an unbounded, densely defined operator on H, then the following statements hold.*

1. *The operator T^* is closed.*
2. *The operator T is closable if and only if* $\mathrm{dom}(T^*)$ *is dense in H in which case $\overline{T} = T^{**}$.*
3. *If T is closable, then $(\overline{T})^* = T^*$.*

The *resolvent set* of T, denoted by $\rho(T)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda\mathrm{id}$ is a bijective mapping from $\mathrm{dom}(T)$ to H whose inverse belongs to $\mathcal{B}(H)$. Like for bounded operators, the *spectrum* of T is then defined to be the complement of the resolvent set and it is denoted by $\sigma(T)$. For every $\lambda \in \rho(T)$ one defines the *resolvent operator* $R_T(\lambda) \coloneqq (T - \lambda\mathrm{id})^{-1}$, which by definition is bounded and linear. We say that T has *compact resolvent* if there exists $\lambda \in \rho(T)$ such that $R_T(\lambda)$ is a compact operator.

The terms *eigenfunction*, *eigenvalue*, *eigenvalue equation* and *eigenspace* are defined and denoted similarly as for bounded operators. Again, the eigenvalues are elements of $\sigma(T)$.

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List of Symbols

Symbol	Descripton, page
$\mathbb{1}_A$	characteristic function of the set A, 9
Δ_μ	μ-Laplace operator, 19, 46, 73
∇_μ	μ-derivative, 15, 42, 63
A_δ	matrix representation of the δ-derivative, 44
$\mathfrak{B}$	Borel σ-algebra, 10
B_δ	matrix representation of the δ-Laplace operator, 48
$\mathbb{C}$	set of complex numbers, 9
C	set of continuous functions, 9
C_B	set of bounded continuous functions, 9
C_c	set of compactly supported continuous functions, 9
C^k	set of k-times continuously differentiable functions, 9
C^∞	set of smooth functions, 9
C_c^∞	set of test functions, 9
$\mathscr{D}_\mu^1$	set μ-differentiable functions, 14, 40, 62
$\mathscr{D}_\mu^2$	set of twice μ-differentiable functions, 18, 46, 73
$\mathscr{D}_{\nu,D}^2$	set of twice ν-differentiable functions fulfilling Dirichlet boundary conditions, 21
$\mathscr{D}_{\nu,N}^2$	set of twice ν-differentiable functions fulfilling von Neumann boundary conditions, 21
$\mathscr{D}_{\nu,P}^2$	set of twice ν-differentiable functions fulfilling periodic boundary conditions, 21
$\mathcal{E}$	energy-form, 45, 71
$\mathrm{Eig}(\lambda)$	eigenspace of the eigenvalue λ, 107
F_μ	distribution function of the measure μ, 10
F_μ^{lc}	left-continuous distribution function of the measure μ, 10

Symbol	Descripton, page
$\check{F}_\nu^{-1}$	pseudoinverse of the distribution function F_ν, 26
$\mathcal{H}_\mu$	set of μ-harmonic functions, 21, 50, 74
$\mathfrak{L}^2_\mu$	set of real-valued square-μ-integrable functions, 11
L^2_μ	set of equivalence classes of $\mathfrak{L}^2_\mu$-functions, 11
$\mathbb{N}$	set of natural numbers, 9
$\mathbb{N}_0$	set of non-negative integers, 9
N_μ	eigenvalue counting function of Δ_μ, 30, 83, 89
$\mathcal{N}_\mu$	set of $\mathfrak{L}^2_\mu$-functions which are constant zero μ-almost everywhere, 11
$\mathbb{Q}$	set of rational numbers, 9
$\mathbb{R}$	set of real numbers, 9
$\overline{\mathbb{R}}$	extended real line, 9
$\mathbb{Z}$	set of integers, 9
z_i	position of the i-th atom, 39, 61
α_i	weight of the i-th atom, 39, 61
Γ	special density function, 64
δ	purely atomic measure, 39
δ_z	Dirac point mass on z, 10
δ_S	sum of Dirac point masses on the elements of S, 10
η	mixed measure, 61
Λ	Lebesgue measure, 10
ν	atomless measure, 13

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