Homogenization of Thermoelasticity Systems Describing Phase Transformations

von Michael Eden

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Gutachter: Prof. Dr. habil. Michael Böhm (Universität Bremen)
Prof. Dr. habil. Adrian Muntean (Karlstad Universitet)

Abstract

This thesis is concerned with the mathematical homogenization of thermoelasticity models with moving boundary describing solid-solid phase transformations occurring in highly heterogeneous, two-phase media.

In the first part of this thesis, existence and uniqueness of weak solutions are established under the assumption that the changes in the geometry, which are due to the moving boundary, are given a priori. This is achieved after a transformation of coordinates to a fixed referential geometry. In addition, uniform a priori estimates are provided. Via an argument utilizing the concept of two-scale convergence, a corresponding homogenized model with distributed time and space dependent microstructures is derived. Quantitative error estimates measuring the accuracy and efficacy of the homogenized model are investigated. While such estimates seem not to be obtainable in the fully coupled setting, optimal convergence rates are proven for some special scenarios where the coupling mechanisms between the mechanical part and the heat part are simplified.

In the second part, a more general scenario, in which the geometric changes are not assumed to be prescribed at the outset, is considered. Starting with the normal velocity of the interface separating the competing phases, a specific transformation of coordinates, the so-called *Hanzawa transformation*, is constructed. This is achieved by (*i*) solving a non-linear system of ODEs characterizing the motion of the interface and (*ii*) using the *Implicit Function Theorem* to arrive at the height function parametrizing this motion. Based on uniform estimates for the functions related to the transformation of coordinates, the strong two-scale convergence of these functions is shown. Finally, these results are used to establish the corresponding homogenized model.

Zusammenfassung

Diese Arbeit befasst sich mit der mathematischen Homogenisierung von Thermo-Elastizitäts-Modellen mit beweglichen Rändern zur Beschreibung von fest-fest Phasentransformationen in äußerst heterogenen zwei-Phasen-Medien.

Im ersten Teil der Arbeit werden Existenz und Einzigkeit einer schwachen Lösung unter der Annahme gezeigt, dass die sich aus den Phasentransformationen ergebenden Geometrieänderungen a priori bekannt sind. Dies gelingt nach einer Koordinatentransformation zu einer festen Referenzgeometrie. Zusätzlich werden gleichmäßige a-priori-Abschätzungen gewonnen. Mittels des Konzeptes der Zwei-Skalen-Konvergenz wird das entsprechende hochskalierte Modell mit verteilten zeit- und ortsabhängigen Mikrostrukturen hergeleitet. Um die Genauigkeit und Effizienz des homogenisierten Problems einzuschätzen, werden quantitative Fehlerabschätzungen untersucht. Auch wenn es so scheint als wenn diese für das vollständig gekoppelte Problem nicht gezeigt werden können, werden solche optimalen Konvergenzraten in speziellen Situationen, in welchen Vereinfachungen bei der Kopplung zwischen der Mechanik und der Wärmeleitung angenommen werden, nachgewiesen.

Im zweiten Teil wird der allgemeinere Fall betrachtet in welchem die Geometrieänderungen nicht als im voraus bekannt vorausgesetzt werden. Mit einer Funktion, die die Normalengeschwindigkeit der Grenzfläche, welche die konkurrierenden Phasen trennt, beschreibt wird eine spezielle Koordinatentransformation, die so genannte *Hanzawa*-Transformation, konstruiert. Hierbei wird zunächst ein nicht-lineares die Bewegung der Grenzfläche charakterisierendes ODE-System gelöst, um dann mit Hilfe des *Satzes über implizite Funktionen* die Existenz einer parametrisierenden Höhenfunktion herzuleiten. Basierend auf zudem gewonnen a-priori-Abschätzungen für die Koordinatentransformation wird weiterhin die starke zwei-Skalen-Konvergenz einiger zugehöriger Funktionen gezeigt. Diese Resultate werden dann abschließend genutzt, um das zugehörige homogenisierte Modell herzuleiten.

Acknowledgements

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I would also like to take the opportunity to extend my thanks to the colleagues at the mathematical department at Karlstad universitet: in the overall six weeks that I was able to work there, I genuinely felt at home and learned a lot – for example, the cultural importance of *fika*.

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Contents

List of Figures

List of Symbols

$a \cdot b$	Scalar product between vectors a and b
$A : B$	Frobenius product between matrices A and B
$a \otimes b$	Tensor product of vectors a and b
$\det(A)$	Determinant of matrix A
$\|a\|$	Euclidean norm of vector a or matrix A
a^T, A^T	Transpose of vector a or matrix A
e_i	The i-th unit vector in $\mathbb{R}^n$
$\mathbb{I}_3$	The unit matrix in $\mathbb{R}^{3\times 3}$
$\\|v\\|_V$	Norm of v in the normed space V
$\\|v\\|_\infty$	the sup or the essential sup norm
V'	Topological dual of a vector space V
$\langle F, v\rangle_{V'V}$	Dual pairing between $F \in V'$ and $v \in V$
$(u, v)_H$	Inner product of $u, v \in H$
∂_t, ∂_i, ∂_{x_i}	Partial derivative with respect to the indicated variable
∇, ∇_x, ∇_y	Gradient operator with respect to the indicated spatial variable
div, div_x, div_y	Divergence operator with respect to the indicated spatial variable
$e(\cdot)$, $e_x(\cdot)$, $e_y(\cdot)$	Linearized strain tensor with respect to the indicated spatial variable
D, D_x, D_y	Overall spatial derivative with respect to the indicated spatial variable
∇_Γ, div_Γ	Surface gradient and surface divergence with respect to a hypersurface Γ
$\mathrm{dist}(x, M)$, $\mathrm{dist}(M_1, M_2)$	Distance between point x and set M and distance beteen two sets M_1 and M_2
$B_r(x)$	Open ball with center x and radius r
$\overline{B}_r(x)$	Closed ball with center x and radius r
λM, $M + x$	The sets $\{\lambda m \,:\, m \in M\}$ and $\{m+x \,:\, m \in M\}$, respectively
$\|M\|$, $\|\Sigma\|$	The Lebesgue measure of a set $M \subset \mathbb{R}^3$, the surface measure of a hypersurface $\Sigma \subset \mathbb{R}^3$
n_Γ	Normal vector field of a surface $\Gamma \subset \mathbb{R}^n$
H_Γ	Mean curvature function of a surface $\Gamma \subset \mathbb{R}^n$

$\kappa_{\Gamma,j}$	Principal curvature functions of a surface $\Gamma \subset \mathbb{R}^n$, $j \in \{1, n-1\}$
$a^{(1)}, a^{(2)}$	Exterior and interior ball radii
L_Γ	Shape tensor field/*Weingarten* map of a surface $\Gamma \subset \mathbb{R}^n$
U_Γ	Tubular neighborhood of a surface $\Gamma \subset \mathbb{R}^n$
W_Γ	Generalized neighborhood of a surface $\Gamma \subset \mathbb{R}^n$
d_Γ	Signed distance function
P_Γ	Projection operator
$\mathbb{1}_M$	Indicator function of a set M
id	Identity operator
$f_{\vert M}$	Restriction of the function f to the set M
$\mathrm{supp}(f)$	Support of the function f
$[\![u]\!]$	Jump across a surface
Y	Unit cell $Y = (0,1)^3$
$d_{jk} = (y_j\delta_{1k}, y_j\delta_{2k}, y_j\delta_{3k})^T$	Auxiliary function for the formulation of cell problems, $j,k = 1,2,3$
$[x]$	Unique $k \in \mathbb{Z}^3$ such that $x - k \in (0,1)^3$
$\{x\}$	$x - [x]$
$\stackrel{2}{\rightharpoonup}$	Two-scale convergence
$\stackrel{2}{\to}$	Strong two-scale convergence
$\mathcal{T}_\varepsilon f = [f]^\varepsilon, \mathcal{T}_{\Gamma_\varepsilon} f_b = [f_b]^\varepsilon$	Periodic unfolding of f, periodic boundary unfolding of f_b
$[f]_\varepsilon$	Folding of function f
δ_{ij}	Kronecker delta
$\mathcal{O}$	Landau symbol

CHAPTER 1

Introduction

The ever increasing demand for mechanically engineered, complex materials like metallic alloys, ceramics, or composites calls for an improved understanding of the underlying physical and chemical processes involved in the manufacturing of these materials. Generally speaking, the properties of a particular material are not fully described only by their chemical composition, i.e., the constituents along with their relative numbers; we also have to account for their structure, i.e., the specific geometric arrangements of the constituents. Changes in that structure, which are usually described as a type of *phase transformations* and which can occur as the result of, for example, changes in temperature or internal stresses, are important processes in the manufacturing of materials.

Often, the spatial scale at which we can observe these structures and their changes is of several orders of magnitude below the size of the materials themselves – in such cases, we refer to these structures as *microstructures* of a *macroscopic* object.[1] Microstructural changes in solid materials are usually accompanied by mechanical effects like local stresses or dislocations which can profoundly influence the properties of the macroscopic object ([Voo04]); see also Section 1.1 for a concrete example. This observation led to the following question motivating the mathematical research of this thesis:

Underlying research question

How can we identify and effectively describe the way in which phase transformations at the microscale influence the macroscopic properties of materials?

To tackle this question, two separate tasks, which lead to different mathematical problems with distinct challenges, have to be considered:

(*i*) First, we have to model the mechanisms of the phase transformations. This is done in the context of *moving boundary problems.*

[1]In the mathematical community, the meaning of the quantifiers *micro* and *macro* is a bit looser then in the physics and engineering community where *atomic*, *nano*, or *meso* are also used and where these quantifiers have fixed ranges. In this work, we use micro and macro only to distinguish between two different length scales.

(*ii*) Second, we have to connect the different spatial scales. Here, we rely on the theory of *mathematical homogenization*.

Moving boundary problems are a special class of nonlinear problems – usually given in the form of partial differential equations – where the (continuous) evolution of a boundary with respect to time is considered and where this evolution is not known at the outset, i.e., determining the precise boundary evolution is part of the mathematical problem. In the case of phase transformations, this boundary represents the contact surface between different phases. Here, the growth of one phase at the expanse of another is the driving force behind this boundary evolution. Due to their intrinsic nonlinear structure, these problems are mathematically challenging and they are actively researched; see, e.g., [CS05, PS16].

In problems that exhibit different scales, it is not feasible to resolve them numerically. However, to the degree that the effects on the microscale are important in describing the macroscopic behavior, ignoring the microscale is also counterproductive. As a consequence, there is a general interest in establishing *effective* models in a trade-off between the two mutually exclusive goals of:

(*i*) *Accuracy*: The model must be as accurate and precise as possible and, as a consequence, be mindful of the microstructure and/or potential micro effects.

(*ii*) *Efficacy*: The model must be simple enough so that it allows for efficient numerical simulations.

In the context of *mathematical homogenization*, such effective models are derived via some specific limit analysis in the framework of *singular perturbed problems* – a procedure which can be interpreted as some sort of averaging. As general references, see [PS08, Tar10].

By proposing the analysis and homogenization of mathematical models describing phase transformations at the microscale, we aim to combine methods from the analysis of moving boundary problem and mathematical homogenization. This leads to several significant mathematical challenges including:

- *Estimates:* The motion of the phase boundary needs to be sufficiently regular and satisfy estimates that are uniform with respect to the scale parameter.
- *Convergence:* As we are considering singular perturbed problems, strong convergence of the involved functions can not be expected and, as the problem is non-linear, weak convergence is not sufficient to pass to the limit.
- *Errors/Correctors:* When measuring the accuracy of the homogenized model, appropriate correctors have to be identified together with certain additional regularity conditions that have to be met.

It is worth noting that, in the context of homogenization, there already are some results for stationary moving boundary problems – often referred to as *free boundary problems*

– like different kinds of obstacle problems [KS14, KS16] or flame propagation [CLM06] but less seems to be known for non-stationary moving boundary problems. Some partial results were obtained in the context of front propagation [BCN11, LS05] or for the chemical degradation of concrete [Pet09].

This thesis is organized as follows:

- In Chapter 2, we introduce the notations specific to this thesis and collect mathematical tools regarding hypersurfaces, coordinate transforms, and homogenization that are frequently used throughout this work.
- In Chapter 3, we propose a mathematical model to describe phase transformations in a thermoelasticity setting for highly heterogeneous two-phase media. This model – and slight variations thereof – are the focus of Chapter 4, Chapter 5, and Chapter 6.
- In Chapter 4, we consider the two-phase thermoelasticity model proposed in Chapter 3 while assuming the a priori knowledge of the geometry changes. For this setting, we study the analysis and homogenization of the problem. Most results of this chapter, Section 4.1 to Section 4.4, are published in [EM17b].
- In Chapter 5, we investigate quantitative error estimates regarding the homogenization procedure outlined in Chapter 4. While comprehensive estimates seem to not be obtainable in the fully-coupled case, explicit rates with respect to the scale parameter are proven under certain reasonable simplifications. These results are published in [EM17a].
- In Chapter 6, we consider a more general scenario where the changes in the geometry are not assumed to be prescribed at the outset. Given the normal velocity of the interface, we establish the corresponding geometry changes and a coordinate transform describing these changes. In addition, we investigate the limit behavior of the functions related to the transformation.
- Finally, in Chapter 7, we briefly summarize our main results and present an outlook for possible future work. In particular, we point out the remaining claims still needed to be proven for a complete treatment of the homogenization of the full moving boundary problem.

1.1 Example: Bainitic phase transformation in steel

Steel, which is an alloy of iron, carbon, and, to a lesser extent, other elements like chromium or nickel, is a prime example of a complex material where the macroscopic properties of the material are highly dependent on the underlying microstructures and on the configuration of different phases. The interplay of processes taking place on different scales in this material is still not sufficiently understood.

Cooling or heating of a steel induces phase transformations which not only influence the macroscopic mechanical properties of the steel but might also induce macroscopic plastic effects. As an example of such a phase transformation, we are particularly interested in the formation of *Bainite* from *Austenite*. *Austenite* is a particular phase of steel where the iron atoms are arranged in a *face-centered cubic* configuration and, depending on the carbon concentration of the particular steel, it is the state of matter of steel in a temperature range of 727℃ up to 1450℃.

Cooling *Austenite* steel into the range of 250℃ up to 550℃ evokes a transformation to *Bainitic* steel, [BH06].[2] In this temperature range, diffusion of carbon, which is an important factor in many transformations in steel, is comparably slow. The Austenite to Bainite transformation is assumed to be partially diffusive – via diffusion of carbon – and partially displacive – via instantaneous, coordinated movement of groups of atoms. The overall process, however, is still not sufficiently understood, cf. [Fie13], which is why we are interested in this specific example. The displacive part is naturally associated with local stresses that are assumed to induce macroscopic plastic effects like *transformation-induced plasticity* (TRIP) ([WBDH08]).

In the context of this specific scenario, the leading research question is:

Research question in the case of Bainite formation

Starting off from a mathematical model describing the formation of *Bainite* and using the framework of mathematical homogenization, is it possible to identify and understand the link to the TRIP-effect as the macroscopic average (in some sense) of the local mechanical effects accompanying the formation of *Bainite*?

[2] Higher temperatures will lead to *Pearlite* and lower temperatures to *Martensite*.

CHAPTER 2

Mathematical preliminaries

In this chapter, we collect some tools which we frequently use in this thesis. After introducing some notations and definitions, we present, in Section 2.2, concepts and results regarding hypersurfaces of $\mathbb{R}^n$. In Section 2.3, we introduce coordinate transforms in the form of motions and related results (transport theorems and transformation formulas). Here, Lemma 2.9 is of particular importance as it ensures the existence of a coordinate transform in our specific setting. Finally, in Section 2.4, we give the basic results on two-scale convergence and its relationship with periodic unfolding.

2.1 Notations and definitions

Let $V = (V, \|\cdot\|)$ be a real Banach space. Its topological dual V' is defined as the space of linear and continuous operators $F : V \to \mathbb{R}$ and, when equipped with the corresponding operator norm, V' is itself a Banach space. The dual pairing between $F \in V'$ and $v \in V$ is defined via $\langle F, v\rangle_{V'V} = F(v)$. The inner product of a Hilbert space H, we denote by $(\cdot,\cdot)$ with two exceptions: we use the dot symbol for the inner product in $\mathbb{R}^3$ and the double dot symbol for the Frobenius inner product in $\mathbb{R}^{3\times 3}$.

For functions that depend on time and on two spatial variables, as in $v\colon S\times\Omega\times Y \to \mathbb{R}^n$, we denote the derivatives of v with respect to the different variables with a subscript t, x, and y indicating differentiation with respect to time, the first, and the second spatial variable, respectively. That is, $\partial_t v\colon S\times\Omega\times Y \to \mathbb{R}^n$ denotes the derivative with respect to time, $D_x v$, $D_y v\colon S\times\Omega\times Y \to \mathbb{R}^{n\times 3}$ denote the overall derivative with respect to the respective space variables, and $D_{(x,y)}v\colon S\times\Omega\times Y \to \mathbb{R}^{n\times 6}$ denotes the overall spatial derivative. For functions that only depend on one spatial variable, we set $D = D_x$. Analogously, we also introduce nabla operators (∇, ∇_x, ∇_y), divergence operators (div, div_x, div_x) as well as linearized strain tensors ($e(\cdot) = {}^1\!/_2(D\cdot + (D\cdot)^T)$, $e_x(\cdot)$, $e_y(\cdot)$). In Section 2.2, we also introduce surface derivatives (∇_Γ, div_Γ). Finally, we indicate composition of different operators as a product, e.g., $\partial_t D_x$, and composition of the same operator as exponentiation, e.g., D_y^2.

Regarding the function spaces that appear in this work (e.g., Lebesgue, Sobolev, and Bochner spaces), we use the notation as introduced in [AF03, List of Spaces and Norms, p. xii] with a few notable exceptions:

- For open sets $S \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^3$ and for $k, l \in \mathbb{N}$, we set

$$W^{(k,l),\infty}(S \times \Omega) \\ = \left\{ u \in L^\infty(S \times \Omega) : \partial_t^i u, D_x^j u \in L^\infty(S \times \Omega)\ (1 \leq i \leq k,\ 1 \leq j \leq l) \right\}.$$

- In Section 2.2, we introduce $L^p(\Gamma)$ and $W^{l,p}(\Gamma)$ as Lebesgue and Sobolev spaces for surfaces.

- We use the number sign (#) subscript to indicate periodicity: for $Y = (0,1)^3$, we set

$$C_\#(Y) := \{u \in C(\mathbb{R})\ :\ u(y) + u(y + e_j) \text{ for all } y \in \mathbb{R} \text{ and } j = 1, 2, 3\}.$$

  Similarly, we take $C_\#^\infty(Y)$, $W_\#^{1,2}(Y)$, and $L_\#^2(Y)$.

We point out some specific notations which are consistently used throughout the thesis:

- A subscript ε denotes dependency on the scale parameter $\varepsilon > 0$.

- Starting in Section 2.2, superscripts (i) $(i = 1, 2)$ always indicate affiliation to the corresponding phase.

- Only in Chapter 3, subscripts *kin* and *c* refer to the *kinematic* and and the *current* configuration

- Starting in Section 4.2, a superscript r indicates a transformation to the reference configuration.

- Starting in Section 4.4, a superscript h refers to a homogenized quantity.

2.2 Hypersurfaces

In this work, particularly in Chapter 6, we repeatedly make use of some rudimentary concepts related to compact sub-manifolds of $\mathbb{R}^n$. For a detailed treatment of manifolds and hypersurfaces in the context of partial differential equations, we refer to [Aub82, PS16].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\Gamma \subset \mathbb{R}^n$ be the boundary of a domain $\Omega^{(2)} \subset \Omega$ (that is $\Gamma = \partial\Omega^{(2)}$) where $\overline{\Omega^{(2)}} \subset \Omega$. We set $\Omega^{(1)} = \Omega \setminus \overline{\Omega^{(2)}}$. The set $\Gamma \subset \mathbb{R}^n$ is called *C^k-hypersurface* $(k \geq 1)$ if, for every $\gamma \in \Gamma$, there is a $\delta > 0$ and a C^k-function

$\rho_\gamma \colon B_\delta(\gamma) \to \mathbb{R}$ such that[1]

(i) $\Gamma \cap B_\delta(\gamma) = \{x \in B_\delta(\gamma) \ : \ \rho_\gamma(x) = 0\}$,

(ii) $|\nabla \rho_\gamma(x)| \neq 0$,

(iii) $\rho_\gamma(x) < 0$ for all $x \in \Omega^{(2)} \cap B_\delta(\gamma)$.

This is to say that the surface can be locally represented as the zero set of a C^k-function. A hypersurface is called bounded or compact if it is bounded or compact as a subset of $\mathbb{R}^n$, respectively. A compact hypersurface whose boundary is empty is called closed.[2] In the following, we are only interested in closed C^2-hypersurfaces. In this setting, the normal vector fields and the curvature fields are well-defined.

The outer *unit normal vector field* $n_\Gamma \colon \Gamma \to \mathbb{R}^n$ is given via $n_\Gamma(\gamma) = \frac{\nabla \rho_\gamma(\gamma)}{|\nabla \rho_\gamma(\gamma)|}$ and the *mean curvature function* $H_\Gamma \colon \Gamma \to \mathbb{R}$ via $H_\Gamma(\gamma) = \frac{1}{n-1} \operatorname{div}\left(\frac{\nabla \rho_\gamma(\gamma)}{|\nabla \rho_\gamma(\gamma)|}\right)$. In addition, we introduce the *shape tensor* (sometimes referred to as *Weingarten map*) $L_\Gamma \colon \Gamma \to \mathbb{R}^{n \times n}$ via $L_\Gamma(\gamma) = D\left(\frac{\nabla \rho_\gamma(\gamma)}{|\nabla \rho_\gamma(\gamma)|}\right)$. Note that $L_\Gamma(\gamma)$ is symmetric and $L_\Gamma(\gamma) n(\gamma) = 0$ for all $\gamma \in \Gamma$. We have $(n-1)H_\Gamma(\gamma) = \operatorname{tr} L_\Gamma(\gamma)$. As continuous functions over a compact set, both $|H_\Gamma|$ and $|L_\Gamma|$ are bounded. More specifically, we have

$$|L_\Gamma(\gamma)| = \max\{|\kappa_{\Gamma,i}(\gamma)| \ : \ i \in \{1, ...n\}\}$$

where $\kappa_{\Gamma,j} \colon \Gamma \to \mathbb{R}$ ($j \in \{1, ..., n-1\}$) denote the functions of eigenvalues of L_Γ (so called *principal curvatures*). Note that these quantities are independent of the choice of $(\rho_\gamma)_{\gamma \in \Gamma}$, we refer to [PS16, Section 2.1].

The following result regarding C^2-hypersurfaces is important because it ensures the existence of tubular neighborhoods; see Figure 2.1.

Lemma 2.1. *Every closed C^2-hypersurface $\Gamma \subset \mathbb{R}^n$ satisfies uniform interior and exterior ball conditions. That is, there exist radii $r^{(i)} > 0$ such that, for every $\gamma \in \Gamma$, there are $x^{(i)} \in \Omega^{(i)}$ for which $B_{r^{(i)}}(x^{(i)}) \subset \Omega^{(i)}$ and $\overline{B}_{r^{(i)}}(x^{(i)}) \cap \Gamma = \{\gamma\}$.*

Proof. More generally, the statement holds true even in the case of $C^{1,1}$-regularity and, even stronger, the uniform ball condition implies $C^{1,1}$-regularity. See [Dal14, Theorem 1.8]. As a simple counterexample for a closed C^1-surface, take $\Gamma = [0,1]^2 \cap \{(x,y) \in \mathbb{R}^2 \ : \ y = |x|^{\frac{3}{2}}\}$. □

Let $\Gamma \subset \mathbb{R}^n$ be a closed C^2-hypersurface and let $s^{(i)}$ be the supremum over all possible

[1] In general, a hypersurface does not need to be the boundary of a domain but we are only interested in this type of hypersurface.

[2] Note that it is not sufficient for the set to be closed in $\mathbb{R}^n$ with the *closed upper hemisphere* – subset of $\partial B_1(0)$ where all coordinates are non-negative – being an obvious counterexample.

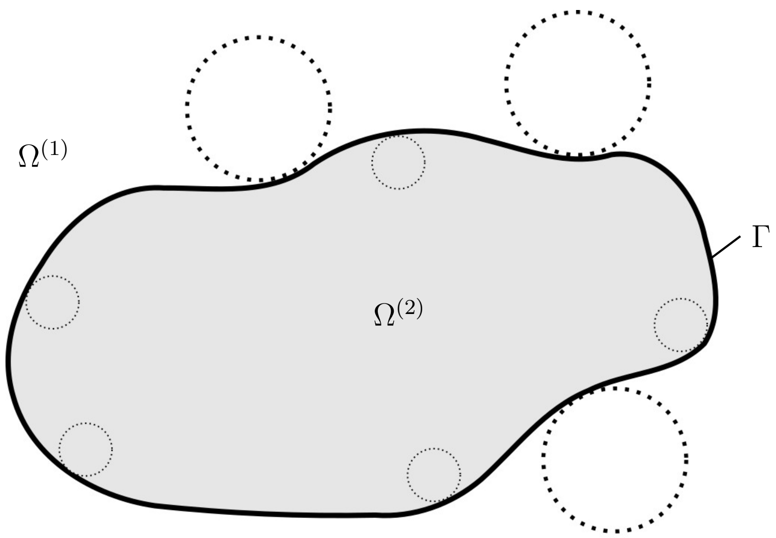


Figure 2.1: Interior and exterior uniform ball condition for a closed C^2-hypersurface Γ bounding a domain $\Omega^{(2)}$.

radii for which it satisfies the ball conditions,[3] i.e.,

$$s^{(i)} = \sup\{r > 0 \ : \ \text{for every } \gamma \in \Gamma \text{ there are } x^{(i)} \in \Omega^{(i)} \\ \text{such that } B_r(x^{(i)}) \subset \Omega^{(i)} \text{ and } \overline{B_r}(x^{(i)}) \cap \Gamma = \{\gamma\}\}. \quad (2.1)$$

The radii $s^{(i)}$ are connected to the principal curvatures of the surface Γ via the estimate

$$\min\{s^{(1)}, s^{(2)}\} \leq \min\{|L_\Gamma(\gamma)|^{-1} \ : \gamma \in \Gamma\}. \quad (2.2)$$

The curvature of Γ is not sufficient to determine this minimum[4] as it is also connected to the topology of Γ. For example, the domain $\Omega^{(2)}$ might be extremely narrow in some region or $\operatorname{dist}(\Gamma, \partial\Omega)$ might be small. For the outer radius, we have $s^{(2)} = \operatorname{dist}(\Gamma, \partial\Omega)$ for $\Omega^{(2)}$ convex.

For small $\delta > 0$, we set $a^{(i)} = (1 - \delta)\, s^{(i)}$ and introduce the function

$$\Lambda \colon \Gamma \times (-a^{(2)}, a^{(1)}) \to U_\Gamma \subset \mathbb{R}^n, \quad \Lambda(\gamma, s) = \gamma + s n_\Gamma(\gamma) \quad (2.3)$$

where U_Γ is a *tubular neighborhood* of Γ given as

$$U_\Gamma = \Gamma \cup \bigcup_{i=1,2} \underbrace{\left\{x \in \Omega^{(i)} \ : \ \operatorname{dist}(x, \Gamma) < a^{(i)}\right\}}_{=:U_\Gamma^{(i)}}.$$

Note that the factor $1 - \delta$ ensures that $\Omega^{(i)} \setminus U_\Gamma$ are non-empty, open sets. See also Figure 2.2.

[3]In general, Γ does not satisfy the uniform ball condition with radii $s^{(i)}$. Take the sphere $\partial B_R(0)$ of radius $R \in \mathbb{R}$ and center $0 \in \mathbb{R}^n$; it satisfies a uniform interior ball condition for all $r < R$ but not for $r = R$.

[4]That is, we do not expect equality in inequality (2.2).

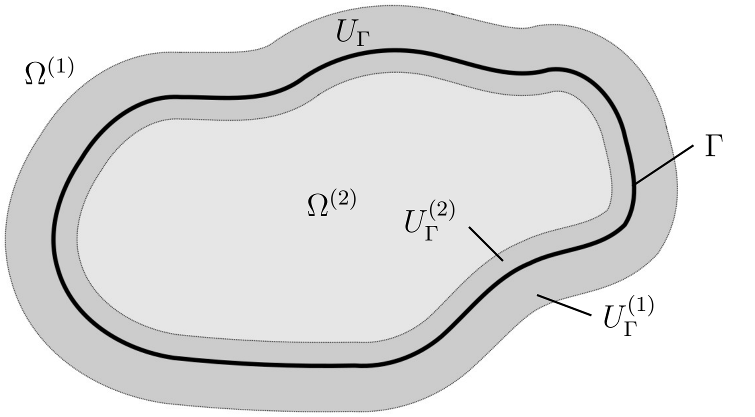


Figure 2.2: Tubular neighborhood U_Γ (dark grey area) and its interior and exterior part, $U_\Gamma^{(2)}$ and $U_\Gamma^{(1)}$, respectively, of a hypersurface Γ bounding a domain $\Omega^{(1)}$. Note that $U_\Gamma = \Gamma \cup U_\Gamma^{(1)} \cup U_\Gamma^{(2)}$.

Lemma 2.2. *Let $\Gamma \subset \mathbb{R}^n$ be a closed C^2-hypersurface for $k \geq 2$. Then, the function $\Lambda\colon \Gamma \times (-a^{(2)}, a^{(1)}) \to U_\Gamma$ as defined in line (2.3) is a C^1-diffeomorphism.*[5]

Proof. We refer to [PS16, Section 3.1, p.65]. □

We introduce the signed distance function

$$d_\Gamma\colon U_\Gamma \to (-a^{(2)}, a^{(1)}), \quad d_\Gamma(y) = \begin{cases} \operatorname{dist}(y,\Gamma), & y \in \Omega^{(1)} \\ -\operatorname{dist}(y,\Gamma), & y \in \overline{\Omega^{(2)}} \end{cases}$$

as well as the projection $P_\Gamma\colon U_\Gamma \to \Gamma$ to the nearest point on the surface. These functions allow for the representation of the inverse of Λ as

$$\Lambda^{-1}\colon U_\Gamma \to \Gamma \times (-a^{(2)}, a^{(1)}), \quad \Lambda(y) = (P_\Gamma(x), d_\Gamma(x))^T.$$

As a consequence, we can infer $d_\Gamma \in C^1(U_\Gamma, (-a^{(2)}, a^{(1)}))$ and $P_\Gamma \in C^1(U_\Gamma, \Gamma)$. It can be shown that the distance function inherits its regularity from the hypersurface.

Lemma 2.3. *Let Γ be a closed C^k-hypersurface ($k \geq 2$) with tubular neighborhood U_Γ. Then, $d_\Gamma \in C^k(U_\Gamma, (-a^{(2)}, a^{(1)}))$. The derivatives of d_Γ and P_Γ are given via*

$$\begin{aligned} Dd_\Gamma(x) &= \left(n_\Gamma\left(P_\Gamma(x)\right)\right)^T, \\ DP_\Gamma(x) &= M\left(P_\Gamma(x), d_\Gamma(x)\right)\left[\mathbb{I} - n_\Gamma(P_\Gamma(x)) \otimes n_\Gamma(P_\Gamma(x))\right] \end{aligned}$$

where

$$M\colon \Gamma \times (-a^{(2)}, a^{(1)}) \to \mathbb{R}^{3\times 3}, \quad M(\gamma, s) = \left(\mathbb{I} - sL_\Gamma(\gamma)\right)^{-1}.$$

Furthermore, for the second derivative of d_Γ, we have the implicit relation

$$D^2 d_\Gamma(x) = -L_\Gamma(P_\Gamma(x))(\mathbb{I} - d_\Gamma(x)L_\Gamma(P_\Gamma(x)))^{-1}.$$

[5]The function is bijective, and both s as well as its inverse s^{-1} are continuously differentiable.

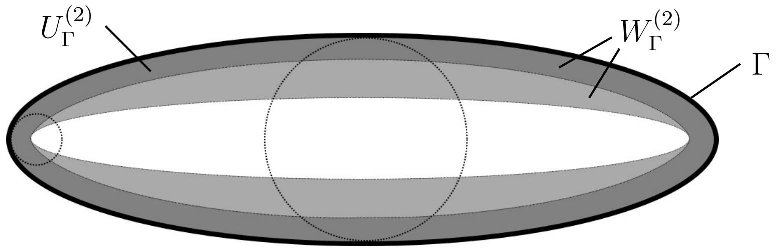


Figure 2.3: Generalized neighborhood $W_\Gamma^{(2)}$ for an ellipsoidal surface Γ in comparison to its tubular neighborhood $U_\Gamma^{(2)}$. In addition, the smallest and biggest interior balls corresponding to the maximal and minimal curvature, respectively, are added. Here, only the interior parts are shown.

Proof. For the regularity of d_Γ, we refer to [Foo84]. The structure of the derivatives can be found in [PS16, Chapter 2, Section 3.1]. The invertibility of $\mathbb{I} - sL_\Gamma(\gamma)$ over $\Gamma \times (-a^{(2)}, a^{(1)})$ is a consequence of the uniform bound $2|L_\Gamma(\gamma)| \leq (\min\{a^{(1)}, a^{(2)}\})^{-1}$. □

Note that Lemma 2.3 implies $\nabla d_\Gamma = n_\Gamma \circ P_\Gamma$.

Remark 2.4. *Note that the $a^{(i)}$ may be far from optimal for some parts of the interface where wider neighborhoods are acceptable. It is possible to generalize the concept of tubular neighborhoods of width $a^{(i)}$ to exploit "regional differences" in curvature and topology. For every $\gamma \in \Gamma$, let – compare with equation (2.1) –*

$$\varsigma^{(i)}(\gamma) = \sup\{r > 0 \ : \ \text{there is } x^{(i)} \in \Omega^{(i)} \text{ s.t. } B_r(x^{(i)}) \subset \Omega^{(i)},\ \overline{B_r}(x^{(i)}) \cap \Gamma = \{\gamma\}\}$$

and introduce functions $\alpha^{(i)} \colon \Gamma \to \mathbb{R}$ via $\alpha^{(i)}(\gamma) = (1-\delta)\varsigma^{(i)}$ for the same $\delta \in (0,1)$ as for the definition of the $a^{(i)}$. Clearly, $\alpha^{(i)}(\gamma) \geq a^{(i)}$ for all $\gamma \in \Gamma$; see Figure 2.3. Moreover, since Γ is a C^2-hypersurface (implying that the curvature field is continuous), the $\alpha^{(i)}$ are continuous. We refer to the set

$$W_\Gamma = \Gamma \cup \underbrace{\bigcup_{\gamma \in \Gamma} \{x \in \Omega^{(i)} \ : \ \operatorname{dist}(x,\gamma) < \alpha^{(i)}(\gamma)\}}_{=:W_\Gamma^{(i)}}$$

as the generalized neighborhood of Γ. This implies $U_\Gamma \subset W_\Gamma$ as well as $W_\Gamma^{(i)} \subset U_\Gamma^{(i)}$.

Definiton 2.5. *Let $\Sigma \subset U_\Gamma$ be a C^1-hypersurface. If there is a function $h \colon \Gamma \to (-a^{(2)}, a^{(1)})$ such that*

$$\Sigma = \{x \in U_\Gamma \ : \ x = \gamma + h(\gamma) n_\Gamma(\gamma),\ \gamma \in \Gamma\},$$

we say that Σ is normally parametrizable with respect to Γ. The function h is called height function. We distinguish between the positive part, $h^{(1)} := h^+ = \max\{0, h\}$, and the negative part $h^{(2)} := h^- = \max\{0, -h\}$.

Note that $h = h^{(1)} - h^{(2)}$. Moreover, we can estimate $h^{(i)} \leq a^{(i)}$ for $i = 1, 2$.

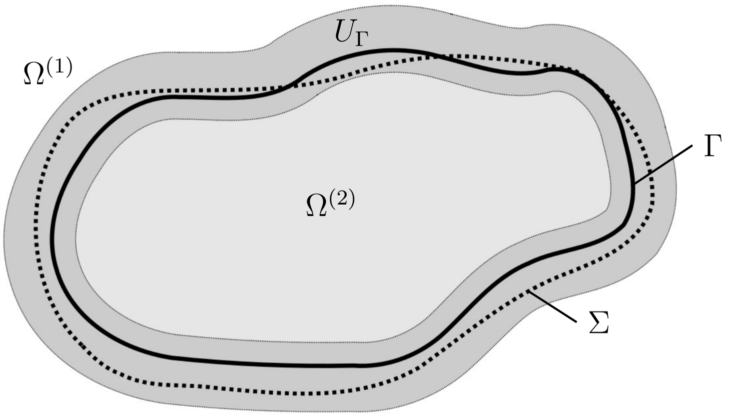


Figure 2.4: C^2- hypersurface Γ with its tubular neighborhood U_Γ and surface $\Sigma \subset U_\Gamma$ (*dotted line*) which is normally parametrizable with respect to Γ.

As C^k-surfaces can be locally represented as the null set of C^k-functions (with non vanishing gradient), the *Implicit Function Theorem*[6] can be used to arrive at the equivalent characterization via C^k-regular *local charts* between open sets of the surface and $\mathbb{R}^{n-1}$. Using the structure of $\mathbb{R}^{n-1}$ and the charts mediating between the surface and $\mathbb{R}^{n-1}$, it is then possible to introduce derivatives (up to order k) of functions $f\colon \Gamma \to \mathbb{R}$. Here, the important step is to ensure that this definition is actually independent on the particular local charts (note that the level set functions pertaining to a particular surface are *not* unique). For a rigorous introduction to derivatives on surfaces, we refer to [PS16, Section 2.1].

Now, let $\Gamma \subset \mathbb{R}^3$ be a C^k-surface ($k \geq 2$), $f_1\colon \Gamma \to \mathbb{R}$ and $f_2\colon \Gamma \to \mathbb{R}^3$. If these functions are sufficiently regular, we denote the *surface gradient* by ∇_Γ (so $\nabla_\Gamma f_1\colon \Gamma \to \mathbb{R}^3$), the *surface divergence* by $\operatorname{div}_\Gamma$ (so $\operatorname{div}_\Gamma f_2\colon \Gamma \to \mathbb{R}$), and the *Laplace-Beltrami operator* by Δ_Γ (so $\Delta_\Gamma f_1 := \operatorname{div}_\Gamma \nabla_\Gamma f_1\colon \Gamma \to \mathbb{R}$).

The surface Γ can also be equipped with a surface measure σ (which it also inherits via the local charts). We introduce the corresponding Lebesgue-spaces $L^p(\Gamma)$ $(1 \leq p \leq \infty)$ as well as Sobolev spaces

$$W^{l,p}(\Gamma) := \{f \in L^p(\Gamma) \ : \ |\nabla_\Gamma^j f| \in L^p(\Gamma) \ (j \leq l)\} \quad (1 \leq p \leq \infty, \ l \leq k).$$

Finally, we introduce the *jump operator* denoting the jump across a hypersurface Γ; for $(u^{(1)}, u^{(2)}) \in C^0(\overline{\Omega^{(1)}}) \times C^0(\overline{\Omega^{(2)}})$, we get $[\![u]\!] \in C^0(\Gamma)$ via $[\![u]\!](\gamma) = u^{(1)}(\gamma) - u^{(2)}(\gamma)$. Using the continuity of the trace operator, we extend this to $(u^{(1)}, u^{(2)}) \in W^{1,2}(\Omega^{(1)}) \times W^{1,1}(\Omega^{(2)})$, where $[\![u]\!] \in L^2(\Gamma)$ is defined as the difference of the traces.

[6]For the statement and a proof, we refer to [Zei86, Theorem 4.B].

2.3 Coordinate transforms

In this section, we collect the important basic tools and concepts regarding coordinate transforms. For a comprehensive treatment, we refer to [MH94]. Note that similar introductions done in the context of homogenization problems can be found in [Mei08, Dob12].

In the following, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $S = (0, T)$ denote a time interval of interest. Moreover, let $\Omega^{(1)}, \Omega^{(2)} \subset \Omega$ be domains where $\overline{\Omega^{(2)}} \subset \Omega$, where $\Gamma := \partial\Omega^{(2)}$ is a C^2-surface, and where $\Omega = \Omega^{(1)} \cup \Omega^{(2)} \cup \Gamma$. Finally, let $U_\Gamma \subset \Omega$ be a tubular neighborhood of Γ and denote by Λ the corresponding C^1-diffeomorphism.

We say that a function $s\colon \overline{\Omega} \to s(\overline{\Omega}) \subset \mathbb{R}^3$ is a *regular C^k-deformation* of the domain Ω if s is a C^k-diffeomorphism and if $\det(Ds(x)) > 0$ for all $x \in \overline{\Omega}$. As $\overline{\Omega}$ is compact, $s(\overline{\Omega})$ is also compact, hence, there is $c > 0$ such that $\det(Ds(x)) > c$ for all $x \in \Omega$. The positivity of the determinant excludes reflections. Consequently, only orientation preserving transformations are considered.

We are especially interested in deformations that are time dependent, often called (regular) motions.

Definiton 2.6. *A regular C^k-motion of a Lipschitz domain Ω over time interval S is a function $s\colon \overline{S \times \Omega} \to \mathbb{R}^3$ such that $s_t := s(t, \cdot)$ is a regular C^k-deformation for every $t \in \overline{S}$ and $s \in C^k(\overline{S \times \Omega}; \mathbb{R}^3)$.*

Now, let $s\colon \overline{S \times \Omega} \to \mathbb{R}^3$ be a regular C^k-motion ($k \geq 2$) and set $Q = \bigcup_{t \in S} \{t\} \times s(t, \Omega)$. We introduce $F\colon \overline{S \times \Omega} \to \mathbb{R}^{3\times 3}$ via $F(t,x) = Ds(t,x)$ and $J\colon \overline{S \times \Omega} \to (0, \infty)$ via $J(t,x) = \det(Ds(t,x))$. It holds *Piola's identity* $\operatorname{div}(JF^{-1}) = 0$. By abuse of notation, we introduce a function $s^{-1}\colon Q \to \Omega$ via $s^{-1}(t,x) = s_t^{-1}(x)$.

For functions $g\colon Q \to \mathbb{R}$ and $h\colon S \times \Omega \to \mathbb{R}$, we define the function corresponding to the initial configuration $\widehat{g}\colon S \times \Omega \to \mathbb{R}$ via $\widehat{g}(t,x) = g(t, s(t,x))$ as well as the function corresponding to the current configuration $\breve{g}\colon Q \to \mathbb{R}$ via $\breve{g}(t,x) = g(t, s^{-1}(t,x))$. Due to the regularity and properties of s, we have $g \in W^{1,2}(Q)$ if and only if $\widehat{g} \in W^{1,2}(S \times \Omega)$. As a relation for the derivatives, a straightforward calculation involving the chain rule leads us to

$$\nabla g = F^{-T} \nabla \widehat{g}, \quad \partial_t g = \partial_t \widehat{g} + \nabla \widehat{g} \cdot F^{-1} \partial_t s.$$

2.3.1 Transport theorems

When deriving mathematical models based on balance laws, we encounter terms like

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{A(t)} \phi(t,x) \,\mathrm{d}x,$$

where ϕ is a density corresponding to some physical quantity (say, mass or energy) and the evolution $t \mapsto A(t)$ is the result of some motion. Resolving above differentiation is standard in the case that $A(0)$ is a domain and ϕ continuously differentiable; the corresponding statement is known as *Reynold's Transport Theorem*. Here, we collect the needed generalizations tailored to the situation of this work.

Theorem 2.7 (Transport theorem (for Sobolev functions))**.** *Let $s\colon \overline{S\times\Omega} \to \mathbb{R}^3$ be a C^1-motion and $u \in W^{1,1}(Q)$, where $Q = \bigcup_{t\in S}(\{t\}\times s(t,\Omega))$. Then, the function $t \mapsto \int_{s(t,\Omega)} u(t,x)\,\mathrm{d}x$ is an element of $W^{1,1}(S)$ and it holds (for a.a. $t \in S$)*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t,\Omega)} u(t,x)\,\mathrm{d}x = \int_{s(t,\Omega)} \partial_t u(t,x) + \operatorname{div}(u(t,x)v(t,x))\,\mathrm{d}x \tag{2.4}$$

where $v(t,x) = \partial_t s(t, s^{-1}(t,x))$.

Proof. In the case of $u \in C^1(Q) \cup C(\overline{Q})$, equation (2.4) is the classical Reynold's transport theorems; we refer to [EGK11, Satz 5.4] and [TM00, Proposition 1.3]. A proof of this more general statement can be found in [BMSR+11]. □

Alternatively, equation (2.4) can also be understood as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t,\Omega)} u(t,x)\,\mathrm{d}x = \int_{s(t,\Omega)} \partial_t u(t,x)\,\mathrm{d}x + \int_{\partial(s(t,\Omega))} u(t,x)v(t,x)\cdot n\,\mathrm{d}\sigma$$

in the sense of traces. Here, $n = n(t,x)$ denotes the outside pointing unit normal vector of the domain $s(t,\Omega)$.

A straightforward consequence of Equation (2.4) is given via:

Corollary 2.8. *Let $s\colon \overline{S\times\Omega} \to \mathbb{R}^3$ be a C^1-motion and $u^{(i)} \in W^{1,1}(Q^{(i)})$, where $Q^{(i)} = \bigcup_{t\in S}\big(\{t\}\times s(t,\Omega^{(i)})\big)$ ($i = 1,2$). Then, the functions $t \mapsto \int_{s(t,\Omega^{(i)})} u(t,x)\,\mathrm{d}x$ are elements of $W^{1,1}(S)$ and it holds (for a.a. $t \in S$)*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t,\Omega)} u(t,x)\,\mathrm{d}x = \sum_{i=1}^{2}\int_{s(t,\Omega^{(i)})} \partial_t u(t,x) + \operatorname{div}(u(t,x)v(t,x))\,\mathrm{d}x \\ - \int_{\Gamma(t)} [\![u]\!]v\cdot n_\Gamma(t,\gamma)\,\mathrm{d}\gamma. \tag{2.5}$$

Proof. Apply Theorem 2.7 to both space-time cylinders $Q^{(i)}$ and add the results. We also point out the reference [WB16] where similar situations are considered. □

2.3.2 Hanzawa transformation

Models describing phase transformations via sharp interfaces, which are the focus of this work, involve changes in the geometry. One possible way to deal with these changes is to perform a change of coordinates to some fixed reference geometry (as would seem natural, one usually chooses the initial geometry). Of particular interest to us is the

Hanzawa transformation which is concerned with transformations between hypersurfaces that are normally parametrizable. Provided that the height function connecting such hypersurfaces is sufficiently regular and satisfies certain estimates, we can show that there exists a corresponding C^1-deformation:

Lemma 2.9. *Let $\Sigma \subset U_\Gamma$ be a closed C^1-hypersurface which is parametrizable with respect to Γ (in the sense of Definition 2.5). If the corresponding height function $h\colon \Gamma \to (-a^{(2)}, a^{(1)})$ satisfies $h \in C^1(\Gamma)$ as well as the estimate*

$$\sup \left\{ \sum_{i=1}^{2} \frac{5}{a^{(i)}} |h^{(i)}(\gamma)| + 2|\nabla_\Gamma h(\gamma)| \ : \ \gamma \in \Gamma \right\} \leq \frac{1}{2}, \tag{2.6}$$

then there exists a regular C^1-deformation $s\colon \overline{\Omega} \to \overline{\Omega}$ such that $\Sigma = s(\Gamma)$.

Proof. Let $\chi \in \mathcal{D}(\mathbb{R}_{\geq 0})$ be a cut-off function such that

$$0 \leq \chi \leq 1, \qquad \chi(r) = 1 \text{ if } r < \frac{1}{3}, \qquad \chi(r) = 0 \text{ if } r > \frac{2}{3}$$

In addition, let $\chi'(r) < 0$ if $1/3 < r < 2/3$ as well as $\|\chi'\|_\infty \leq 4$. We introduce a function $s\colon \overline{\Omega} \to \mathbb{R}^3$ via

$$s(x) = \begin{cases} x + h(P_\Gamma(x)) n(P_\Gamma(x)) \chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(1)}}\right), & x \in U_\Gamma^{(1)} \cup \Gamma \\ x + h(P_\Gamma(x)) n(P_\Gamma(x)) \chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(2)}}\right), & x \in U_\Gamma^{(2)} \\ x, & x \notin U_\Gamma \end{cases}.$$

Note that, we then have $s(\gamma) = \gamma + h(\gamma) n_\Gamma(\gamma)$ for $\gamma \in \Gamma$. We show that s has the desired properties.

(i) Tracking of the interface ($\Sigma = s(\Gamma)$). This follows directly via the definition of the function s and the definition of the height function given by Definition 2.5.

(ii) Regularity. As the pieces of s are composed of C^1-functions, $s_{|U_\Gamma^{(i)}} \in C^1(U_\Gamma^{(i)})$ and, since $\chi(r) = 1$ for all $r < 1/3$, we infer $s_{|U_\Gamma} \in C^1(U_\Gamma)$. Also, $s(x) = x$ if either $d_\Gamma^{(1)} > 2/3 a^{(1)}$ or $d_\Gamma^{(2)} > 2/3 a^{(2)}$, which implies $s \in C^1(\overline{\Omega})$.

(iii) Invertibility. We show that s is injective and $s(\overline{\Omega}) = \overline{\Omega}$. Due to $s(U_\Gamma) \subset U_\Gamma$ and $s_{|(\overline{\Omega} \setminus U_\Gamma)} = \mathrm{Id}$, we only have to consider $s_{|U_\Gamma}$. For fixed $\gamma \in \Gamma$, we introduce the function

$$f_\gamma\colon [-a^{(2)}, a^{(1)}] \to [-a^{(2)}, a^{(1)}], \quad f_\gamma(r) = \begin{cases} r + h(\gamma)\chi\left(\frac{r}{a^{(1)}}\right), & r \geq 0 \\ r + h(\gamma)\chi\left(\frac{-r}{a^{(2)}}\right), & r < 0. \end{cases}$$

Then, $f_\gamma \in C^1([-a^{(2)}, a^{(1)}])$, where

$$f_\gamma'(r) = \begin{cases} 1 + \frac{h(\gamma)}{a^{(1)}} \chi'\left(\frac{r}{a^{(1)}}\right), & r \geq 0, \\ \\ 1 - \frac{h(\gamma)}{a^{(2)}} \chi'\left(\frac{r}{a^{(2)}}\right), & r < 0. \end{cases}$$

Positivity of this derivative implies that f_γ is injective as long as $|h^{(i)}(\gamma)| < a^{(i)}/4$ (note that $\chi' \leq 0$ and $|\chi| \leq 4$). This estimate holds true by assumption; see inequality (2.6). In addition, as f is continuous and both $f(-a^{(2)}) = -a^{(2)}$ and $f(a^{(1)}) = a^{(1)}$, f is also onto.

Now, let $x_1, x_2 \in U_\Gamma$ such that $s(x_1) = s(x_2)$. From $P_\Gamma \circ s_{|U_\Gamma} = P_\Gamma$, we infer $P_\Gamma(x_1) = P_\Gamma(x_2) =: \gamma$. Now, there are three different scenarios: either both points are in $U_\Gamma^{(1)} \cup \Gamma$, both points are in $U_\Gamma^{(2)}$, or there is one point in each of these sets. The last scenario, however, is absurd because $d_\Gamma(x_1)$ and $d_\Gamma(x_2)$ have different signs but still $d_\Gamma(s(x_1)) = d_\Gamma(s(x_2))$. For the other two scenarios, $d_\Gamma(s(x_1)) = d_\Gamma(s(x_2))$ implies

$$d_\Gamma(x_1) + h(\gamma)\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right) = d_\Gamma(x_2) + h(\gamma)\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)$$

for some $i = 1, 2$, or, equivalently,

$$f\left(d_\Gamma(x_1)\right) = f\left(d_\Gamma(x_2)\right).$$

Since $d_\Gamma(x_1) = d_\Gamma(x_2)$ (injectivity of the function f) and $P_\Gamma(x_1) = P_\Gamma(x_2)$, it follows that $x_1 = x_2$. Now, let $x \in U_\Gamma$ and set $\gamma = P_\Gamma(x)$. Since f is onto, there is $\beta \in [-a^{(2)}, a^{(1)}]$ such that $f(\beta) = d_\Gamma(x)$. This implies $s(\gamma + \beta n(\gamma)) = x$.

(iv) Positivity of the Jacobian determinant and regularity of the inverse. We introduce $F := Ds\colon \overline{\Omega} \to \mathbb{R}^{3\times 3}$ and claim that

$$\|F - \mathrm{Id}\|_\infty \leq \frac{1}{2}. \tag{2.7}$$

If inequality (2.7) holds, $F(x)$ is invertible and $\det(F(x)) > 0$ for all $x \in \overline{\Omega}$. Via the *Inverse Function Theorem*, we are then able to infer $s^{-1} \in C^1(\overline{\Omega})$.

We show that inequality (2.7) is valid. We have $F_{|\overline{\Omega}\setminus U_\Gamma} = \mathrm{Id}$ as well as $F_{|U_\Gamma^{(i)}} = \mathrm{Id} + D\psi^{(i)}$ $(i = 1, 2)$, where $\psi^{(i)}\colon \overline{\Omega} \to \mathbb{R}^3$ are given via

$$\psi^{(i)}(x) = h(P_\Gamma(x))n(P_\Gamma(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right).$$

For $x \in U_\Gamma^{(i)}$, we calculate

$$\begin{aligned}\nabla\psi^{(i)}(x) = \nabla\left(h(P_\Gamma(x))\right)^T n(P_\Gamma(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\\ + \nabla\left(n(P_\Gamma(x))\right) h(P_\Gamma(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\\ + h(P_\Gamma(x))\nabla\left(\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\right)^T n(P_\Gamma(x)).\end{aligned} \tag{2.8}$$

Taking into consideration Lemma 2.3 and the fact that $\operatorname{dist}(x,\Gamma) = d_\Gamma(x)$ for $x \in U_\Gamma^{(1)}$

and $\operatorname{dist}(x,\Gamma) = -d_\Gamma(x)$ for $x \in U_\Gamma^{(2)}$, this leads to

$$\begin{aligned}\nabla\psi^{(i)}(x) &= (-1)^{i+1}\frac{h(P_\Gamma(x))}{a^{(i)}}\chi'\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)[n(P_\Gamma(x))\otimes n(P_\Gamma(x))] \\ &- h(P_\Gamma(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)L_\Gamma(P_\Gamma(x))M\left(d_\Gamma(x),P_\Gamma(x)\right)\left(\mathbb{I}_3 - n(P_\Gamma(x))\otimes n(P_\Gamma(x))\right) \\ &+ \chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\Big[n(P_\Gamma(x))\otimes M\left(d_\Gamma(x),P_\Gamma(x)\right)\nabla_\Gamma h(P_\Gamma(x))\Big]. \qquad (2.9)\end{aligned}$$

Due to $2|L_\Gamma(\gamma)| \leq (\min\{a^{(1)},a^{(2)}\})^{-1}$, we estimate

$$|M(r,\gamma)| \leq \frac{1}{|1-|rL_\Gamma(\gamma)||} \leq 2 \quad (r\in[-a^{(2)},a^{(1)}],\ \gamma\in\Gamma).$$

For the individual terms in equation (2.9), we get

$$\left|\frac{h(P_\Gamma(x))}{\varepsilon a}\chi'\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\big[n(P_\Gamma(x))\otimes n(P_\Gamma(x))\big]\right| \leq \frac{4}{a^{(i)}}\|h\|_\infty\,,$$

$$\begin{aligned}\Bigg|h(P_\Gamma(x))\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)L_\Gamma(P_\Gamma(x)) \\ M(d_\Gamma(x),P_\Gamma(x))\left(\mathbb{I}-n(P_\Gamma(x))\otimes n(P_\Gamma(x))\right)\Bigg| \leq \frac{1}{a^{(i)}}\|h\|_\infty\,,\end{aligned}$$

$$\left|\chi\left(\frac{\operatorname{dist}(x,\Gamma)}{a^{(i)}}\right)\Big[n(P_\Gamma(x))\otimes M(d_\Gamma(x),P_\Gamma(x))\nabla_\Gamma h(P_\Gamma(x))\Big]\right| \leq 2\,\|\nabla_\Gamma h\|_\infty\,.$$

and, in summary,

$$|\nabla\psi^{(i)}(x)| \leq \frac{1}{2} \quad (x\in U_\Gamma^{(i)}).$$

□

2.4 Mathematical homogenization

On an abstract level, *mathematical homogenization* is concerned with the mathematically rigorous development of tools and methods that are able to distill (in some sense) the effective or averaged properties of a parametrized family of problems. More concretely, the goal is to derive averaged representations of the properties of complex materials and processes.

The general approach in homogenization is a two-step process:

(i) The *real* problem with the *real* parameter ε_0 is embedded into an family of problems with ε_0 replaced by $\varepsilon > 0$.

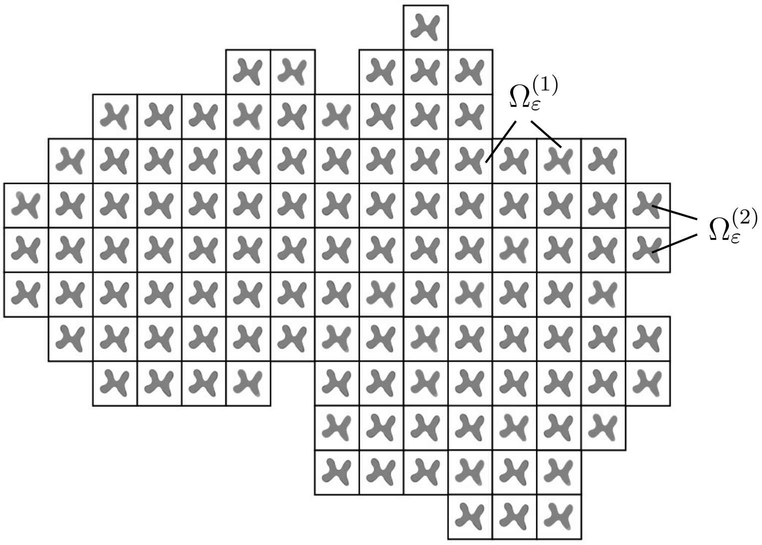


Figure 2.5: Rectilinear domain Ω with εY-periodic subdomains $\Omega_\varepsilon^{(1)}$ (*white*) and $\Omega_\varepsilon^{(2)}$ (*grey*).

(*ii*) A limit procedure $\varepsilon \to 0$ (in some sense) is employed.

Note that such a limit procedure is not really meaningful from a physical point of view, since the size of the micro structure is fixed and since values of ε that are, e.g., smaller than the *Planck length* do not carry physical meaning.[7] Nevertheless, it is a very powerful tool for deriving models that are both accurate and efficient; we refer to [Tar10] where a multitude of examples is provided.

Let $S = (0, T)$ denote the time interval of interest and $\Omega \subset \mathbb{R}^3$ a *rectilinear*[8] domain whose corner coordinates are rational. From this we can infer, that there is a maximal $e_0 > 0$ and a set $Z_{\varepsilon_0} \subset \mathbb{Z}^3$ such that

$$\Omega = \operatorname{int}\left(\bigcup_{k \in Z_{\varepsilon_0}} \varepsilon_0(\overline{Y} + k) \right),$$

i.e., Ω can be covered by translated cells of radius ε_0, see Figure 2.5. We take $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} = 2^{-n}\varepsilon_0$ ensuring that Ω can be covered by cells of radius ε_n for all $n \in \mathbb{N}$.

Let $Y = (0,1)^3$ be the open unit cell in $\mathbb{R}^3$. Take $Y^{(1)}$, $Y^{(2)} \subset Y$ two disjoint open sets, such that $Y^{(1)}$ is connected, such that $\Gamma := \overline{Y^{(1)}} \cap \overline{Y^{(2)}}$ is a C^2 interface, $\Gamma = \partial Y^{(2)}$, $\overline{Y^{(2)}} \subset Y$, and $Y = Y^{(1)} \cup Y^{(2)} \cup \Gamma$. With $n = n(y)$, $y \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

For $\varepsilon > 0$, we introduce the εY-periodic domains $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ and the interface Γ_ε

[7]We also refer to [XYZ16], where it is argued that in the physical and engineering literature homogenization usually refers to a limit procedurce where the representative size of the macroscale is assumed to tend to infinity.

[8]A polygon where all sides are parallel to the axes, sometimes called *general rectangular domain* (e.g., [Höp16]).

representing the two phases and the phase boundary, respectively, via ($i = 1, 2$)

$$\Omega_\varepsilon^{(i)} = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^3} \varepsilon(Y^{(i)} + k) \right), \qquad \Gamma_\varepsilon = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^3} \varepsilon(\Gamma + k) \right).$$

For a set $M \subset \mathbb{R}^3$, $k \in \mathbb{Z}^3$, and $\varepsilon > 0$, we employ the notation

$$\varepsilon(M + k) := \left\{ x \in \mathbb{R}^3 \ : \ \frac{x}{\varepsilon} - k \in M \right\}.$$

2.4.1 Two-scale convergence

The concept of *two-scale convergence* was first introduced in [Ngu89] and developed further in, e.g., [All92, LNW02]. In some sense, the two-scale convergence is an adaptation of the L^2-weak convergence to periodic settings.

Definiton 2.10 (Two-scale convergence)**.** *A sequence (v_ε) in $L^2(\Omega)$ is said to two-scale converge to a limit $v_0 \in L^2(\Omega \times Y)$ if, for all $\varphi \in L^2(\Omega; C_\#(Y))$, it holds*

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} v_0(x, y) \varphi(x, y) \, \mathrm{d}(x, y). \tag{2.10}$$

Similarly, a sequence (v_ε) in $L^2(S \times \Omega)$ is said to two-scale converge to a limit $v_0 \in L^2(S \times \Omega \times Y)$ if, for all $\varphi \in L^2(S \times \Omega; C_\#(Y))$,

$$\lim_{\varepsilon \to 0} \int_{S \times \Omega} v_\varepsilon(t, x) \varphi\left(t, x, \frac{x}{\varepsilon}\right) \mathrm{d}(t, x) = \int_{S \times \Omega \times Y} v_0(t, x, y) \varphi(t, x, y) \, \mathrm{d}(t, x, y).$$

In both cases, we write $v_\varepsilon \overset{2}{\rightharpoonup} v_0$.

Note that two-scale limits are unique (w.r.t. the Lebesgue measure). If $v \in L^2(\Omega)$ and $v_\varepsilon \to v$ in $L^2(\Omega)$, then $v_\varepsilon \overset{2}{\rightharpoonup} v$, where we identified v as an element of $L^2(\Omega \times Y)$ that is constant over Y. Furthermore, two-scale convergence implies weak convergence in the sense that, if $v \overset{2}{\rightharpoonup} v_0$, we can infer $v \rightharpoonup \int_y v_0 \, \mathrm{d}y$. As a simple example for a two-scale convergent sequence, take any $v_0 \in C(\overline{\Omega}; C_\#(Y))$ and $(v_\varepsilon) \subset L^2(\Omega)$ given via $v_\varepsilon(x) = v_0\left(x, \frac{x}{\varepsilon}\right)$. Then $v_\varepsilon \overset{2}{\rightharpoonup} v_0$; see, e.g., [PS08, Lemma 2.34].

Remark 2.11. *Some care is necessary with respect to the choice of test functions $\varphi \in L^2(\Omega; C_\#(Y))$ in the definition of two-scale convergence. One the one hand, it is not possible to work with, say, $\varphi \in L^2(\Omega; H^1_\#(Y))$ or even $\varphi \in L^2(\Omega; L^2_\#(Y))$, as $x \mapsto \varphi\left(x, \frac{x}{\varepsilon}\right)$ is generally not well-defined. More importantly, though, taking to the smaller set of smooth functions, i.e, $\varphi \in C_0^\infty(\Omega; C_\#^\infty(Y))$ is too restrictive.*[9] *However, in the case of a sequence (v_ε) that is bounded in $L^2(\Omega)$, testing with smooth functions is sufficient, we refer to [LNW02, Proposition 13].*

[9] The standard example is given for $\Omega = (0, 1)$ with $f_\varepsilon = \varepsilon^{-1} \mathbb{1}_{[0,\varepsilon)}$. Equation (2.10) holds for all smooth functions but is not two-scale convergent (and not even weakly convergent). We refer to [LNW02, Example 11].

In the following, the results are only formulated for the time dependent case, but they are valid in both cases. As the time variable only acts as a parameter, the proofs are almost identical, cf. [Pet06, Remark 3.1.11].

Theorem 2.12.

(i) Let (v_ε) be a bounded sequence in $L^2(S\times\Omega)$. Then, there exists $v \in L^2(S\times\Omega\times Y)$ and a subsequence of v_ε which two-scale converges to v.

(ii) Let (v_ε) be a bounded sequence in $L^2(S;H^1(\Omega))$ and $v_0 \in L^2(S;H^1(\Omega))$ such that $v_\varepsilon \rightharpoonup v_0$ in $L^2(S;H^1(\Omega))$. Then, $v_\varepsilon \overset{2}{\rightharpoonup} v$ and there exists $v_1 \in L^2(S\times\Omega;H^1_\#(Y))$ and a subsequence of ∇v_ε that two-scale converges to $\nabla v_0 + \nabla_y v_1$.

(iii) Let $(v_\varepsilon) \subset L^2(S;H^1(\Omega))$ such that

$$\sup_{0<\varepsilon<\varepsilon_0}\left(\|v_\varepsilon\|_{L^2(S\times\Omega)} + \varepsilon\|\nabla v_\varepsilon\|^3_{L^2(S\times\Omega)}\right) < \infty.$$

Then, there exists a limit $v_0 \in L^2(S\times\Omega;H^1_\#(Y))$ such that $v_\varepsilon \overset{2}{\rightharpoonup} v_0$ and $\varepsilon\nabla v_\varepsilon \overset{2}{\rightharpoonup} \nabla_y v_0$ at least up to a subsequence. In addition, we have (along the same subsequence)

$$\lim_{\varepsilon\to 0}\int_{S\times\Gamma_\varepsilon} v_\varepsilon(t,x)\varphi\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}(t,\sigma) = \int_{S\times\Omega\times\Gamma} v_0(t,x,y)\varphi(t,x,y)\,\mathrm{d}(t,x,\sigma).$$

Proof. (i) See [All92, Theorem 1.2] or [PS08, Theorem 2.38].

(ii) See [All92, Proposition 1.14 (i)] or [PS08, Theorem 2.39 (i)].

(iii) See [All92, Proposition 1.14 (ii)] or [PS08, Theorem 2.39 (iii)].

□

As it is the case with weak convergence, relying only on the two-scale convergence of two sequences v_ε, w_ε to their limit functions v_0 and w_0, respectively, is not sufficient to infer convergence of their product.

Definiton 2.13 (Strong two-scale convergence)**.** *A sequence (v_ε) in $L^2(S\times\Omega)$ is said to strongly two-scale converge to a limit $v_0 \in L^2(S\times\Omega\times Y)$ if $v_\varepsilon \overset{2}{\rightharpoonup} v_0$ and*

$$\lim_{\varepsilon\to 0}\|v_\varepsilon\|_{L^2(S\times\Omega)} = \|v_0\|_{L^2(S\times\Omega\times Y)}.$$

We write $v_\varepsilon \overset{2}{\to} v_0$.

Note that the strong convergence of a sequence (v_ε) implies its strong two-scale convergence. For any $v_0 \in L^2(\Omega;C_\#(Y))$ and $(v_\varepsilon) \subset L^2(\Omega)$ given via $v_\varepsilon(x) = v_0\left(x,\frac{x}{\varepsilon}\right)$, we have $v_\varepsilon \overset{2}{\to} v_0$, see [All92, Remark 1.9].

Theorem 2.14. *Let* $(v_\varepsilon), (w_\varepsilon) \subset L^2(S \times \Omega)$ *and* $v_0, w_0 \in L^2(S \times \Omega \times Y)$ *such that* $v_\varepsilon \xrightarrow{2} v_0$ *and* $w_\varepsilon \xrightharpoonup{2} w_0$. *Then,*

$$\lim_{\varepsilon \to 0} \int_{S\times\Omega} v_\varepsilon(x) w_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}(t,x) = \int_{S\times\Omega\times Y} v_0(x,y) w_0(x,y) \varphi(x,y) \,\mathrm{d}(t,x,y)$$

for all $\varphi \in C^\infty(\overline{S \times \Omega}; C_\#(Y))$. *Furthermore, if* $(v_\varepsilon) \subset L^\infty(S \times \Omega)$, $v_\varepsilon w_\varepsilon \xrightharpoonup{2} v_0 w_0$.

Proof. See [LNW02, Theorem 18 and the remark succeeding this theorem]. □

2.4.2 Periodic unfolding

An alternative way to tackle homogenization problems and to introduce a notion of two-scale convergence is the method of periodic unfolding as introduced in [CDG02]. We also refer to the later works [CDZ06, CDG08] where perforated domains and the connection to the two-scale convergence as introduced in Definition 2.10 are treated.

The main advantage of the periodic unfolding method is given by the fact that it maps (via the unfolding transformation) the concepts of two-scale and strong two-scale convergence to the usual weak and strong convergence. As a consequence, the aforementioned potential problems regarding the admissible choices of test functions, see Remark 2.11, are circumvented.

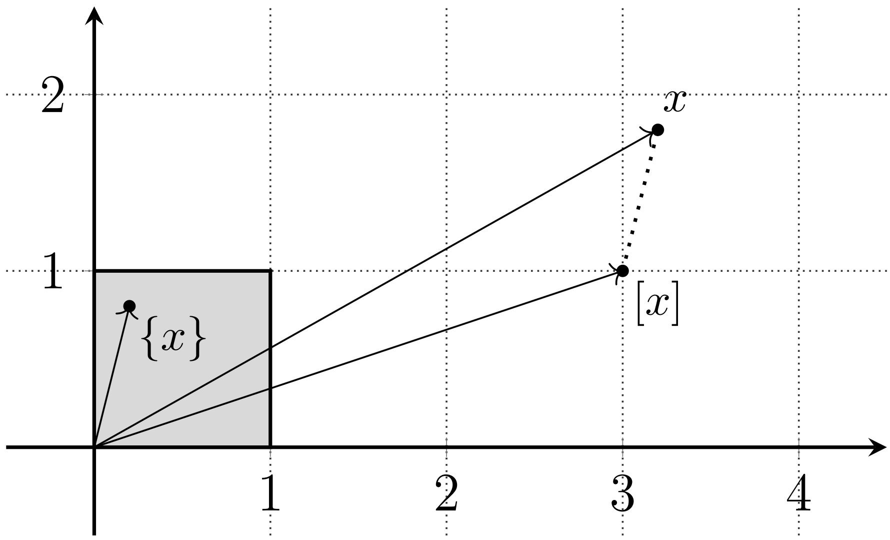


Figure 2.6: Simple example demonstrating the construction of $[x]$ and $\{x\}$.

For every $x \in \mathbb{R}^3$, there is a unique $k \in \mathbb{Z}^3$ such that $x - k \in [0,1)^n$. We introduce the two operations (see also Figure 2.6)

$$[\cdot]\colon \mathbb{R}^3 \to \mathbb{Z}^3, \quad [x] = k \text{ such that } x - [x] \in [0,1)^n,$$
$$\{\cdot\}\colon \mathbb{R}^3 \to Y, \quad \{x\} = x - [x]$$

and observe that $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right] + \left\{\frac{x}{\varepsilon}\right\}\right)$ for all $\varepsilon > 0$.

Definiton 2.15 (Periodic unfolding)**.** *For measurable functions* $\varphi\colon \Omega \to \mathbb{R}$ *as well as* $\varphi_\Gamma\colon \Gamma_\varepsilon \to \mathbb{R}$*, we set*

$$\mathcal{T}_\varepsilon(v)\colon \Omega \times Y \to \mathbb{R}, \quad (\mathcal{T}_\varepsilon(v))(x,y) = v\left(\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]\right),$$
$$\mathcal{T}_{\Gamma_\varepsilon}(v)\colon \Omega \times \Gamma \to \mathbb{R}, \quad (\mathcal{T}_{\Gamma_\varepsilon}(v))(x,y) = v_\Gamma\left(\varepsilon y + \varepsilon\left[\frac{x}{\varepsilon}\right]\right),$$

and note that $\mathcal{T}_\varepsilon(v)$ *and* $\mathcal{T}_{\Gamma_\varepsilon}(v)$ *are also measurable.*

As both operators are clearly linear, we sometimes abbreviate via $\mathcal{T}_\varepsilon v = \mathcal{T}_\varepsilon(v)$. We also have the product rule $\mathcal{T}_\varepsilon(v_1 \cdot v_2) = \mathcal{T}_\varepsilon v_1 \cdot \mathcal{T}_\varepsilon v_2$.

In the following lemma, we collect some important standard results regarding the periodic unfolding operators.

Lemma 2.16. *The unfolding operations defined in Definition 2.15 induce linear and continuous operators* $\mathcal{T}_\varepsilon\colon L^p(\Omega) \to L^p(\Omega\times Y)$ *and* $\mathcal{T}_{\Gamma_\varepsilon}\colon L^p(\Gamma_\varepsilon) \to L^p(\Omega\times\Gamma)$ *(*$p \in [1,\infty)$*) with the following properties:*

(i) Integral identities: For all $v \in L^1(\Omega)$ *and* $v_\Gamma \in L^1(\Gamma_\varepsilon)$*, it holds*

$$\int_\Omega v(x)\,\mathrm{d}x = \int_{\Omega\times Y} \mathcal{T}_\varepsilon v(x,y)\,\mathrm{d}(x,y), \tag{2.11a}$$
$$\int_{\Gamma_\varepsilon} v_\Gamma(x)\,\mathrm{d}\sigma = \varepsilon^{-1}\int_{\Omega\times\Gamma} \mathcal{T}_{\Gamma_\varepsilon} v_\Gamma(x,y)\,\mathrm{d}(x,\sigma). \tag{2.11b}$$

(ii) Continuity estimates:

$$\|\mathcal{T}_\varepsilon\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega\times Y))} \le 1, \qquad \|\mathcal{T}_{\Gamma_\varepsilon}\|_{\mathcal{L}(L^p(\Gamma),L^p(\Omega\times\Gamma))} \le \varepsilon.$$

Proof. We refer to [CDG02, Proposition 1] and [CDZ06, Proposition 2.5 and Proposition 5.2]. □

Please note that the validity of integral identities (2.11a) and (2.11a) heavily relies on the particular rectilinear structure of our domain Ω. For general domains, corrections are needed, cf. [CDG08, Proposition 2.5].

Using these unfolding operators it is then common to define the two-scale convergence and strong two-scale convergence via the weak and strong convergence in $L^2(\Omega\times\Omega)$. The following lemma deals with the equivalence of this approach with the convergences defined in Section 2.4.1.

Lemma 2.17. *Let* $(v_\varepsilon) \subset L^2(\Omega)$ *and* $v_0 \in L^2(\Omega\times Y)$*. Then,*

(i) $v_\varepsilon \overset{2}{\rightharpoonup} v_0$ *if and only if* $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup v_0$ *in* $L^2(S\times Y)$*.*

(ii) $v_\varepsilon \overset{2}{\to} v_0$ *if and only if* $\mathcal{T}_\varepsilon v_\varepsilon \to v_0$ *in* $L^2(S\times Y)$*.*

Proof. For (i), we refer to [CDG08, Proposition 2.14]. Due to the integral identity (2.11a) and the product rule, we see that $\|v_\varepsilon\|_{L^2(\Omega)} = \|\mathcal{T}_\varepsilon v_\varepsilon\|_{L^2(\Omega\times Y)}$. Moreover, $\mathcal{T}_\varepsilon v_\varepsilon \to v$ if and only if both $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup v$ and $\|\mathcal{T}_\varepsilon v_\varepsilon\|_{L^2(\Omega\times Y)} \to \|v_0\|_{L^2(\Omega\times Y)}$. Statement (ii) follows via (i). □

CHAPTER 3

Modeling of phase transformations

In this chapter, we establish a mathematical model that describes phase transformations in a thermoelasticity setting for highly heterogeneous two-phase media; see equations (3.10a) to (3.10e). In this chapter, we establish a mathematical model that describes phase transformations in a thermoelasticity setting for highly heterogeneous two-phase media; see equations (3.10a) to (3.10e). The analysis and homogenization of this model is the focus of Chapters 4 to 6.

This chapter is organized as follows: After an introduction in which we give the physical context for our problem and a couple of relevant literature hints, we start out, in Section 3.2, by establishing a rather general framework for phase transformation problems. Our approach is based on *rational mechanics*. In Section 3.3, we propose a simplified linear thermoelasticity problem with free boundary that is the focus of the mathematical analysis in the subsequent chapters. In Section 3.4, this simplified model is embedded into a periodic homogenization setting where phase transformations occur in a highly heterogeneous two-phase medium.

3.1 Introduction

The properties of complex, solid materials like metallic alloys or ceramics are highly dependent on both their chemical composition, i.e., the constituents along with their relative numbers, as well as their microstructure, i.e., the specific arrangement of the constituents. Frequently, several distinct microstructures with their own particular chemical compositions are present in a given material; we refer to them as *phases*. Steel, for example, which is an alloy of iron, carbon, and, to a lesser extent, other elements like chromium or nickel, is known for the variety of different phases that can form based on its chemical composition, the temperature distribution, the history of cooling and heating, and its stresses. In Figure 3.1 some typical phases in steel are depicted. For details on the modeling of steel, we refer the reader, for instance, to [ARH08, Hor85].

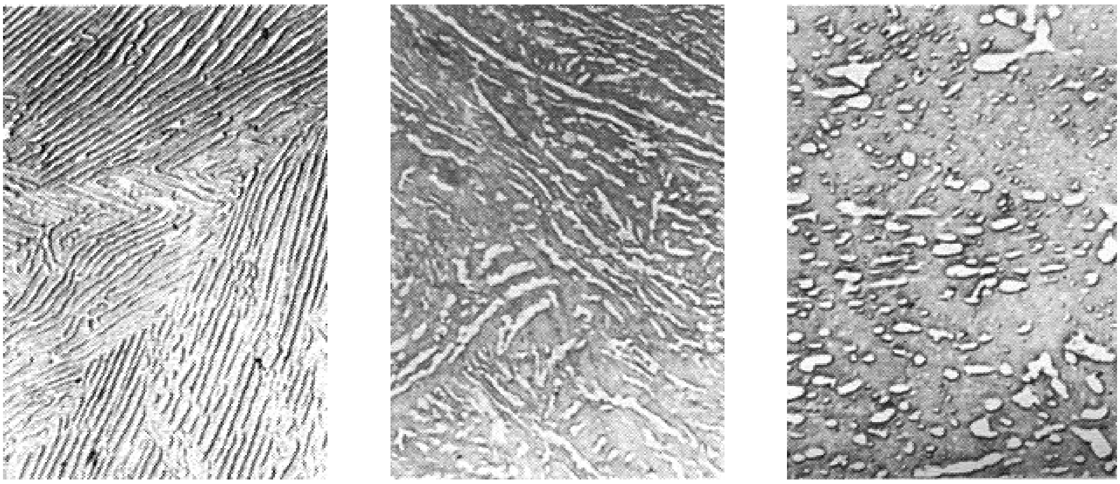


Figure 3.1: Examples of phases of steel: (a) Pearlite, (b) Bainite, (c) Martensite. These images were recorded using a transmission electron microscope and the resolution is in the μm-range. These figures are taken from [Föl99, Section 8.4.1].

Due to changes in, e.g., the temperature or the internal stresses, phases transform[1]; that is one phase grows at the expense of another phase. Going back to the example of steel, *Bainite* steel can form from *Austenite* steel when the Austenite is cooled past a critical temperature; details are given in Section 1.1. Although such a process is conceptually similar to the freezing of water, there are important differences in the physical processes and, as a consequence, also in the mathematical models describing them: As a transformation between different solid phases, which are generally more rigid than fluids, mechanical effects like stresses or distortions in the crystal lattice structures are important effects to consider. By contrast, most models for ice-water phase transformation do not account for these effects.

Phase transformations naturally lead to changes in the underlying geometry of the arrangement of the phases. They are characterized by the motion of the interfaces separating the competing phases. Note that, in reality, there might not be well-defined interfaces but rather thin intermediate regions. In this work, we assume that the borders between different phases can be represented by hypersurfaces; this leads to so-called *sharp interface* models. For simplicity, we focus on two-phase systems where the interface separating the two phases is assumed to be *(thermodynamically) inactive*. This is to say that the interface does not, by itself, carry any mass, momentum, or energy.

A comprehensive survey with a strong mathematical flavor for the modeling of two-phase systems with sharp interfaces is given by [WB16], where a particular emphasis is given on different concepts of interfaces (e.g., material vs. singular, inactive vs. active) and references to both the mathematical as well as to the engineering literature are provided. The classical references we have in mind here are [TT60, Gur99, Nol74]. Sharp interface models in situations that are somewhat similar to the ones investigated in this thesis, can be found in [PS16, PSZ13]. The modeling approach outlined in the following sections follows closely [WB16].

[1] As some authors reserve *transition* for phase changes that involve a change in the state of matter, e.g., liquid to solid, and since we are interested in solid-solid changes, we use *transformation*.

As an alternative approach to the direct modeling of phase transformation, *phase-field* models,[2] are often considered. Via an indicator variable, the phase-field variable, the moving interface is approximated via an interfacial region of small thickness. In this work, we prefer the direct moving-interface modeling and do not focus on phase field approaches. We refer the interested reader to [Höp16, MSA+15, MBW08] for related developments in the phase-field direction.

3.2 General thermomechanics model

We derive a general model describing the thermomechanical properties of a two-phase system with sharp interface allowing for phase transformations.

Within this chapter, we rely on several important fundamental principles of *rational mechanics* which are usually only implicitly assumed but merit to be mentioned. For a comprehensive overview of the underlying principles in rational mechanics, we refer to [Nol74, TN92].

Fundamental Principles

(P1) *Continuum principle:* All physical bodies can be identified with Lebesgue measurable subsets of $\mathbb{R}^n$ ($n \in \mathbb{N}$; usually $n = 3$).

(P2) *Principle of non-singularity:* All extensive[3]physical quantities can be represented by measures that are absolute continuous with respect to the Lebesgue measure.

(P3) *Cauchy hypothesis:* All fluxes satisfy the *Cauchy* hypothesis and are well-defined functions of time and space; see Remark 3.1.

Note that there are also a couple of additional foundational presuppositions like the *principle of determinism* and *principle of frame indifference* which are pretty much agreed upon in the context of rational mechanics.

The mathematical operator problems based on the models developed in this chapter do not need to satisfy Principle (P2). In fact, relying on the concept of weak solutions, it is possible to consider distributional (in particular, non-local) effects.

3.2.1 Changes in the geometry

Let $\Omega \subset \mathbb{R}^3$ be a bounded *Lipschitz domain* that represents the overall two-phase system and let $S = (0, T)$, $T > 0$, represent a time interval of interest. We denote the outer

[2] Some authors prefer the term *diffuse interface* instead of phase-field.

[3] An additive quantity, i.e., the value for the overall system is the sum of the values for any partition.

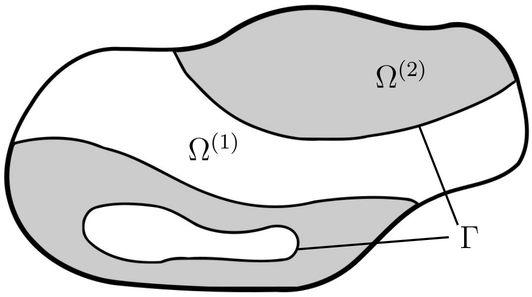


Figure 3.2: General domain where all of $\Omega^{(1)}$, $\Omega^{(2)}$, and Γ are disconnected.

unit normal vector field of Ω with $\nu = \nu(x)$. Moreover, let $\Omega^{(1)}, \Omega^{(2)} \subset \Omega$ be two disjoint subdomains representing the initial distribution of the two distinct phases. We therefore assume

$$(i)\ \Omega^{(1)} \cap \Omega^{(2)} = \emptyset \ \text{ and } (ii)\ \operatorname{int}\left(\overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}\right) = \Omega.$$

We introduce the interface $\Gamma := \overline{\Omega^{(1)}} \cap \overline{\Omega^{(2)}}$. Note that Γ is a hypersurface such that $\Gamma \cap \partial\Omega$, while not necessarily empty, is a null set with respect to the surface measure on $\partial\Omega$. Note that, in this setting, neither of $\Omega^{(1)}$, $\Omega^{(2)}$, and Γ need to be connected, see Figure 3.2.

In the following, we account for two different mechanisms of geometric changes: (i) *moving interface:* the interface Γ might move due to phase transformations, and (ii) *kinematic motion:* stresses might induce deformations.

To separate these two mechanisms, we differentiate between three different configurations: the *initial configuration* (initial state without moving interface and without kinematic motion), the *referential configuration* (including moving interface but without kinematic motion), and the *current configuration* (current state due to moving interface and kinematic motion). This view is in line with the framework described in [WB16].

We expect the overall changes in geometry to be coherent. In particular, this means that no cracks and holes can form.

Moving interface (geometry changes due to the phase transformations). For $t \in \overline{S}$, let $\Omega^{(1)}(t), \Omega^{(2)}(t) \subset \Omega$ denote the subdomains representing the two distinct phases and $\Gamma(t)$ the interface between these domains at time t.

Assumptions on the Interface Motion

(A1) There is a C^1-motion $s_\Gamma\colon \overline{S \times \Omega} \to \overline{\Omega}$ such that $\Gamma(t) = s_\Gamma(t, \Gamma)$.

(A2) $\Gamma(t)$ is a C^2-hypersurface for all $t \in \overline{S}$.

Note that by Assumption (A2), the existence of the curvature of $\Gamma(t)$ is guaranteed.

We denote the unit normal vector field at $\gamma \in \Gamma(t)$ pointing outwards $\Omega^{(2)}(t)$ into $\Omega^{(1)}(t)$

via $n_\Gamma = n_\Gamma(t,\gamma)$ and the mean curvature function via $H_\Gamma = H_\Gamma(t,\gamma)$. Furthermore, we introduce the normal velocity of the interface at $\gamma \in \Gamma(t)$ via $V_\Gamma(t,\gamma) = \partial_t s_\Gamma(t,\gamma) \cdot n_\Gamma(t,\gamma)$. Finally, we introduce the time-space cylinders

$$Q = S \times \Omega, \qquad Q^{(i)} = \bigcup_{t\in S} \{t\} \times \Omega^{(i)}(t), \qquad \Xi = \bigcup_{t\in S} \{t\} \times \Gamma(t).$$

These sets refer to the reference configuration as they account for the interface motion but not for any kinematic motion.

Kinematic motion (deformations due to stresses). As a response to possible stresses due to applied forces, the material body deforms. This effect is in addition to the (non-material) interface motion described above.

For a point $x \in \overline{\Omega}$, let $s_{kin}(t,x) \in \mathbb{R}^3$ denote its deformed position at time $t \in \overline{S}$. Obviously, $s_{kin}(0,x) = x$ for all $x \in \overline{\Omega}$. We introduce the corresponding function $s_{kin} \colon \overline{S \times \Omega} \to \mathbb{R}^3$ and assume

Kinematic motion

(A3) *Coherence:* s_{kin} is a continuous function.

(A4) *Regularity:* $s_{kin}(\cdot,x)$ is $C^2(\overline{S})$ for all $x \in \overline{Q} \cap \Xi$ and $s_{kin|Q^{(i)}}$ is a C^1-motion $(i = 1,2)$.

(A5) *Fixed overall domain:* $s_{kin}(t,\overline{\Omega}) = \overline{\Omega}$ for all $t \in \overline{S}$.

(A6) *Interface compatibility:* $s_{kin}(t,\Gamma(t))$ is a C^2-hypersurface for all $t \in \overline{S}$.

Less restrictive sets of assumptions are possible, we refer to [WB16, Section 2.5] for a more detailed discussion on the minimal possible assumption on the interface and its motion in relation to the deformation. While Assumptions (A3) and (A4) are necessary for some of the involved quantities (e.g., acceleration) to be well-defined, relaxing Assumption (A5) would only add a few technical details . Since we are mainly interested in the interface motion as opposed to the motion of the outer boundary, we assume the overall domain to be fixed. By Assumption (A6), the existence of the curvature in the current configuration is guaranteed; without it, we would have to restrict ourselves to working in the reference configuration only.

We set the deformed time-space sets (here, and in the following, a subscript c indicates reference to the current configuration accounting for both the interface motion and the deformation)

$$\Omega_c^{(i)}(t) = s_{kin}(t,\Omega^{(i)}(t)), \quad Q_c^{(i)} = \bigcup_{t\in S} \{t\} \times \Omega_c^{(i)}(t)$$
$$\Gamma_c(t) = s_{kin}(t,\Gamma(t)), \qquad \Xi_c = \bigcup_{t\in S} \{t\} \times \Gamma_c(t).$$

For $(t,x) \in \overline{Q^{(i)}}$, we introduce the kinematic velocity $v_{kin}^{(i)}(t,x) = \partial_t s_{kin}(t, s_{kin}^{-1}(t,x))$ and the overall (or current) velocity $v_c^{(i)}(t,x) = \partial_t(s_{kin} \circ s_\Gamma)\left(t, s_\Gamma^{-1}(t, s_{kin}^{-1}(t,x))\right)$. Setting $s_c = s_{kin} \circ s_\Gamma$, we infer $v_c^{(i)}(t,x) = \partial_t s_c\left(t, s_c^{-1}(t,x)\right)$. For $(t,x)) \in \Xi_c$, we introduce the corresponding unit normal vector $n_{\Gamma_c} = n_{\Gamma_c}(t,x)$, mean curvature $H_{\Gamma_c} = H_{\Gamma_c}(t,x)$ and normal velocity $V_{\Gamma_c}(t,x) = \partial_t s_c(t,x) \cdot n_c(t,x)$. Note, that the velocities $v_{kin}^{(i)}$ and $v_c^{(i)}$ do not need to be continuous across the interface.

3.2.2 Balance equations

Based on first principles, we derive a general system of equations and exchange conditions describing the thermomechanical properties of a two-phase systems undergoing phase transitions.

From here on, a superscript (i) denotes the affiliation to the corresponding phase, i.e., $i = 1,2$. In the following, let $\omega \subset \Omega$ be a Lipschitz domain and set $\omega^{(i)}(t) := s_{kin}(t,\omega) \cap \Omega_c^{(i)}(t)$ $(t \in S,\ i = 1,2)$. Moreover, let $t_0 \in S$ and $\delta > 0$ such that $t_0 + \delta \in S$. We set

$$Q_\delta^{(i)} = \bigcup_{t \in (t_0, t_0+\delta)} \{t_0\} \times \omega^{(i)}(t) \quad (i = 1,2).$$

Generic balance equation. Take any scalar valued *extensive* physical quantity, e.g., mass or internal energy, which we then represent as a time-parametrized Ξ-finite signed-measure $\Phi^{(i)}(t)\colon \mathcal{L}(\Omega_c^{(i)}(t)) \to \mathbb{R}$. Here, $\mathcal{L}(\Omega_c^{(i)}(t))$ denotes the Ξ-algebra of Lebesgue measurable subsets of $\Omega_c^{(i)}(t)$. In the case of vector-valued properties, e.g., momentum, we consider vector-measures, see also [WB16].

Due to Principle (P3) and the Radon-Nikodym theorem, see [Zei89, Appendix (82a)], we can find corresponding densities $\phi_c^{(i)}(t)\colon \Omega_c^{(i)}(t) \to \mathbb{R}$ such that

$$\Phi^{(i)}(t, \omega^{(i)}(t)) = \int_{\omega^{(i)}(t)} \phi_c^{(i)}(t,x)\,\mathrm{d}x.$$

We can also introduce the overall measure of this particular quantity, $\Phi(t)\colon \mathcal{L}(\Omega) \to \mathbb{R}$, via

$$\Phi(t,\omega) = \sum_{i=1}^{2} \Phi^{(i)}(t, \omega^{(i)}(t)) = \sum_{i=1}^{2} \int_{\omega^{(i)}(t)} \phi_c^{(i)}(t,x)\,\mathrm{d}x.$$

There might be production (both positive or negative) of the physical quantity represented by Φ inside Ω. This, we represent as signed-measures $F_\Phi^{(i)}\colon \mathcal{L}(Q_c^{(i)}) \to \mathbb{R}$ $(i = 1,2)$, where we, again, can identify the corresponding production densities $f_{\phi_c}^{(i)}\colon Q_c^{(i)} \to \mathbb{R}$.

The change of the overall quantity Φ in the domain ω over the time interval $(t_0, t_0 + \delta)$ is given via $\Phi(t_0 + \delta, \omega) - \Phi(t_0, \omega)$ and the individual changes are, analogously, given as

$$\Phi^{(i)}(t_0 + \delta, \omega^{(i)}(t + \delta)) - \Phi^{(i)}(t_0, \omega^{(i)}(t)) \quad (i = 1,2).$$

These changes can be attributed either to the production inside of ω or to an exchange with the neighborhood of ω via fluxes. Similar as with the production measures, we represent the fluxes via signed-measures $J_\Phi^{(i)}\colon \mathcal{L}(Q_c^{(i)}) \to \mathbb{R}$ $(i = 1, 2)$. The overall flux in time interval $(t_0, t_0 + \delta)$ into ω is therefore given as

$$J_\phi^{(i)}(Q_\delta^{(i)}) = \Phi^{(i)}(t_0 + \delta, \omega^{(i)}) - \Phi^{(i)}(t_0, \omega^{(i)}) - F_\Phi^{(i)}(Q_\delta^{(i)}) \quad (i = 1, 2).$$

Generally, we would have to distinguish between the convective part of the fluxes (due to the kinematic motion) and the non-convective parts (e.g., diffusion). However, as we are considering *material control volumes*, there is no mass flux across the boundary $\partial\omega(t)$, i.e., the convective flux is zero, for more details we, again, refer to [WB16].

Remark 3.1 (Cauchy fluxes). *For every $t \in S$ and $i = 1, 2$, there is a function $q_{\omega^{(i)}}^{(i)}(t)\colon \partial\omega^{(i)} \to \mathbb{R}$ such that*

$$J_\phi^{(i)}(l) = \int_l q_{\omega^{(i)}}^{(i)}(t, x)\,\mathrm{d}\sigma \quad \textit{for all } l \in \mathcal{L}(\partial\omega^{(i)}).$$

If there is a function $j_{\phi_c}^{(i)}\colon Q_c^{(i)} \to \mathbb{R}^3$ such that $j_{\phi_c}^{(i)}(t, x) n_{\omega^{(i)}} = q_{\omega^{(i)}}^{(i)}(t, x)$ for all Lipschitz continuous sets $\omega^{(i)} \subset \mathcal{L}(\Omega^{(i)}(t))$, the flux is called a Cauchy flux. We refer to [RS04].

As we are considering Cauchy fluxes (Principle (P3)), there are $j_{\phi_c}^{(i)}\colon Q_c^{(i)} \to \mathbb{R}^3$ such that

$$J_\phi^{(i)}(\omega^{(i)}) = \int_{\partial\omega^{(i)}} j_{\phi_c}^{(i)}(t, x) \cdot n_{\omega^{(i)}}(t, x)\,\mathrm{d}x \quad (i = 1, 2).$$

We can infer that

$$\sum_{i=1,2}^{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega^{(i)}(t)} \phi_c^{(i)}\,\mathrm{d}x = \sum_{i=1}^{2} \int_{\partial\omega^{(i)}(t)} j_{\phi_c}^{(i)} \cdot n_{\omega^{(i)}}\,\mathrm{d}x + \sum_{i=1}^{2} \int_{\omega^{(i)}(t)} f_{\phi_c}^{(r)}\,\mathrm{d}x. \tag{3.1}$$

Using Reynold's transport theorem, Corollary 2.8, and applying the divergence theorem, we are led to

$$\begin{aligned} \sum_{i=1}^{2} \int_{\omega^{(i)}(t)} \partial_t \phi_c^{(i)} + \operatorname{div}\left(\phi_c^{(i)} v\right)\,\mathrm{d}x - \int_{\omega^{(\Gamma)}(t)} [\![\phi_c]\!] v_{\Gamma_c} \cdot n_{\Gamma_c}\,\mathrm{d}\sigma \\ = \sum_{i=1}^{2} \int_{\omega^{(i)}(t)} \operatorname{div}\left(j_{\phi_c}^{(i)}\right)\,\mathrm{d}x - \int_{\omega^{(\Gamma)}(t)} [\![j_{\phi_c}]\!] \cdot n_{\Gamma_c}\,\mathrm{d}\sigma \\ + \int_{\omega^{(\Gamma)}(t)} [\![\phi_c(V_{\Gamma_c} - v_c)]\!] \cdot n_\Gamma\,\mathrm{d}\sigma + \sum_{i=1}^{2} \int_{\omega^{(i)}(t)} f_{\phi_c}^{(i)}\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Making use of the invertibility of the motion s_{kin} and relying on the *fundamental lemma of calculus of variations*[4], we arrive at localized balance equations with respect to the reference configuration:

[4]For a statement and the proof, we refer to [JLJ08, Lemma 1.1.1].

Generic balance system - reference configuration

$$\partial_t \phi^{(i)} - \operatorname{div}\left(j_\phi^{(i)}\right) = f_\phi^{(1)} \quad \text{in } Q^{(i)},\ i = 1, 2, \tag{3.3a}$$

$$[\![\phi]\!] V_\Gamma - [\![j_\phi]\!] \cdot n_\Gamma = f_\phi^{(\Gamma)} \quad \text{on } \Xi. \tag{3.3b}$$

Based on this system, we now formulate the balance systems for the mass, momentum, energy, and for the tracer substance. Here, tracer substance refers to any solute whose mass density is insignificant to the overall mass density. In steel, e.g., this could be carbon or other dissolved elements.

Mass balance. We introduce the mass densities $\rho^{(i)}$ $(i = 1, 2)$ and make the following natural assumptions:

Assumptions for the mass balance

(A7) There is no production of mass, i.e., $f_\rho^{(i)} = 0$.

(A8) There is no advective flux of mass, i.e., $j_\rho^{(i)} = 0$.

(A9) The mass densities are constant in the individual phases.

Note that Assumption (A9) is quite restrictive. However, we are primarily interested in allowing for phases with differing densities; so this is not an issue.

Taking into account Assumption (A9) and equation (3.3b), we infer

$$[\![\rho]\!] V_\Gamma(t, \gamma) = f_\rho^{(\Gamma)} \tag{3.4}$$

for all $\gamma \in \Gamma(t)$ and all $t \in S$. Therefore, the absence of mass production at the interface, i.e., $f_\rho^{(\Gamma)} = 0$, implies that either the densities are equal for both phases or that there is no interface motion. Kinematic motion would still be possible for non-equal densities albeit with the condition that $|\Omega_c^{(i)}(t)|$ are constant for $i = 1, 2$; for details, see [PSSS12].

Momentum balance. We introduce the momentum densities $\rho^{(i)} v^{(i)}$ and the flux densities $-P^{(i)}$ (first Piola-Kirchhoff tensors) $(i = 1, 2)$. Inserting these quantities into equations (3.3a) and (3.3b), we are led to

$$\rho^{(i)} \partial_t v^{(i)} - \operatorname{div}\left(P^{(i)}\right) = f_v^{(i)} \quad \text{in } Q^{(i)},\ i = 1, 2, \tag{3.5a}$$

$$[\![\rho v]\!] V_\Gamma + [\![P]\!] n_\Gamma = f_v^{(\Gamma)} \quad \text{on } \Xi. \tag{3.5b}$$

Internal energy balance. For the energy balance, we make the following assumptions:

Assumptions for the energy balance

(A10) The overall energy densities, for $i = 1, 2$, are the sums of the kinetic energy densities, which are given via $\frac{1}{2}\rho^{(i)} v^{(i)} \cdot v^{(i)}$, and the internal energy densities, which are given via $\rho^{(i)} e^{(i)}$.

(A11) The flux densities are given as the sums of the heat flux densities, which we denote by $q^{(i)}$, and the dissipative energy flux densities, which are given via $-\left(P^{(i)}\right)^T v^{(i)}$.

Taking into account Assumptions (A10) and (A11) as well as Equation (3.5a), the corresponding system of balance equations for the internal energy is given via

$$\rho^{(i)} \partial_t e^{(i)} - \operatorname{div}\left(q^{(i)}\right) = P^{(i)} \colon Dv^{(i)} + f^{(i)} \quad \text{in } Q^{(1)},\ i = 1, 2 \tag{3.6a}$$

$$[\![\rho e]\!] V_\Gamma + \langle P \rangle n_\Gamma \cdot [\![\partial_t u]\!] - [\![q]\!] n_\Gamma = f^{(\Gamma)} \quad \text{on } \Xi. \tag{3.6b}$$

Here, $\langle P \rangle = 1/2(P^{(1)} + P^{(2)})$ which results from the product rule $[\![ab]\!] = \langle a \rangle [\![b]\!] + [\![a]\!] \langle b \rangle$. For more details regarding the calculations leading to this specific systems of equations and other representations of the same equations, we refer to [WB16].

Balance of tracer substance. For $i = 1, 2$, we introduce tracer densities $c^{(i)}$ with corresponding flux densities $j_c^{(i)} = 0$ and assume:

Assumptions for the tracer substance balance

(A12) The tracer densities are small in comparison with the body densities $\rho^{(i)}$ so they do not influence the mass balance.

The balance system is then given as

$$\partial_t c^{(i)} - \operatorname{div}\left(j_c^{(i)}\right) = f_c^{(i)} \quad \text{in } Q^{(1)},\ i = 1, 2, \tag{3.7a}$$

$$[\![c]\!] V_\Gamma - [\![j_c]\!] n_\Gamma = f_c^{(\Gamma)} \quad \text{on } \Xi. \tag{3.7b}$$

Note on thermodynamic consistency. In addition to satisfying the local balance laws, physical processes are also expected to respect the second law of thermodynamics which essentially stipulates that the *entropy* of an isolated system does not decrease with time. Starting with the balance formulations for the entropy and incorporating some concepts from thermodynamics, it is possible to obtain the so called *Clausius-Duhem inequality* as a way to express this law. This inequality gives a constraint for the class of admissible *constitutive relations*: Mathematical models where the choice of constitutive relations leads to the Clausius-Duhem inequality being satisfied (for all possible processes) are called *thermodynamically consistent*. For details, we refer to, e.g., [Hau02, TN04, WB16]. In this work, we do not account for thermodynamic consistency although we expect it this to hold.

Summary of balance equations. Considering the balances for mass, momentum, energy, and tracer substance, we obtain the overall balance system

General system of balances - reference configuration

Balances in the bulk phases $(i = 1, 2)$

$$\rho^{(i)}\partial_t v^{(i)} - \operatorname{div}\left(P^{(i)}\right) = f_v^{(i)} \quad \text{in } Q^{(i)}, \tag{3.8a}$$

$$\rho^{(i)}\partial_t e^{(i)} - \operatorname{div}\left(q^{(i)}\right) = P^{(i)} : Dv^{(i)} + f_e^{(i)} \quad \text{in } Q^{(i)}, \tag{3.8b}$$

$$\partial_t c^{(z)} - \operatorname{div}\left(j_c^{(i)}\right) = f_c^{(i)} \quad \text{in } Q^{(i)} \tag{3.8c}$$

Balances on the moving interface

$$[\![\rho]\!]V_\Gamma = f_\rho^{(\Gamma)} \quad \text{on } \Xi, \tag{3.8d}$$

$$[\![\rho v]\!]V_\Gamma - [\![P]\!]n_\Gamma = f_v^{(\Gamma)} \quad \text{on } \Xi, \tag{3.8e}$$

$$[\![\rho e]\!]V_\Gamma + \langle P\rangle n_\Gamma \cdot [\![\partial_t u]\!] - [\![q]\!]n_\Gamma = f_e^{(\Gamma)} \quad \text{on } \Xi, \tag{3.8f}$$

$$[\![c]\!]V_\Gamma - [\![j_c]\!]n_\Gamma = f_c^{(\Gamma)} \quad \text{on } \Xi. \tag{3.8g}$$

This system has to completed with constitutive relations, initial conditions, transmision conditons, as well as boundary conditions.

3.3 Linear thermoelasticity

In this section, based on the general system given via equations (3.8a) to (3.8g), we derive a simplified model for the thermomechanic behavior of a two-phase system undergoing phase transformations. Fundamentally, we perform the following two linearizations:

(i) *Geometrical linearization:* Assuming the deformations and their gradients to be small, the geometry can be assumed (as a first order approximation) to be unchanged by the deformation. That is, the kinematic motion is neglegible.

(ii) *Physical linearization:* Assuming linear constitutive laws, we are led to the standard model for linear thermoelasticity.

Linear thermoelasticity models are widely used to describe the interplay between mechanical and heat effects in solids. It is worth pointing out the structural similarity between thermoelasticity models and models for poroelasticity, in particular, *Biot's theory of linear poroelasticity*; see [Bio41, SM02].

For $i = 1, 2$, $t \in S$, and $x \in \Omega^{(i)}(t)$, let $u^{(i)} = u^{(i)}(t, x)$ denote the deformation and $\theta^{(i)} = \theta^{(i)}(t, x)$ the temperature in the respective phase.

Assumptions of quasi-stationary linear thermoelasticity

(A13) *Geometrical linearization:* The deformations are small, that is, $\|\nabla u\| \ll 1$.[5]

(A14) *Thermoelasticity:* The constitutive relations $\sigma^{(i)} = \mathcal{C}^{(i)} e(u^{(i)}) - \alpha^{(i)}\theta^{(i)}$, $e^{(i)} = c^{(i)}\theta^{(i)}$, and $\sigma^{(i)} : \partial_t u^{(i)} = -\gamma^{(i)} \operatorname{div} u^{(i)}$ hold true.

(A15) *Fourier's law of conductivity:* T heat flux is proportional to the negative gradient of the temperature; that is, $q^{(i)} = -K^{(i)}\nabla\theta^{(i)}$.

(A16) *Quasi-stationary mechanics:* The mechanical behavior is quasi-static and therefore always in equilibrium. As a consequence, $\partial_t v^{(1)}$ is negligible.

(A17) *Surface stresses and latent heat:* The surface stresses are proportional to the curvature via $f_u^{(\Gamma)} = -\sigma_0 H_\Gamma n_\Gamma$ while the surface energy is given via $f_e^{(\Gamma)} = -LV_\Gamma$.

Here, $\Xi^{(i)} : Q^{(i)} \to \mathbb{R}^3$ denote the Cauchy stress tensors, $\mathcal{C}^{(i)} \in \mathbb{R}^{3\times3\times3\times3}$ are the *stiffness* tensors, $\alpha^{(i)} > 0$ the *thermal expansion* coefficients, $c^{(i)} > 0$ the *heat capacities*, $\gamma^{(i)} > 0$ are the *dissipation coefficients*, $K^{(i)} \in \mathbb{R}^{3\times3}$ the *thermal conductivities*, and $\sigma_0 > 0$ is the coefficient of surface tension. In addition, $e(v) = 1/2(Dv + (Dv)^T)$ denotes the linearized strain tensor and $\mathbb{I}_3$ the identity matrix. Finally, we take a function v_Γ to be the normal velocity of Γ.

Under the assumptions of geometric linearity, it can be shown that the reference representation and the current representation coincide: the Piola-Kirchhoff stress tensors reduce to the Cauchy stress tensors. As a consequence, the model simplifies to:

Two-phase thermoelasticity model

Balances in the bulk phases $(i = 1, 2)$

$$-\operatorname{div}\left(\mathcal{C}^{(i)} e(u^{(i)}) - \alpha^{(i)}\theta^{(i)}\mathbb{I}_3\right) = f_u^{(i)} \quad \text{in } Q^{(i)}, \tag{3.9a}$$

$$\rho^{(i)} c^{(i)} \partial_t\theta^{(i)} + \gamma^{(i)} \operatorname{div} \partial_t u^{(i)} - \operatorname{div}\left(K^{(i)}\nabla\theta^{(i)}\right) = f_e^{(i)} \quad \text{in } Q^{(i)}, \tag{3.9b}$$

Balances on the moving interface

$$-[\![\mathcal{C}e(u) - \alpha\theta\mathbb{I}_3]\!] n_\Gamma = -\sigma_0 H_\Gamma n_\Gamma \quad \text{on } \Xi, \tag{3.9c}$$

$$[\![\rho c\theta]\!] V_\Gamma + [\![\gamma \operatorname{div} u]\!] V_\Gamma - [\![K\nabla\theta]\!] n_\Gamma = LV_\Gamma \quad \text{on } \Xi. \tag{3.9d}$$

Motion of the interface

$$V_\Gamma = v_\Gamma \quad \text{on } \Xi. \tag{3.9e}$$

In addition, we have to formulate transmission conditions, boundary conditions, and

[5] Actually, it is the deformation gradient that is assumed to be small, but, nevertheless, the assumption is usually called *the assumption of small deformations*.

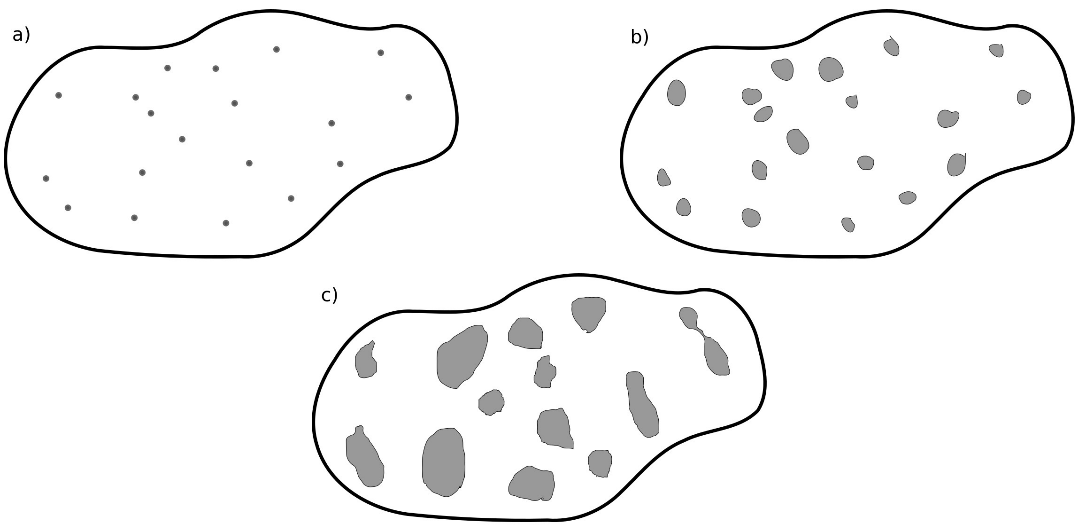


Figure 3.3: (*a*) Small phase nuclei, (*b*) grown nuclei, (*c*) coalescing nuclei. With the method of interface motions as described in the previous sections, it is only possible to model the process of getting from (*a*) to (*b*).

initial conditions. For the transmission, we usually expect both the deformations and the temperatures to have continuous transmission; these conditions are sometimes called *coherence* and *homothermal*. However, other conditions are also possible. For example, an imperfect heat transmission between the phases can be expressed via a *Robin type* interface conditions (see, e.g., [DLN15])

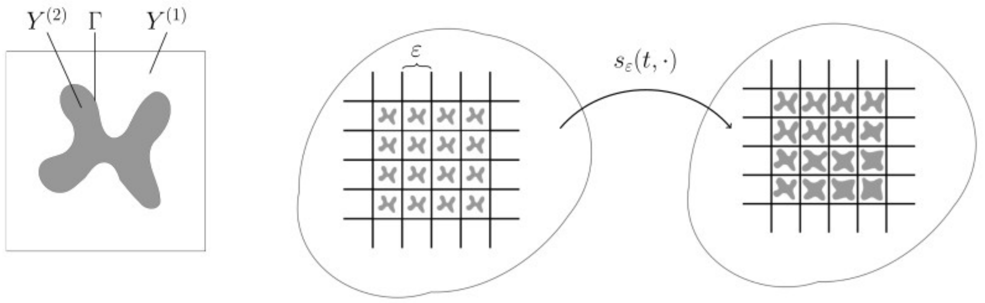
$$-K^{(2)}\nabla\theta^{(2)}\cdot n_\Gamma = \delta(\theta^{(2)} - \theta^{(1)}).$$

3.4 Homogenization setting

In many phase-change problems, there is the additional challenge that the transformations occur at a different scale than the scale of interest. Instead of a material consisting of two phases that are each made up of a small number of connected components which then might grow or shrink, similar as depicted in Figure 3.2, phase transformations are often better described as the growth of a lot of very small and, more or less, evenly distributed *phase nuclei*, cf. Figure 3.3.

The main challenge, here, is the complex structure of the interface Γ and of the motion s_Γ. Tackling numerically the thermoelasticity problem given via equations (3.9a) to (3.9d) comes at a high computational cost as the mesh used in the discretization has to resolve the geometry of the problem. Now, let $\varepsilon_0 > 0$ be representative for the size of the phase nuclei; as the nuclei are assumed to be small in comparison to the overall system, we expect $\varepsilon_0 \ll |\Omega|$.[6] The distribution of the nuclei is, in general, not perfectly uniform; but as an approximation, we assume the initial distribution of nuclei

[6]Here, we have made the hidden assumption that all of the nuclei are, at least approximately, of the same size. In general, this does not have to be the case.



(a) The unit cell Y (b) The periodic reference geometry (c) The actual geometry at time t

Figure 3.4: Reference geometry and the resulting ε-periodic initial configuration. Note that for $t \neq 0$, these domains typically loose their periodicity. The progress from (b) to (c) corresponds to the growing of nuclei presented in Figure 3.3.

to be periodic. This is one of the fundamental assumptions of the, therefore quite aptly named, theory of *periodic homogenization*.

We set $Y = (0,1)^3 \subset \mathbb{R}^3$ and take $Y^{(1)}$, $Y^{(2)} \subset Y$ two disjoint open sets, such that $Y^{(1)}$ is connected, such that $\Gamma := \overline{Y^{(1)}} \cap \overline{Y^{(2)}}$ is a C^2-hypersurface. Moreover, we expect $\Gamma = \partial Y^{(2)}$, $\overline{Y^{(2)}} \subset Y$, and $Y = Y^{(1)} \cup Y^{(2)} \cup \Gamma$, see Figure 3.4$(a)$. With $n_\Gamma = n_\Gamma(\gamma)$, $\gamma \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

For $\varepsilon > 0$, we introduce the εY-periodic, initial domains $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ and interface Γ_ε representing the two phases and the phase boundary, respectively, via $(i = 1,2)$

$$\Omega_\varepsilon^{(i)} = \Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(Y^{(i)} + k) \right), \qquad \Gamma_\varepsilon = \Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(\Gamma + k) \right).$$

Here, for a set $M \subset \mathbb{R}^3$, $k \in \mathbb{Z}^3$, and $\varepsilon > 0$, we employ the notation

$$\varepsilon(M + k) := \left\{ x \in \mathbb{R}^3 \ : \ \frac{x}{\varepsilon} - k \in M \right\}.$$

With $n_{\Gamma_\varepsilon} = n_{\Gamma_\varepsilon}(\frac{x}{\varepsilon})$, $x \in \Gamma_\varepsilon$, we denote the unit normal vector (extended by periodicity) pointing outwards $\Omega_\varepsilon^{(2)}$ into $\Omega_\varepsilon^{(1)}$. The above construction ensures that $\Omega_\varepsilon^{(1)}$ is connected and that $\Omega_\varepsilon^{(2)}$ is disconnected. We also have that $\partial\Omega_\varepsilon^{(2)} \cap \partial\Omega = \emptyset$.

We also introduce the corresponding, non-cylindrical space time domains

$$\begin{aligned} \Omega_\varepsilon^{(i)}(t) &= s_\varepsilon(t, \Omega_\varepsilon^{(i)}), \qquad & Q_\varepsilon^{(i)} &= \bigcup_{t\in S} \{t\} \times \Omega_\varepsilon^{(i)}(t), \\ \Gamma_\varepsilon(t) &= s_\varepsilon(t, \Gamma_\varepsilon), \qquad & \Xi_\varepsilon &= \bigcup_{t\in S} \{t\} \times \Gamma_\varepsilon(t), \end{aligned}$$

For any given $\varepsilon > 0$, in particular also for $\varepsilon = \varepsilon_0$, the corresponding balance equations and exchange conditions for the two-phase thermoelasticity problem are given via:

Two-phase thermoelasticity model - homogenization setting

$$-\operatorname{div}(\mathcal{C}_\varepsilon^{(i)}e(u_\varepsilon^{(i)}) - \alpha_\varepsilon^{(i)}\theta_\varepsilon^{(i)}\mathbb{I}_3) = f_\varepsilon^{(i)} \quad \text{in } Q_\varepsilon^{(i)}, \tag{3.10a}$$

$$\partial_t\left(\rho^{(i)}c^{(i)}\theta_\varepsilon^{(i)} + \gamma_\varepsilon^{(i)}\operatorname{div} u_\varepsilon^{(i)}\right) - \operatorname{div}(K_\varepsilon^{(i)}\nabla\theta_\varepsilon^{(i)}) = g_\varepsilon^{(i)} \quad \text{in } Q_\varepsilon^{(i)}, \tag{3.10b}$$

$$-[\![\mathcal{C}_\varepsilon e(u_\varepsilon) - \alpha_\varepsilon\theta_\varepsilon\mathbb{I}_3]\!]n_{\Gamma_\varepsilon} = -\sigma_0 H_{\Gamma_\varepsilon}n_{\Gamma_\varepsilon} \quad \text{on } \Xi_\varepsilon, \tag{3.10c}$$

$$[\![\rho c\theta_\varepsilon]\!]V_{\Gamma_\varepsilon} + [\![\gamma_\varepsilon \operatorname{div} u_\varepsilon]\!]V_{\Gamma_\varepsilon} - [\![K_\varepsilon\nabla\theta_\varepsilon]\!]n_{\Gamma_\varepsilon} = LV_{\Gamma_\varepsilon} \quad \text{on } \Xi_\varepsilon, \tag{3.10d}$$

$$V_{\Gamma_\varepsilon} = v_{\Gamma_\varepsilon} \quad \text{on } \Xi_\varepsilon. \tag{3.10e}$$

This model has to be completed with initial condition, transmission conditions, and boundary conditions. Regarding the transmission across the interface, both the temperatures and the deformations are assumed to be continuous across the interface, these conditions are sometimes called *homothermal* and *coherent*, respectively, see [BM05]. For the normal velocity, a very general *ansatz* would be given via $v_{\Gamma_\varepsilon} = v_{\Gamma_\varepsilon}(\theta_\varepsilon, H_{\Gamma_\varepsilon}, e(u_\varepsilon))$. Some common choices are $v_{\Gamma_\varepsilon}(\theta_\varepsilon, H_{\Gamma_\varepsilon}, e(u_\varepsilon)) = \beta(\theta_\varepsilon - \theta_{crit})$ (*kinetic undercooling*) or $v_{\Gamma_\varepsilon}(\theta_\varepsilon, H_{\Gamma_\varepsilon}, e(u_\varepsilon)) = \beta(-\Xi_0 H_{\Gamma_\varepsilon} + \theta_\varepsilon - \theta_{crit})$ (*Gibbs-Thomson undercooling*).

CHAPTER 4

Thermoelasticity problem with fixed interface

In this chapter, the analysis and homogenization of a two-phase thermoelasticity problem with prescribed interface motion are considered. The main difficulties here are the coupling between the mechanical part and the heat part as well as the time dependency, which is a consequence of the interface motion, of the involved operators.

Note that the following, Sections 4.1 to 4.4 to be precise, is published in [EM17b].[1] Some cosmetic changes regarding the typesetting as well as some changes to the notation (e.g., indicating the parameter ε via subscripts instead of superscripts) were done to ensure compatibility throughout the thesis. In addition, some references to other parts of the thesis as well as to Section 4.5 were added. In Section 4.5, some additional results, which were out of the scope of the article, connected to the thermoelasticity problem considered in [EM17b] are presented.

The main results of this chapter are:

- Theorems 4.7 and 4.8 where the solvability of the ε-problem and corresponding, ε-independent a priori estimates are established,
- the PDE-System given by equations (4.30a) to (4.30d) which is established as the limit problem,
- Theorem 4.18 where it is established that the limit problem admits exactly one solution.

4.1 Introduction

In this chapter, we consider a heterogeneous medium where the two building components are different solid phases of the same material (like *Austenite* and *Bainite* phases in steel, e.g.) separated by a sharp interface. One phase is assumed to be a connected matrix

[1] The results presented are due to the first author.

in which finely interwoven, periodically distributed inclusion of the second phase are embedded. The second phase is therefore disconnected. We refer to these phases as microstructures.

Our interest is the case where phase transformations are possible, e.g., one phase might grow at the expanse of the other phase, thereby leading to a motion of the interface and, as a consequence, to time dependent domains that are not necessarily periodic anymore. However, we assume to have *a priori* knowledge of the phase transformation, i.e., the motion of the interface is prescribed. For a rather general modeling of phase transformations (including a possible mathematical treatment), we refer the reader to [Vis96], and for the metallurgical perspective on phase transformation in steel (especially, with respect to the Bainite transformation), we refer to [Fie13, PE12, Sol08]. Looking at such a highly-heterogeneous medium, we study the coupling between the mechanics of the material and the thermal conduction effect (*thermomechanics*) under the influence of the phase transformation. In particular, we explore the interplay between *surface stresses* and *latent heat*, see for instance [Kup79] for related thermoelasticity scenarios. In this work, we start of from the quasi-static assumption that the mechanical processes are reversible. Furthermore, the constitutive laws are taken to be linear. Our main contribution here is the treatment of the mechanical dissipation and of *a priori* prescribed phase transformations in the thermoelasticity setting.

It is worth noting that the homogenization of different thermoelasticity problems has already been addressed in the literature. In one of the earlier works, [Fra83], a one-phase linear thermoelasticity problem is homogenized via a semi-group approach. In [TW11], a formal homogenization via asymptotic expansion for a similar model (but for a one-dimensional geometry) was conducted. A two-phase problem including transmission conditions and discontinuities at the interface has been investigated in the context of homogenization (using *periodic unfolding*) in [ETŢ15]. A similar situation of a highly heterogeneous two-phase medium with *a priori* given phase transformation was considered in [EKK02]. Here, the authors use formal asymptotic expansions to derive a homogenized model. We also want to point out the structural similarity between the thermoelasticity models and models for poroelasticity, cf. *Biot's linear poroelasticity* [Bio41, SM02]; for a reference of the derivation of the Biot model via two-scale homogenization, we refer to [Mik03, Section 5.2]. Examples for homogenization in the context of two-phase poroelasticity, so called double poroelasticity, can be found in [Ain13, EB14]. For some homogenization results via formal asymptotics for problems where the micro-structural changes are not prescribed, we refer to [BBPR16, KvNP14, Mei08].

As an alternative approach in the modeling of phase transformation, in particular in the case of phase transformations in steel, phase-field models are often considered, we refer to, e.g., [MSA+15, MBW08]. Some thoughts regarding possible numerical simulations of a similar one-phase problem for a highly heterogeneous media are given in [PL93]. In [TW11], a numerical framework based on homogenization (via averaging) for a thermoelasticity problem in highly heterogeneous media is developed and investigated.

The chapter is organized as follows: In Section 4.2, we introduce the ε-microscopic geometry and the thermoelasticity problem and, then, transform this to a fixed reference domain. The well-posedness of our microscopic model is investigated in Section 4.3. In addition, ε-independent estimates necessary for the homogenization process are established. Finally, in Section 4.4, we perform the homogenization procedure relying on the two-scale convergence technique.

4.2 Setting and transformation to fixed domain

We start by describing the geometrical setting of the ε-parametrized microscopic problem including the transformation characterizing the interface motion. After that, we go on with formulating the microscopic problem for a highly heterogeneous media – first for the moving interface and then for the back-transformed, fixed interface.

We note that our setting (with the transformation) is closely related to the notion of *locally periodic* domains, see [FAZM11, vNM11]. In addition, we also refer to [Dob14, Mei08], where similar transformation settings are introduced.

Let $S = (0, T)$, $T > 0$, be a time interval. Let Ω be the interior of a union of a finite number of closed cubes Q_j, $1 \leq j \leq n$, $n \in \mathbb{N}$, whose vertices are elements of $\mathbb{Q}^3$ (i.e., have rational coefficients) such that, in addition, Ω is a Lipschitz domain. At the cost of additional technical difficulties, Ω could be of much more general structure, see also Section 4.5. By this particular choice, we avoid the inherent technical difficulties that would arise in the homogenization process due to the involvement of general geometries; we are focusing instead on the technical difficulties arising (a) due to the strong coupling in the structure of the governing partial differential equations and (b) due to the time-dependency of the geometry.

In addition, we denote the outer normal vector of Ω with $\nu = \nu(x)$. Let $Y = (0,1)^3$ be the open unit cell in $\mathbb{R}^3$. Take $Y^{(1)}$, $Y^{(2)} \subset Y$ two disjoint open sets, such that $Y^{(1)}$ is connected, such that $\Gamma := \overline{Y^{(1)}} \cap \overline{Y^{(2)}}$ is a C^3 interface, $\Gamma = \partial Y^{(2)}$, $\overline{Y^{(2)}} \subset Y$, and $Y = Y^{(1)} \cup Y^{(2)} \cup \Gamma$, see Figure 4.1. With $n_\Gamma = n_\Gamma(y)$, $y \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

For $\varepsilon > 0$, we introduce the εY-periodic, initial domains $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ and interface Γ_ε representing the two phases and the phase boundary, respectively, via $(i = 1, 2)$

$$\Omega_\varepsilon^{(i)} = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^3} \varepsilon (Y^{(i)} + k) \right), \qquad \Gamma_\varepsilon = \Omega \cap \left(\bigcup_{k \in \mathbb{Z}^3} \varepsilon (\Gamma + k) \right).$$

Here, for a set $M \subset \mathbb{R}^3$, $k \in \mathbb{Z}^3$, and $\varepsilon > 0$, we employ the notation

$$\varepsilon(M + k) := \left\{ x \in \mathbb{R}^3 \ : \ \frac{x}{\varepsilon} - k \in M \right\}.$$

From now on, we take $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ to be a sequence of monotonically decreasing positive numbers converging to zero such that Ω can be represented as the union of cubes of size ε_n. Note that this is possible due to the particular structure of Ω; we refer to Section 2.4.

The above construction ensures that $\Omega_\varepsilon^{(1)}$ is connected and that $\Omega_\varepsilon^{(2)}$ is disconnected. We also have that $\partial\Omega_\varepsilon^{(2)} \cap \partial\Omega = \emptyset$. In the different case that both $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ are connected, we additionally would need to rely on special uniform extension operators, see [HB14], in order to pass to the homogenization limit. We also refer to Section 4.5, particularly Corollary 4.16.

Assumptions on the motion of the interface

We assume that we are given a function $s\colon \overline{S} \times \overline{\Omega} \times \mathbb{R}^3 \to \overline{Y}$ such that

(1) *Regularity*: $s \in C^1(\overline{S}; C^2(\overline{\Omega}) \times C^2_\#(Y))$,[2]

(2) *Invertibility*: $s(t,x,\cdot)_{|\overline{Y}}\colon \overline{Y} \to \overline{Y}$ is bijective for every $(t,x) \in \overline{S} \times \overline{\Omega}$,

(3) *Regularity of the inverse*: $s^{-1} \in C^1(\overline{S}; C^2(\overline{\Omega}) \times C^2_\#(Y))$,[3]

(4) *Initial condition*: $s(0,x,y) = y$ for all $x \in \overline{\Omega}$ and all $y \in \overline{Y}$,

(5) there is a constant $c > 0$ with $\operatorname{dist}(\partial Y, \gamma) > c$ for all $(t,x) \in \overline{S} \times \overline{\Omega}$ and $\gamma \in s(t,x,\Gamma)$,

(6) $s(t,x,y) = y$ for all $(t,x) \in \overline{S} \times \overline{\Omega}$ and for all $y \in \overline{Y}$ with $\operatorname{dist}(\partial Y, y) < \frac{c}{2}$,

(7) there are constants $c_s, C_s > 0$ satisfying

$$c_s \leq \det(D_y s(t,x,y)) \leq C_s, \quad (t,x,y) \in \overline{S} \times \overline{\Omega} \times \mathbb{R}^3.$$

We introduce the (t,x)-parametrized, deformed sets

$$Y^{(1)}(t,x) = s(t,x,Y^{(1)}), \quad Y^{(2)}(t,x) = s(t,x,Y^{(2)}), \quad \Gamma(t,x) = s(t,x,\Gamma).$$

Here, Assumptions (1)-(3) ensure that the transformation is regular enough for our further considerations (e.g., to guarantee that the curvature of the deformed domains is well-defined). For the initial configuration, we have $Y^{(1)} = Y^{(1)}(0,x)$, $Y^{(2)} = Y^{(2)}(0,x)$, and $\Gamma = \Gamma(0,x)$ (Assumption (4)). In addition, by Assumption (5), we get a uniform (with respect to $(t,x) \in \overline{S} \times \overline{\Omega}$) minimum distance between the interface $\Gamma(t,x)$ and the boundary of the Y-cell and, with Assumption (6), make sure that points near the boundary of the unit cell Y are not deformed. Finally, Assumption (7) is of particular importance when it comes to proving ε-independent estimates.

We introduce the operations

$$\begin{aligned} [\cdot]&\colon \mathbb{R}^3 \to \mathbb{Z}^3, \quad [x] = k \text{ such that } x - [x] \in Y, \\ \{\cdot\}&\colon \mathbb{R}^3 \to Y, \quad \{x\} = x - [x] \end{aligned}$$

[2] The # subscript denotes periodicity, i.e., for $k \in \mathbb{N}$, we have $C^k_\#(Y) = \{f \in C^k(\mathbb{R}^3) : f(x+e_i) = f(x) \text{ for all } x \in \mathbb{R}^3\}$, e_i basis vector of $\mathbb{R}^3$.

[3] Here, $s^{-1}\colon \overline{S} \times \overline{\Omega} \times \mathbb{R}^3 \to \overline{Y}$ is the unique function such that $s(t,x,s^{-1}(t,x,y)) = y$ for all $(t,x,y) \in \overline{S} \times \overline{\Omega} \times \overline{Y}$ extended by periodicity to all $y \in \mathbb{R}^3$.

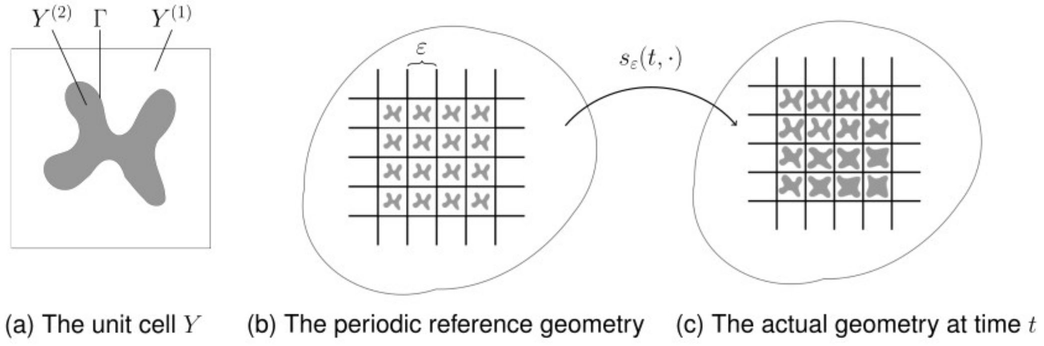


Figure 4.1: Reference geometry and the resulting ε-periodic initial configuration. Note that for $t \neq 0$, these domains typically loose their periodicity.

and define the ε-dependent function[4]

$$s_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^3, \quad s_\varepsilon(t,x) := \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon s\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right], \left\{\frac{x}{\varepsilon}\right\}\right). \tag{4.1}$$

The function s_ε is well-defined as $\left\{\frac{x}{\varepsilon}\right\} \in Y$ and $\varepsilon\left[\frac{x}{\varepsilon}\right] \in \overline{\Omega}$. Since $s(t,x,y) = y$ for all $(t,x) \in \overline{S} \times \overline{\Omega}$ and for all $y \in Y$ such that $\operatorname{dist}(\partial Y, y) > \frac{c}{2}$, we see that

$$s_\varepsilon \in C^1(\overline{S}; C^2(\overline{\Omega})).$$

For $i = 1,2$ and $t \in \overline{S}$, we set the time dependent sets $\Omega_\varepsilon^{(i)}(t)$ and $\Gamma_\varepsilon(t)$ and the corresponding non-cylindrical space-time domains $Q_\varepsilon^{(i)}$ and space-time phase boundary Ξ_ε via

$$\Omega_\varepsilon^{(i)}(t) = s_\varepsilon(t, \Omega_\varepsilon^{(i)}), \qquad Q_\varepsilon^{(i)} = \bigcup_{t\in S}\{t\} \times \Omega_\varepsilon^{(i)}(t),$$
$$\Gamma_\varepsilon(t) = s_\varepsilon(t, \Gamma_\varepsilon), \qquad \Xi_\varepsilon = \bigcup_{t\in S}\{t\} \times \Gamma_\varepsilon(t),$$

and denote by $n_{\Gamma_\varepsilon} = n_{\Gamma_\varepsilon}(t,x)$, $t \in S$, $x \in \Gamma_\varepsilon(t)$, the unit normal vector pointing outwards $\Omega_\varepsilon^{(2)}(t)$ into $\Omega_\varepsilon^{(1)}(t)$. The time-dependent domains $\Omega_\varepsilon^{(i)}(t)$ host the phases at time $t \in \overline{S}$ and model the motion of the interface Γ_ε. We emphasize again that, for any $t > 0$, the sets $\Omega_\varepsilon^{(1)}(t)$, $\Omega_\varepsilon^{(2)}(t)$, and $\Gamma_\varepsilon(t)$ do not need to be periodic.

For all $(t,x) \in \overline{S} \times \overline{\Omega}$, we introduce the functions

$$F_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^{3\times 3}, \qquad F_\varepsilon(t,x) := \nabla s_\varepsilon(t,x), \tag{4.2a}$$
$$J_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \mathbb{R}, \qquad J_\varepsilon(t,x) := \det\left(\nabla s_\varepsilon(t,x)\right), \tag{4.2b}$$
$$v_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^3, \qquad v_\varepsilon(t,x) := \partial_t s_\varepsilon(t,x), \tag{4.2c}$$

and see that[5]

$$n_{\Gamma_\varepsilon}(t, s_\varepsilon(t,\gamma)) = \frac{F_\varepsilon^{-T}(t,\gamma) n_{\Gamma_\varepsilon}(\gamma)}{|F_\varepsilon^{-T}(t,\gamma) n_{\Gamma_\varepsilon}(\gamma)|} \quad (\gamma \in \Gamma_\varepsilon). \tag{4.2d}$$

[4]This is the typical notation in the context of homogenization via the *periodic unfolding method*, see, e.g., [CDG08, Dob12].

[5]Here, F_ε^{-T} is defined via $F_\varepsilon^{-T}(t,x) := \left((F_\varepsilon(t,x))^{-1}\right)^T$.

In addition, we need the *normal velocity* (inwards $\Omega_\varepsilon^{(1)}(t)$) $\widehat{V}_{\Gamma_\varepsilon}$ and the *mean curvature* $\widehat{H}_{\Gamma_\varepsilon}$ of the interface $\Gamma_\varepsilon(t)$ (with respect to the coordinates of the initial configuration!):

$$\widehat{V}_{\Gamma_\varepsilon}\colon \overline{S}\times\Gamma_\varepsilon\to\mathbb{R},\quad \widehat{V}_{\Gamma_\varepsilon}(t,\gamma):=v_\varepsilon(t,\gamma)\cdot n_{\Gamma_\varepsilon}(t,s_\varepsilon(t,\gamma)), \tag{4.2e}$$

$$\widehat{H}_{\Gamma_\varepsilon}\colon \overline{S}\times\Gamma_\varepsilon\to\mathbb{R},\quad \widehat{H}_{\Gamma_\varepsilon}(t,\gamma):=-\operatorname{div}\left((F_\varepsilon)^{-1}(t,\gamma)n_{\Gamma_\varepsilon}(t,s_\varepsilon(t,\gamma))\right). \tag{4.2f}$$

We note that, via this definition, H_{Γ_ε} is non positive at points $\gamma\in\Gamma_\varepsilon$, where the intersection of $\Omega_\varepsilon^{(2)}$ and a sufficiently small ball with center x is a convex set, and that H_{Γ_ε} is non negative when this holds true for $\Omega_\varepsilon^{(1)}$.

Under this given transformation describing the phase transformation, i.e., the function s_ε and the resulting time dependent domains $\Omega_\varepsilon^{(i)}$, we consider a fully coupled thermoelasticity problem where we assume the mechanical response to be quasi-stationary and the constitutive laws to be linear.

For $i=1,2$, $t\in S$, and $x\in\Omega_\varepsilon^{(i)}(t)$, let $u_\varepsilon^{(i)}=u_\varepsilon^{(i)}(t,x)$ denote the deformation and $\theta_\varepsilon^{(i)}=\theta_\varepsilon^{(i)}(t,x)$ the temperature in the respective phase.

The bulk equations of thermoelasticity are given as

$$-\operatorname{div}(\mathcal{C}_\varepsilon^{(1)}e(u_\varepsilon^{(1)})-\alpha_\varepsilon^{(1)}\theta_\varepsilon^{(1)}\mathbb{I}_3)=f_\varepsilon^{(1)}\quad\text{in } Q_\varepsilon^{(1)}, \tag{4.3a}$$

$$-\operatorname{div}(\mathcal{C}_\varepsilon^{(2)}e(u_\varepsilon^{(2)})-\alpha_\varepsilon^{(2)}\theta_\varepsilon^{(2)}\mathbb{I}_3)=f_\varepsilon^{(2)}\quad\text{in } Q_\varepsilon^{(2)}, \tag{4.3b}$$

$$\partial_t\left(\rho^{(1)}c^{(1)}\theta_\varepsilon^{(1)}+\gamma_\varepsilon^{(1)}\operatorname{div}u_\varepsilon^{(1)}\right)-\operatorname{div}(K_\varepsilon^{(1)}\nabla\theta_\varepsilon^{(1)})=g_\varepsilon^{(1)}\quad\text{in } Q_\varepsilon^{(1)}, \tag{4.3c}$$

$$\partial_t\left(\rho^{(2)}c^{(2)}\theta_\varepsilon^{(2)}+\gamma_\varepsilon^{(2)}\operatorname{div}u_\varepsilon^{(2)}\right)-\operatorname{div}(K_\varepsilon^{(2)}\nabla\theta_\varepsilon^{(2)})=g_\varepsilon^{(2)}\quad\text{in } Q_\varepsilon^{(2)}. \tag{4.3d}$$

Here, $\mathcal{C}_\varepsilon^{(i)}\in\mathbb{R}^{3\times3\times3\times3}$ are the *stiffness* tensors, $\alpha_\varepsilon^{(i)}>0$ the *thermal expansion* coefficients, $\rho^{(i)}>0$ the *mass densities*, $c^{(i)}>0$ the *heat capacities*, $\gamma_\varepsilon^{(i)}>0$ are the *dissipation coefficients*, $K_\varepsilon^{(i)}\in\mathbb{R}^{3\times3}$ the *thermal conductivities*, and $f_\varepsilon^{(i)}$, $g_\varepsilon^{(i)}$ are volume densities. In addition, $e(v)=1/2(Dv+(Dv)^T)$ denotes the linearized strain tensor and $\mathbb{I}_3$ the identity matrix. For more details regarding the modeling, we refer to [Bio56, Kup79, WB16] and also to Chapter 3.

At the interface between the phases, the transmission of both the temperature and deformation is assumed to be continuous,[6] i.e.,

$$[\![u_\varepsilon]\!]=0,\quad [\![\theta_\varepsilon]\!]=0\quad\text{on }\ \Xi_\varepsilon, \tag{4.3e}$$

where $[\![v]\!]:=v^{(1)}-v^{(2)}$ denotes the jump across the interface separating the phases.

The jump in the flux of force densities across the interface is assumed to be proportional to the mean curvature of the interface leading to

$$[\![\mathcal{C}_\varepsilon\varepsilon(u_\varepsilon)-\alpha_\varepsilon\theta_\varepsilon\mathbb{I}_3]\!]n_{\Gamma_\varepsilon}=-\varepsilon^2H_{\Gamma_\varepsilon}\sigma_0n_{\Gamma_\varepsilon}\quad\text{on }\ \Xi_\varepsilon, \tag{4.3f}$$

where $\sigma_0>0$ is the coefficient of surface tension and where H_{Γ_ε} is the mean curvature of the interface with respect to the moving coordinates.[7] Here, the scaling via ε^2 counters

[6] These conditions are sometimes called *coherent* and *homothermal*, see [BM05].

[7] I.e., $H_{\Gamma_\varepsilon}(t,s_\varepsilon(t,x))=\widehat{H}_{\Gamma_\varepsilon}(t,x)$.

the effects of both the interface surface area, note that $\varepsilon|\Gamma_\varepsilon| \in \mathcal{O}(1)$, and the curvature itself, note that $\varepsilon|H_{\Gamma_\varepsilon}| \in \mathcal{O}(1)$.

In a similar way, the jump of the heat across the interface is assumed to be given via the constant of *latent heat* $L \in \mathbb{R}$:

$$[\![\rho c_d]\!]\theta_\varepsilon V_{\Gamma_\varepsilon} + [\![\gamma_\varepsilon \operatorname{div} u_\varepsilon]\!]V_{\Gamma_\varepsilon} - [\![K_\varepsilon \nabla\theta_\varepsilon]\!] \cdot n_{\Gamma_\varepsilon} = LV_{\Gamma_\varepsilon} \quad \text{in} \quad \Xi_\varepsilon, \tag{4.3g}$$

where V_{Γ_ε} denotes the normal velocity of the interface with respect to the moving coordinates. Note that, if we neglect the dissipation and if we have equal densities and heat capacities in both phases (or, a bit more general, $[\![\rho c_d]\!] = 0$), equation (4.3g) reduces to the usual *Stefan condition*. More complex interface conditions than equations (4.3f), (4.3f) would arise, if the interface were allowed to be thermodynamically active thereby requiring us to formulate separate balance equations for surface stress and surface heat, we refer to [WB16] and Chapter 3 as well as Section 4.5.

Finally, we pose homogeneous *Dirichlet conditions* for the momentum equation and homogeneous *Neumann conditions* for the heat equation as well as initial conditions for the temperature:

$$u_\varepsilon^{(1)} = 0 \qquad \text{on} \quad S \times \partial\Omega_\varepsilon^{(1)}, \tag{4.3h}$$

$$-K_\varepsilon^{(1)}\nabla\theta_\varepsilon^{(1)} \cdot \nu = 0 \qquad \text{on} \quad S \times \partial\Omega_\varepsilon^{(1)}, \tag{4.3i}$$

$$\theta_\varepsilon^{(i)}(0) = \vartheta_\varepsilon^{(i)} \quad \text{in} \quad \Omega_\varepsilon^{(i)}, \tag{4.3j}$$

where $\vartheta_\varepsilon^{(i)}$ are some (possibly highly heterogeneous) initial temperature distributions.

To summarize, we are considering a highly heterogeneous medium that is composed of two different phases/microstructures where one phase is a connected matrix in which small inclusions of the other phase are (in the beginning, periodically) embedded (see Figure 4.1), e.g., *bainitic* inclusions in *austenite* steel. Due to phase transformations (in our example, the bainitic inclusions might grow at the cost of the austenite phase) which are assumed to be completely known *a priori*, the phase domains change with time. In this geometrical setting, we then investigate the thermomechanical response of the two-phase medium to the surface stresses exerted by the phase interface due to its curvature (equation (4.3f)) and the latent heat released via the phase transformation (equation (4.3g)). We note that this situation has some similarity with the one considered in [EKK02].

Now, we choose a particular scaling of some coefficients with respect to the ε-parameter: For $i = 1, 2$, we assume that there are constants $\mathcal{C}^{(i)} \in \mathbb{R}^{3\times3\times3\times3}$, $K^{(i)} \in \mathbb{R}^{3\times3}$, $\alpha^{(i)}, \gamma^{(i)} > 0$ such that

$$\mathcal{C}_\varepsilon^{(1)} = \mathcal{C}^{(1)}, \qquad K_\varepsilon^{(1)} = K^{(1)}, \qquad \alpha_\varepsilon^{(1)} = \alpha^{(1)}, \qquad \gamma_\varepsilon^{(1)} = \gamma^{(1)},$$
$$\mathcal{C}_\varepsilon^{(2)} = \varepsilon^2\mathcal{C}^{(2)}, \quad K_\varepsilon^{(2)} = \varepsilon^2 K^{(2)}, \quad \alpha_\varepsilon^{(2)} = \varepsilon\alpha^{(2)}, \quad \gamma_\varepsilon^{(2)} = \varepsilon\gamma^{(2)}.$$

These specific ε-scalings are quite common in the modeling of two-phase media, see, e.g., [Ain13, CS99, EB14, FAZM11, Yeh11], and are usually justified (albeit only heuristically) by assuming different orders of magnitude of the characteristic time scales of the involved physical processes in the respective domains. In our case, this means that the effect of heat conduction, dissipation, stresses, and thermal expansion are assumed to be smaller/slower in the inclusions when compared to the matrix. By two-scale convergence results, this scaling leads to a distributed microstructure model, cf. [All92, Proposition 1.14. (ii)].

Other ε-scalings are, of course, possible and, depending on the underlying assumptions regarding the orders of magnitude of the involved processes, might be sensible. Without the scalings in the bulk equations (i.e., for $\mathcal{C}_\varepsilon^{(2)}$, $K_\varepsilon^{(2)}$, $\alpha_\varepsilon^{(2)}$, and $\gamma_\varepsilon^{(2)}$)), e.g., we would expect to get a purely macroscopical limit problem, where only some of the information of the microstructure (and their changes) are coded into the averaged coefficients, similar to the results in [Ain11]. Related problems without the ε^2-scaling of $\mathcal{C}_\varepsilon^{(2)}$ but otherwise the same scaling for similar scenarios were investigated in [Ain13, EB14] in the context of double poroelasticity. For a more holistic approach to different sets of scalings and their effect on the homogenization procedure, we refer to [PB08].

We assume that the tensors $\mathcal{C}^{(i)}$ and matrices $K^{(i)}$ are symmetric and have constant entries and also that there is a constant $c > 0$ such that $\mathcal{C}^{(i)}M : M \geq c\,|M|^2$ for all symmetric matrices $M \in \mathbb{R}^{3\times 3}$ and $K^{(i)}v \cdot v \geq c\,|v|^2$ for all $v \in \mathbb{R}^3$. Note that it would also be possible to treat non-constant coefficients as long as estimates (4.6a)-(4.6e) hold uniformly in time and space and as long as the functions are sufficiently regular for the analysis part to hold.[8] We also refer to Section 4.5 where some generalizations regarding the coefficients are considered.

Now, from the construction and the regularity of s, we have the following estimates available concerning the quantities related to the transformation that are defined by equations (4.2a)-(4.2f):

$$\sup_{\varepsilon>0}\Big(\|F_\varepsilon\|_{L^\infty(S\times\Omega)^{3\times3}} + \left\|F_\varepsilon^{-T}\right\|_{L^\infty(S\times\Omega)^{3\times3}} + \|J_\varepsilon\|_{L^\infty(S\times\Omega)} \\ + \varepsilon^{-1}\|v_\varepsilon\|_{L^\infty(S\times\Omega)^3} + \varepsilon^{-1}\|\widehat{V}_{\Gamma_\varepsilon}\|_{L^\infty(S\times\Gamma_\varepsilon)} + \varepsilon\|\widehat{H}_{\Gamma_\varepsilon}\|_{L^\infty(S\times\Gamma_\varepsilon)}\Big) < \infty,. \tag{4.4}$$

In addition, we also see that there is an ε-independent $c > 0$ such that $J_\varepsilon(t,x) \geq c$ for all $(t,x) \in \overline{S}\times\overline{\Omega}$.

For a given function $\varphi = \varphi(t,x)$, we denote the corresponding pull-back function by $\widehat{\varphi}_\varepsilon(t,x) = \varphi(t, s_\varepsilon^{-1}(t,x))$. We introduce the transformed coefficient functions needed to transform equations (4.3a)-(4.3i) in a fixed domain, i.e., to a formulation without a

[8] E.g., we would need $\rho^{(i)}c^{(i)}$, $\mathcal{C}^{(i)}$, and $\alpha^{(i)}$ to be differentiable with respect to time in order for Lemma 4.5 to hold.

motion of the phase interface:

$$\mathbb{A}_\varepsilon \colon S \times \Omega \to \mathbb{R}^{3\times3\times3\times3}, \qquad \mathbb{A}_\varepsilon B = \frac{1}{2}\left(F_\varepsilon^{-T} B + \left(F_\varepsilon^{-T} B\right)^T\right), \tag{4.5a}$$

$$\mathcal{C}_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^{3\times3\times3\times3}, \qquad \mathcal{C}_\varepsilon^{r,i} = J_\varepsilon \mathbb{A}_\varepsilon^T \mathcal{C}_\varepsilon^{(i)} \mathbb{A}_\varepsilon, \tag{4.5b}$$

$$\alpha_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^{3\times3}, \qquad \alpha_\varepsilon^{r,i} = J_\varepsilon \alpha_\varepsilon^{(i)} F_\varepsilon^{-T}, \tag{4.5c}$$

$$H_{\Gamma_\varepsilon}^r \colon S \times \Gamma_\varepsilon \to \mathbb{R}^{3\times3}, \qquad H_{\Gamma_\varepsilon}^r = J_\varepsilon \sigma_0 \widehat{H}_{\Gamma_\varepsilon} F_\varepsilon^{-1}, \tag{4.5d}$$

$$c_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}, \qquad c_\varepsilon^{r,i} = J_\varepsilon \rho^{(i)} c^{(i)}, \tag{4.5e}$$

$$\gamma_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}, \qquad \gamma_\varepsilon^{r,i} = J_\varepsilon \gamma_\varepsilon^{(i)} F_\varepsilon^{-T}, \tag{4.5f}$$

$$v_\varepsilon^r \colon S \times \Omega \to \mathbb{R}^3, \qquad v_\varepsilon^r = F_\varepsilon^{-1} v_\varepsilon, \tag{4.5g}$$

$$K_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^{3\times3}, \qquad K_\varepsilon^{r,i} = J_\varepsilon F_\varepsilon^{-1} K_\varepsilon^{(i)} F_\varepsilon^{-T}, \tag{4.5h}$$

$$V_{\Gamma_\varepsilon}^r \colon S \times \Gamma_\varepsilon \to \mathbb{R}, \qquad V_{\Gamma_\varepsilon}^r = J_\varepsilon \widehat{V}_{\Gamma_\varepsilon}, \tag{4.5i}$$

$$f_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^3, \qquad f_\varepsilon^{r,i} = J_\varepsilon \widehat{f_\varepsilon^{(i)}}, \tag{4.5j}$$

$$g_\varepsilon^{r,i} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}, \qquad g_\varepsilon^{r,i} = J_\varepsilon \widehat{g_\varepsilon^{(i)}}. \tag{4.5k}$$

Then, as a consequence of the estimate (4.4), we have

$$\sup_{\varepsilon>0} \sum_{i=1,2} \Bigg(\left\|\mathcal{C}_\varepsilon^{r,i}\right\|_{L^\infty(S\times\Omega)} + \left\|\alpha_\varepsilon^{r,i}\right\|_{L^\infty(S\times\Omega)} + \varepsilon \left\|H_{\Gamma_\varepsilon}^r\right\|_{L^\infty(S\times\Gamma)} + \left\|c_\varepsilon^{r,i}\right\|_{L^\infty(S\times\Omega)} + \varepsilon^{-1} \left\|v_\varepsilon^r\right\|_{L^\infty(S\times\Omega)} + \left\|K_\varepsilon^{r,i}\right\|_{L^\infty(S\times\Omega)} + \varepsilon^{-1} \left\|V_{\Gamma_\varepsilon}^r\right\|_{L^\infty(S\times\Gamma)} \Bigg) < \infty. \tag{4.6a}$$

Furthermore, using the uniform positivity of J_ε, we get the following uniform positivity estimates

$$\mathcal{C}_\varepsilon^{r,i} M : M \geq c\,|M|^2 \quad \text{for all} \quad M \in \operatorname{Sym}(3), \tag{4.6b}$$

$$\alpha_\varepsilon^{r,i} v \cdot v \geq c\,|v|^2 \quad \text{for all} \quad v \in \mathbb{R}^3, \tag{4.6c}$$

$$c_\varepsilon^{r,i} \geq c, \tag{4.6d}$$

$$K_\varepsilon^{r,i} v \cdot v \geq c\,|v|^2 \quad \text{for all} \quad v \in \mathbb{R}^3. \tag{4.6e}$$

Taking the back-transformed quantities (defined on the initial periodic domains $\Omega_\varepsilon^{(i)}$) $U_\varepsilon^{(i)} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^3$ and $\Theta_\varepsilon^{(i)} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^3$ given via $U_\varepsilon^{(i)}(t,x) = u_\varepsilon^{(i)}(t, s_\varepsilon^{-1}(t,x))$ and $\Theta_\varepsilon^{(i)}(t,x) = \theta_\varepsilon^{(i)}(t, s_\varepsilon^{-1}(t,x))$, we get the following problem in fixed coordinates:

Thermoelasticity problem - fixed coordinates

$$-\operatorname{div}\left(\mathcal{C}_\varepsilon^{r,1}(U_\varepsilon^{(1)}) - \Theta_\varepsilon^{(1)} \alpha_\varepsilon^{r,1}\right) = f_\varepsilon^{r,1} \quad \text{in} \quad S \times \Omega_\varepsilon^{(1)}, \tag{4.7a}$$

$$-\operatorname{div}\left(\varepsilon^2 \mathcal{C}_\varepsilon^{r,2} e(U_\varepsilon^{(2)}) - \varepsilon \Theta_\varepsilon^{(2)} \alpha_\varepsilon^{r,2}\right) = f_\varepsilon^{r,2} \quad \text{in} \quad S \times \Omega_\varepsilon^{(2)}, \tag{4.7b}$$

$$\partial_t \left(c_\varepsilon^{r,1} \Theta_\varepsilon^{(1)} + \gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) - \operatorname{div}\left(K_\varepsilon^{r,1} \nabla \Theta_\varepsilon^{(1)}\right) - \operatorname{div}\left(\left(c_\varepsilon^{r,1} \Theta_\varepsilon^{(1)} + \gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) v_\varepsilon^r\right) = g_\varepsilon^{r,1} \quad \text{in} \quad S \times \Omega_\varepsilon^{(1)}, \tag{4.7c}$$

$$\partial_t \left(c_\varepsilon^{r,2} \Theta_\varepsilon^{(2)} + \varepsilon \gamma_\varepsilon^{r,2} : DU_\varepsilon^{(2)}\right) - \operatorname{div}\left(\varepsilon^2 K_\varepsilon^{r,2} \nabla \Theta_\varepsilon^{(2)}\right) - \operatorname{div}\left(\left(c_\varepsilon^{r,2} \Theta_\varepsilon^{(2)} + \varepsilon \gamma_\varepsilon^{r,2} : DU_\varepsilon^{(2)}\right) v_\varepsilon^r\right) = g_\varepsilon^{r,2} \quad \text{in} \quad S \times \Omega_\varepsilon^{(2)}, \tag{4.7d}$$

complemented with interface transmission, boundary, and initial conditions.

For more details regarding the transformation to a fixed domain, we refer to [Dob12, Mei08, PSZ13] as well as to Section 2.3.

Note that the structure of this system is similar to the moving interface problem given via equations (4.3a)-(4.3i), except for the advection terms, some additional non-isotropic effects, and the time/space dependency of all coefficients.

4.3 Analysis of the micro problem

We introduce the functional spaces

$$V_u := W_0^{1,2}(\Omega)^3, \quad V_\theta := W^{1,2}(\Omega), \quad H := L^2(\Omega)$$

and obtain, after identifying H with their dual via *Riesz's representation map*, the *Gelfand triple*, $V_\theta \hookrightarrow H \hookrightarrow V_\theta{}'$. By $(,)_H$ and $\langle , \rangle_{V'V}$, we denote the inner product of a Hilbert space H and the dual product of a Banach space V, respectively.

Using the well-known *Korn inequality* (see, e.g., [DL76]), we see that we can use

$$\|u\|_{V_u} := \|e(v)\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + \varepsilon \|e(v)\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}},$$

where $e(u) = 1/2(Du + (Du)^T)$, instead of the standard Sobolev norm for V_u. In the following lemma, we establish some control on the parameter ε

Lemma 4.1. *There is a $C > 0$ independent of $\varepsilon > 0$ such that*

$$\|v\|_{L^2(\Omega)^3} + \|Dv\|_{L^2(\Omega_\varepsilon^{(1)})^3} + \varepsilon \|Dv\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \leq C \left(\|e(v)\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + \varepsilon \|e(v)\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \right), \quad v \in V_u. \quad (4.8)$$

Proof. Let $v \in V_u$ and set $v_\varepsilon^{(i)} := v_{|\Omega_\varepsilon^{(i)}}$, $i = 1, 2$. Then, via extending $v_\varepsilon^{(1)}$ appropriately to the whole of Ω (for this, we refer to, e.g., [OSY92, Chapter 1.4]), we call that respective extension $\widetilde{v_\varepsilon^{(1)}}$. Then, using Korn's inequality, we get the ε-independent estimate

$$\|\widetilde{v_\varepsilon^{(1)}}\|_{W^{1,2}(\Omega)} \leq C_1 \|e(v_\varepsilon^{(1)})\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}}.$$

Now, since $v \in V_u$, we have $w_\varepsilon := (v - \widetilde{v_\varepsilon^{(1)}})_{|\Omega_\varepsilon^{(2)}} \in W_0^{1,2}(\Omega_\varepsilon^{(2)})$. Via a scaling argument (and using Korn's inequality for functions in $W_0^{1,2}(Y^{(2)})$), we see that

$$\|w_\varepsilon\|_{L^2(\Omega_\varepsilon^{(2)})^3} + \varepsilon \|Dw_\varepsilon\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \leq \varepsilon C_2 \|e(w_\varepsilon)\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}}.$$

This leads to

$$\|v_\varepsilon^{(2)}\|_{L^2(\Omega_\varepsilon^{(2)})^3} + \varepsilon \|Dv_\varepsilon^{(2)}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \leq (1 + C_2) \left(\|\widetilde{v_\varepsilon^{(1)}}\|_{W^{1,2}(\Omega)^3} + \varepsilon C_2 \|e(v_\varepsilon^{(2)})\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \right).$$

Finally, setting $C = \max\{C_1(2 + C_2), C_2\}$, we get the desired estimate. $\square$

Our concept of weak formulation corresponding to the problem in fixed domain, equations (4.7a)-(4.7d), is given as:

Variational formulation

Find $(U_\varepsilon, \Theta_\varepsilon) \in L^2(S; V_u \times V_\theta)$ such that $(\partial_t U_\varepsilon, \partial_t \Theta_\varepsilon) \in L^2(S; V_u' \times V_\theta')$ and $\Theta_\varepsilon(0) = \theta_{0\varepsilon}$ satisfying

$$\begin{aligned}\int_{\Omega_\varepsilon^{(1)}} \mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{(1)}) : e(v_u)\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} \mathcal{C}_\varepsilon^{r,2} e(U_\varepsilon^{(2)}) : e(v_u)\,\mathrm{d}x \\ - \int_{\Omega_\varepsilon^{(1)}} \Theta_\varepsilon^{(1)} \alpha_\varepsilon^{r,1} : \nabla v_u\,\mathrm{d}x - \varepsilon \int_{\Omega_\varepsilon^{(2)}} \Theta_\varepsilon^{(2)} \alpha_\varepsilon^{r,2} : \nabla v_u\,\mathrm{d}x \\ = \int_{\Omega_\varepsilon^{(1)}} f_\varepsilon^{r,1} \cdot v_u\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} f_\varepsilon^{r,2} \cdot v_u\,\mathrm{d}x + \varepsilon^2 \int_{\Gamma_\varepsilon} H_{\Gamma_\varepsilon}^r n_{\Gamma_\varepsilon} \cdot v_u\,\mathrm{d}s, \end{aligned} \tag{4.9a}$$

$$\begin{aligned}\int_{\Omega_\varepsilon^{(1)}} \partial_t \left(c^{r,1} \Theta_\varepsilon^{(1)}\right) v_\theta\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(1)}} v_\varepsilon^r \Theta_\varepsilon^{(1)} \cdot \nabla v_\theta\,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(2)}} \partial_t \left(c^{r,2} \Theta_\varepsilon^{(2)}\right) v_\theta\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} v_\varepsilon^r \Theta_\varepsilon^{(2)} \cdot \nabla v_\theta\,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(\gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) v_\theta\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(1)}} v_\varepsilon^r \left(\gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) \cdot \nabla v_\theta\,\mathrm{d}x \\ + \varepsilon \int_{\Omega_\varepsilon^{(2)}} \partial_t \left(\gamma_\varepsilon^{r,2} : DU_\varepsilon^{(2)}\right) v_\theta\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon^{(2)}} v_\varepsilon^r \left(\gamma_\varepsilon^{r,2} : DU_\varepsilon^{(2)}\right) \cdot \nabla v_\theta\,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(1)}} K_\varepsilon^{r,1} \nabla \Theta_\varepsilon^{(1)} \cdot \nabla v_\theta\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} K_\varepsilon^{r,2} \nabla \Theta_\varepsilon^{(2)} \cdot \nabla v_\theta\,\mathrm{d}x \\ + \int_{\Gamma_\varepsilon} V_{\Gamma_\varepsilon}^r v_\theta\,\mathrm{d}s = \int_{\Omega_\varepsilon^{(1)}} g_\varepsilon^{r,1} v_\theta\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} g_\varepsilon^{r,2} v_\theta\,\mathrm{d}x \end{aligned} \tag{4.9b}$$

for all $(v_u, v_\theta) \in L^2(S; V_u \times V_\theta)$.

We start off with the mechanical part, i.e., equation (4.9a), and define, for $t \in S$, the linear operators

$$\begin{aligned} E_\varepsilon(t) \colon V_u \to {V_u}', \qquad & F_\varepsilon(t) \in L^2(\Omega), \\ e_\varepsilon^{th}(t) \colon H \to {V_u}', \qquad & \mathcal{H}_\varepsilon(t) \in {V_u}' \end{aligned}$$

via

$$\begin{aligned} \langle E_\varepsilon(t) u, v \rangle_{{V_u}' V_u} &= \int_{\Omega_\varepsilon^{(1)}} \mathcal{C}_\varepsilon^{r,1}(t) e(u) : e(v)\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} \mathcal{C}_\varepsilon^{r,2}(t) e(u) : e(v)\,\mathrm{d}x, \\ \left\langle e_\varepsilon^{th}(t) \varphi, v \right\rangle_{{V_u}' V_u} &= \int_{\Omega_\varepsilon^{(1)}} \varphi \alpha_\varepsilon^{r,1}(t) : Dv\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon^{(2)}} \varphi \alpha_\varepsilon^{r,2}(t) : Dv\,\mathrm{d}x, \\ F_\varepsilon(t) &= \begin{cases} f_\varepsilon^{r,1}(t), & x \in \Omega_\varepsilon^{(1)} \\ f_\varepsilon^{r,2}(t), & x \in \Omega_\varepsilon^{(2)} \end{cases}, \\ \langle \mathcal{H}_\varepsilon(t), v \rangle_{{V_u}' V_u} &= \varepsilon^2 \int_{\Gamma_\varepsilon} H_{\Gamma_\varepsilon}^r(t) n_{\Gamma_\varepsilon} \cdot v\,\mathrm{d}s. \end{aligned}$$

The weak form (4.9a) is then equivalent to the operator equation

$$E_\varepsilon(t)U_\varepsilon - e_\varepsilon^{th}(t)\Theta_\varepsilon = F_\varepsilon(t) + \mathcal{H}_\varepsilon(t) \quad \text{in} \quad V_u'. \tag{4.10}$$

Lemma 4.2. *The operator $E_\varepsilon(t)$, $t \in \overline{S}$, is coercive, continuous (both uniformly in time and in the parameter ε), and symmetric.*

Proof. Let $u, v \in W^{1,2}(\Omega)$. Due to the estimates (4.6b), (4.6c), we have

$$\langle E_\varepsilon(t)v, v\rangle_{V_u'V_u} \geq c_{\mathcal{C}} \|v\|_{V_u}^2 ,$$
$$\left| \langle E_\varepsilon(t)u, v\rangle_{V_u'V_u} \right| \leq C_{\mathcal{C}} \|u\|_{V_u} \|v\|_{V_u}$$

for all $t \in \overline{S}$ and all $\varepsilon > 0$. The symmetry follows, after transforming to moving coordinates, from the symmetry of the $\mathcal{C}_i$. ☐

Since $E_\varepsilon(t)$ is coercive and continuous, we therefore established, via *Lax-Milgram's Lemma*, that, for all $F_\varepsilon \in L^2(\Omega)$, $\mathcal{H}_\varepsilon \in L^2(\Gamma_\varepsilon)$, $\Theta_\varepsilon \in L^2(\Omega)$, and $t \in \overline{S}$, there is a unique weak solution $U_\varepsilon(t) \in V_u$ to Problem 4.10. In particular, the inverse operator $E_\varepsilon^{-1}(t)$ is well defined as well as linear, bounded, and coercive.

Now, we turn our attention to the heat related part of our system, i.e., equation (4.9b), where we deal with the coupling[9] due to the dissipation term. This is done by combining the structure of the full problem (the *coupling operators* are basically dual to one another) and the just investigated properties of the operators of the mechanical part. Note that the following considerations regarding the *thermal stress operator* e_ε^{th} are (in spirit) quite similar to those presented in [SM02]. We see that, for $v_\theta \in V_\theta$ and $v_u \in V_u$, we have[10]

$$\begin{aligned}
\left\langle e_\varepsilon^{th}(t)v_\theta, v\right\rangle_{V_u'V_u} &= \int_{\Omega_\varepsilon^{(1)}} v_\theta \alpha^{r,1}(t) : Dv_u \,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon^{(2)}} v_\theta \alpha^{r,2}(t) : Dv_u \,\mathrm{d}x \\
&= -\int_{\Omega_\varepsilon^{(1)}} \alpha^{r,1}(t) Dv_\theta \cdot v_u \,\mathrm{d}x - \varepsilon \int_{\Omega_\varepsilon^{(2)}} \alpha^{r,2}(t) Dv_\theta \cdot v_u \,\mathrm{d}x \\
&\quad + \int_{\Gamma_\varepsilon} [\![\alpha^r]\!](t) v_\theta n_{\Gamma_\varepsilon}(t) \cdot v_u \,\mathrm{d}s,
\end{aligned}$$

and, as a result,

$$e_\varepsilon^{th}(t)_{|V_\theta} \colon V_\theta \to L^2(\Omega)^3 \times L^2(\Gamma_\varepsilon)^3 \subset V_u{}'.$$

In addition, we take a look at the corresponding dual operator $\left(e_\varepsilon^{th}(t)_{|V_\theta}\right)' : L^2(\Omega)^3 \times L^2(\Gamma_\varepsilon)^3 \to V_\theta{}'$ given via

$$\left\langle \left(e_\varepsilon^{th}(t)_{|V_\theta}\right)' [f,g], v_\theta \right\rangle_{V_\theta' V_\theta} = \left(e_\varepsilon^{th}(t)_{|V_\theta} v_\theta, [f,g]\right)_{L^2(\Omega)\times L^2(\Gamma_\varepsilon)} .$$

[9] And therefore the mixed derivative term for the deformations $u_\varepsilon^{(i)}$.
[10] Here, we used that $[\![\alpha^r(t)]\!] = [\![\alpha]\!]$ and $\operatorname{div}(J_\varepsilon F_\varepsilon) = 0$.

For functions $v_u \in V_u$, we have $v_u = [v_u, v_{u|\Gamma_\varepsilon}] \in L^2(\Omega)^3 \times L^2(\Gamma_\varepsilon)^3$ and see that

$$\left\langle \left(e_\varepsilon^{th}(t)_{|V_\theta}\right)' v_u, v_\theta \right\rangle_{V_\theta' V_\theta} = \int_{\Omega_\varepsilon^{(1)}} v_\theta \alpha^{r,1}(t) : Dv_u \,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} v_\theta \alpha^{r,2}(t) : Dv_u \,\mathrm{d}x.$$

As a consequence, we have $\left(e_\varepsilon^{th}(t)_{|V_\theta}\right)'_{|V_u} : V_u \to H \subset V_\theta{}'$. For $v_u \in V_u$ and $f \in H$,

$$\begin{aligned}\left(\left(e_\varepsilon^{th}(t)_{|V_\theta}\right)'_{|V_u} v_u, f\right)_H &= \int_{\Omega_\varepsilon^{(1)}} f\alpha^{r,1}(t) : Dv_u \,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} f\alpha^{r,2}(t) : Dv_u \,\mathrm{d}x \\ &= \left\langle e_\varepsilon^{th}(t) f, v_u \right\rangle_{V_u' V_u},\end{aligned}$$

which implies $\left(e_\varepsilon^{th}(t)_{|V_\theta}\right)'_{|V_u} = \left(e_\varepsilon^{th}(t)\right)'$ for all $t \in S$.

From the definition of the operator $e_\varepsilon^{th}(t)$, we have the following uniform estimate:

Lemma 4.3. *For $v_u \in V_u$ and $f \in H$, it holds (uniform in $t \in \overline{S}$ and $\varepsilon > 0$)*

$$\left|\left\langle e_\varepsilon^{th}(t) f, v_u \right\rangle_{V_u' V_u}\right| \leq C\|f\|_H \left(\|\nabla v_u\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + \varepsilon \|\nabla v_u\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}}\right). \tag{4.11}$$

Now, we introduce some linear, t-parametrized functions:[11]

$$\begin{aligned}
&B_\varepsilon^{(1)}(t)\colon H \to H, && B_\varepsilon^{(1)}(t) f = c_\varepsilon^r(t) f, \\
&B_\varepsilon^{(2)}(t)\colon H \to H, && B_\varepsilon^{(2)}(t) f = \frac{\gamma}{\alpha}\left(e_\varepsilon^{th}(t)\right)' E_\varepsilon^{-1}(t) e_\varepsilon^{th}(t) f, \\
&A_\varepsilon^{(1)}(t)\colon V_\theta \to V_\theta{}', && \left\langle A_\varepsilon^{(1)}(t) v_\theta, v_\theta \right\rangle_{V_\theta' V_\theta} = (\rho c v_\varepsilon^r(t) v_\theta, \nabla v_\theta)_H \\
&A_\varepsilon^{(2)}(t)\colon V_\theta \to V_\theta{}', && \langle A_\varepsilon^{(2)}(t) v_\theta, v_\theta \rangle_{V_\theta' V_\theta} = (K_\varepsilon^r(t) \nabla v_\theta, \nabla v_\theta)_H, \\
&A_\varepsilon^{(3)}(t)\colon V_\theta \to V_\theta{}', && \left\langle A_\varepsilon^{(3)}(t) v_\theta, v_\theta \right\rangle_{V_\theta' V_\theta} = \left(v_\varepsilon^r(t) B_\varepsilon^{(2)}(t) v_\theta, \nabla v_\theta\right)_H.
\end{aligned}$$

and $\mathcal{F}_\varepsilon(t) \in V_\theta{}'$, $G_\varepsilon(t) \in L^2(S; H)$ via

$$\begin{aligned}
\langle \mathcal{F}_\varepsilon(t), v_\theta \rangle_{V_\theta' V_\theta} &= \left(\partial_t \left(\frac{\gamma}{\alpha}\left(e_\varepsilon^{th}(t)\right)' E_\varepsilon^{-1}\left(F_\varepsilon(t) + \mathcal{H}_\varepsilon(t)\right)\right), v_\theta\right)_H \\
&\quad + \left(v_\varepsilon^r \frac{\gamma}{\alpha}\left(e_\varepsilon^{th}(t)\right)' E_\varepsilon^{-1}\left(F_\varepsilon(t) + \mathcal{H}_\varepsilon(t)\right), \nabla v_\theta\right)_H, \\
G_\varepsilon(t) &= \begin{cases} g_\varepsilon^r(t), & x \in \Omega_\varepsilon^{(1)} \\ g_\varepsilon^r(t), & x \in \Omega_\varepsilon^{(2)} \end{cases}.
\end{aligned}$$

We note that $\mathcal{F}_\varepsilon(t)$ is well-defined if, for example, $F_\varepsilon \in C^1(S; H)^3$.

The variational formulation (4.9b) can then be rewritten as an abstract operator problem:

Operator formulation

Find $\Theta_\varepsilon \in L^2(S; V_\theta)$ such that $\partial_t \Theta_\varepsilon \in L^2(S; V_\theta{}')$, such that $\Theta_\varepsilon(0) = \theta_{\varepsilon 0}$, and such that

$$\partial_t\left(\sum_{i=1}^{2} B_\varepsilon^{(i)}(t)\Theta_\varepsilon\right) + \sum_{i=1}^{3} A_\varepsilon^{(i)}(t)\Theta_\varepsilon + \mathcal{V}_{\Gamma_\varepsilon}(t) = G_\varepsilon(t) - \mathcal{F}_\varepsilon(t) \quad \text{in } V_\theta', \ t \in S. \tag{4.12}$$

[11] Note that $\gamma_\varepsilon^r = \frac{\gamma}{\alpha}\alpha_\varepsilon^r$.

To tackle above problem, we first have to investigate some properties of the involved operators.

Lemma 4.4. *The operator $B_\varepsilon^{(2)}$ is continuous (uniformly in $t \in \overline{S}$ and $\varepsilon > 0$), self-adjoint, and strictly monotone. In addition, for every $f, g \in H$, we have $(B_\varepsilon^{(2)}(\cdot)f, g)_H \in L^\infty(S)$.*

Proof. We start off with proving the continuity property. Let $f \in L^2(\Omega)$ and $u_\varepsilon(f) := E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f$, i.e., the unique solution of

$$\langle E_\varepsilon(t)u_\varepsilon(f), v_u\rangle_{V_u'V_u} = (e_\varepsilon^{th}(t)f, v_u)_H, \quad v_u \in V_u.$$

Due to the estimates from Lemmas 4.2 and 4.3, we have $\|u_\varepsilon(f)\|_{V_u} \leq C\|f\|_H$, which implies, for all $g \in H$,

$$\left|\left(B_\varepsilon^{(2)} f, g\right)_H\right| \leq \frac{\gamma}{\alpha}\|e_\varepsilon^{th}(t)g\|_{V_u'}\|u_\varepsilon(f)\|_{V_u} \leq C\|g\|_H\|f\|_H,$$

where $C > 0$ is independent of both $t \in \overline{S}$ and $\varepsilon > 0$. As an immediate consequence, $(B_\varepsilon^{(2)}(\cdot)f, g)_H \in L^\infty(S)$. Furthermore, since

$$\left(B_\varepsilon^{(2)} f, g\right)_\Omega, = \left\langle e_\varepsilon^{th}(t)g, E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f\right\rangle_{V_u'V_u}$$

and since E_ε^{-1} is strictly monotone and symmetric, we also have that $B_\varepsilon^{(2)}$ is monotone and self-adjoint. □

We establish some further regularity (with respect to time) of the following operator:

$$B_\varepsilon(t)\colon L^2(\Omega) \to L^2(\Omega) \quad \text{via} \quad B_\varepsilon(t) = B_\varepsilon^{(1)}(t) + B_\varepsilon^{(2)}(t).$$

Lemma 4.5. *There is a $C > 0$ independent of $t \in S$ and $\varepsilon > 0$ such that*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon(t)f, g\right)_H\right| \leq C\,\|f\|_H\,\|g\|_H$$

for all $f, g \in H$.

Proof. Let $f, g \in H$ be given. Then,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon(t)f, g\right)_H\right| \leq \left|\partial_t\left(c^r(t)\right)\right|\|f\|_H\,\|g\|_H + \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(e_\varepsilon^{th}(t)\right)' E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f, g\right)_H\right|.$$

In addition,

$$\left(\left(e_\varepsilon^{th}(t)\right)' E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f, g\right)_H = \left\langle e_\varepsilon^{th}(t)g, E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f\right\rangle_{V_u'V_u},$$

where $u_\varepsilon^f(t) := E_\varepsilon^{-1}(t)e_\varepsilon^{th}(t)f$ admits the ε-uniform bound

$$\|u_\varepsilon(f)\|_{V_u} \leq C\|f\|_H.$$

Formally, provided that all derivatives exist, we have

$$\partial_t u_\varepsilon^f(t) = \partial_t E_\varepsilon^{-1}(t) e_\varepsilon^{th}(t) f + E_\varepsilon^{-1}(t)\partial_t e_\varepsilon^{th}(t) f.$$

Introducing the operators $\widetilde{E_\varepsilon}(t)\colon V_u \to {V_u}'$ and $\widetilde{e_\varepsilon^{th}}(t)\colon H \to {V_u}'$ via

$$\begin{aligned}
\left\langle \widetilde{E_\varepsilon}(t)u, v\right\rangle_{{V_u}'V_u} &= \int_{\Omega_\varepsilon^{(1)}} \partial_t \mathcal{C}^{r,1}(t) e(u) : e(v)\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} \partial_t \mathcal{C}^{r,2}(t) e(u) : e(v)\,\mathrm{d}x,\\
\left\langle \widetilde{e_\varepsilon^{th}}(t) f, v\right\rangle_{{V_u}'V_u} &= \int_{\Omega_\varepsilon^{(1)}} f \partial_t \alpha^{r,1}(t) : Dv\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon^{(2)}} f \partial_t \alpha^{r,2}(t) : Dv\,\mathrm{d}x,
\end{aligned}$$

and $\widetilde{u_\varepsilon^f} \in V_u$ as the unique solution to

$$\left\langle E_\varepsilon(t)\widetilde{u_\varepsilon^f}, v_u\right\rangle_{{V_u}'V_u} = \left\langle \widetilde{e_\varepsilon^{th}} f, v_u\right\rangle_{{V_u}'V_u} - \left\langle \widetilde{E_\varepsilon} u_\varepsilon^f, v_u\right\rangle_{{V_u}'V_u}, \quad v_u \in V_u, \tag{4.13}$$

we see that this is justified and $\partial_t u_\varepsilon^f = \widetilde{u_\varepsilon^f}$. Furthermore, in testing the weak formulation given via equation (4.13) with $\partial_t u_\varepsilon(f)$ and using both the uniform bounds on the coefficients and the estimate on $u_\varepsilon(f)$, inequality (4.6a), we see that

$$\left\|\partial_t u_\varepsilon^f(t)\right\|_{V_u} \leq C \left\|f\right\|_H,$$

where $C > 0$ is independent of $t \in \overline{S}$ and $\varepsilon > 0$, and, due to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon^{(2)}(t)f, g\right)_H = &\int_{\Omega_\varepsilon^{(1)}} g\partial_t \alpha^{r,1}(t) : Du_\varepsilon^f(t)\,\mathrm{d}x + \int_{\Omega_\varepsilon^{(1)}} g\alpha^{r,1}(t) : D\partial_t u_\varepsilon^f(t)\,\mathrm{d}x\\
&+ \varepsilon \int_{\Omega_\varepsilon^{(2)}} g\partial_t \alpha^{r,2}(t) : Du_\varepsilon^f(t)\,\mathrm{d}x + \varepsilon \int_{\Omega_\varepsilon^{(2)}} g\alpha^{r,2}(t) : D\partial_t u_\varepsilon^f(t)\,\mathrm{d}x,
\end{aligned}$$

we then get the proposed estimate. □

We introduce the operator $A_\varepsilon(t)\colon V_\theta \to {V_\theta}'$ via $A_\varepsilon(t) = \sum_{i=1}^3 A_\varepsilon^{(i)}(t)$.

Lemma 4.6. *There are $\lambda_1, \lambda_2 > 0$ (independent of $t \in \overline{S}$ and $\varepsilon > 0$) such that*

$$\left\langle A_\varepsilon(t)v_\theta, v_\theta\right\rangle_{{V_\theta}'V_\theta} + \lambda_1 \left(B_\varepsilon(t)v_\theta, v_\theta\right)_H \geq \lambda_2 \left\|v_\theta\right\|_{V_\theta}, \quad v_\theta \in V_\theta.$$

Proof. Let $v_\theta \in V_\theta$. Due to the positivity of c^r, equation (4.6d), and the strict monotonicity of $B_\varepsilon^{(2)}(t)$, cf. Lemma 4.4, we have

$$\left(B_\varepsilon(t)v_\theta, v_\theta\right)_H \geq c\left\|v_\theta\right\|_H^2, \quad v_\theta \in V_\theta.$$

Using the positivity of $K_\varepsilon^{r,i}$ (4.6e), the boundedness of $\varepsilon^{-1}|v_\varepsilon^r|$ (4.5), and the continuity estimate for $B_\varepsilon^{(2)}$ established in Lemma 4.4, we get

$$\left\langle A_\varepsilon(t)v_\theta, v_\theta\right\rangle_{{V_\theta}'V_\theta} \geq C_1\left(\left\|\nabla v_\theta\right\|_{L^2(\Omega_\varepsilon^{(1)})}^2 + \varepsilon^2\left\|\nabla v_\theta\right\|_{L^2(\Omega_\varepsilon^{(2)})}^2\right) - C_2\left\|v_\theta\right\|_H^2, \quad v_\theta \in V_\theta.$$

From those estimates, we see that the statement holds. □

Having now these results available, we are finally able to prove the main existence theorem for the coupled thermoelasticity problem formulated in fixed coordinates.

Theorem 4.7 (Existence Theorem). *Let $F_\varepsilon \in C^1(S;H)^3$, $G_\varepsilon \in L^2(S\times\Omega)$, and $\vartheta_\varepsilon^{(i)} \in L^2(\Omega_\varepsilon^{(i)})$. Then, there exists a unique $(U_\varepsilon, \Theta_\varepsilon) \in L^2(S; V_u \times V_\theta)$ such that $\partial_t(U_\varepsilon, \Theta_\varepsilon) \in L^2(S; {V_u}' \times {V_\theta}')$, such that $\Theta_\varepsilon(0)_{|\Omega_\varepsilon^{(i)}} = \vartheta_\varepsilon^{(i)}$ solving the variational system (4.9) for fixed coordinates.*

Proof. In light of the coercivity-type estimate established in Lemma 4.6 and the estimate for $B_\varepsilon^{(2)}(t)$ given with Lemma 4.5, we see ([Sho96, Chapter III, Proposition 3.2 and Proposition 3.3]) that there is a unique $\Theta_\varepsilon \in L^2(S;V_\theta)$ such that $\partial_t\Theta_\varepsilon \in L^2(S;{V_\theta}')$, $\Theta_\varepsilon(0)_{|\Omega_\varepsilon^{(i)}} = \vartheta_\varepsilon^{(i)}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon(t)\Theta_\varepsilon\right) + A_\varepsilon(t)\Theta_\varepsilon + \mathcal{V}_{\Gamma_\varepsilon}(t) = G_\varepsilon(t) - \mathcal{F}_\varepsilon(t) \qquad \text{in} \quad {V_\theta}'.$$

Defining, for every $t \in \overline{S}$, [12]

$$U_\varepsilon(t) := E_\varepsilon^{-1}(t)\left(e_\varepsilon^{th}(t)\Theta_\varepsilon(t) + F_\varepsilon(t) + \mathcal{H}_\varepsilon(t)\right) \in V_u,$$

we see that $\partial_t U_\varepsilon \in L^2(S;{V_u}')$ and that $U_\varepsilon(t)$ solves the mechanical part given via the variational equation (4.10) for $t \in S$. □

Transforming the solution $(U_\varepsilon, \Theta_\varepsilon)$ back to moving coordinates, i.e., setting $u_\varepsilon(t,x) = U_\varepsilon(t, s_\varepsilon(t,x))$ and $\theta_\varepsilon(t,x) = \Theta_\varepsilon(t, s_\varepsilon(t,x))$, we then get the solution to the original problem given by equations (4.3a)-(4.3i). In the following theorem, we establish the *a priori* estimates needed to justify the homogenization process.

Theorem 4.8 (ε-independent *a priori* estimates). *Assuming that*

$$\sup_{\varepsilon>0}\left(\|F_\varepsilon\|_{C^1(S;L^2(\Omega))^3} + \|G_\varepsilon\|_{L^2(S\times\Omega)} + \sum_{i=1,2}\|\vartheta_\varepsilon^{(i)}\|_{L^2(\Omega)}\right) < \infty,$$

we have

$$\begin{aligned}\|\Theta_\varepsilon\|_{L^\infty(S;H)} &+ \|\nabla\Theta_\varepsilon\|_{L^2(S\times\Omega_\varepsilon^{(1)})^3} + \varepsilon\|\nabla\Theta_\varepsilon\|_{L^2(S\times\Omega_\varepsilon^{(2)})^3} \\ &+ \|U_\varepsilon\|_{L^\infty(S;H)^3} + \|DU_\varepsilon\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(1)}))^{3\times3}} + \varepsilon\|DU_\varepsilon\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(2)}))^{3\times3}} \leq \widetilde{C}, \quad (4.14)\end{aligned}$$

where $C, \widetilde{C}$ are independent of the choice of ε.

Proof. Testing the variational equality (4.12) with Θ_ε, using the identity

$$\left(\partial_t\left(B_\varepsilon(t)v_\theta\right), v_\theta\right)_H = \left(\partial_t\left(B_\varepsilon(t)\right)v_\theta, v_\theta\right)_H + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon(t)v_\theta, v_\theta\right)_H,$$

[12] Note that, since $\Theta_\varepsilon \in C(\overline{S};H)$, this is well-defined.

and the uniform operator estimates established in Lemmas 4.2-4.4 and in Lemma 4.6, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(B_\varepsilon(t)\Theta_\varepsilon,\Theta_\varepsilon\right)_H + \|\nabla\Theta_\varepsilon\|^2_{L^2(\Omega_\varepsilon^{(1)})} + \varepsilon^2\|\nabla\Theta_\varepsilon\|^2_{L^2(\Omega_\varepsilon^{(2)})}$$
$$\leq C\left(\|\Theta_\varepsilon\|_H^2 + \|\mathcal{F}_\varepsilon(t)\|_H^2 + \|G_\varepsilon(t)\|^2_{V_\theta'} + \|\mathcal{V}_{\Gamma_\varepsilon}(t)\|_{L^2(\Gamma_\varepsilon)}\|\Theta_\varepsilon\|_{L^2(\Gamma_\varepsilon)}\right). \quad (4.15)$$

For the temperature on Γ_ε, we have the following ε-trace estimate, see, e.g., [ADH95],

$$\varepsilon\|\Theta_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \leq C\left(\|\Theta_\varepsilon\|_H^2 + \varepsilon^2\|\nabla\Theta_\varepsilon\|_H^2\right). \quad (4.16)$$

Integrating inequality (4.15) over $(0,t)$ and using the positivity of $B_\varepsilon^{(1)}$ and the monotonicity of $B_\varepsilon^{(2)}$, we then get

$$\|\Theta_\varepsilon(t)\|_H^2 + \int_0^t \|\nabla\Theta_\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon^{(1)})}\,\mathrm{d}\tau + \varepsilon^2\int_0^t \|\nabla\Theta_\varepsilon(\tau)\|^2_{L^2(\Omega_\varepsilon^{(2)})}\,\mathrm{d}\tau$$
$$\leq C\Bigg(\|\Theta_\varepsilon(0)\|_H^2 + \int_0^t\|\Theta_\varepsilon(\tau)\|_H^2\,\mathrm{d}\tau + \int_0^t\|\mathcal{F}_\varepsilon(\tau)\|_H^2\,\mathrm{d}\tau$$
$$+ \int_0^t\|G_\varepsilon(\tau)\|^2_{V_\theta'}\,\mathrm{d}\tau + \int_0^t\|\mathcal{V}_{\Gamma_\varepsilon}(\tau)\|^2_{L^2(\Gamma_\varepsilon)}\,\mathrm{d}\tau\Bigg).$$

A direct application of *Gronwall's inequality* yields the desired estimates for the temperatures. Testing equation (4.10) with U_ε and using the trace estimate (4.16), we get

$$\|U_\varepsilon(t)\|^2_{V_u} \leq C\left(\|\Theta_\varepsilon(t)\|_H^2 + \|F_\varepsilon(t)\|_H^2 + \varepsilon^2\|H^r_{\Gamma_\varepsilon}(t)\|^2_{L^2(\Gamma_\varepsilon)}\right).$$

Via the Korn-type estimate given by Lemma 4.1, we see that the estimates for the deformations are valid. □

4.4 Homogenization

In the following, we use the notion of two-scale convergence to derive a homogenized model. Our basic references for homogenization, in general, and two-scale convergence, in particular, are [All92, LNW02, Ngu89, Tar09].[13] For the convenience of the reader, we recall the definition of two-scale convergence:

Definiton 4.9 (Two-scale convergence)**.** *A sequence $v_\varepsilon \in L^2(S\times\Omega)$ is said to two scale converge two a limit function $v \in L^2(S\times\Omega\times Y)$ ($v_\varepsilon \overset{2}{\rightharpoonup} v$) if*

$$\lim_{\varepsilon\to 0}\int_S\int_\Omega v_\varepsilon(t,x)\varphi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t = \int_S\int_\Omega\int_Y v(t,x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi \in L^2(S\times\Omega;C_\#(Y))$.

[13] We also refer to Section 2.4, where the relevant definitions and results are collected.

In addition to the two-scale convergence, we recall the notion of *strong two-scale convergence*. This concept is needed to pass to the limit for some products of two-scale convergent sequences.

Definiton 4.10 (Strong two-scale convergence). *A sequence $v_\varepsilon \in L^2(S\times\Omega)$ is said to strongly two scale converge to a limit function $v \in L^2(S\times\Omega\times Y)$ ($v_\varepsilon \overset{2}{\to} u$) if both $v_\varepsilon \overset{2}{\rightharpoonup} v$ and*

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^2(S\times\Omega)} = \|v\|_{L^2(S\times\Omega\times Y)}.$$

It can be shown, see, e.g., [LNW02, Theorem 18][14], that if $u_\varepsilon \overset{2}{\rightharpoonup} u$ and $v_\varepsilon \overset{2}{\to} v$, we then have

$$\int_S\int_\Omega u_\varepsilon(t,x)v_\varepsilon(t,x)\varphi\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x\,\mathrm{d}t \to \int_S\int_\Omega\int_Y u(t,x,y)v(t,x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi \in C_0^\infty(\Omega; C_\#^\infty(Y))$.

For a function $v_\varepsilon \in \Omega_\varepsilon^{(i)}$ $(i=1,2)$, we denote, by abuse of notation, its zero extension to the whole of Ω with $\mathbb{1}_{|\Omega_\varepsilon^{(i)}} v_\varepsilon$. Furthermore, $W_\#^{1,2}(Y)$ is defined as the closure of $C_\#^1(Y)$ with respect to the $W^{1,2}$-Norm, and $W_\#^{1,2}(Y^{(1)})$ as the subspace of $W_\#^{1,2}(Y)$ with zero average. For functions depending on both $x\in\Omega$ and $y \in Y$, we denote derivatives with respect to $y\in Y$ with the subscript Y, i.e., e_y, ∇_y, div_y.

By the ε-independent estimates established in Theorem 4.8, we have the following two-scale limits.

Theorem 4.11 (Two-scale limits). *There are functions*

$$\begin{aligned} &u \in L^2(S;V_u), && U^{(2)} \in L^2(S\times\Omega; W_\#^{1,2}(Y)^3),\\ &\theta \in L^2(S;V_\theta), && \Theta^{(2)} \in L^2(S\times\Omega; W_\#^{1,2}(Y)),\\ &\widetilde{U} \in L^2(S\times\Omega; W_\#^{1,2}(Y)^3), && \widetilde{\Theta} \in L^2(S\times\Omega; W_\#^{1,2}(Y)) \end{aligned}$$

such that

$$\begin{aligned} &\mathbb{1}_{\Omega_\varepsilon^{(1)}} U_\varepsilon^{(1)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(1)}} u, && \mathbb{1}_{\Omega_\varepsilon^{(1)}} DU_\varepsilon^{(1)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(1)}} Du + \mathbb{1}_{Y^{(1)}} D_y\widetilde{U},\\ &\mathbb{1}_{\Omega_\varepsilon^{(2)}} U_\varepsilon^{(2)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(2)}} U^{(2)}, && \mathbb{1}_{\Omega_\varepsilon^{(2)}} DU_\varepsilon^{(2)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(2)}} D_y U^{(2)},\\ &\mathbb{1}_{\Omega_\varepsilon^{(1)}} \Theta_\varepsilon^{(1)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(1)}} \theta, && \mathbb{1}_{\Omega_\varepsilon^{(1)}} \nabla\Theta_\varepsilon^{(1)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(1)}} \nabla\theta + \mathbb{1}_{Y^{(1)}} \nabla_y\widetilde{\Theta},\\ &\mathbb{1}_{\Omega_\varepsilon^{(2)}} \Theta_\varepsilon^{(2)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(2)}} \Theta^{(2)}, && \mathbb{1}_{\Omega_\varepsilon^{(2)}} \nabla\Theta_\varepsilon^{(2)} \overset{2}{\rightharpoonup} \mathbb{1}_{Y^{(2)}} \nabla_y\Theta^{(2)}. \end{aligned}$$

Remark 4.12. *We distinguish between functions that depend on $y \in Y$ and functions independent of $y\in Y$, by using capitalized letters for the former and lowercase letters for the other.*

For a function $v = v(t,x,y)$, we denote the corresponding transformed function as $\widehat{v}(t,x,y) = v(t,x,s(t,x,y))$. To keep the notation consistent, we also set $u^{(2)}(t,x,y) = U^{(2)}(t,x,s(t,x,y))$ and $\theta^{(2)}(t,x,y) = \Theta^{(2)}(t,x,s(t,x,y))$.

[14] Combined with the remark succeeding the proof of Theorem 18.

Now, we introduce the homogenized transformation related quantities (all elements of $L^\infty(S\times\Omega\times Y)$)

$$F\colon \overline{S}\times\overline{\Omega}\times\overline{Y}\to\mathbb{R}^{3\times 3},\quad F(t,x,y):=D_ys(t,x,y), \tag{4.17a}$$
$$J\colon \overline{S}\times\overline{\Omega}\times\overline{Y}\to\mathbb{R},\quad J(t,x,y):=\det\left(D_ys(t,x,y)\right), \tag{4.17b}$$
$$v\colon \overline{S}\times\overline{\Omega}\times\overline{Y}\to\mathbb{R}^{3},\quad v(t,x,y):=\partial_t s(t,x,y), \tag{4.17c}$$
$$\widehat{V}_\Gamma\colon \overline{S}\times\overline{\Omega}\times\Gamma\to\mathbb{R},\quad \widehat{V}_\Gamma(t,x,y):=v(t,x,y)\cdot n_\Gamma(t,s(t,x,y)), \tag{4.17d}$$
$$\widehat{H}_\Gamma\colon \overline{S}\times\overline{\Omega}\times\Gamma\to\mathbb{R},\quad \widehat{H}_\Gamma(t,x,y):=-\operatorname{div}_y\left(F^{-1}(t,x,y)n_\Gamma(t,s(t,x,y))\right) \tag{4.17e}$$

and see that they are strong two-scale limits of their ε-periodic counterpart

$$F_\varepsilon\overset{2}{\to}F,\quad J_\varepsilon\overset{2}{\to}J,\quad \frac{1}{\varepsilon}v_\varepsilon\overset{2}{\to}v,\quad \frac{1}{\varepsilon}\widehat{V}_{\Gamma_\varepsilon}\overset{2}{\to}\widehat{V}_\Gamma,\quad \varepsilon\widehat{H}_{\Gamma_\varepsilon}\overset{2}{\to}\widehat{H}_\Gamma.$$

This can be seen by using the regularity of the function s, the fact that $\varepsilon\left[\frac{x}{\varepsilon}\right]\to x$, and using [All92, Lemma 1.3.].[15]. For a similar situation in the case of periodic unfolding, we refer to [Dob12, Lemma 3.4.6]. As a consequence, we also have strong two-scale convergence for the transformed coefficients, see (4.5a)-(4.5i), the limits of whose are labeled via a r-superscript.

We assume that, for $i=1,2$ and almost all $t\in S$, there are functions $f^{(i)}(t)$, $g^{(i)}(t)$, and $\vartheta^{(i)}\in L^2(\Omega\times Y^{(i)}$, such that $\widehat{f^{(i)}}\in C^1(S;L^2(\Omega\times Y)^3)$, $\widehat{g^{(i)}}\in L^2(S\times\Omega\times Y)$, and such that

$$\mathbb{1}_{|\Omega_\varepsilon^{(i)}}\widehat{f_\varepsilon^{(i)}}\overset{2}{\rightharpoonup}\mathbb{1}_{|Y^{(i)}}\widehat{f^{(i)}},\quad \mathbb{1}_{|\Omega_\varepsilon^{(i)}}\widehat{g_\varepsilon^{(i)}}\overset{2}{\rightharpoonup}\mathbb{1}_{|Y^{(i)}}\widehat{g^{(i)}},\quad \mathbb{1}_{|\Omega_\varepsilon^{(i)}}\vartheta_\varepsilon^{(i)}\overset{2}{\rightharpoonup}\mathbb{1}_{|Y^{(i)}}\vartheta^{(i)}.$$

In particular, this implies

$$\mathbb{1}_{|\Omega_\varepsilon^{(i)}}f^{r,i}\overset{2}{\rightharpoonup}\mathbb{1}_{|Y^{(i)}}J\widehat{f^{(i)}},\quad \mathbb{1}_{|\Omega_\varepsilon^{(i)}}g_\varepsilon^{r,i}\overset{2}{\rightharpoonup}\mathbb{1}_{|Y^{(i)}}J\widehat{g^{(i)}}.$$

We set $f^{r,i}=J\widehat{f^{(i)}}$ and $g^{r,i}=J\widehat{g^{(i)}}$.

4.4.1 Homogenization of the mechanical part

Let $v\in C_0^\infty(\Omega)^3$ and $v^{(2)}\in C^\infty(\overline{\Omega};C_\#^\infty(Y))^3$ such that $v(x)=v^{(2)}(x,y)$ for all $(x,y)\in\Omega\times\Gamma$. Furthermore, let $\widetilde{v}\in C^\infty(\overline{\Omega};C_\#^\infty(Y))^3$. We introduce functions

$$v_\varepsilon^{(1)}\colon\Omega\to\mathbb{R}^3,\quad v_\varepsilon^{(1)}(x):=v(x)+\varepsilon\widetilde{v}\left(x,\frac{x}{\varepsilon}\right),$$
$$v_\varepsilon^{(2)}\colon\Omega\to\mathbb{R}^3,\quad v_\varepsilon^{(2)}(x):=v^{(2)}\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\widetilde{v}\left(x,\frac{x}{\varepsilon}\right),$$
$$v_\varepsilon\colon\Omega\to\mathbb{R}^3,\quad v_\varepsilon(x):=\begin{cases}v_\varepsilon^{(1)}(x),\ x\in\Omega^{(1)},\\ v_\varepsilon^{(2)}(x),\ x\in\Omega^{(2)}.\end{cases}$$

[15]Note that ignoring the "mismatch" $x-\varepsilon\left[\frac{x}{\varepsilon}\right]$, we basically have $F_\varepsilon(t,x)\approx F\left(t,x,\frac{x}{\varepsilon}\right)$

As a consequence, $v_\varepsilon \in W_0^{1,2}(\Omega)^3$. Choosing v_ε as a test function and letting $\varepsilon \to 0$, we then get, up to a subsequence, the following limit problem:

$$\begin{aligned}
\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(u) + e_y(\widetilde{U}^{(1)})\big) : \big(e(v) + e_y(\widetilde{v})\big)\,\mathrm{d}y\,\mathrm{d}x \\
+ \int_\Omega \int_{Y^{(2)}} \mathcal{C}^{r,2} e_y(U^{(2)}) : e_y(v^{(2)})\,\mathrm{d}y\,\mathrm{d}x \\
- \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\theta : \big(Dv + D_y\widetilde{v}\big)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(2)}} \alpha^{r,2}\Theta^{(2)} : D_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x \\
= \int_\Omega \int_{Y^{(1)}} f^{(1)} \cdot v\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_{Y^{(2)}} f^{(2)} \cdot v^{(2)}\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_\Gamma H_\Gamma^r n_\Gamma \cdot v\,\mathrm{d}s\,\mathrm{d}x \quad (4.18)
\end{aligned}$$

for all

$$(v, \widetilde{v}, v^{(2)}) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega; C_\#^\infty(Y)) \times C_0^\infty(\Omega; C_\#^\infty(Y))$$

such that $v(x) = v^{(2)}(x, y)$ for all $(x, y) \in \Omega \times \Gamma$. By density arguments, equation (4.18) holds also true for all $(v, \widetilde{v}, v^{(2)})$, where $v \in W_0^{1,2}(\Omega)^3$ and $\widetilde{v}, v^{(2)} \in L^2(\Omega; W_\#^{1,2}(Y))^3$ such that $v(x) = v^{(2)}(x, y)$ for almost all $(x, y) \in \Omega \times \Gamma$. As a next step, we are going to decouple the limit problem (4.18). For this goal, we choose $v \equiv 0$ and $v^{(2)} \equiv 0$. We obtain:

$$\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(u) + e_y(\widetilde{U}^{(1)})\big) : e_y(\widetilde{v})\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\theta : D_y\widetilde{v}\,\mathrm{d}y\,\mathrm{d}x = 0 \qquad (4.19)$$

for all $\widetilde{v} \in L^2(\Omega; W_\#^{1,2}(Y))^3$.

Now, letting $v \equiv 0$ and forcing $v^{(2)} = 0$ a.e. on $\Omega \times \Gamma$, we get

$$\begin{aligned}
\int_\Omega \int_{Y^{(2)}} \mathcal{C}^{r,2} e_y(U^{(2)}) : e_y(v^{(2)})\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(2)}} \alpha^{r,2}\Theta^{(2)} : D_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x \\
= \int_\Omega \int_{Y^{(2)}} f^{(2)} \cdot v^{(2)}\,\mathrm{d}y\,\mathrm{d}x \quad (4.20)
\end{aligned}$$

for all $v^{(2)} \in L^2(\Omega; W_0^{1,2}(Y^{(2)}))^3$. Next, while keeping $\widetilde{v} \equiv 0$, we choose test functions such that $v(x) = v^{(2)}(x, y)$ for almost all $(x, y) \in \Omega \times Y^{(2)}$ (in particular, we have that $v^{(2)}$ is constant in $y \in Y$) and see that

$$\begin{aligned}
\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(u^{(1)}) + e_y(\widetilde{U}^{(1)})\big) : e(v)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\theta^{(1)} : Dv\,\mathrm{d}y\,\mathrm{d}x \\
= \int_\Omega \int_{Y^{(1)}} f^{(1)} \cdot v\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_{Y^{(2)}} f^{(2)} \cdot v\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_\Gamma H_\Gamma^r n_\Gamma \cdot v\,\mathrm{d}s\,\mathrm{d}x. \quad (4.21)
\end{aligned}$$

Summarizing, we obtain the following system of variational equalities:

$$\begin{aligned}
\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(u^{(1)}) + e_y(\widetilde{U}^{(1)})\big) : e(v)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\theta^{(1)} : Dv\,\mathrm{d}y\,\mathrm{d}x \\
= \int_\Omega \int_{Y^{(1)}} f^{(1)} \cdot v\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_{Y^{(2)}} f^{(2)} \cdot v\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_\Gamma H_\Gamma^r n_\Gamma \cdot v\,\mathrm{d}s\,\mathrm{d}x, \quad (4.22a)
\end{aligned}$$

$$\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(u^{(1)}) + e_y(\widetilde{U}^{(1)})\big) : e_y(\widetilde{v}) \,\mathrm{d}y \,\mathrm{d}x \\ - \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\theta^{(1)} : D_y\widetilde{v} \,\mathrm{d}y \,\mathrm{d}x = 0, \quad (4.22b)$$

$$\int_\Omega \int_{Y^{(2)}} \mathcal{C}^{r,2} e_y(U^{(2)}) : e_y(v^{(2)}) \,\mathrm{d}y \,\mathrm{d}x - \int_\Omega \int_{Y^{(2)}} \alpha^{r,2}\Theta^{(2)} : D_y v^{(2)} \,\mathrm{d}y \,\mathrm{d}x \\ = \int_\Omega \int_{Y^{(2)}} f^{(2)} \cdot v^{(2)} \,\mathrm{d}y \,\mathrm{d}x \quad (4.22c)$$

for all $\big(v, \widetilde{v}, v^{(2)}\big) \in W_0^{1,2}(\Omega)^3 \times L^2(\Omega; W_\#^{1,2}(Y))^3 \times L^2(\Omega; W_0^{1,2}(Y^{(2)}))^3$. In addition to equations (4.22a)-(4.22c), we have the additional constraint $u^{(1)}(t,x) = U^{(2)}(t,x,y)$ for almost all $(t,x,y) \in S \times \Omega \times \Gamma$.

We go on by introducing cell problems and effective quantities to get a more accessible form of the homogenization limit. For $j,k = \{1,2,3\}$ and $y \in Y$, set $d_{jk} = (y_j\delta_{1k}, y_j\delta_{2k}, y_j\delta_{3k})^T$, where δ is the *Kronecker* delta. For $t \in S$, $x \in \Omega$, let $\tau_{jk}(t,x,\cdot)$, $\tau(t,x,\cdot) \in W_\#^{1,2}(Y^{(1)})^3$ are the solutions to

$$0 = \int_{Y^{(1)}} \mathcal{C}^{r,1} e_y(\tau_{jk} + d^{jk}) : e_y(\widetilde{v}) \,\mathrm{d}y, \quad (4.23a)$$

$$0 = \int_{Y^{(1)}} \mathcal{C}^{r,1} e_y(\tau) : e_y(\widetilde{v}) \,\mathrm{d}y - \int_{Y^{(1)}} \alpha^{r,1} : D_y\widetilde{v} \,\mathrm{d}y \quad (4.23b)$$

for all $\widetilde{v} \in W_\#^{1,2}(Y^{(1)})^3$. In addition, we introduce the effective elasticity tensor $\mathcal{C}^h \colon S \times \Omega \to \mathbb{R}^{3\times3\times3\times3}$, $\mathcal{C}^h(t,x) = \big(\mathcal{C}^h(t,x)\big)_{1\le i,j,k,l\le 3}$, via

$$\big(\mathcal{C}^h\big)_{j_1j_2j_3j_4} = \int_{Y^{(1)}} \mathcal{C}^{r,1} e_y\left(\tau_{j_1j_2} + d_{j_1j_2}\right) : e_y\left(\tau_{j_3j_4} + d_{j_3j_4}\right) \,\mathrm{d}y. \quad (4.23c)$$

Furthermore, we introduce the following effective functions:

$$H^h \colon S \times \Omega \to \mathbb{R}, \qquad H_\Gamma^h(t,x) = \int_\Gamma H_\Gamma^r(t,x,s) n(t,x,s) \,\mathrm{d}s, \quad (4.23d)$$

$$f^h \colon S \times \Omega \to \mathbb{R}, \qquad f^h(t,x) = \int_{Y^{(1)}} f^{r,1}(t,x,y) \,\mathrm{d}y + \int_{Y^{(2)}} f^{r,2}(t,x,y) \,\mathrm{d}y, \quad (4.23e)$$

$$\alpha^h \colon S \times \Omega \to \mathbb{R}^{3\times3}, \quad \alpha^h(t,x) = \int_{Y^{(1)}} \big(\alpha^{r,1} - C^{r,1} e_y\big(\tau^u\big)\big) \,\mathrm{d}y. \quad (4.23f)$$

We see that, at least up to a function independent of $y \in Y$, it holds

$$\widetilde{U}(t,x,y) = \sum_{j,k=1}^{3} \tau_{jk}(t,x,y)(e(u)(t,x))_{jk} + \tau(t,x,y)\theta(t,x).$$

After transforming the microscopic mechanical part to moving coordinates, we are led to

$$\int_\Omega C^h e(u) : e(v) \,\mathrm{d}x - \int_\Omega \alpha^h\theta : Dv \,\mathrm{d}x = \int_\Omega f^h \,\mathrm{d}x + \int_\Omega H_\Gamma^h \,\mathrm{d}x, \quad (4.24a)$$

$$\int_{Y^{(2)}(t,x)} \mathcal{C}^{(2)} e_y(U^{(2)}) : e_y(v^{(2)})\,\mathrm{d}y - \int_{Y^{(2)}(t,x)} \alpha^{(2)}\Theta^{(2)} \operatorname{div}_y v^{(2)}\,\mathrm{d}y = \int_{Y^{(2)}(t,x)} f^{(2)}\cdot v^{(2)}\,\mathrm{d}y \quad (4.24b)$$

for all $v \in W_0^{1,2}(\Omega)^3$, $v^{(2)} \in W_0^{1,2}(Y^{(2)}(t,x))^3$ and almost all $t \in S$.

4.4.2 Homogenization of the heat part

Let $(v,\widetilde{v}) \in C^\infty(\overline{S}\times\overline{\Omega}) \times C^\infty(\overline{S}\times\overline{\Omega}; C^\infty_\#(Y))$ and $v^{(2)} \in C^\infty(\overline{S}\times\overline{\Omega}; C^\infty_\#(Y))$ such that $v(T) = \widetilde{v}(T) = v^{(2)}(T) = 0$ and such that $v(t,x) = v^{(2)}(t,x,y)$ for all $(t,x,y) \in S\times\Omega\times\Gamma$. We introduce the functions

$$\begin{aligned}
v_\varepsilon^{(1)}\colon S\times\Omega\to\mathbb{R}^3, &\quad v_\varepsilon^{(1)}(t,x) = v(t,x) + \varepsilon\widetilde{v}\left(t,x,\frac{x}{\varepsilon}\right),\\
v_\varepsilon^{(2)}\colon S\times\Omega\to\mathbb{R}^3, &\quad v_\varepsilon^{(2)}(t,x) = v^{(2)}\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon\widetilde{v}\left(t,x,\frac{x}{\varepsilon}\right),\\
v_\varepsilon\colon S\times\Omega\to\mathbb{R}^3, &\quad v_\varepsilon(t,x) = \begin{cases} v_\varepsilon^{(1)}(t,x), & x\in\Omega^{(1)},\\ v_\varepsilon^{(2)}(t,x), & x\in\Omega^{(2)}.\end{cases}
\end{aligned}$$

Then, $v_\varepsilon \in W^{1,2}(\Omega)$. Choosing v_ε as a test function and letting $\varepsilon\to 0$, we get, up to a subsequence, the following limit problem:

$$\begin{aligned}
&-\int_S\int_\Omega \rho^{(1)}c^{(1)}\left|Y^{(1)}\right|\theta\partial_t v\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho^{(1)}c^{(1)}\left|Y^{(1)}\right|\vartheta^{(1)}\partial_t v(0)\,\mathrm{d}x\\
&\quad - \int_S\int_\Omega\int_{Y^{(2)}} c^{r,2}\Theta^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega\int_{Y^{(2)}} c^{(2)}\vartheta^{(2)}v^{(2)}(0)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad + \int_S\int_\Omega\int_{Y^{(2)}} c^{r,2}v^r\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_S\int_\Omega\int_{Y^{(1)}} \gamma^{r,1} : \left(Du + D_y\widetilde{U}\right)\partial_t v\,\mathrm{d}x\,\mathrm{d}t - \int_S\int_\Omega\int_{Y^{(2)}} \gamma^{r,2} : D_yU^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_S\int_\Omega\int_{Y^{(2)}} v^r\left(\gamma^{r,2} : D_yU_\varepsilon^{(2)}\right)\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_S\int_\Omega\int_{Y^{(1)}} K^{r,1}\left(\nabla\theta + \nabla_y\widetilde{\Theta}\right)\cdot\left(\nabla v + \nabla_y\widetilde{v}\right)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_S\int_\Omega\int_{Y^{(2)}} K^{r,2}\nabla_y\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_S\int_\Omega\int_\Gamma V_\Gamma^r v\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \int_S\int_\Omega\int_{Y^{(1)}} g^{r,1}v\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_S\int_\Omega\int_{Y^{(2)}} g^{r,2}v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \quad (4.25)
\end{aligned}$$

for all $(v,\widetilde{v}) \in C^\infty(\overline{S}\times\overline{\Omega}) \times C^\infty(\overline{S}\times\overline{\Omega}; C^\infty_\#(Y))$ and $v^{(2)} \in C^\infty(\overline{S}\times\overline{\Omega}; C^\infty_\#(Y))$ such that $v(T) = \widetilde{v}(T) = v^{(2)}(T) = 0$ and such that $v(t,x) = v^{(2)}(t,x,y)$ for all $(t,x,y)\in S\times\Omega\times\Gamma$. Here, $\left|Y^{(1)}\right| = \left|Y^{(1)}(t,x)\right|$.

Using the same decoupling strategy as for the mechanical part, we obtain the following system of variational equalities:

$$
\begin{aligned}
&-\int_S\int_\Omega \rho^{(1)}c^{(1)}\left|Y^{(1)}\right|\theta\partial_t v\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega \rho^{(1)}c^{(1)}\left|Y^{(1)}\right|\vartheta^{(1)}\partial_t v(0)\,\mathrm{d}x\\
&\quad-\int_S\int_\Omega\left(\int_{Y^{(2)}}c^{r,2}\Theta^{(2)}\,\mathrm{d}y\right)\partial_t v\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\left(\int_{Y^{(2)}}c^{(2)}\vartheta^{(1)}\,\mathrm{d}y\right)v(0)\,\mathrm{d}x\\
&\quad-\int_S\int_\Omega\left(\int_{Y^{(1)}}\gamma^{r,1}:\left(Du+D_y\widetilde{U}\right)\,\mathrm{d}y+\int_{Y^{(2)}}\gamma^{r,2}:D_yU^{(2)}\,\mathrm{d}y\right)\partial_t v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_S\int_\Omega\int_{Y^{(1)}}K^{r,1}\left(\nabla\theta+\nabla_y\widetilde{\Theta}\right)\cdot\nabla v\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t+\int_S\int_\Omega\left(\int_\Gamma V_\Gamma^r\,\mathrm{d}s\right)v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_S\int_\Omega\left(\int_{Y^{(1)}}g^{r,1}\,\mathrm{d}y\right)v\,\mathrm{d}x\,\mathrm{d}t+\int_S\int_\Omega\left(\int_{Y^{(2)}}g^{r,2}\,\mathrm{d}y\right)v\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{4.26a}
$$

$$
\int_S\int_\Omega\int_{Y^{(1)}}K^{r,1}\left(\nabla\theta+\nabla_y\widetilde{\Theta}\right)\cdot\nabla_y\widetilde{v}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0,
\tag{4.26b}
$$

$$
\begin{aligned}
&-\int_S\int_\Omega\int_{Y^{(2)}}c^{r,2}\Theta^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\int_{Y^{(2)}}c^{(2)}\vartheta^{(2)}v^{(2)}(0)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\int_S\int_\Omega\int_{Y^{(2)}}\rho^{(2)}c^{(2)}v^r\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_S\int_\Omega\int_{Y^{(2)}}\gamma^{r,2}:D_yU^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t+\int_S\int_\Omega\int_{Y^{(2)}}v^r\left(\gamma^{r,2}:D_yU^{(2)}\right)\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_S\int_\Omega\int_{Y^{(2)}}K^{r,2}\nabla_y\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=\int_S\int_\Omega\int_{Y^{(2)}}g^{r,2}v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{4.26c}
$$

for all $(v,\widetilde{v},v^{(2)})\in L^2(S;W^{1,2}(\Omega))\times L^2(S\times\Omega;W^{1,2}_\#(Y))\times L^2(S\times\Omega;W^{1,2}_0(Y^{(2)}))$ such that $(\partial_t v,\partial_t v^{(2)})\in L^2(S;(W^{1,2}(\Omega)'))\times L^2(S\times\Omega;W^{-1,2}(Y^{(2)}))$ and such that $v(T)=v^{(2)}(T)=0$.

Now, we want to find a more accessible description of the homogenized problem given via equations (4.26a)-(4.26c). With that in mind, for $j\in\{1,2,3\}$, $t\in S$, $x\in\Omega$, let $\tau_j(t,x,\cdot)\in W^{1,2}_\#(Y^{(1)})$ be the solution to

$$
\int_{Y^{(1)}}K^{r,1}\left(\nabla_y\tau_j+e_j\right)\cdot\nabla_y\widetilde{v}\,\mathrm{d}y=0,\quad \widetilde{v}\in W^{1,2}_\#(Y^{(1)}).
\tag{4.27}
$$

We introduce the following effective functions

$$
\begin{aligned}
c^h\colon S\times\Omega\to\mathbb{R},\qquad V^h\colon S\times\Omega\to\mathbb{R},\quad \gamma^h\colon S\times\Omega\to\mathbb{R}^{3\times3},\\
K^h\colon S\times\Omega\to\mathbb{R}^{3\times3},\quad g^h\colon S\times\Omega\to\mathbb{R}
\end{aligned}
$$

defined via

$$c^h(t,x) = \rho^{(1)}c^{(1)}\left|Y^{(1)}(t,x)\right| + \alpha^{(1)}\int_{Y^{(1)}} \operatorname{div}_y(\tau)(t,x,y)\,\mathrm{d}y, \tag{4.28a}$$

$$K^h(t,x)_{ij} = \int_{Y^{(1)}} K^{r,1}(t,x,y)\left(\nabla_y\tau_j(t,x,y)+e_j\right)\cdot\left(\nabla_y\tau_i(t,x,y)+e_i\right), \tag{4.28b}$$

$$V^h(t,x) = \int_\Gamma V_\Gamma^r(t,x,s)\,\mathrm{d}s, \tag{4.28c}$$

$$g^h(t,x) = \int_{Y^{(1)}} g^{r,1}(t,x,y)\,\mathrm{d}y + \int_{Y^{(2)}} g^{r,2}(t,x,y)\,\mathrm{d}y \tag{4.28d}$$

$$\gamma^h(t,x)_{ij} = \int_Y \left(\gamma^{r,1}+\gamma^{(1)}\nabla_y\tau_{jk}(t,x,y)\right)\,\mathrm{d}y + \gamma^{(2)}\left|Y^{(2)}(t,x)\right|\mathbb{I}_3. \tag{4.28e}$$

The system of variational equalities (4.26a)-(4.26c) then reads

$$\begin{aligned}
&-\int_S\int_\Omega c^h\theta^{(1)}\partial_t v\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega c^h(0)\vartheta_0^{(1)}\partial_t v(0)\,\mathrm{d}x \\
&-\int_S\int_\Omega\left(\int_{Y^{(2)}(t,x)}\rho^{(2)}c^{(2)}\Theta^{(2)}\,\mathrm{d}y\right)\partial_t v\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega\left(\int_{Y^{(2)}(0)}\rho^{(2)}c^{(2)}\vartheta^{(2)}\,\mathrm{d}y\right)v(0)\,\mathrm{d}x \\
&\qquad -\int_S\int_\Omega\gamma^h : Du\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_S\int_\Omega K^h\nabla\theta\cdot\nabla v\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad = -\int_S\int_\Omega V^h v\,\mathrm{d}x\,\mathrm{d}t + \int_S\int_\Omega g^h\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.29a}$$

$$\begin{aligned}
&-\int_S\int_\Omega\int_{Y^{(2)}} c^{r,2}\Theta^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}t + \int_S\int_\Omega\int_{Y^{(2)}}\rho^{(2)}c^{(2)}v^r\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}t \\
&\quad -\int_\Omega\int_{Y^{(2)}} c^{(2)}\vartheta^{(2)}v^{(2)}(0)\,\mathrm{d}y - \int_S\int_\Omega\int_{Y^{(2)}}\gamma^{r,2} : D_yU^{(2)}\partial_t v^{(2)}\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \int_S\int_\Omega\int_{Y^{(2)}} v^r\left(\gamma^{r,2} : D_yU_\varepsilon^{(2)}\right)\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_S\int_\Omega\int_{Y^{(2)}} K^{r,2}\nabla_y\Theta^{(2)}\cdot\nabla_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_S\int_\Omega\int_{Y^{(2)}} g^{r,2}v^{(2)}\,\mathrm{d}y\,\mathrm{d}t
\end{aligned} \tag{4.29b}$$

for all $(v, v^{(2)}) \in L^2(S; W^{1,2}(\Omega)) \times L^2(S\times\Omega; W_0^{1,2}(Y^{(2)}))$ such that $(\partial_t v, \partial_t v^{(2)}) \in L^2(S; (W^{1,2}(\Omega)')) \times L^2(S\times\Omega; W^{-1,2}(Y^{(2)}))$ and such that $v(T) = v^{(2)}(T) = 0$.

Finally, we are able to present the complete homogenized problem of the initial highly heterogeneous ε-problem given by equations (4.3a)-(4.3i). We transform the variational equations (4.29b) to the moving domain formulation and combine the homogenized mechanical system (equations (4.24a), (4.24b)) and the homogenized thermo system (equations (4.29a), (4.29b)). Via localization, this results in the following two-scale system of partial differential equations (complemented by initial conditions and macroscopic boundary conditions)

Homogenization limit with distributed microstructures

Effective, macroscopic thermoelasticity - cylindrical coordinates

$$-\operatorname{div}\left(\mathcal{C}^h e(u) - \alpha^h \theta\right) = f^h + H^h \quad \text{in} \quad S \times \Omega, \tag{4.30a}$$

$$\partial_t\left(c^h\theta + \rho^{(2)}c^{(2)}\int_{Y^{(2)}(t,x)} \theta^{(2)}\,\mathrm{d}y + \gamma^h : Du\right) - \operatorname{div}\left(K^h \nabla\theta\right) = g^h - V^h \quad \text{in} \quad S \times \Omega, \tag{4.30b}$$

Parametrized microscopic problem - non-cylindrical coordinates

$$-\operatorname{div}_y\left(\mathcal{C}^{(2)} e_y(u^{(2)}) - \alpha^{(2)}\theta^{(2)}\mathbb{I}_3\right) = f^{(2)} \text{ in } Y^{(2)}(t,x), \tag{4.30c}$$

$$\rho^{(2)}c^{(2)}\partial_t\theta^{(2)} + \gamma^{(2)}\partial_t \operatorname{div}_y u^{(2)} - \operatorname{div}_y\left(K^{(2)}\nabla_y\theta^{(2)}\right) = g^{(2)} \text{ in } Y^{(2)}(t,x), \tag{4.30d}$$

$$u^{(2)} = u, \quad \theta^{(2)} = \theta \quad \text{on } \Gamma(t,x). \tag{4.30e}$$

This homogenized model is a typical example of what is usually called a *distributed-microstructure* model [Sho93]. In simple words, this means that on the one hand, we have obtained an averaged macroscopic description of the coupled thermoelasticity, that is equations (4.30a) and (4.30b), while on the other hand, these averaged equations are, at every point $x \in \Omega$, coupled with an x-parametrized microscopic problem, see equations (4.30c)-(4.30e).

The coupling between the two-scales (microscopic and macroscopic), again, is two-fold: a) Via the *Dirichlet*-boundary condition on $\partial Y^{(2)}(t,x)$ (equation (4.30e)), which is a direct consequence of the continuity conditions posed on the phase-interface of the ε-microproblem, the macroscopic quantities determine the boundary values of the microscopic quantities. b) In contrast, in the macroscopic heat equation, we see that the average of the microscopic heat density, i.e.,

$$\rho^{(2)}c^{(2)}\int_{Y^{(2)}(t,x)} \theta^{(2)}\,\mathrm{d}y$$

is part of the overall heat density. In the case of $\gamma_i = 0$, i.e., when there is no dissipation, the *overall effective heat density* $e^h = e^h(t,x)$ would then be given as

$$e^h = c^h\theta + \rho^{(2)}c^{(2)}\int_{Y^{(2)}(t,x)} \theta^{(2)}\,\mathrm{d}y$$

This seems to suggest that equation (4.30b) should, actually, be interpreted as a balance equation for the so-called overall heat density, where part of the balanced quantity, the microscopic temperature $\Theta^{(2)}$, is given as a solution to the microscopic heat balance equation.

In the homogenization limit, the phase transformation is a purely microscopic phenomenon, where we have the free boundary $\Gamma(t,x) = \partial Y^{(2)}(t,x)$. However, the transformation does also turn up in the macroscopic part, where it enters via the volume force densities *effective mean curvature* H^h and the *effective normal velocity* W^h.

4.5 Remarks on and additions to [EM17b]

In this section, we present some additional results and highlight some possible generalizations of the results presented in [EM17b].

Roughly speaking, these can be subdivided into three different aspects which are

(*i*) *Mathematical Model:* Different generalization of the thermoelasticity model are considered including time and space dependent coefficients (Corollary 4.13 and Corollary 4.14) and lower order non-linearities.

(*ii*) *Geometry and Transformation:* Generalizations of the underlying geometry are considered, e.g., non-rectangular domain (Corollary 4.15), both phases connected (Corollary 4.16).

(*iii*) *Homogenization:* Uniqueness for the homogenization limit is established (Theorem 4.18).

Mathematical Model. Using the results presented in Section 4.3, it is possible to also account for non-constant coefficients as long as some regularity and boundedness conditions are met:

Corollary 4.13. *For $i = 1, 2$, let*

$$K^{(i)} \in L^\infty\left(Q_\varepsilon^{(i)}\right)^{3\times3},\ \mathcal{C}^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)^{3\times3\times3\times3},\ \alpha^{(i)},\, \gamma^{(i)},\, \rho^{(i)},\, c^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)$$

such that $\mathcal{C}^{(i)}$ and $K^{(i)}$ are symmetric and uniformly positive definite[16] *and such that $\alpha^{(i)}$, $\gamma^{(i)}, \rho^{(i)}, c^{(i)}$ are positive and bounded away by zero. Then, Theorems 4.7 and 4.8 hold true. The homogenization result is unaffected.*

Proof. Due to the interface motion, all reference-based coefficient functions ($K^{r,i}$ and so on) are already time and space dependent so the only potential problem arising relates to the properties of these functions. Taking into consideration the C^1-regularity of $\mathcal{C}^{(i)}$, $\alpha^{(i)}$, $\gamma^{(i)}$, $\rho^{(i)}$, $c^{(i)}$, it is easy to see that Lemma 4.5 still holds. The rest follows via the uniform bounds on the coefficients. Regarding the homogenization part: multiplying with ε-independent, L^∞ functions does not change the limit process. The only small and rather obvious difference is in the final depiction of some terms, e.g., we end with $\partial_t(\rho^{(2)}c^{(2)}\Theta^{(2)})$ instead of $\rho^{(2)}c^{(2)}\partial_t\Theta^{(2)}$ □

One might also be interested in the case where the coefficients explicitly (not only via their domain) depend on the scale parameter ε. Here, some additional care is needed to ensure that the limit process is still valid. We introduce the sets

$$Q_Y^{(i)} = \bigcup_{(t,x)\in S\times\Omega} \{(t,x)\} \times Y^{(i)}(t,x).$$

[16] There is a constant $c > 0$ such that $\mathcal{C}^{(i)}(t,x)M : M \geq c\,|M|^2$ for all symmetric matrices $M \in \mathbb{R}^{3\times3}$ and such that $K^{(i)}(t,x)v \cdot v \geq c\,|v|^2$ for all $v \in \mathbb{R}^3$ and for all $(t,x) \in Q_\varepsilon^{(i)}$.

Corollary 4.14. *For $i = 1,2$ and $\varepsilon > 0$, let*

$$K_\varepsilon^{(i)} \in L^\infty\left(Q_\varepsilon^{(i)}\right)^{3\times3}, \ \mathcal{C}_\varepsilon^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)^{3\times3\times3\times3}, \ \alpha_\varepsilon^{(i)}, \gamma_\varepsilon^{(i)}, \rho_\varepsilon^{(i)}, c_\varepsilon^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)$$

such that $\mathcal{C}_\varepsilon^{(i)}$ and $K_\varepsilon^{(i)}$ are symmetric and uniformly (also with respect to the parameter ε) positive definite and such that $\alpha_\varepsilon^{(i)}$, $\gamma_\varepsilon^{(i)}, \rho_\varepsilon^{(i)}, c_\varepsilon^{(i)}$ are positive and bounded away by zero. Moreover, we suppose

$$\sup_{\varepsilon>0}\Bigg(\|K_\varepsilon^{(i)}\|_{L^\infty(Q_\varepsilon^{(i)})^{3\times3}} + \|\mathcal{C}_\varepsilon^{(i)}\|_{W^{1,\infty}(Q_\varepsilon^{(i)})^{3\times3\times3\times3}} + \|\alpha_\varepsilon^{(i)}\|_{W^{1,\infty}(Q_\varepsilon^{(i)})} + \|\gamma_\varepsilon^{(i)}\|_{W^{1,\infty}(Q_\varepsilon^{(i)})} + \|\rho_\varepsilon^{(i)}\|_{W^{1,\infty}(Q_\varepsilon^{(i)})} + \|c_\varepsilon^{(i)}\|_{W^{1,\infty}(Q_\varepsilon^{(i)})}\Bigg) < \infty. \quad (4.31)$$

Then, Theorems 4.7 and 4.8 hold true. In addition, if we assume that there are functions

$$K^{(i)} \in L^\infty\left(Q_Y^{(i)}\right)^{3\times3}, \ \mathcal{C}^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)^{3\times3\times3\times3}, \ \alpha^{(i)}, \gamma^{(i)}, \rho^{(i)}, c^{(i)} \in W^{1,\infty}\left(Q_\varepsilon^{(i)}\right)$$

such that $f_\varepsilon^{r,i} \xrightarrow{2} f^{r,i}$ for $f = K, \mathcal{C}, \alpha, \gamma, \rho, c$, then the homogenization result is also valid. Here, the superscript r denotes the pullback to the reference configuration.

Proof. Taking into consideration the uniform estimate given via equation (4.31) and Corollary 4.13, the validity of Theorems 4.7 and 4.8 follows. Finally, with the assumed strong two-scale convergence, we can pass to the limit in the same way as outlined in Section 4.4. □

We point out that it is also possible to include some lower order non-linearities without much difficulty. Generally speaking, non-linearities in the disconnected $\Omega_\varepsilon^{(2)}$-part are very difficult to treat as we can only expect weak convergence here. In the connected domain $\Omega_\varepsilon^{(1)}$, however, they are quite manageable. As a rather simple example, let us assume that we have a Caratheodory function[17] $g\colon S \times \Omega \times \mathbb{R} \to \mathbb{R}$ and $a \in L^2(Q)$ such that $|g(t,x,r)| \leq C(a(t,x) + r)$ for a.a. $(t,x,r) \in S \times \Omega \times \mathbb{R}$. Now, introducing

$$G_\varepsilon^{(1)}\colon L^2(Q) \to L^2(Q) \quad \text{via} \quad G_\varepsilon^{(1)}(\varphi)(t,x) = g\left(t, x, \mathbb{1}_{\Omega_\varepsilon^{(1)}}(x)\theta(t,x)\right),$$

we might propose $G_\varepsilon^{(1)}(\Theta_\varepsilon^{(1)})$ as the heat production density, i.e.,

$$\partial_t\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} + \gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) - \operatorname{div}\left(K_\varepsilon^{r,1}\nabla\Theta_\varepsilon^{(1)}\right) - \operatorname{div}\left(\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} + \gamma_\varepsilon^{r,1} : DU_\varepsilon^{(1)}\right) v_\varepsilon^r\right) = G_\varepsilon^{(1)}(\Theta_\varepsilon^{(1)}) \quad \text{in} \quad S \times \Omega_\varepsilon^{(1)}.$$

Using the compact (and uniform in ε!) embedding $W^{1,2}(\Omega_\varepsilon^{(1)}) \hookrightarrow\hookrightarrow L^2(\Omega)$, it can be shown that the resulting system admits a (generally non-unique) solution via Schauder's fixed point theorem. For uniqueness, an additional property, e.g., Lipschitz continuity of g with respect to $r \in \mathbb{R}$, is needed.

[17] Continuous over $\mathbb{R}$ for almost all $(t,x) \in S \times \Omega$ and measurable over $S \times \Omega$ for all $r \in \mathbb{R}$.

Geometry and transformation. In [EM17b], a very special initial configuration is considered: Ω is assumed to be a connected domain such that $\overline{\Omega}$ is an axis-aligned, rectilinear polygon and the periodicity cell $Y = (0,1)^3$ is assumed to be decomposable as $Y = Y^{(1)} \cup Y^{(2)} \cup \Gamma$ where $Y^{(1)}, Y^{(2)}$ are disjoint open domains and where $\Gamma = \partial Y^{(2)}$ is a C^2-hypersurface. The ε-dependent motion function s_ε describing the phase transformation is assumed to be given via equation (4.1) where the function s has to obey Assumptions (1)-(7).

Now, there are three different aspects of this setting where potential generalizations could be investigated: (i) the rectilinear structure of the domain Ω, (ii) the configuration of the underlying periodicity cell, and (iii) the transformation describing the interface motion.

(i). In the case where $\overline{\Omega}$ is not a (rectilinear) polygon, one problem is given by the fact that, for some $\varepsilon > 0$, we have $\Gamma_\varepsilon^{ext} := \partial\Omega_\varepsilon^{(2)} \cap \partial\Omega \neq \emptyset$. In addition, $\Omega_\varepsilon^{(1)}$ may not be connected anymore. In this general setting, the analysis of Section 4.3 already breaks with Lemma 4.1, as uniform extension estimates are not available due to the complex structure at the boundary. We refer to [ACPDMP92], where it is argued that extension estimates can be uniform only at a distance from $\partial\Omega$.

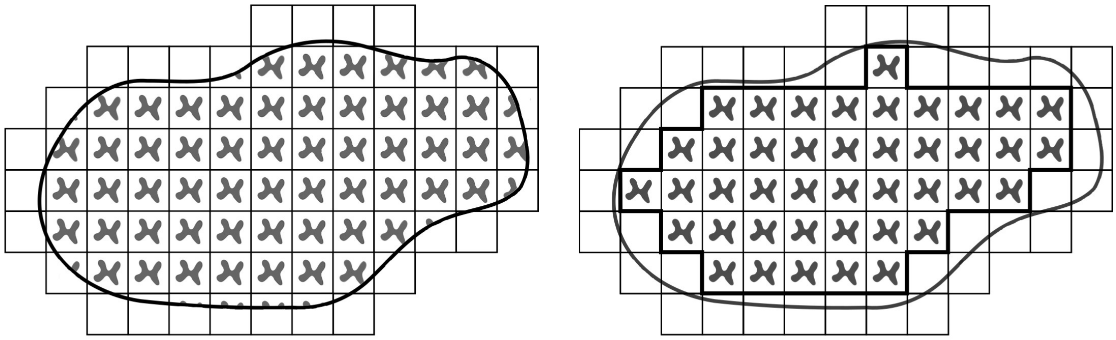


Figure 4.2: (Left) Domain Ω with complex boundary structure; (Right) Domain $\widetilde{\Omega}_\varepsilon$ with removed layer.

A simple and common way, see, e.g., [CP79, ETŢ15], to circumvent the problems arising due to the complex structure at the boundary is to remove the problematic boundary layer of dimension ε and investigate the problem for this simplified geometry.

To make things more precise, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $Y^{(i)}$ $(i = 1,2)$ be as above. We now only include the ε-scaled cells that are compactly embedded in Ω, i.e., we take

$$\widetilde{\Omega}_\varepsilon = \Omega \cap \left(\bigcup_{k \in Z_\varepsilon} \varepsilon(Y + k) \right), \quad \text{where } \; Z_\varepsilon = \left\{ k \in \mathbb{Z}^3 \;:\; \varepsilon(k + Y) \subset \Omega \right\}$$

and set

$$\widetilde{\Omega}_\varepsilon^{(2)} = \widetilde{\Omega}_\varepsilon \cap \left(\bigcup_{k \in \mathbb{Z}^3} \varepsilon(Y^{(2)} + k) \right), \quad \widetilde{\Gamma}_\varepsilon = \partial\widetilde{\Omega}_\varepsilon^{(2)}$$

as well as $\widetilde{\Omega}_\varepsilon^{(1)} = \Omega \setminus \overline{\widetilde{\Omega}_\varepsilon^{(2)}}$, illustrated on the right in figure 4.2. It is easy to see that

$\lim_{\varepsilon\to 0}|\Omega\setminus\widetilde{\Omega}_\varepsilon| = 0$ (see, e.g., [Han11, Eq. (2.3)]), which suggests that this simplification is of low significance in the limit process.

Corollary 4.15. *Replacing $\Omega_\varepsilon^{(i)}$ and Γ_ε with $\widetilde{\Omega}_\varepsilon^{(i)}$ and $\widetilde{\Gamma}_\varepsilon$, respectively, and assuming that $s_\varepsilon(t,x) = x$ for all x in $\Omega\setminus\widetilde{\Omega}_\varepsilon$, Theorems 4.7 and 4.8 still hold true and the homogenization result is valid.*

Proof. Taking into consideration [CP99, Chapter 2, Remark 2.13], Lemma 4.1 can easily be salvaged. The rest of the analysis and the homogenization is not affected by this change in geometry. □

(ii). Now, we loosen the assumptions placed on the underlying periodic structure: We still expect $Y^{(1)}, Y^{(2)} \subset Y$ to be disjoint open Lipschitz domains such that $\Gamma := \partial Y^{(2)}$ is a C^2-hypersurface and such that $\overline{Y} = \overline{Y^{(1)}} \cup \overline{Y^{(2)}}$. However, we do not assume that $\Gamma\cap\partial Y = \emptyset$; but note that we still expect $\Omega_\varepsilon^{(1)}$ to be connected. As a consequence, $\Omega_\varepsilon^{(2)}$ may or may not be connected; see Figure 4.3 for an example where both phases are connected.[18]

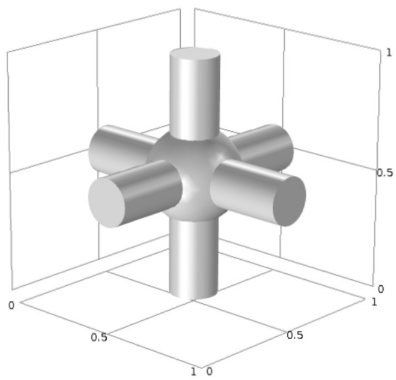


Figure 4.3: A unit cell leading to a pipe-like system; figure taken from [Höp16, P. 14].

Note that, due to the Lipschitz regularity of $Y^{(i)}$, $i = 1, 2$, pathological cases as presented in [ACPDMP92, Fig. 1] are excluded.

Again, we have to deal with a complex boundary as $\partial\Omega\cap\Gamma_\varepsilon \neq \emptyset$ where the proof of Lemma 4.1 given above is not valid (since the referenced extension operator is not valid in this case). Due to the rectilinear structure of the overall domain Ω, however, it is still possible to control the behavior at the boundary and recover corresponding extension operators.

Corollary 4.16. *Let the underlying microstructure be given as described in this paragraph (ii). Then, Theorems 4.7 and 4.8 still hold true. The homogenization procedure also holds with some changes in the boundary conditions for the microscopic problem, where we get*

$$u^{(2)} = u, \quad \theta^{(2)} = \theta \quad \text{on} \quad \Gamma^{in}(t,x), \tag{4.32a}$$

$$y \mapsto u^{(2)}, \theta^{(2)} \qquad Y-\text{periodic}. \tag{4.32b}$$

[18] This is not possible in $\mathbb{R}^2$.

Here, $\Gamma^{(in)}(t,x) = \Gamma(t,x) \setminus \partial Y$.

Proof. To recover Lemma 4.1, we have to rely on more sophisticated extension operators as developed in [Höp16, Theorem 3.5]. With that in mind, the rest of the analysis part is still valid. Regarding the limit process, the existence of two-scale limits, i.e., Theorem 4.11, is clear and passing to the limit leads to equation (4.18) (momentum equation) and equation (4.25) (heat equation). At this point, some additional care is needed in decoupling these problems. For the mechanical part, while equation (4.19) is, of course, still valid for all $\widetilde{v} \in L^2(\Omega; W^{1,2}_{\#}(Y))^3$, variational equation (4.20) now holds for all $v^{(2)} \in L^2(\Omega; W^{1,2}_{\#}(Y^{(2)}))^3$, which leads to the periodicity condition (4.32b) for $U^{(2)}$ (and, analogously, for $\Theta^{(2)}$). □

Note that, although $\Gamma^{ext} = \Gamma(t,x) \cap \partial Y \neq 0$ is allowed in this setting, this set is necessarily constant independent of space and time due to Assumption (5).

(iii). In general, the motion function $s_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \overline{\Omega}$ does not have to be explicitly given as a folded periodic two-scale motion function $s \colon \overline{S} \times \overline{\Omega} \times \overline{Y} \to \overline{Y}$. The minimal requirements for the analysis and the limit process to still be valid are:

Minimal requirements - transformation function

Let $s_\varepsilon \colon \overline{S} \times \overline{\Omega} \to \overline{\Omega}$ such that

(1) *Regularity*: $s_\varepsilon \in C^1(\overline{S}; C^2(\overline{\Omega}))$.

(2) *Invertability*: there is $s_\varepsilon^{-1} = s_\varepsilon^{-1}(t,x)$ satisfying $s_\varepsilon(t, s_\varepsilon^{-1}(t,x)) = x$ for all $(t,x) \in S \times \Omega$.

(3) *Regularity of the inverse*: $s_\varepsilon^{-1} \in C^1(\overline{S}; C^2(\overline{\Omega}))$.

(4) *Initial condition*: $s_\varepsilon(0,x) = x$ for all $x \in \Omega$.

(5) *Estimates*: There is $c > 0$ such that $\det Ds_\varepsilon > c$ and it holds equation (4.4) as well as

$$\sup_{0<\varepsilon<\varepsilon_0} \left(\|\partial_t D s_\varepsilon\|_{L^\infty(S\times\Omega)} \right) < \infty.$$

(6) *Strong two-scale limits*: there is $s \in C^1(\overline{S}; C^0(\overline{\Omega}; C^2_{\#}(Y)))$ such that

$$\frac{1}{\varepsilon}\partial_t s_\varepsilon \xrightarrow{2} \partial_t s, \quad Ds_\varepsilon \xrightarrow{2} D_y s, \quad \varepsilon D^2 s_\varepsilon \xrightarrow{2} D_y^2 s.$$

(7) *Motion properties of limit*: there is a function $s^{-1} = s^{-1}(t,x,y)$ satisfying $s(t,x,s^{-1}(t,x,y)) = y$ for all $(t,x,y) \in S \times \Omega \times Y$ as well as $s_\varepsilon^{-1} \in C^1(\overline{S}; C^0(\overline{\Omega}; C^2_{\#}(Y)))$.

It is not difficult to see that these requirements are already sufficient for the analysis and the limit process, see the following corollary. In this general setting, we do not assume that inclusions stay in their respective cell, i.e., there may be $k \in \mathbb{Z}^3$ and $\varepsilon > 0$

such that $\varepsilon(Y+k) \subset \Omega$ but $s_\varepsilon(t, \varepsilon(Y^{(2)}+k)) \not\subset \varepsilon(Y^{(2)}+k)$. This is not possible in the setting described in Section 4.2 as this violates Assumption (5).

Corollary 4.17. *Let the set of minimal requirements be fulfilled. Then, Theorems 4.7 and 4.8 as well as the limit process are still valid.*

Proof. This is straightforward as the assumptions are specifically chosen so that all requirements for the analysis and homogenization are satisfied. □

Finally, we point out that it is possible to combine Corollaries 4.16 and 4.17 to account for situations where both phases are connected and the phase transformations are not restricted to the interior of Y, i.e., $\Gamma^{ext} = \Gamma^{ext}(t,x)$ might also evolve with respect to $(t,x) \in S \times \Omega$.

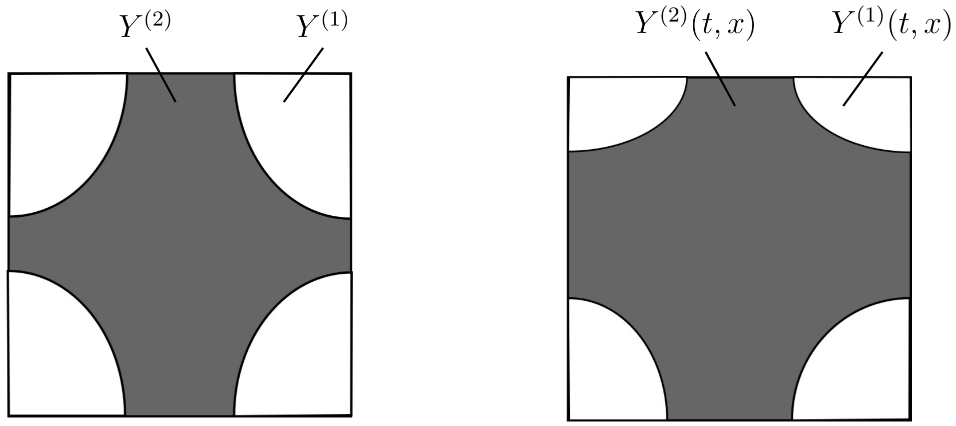


Figure 4.4: (left) Cross section of a unit cell where both phases are connected (as in Figure 4.3), (right) deformed cross section for some $(t,x) \in S \times \Omega$. Note that the deformed unit cell is still periodic since $s(t,x,\cdot)$ is periodic.

Homogenization. Here, we present a complementary result establishing the uniqueness of the limit problem established in Section 4.4 and given by equations (4.30a) to (4.30d). This result is obtained via energy estimates where some particular focus on the coupling terms is vital.

Theorem 4.18 (Uniqueness of limit problem)**.** *The homogenized problem given by equations* (4.30a) *to* (4.30d) *has exactly one solution.*

Proof. With the limit process outlined in Section 4.4, we know that there is a set of functions $(u, \theta, U^{(2)}, \Theta^{(2)})$ where

$$u \in L^\infty\left(S; W_0^{1,2}(\Omega)\right)^3, \quad U^{(2)} \in L^\infty\left(S; L^2\left(\Omega; W_\#^{1,2}\left(Y^{(2)}\right)\right)\right)^3,$$
$$\theta \in L^2\left(S; W^{1,2}(\Omega)\right), \qquad \Theta^{(2)} \in L^2\left(S; L^2\left(\Omega; W_\#^{1,2}\left(Y^{(2)}\right)\right)\right),$$

such that $(u, \theta, u^{(2)}, \theta^{(2)})$ is a solution to the limit problem given via equations (4.30a) to (4.30e). Here, again,

$$f^{(2)}(t,x,y) = F^{(2)}(t,x,s(t,x,y)) \quad (f = u, \theta).$$

Assume that we have two different sets of solutions and denote their pair-wise difference as $(w, \xi, W^{(2)}, \Xi^{(2)})$. We introduce

$$\widetilde{W}(t,x,y) = \sum_{j,k=1}^{3} \tau_{jk}(t,x,y) e(w)(t,x) e_j \cdot e_k + \tau(t,x,y)\xi(t,x),$$
$$\widetilde{\Xi}(t,x,y) = \sum_{j=1}^{3} \tau_j(t,x,y) \nabla\xi(t,x) \cdot e_j$$

and get (as both sets of solutions satisfy equations (4.18) and (4.25))

$$\begin{aligned}\int_\Omega \int_{Y^{(1)}} \mathcal{C}^{r,1}\big(e(w) + e_y(\widetilde{W})\big) : \big(e(v) + e_y(\widetilde{v})\big)\,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^{(2)}} \mathcal{C}^{r,2} e_y(W^{(2)}) : e_y(v^{(2)})\,\mathrm{d}y\,\mathrm{d}x \\ - \int_\Omega \int_{Y^{(1)}} \alpha^{r,1}\xi : \big(Dv + D_y\widetilde{v}\big)\,\mathrm{d}y\,\mathrm{d}x - \int_\Omega \int_{Y^{(2)}} \alpha^{r,2}\Xi^{(2)} : D_y v^{(2)}\,\mathrm{d}y\,\mathrm{d}x = 0, \quad (4.33a)\end{aligned}$$

$$\begin{aligned}\int_\Omega \partial_t(c^{r,1}\xi)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \int_{Y^{(2)}} \partial_t(c^{r,2}\Xi^{(2)})\varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^{(1)}} \partial_t\left(\gamma^{r,1} : \left(Dw + D_y\widetilde{W}\right)\right)\varphi\,\mathrm{d}x + \int_\Omega \int_{Y^{(2)}} \partial_t\left(\gamma^{r,2} : D_y W^{(2)}\right)\varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^{(2)}} c^{r,2} v^r \Xi^{(2)} \cdot \nabla_y \varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega \int_{Y^{(2)}} v^r\left(\gamma^{r,2} : D_y W_\varepsilon^{(2)}\right) \cdot \nabla_y\varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^{(1)}} K^{r,1}\left(\nabla\xi + \nabla_y\widetilde{\Xi}\right) \cdot \left(\nabla\varphi + \nabla_y\widetilde{\varphi}\right)\,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^{(2)}} K^{r,2}\nabla_y\Xi^{(2)} \cdot \nabla_y\varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x = 0 \quad (4.33b)\end{aligned}$$

for all admissible sets of test functions $(v, \widetilde{v}, v^{(2)}, \varphi, \widetilde{\varphi}, \varphi^{(2)})$. For details, we refer to Section 4.4. Now, choosing $(\partial_t w, \partial_t\widetilde{W}, \partial_t W^{(2)}, \xi, \widetilde{\Xi}, \Xi^{(2)})$ as the set of test functions and using the coercivity and boundedness properties of the involved coeeficients, standard energy estimates lead to

$$\begin{aligned}c\frac{\mathrm{d}}{\mathrm{d}t}\left(\|e(w) + e_y(\widetilde{W})\|^2_{L^2(\Omega\times Y^{(1)})^{3\times3}} + \|e_y(W^{(2)})\|^2_{L^2(\Omega\times Y^{(2)})^{3\times3}}\right) \\ \leq C\left(\|e(w) + e_y(\widetilde{W})\|^2_{L^2(\Omega\times Y^{(1)})^{3\times3}} + \|e_y(W^{(2)})\|^2_{L^2(\Omega\times Y^{(2)})^{3\times3}}\right) \\ + \underbrace{\int_\Omega \int_{Y^{(1)}} \gamma^{r,1}\xi : \partial_t\big(Dw + D_y\widetilde{W}\big)\,\mathrm{d}y\,\mathrm{d}x}_{A_1} + \underbrace{\int_\Omega \int_{Y^{(2)}} \gamma^{r,2}\Xi^{(2)} : \partial_t D_y W^{(2)}\,\mathrm{d}y\,\mathrm{d}x}_{B_1},\end{aligned}$$

$$
\begin{aligned}
c\bigg(\frac{\mathrm{d}}{\mathrm{d}t}&\Big(\|\xi\|^2_{L^2(\Omega)} + \|\Xi^{(2)}\|^2_{L^2(\Omega\times Y^{(2)})}\Big) + \|\nabla\xi + \nabla_y\widetilde{\Xi}\|^2_{L^2(\Omega\times Y^{(1)})} + \|\nabla_y\Xi^{(2)}\|^2_{L^2(\Omega\times Y^{(2)})}\bigg)\\
&\leq C\Big(\|\xi\|^2_{L^2(\Omega\times Y^{(1)})} + \|\Xi^{(2)}\|^2_{L^2(\Omega\times Y^{(2)})} + \|e_y(W^{(2)})\|^2_{L^2(\Omega\times Y^{(2)})}\Big)\\
&\quad - \underbrace{\int_\Omega\int_{Y^{(1)}} \partial_t\Big(\gamma^{r,1} : \Big(Dw + D_y\widetilde{W}\Big)\Big)\varphi\,\mathrm{d}x}_{A_2} - \underbrace{\int_\Omega\int_{Y^{(2)}} \partial_t\left(\gamma^{r,2} : D_yW^{(2)}\right)\varphi^{(2)}\,\mathrm{d}y\,\mathrm{d}x}_{B_2}
\end{aligned}
$$

where c, C are generic uniform constants independent of the functions. Regarding the A-terms, we see that

$$
\begin{aligned}
&\frac{\alpha^{(1)}}{\gamma^{(1)}}A_1 - A_2\\
&\quad= \int_\Omega\int_{Y^{(1)}} \gamma^{r,1}\xi : \partial_t\big(Dw + D_y\widetilde{W}\big)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_{Y^{(1)}} \gamma^{r,1} : \Big(Dw + D_y\widetilde{W}\Big)\,\partial_t\xi\,\mathrm{d}x\\
&\quad= \int_\Omega\int_{Y^{(1)}} \partial_t\gamma^{r,1}\xi : \big(Dw + D_y\widetilde{W}\big)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad\leq C\|\xi\|_{L^2(\Omega)}\|Dw + D_y\widetilde{W}\|_{L^2(\Omega\times Y^{(1)})^3}
\end{aligned}
$$

which, by analogy, holds similarly for the B-terms. Combining these results and employing Gronwall's inequality, we can infer that $\xi, \Xi^{(2)}, W^{(2)}$ almost everywhere. In addition, we conclude that

$$
\|e(w) + e_y(\widetilde{W})\|^2_{L^\infty(S;L^2(\Omega\times Y^{(1)}))^{3\times3}} = 0,
$$

i.e., $e(w) = -e_y(\widetilde{W})$ a.e. in $S\times\Omega\times Y^{(1)}$, which implies that $e_y(\widetilde{W})$ is Y-independent. Therefore, as $\widetilde{W}$ is Y-periodic, we see that $\widetilde{W}$ also is constant in $y\in Y$. As a consequence, $e(w) = 0$ a.e., and, due to Dirichlet boundary conditions, also $w = 0$ a.e. □

CHAPTER 5

Corrector estimates

In this chapter, quantitative error estimates regarding the homogenization procedure outlined in Chapter 4 are considered. While such estimates seem to not be obtainable in the fully-coupled thermoelasticity (due to the interplay of the scale coupling, micro vs. macroscale, and the physical coupling, momentum equation vs. heat balance), explicit rates are provem under certain reasonable simplifications to the model.

Please note that the following, Sections 5.1 to 5.4 to be precise, is already published, [EM17a],[1] where some minor cosmetic changes regarding the typesetting as well as some small changes in some notations (to ensure coherence throughout the thesis) were performed. Also, we added a short paragraph *Estimates for the time derivatives* (see Section 5.4.1) and some references to other parts of the thesis.

The main results of this chapter are:

- Theorem 5.9: corrector estimates for a *weakly coupled* problem, where the assumption is made that either mechanical dissipation or thermal stresses negligible,
- Theorem 5.10: corrector estimates for *microscale coupling*, where the assumption is made that mechanical dissipation and thermal stresses only significant in the slow-conducting phase (and, therefore, negligible in the fast-conducting phase).

5.1 Introduction

We aim to derive quantitative estimates that show the quality of the upscaling process of a coupled linear thermoelasticity system with a priori known phase transformations posed in a high-contrast media (as given by equations (4.3a) to (4.3j)) to its corresponding two-scale thermoelasticity system (as given by equations (4.30a) to (4.30e)).

The problem we have in mind is posed in a medium where the two building components,

[1] The results presented are due to the first author.

initially assumed to be periodically distributed, are different solid phases of the same material in which phase transformations that are a priori known occur. As the main effect, the presence of phase transformations leads to evolution problems in time-dependent domains that are not necessarily periodic anymore.

In our earlier paper [EM17b], we studied the well-posedness of such a thermoelasticity problem and conducted a homogenization procedure via the two-scale convergence technique (cf. [All92] for details). Those results were obtained after transforming the problem to a fixed reference geometry. In this work, our goal is to further investigate the connection of those problems and to derive an upper bound for the convergence rate (in some yet to be defined sense) of their solutions. While historically a tool to also justify the homogenization (via asymptotic expansions) in the first place, such estimates, which in the homogenization literature are usually called *error* and *corrector estimates*, provide a means to evaluate the accuracy of the upscaled model. Also, such estimates are especially interesting from a computational point of view. In the context of *Multiscale FEM*, for example, they are needed to ensure/control the convergence of the method, we refer to, e.g., [AB05, HW97].

The basic idea is to estimate the L^2- and $W^{1,2}-$errors of the solutions of the problems using energy-like estimates, additional regularity results, and special operator estimates for functions with zero average. Since, in general, the solutions of the ε-problem do not have the same domain as the solutions of the two-scale problem, we additionally rely on so-called *macroscopic reconstructions* (we refer to Section 5.3). The difficulty in getting such estimates in our specific scenario is twofold: First, the coupling between the quasi-stationary momentum equation and the heat equation and, second, the interface motion which (after transforming to a reference domain) leads to additional terms as well as time-dependent and non-periodic coefficients functions.

As typical for a corrector estimate result, the main goal is to show that there is a constant $C > 0$ which is independent on the particular choice of ε such that

Targeted Corrector Estimate Result

$$\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^\infty(S\times\Omega)} + \|U_\varepsilon^{\mathrm{err}}\|_{L^\infty(S;L^2(\Omega))^3} + \|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^3} + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(1)}))^{3\times3}} \\ + \varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^3} + \varepsilon\,\|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(2)}))^{3\times3}} \leq C(\sqrt{\varepsilon}+\varepsilon). \quad (5.1)$$

Here, Ω represents the full medium, $\Omega_\varepsilon^{(1)}$ the fast-heat-conducting connected matrix and $\Omega_\varepsilon^{(2)}$ the slow-heat-conducting inclusions. For the definitions of the (error and corrector) functions regarding the temperature Θ_ε and the deformation U_ε, we refer the reader to the beginning of Section 5.4.

Unfortunately, in the general setting of a fully-coupled thermoelasticity problem with moving interface, such corrector estimates as stated in (5.1) seem not to be obtainable; in Section 5.4.3, we point out where and why the usual strategy for establishing such estimates is bound to fail.

Instead, we show that there are a couple of possible simplifications of the full model in which (5.1) holds:

(a) *Weakly coupled problem*: If we assume either the *mechanical dissipation* or the *thermal stresses* to be negligible, we are led to weakly coupled problems, where the desired estimates can be established successively, see Theorem 5.9. We note that the regularity requirements are higher in the case of no thermal stress compared to the case of mechanical dissipation.

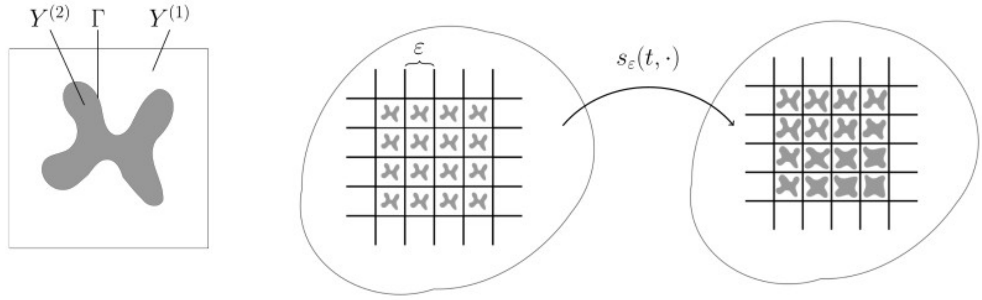
(b) *Microscale coupling*: If *mechanical dissipation* and *thermal stress* are only really significant in the slow-conducting component and negligible in the connected matrix part, the estimates hold, see Theorem 5.10.

As pointed out in [Wan99], neglecting the effect of mechanical dissipation is a step that is quite usual in modeling thermoelasticity problems.

Convergence rates for specific one-phase problems with periodic constants (some of them posed in perforated domains) were investigated in, e.g., [BP13, BPC98, CP98]. In [Eck05], convergence rates for a complex nonlinear problem modeling liquid-solid phase transitions via a phase-field approach were derived. A homogenization result including corrector estimates for a two-scale diffusion problem posed in a locally-periodic geometry was proven in [MvN13, vNM11]. Here, similar to our scenario, the microstructures are non-uniform, non-periodic, and assumed to be *a priori* known; the microstructures are however time independent and there are no coupling effects. For some corrector estimate results in the context of thermo and elasticity problems, we refer to [BP13, STV13], e.g. We also want to point out to the newer and different philosophy in which the solutions are compared in the two-scale spaces (e.g., $L^2(\Omega; W^{1,2}_{\#}(Y^{(2)}))$ as opposed to the, possibly ε-dependent, spaces for the ε-problem (e.g., $L^2(\Omega^{(2)}_\varepsilon))$,[2] a method which requires considerably less regularity on part of the solutions of the homogenized problems, we refer to [Gri04, MR16, Rei15].

The paper is organized as follows: In Section 5.2, we introduce the ε-microscopic geometry and formulate the thermoelasticity micro-problem in the moving geometry and transformed to a fixed reference domain and also state the homogenized two-scale problem. The assumptions on our data, some regularity statements, and auxiliary estimate results are then collected in Section 5.3. Finally, in Section 5.4, we focus on establishing convenient ε-control for the terms arising in the *error formulation* (see equation (5.7)). Based on these estimates, the corrector estimate (5.1) is then shown to hold for the above described cases (a) and (b).

[2]Here, we have used notation and domains as introduced in Section 5.2.



(a) The unit cell Y (b) The periodic reference geometry (c) The actual geometry at time t

Figure 5.1: Reference geometry and the resulting ε-periodic initial configuration. Note that for $t \neq 0$, these domains typically loose their periodicity.

5.2 Setting

5.2.1 Interface motion

The following notation is taken from [EM17b].

Let $S = (0,T)$, $T > 0$, be a time interval. Let Ω be the interior of a union of a finite number of closed cubes Q_j, $1 \leq j \leq n$, $n \in \mathbb{N}$, whose corner coordinates are rational such that, in addition, Ω is a Lipschitz domain.[3]

In addition, we denote the outer normal vector of Ω with $\nu = \nu(x)$. Let $Y = (0,1)^3$ be the open unit cell in $\mathbb{R}^3$. Take $Y^{(1)}, Y^{(2)} \subset Y$ two disjoint open sets, such that $Y^{(1)}$ is connected, such that $\Gamma := \overline{Y^{(1)}} \cap \overline{Y^{(2)}}$ is a C^3 interface, $\Gamma = \partial Y^{(2)}$, $\overline{Y^{(2)}} \subset Y$, and $Y = Y^{(1)} \cup Y^{(2)} \cup \Gamma$. With $n_\Gamma = n_\Gamma(y)$, $y \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

For $\varepsilon > 0$, we introduce the εY-periodic, initial domains $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ and interface Γ_ε representing the two phases and the phase boundary, respectively, via ($i = 1,2$)

$$\Omega_\varepsilon^{(i)} = \Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(Y^{(i)} + k) \right), \qquad \Gamma_\varepsilon = \Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(\Gamma + k) \right).$$

Here, for a set $M \subset \mathbb{R}^3$, $k \in \mathbb{Z}^3$, and $\varepsilon > 0$, we employ the notation

$$\varepsilon(M + k) := \left\{ x \in \mathbb{R}^3 \ : \ \frac{x}{\varepsilon} - k \in M \right\}.$$

From now on, we take $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ to be a sequence of monotonically decreasing positive numbers converging to zero such that Ω can be represented as the union of cubes of size ε. Note that this is possible due to the assumed structure of Ω.

Here $n_{\Gamma_\varepsilon} = n_\Gamma(\frac{x}{\varepsilon})$, $x \in \Gamma_\varepsilon$, denotes the unit normal vector (extended by periodicity)

[3] That is, Ω is *rectilinear* in the sense of Section 2.4.

pointing outwards $\Omega_\varepsilon^{(2)}$ into $\Omega_\varepsilon^{(1)}$. The above construction ensures that $\Omega_\varepsilon^{(1)}$ is connected and that $\Omega_\varepsilon^{(2)}$ is disconnected. We also have that $\partial\Omega_\varepsilon^{(2)} \cap \partial\Omega = \emptyset$.

Assumptions on the motion of the interface

We assume that we are given a function $s\colon \overline{S} \times \overline{\Omega} \times \mathbb{R}^3 \to \overline{Y}$ such that

(1) *Regularity*: $s \in C^1(\overline{S}; C^2(\overline{\Omega}) \times C^2_\#(Y))$,[4]

(2) *Invertability*: $s(t,x,\cdot)_{|\overline{Y}}\colon \overline{Y} \to \overline{Y}$ is bijective for every $(t,x) \in \overline{S} \times \overline{\Omega}$,

(3) *Regularity of inverse*: $s^{-1} \in C^1(\overline{S}; C^2(\overline{\Omega}) \times C^2_\#(Y))$,[5]

(4) *Initial condition*: $s(0,x,y) = y$ for all $x \in \overline{\Omega}$ and all $y \in \overline{Y}$,

(5) there is a constant $c > 0$ such that $\operatorname{dist}(\partial Y, \gamma) > c$ for all $(t,x) \in \overline{S} \times \overline{\Omega}$ and $\gamma \in s(t,x,\Gamma)$,

(6) $s(t,x,y) = y$ for all $(t,x) \in \overline{S} \times \overline{\Omega}$ and for all $y \in \overline{Y}$ such that $\operatorname{dist}(\partial Y, y) < \frac{c}{2}$,

(7) there are constants $c_s, C_s > 0$ such that

$$c_s \le \det(D_y s(t,x,y)) \le C_s, \quad (t,x,y) \in \overline{S} \times \overline{\Omega} \times \mathbb{R}^3.$$

We set the (t,x)-parametrized sets

$$Y^{(1)}(t,x) = s(t,x,Y^{(1)}), \quad Y^{(2)}(t,x) = s(t,x,Y^{(2)}), \quad \Gamma(t,x) = s(t,x,\Gamma).$$

We introduce the operations

$$\begin{aligned} [\cdot]\colon \mathbb{R}^3 \to \mathbb{Z}^3, &\quad [x] = k \text{ such that } x - [x] \in Y, \\ \{\cdot\}\colon \mathbb{R}^3 \to Y, &\quad \{x\} = x - [x] \end{aligned}$$

and define the ε-dependent function[6]

$$s_\varepsilon\colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^3, \quad s_\varepsilon(t,x) := \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon s\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right], \frac{x}{\varepsilon}\right).$$

For $i = 1,2$ and $t \in \overline{S}$, we set the time dependent sets $\Omega_\varepsilon^{(i)}(t)$ and $\Gamma_\varepsilon(t)$ and the corresponding non-cylindrical space-time domains $Q_\varepsilon^{(i)}$ and space-time phase boundary Ξ_ε via

$$\begin{aligned} \Omega_\varepsilon^{(i)}(t) = s_\varepsilon(t, \Omega_\varepsilon^{(i)}), &\qquad Q_\varepsilon^{(i)} = \bigcup_{t\in S} \left(\{t\} \times \Omega_\varepsilon^{(i)}(t)\right), \\ \Gamma_\varepsilon(t) = s_\varepsilon(t, \Gamma_\varepsilon), &\qquad \Xi_\varepsilon = \bigcup_{t\in S} \left(\{t\} \times \Gamma_\varepsilon(t)\right), \end{aligned}$$

[4] The # subscript denotes periodicity, i.e., for $k \in \mathbb{N}$, we have $C^k_\#(Y) = \{f \in C^k(\mathbb{R}^3) : f(x + e_i) = f(x) \text{ for all } x \in \mathbb{R}^3\}$, e_i basis vector of $\mathbb{R}^3$.

[5] Here, $s^{-1}\colon \overline{S} \times \overline{\Omega} \times \mathbb{R}^3 \to \overline{Y}$ is the unique function such that $s(t,x,s^{-1}(t,x,y)) = y$ for all $(t,x,y) \in \overline{S} \times \overline{\Omega} \in \overline{Y}$ extended by periodicity to all $y \in \mathbb{R}^3$.

[6] This is the typical notation in the context of homogenization via the *periodic unfolding method*, see, e.g., [CDG08, Dob12].

and denote by $n_{\Gamma_\varepsilon} = n_{\Gamma_\varepsilon}(t,x)$, $t \in S$, $x \in \Gamma_\varepsilon(t)$, the unit normal vector pointing outwards $\Omega_\varepsilon^{(2)}(t)$ into $\Omega_\varepsilon^{(1)}(t)$. The time-dependent domains $\Omega_\varepsilon^{(i)}$ host the phases at time $t \in \overline{S}$ and model the motion of the interface Γ_ε. We emphasize that, for any $t \neq 0$, the sets $\Omega_\varepsilon^{(1)}(t)$, $\Omega_\varepsilon^{(2)}(t)$, and $\Gamma_\varepsilon(t)$ are, in general, not periodic.

We introduce the transformation-related functions (here, $\widehat{V}_{\Gamma_\varepsilon}$ is the normal velocity and $\widehat{H}_{\Gamma_\varepsilon}$ the mean curvature of the interface) via

$$
\begin{aligned}
F_\varepsilon &\colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^{3\times 3}, & F_\varepsilon(t,x) &:= Ds_\varepsilon(t,x),\\
J_\varepsilon &\colon \overline{S} \times \overline{\Omega} \to \mathbb{R}, & J_\varepsilon(t,x) &:= \det\left(Ds_\varepsilon(t,x)\right),\\
v_\varepsilon &\colon \overline{S} \times \overline{\Omega} \to \mathbb{R}^3, & v_\varepsilon(t,x) &:= \partial_t s_\varepsilon(t,x),\\
\widehat{V}_{\Gamma_\varepsilon} &\colon \overline{S} \times \Gamma_\varepsilon \to \mathbb{R}, & \widehat{V}_{\Gamma_\varepsilon}(t,x) &:= v_\varepsilon(t,x) \cdot n_{\Gamma_\varepsilon}(t, s_\varepsilon(t,x)),\\
\widehat{H}_{\Gamma_\varepsilon} &\colon \overline{S} \times \Gamma_\varepsilon \to \mathbb{R}, & \widehat{H}_{\Gamma_\varepsilon}(t,x) &:= -\operatorname{div}\left((F^\varepsilon)^{-1}(t,x) n_{\Gamma_\varepsilon}(t, s_\varepsilon(t,x))\right)
\end{aligned}
$$

for which we have the following estimates

$$
\begin{aligned}
&\left\|F_\varepsilon\right\|_{L^\infty(S\times\Omega)^{3\times 3}} + \left\|F_\varepsilon^{-1}\right\|_{L^\infty(S\times\Omega)^{3\times 3}} + \left\|J_\varepsilon\right\|_{L^\infty(S\times\Omega)}\\
&\qquad + \varepsilon^{-1}\left\|v_\varepsilon\right\|_{L^\infty(S\times\Omega)^3} + \varepsilon^{-1}\left\|V_{\Gamma_\varepsilon}\right\|_{L^\infty(S\times\Gamma_\varepsilon)} + \varepsilon\left\|H_{\Gamma_\varepsilon}\right\|_{L^\infty(S\times\Gamma_\varepsilon)} \leq C. \qquad (5.2)
\end{aligned}
$$

By design, the constant C entering (5.2) is independent on the choice of ε. Note that the same estimate also holds for the time derivatives of these functions.

5.2.2 Micro problem and homogenization result

The bulk equations of the coupled thermoelasticity problem are given as (we refer to [Bio56, EM17b, Kup79])

$$
\begin{aligned}
-\operatorname{div}(\mathcal{C}_\varepsilon^{(1)} e(u_\varepsilon^{(1)}) - \alpha_\varepsilon^{(1)}\theta_\varepsilon^{(1)}\mathbb{I}_3) &= f_\varepsilon^{(1)} \quad \text{in } Q_\varepsilon^{(1)}, && (5.3a)\\
-\operatorname{div}(\mathcal{C}_\varepsilon^{(2)} e(u_\varepsilon^{(2)}) - \alpha_\varepsilon^{(2)}\theta_\varepsilon^{(2)}\mathbb{I}_3) &= f_\varepsilon^{(2)} \quad \text{in } Q_\varepsilon^{(2)}, && (5.3b)\\
\partial_t\left(\rho^{(1)}c^{(1)}\theta_\varepsilon^{(1)} + \gamma_\varepsilon^{(1)}\operatorname{div} u_\varepsilon^{(1)}\right) - \operatorname{div}(K_\varepsilon^{(1)}\nabla\theta_\varepsilon^{(1)}) &= g_\varepsilon^{(1)} \quad \text{in } Q_\varepsilon^{(1)}, && (5.3c)\\
\partial_t\left(\rho^{(2)}c^{(2)}\theta_\varepsilon^{(2)} + \gamma_\varepsilon^{(2)}\operatorname{div} u_\varepsilon^{(2)}\right) - \operatorname{div}(K_\varepsilon^{(2)}\nabla\theta_\varepsilon^{(2)}) &= g_\varepsilon^{(2)} \quad \text{in } Q_\varepsilon^{(2)}. && (5.3d)
\end{aligned}
$$

Here, $\mathcal{C}_\varepsilon^{(i)} \in \mathbb{R}^{3\times3\times3\times3}$ are the *stiffness* tensors, $\alpha_\varepsilon^{(i)} > 0$ the *thermal expansion* coefficients, $\rho^{(i)} > 0$ the *mass densities*, $c^{(i)} > 0$ the *heat capacities*, $\gamma_\varepsilon^{(i)} > 0$ are the *dissipation coefficients*, $K_\varepsilon^{(i)} \in \mathbb{R}^{3\times 3}$ the *thermal conductivities*, and $f_\varepsilon^{(i)}$, $g_\varepsilon^{(i)}$ are volume densities. In addition, $e(v) = 1/2(\nabla v + \nabla v^T)$ denotes the linearized strain tensor and $\mathbb{I}_3$ the identity matrix.

At the interface between the phases, we assume continuity of both the temperature and the deformation. We also expect the fluxes of force and heat densities to be given via the mean curvature and the interface velocity, resp.:[7]

$$\begin{aligned}
[\![u_\varepsilon]\!] &= 0, && \text{on } \Xi_\varepsilon, && (5.3e)\\
[\![\mathcal{C}_\varepsilon \varepsilon(u_\varepsilon) - \alpha_\varepsilon \theta_\varepsilon \mathbb{I}_3]\!] n_{\Gamma_\varepsilon} &= -\varepsilon^2 H_{\Gamma_\varepsilon} n_{\Gamma_\varepsilon} && \text{on } \Xi_\varepsilon, && (5.3f)\\
[\![\theta_\varepsilon]\!] &= 0 && \text{on } \Xi_\varepsilon, && (5.3g)\\
[\![\rho c]\!] \theta_\varepsilon V_{\Gamma_\varepsilon} + [\![\gamma_\varepsilon \operatorname{div} u_\varepsilon]\!] V_{\Gamma_\varepsilon} - [\![K_\varepsilon \nabla \theta_\varepsilon]\!] \cdot n_{\Gamma_\varepsilon} &= L V_{\Gamma_\varepsilon} && \text{on } \Xi_\varepsilon. && (5.3h)
\end{aligned}$$

Here, $[\![v]\!] := v^{(1)} - v^{(2)}$ denotes the jump across the boundary separating the phases, $\sigma_0 > 0$ is the coefficient of surface tension, and $L \in \mathbb{R}$ is the latent heat corresponding to the phase transformation.

Finally, at the boundary of Ω and for the initial condition, we pose (note that $\partial\Omega = \partial\Omega_\varepsilon^{(1)}$)

$$\begin{aligned}
u_\varepsilon^{(1)} &= 0 && \text{on } S \times \partial\Omega_\varepsilon^{(1)}, && (5.3i)\\
\theta_\varepsilon^{(1)} &= 0 && \text{on } S \times \partial\Omega_\varepsilon^{(1)}, && (5.3j)\\
\theta_\varepsilon^{(i)}(0) &= \vartheta_\varepsilon^{(i)} && \text{in } \Omega_\varepsilon^{(i)}, && (5.3k)\\
u_\varepsilon^{(i)}(0) &= 0 && \text{in } \Omega^{(i)}, && (5.3l)
\end{aligned}$$

where $\vartheta_\varepsilon^{(i)}$ are initial temperature distributions that might be highly oscillating. The scaling of the coefficients is chosen as

$$\begin{aligned}
&\mathcal{C}_\varepsilon^{(1)} = \mathcal{C}^{(1)}, && K_\varepsilon^{(1)} = K^{(1)}, && \alpha_\varepsilon^{(1)} = \alpha^{(1)}, && \gamma_\varepsilon^{(1)} = \gamma^{(1)},\\
&\mathcal{C}_\varepsilon^{(2)} = \varepsilon^2 \mathcal{C}^{(2)}, && K_\varepsilon^{(2)} = \varepsilon^2 K^{(2)}, && \alpha_\varepsilon^{(2)} = \varepsilon \alpha^{(2)}, && \gamma_\varepsilon^{(2)} = \varepsilon \gamma^{(2)}.
\end{aligned}$$

Remark 5.1. *The simplified models described in the introduction (for which corrector estimates can be established) correspond to $\alpha^{(1)} = \alpha^{(2)} = 0$ or $\gamma^{(1)} = \gamma^{(2)} = 0$ (case (1), weakly coupled problem) and $\alpha^{(1)} = \gamma^{(1)} = 0$ (case (2), mirco coupled problem).*

Now, taking the pulled-back quantities (defined on the initial, periodic domains $\Omega_\varepsilon^{(i)}$) $U_\varepsilon^{(i)} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^3$ and $\Theta_\varepsilon^{(i)} \colon S \times \Omega_\varepsilon^{(i)} \to \mathbb{R}^3$ given via $U_\varepsilon^{(i)}(t,x) = u_\varepsilon^{(i)}(t, s_\varepsilon^{-1}(t,x))$ and $\Theta_\varepsilon^{(i)}(t,x) = \theta_\varepsilon^{(i)}(t, s_\varepsilon^{-1}(t,x))$,[8] we get the following problem in fixed coordinates (for more details regarding the transformation to a fixed domain, we refer to [Dob12, Mei08, PSZ13]):[9]

[7] Here, the scaling via ε^2 counters the effects of both the interface surface area, note that $\varepsilon|\Gamma_\varepsilon| \in \mathcal{O}(1)$, and the curvature itself, note that $\varepsilon|\widehat{H}_{\Gamma_\varepsilon}| \in \mathcal{O}(1)$.

[8] Here, $s_\varepsilon^{-1} \colon \overline{S} \times \overline{\Omega} \to \overline{\Omega}$ is the inverse function of s_ε.

[9] Here, the superscript r,ε denotes the transformed quantities, for example, $K_\varepsilon^{r,1} = J_\varepsilon F_\varepsilon^{-1} K^{(1)} F_\varepsilon^{-T}$ (cf. [EM17b]).

ε-Thermoelasticity problem - fixed coordinates

Phase 1 - Balances of momentum and heat

$$-\operatorname{div}\left(\mathcal{C}_\varepsilon^{r,1}e(U_\varepsilon^{(1)})-\Theta_\varepsilon^{(1)}\alpha_\varepsilon^{r,1}\right)=f_\varepsilon^{r,1} \quad \text{in} \quad S\times\Omega_\varepsilon^{(1)}, \tag{5.4a}$$

$$\begin{aligned}\partial_t\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)}+\gamma_\varepsilon^{r,1}:Du_\varepsilon^{(1)}\right)-\operatorname{div}\left(K_\varepsilon^{r,1}\nabla\Theta_\varepsilon^{(1)}\right)&\\ -\operatorname{div}\left(\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)}+\gamma_\varepsilon^{r,1}:Du_\varepsilon^{(1)}\right)v_\varepsilon\right)&=g_\varepsilon^{r,1}\end{aligned} \quad \text{in} \quad S\times\Omega_\varepsilon^{(1)}, \tag{5.4b}$$

Phase 2 - Balances of momentum and heat

$$-\operatorname{div}\left(\varepsilon^2\mathcal{C}_\varepsilon^{r,2}e(U_\varepsilon^{(2)})-\varepsilon\Theta_\varepsilon^{(2)}\alpha_\varepsilon^{r,2}\right)=f_\varepsilon^{r,2} \quad \text{in} \quad S\times\Omega_\varepsilon^{(2)}, \tag{5.4c}$$

$$\begin{aligned}\partial_t\left(c_\varepsilon^{r,2}\Theta_\varepsilon^{(2)}+\varepsilon\gamma_\varepsilon^{r,2}:Du_\varepsilon^{(2)}\right)-\operatorname{div}\left(\varepsilon^2K_\varepsilon^{r,2}\nabla\Theta_\varepsilon^{(2)}\right)&\\ -\operatorname{div}\left(\left(c_\varepsilon^{r,2}\Theta_\varepsilon^{(2)}+\varepsilon\gamma_\varepsilon^{r,2}:Du_\varepsilon^{(2)}\right)v_\varepsilon\right)&=g_\varepsilon^{r,2}\end{aligned} \quad \text{in} \quad S\times\Omega_\varepsilon^{(2)}, \tag{5.4d}$$

complemented with interface transmission, boundary, and initial conditions.

Now, for $j,k=1,2,3$ and $y\in Y$, set $d_{jk}=(y_j\delta_{1k},y_j\delta_{2k},y_j\delta_{3k})^T$ where δ denotes the *Kronecker delta.* For $t\in S$, $x\in\Omega$, let $\tau_j(t,x,\cdot)\in W_\#^{1,2}(Y^{(1)})$, $\tau_{jk}(t,x,\cdot)$, $\tau(t,x,\cdot)\in W_\#^{1,2}(Y^{(1)})^3$ be the solutions to the following variational cell problems[10]

$$0=\int_{Y^{(1)}}K^{r,1}\left(\nabla_y\tau_j+e_j\right)\cdot\nabla_y v\,\mathrm{d}y \quad \text{for all} \quad v\in W_\#^{1,2}(Y^{(1)}), \tag{5.5a}$$

$$0=\int_{Y^{(1)}}\mathcal{C}^{r,1}e_y(\tau_{jk}+d_{jk}):e_y(v)\,\mathrm{d}y \quad \text{for all} \quad v\in W_\#^{1,2}(Y^{(1)})^3, \tag{5.5b}$$

$$0=\int_{Y^{(1)}}\mathcal{C}^{r,1}e_y(\tau):e_y(v^{(1)})\,\mathrm{d}y-\int_{Y^{(1)}}\alpha^{r,1}:D_yv\,\mathrm{d}y \quad \text{for all} \quad v\in W_\#^{1,2}(Y^{(1)})^3. \tag{5.5c}$$

Using these functions, we introduce the fourth rank tensor $\mathcal{C}=\mathcal{C}(t,x,y)$ and the matrix $K=K(t,x,y)$ via

$$\begin{aligned}(\mathcal{C})_{i_1i_2j_1j_2}&=\mathcal{C}^{r,1}e_y\left(\tau_{i_1i_2}+d_{i_1i_2}\right):e_y\left(\tau_{j_1j_2}+d_{j_1j_2}\right),\\ (K)_{ij}&=K^{r,1}\left(\nabla_y\tau_j+e_j\right)\cdot\left(\nabla_y\tau_i+e_i\right)\end{aligned}$$

and define the averaged tensors

$$C^h=\int_{Y^{(1)}}C\,\mathrm{d}y, \quad K^h=\int_{Y^{(1)}}K\,\mathrm{d}y.$$

Furthermore, we define the following set of averaged coefficients

$$\begin{aligned}\alpha^h&=\int_{Y^{(1)}}\left(\alpha^{r,1}C^{r,1}e_y(\tau)\right)\,\mathrm{d}y,\\ c^h&=\rho^{(1)}c^{(1)}\left|Y^{(1)}\right|+\int_{Y^{(1)}}\gamma^{r,1}:D_y\tau\,\mathrm{d}y,\\ \gamma^h&=\int_{Y^{(1)}}\left(\gamma^{r,1}+\gamma^{r,1}D_y\tau_{jk}\right)\,\mathrm{d}y.\end{aligned}$$

[10] Here, and in the following, the superscript r denotes the transformed quantities, e.g., $K^{r,1}=JF^{-1}K^{(1)}F^{-T}$.

For the averaged data (right hand sides and transformation related functions), we set

$$f^h = \int_{Y^{(1)}} f^{r,1}\,\mathrm{d}y + \int_{Y^{(2)}} f^{r,2}\,\mathrm{d}y, \qquad H^h = \int_\Gamma \widehat{H}_\Gamma n_\Gamma\,\mathrm{d}s,$$
$$g^h = \int_{Y^{(1)}} g^{r,1}\,\mathrm{d}y + \int_{Y^{(2)}} g^{r,2}\,\mathrm{d}y, \qquad V^h = \int_\Gamma \widehat{V}_\Gamma^r\,\mathrm{d}s.$$

Finally, we introduce the operator $A^h\colon L^2(S\times\Omega;L^1(Y)) \to L^2(S\times\Omega)$ via

$$A^h(\Theta^{(2)},U^{(2)}) = \int_{Y^{(2)}} \left(c^{r,2}\Theta^{(2)} + \gamma^{r,2}U^{(2)}\right)\,\mathrm{d}y$$

After a homogenization procedure (the details of which are presented in [EM17b] and Chapter 4), we get the following upscaled two-scale model

Limit problem

Effective, macroscopic thermoelasticity

$$-\operatorname{div}\left(\mathcal{C}^h e(u) - \alpha^h\theta\right) = f^h + H^h \text{ in } S\times\Omega, \tag{5.6a}$$

$$\partial_t\left(c^h\theta + \gamma^h : Du + A^h(\Theta^{(2)},U^{(2)})\right) - \operatorname{div}\left(K^h\nabla\theta\right) = g^h - V^h \text{ in } S\times\Omega, \tag{5.6b}$$

Parametrized microscopic problem - reference coordinates

$$-\operatorname{div}_y\left(\mathcal{C}^{r,2}e_y(U^{(2)}) - \alpha^{r,2}\Theta^{(2)}\right) = f^r \quad \text{in} \quad S\times Y^{(2)}, \tag{5.6c}$$

$$\partial_t\left(c^{r,2}\Theta^{(2)} + \gamma^{r,2} : D_y u^{(2)}\right) - \operatorname{div}_y\left(K^{r,2}\nabla_y\Theta^{(2)}\right) - \operatorname{div}_y\left(\left(c^{r,2}\Theta^{r,2} + \gamma^{r,2} : D_y u^{(2)}\right)v^r\right) = g^r \quad \text{in} \quad S\times Y^{(2)}, \tag{5.6d}$$

$$U^{(2)} = u, \quad \Theta^{(2)} = \theta \quad \text{on} \quad S\times\Gamma, \tag{5.6e}$$

again, complemented with corresponding initial and boundary values.

5.3 Preliminaries

In this section, we lay the groundwork for the corrector estimations that are done in Section 5.4 by stating the existence and regularity results of the solutions and by providing some auxiliary estimates.

We introduce the spaces

$$W^{1,2}(\Omega_\varepsilon^{(1)};\partial\Omega) := \left\{u\in W^{1,2}(\Omega_\varepsilon^{(1)}) \ : \ u = 0 \text{ on } \partial\Omega\right\},$$
$$\mathcal{W}(S;X) := \left\{u\in L^2(S;X) \ : \ \partial_t u \in L^2(S;X')\right\},$$

where X is a Banach space. In general, we do not differentiate (in the notation) between a function defined on Ω and its restriction to $\Omega_\varepsilon^{(1)}$ or $\Omega_\varepsilon^{(2)}$ or between a function defined on one of those subdomains and its trivial extension to the whole of Ω. Here, and in the following, C, C_1, C_2 denote generic constants which are independent of ε. Their values might change even from line to line.

For a function $f = f(x,y)$, we introduce the so called macroscopic reconstruction $[f]_\varepsilon = [f]_\varepsilon(x) = f(x, x/\varepsilon)$. Note that, for general $f \in L^\infty(\Omega; W^{1,2}(Y))$, $[f]_\varepsilon$ may not even be measurable (see [All92]); continuity in one variable, e.g., $f \in L^\infty(\Omega; C_\#(Y))$ is sufficient, though. Applying the chain rule leads to $D[f]_\varepsilon = [D_x f]_\varepsilon + 1/\varepsilon [D_y f]_\varepsilon$, where $D = \nabla, e(\cdot), \operatorname{div}(\cdot)$, for sufficiently smooth functions f.

Assumption (A1). *We assume that* $\vartheta_\varepsilon \in W^{1,2}(\Omega)$,[11] $f_\varepsilon^{r,i} \in C^1(S; L^2(\Omega_\varepsilon^{(i)}))$, $g_\varepsilon^{r,i} \in L^2(S \times \Omega_\varepsilon^{(i)})$. *Furthermore, let* $\vartheta \in W^{1,2}(\Omega)$, $f^h \in C^1(S; L^2(\Omega))$, $g^h \in L^2(S \times \Omega)$ *for the macroscopic homogenized part and* $\vartheta^{(2)} \in C(\Omega; W^{1,2}(Y^{(2)}))$, $f^{r,1} \in C^1(S; L^2(\Omega))$, $g^{r,2} \in L^2(S \times \Omega)$ *for the two-scale part.*

We expect the following convergence rates to hold for our data:

$$\left\|\vartheta_\varepsilon^{(1)} - \vartheta\right\|_{L^2(\Omega_\varepsilon^{(1)})} \leq C\sqrt{\varepsilon},$$
$$\left\|\vartheta_\varepsilon^{(2)} - \left[\vartheta^{(2)}\right]_\varepsilon\right\|_{L^2(\Omega_\varepsilon^{(2)})} + \left\|f_\varepsilon^{r,2} - \left[f^{r,2}\right]_\varepsilon\right\|_{L^2(\Omega_\varepsilon^{(2)})^3} + \left\|g_\varepsilon^{r,2} - \left[g^{r,2}\right]_\varepsilon\right\|_{L^2(\Omega_\varepsilon^{(2)})} \leq C\varepsilon.$$

In addition, we assume that

$$\int_{\Omega_\varepsilon^{(1)}} \left|(f_\varepsilon^{r,1} - f^h)\varphi(x)\right| \,\mathrm{d}x \leq C\varepsilon\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \quad \text{for all } \varphi \in W^{1,2}(\Omega_\varepsilon^{(1)}; \partial\Omega)^3,$$
$$\int_{\Omega_\varepsilon^{(1)}} \left|(g_\varepsilon^{r,1} - g^h)\varphi(x)\right| \,\mathrm{d}x \leq C\varepsilon\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \quad \text{for all } \varphi \in W^{1,2}(\Omega_\varepsilon^{(1)}; \partial\Omega).$$

If we are also interested in developing estimates for the time derivatives, we need stronger regularity assumptions.

Assumption (A2). *Additionally to Assumption (A1), we also expect the following convergence rates to hold:*

$$\left\|\vartheta_\varepsilon^{(1)} - \vartheta\right\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \leq C\sqrt{\varepsilon},$$
$$\left\|\vartheta_\varepsilon^{(2)} - \left[\vartheta^{(2)}\right]_\varepsilon\right\|_{W^{1,2}(\Omega_\varepsilon^{(2)})} + \left\|\partial_t(f_\varepsilon^{r,2} - [f^r]_\varepsilon)\right\|_{L^2(\Omega_\varepsilon^{(2)})^3} + \left\|\partial_t(g_\varepsilon^{r,2} - [g^r]_\varepsilon)\right\|_{L^2(\Omega_\varepsilon^{(2)})} \leq C\varepsilon.$$

Moreover, we assume

$$\int_{\Omega_\varepsilon^{(1)}} \left|\partial_t(f_\varepsilon^{r,1} - f^h)\varphi(x)\right| \,\mathrm{d}x \leq C\varepsilon\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \quad \text{for all } \varphi \in W^{1,2}(\Omega_\varepsilon^{(1)}; \partial\Omega)^3,$$
$$\int_{\Omega_\varepsilon^{(1)}} \left|\partial_t(g_\varepsilon^{r,1} - g^h)\varphi(x)\right| \,\mathrm{d}x \leq C\varepsilon\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \quad \text{for all } \varphi \in W^{1,2}(\Omega_\varepsilon^{(1)}; \partial\Omega).$$

[11] Here, $\vartheta_\varepsilon = \sum_{i=1,2} \mathbb{1}_{|\Omega_\varepsilon^{(i)}} \vartheta_\varepsilon^{(i)}$ (after trivially extending $\vartheta_\varepsilon^{(i)}$).

5.3.1 Regularity results

To be able to justify the steps in the estimates shown in Section 5.4, some of the involved functions need to be of higher regularity than it is guaranteed via the standard $W^{1,2}$-theory for elliptic/parabolic problems. In the following lemmas, we collect the appropriate regularity results.

Lemma 5.2 (Regularity of cell problem solutions)**.** *The solutions of the problems* (5.5a)-(5.5c) *possess the regularity (for some $p > 3$)*

$$\tau_j \in C^2(\overline{S}; C^1(\overline{\Omega}; W^{2,p}(Y^{(1)}))), \quad \tau_{jk}, \tau \in C^2(\overline{S}; C^1(\overline{\Omega}; W^{2,p}(Y^{(1)})^3)).$$

Proof. The regularity with respect to $y \in Y^{(1)}$ can be derived from standard elliptic regularity theory (we refer to [GT13] for the general results and [Eck05] for the application to our case of the cell problems). The rest is a direct consequence of the regularity (with respect to $t \in S$ and $x \in \Omega$) of the involved coefficients. □

Note that this implies, in particular, that the cell problem functions and their gradients (with respect to $y \in Y$) are bounded and that their macroscopic reconstructions are well-defined measurable functions.

In the following, we denote $U_\varepsilon = (U_\varepsilon^{(1)}, U_\varepsilon^{(2)})$ and $\Theta_\varepsilon = (\Theta_\varepsilon^{(1)}, \Theta_\varepsilon^{(2)})$.

Lemma 5.3 (Existence and Regularity Theorem for the ε-Problem)**.** *There is a unique* $(U_\varepsilon, \Theta_\varepsilon) \in \mathcal{W}(S; W_0^{1,2}(\Omega)^3 \times W_0^{1,2}(\Omega))$ *solving the variational system* (5.4) *for which standard energy estimates hold independently of the parameter ε. Furthermore, this solution possesses the regularity $(U_\varepsilon, \Theta_\varepsilon) \in C^1(S; W^{2,2}(\Omega_\varepsilon^{(1)})^3 \times W^{2,2}(\Omega_\varepsilon^{(2)})^3) \times L^2(S; W^{2,2}(\Omega_\varepsilon^{(1)}) \times W^{2,2}(\Omega_\varepsilon^{(2)}))$ with $\partial_t \Theta_\varepsilon \in L^2(S; W^{1,2}(\Omega))$.*

Proof. The proof of the existence of a unique solution and of the energy estimates is given in [EM17b, Theorem 3.7, Theorem 3.8]. As a linear transmission problem (with sufficiently regular coefficients), regularity results apply (we refer to, e.g., [Eva10]; see, also, [SM02] for a similar coupling problem). □

Since s_ε is a diffeomorphism, this leads to a unique solution to the moving interface problem, also. However, while the solution has $W^{2,2}$-regularity, its second derivatives are not necessarily bounded independently of $\varepsilon > 0$.

Lemma 5.4 (Existence and Regularity Theorem for the Homogenized Problem)**.** *There is a unique*

$$(u, \theta, U^{(2)}, \Theta^{(2)}) \in \mathcal{W}(S; W_0^{1,2}(\Omega)^3 \times W_0^{1,2}(\Omega) \times L^2(\Omega; W^{1,2}(Y^{(2)})^3) \times L^2(\Omega; W^{1,2}(Y^{(2)})))$$

solving the variational system (5.6)*. Furthermore,*

$$(u, \theta) \in C^1(S; W^{2,2}(\Omega)^3) \times L^2(S; W^{2,2}(\Omega)) \quad \textit{such that} \quad \partial_t \theta \in L^2(S; W^{1,2}(\Omega)),$$

as well as

$$(U^{(2)}, \Theta^{(2)}) \in C^1(S; W^{2,2}(\Omega; W^{2,2}(Y^{(2)})^3)) \times L^2(S; W^{2,2}(\Omega; W^{2,2}(Y^{(2)})))$$
$$\text{such that } \partial_t \Theta^{(2)} \in L^2(S; W^{2,2}(\Omega; W^{1,2}(Y^{(2)}))).$$

Proof. The existence of a solution is given via the two-scale homogenization procedure outlined in [EM17b] and uniqueness for this linear coupled transmission problem can then be shown using energy estimates. As to the higher regularity, this follows, again, via the regularity of domain, coefficients, and data, we refer to results outlined in [Eva10, SM02, vNM11]. □

5.3.2 Auxiliary estimates

As far as the quantities related to the transformation are concerned, we have the following estimates available as stated in Lemma 5.5 and Lemma 5.6.

Lemma 5.5. *There is a constant $C > 0$ independent of the parameter ε such that*

$$\|F_\varepsilon - [F]_\varepsilon\|_{L^\infty(S\times\Omega)^{3\times3}} + \|J_\varepsilon - [J]_\varepsilon\|_{L^\infty(S\times\Omega)} + \|v_\varepsilon - \varepsilon [v]_\varepsilon\|_{L^\infty(S\times\Omega)}$$
$$+ \|\widehat{V}_{\Gamma_\varepsilon} - \varepsilon \left[\widehat{V}_\Gamma\right]_\varepsilon\|_{L^\infty(S\times\Gamma)} + \|\widehat{H}_{\Gamma_\varepsilon} - \left[\widehat{H}_\Gamma\right]_\varepsilon\|_{L^\infty(S\times\Gamma)} \le C\varepsilon$$

The same estimates hold for the time derivatives of those functions.

Proof. We show this only for F_ε; the other estimates follow in the same way:

$$\|F_\varepsilon - [F]_\varepsilon\|_{L^\infty(S\times\Omega)^{3\times3}} = \operatorname*{ess\,sup}_{(t,x)\in S\times\Omega} \left| D_y s\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right], \frac{x}{\varepsilon}\right) - D_y s\left(t, x, \frac{x}{\varepsilon}\right)\right\| \le L\frac{\varepsilon}{\sqrt{2}}.$$

Here, L is the Lipschitz constant of F_ε with respect to $x \in \Omega$ (uniform in $S \times Y$). □

Based on the estimates provided in Lemma 5.5 and due to the fact that all material parameters are assumed to be constant in the moving geometry, we get the same estimates for the material parameters ($K_\varepsilon^{r,i}$, $\alpha_\varepsilon^{r,i}$ and so on) in the reference configuration.

The following lemma is concerned with ε-independent estimates for the macroscopic reconstruction of periodic functions with zero average. There are several different but similar theorems that can be found in the literature regarding corrector estimates in the context of homogenization, we refer to, e.g., [CPS07, CP98, Eck05, MvN13], but for our purposes the following version suffices:

Lemma 5.6. *Let $f \in L^2(S \times \Omega_\varepsilon^{(1)}; C_\#(Y))$ such that*

$$\int_{Y^{(1)}} f(t,x,y)\,\mathrm{d}y = 0 \quad \textit{a.e. in} \quad S \times \Omega_\varepsilon^{(1)}.$$

Then, there is a constant $C > 0$ such that, independently of ε,

$$\int_{\Omega_\varepsilon^{(1)}} |[f]_\varepsilon (t,x)\varphi(x)| \, \mathrm{d}x \le C\varepsilon \|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})}.$$

for all $\varphi \in W^{1,2}(\Omega_\varepsilon^{(1)}; \partial\Omega)$.

Proof. This can be proven similarly to the corresponding statements in [CP98, Lemma 3] and [MvN13, Lemma 5.2]. □

5.4 Corrector estimates

In this section, we are concerned with the actual corrector estimates. Reconstructing micro-solutions from the homogenized functions via the $[\cdot]_\varepsilon$-operation and calculate the differences of the momentum equations (5.3a), (5.3b), (5.6a), and (5.6c) and heat equations (5.3c), (5.3d), (5.6b), and (5.6d), respectively, we get

Difference of ε-problem and homogenization limit

Momentum equations

$$-\operatorname{div}\left(\mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{(1)}) - \kappa\mathcal{C}^h e(u) - \alpha_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} + \kappa\alpha^h\theta\right) + H_\Gamma^h = f_\varepsilon^{r,1} - \kappa f^h, \tag{5.7a}$$

$$\begin{aligned} &-\operatorname{div}\left(\varepsilon^2\mathcal{C}_\varepsilon^{r,2} e(U_\varepsilon^{(2)}) - \varepsilon\alpha_\varepsilon^{r,2}\Theta_\varepsilon^{(2)}\right) \\ &\quad + \left[\operatorname{div}_y\left(\mathcal{C}^{r,2}e_y(U^{(2)}) - \alpha^{r,2}\Theta^{(2)}\mathbb{I}_3\right)\right]_\varepsilon = f_\varepsilon^{r,2} - \left[f^{r,2}\right]_\varepsilon, \end{aligned} \tag{5.7b}$$

Heat equations

$$\begin{aligned} &\partial_t\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} - \kappa c^h\theta\right) + \partial_t\left(\gamma_\varepsilon^{r,1} : Du_\varepsilon^{(1)} - \kappa\gamma^h : Du\right) \\ &\quad - \operatorname{div}\left(K_\varepsilon^{r,1}\nabla\Theta_\varepsilon^{(1)} - \kappa K^h\nabla\theta\right) - \kappa W_\Gamma^h = g_\varepsilon^{r,1} - \kappa g^h, \end{aligned} \tag{5.7c}$$

$$\begin{aligned} &\partial_t\left(c_\varepsilon^{r,2}\Theta_\varepsilon^{(2)} - \left[c^{r,2}\Theta^{(2)}\right]_\varepsilon\right) \\ &\quad +\partial_t\left(\varepsilon\gamma_\varepsilon^{r,2} : Du_\varepsilon^{(2)} - \left[\gamma^{r,2} : D_y u^{(2)}\right]_\varepsilon\right) \\ &\quad - \operatorname{div}\left(\varepsilon^2 K_\varepsilon^{r,2}\nabla\Theta_\varepsilon^{(2)}\right) + \left[\operatorname{div}_y\left(K^{r,2}\nabla_y\Theta^{(2)}\right)\right]_\varepsilon = g_\varepsilon^{r,2} - \left[g^{r,2}\right]_\varepsilon. \end{aligned} \tag{5.7d}$$

These equations hold in $S\times\Omega_\varepsilon^{(1)}$ and $S\times\Omega_\varepsilon^{(2)}$, respectively, where κ is given via $\kappa(t,x) = |Y^{(1)}(t,x)|^{-1}$. Using the interface and boundary conditions for both the ε-problem and the homogenized problem and then performing an integration by parts, these equations correspond to a variational problem in $W^{-1,2}(\Omega)^3 \times W^{-1,2}(\Omega)$.

Our strategy in establishing the estimates is as follows: After introducing error and corrector functions and doing some further preliminary estimates, we first, in Section 5.4.1, concentrate on the momentum part, i.e., equations (5.7a) and (5.7b). Here, we take the different terms arising in the weak formulation and estimate them individually using the results from Section 5.3 and usual energy estimation techniques. Combining those estimates, it is shown that the *mechanical error* can be controlled by the *heat error*, see Remark 5.7. Then, in Section 5.4.2, we basically do the same for the heat conduction part, i.e., equations (5.7c) and (5.7d), thereby arriving at the corresponding result that the *heat error* is controlled by the *mechanical error* (and the time derivative of the *mechanical error*), see Remark 5.8.

Finally, in Section 5.4.3, we go about combining those individual estimates. Here, we show that for the scenarios (*a*) (Theorem 5.9) and (*b*) (Theorem 5.10) (as described in the introduction), we get the desired estimates, i.e., equation (5.1). Moreover, we point out why the same strategy does not work for the full problem.

Now, we introduce the functions[12]

$$U_\varepsilon^{\mathrm{err}} = \begin{cases} U_\varepsilon^{(1)} - u & \text{in } S \times \Omega_\varepsilon^{(1)} \\ U_\varepsilon^{(2)} - \left[U^{(2)}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases}, \qquad \Theta_\varepsilon^{\mathrm{err}} = \begin{cases} \Theta_\varepsilon^{(1)} - \theta & \text{in } S \times \Omega_\varepsilon^{(1)} \\ \Theta_\varepsilon^{(2)} - \left[\Theta^{(2)}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases},$$

$$U_\varepsilon^{\mathrm{cor}} = \begin{cases} U_\varepsilon^{\mathrm{err}} - \varepsilon \left[\widetilde{U}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(1)} \\ U_\varepsilon^{\mathrm{err}} & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases}, \qquad \Theta_\varepsilon^{\mathrm{cor}} = \begin{cases} \Theta_\varepsilon^{\mathrm{err}} - \varepsilon \left[\widetilde{\Theta}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(1)} \\ \Theta_\varepsilon^{\mathrm{err}} & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases}.$$

The functions $\widetilde{U}$ and $\widetilde{\theta}$ are the functions arising in the two-scale limits of the gradients of $U_\varepsilon^{(1)}$ and $\Theta_\varepsilon^{(1)}$, respectively, and are given by (cf. [EM17b])

$$\widetilde{U} = \sum_{j,k=1}^{3} \tau_{jk} e(u) e_j \cdot e_k + \tau\theta, \qquad \widetilde{\Theta} = \sum_{j=1}^{3} \tau \nabla\theta^{(1)} \cdot e_j.$$

Due to the corrector parts (namely, $\varepsilon\left[\widetilde{\Theta}\right]_\varepsilon$ and $\varepsilon\left[\widetilde{U}\right]_\varepsilon$), the corrector functions does, in general, not vanish at $\partial\Omega$ and are therefore not valid choices of test functions for a weak variational formulation of the system given via equations (5.7a) to (5.7d). Because of that, we introduce a smooth cut-off function $m_\varepsilon \colon \Omega \to [0,1]$ with $m_\varepsilon(x) = 0$ for all $x \in \Omega_\varepsilon^{(1)}$ with $\operatorname{dist}(x, \partial\Omega) \leq \varepsilon c/2$ and $m_\varepsilon(x) = 1$ for all $x \in \Omega_\varepsilon^{(1)}$ with $\operatorname{dist}(x, \partial\Omega) \geq \varepsilon c$. Furthermore, we require the estimate

$$\sqrt{\varepsilon}\, \|\nabla m_\varepsilon\|_{L^2(\Omega)} + \sqrt{\varepsilon^3}\, \|\Delta m_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}} \|1 - m_\varepsilon\|_{L^2(\Omega)} \leq C \tag{5.8}$$

to hold independently of the parameter ε. For this cut-off function m_ε, we set

$$U_\varepsilon^{\mathrm{cor0}} = \begin{cases} U_\varepsilon^{\mathrm{cor}} + (1 - m_\varepsilon)\varepsilon \left[\widetilde{U}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(1)} \\ U_\varepsilon^{\mathrm{cor}} - \varepsilon \left[\widetilde{U}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases},$$

$$\Theta_\varepsilon^{\mathrm{cor0}} = \begin{cases} \Theta_\varepsilon^{\mathrm{cor}} + (1 - m_\varepsilon)\varepsilon \left[\widetilde{\Theta}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(1)} \\ \Theta_\varepsilon^{\mathrm{cor}} - \varepsilon \left[\widetilde{\Theta}\right]_\varepsilon & \text{in } S \times \Omega_\varepsilon^{(2)} \end{cases}.$$

[12]The subscripts "err" and "cor" for *error* and *corrector*, respectively.

We then have $\Theta_\varepsilon^{\mathrm{cor0}} \in W_0^{1,2}(\Omega)$ and $U_\varepsilon^{\mathrm{cor0}} \in W_0^{1,2}(\Omega)^3$. Owing to the regularity of $\widetilde{U}$ and $\widetilde{\Theta}$ (Lemma 5.2) and the estimate (5.8) for m_ε, these modified correctors admit the following ε-uniform estimates for the deformation correctors[13]

$$\begin{aligned}
\|U_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(i)})^3} &\leq \left\|U_\varepsilon^{\mathrm{cor0}}\right\|_{L^2(\Omega_\varepsilon^{(i)})^3} + C\varepsilon \\
&\leq \|U_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(i)})^3} + 2C\varepsilon, \\
\|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times3}} &\leq \left\|e(U_\varepsilon^{\mathrm{cor0}})\right\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times3}} + C(\sqrt{\varepsilon}+\varepsilon) \\
&\leq \|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times3}} + 2C(\sqrt{\varepsilon}+\varepsilon), \\
\varepsilon\|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} &\leq \varepsilon\left\|e(U_\varepsilon^{\mathrm{cor0}})\right\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + C(\sqrt{\varepsilon}+\varepsilon) \\
&\leq \varepsilon\|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + 2C(\sqrt{\varepsilon}+\varepsilon),
\end{aligned}$$

as well as for the temperature correctors

$$\begin{aligned}
\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(i)})} &\leq \left\|\Theta_\varepsilon^{\mathrm{cor0}}\right\|_{L^2(\Omega_\varepsilon^{(i)})} + C\varepsilon \\
&\leq \|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(i)})} + 2C\varepsilon, \\
\|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^3} &\leq \left\|\nabla\Theta_\varepsilon^{\mathrm{cor0}}\right\|_{L^2(\Omega_\varepsilon^{(1)})^3} + C(\sqrt{\varepsilon}+\varepsilon) \\
&\leq \|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^3} + 2C(\sqrt{\varepsilon}+\varepsilon), \\
\varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(2)})^3} &\leq \varepsilon\left\|\nabla\Theta_\varepsilon^{\mathrm{cor0}}\right\|_{L^2(\Omega_\varepsilon^{(2)})^3} + C(\sqrt{\varepsilon}+\varepsilon) \\
&\leq \varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(2)})^3} + 2C(\sqrt{\varepsilon}+\varepsilon).
\end{aligned}$$

Applying Korns inequality to $U_\varepsilon^{\mathrm{cor0}}$ (see [EM17b, Lemma 3.1]) and using the above estimates, we then get

$$\begin{aligned}
&\|U_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega)} + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^3} + \varepsilon\|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} \\
&\qquad\leq \|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(1)})^3} + \varepsilon\|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + C(\sqrt{\varepsilon}+\varepsilon). \qquad (5.9)
\end{aligned}$$

5.4.1 Estimates for the momentum equations

Let us first concentrate on the mechanical part of the corrector equations, namely equations (5.7a) and (5.7b). Starting with a variational form of said equations, we employ usual energy estimation techniques as well as the results and assumptions collected in Section 5.3 in order to estimate the *momentum error* in terms of the *heat error* and the parameter ε.

To that end, for $j = 1, ..9$, we introduce time-parametrized linear functionals $I_\varepsilon^{(j)} \colon S \times$

[13]The same estimates hold when replacing the linearized strain tensor with the gradient operator.

$W_0^{1,2}(\Omega)^3 \to \mathbb{R}$ defined via

$$
\begin{aligned}
I_\varepsilon^{(1)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(1)}} \big(\mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{(1)}) - \kappa \mathcal{C}^h e(u)\big) : e(\varphi)\,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa \mathcal{C}^h e(u) n_{\Gamma_\varepsilon} \cdot \varphi \,\mathrm{d}s,\\
I_\varepsilon^{(2)}(t,\varphi) &= \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} \big(\mathcal{C}_\varepsilon^{r,2} e(U_\varepsilon^{(2)}) - \big[\mathcal{C}^{r,2}\big]_\varepsilon e(\big[U^{(2)}\big]_\varepsilon)\big) : e(\varphi)\,\mathrm{d}x,\\
I_\varepsilon^{(3)}(t,\varphi) &= -\int_{\Omega_\varepsilon^{(1)}} \big(\alpha_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} - \kappa \alpha^h \theta\big) : D\varphi\,\mathrm{d}x + \int_{\Gamma_\varepsilon} \kappa \alpha^h \theta n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s,\\
I_\varepsilon^{(4)}(t,\varphi) &= \varepsilon \int_{\Omega_\varepsilon^{(2)}} \big(\alpha_\varepsilon^{r,2}\Theta_\varepsilon^{(2)} - \big[\alpha^{r,2}\Theta^{(2)}\big]_\varepsilon\big) : D\varphi\,\mathrm{d}x,\\
I_\varepsilon^{(5)}(t,\varphi) &= -\int_{\Omega_\varepsilon^{(1)}} \left(f_\varepsilon^{r,1} - \kappa \int_{Y^{(1)}} f^{r,1}\,\mathrm{d}y\right) \cdot \varphi\,\mathrm{d}x,\\
I_\varepsilon^{(6)}(t,\varphi) &= -\varepsilon^2 \int_{\Gamma_\varepsilon} \big[\mathcal{C}^{r,2}\big]_\varepsilon e(\big[U^{(2)}\big]_\varepsilon) n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s - \int_{\Omega_\varepsilon^{(1)}} \kappa \int_{Y^{(2)}} f^{r,2}\,\mathrm{d}y \cdot \varphi\,\mathrm{d}x,\\
I_\varepsilon^{(7)}(t,\varphi) &= -\varepsilon^2 \int_{\Gamma_\varepsilon} H_{\Gamma_\varepsilon}^r n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s - \int_{\Omega_\varepsilon^{(1)}} \kappa H^h \cdot \varphi\,\mathrm{d}x,\\
I_\varepsilon^{(8)}(t,\varphi) &= -\int_{\Omega_\varepsilon^{(2)}} \big(f_\varepsilon^{r,2} - \big[f^{r,2}\big]_\varepsilon\big) \cdot \varphi\,\mathrm{d}x,
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
I_\varepsilon^{(9)}(t,\varphi) = -\int_{\Omega_\varepsilon^{(2)}} \Big(&\varepsilon^2 \operatorname{div}\big(\big[\mathcal{C}^{r,2} e_x(U^{(2)})\big]_\varepsilon\big) + \varepsilon \big[\operatorname{div}_x \big(\mathcal{C}^{r,2} e_y(U^{(2)})\big]_\varepsilon \\
&+ \varepsilon \big[\operatorname{div}_x \big(\alpha^{r,2}\Theta^{(2)}\big)\big]_\varepsilon\big)\Big) \cdot \varphi\,\mathrm{d}x.
\end{aligned}
$$

After integration by parts, a variational form of equations (5.7a) and (5.7b) is given via

$$
\sum_{j=1}^{9} I_\varepsilon^{(j)}(t,\varphi) = 0 \quad \big(\varphi \in W_0^{1,2}(\Omega)^3\big). \tag{5.10}
$$

We now go on estimating these $I_\varepsilon^{(j)}$-terms individually and then combine the resulting estimates. This is done for the particular choices of test functions $\varphi = U_\varepsilon^{\mathrm{cor}0}$ as well as $\varphi = \partial_t U_\varepsilon^{\mathrm{cor}0}$. While the first choice is the natural one for energy estimates, the second is needed in order to merge the resulting estimates with the heat equations.

Estimates for the deformations and stresses. Taking a look at $I_\varepsilon^{(1)}$, we split

$$
\begin{aligned}
I_\varepsilon^{(1)}(t,\varphi) = &\int_{\Omega_\varepsilon^{(1)}} \mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{\mathrm{cor}}) : e(\varphi)\,\mathrm{d}x \\
&+ \int_{\Omega_\varepsilon^{(1)}} \big(\mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{(1)} - U_\varepsilon^{\mathrm{cor}}) - \kappa \mathcal{C}^h e(u)\big) : e(\varphi)\,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa \mathcal{C}^h e(u) n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s.
\end{aligned}
$$

We also see (via the definitions of $\mathcal{C}$ and $\widetilde{U}$)

$$\mathcal{C}^{r,1}_\varepsilon e(U^{(1)}_\varepsilon - U^{\mathrm{cor}}_\varepsilon) = [\mathcal{C}]_\varepsilon e(u) + \left[\mathcal{C}^{r,1} e_y(\tau)\right]_\varepsilon \theta \\ + \underbrace{\left(\mathcal{C}^{r,1}_\varepsilon - \left[\mathcal{C}^{r,1}\right]_\varepsilon\right)\left(e(u) + \left[e_y\left(\widetilde{U}\right)\right]_\varepsilon\right) + \varepsilon \mathcal{C}^{r,1}_\varepsilon \left[e_x\left(\widetilde{U}\right)\right]_\varepsilon}_{=:R^{(1)}_\varepsilon(t)},$$

where, using Lemma 5.6, the estimate

$$\left|\int_{\Omega^{(1)}_\varepsilon} R^{(1)}_\varepsilon(t)\varphi \,\mathrm{d}x\right| \le C\varepsilon \|\varphi\|_{W^{1,2}(\Omega^{(1)}_\varepsilon)}$$

holds. Now, seeing that $\int_\Gamma \mathcal{C}e(u)n\,\mathrm{d}s = 0$ and $\mathrm{div}_y(\mathcal{C}e(u)) = 0$ a.e. in $S \times \Omega$ (note that u is constant over Y) and using Lemma 5.6, we get

$$\left|\int_{\Omega^{(1)}_\varepsilon} \left([\mathcal{C}]_\varepsilon - \kappa(0)\mathcal{C}^h\right) e(u) : e(\varphi)\,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa(0)\mathcal{C}e(u)n_{\Gamma_\varepsilon}\cdot\varphi\,\mathrm{d}x\right| \\ = \left|\int_{\Omega^{(1)}_\varepsilon} \mathrm{div}\left(\left([\mathcal{C}]_\varepsilon - \kappa(0)\mathcal{C}^h\right)e(u)\right)\cdot\varphi\,\mathrm{d}x\right| \le C\varepsilon\|\varphi\|_{W^{1,2}(\Omega^{(1)}_\varepsilon)}$$

for all $\varphi \in W^{1,2}(\Omega^{(1)}_\varepsilon;\partial\Omega)$. Also,

$$\int_{\Omega^{(1)}_\varepsilon} \mathcal{C}^{r,1}_\varepsilon e(U^{\mathrm{cor}}_\varepsilon) : e(U^{\mathrm{cor}0}_\varepsilon)\,\mathrm{d}x \\ \ge C_1\|e(U^{\mathrm{cor}}_\varepsilon)\|^2_{L^2(\Omega^{(1)}_\varepsilon)^{3\times3}} + \varepsilon\int_{\Omega^{(1)}_\varepsilon} \mathcal{C}^{r,1}_\varepsilon e(U^{\mathrm{cor}}_\varepsilon) : e\left((1-m_\varepsilon)\left[\widetilde{U}\right]_\varepsilon\right)\,\mathrm{d}x \\ \ge \frac{C_1}{2}\|e(U^{\mathrm{cor}}_\varepsilon)\|^2_{L^2(\Omega^{(1)}_\varepsilon)^{3\times3}} - C_2(\varepsilon+\varepsilon^2).$$

Here, the second inequality can be shown using the assumptions on m_ε and the known estimates of the involved functions. As a result, we arrive at

$$I^{(1)}_\varepsilon(t, U^{\mathrm{cor}0}_\varepsilon) - \int_{\Omega^{(1)}_\varepsilon} \left[\mathcal{C}^{r,1}e_y(\tau)\right]_\varepsilon \theta : e(U^{\mathrm{cor}0}_\varepsilon)\,\mathrm{d}x \\ \ge C_1\|e(U^{\mathrm{cor}}_\varepsilon)\|^2_{L^2(\Omega^{(1)}_\varepsilon)^{3\times3}} - C_2\left(\varepsilon+\varepsilon^2\right). \quad (5.11)$$

Now, going on with $I^{(2)}_\varepsilon$, it is easy to see that

$$I^{(2)}_\varepsilon(t, U^{\mathrm{cor}0}_\varepsilon) = \varepsilon^2\int_{\Omega^{(2)}_\varepsilon} \left[\mathcal{C}^{r,2}\right]_\varepsilon \left(e(U^{(2)}_\varepsilon) - e(\left[U^{(2)}\right]_\varepsilon)\right) : e(U^{\mathrm{cor}}_\varepsilon)\,\mathrm{d}x \\ + \varepsilon^2\int_{\Omega^{(2)}_\varepsilon} \left(\mathcal{C}^{r,2}_\varepsilon - \left[\mathcal{C}^{r,2}\right]_\varepsilon\right) e(U^{(2)}_\varepsilon) : e(U^{\mathrm{cor}}_\varepsilon)\,\mathrm{d}x \\ \ge C_1\varepsilon^2\left\|e(U^{\mathrm{cor}}_\varepsilon)\right\|^2_{L^2(\Omega^{(1)}_\varepsilon)^{3\times3}} - C_2\varepsilon^2. \quad (5.12)$$

Going forward with term $I^{(3)}_\varepsilon$, we decompose

$$\alpha^{r,1}_\varepsilon\Theta^{(1)}_\varepsilon - \kappa\alpha^h\theta = \left(\alpha^{r,1}_\varepsilon - \left[\alpha^{r,1}\right]_\varepsilon\right)\Theta^{(1)}_\varepsilon + \left[\alpha^{r,1}\right]_\varepsilon\Theta^{\mathrm{err}}_\varepsilon + \left(\left[\alpha^{r,1}\right]_\varepsilon - \kappa\alpha^h\right)\theta \quad (5.13)$$

from which we can estimate the first two terms as

$$\left|\int_{\Omega_\varepsilon^{(1)}} \left(\left(\alpha_\varepsilon^{r,1} - \left[\alpha^{r,1}\right]_\varepsilon\right)\Theta_\varepsilon^{(1)} + \left[\alpha^{r,1}\right]_\varepsilon \Theta_\varepsilon^{\mathrm{err}}\right) : D\varphi\right| \leq C\left(\varepsilon + \|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(1)})}\right)\|D\varphi\|_{L^2(\Omega_\varepsilon^{(1)})}.$$

For the remaining term of equation (5.13) combined with the interface integral part of $I_\varepsilon^{(3)}$, we get

$$\begin{aligned}
&\int_{\Omega_\varepsilon^{(1)}} \left(\left[\alpha^{r,1}\right]_\varepsilon - \kappa\alpha^h\right)\theta : D\varphi \,\mathrm{d}x + \int_{\Gamma_\varepsilon} \kappa\alpha^h\theta n_{\Gamma_\varepsilon}\cdot\varphi\,\mathrm{d}s \\
&\qquad = -\int_{\Omega_\varepsilon^{(1)}} \left[\mathrm{div}_x\left(\left(\alpha^{r,1} - \kappa\int_{Y^{(1)}}\alpha^{r,1}\,\mathrm{d}y\right)\theta\right)\right]_\varepsilon \cdot\varphi\,\mathrm{d}x \\
&\qquad\quad - \int_{\Omega_\varepsilon^{(1)}} \kappa\,\mathrm{div}\left(\int_{Y^{(1)}} C^{r,1}e_y(\tau)\,\mathrm{d}y\,\theta\right)\cdot\varphi\,\mathrm{d}x \\
&\qquad\quad - \int_{\Gamma_\varepsilon}\left[\alpha^{r,1}\right]_\varepsilon \theta n_{\Gamma_\varepsilon}\cdot\varphi\,\mathrm{d}s - \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^{(1)}}\left[\mathrm{div}_y\left(\alpha^{r,1}\theta\right)\right]_\varepsilon\cdot\varphi\,\mathrm{d}x
\end{aligned}$$

We apply Lemma 5.6 to

$$\begin{aligned}
f_1 &= \mathrm{div}_x\left(\left(\alpha^{r,1} - \kappa\int_{Y^{(1)}}\alpha^{r,1}\,\mathrm{d}y\right)\theta^{(1)}\right), \\
f_2 &= \mathrm{div}_x\left(\left(\left(\mathcal{C}^{r,1}e_y(\tau)\right) - \kappa\int_{Y^{(1)}} C^{r,1}e_y(\tau)\,\mathrm{d}y\right)\theta^{(1)}\right)
\end{aligned}$$

and recall that τ is a solution of the cell problem (5.5c) (because of this the "$\frac{1}{\varepsilon}\left[\mathrm{div}_y\right]_\varepsilon$"-terms vanish) which leads to

$$\left|I_\varepsilon^{(3)}(t,\varphi) + \int_{\Omega_\varepsilon^{(1)}}\left[\mathcal{C}^{r,1}e_y(\tau)\right]_\varepsilon\theta : e(\varphi)\,\mathrm{d}x\right| \leq C\left(\varepsilon + \|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(1)})}\right)\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})}. \quad (5.14)$$

Similarly as with $I_\varepsilon^{(2)}$, for the thermo-elasticity term $I_\varepsilon^{(4)}$, we get

$$\left|I_\varepsilon^{(4)}(t,\varphi)\right| \leq C\|\Theta_\varepsilon^{\mathrm{err}}\|^2_{L^2(\Omega_\varepsilon^{(2)})} + \varepsilon^2\|D\varphi\|^2_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}}. \quad (5.15)$$

Next, we tackle the mean curvature error term $I_\varepsilon^{(7)}$:

$$\begin{aligned}
\left|I_\varepsilon^{(7)}(t,\varphi)\right| \leq \varepsilon^2 &\left|\int_{\Gamma_\varepsilon}\left(H_{\Gamma_\varepsilon}^r - [H_\Gamma^r]_\varepsilon\right)n_{\Gamma_\varepsilon}\cdot\varphi\,\mathrm{d}s\right| \\
&+ \left|\varepsilon^2\int_{\Gamma_\varepsilon}[H_\Gamma^r]_\varepsilon n_{\Gamma_\varepsilon}\cdot\varphi\,\mathrm{d}s - \int_{\Omega_\varepsilon^{(1)}}\kappa H^h\cdot\varphi\,\mathrm{d}x\right|.
\end{aligned}$$

Now, in view of the assumptions on our data and for the source density errors (stated in Assumption (A1)) and the curvature estimate, using Lemma 5.5 (for the functional $I_\varepsilon^{(6)}(t,\varphi)$), and the boundedness of the functions involved in $I_\varepsilon^{(9)}$, we estimate

$$\sum_{j=5}^{9}\left|I_\varepsilon^{(j)}(t,\varphi)\right| \leq C\varepsilon\left(\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} + \|\varphi\|_{L^2(\Omega_\varepsilon^{(2)})}\right). \quad (5.16)$$

Finally, merging the individual estimates for the error terms (namely, inequalities (5.11), (5.12), and (5.14) to (5.16) and equation (5.9), we conclude

Corrector estimate for the momentum error

$$\|U_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega)^3} + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + \varepsilon \|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times 3}} \leq C\left(\sqrt{\varepsilon} + \varepsilon + \|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega)}\right). \tag{5.17}$$

Remark 5.7. *With inequality* (5.17) *it is clear that the error in the mechanical part inherits the convergence rate from the heat-error (at least if it is not faster than* $\sqrt{\varepsilon}$*).*

Estimates for the time derivatives. Under stronger assumptions, namely Assumption (A2), corrector estimates for the time derivatives can be established. Since this is done quite analogously to the estimates leading to inequality (5.17), except for a few terms arising due to the time differentiation, we omit most of the details of this calculations.

Now, if Assumption (A2) is fulfilled, it is possible to differentiate the variational equation (5.10) (based on equations (5.7a) and (5.7b)) with respect to time and arrive at

$$\sum_{j=1}^{9} \partial_t I_\varepsilon^{(j)}(t,\varphi) = 0 \quad \left(\varphi \in W_0^{1,2}(\Omega)^3\right). \tag{5.18}$$

Starting with the first term, we calculate

$$\begin{aligned}\partial_t I_\varepsilon^{(1)}(t,\varphi) = &\underbrace{\int_{\Omega_\varepsilon^{(1)}} \left(\partial_t \mathcal{C}_\varepsilon^{r,1} e(U_\varepsilon^{(1)}) - \kappa \partial_t \mathcal{C}^h e(u)\right) : e(\varphi)\,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa \partial_t \mathcal{C}^h e(u) n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s}_{=: I_\varepsilon^{(1a)}(t,\varphi)} \\ &+ \underbrace{\int_{\Omega_\varepsilon^{(1)}} \left(\mathcal{C}_\varepsilon^{r,1} e(\partial_t U_\varepsilon^{(1)}) - \kappa \mathcal{C}^h e(\partial_t u)\right) : e(\varphi)\,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa \mathcal{C}^h e(\partial_t u) n_{\Gamma_\varepsilon} \cdot \varphi\,\mathrm{d}s}_{=: I_\varepsilon^{(1b)}(t,\varphi)}. \end{aligned} \tag{5.19}$$

For the second term on the right hand side of equation (5.19), $I_\varepsilon^{(1a)}(t,\varphi)$, we can infer from Lemmas 5.5 and 5.6 as well as the arguments leading to inequality (5.11) that

$$I_\varepsilon^{(1b)}\left(t, \partial_t U_\varepsilon^{\mathrm{cor}0}(t)\right) - \int_{\Omega_\varepsilon^{(1)}} \left[\mathcal{C}^{r,1} e_y(\partial_t \tau)\right]_\varepsilon \partial_t \theta : e(\partial_t U_\varepsilon^{\mathrm{cor}0})\,\mathrm{d}x \geq C_1 \|e(\partial_t U_\varepsilon^{\mathrm{cor}})\|^2_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} - C_2\left(\varepsilon + \varepsilon^2\right). \tag{5.20}$$

For the first term, $I_\varepsilon^{(1a)}(t,\varphi)$, we can estimate

$$\left| I_\varepsilon^{(1a)}(t, \partial_t U_\varepsilon^{\mathrm{cor}0}(t)) - \int_{\Omega_\varepsilon^{(1)}} \left[\partial_t \mathcal{C}^{r,1} e_y(\tau)\right]_\varepsilon \theta : e(\partial_t U_\varepsilon^{\mathrm{cor}0})\,\mathrm{d}x \right| \leq C_1 \|e(U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} \|e(\partial_t U_\varepsilon^{\mathrm{cor}})\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + C_2\left(\varepsilon + \varepsilon^2\right).$$

Taking into account the assumptions on our data and on the error of the time derivatives of the source densities as stated in Assumption (A2) as well as the bounds formulated in Lemma 5.5, we can also estimate

$$\sum_{j=5}^{9} \left|\partial_t I_\varepsilon^{(j)}(t,\varphi)\right| \leq C\varepsilon \left(\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})^3} + \|\varphi\|_{L^2(\Omega_\varepsilon^{(2)})^3}\right) \quad (\varphi \in W_0^{1,2}(\Omega)^3)\,.$$

For the remaining terms, $I_\varepsilon^{(j)}$ ($j = 2,3,4$), the strategy is the same as in the preceding paragraph (leading to inequalities (5.12), (5.14), and (5.15)) with some additional time derivative terms which have to be estimated in a similar fashion as the term $I_\varepsilon^{(1a)}$ above.

Combining these estimates, we are led to:

Corrector estimate for the momentum error - time derivatives

$$\begin{aligned}\|\partial_t U_\varepsilon^{\text{err}}\|_{L^2(\Omega)^3} + \|D\partial_t U_\varepsilon^{\text{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times3}} + \varepsilon\,\|D\partial_t U_\varepsilon^{\text{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times3}}\\ \leq C\left(\sqrt{\varepsilon} + \varepsilon + \|\Theta_\varepsilon^{\text{err}}\|_{L^2(\Omega)} + \|\partial_t\Theta_\varepsilon^{\text{err}}\|_{L^2(\Omega)}\right). \quad (5.21)\end{aligned}$$

Finally, we present some additional estimates that can be established in the same way and which are useful when combining the estimates for the momentum equations and the heat equations. If we take $\partial_t U^{\text{cor}_\varepsilon}$ as a test function and follow the same strategy as in (5.11), we can estimate

$$\begin{aligned}&\int_0^t I_\varepsilon^{(1)}(\tau, \partial_t U_\varepsilon^{\text{cor0}})\,\mathrm{d}\tau - \int_0^t \int_{\Omega_\varepsilon^{(1)}} \left[\mathcal{C}^{r,1} e_y(\tau)\right]_\varepsilon \theta : e(\partial_t U_\varepsilon^{\text{cor0}})\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad\geq C_1\left(\|e(U_\varepsilon^{\text{cor}})(t)\|^2_{L^2(\Omega_\varepsilon^{(1)})} - \|e(U_\varepsilon^{\text{cor}})(0)\|^2_{L^2(\Omega_\varepsilon^{(1)})}\right) - C_2\int_0^t \|U_\varepsilon^{\text{cor}}\|^2_{W^{1,2}(\Omega_\varepsilon^{(1)})^3}\,\mathrm{d}\tau\\ &\qquad - C_3(\varepsilon+\varepsilon^2), \quad (5.22)\end{aligned}$$

where some integration by parts with respect to time was done. Similarly, we obtain

$$\begin{aligned}\int_0^t I_\varepsilon^{(2)}(\tau, \partial_t U_\varepsilon^{\text{cor0}})\,\mathrm{d}\tau &\geq C_1\varepsilon^2\left(\|e(U_\varepsilon^{\text{cor}})(t)\|^2_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} - \|e(U_\varepsilon^{\text{cor}})(0)\|^2_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}}\right)\\ &\quad - C_2\varepsilon^2\int_0^t \|e(U_\varepsilon^{\text{cor}})\|^2_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}}\,\mathrm{d}\tau - C_3\varepsilon^2. \quad (5.23)\end{aligned}$$

5.4.2 Estimates for the heat conduction equations

In this section, we go on with establishing some control on the error terms in the heat conduction equations, where the overall strategy is the same as with the momentum error estimates established in Section 5.4.

To that end, for $j = 1, .., 13$, we introduce time-parametrized linear functionals $E_\varepsilon^{(j)}\colon S\times W_0^{1,2}(\Omega) \to \mathbb{R}$ defined via

$$
\begin{aligned}
E_\varepsilon^{(1)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} - \kappa \int_{Y^{(1)}} c^r \,\mathrm{d}y\, \theta \right) \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(2)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(\gamma_\varepsilon^{r,1} : Du_\varepsilon^{(1)} - \kappa\gamma^h : Du - \kappa\theta \int_{Y^{(1)}} \gamma^{r,1} : D_y\tau \,\mathrm{d}y \right) \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(3)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(1)}} \left(\left(c_\varepsilon^{r,1}\Theta_\varepsilon^{(1)} + \gamma_\varepsilon^{r,1} : Du_\varepsilon^{(1)}\right) v_\varepsilon \right) \cdot \nabla\varphi \,\mathrm{d}x,\\
E_\varepsilon^{(4)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(1)}} \left(K_\varepsilon^{r,1}\nabla\Theta_\varepsilon^{(1)} - \kappa K^h \nabla\theta \right) \cdot \nabla\varphi \,\mathrm{d}x - \int_{\Gamma_\varepsilon} \kappa K^h \nabla\theta \cdot n_{\Gamma_\varepsilon} \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(5)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(2)}} \left(\partial_t \left(c_\varepsilon^{r,2}\Theta_\varepsilon^{(2)}\right) - \left[\partial_t(c^{r,2}\Theta^{(2)})\right]_\varepsilon \right) \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(6)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(2)}} \left(\partial_t \left(\varepsilon\gamma_\varepsilon^{r,2} : Du_\varepsilon^{(2)}\right) - \left[\partial_t(\gamma^{r,2} : D_y u^{(2)})\right]_\varepsilon \right) \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(7)}(t,\varphi) &= \int_{\Omega_\varepsilon^{(2)}} \left(c_\varepsilon^{r,2}\Theta_\varepsilon^{(2)} v_\varepsilon - \left[c^{r,2}\Theta^{(2)}\right]_\varepsilon [v^r]_\varepsilon \right) \cdot \nabla\varphi \,\mathrm{d}x,\\
E_\varepsilon^{(8)}(t,\varphi) &= \varepsilon \int_{\Omega_\varepsilon^{(2)}} \left(\gamma_\varepsilon^{r,2} : Du_\varepsilon^{(2)} v_\varepsilon - \left[\gamma^{r,2}\right]_\varepsilon : D\left[U^{(2)}\right]_\varepsilon [v^r]_\varepsilon \right) \cdot \nabla\varphi \,\mathrm{d}x,\\
E_\varepsilon^{(9)}(t,\varphi) &= \varepsilon^2 \int_{\Omega_\varepsilon^{(2)}} \left(K_\varepsilon^{r,2}\nabla\Theta_\varepsilon^{(2)} - \left[K^{r,2}\right]_\varepsilon \nabla\left[\Theta^{(2)}\right]_\varepsilon \right) \cdot \nabla\varphi \,\mathrm{d}x,\\
E_\varepsilon^{(10)}(t,\varphi) &= \int_{\Gamma_\varepsilon} V_{\Gamma_\varepsilon}^r \varphi \,\mathrm{d}\sigma + \int_{\Omega_\varepsilon^{(1)}} \kappa V^h \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(11)}(t,\varphi) &= -\int_{\Omega_\varepsilon^{(1)}} \kappa \int_\Gamma K^{r,2}\nabla_y\Theta^{(2)} \cdot n \,\mathrm{d}s\, \varphi \,\mathrm{d}x - \varepsilon^2 \int_{\Gamma_\varepsilon} \left[K^{r,2}\right]_\varepsilon \nabla\left[\Theta^{(2)}\right]_\varepsilon \cdot n_{\Gamma_\varepsilon} \varphi \,\mathrm{d}\sigma,\\
E_\varepsilon^{(12)}(t,\varphi) &= -\int_{\Omega_\varepsilon^{(1)}} \left(g_\varepsilon^{r,1} - \kappa \int_{Y^{(1)}} g^{r,1} \,\mathrm{d}y \right) \varphi \,\mathrm{d}x + \int_{\Omega_\varepsilon^{(2)}} \left(g_\varepsilon^{r,2} - \left[g^{r,2}\right]_\varepsilon \right) \varphi \,\mathrm{d}x,\\
E_\varepsilon^{(13)}(t,\varphi) &= -\varepsilon \int_{\Omega_\varepsilon^{(2)}} \left(\left[\operatorname{div}_x \left(K^{r,2}\nabla\Theta^{(2)}\right)\right]_\varepsilon + \operatorname{div}\left[K^{r,2}\nabla_x\Theta^{(2)}\right]_\varepsilon \right) \varphi \,\mathrm{d}x.
\end{aligned}
$$

Multiplying equations (5.7c) and (5.7d) with test functions $\varphi \in W_0^{1,2}(\Omega)$, integrating over Ω, and then integrating by parts while using the interface conditions, we are lead to

$$
\sum_{j=1}^{13} E_\varepsilon^{(j)}(t,\varphi) = 0 \quad \left(\varphi \in W_0^{1,2}(\Omega)\right). \tag{5.24}
$$

For the first functional, we see that

$$
\begin{aligned}
E_\varepsilon^{(1)}(t,\Theta_\varepsilon^{\mathrm{cor0}}) = \int_{\Omega_\varepsilon^{(1)}} \rho^{(1)}c^{(1)}\partial_t \left(\Theta_\varepsilon^{(1)}(J_\varepsilon - \frac{1}{|Y^{(1)}(0)|}\left|Y^{(1)}\right|) \right) \Theta_\varepsilon^{\mathrm{cor0}} \,\mathrm{d}x \\
+ \int_{\Omega_\varepsilon^{(1)}} \rho^{(1)}c^{(1)} \frac{1}{|Y^{(1)}(0)|} \partial_t \left(\left|Y^{(1)}\right| \Theta_\varepsilon^{\mathrm{err}} \right) \Theta_\varepsilon^{\mathrm{err}} \,\mathrm{d}x + R_\varepsilon^{(2)}(t),
\end{aligned}
$$

where

$$R_\varepsilon^{(2)}(t) = \varepsilon \int_{\Omega_\varepsilon^{(1)}} \rho^{(1)} c^{(1)} \frac{1}{|Y^{(1)}(0)|} \partial_t \left(\left|Y^{(1)}\right| \Theta_\varepsilon^{\mathrm{err}} \right) m_\varepsilon \left[\widetilde{\Theta} \right]_\varepsilon .$$

Using the regularity estimates for $\widetilde{\Theta}$ and $\Theta_\varepsilon^{(1)}$, we see that there is a constant $c > 0$ independent of ε such that (for every $\delta > 0$)

$$\left| \int_0^t R_\varepsilon^{(2)}(\tau) \,\mathrm{d}\tau \right| \leq \int_0^t \|\Theta_\varepsilon^{\mathrm{err}}(\tau)\|^2 \,\mathrm{d}\tau + \delta \left(\|\Theta_\varepsilon^{\mathrm{err}}(t)\|^2 - \|\Theta_\varepsilon^{\mathrm{err}}(0)\|^2 \right) + C_\delta t \varepsilon^2.$$

With this estimate and Lemma 5.6, we then get

$$\begin{aligned} \int_0^t E_\varepsilon^{(1)}(\tau, \Theta_\varepsilon^{\mathrm{cor}0}) \,\mathrm{d}\tau \geq C_1 \left(\|\Theta_\varepsilon^{\mathrm{err}}(t)\|^2_{\Omega_\varepsilon^{(1)}} - \|\Theta_\varepsilon^{\mathrm{err}}(0)\|^2_{\Omega_\varepsilon^{(1)}} \right) \\ - C_2 \left(\int_0^t \|\Theta_\varepsilon^{\mathrm{err}}(\tau)\|^2_{\Omega_\varepsilon^{(1)}} + \varepsilon \left\|\Theta_\varepsilon^{\mathrm{cor}}(\tau)\right\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \,\mathrm{d}\tau + t\varepsilon^2 \right). \end{aligned} \tag{5.25}$$

For the dissipation term of $\Omega_\varepsilon^{(1)}$, namely $E_\varepsilon^{(2)}$, we start by noticing that

$$\kappa \gamma^h : Du + \kappa \int_{Y^{(1)}} \gamma^{r,1} : D_y \tau \,\mathrm{d}y \,\theta = \kappa \int_{Y^{(1)}} \gamma^{r,1} : \left(Du + \nabla_y \widetilde{U} \right) \mathrm{d}y \tag{5.26}$$

and decompose

$$\begin{aligned} E_\varepsilon^{(2)}(t,\varphi) = \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(\gamma_\varepsilon^{r,1} : Du_\varepsilon^{\mathrm{cor}} \right) \varphi \,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(\left(\gamma_\varepsilon^{r,1} - \left[\gamma^{r,1}\right]_\varepsilon \right) : D(U_\varepsilon^{(1)} - U_\varepsilon^{\mathrm{cor}}) \right) \varphi \,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(1)}} \partial_t \left(\left[\gamma^{r,1}\right]_\varepsilon : D(U_\varepsilon^{(1)} - U_\varepsilon^{\mathrm{cor}}) - \kappa \int_{Y^{(1)}} \gamma^{r,1} : \left(Du + \nabla_y \widetilde{U} \right) \mathrm{d}y \right) \varphi \,\mathrm{d}x. \end{aligned}$$

Applying Lemma 5.6 to

$$f = \partial_t \left(\gamma^{r,1} : \left(Du^{(1)} + \nabla_y \widetilde{U} \right) - \kappa \int_{Y^{(1)}} \gamma^{r,1} : \left(Du + \nabla_y \widetilde{U} \right) \mathrm{d}y \right),$$

leads to

$$\begin{aligned} \left| E_\varepsilon^{(2)}(t,\varphi) \right| \\ \leq C \left(\left(\|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} + \|\nabla \partial_t U_\varepsilon^{\mathrm{cor}}\|_{L^2(\Omega_\varepsilon^{(1)})^{3\times 3}} \right) \|\varphi\|_{L^2(\Omega_\varepsilon^{(1)})} + \varepsilon \|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} \right). \end{aligned} \tag{5.27}$$

In the case of $E_\varepsilon^{(3)}$, the estimate $\varepsilon^{-1}\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C$ (see equation (5.2)) implies

$$\left| E_\varepsilon^{(3)}(\tau,\varphi) \right| \leq C\varepsilon \|\nabla\varphi\|_{L^2(\Omega_\varepsilon^{(1)})} \quad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{5.28}$$

For handling the heat conduction functional,

$$\begin{aligned} E_\varepsilon^{(4)}(t,\varphi) = \int_{\Omega_\varepsilon^{(1)}} K_\varepsilon^{r,1} \nabla \Theta_\varepsilon^{\mathrm{cor}} \cdot \nabla\varphi \,\mathrm{d}x \\ + \int_{\Omega_\varepsilon^{(1)}} \left(K_\varepsilon^{r,1} \nabla (\Theta_\varepsilon^{(1)} - \Theta_\varepsilon^{\mathrm{cor}}) - \frac{1}{|Y^{(1)}(0)|} K^h \nabla\theta \right) \cdot \nabla\varphi \,\mathrm{d}x \\ - \int_{\Gamma_\varepsilon} \frac{1}{|Y^{(1)}(0)|} K^h \nabla\theta \cdot n_{\Gamma_\varepsilon} \varphi \,\mathrm{d}x, \end{aligned}$$

the strategy is exactly the same as with dealing with the $I_\varepsilon^{(1)}$-estimate of the mechanical part, see inequality (5.11), which then leads to

$$E_\varepsilon^{(4)}(t,\Theta_\varepsilon^{\mathrm{cor}0}) \geq C_1 \left\|\nabla\Theta_\varepsilon^{\mathrm{cor}}\right\|^2_{L^2(\Omega_\varepsilon^{(1)})} - C_2\varepsilon\left(\left\|\Theta_\varepsilon^{\mathrm{cor}0}\right\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} + 1 + \varepsilon\right). \tag{5.29}$$

Now, turning our attention to the next functional, $E_\varepsilon^{(5)}$, it follows from Lemma 5.5 that

$$\begin{aligned}\int_0^t & E_\varepsilon^{(5)}(\tau,\Theta_\varepsilon^{\mathrm{cor}0})\,\mathrm{d}\tau \\ &\geq C_1\left(\|\Theta_\varepsilon^{\mathrm{err}}(t)\|^2_{L^2(\Omega_\varepsilon^{(2)})} - \|\Theta_\varepsilon^{\mathrm{err}}(0)\|^2_{L^2(\Omega_\varepsilon^{(2)})}\right) - \int_0^t \|\Theta_\varepsilon^{\mathrm{err}}(\tau)\|^2_{L^2(\Omega_\varepsilon^{(2)})}\,\mathrm{d}\tau - \varepsilon^2. \end{aligned}\tag{5.30}$$

Estimates for the dissipation error terms, $E_\varepsilon^{(6)}$-$E_\varepsilon^{(8)}$, are given by

$$\left|E_\varepsilon^{(6)}(t,\varphi)\right| \leq C\varepsilon\,\|\varphi\|_{L^2(\Omega_\varepsilon^{(2)})}\left(\|Du_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + \|\nabla\partial_t U_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + \varepsilon\right), \tag{5.31}$$

$$\left|E_\varepsilon^{(7)}(t,\varphi)\right| \leq C\varepsilon\,\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^{(2)})}\left(\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(2)})} + \varepsilon\right), \tag{5.32}$$

$$\left|E_\varepsilon^{(8)}(t,\varphi)\right| \leq C\varepsilon^2\,\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^{(2)})}\left(\|Du_\varepsilon^{\mathrm{err}}\|_{L^2(\Omega_\varepsilon^{(2)})^{3\times3}} + \varepsilon\right). \tag{5.33}$$

Similarly, we obtain

$$E_\varepsilon^{(9)}(t,\Theta_\varepsilon^{\mathrm{cor}0}) \geq C_1\varepsilon^2\,\|\nabla\Theta_\varepsilon^{\mathrm{err}}\|^2_{L^2(\Omega_\varepsilon^{(2)})} - C_2\varepsilon^2. \tag{5.34}$$

Taking a look at the interface velocity terms, we get

$$\left|E_\varepsilon^{(10)}(t,\varphi)\right| \leq \left|\int_{\Gamma_\varepsilon}\left(V_{\Gamma_\varepsilon}^r - \varepsilon\,[W_\Gamma^r]_\varepsilon\,\varphi\right)\mathrm{d}\sigma\right| + \left|\int_{\Gamma_\varepsilon}\varepsilon\,[W_\Gamma^r]_\varepsilon\,\varphi\,\mathrm{d}\sigma + \int_{\Omega_\varepsilon^{(1)}}\kappa W^h\varphi\,\mathrm{d}x\right|.$$

Using Lemma 5.5 for the functional $E_\varepsilon^{(11)}$, cf. [MvN13], the estimates on the functions that are involved, and our assumptions on the data, it is straightforward to show

$$\sum_{j=10}^{12}\left|E_\varepsilon^{(j)}(t,\varphi)\right| \leq C\varepsilon\left(\|\varphi\|_{W^{1,2}(\Omega_\varepsilon^{(1)})} + \|\varphi\|_{L^2(\Omega_\varepsilon^{(2)})}\right). \tag{5.35}$$

Finally, for the functional $E_\varepsilon^{(13)}$ catching some of the terms arising in the elliptic part for Θ_{err}, we get

$$\left|E_\varepsilon^{13}(t,\Theta_{\mathrm{cor}0})\right| \leq C\varepsilon\left(\|\Theta_\varepsilon^{\mathrm{err}}\| + \varepsilon\right). \tag{5.36}$$

Summarizing those inequalities (5.25) and (5.27) to (5.36) and using Young's and Gronwall's inequalities, we arrive at

Corrector estimate for the heat error

$$\begin{aligned}&\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^\infty(S\times\Omega)} + \|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^3} + \varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^3} \\ &\quad\leq C\Bigg(\sqrt{\varepsilon} + \varepsilon + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^{3\times3}} + \varepsilon\|Du_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^{3\times3}} \\ &\qquad\qquad + \|\nabla\partial_t U_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^{3\times3}} + \varepsilon\|\nabla\partial_t U_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^{3\times3}}\Bigg). \end{aligned}\tag{5.37}$$

Remark 5.8. *With inequality* (5.37) *at hand, we conclude that estimates for* $Du_\varepsilon^{\mathrm{cor}}$ *and* $\nabla\partial_t U_\varepsilon^{\mathrm{cor}}$ *also lead to corresponding corrector estimates for the heat part.*

5.4.3 Overall estimates

In this section, we combine the estimates from the preceding sections, Section 5.4.1 and Section 5.4.2. It is clear that the following statement now follows directly from inequalities (5.17) and (5.37).

Theorem 5.9 (Corrector for Weakly Coupled Problem)**.** *If we reduce our problem to a weakly coupled problem, that is, if we assume either* $\alpha^{(1)} = \alpha^{(2)} = 0$ *(together with Assumption (A1)) or* $\gamma^{(1)} = \gamma^{(2)} = 0$ *(together with Assumption (A2)), we have the following corrector estimate:*

$$\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^\infty(S\times\Omega)} + \|U_\varepsilon^{\mathrm{err}}\|_{L^\infty(S;L^2(\Omega))^3} + \|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^3} + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(1)}))^{3\times 3}} \\ + \varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^3} + \varepsilon\,\|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(2)}))^{3\times 3}} \leq C(\sqrt{\varepsilon}+\varepsilon).$$

Moreover, for the heat part, we take $\Theta_\varepsilon^{\mathrm{cor}0}$ and, for the mechanical part, $\partial_t U_\varepsilon^{\mathrm{cor}0}$ as a test function, sum the weak formulations, integrate over $(0,t)$ and get

$$\int_0^t \left(\sum_{j=1}^{9} I_\varepsilon^{(j)}(\tau, \partial_t U_\varepsilon^{\mathrm{cor}0}(\tau)) + \sum_{j=1}^{13} E_\varepsilon^{(j)}(\tau, \Theta_\varepsilon^{\mathrm{cor}0}(\tau)) \right) \mathrm{d}\tau = 0. \tag{5.38}$$

Now, we first take a view on the error terms corresponding to the coupling terms for the $\Omega_\varepsilon^{(2)}$ part for both the mechanical and the heat part, namely $I_\varepsilon^{(4)}$, $E_\varepsilon^{(6)}$, and $E_\varepsilon^{(8)}$. While $E_\varepsilon^{(8)}$ can be controlled in terms of $\varepsilon Du_\varepsilon^{\mathrm{err}}$ and $\varepsilon\nabla\Theta_\varepsilon^{\mathrm{err}}$ (see inequality (5.33)), this is not possible for either $I_\varepsilon^{(4)}$ or $E_\varepsilon^{(6)}$ due to the involved time derivatives. If we take a look at the sum of those (appropriately scaled)[14] two terms, however, we see that they counterbalance each other leading to

$$\left| \frac{\gamma^{(2)}}{\alpha^{(2)}} I_\varepsilon^{(4)}(\tau, \partial_t U_\varepsilon^{\mathrm{cor}0}) + E_\varepsilon^{(6)}(\tau, \Theta_\varepsilon^{\mathrm{cor}0}) \right| \leq C\|\Theta_\varepsilon^{\mathrm{err}}\|^2_{L^2(\Omega_\varepsilon^{(2)})} + \varepsilon^2\|Du_\varepsilon^{\mathrm{cor}}\|^2_{L^2(\Omega_\varepsilon^{(2)})} + \varepsilon^2. \tag{5.39}$$

Note that with inequality (5.23), the term $\varepsilon^2\|Du_\varepsilon^{\mathrm{cor}}\|^2_{L^2(\Omega_\varepsilon^{(2)})}$ is resolvable via Gronwall's inequality.

This, unfortunately, does not work for the coupling parts in $\Omega_\varepsilon^{(1)}$, i.e., $I_\varepsilon^{(3)}$ and $E_\varepsilon^{(2)}$: Here, we would have to apply Lemma 5.6 at the cost of additional derivatives (we only get control in $W^{1,2}$ and not in L^2), which, in general, can not be compensated without additional structural assumptions.

As a result of this observation and the estimates collected in the previous sections, we get:

[14] Assuming $\alpha^{(2)} \neq 0$.

Theorem 5.10 (Corrector for Microscale Coupled Problem)**.** *If we simplify our problem so that there is only coupling in the* $\Omega_\varepsilon^{(2)}$ *part, that is, if we assume* $\alpha^{(1)} = \gamma^{(1)} = 0$, *we have the following corrector estimate:*

$$\begin{aligned}\|\Theta_\varepsilon^{\mathrm{err}}\|_{L^\infty(S\times\Omega)} + \|U_\varepsilon^{\mathrm{err}}\|_{L^\infty(S;L^2(\Omega))^3} + \|\nabla\Theta_\varepsilon^{\mathrm{cor}}\|_{L^2(S\times\Omega_\varepsilon^{(1)})^3} + \|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(1)}))^{3\times3}} \\ + \varepsilon\|\nabla\Theta_\varepsilon^{\mathrm{err}}\|_{L^2(S\times\Omega_\varepsilon^{(2)})^3} + \varepsilon\,\|Du_\varepsilon^{\mathrm{cor}}\|_{L^\infty(S;L^2(\Omega_\varepsilon^{(2)}))^{3\times3}} \leq C(\sqrt{\varepsilon}+\varepsilon).\end{aligned}$$

CHAPTER 6

Moving boundary problem with prescribed normal velocity

In this chapter, the analysis and homogenization of a linear parabolic two-phase problem with moving boundary is considered. In this context, we assume the normal velocity to be prescribed, i.e., it is a given datum of the problem. The main challenges are: (i) deduce a corresponding motion with which to arrive at a fixed domain formulation and (ii) characterize the limit behavior of the functions related to the transformation.

The main new results of this chapter are:

- Theorem 6.1: The existence and regularity of a height function and the corresponding *Hanzawa* transformation characterizing the interface movement is established. Section 6.4 is devoted to the proof of this theorem.

- Theorem 6.2: The strong two-scale convergence of the functions related to the transformation (e.g., the Jacobi determinant) is proved. The proof is given in Section 6.5.

Based on these main results and taking into considerations the analysis and homogenization of the thermoelasticity problem considered in Chapter 4, it is then straight forward to arrive at:

- Theorem 6.3: There is unique solution to the moving boundary problem with prescribed normal velocity as well as corresponding a priori estimates.

- Theorem 6.4: The two-scale limit problem is given.

6.1 Introduction

In this chapter, we investigate the heat equation posed for a highly heterogeneous two-phase medium where the building components are different solid phases of the same material separated by a sharp interface. We assume a periodic setup of the geometry representing the two-phase medium. Here, one phase forms a connected matrix in which periodically distributed inclusions of the second phase are embedded. We are particularly interested in scenarios where phase transformations are possible which leads to time dependent domains that are not necessarily periodic anymore.

In our earlier work [EM17b], the rigorous homogenization of a similar problem (including the coupling with the elastic behaviour of the medium) was considered, where having the a priori knowledge of the mathematical deformation describing the changes in the geometry was essential for both the analysis of the microproblem as well as the limit procedure. Building on that, we do not assume the changes in geometry to be given a priori but still to be independent of the temperatures. This can be seen as an important intermediate step for the full free boundary problem outlined in Section 3.4, where thermoelastic behavior and geometric changes are coupled via an interface condition linking the normal velocity to the temperature and surface stresses.

Using the solutions of a nonlinear system of ordinary differential equations describing this interface movement, we prove the existence of a height function parameterizing $\Gamma_\varepsilon(t)$ in terms of its distance to Γ_ε in a particular point; we refer to Section 2.2 for details. With this height function at hand, we are then able to introduce the transformation needed to reformulate the whole problem with respect to fixed domains. Note that the procedure follows [Che92], where a similar problem is considered. The main difference is given by the homogenization context, which forces us to be especially careful when it comes to the influence of the parameter ε which, in turn, leads to much more involved estimations. We also refer to [PS16, PSZ13], where similar interface movement problems are considered. To be able to talk about the limit $\varepsilon \to 0$, strong two-scale convergence (in the sense of Definition 2.13) of some sequences related to the interface movement has to be established. Similar considerations (not related to moving boundary problems but also dealing with the two-scale limits of products) can be found, e.g., in [MCP08].

This chapter is organized as follows: In Section 6.2, we introduce the ε-periodic geometry, the moving boundary problem with prescribed normal velocity as well as the level set equation associated with the normal velocity. The main results of this chapter regarding the moving boundary problem, Theorems 6.1 to 6.4, are given in Section 6.3. Finally, Sections 6.4 and 6.5 are dedicated to the detailed proofs of Theorem 6.1 and Theorem 6.2, respectively.

6.2 Setting and problem statement

We consider the same geometric setup as introduced in Section 2.4 which we briefly recall for the convenience of the reader: Let $S = (0,T)$, $T > 0$, be a time interval. Let $\Omega \subset \mathbb{R}^3$ be a rectilinear domain whose corner coordinates are rational and let ε_0 be the maximal ε such that Ω can be parqueted by cubes of side length ε_0. We set $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} = \varepsilon_0 2^{-n}$. We denote the outer normal vector of Ω with $\nu = \nu(x)$.

Take open and disjoint sets $Y^{(1)}$, $Y^{(2)} \subset Y = (0,1)^3$ such that $Y^{(1)}$ is connected, such that $\overline{Y^{(2)}} \subset Y$, and such that $Y = Y^{(1)} \cup \overline{Y^{(2)}}$. Moreover, let $\Gamma := \partial Y^{(2)}$ be a C^3-hypersurface. By $n_\Gamma = n_\Gamma(y)$, $y \in \Gamma$, we denote the normal vector of Γ pointing outwards of $Y^{(2)}$.

As a C^3-hypersurface, Γ admits a tubular neighborhood U_Γ (see Lemma 2.1). In addition, there are $a^{(1)}$, $a^{(2)} > 0$ such that the function

$$\Lambda\colon \Gamma \times (-a^{(2)}, a^{(1)}) \to U_\Gamma, \quad \Lambda(\gamma, s) := \gamma + s n_\Gamma(\gamma)$$

is a C^2-diffeomorphism satisfying $\Lambda(\Gamma \times (-a^{(2)}, a^{(1)})) \subset Y$.[1]

We introduce the εY-periodic domains $\Omega_\varepsilon^{(1)}$ and $\Omega_\varepsilon^{(2)}$ and the interface Γ_ε representing the two phases and the phase boundary, respectively, via ($i = 1, 2$)

$$\Omega_\varepsilon^{(i)} := \operatorname{int}\left(\Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(\overline{Y^{(i)}} + k)\right)\right), \qquad \Gamma_\varepsilon := \Omega \cap \left(\bigcup_{k\in\mathbb{Z}^3} \varepsilon(\Gamma + k)\right).$$

Moreover, we introduce the rescaled C^2-diffeomorphism

$$\Lambda_\varepsilon\colon \Gamma_\varepsilon \times (-\varepsilon a^{(2)}, \varepsilon a^{(1)}) \to \Omega, \quad \Lambda_\varepsilon(\gamma, r) = \gamma + r n_{\Gamma_\varepsilon}(\gamma).$$

and the family of interfaces, $l \in [-\varepsilon a^{(2)}, \varepsilon a^{(1)}]$,

$$\Gamma_\varepsilon^{(l)} := \{\Lambda_\varepsilon(\gamma, l) \ : \ \gamma \in \Gamma_\varepsilon\} \tag{6.1}$$

as well as the family of tubes around Γ

$$U_{\Gamma_\varepsilon}(r) := \bigcup_{l\in(-\varepsilon r a^{(2)}, \varepsilon r a^{(1)})} \Gamma_\varepsilon^{(l)}.$$

for $r \in (0, 1]$. Note that $U_{\Gamma_\varepsilon} = U_{\Gamma_\varepsilon}(1)$. Moreover, for $\gamma \in \Gamma_\varepsilon$, let $L_{\Gamma_\varepsilon}(\gamma) = -\nabla_{\Gamma_\varepsilon} n_{\Gamma_\varepsilon}(\gamma)$ denote the *Weingarten map*, where we have, by inequality (2.2),

$$|L_{\Gamma_\varepsilon}(\gamma)| \le \frac{1}{2\varepsilon \max\{a^{(1)}, a^{(2)}\}}. \tag{6.2}$$

For $l \in [-\varepsilon a^{(2)}, \varepsilon a^{(1)}]$ and $\gamma \in \Gamma_\varepsilon^{(l)}$, the normal vector of $\Gamma_\varepsilon^{(l)}$ in γ is given as $n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(\gamma))$, where $P_{\Gamma_\varepsilon}\colon U_{\Gamma_\varepsilon} \to \Gamma_\varepsilon$ is the projection operator restricted to U_{Γ_ε}. On its range, Λ_ε is invertible via

$$\Lambda_\varepsilon^{-1}\colon U_{\Gamma_\varepsilon} \to \Gamma_\varepsilon \times [-\varepsilon a^{(2)}, \varepsilon a^{(1)}], \quad \Lambda_\varepsilon^{-1}(x) = \left(P_{\Gamma_\varepsilon}(x), d_{\Gamma_\varepsilon}(x)\right)^T.$$

Here, $d_{\Gamma_\varepsilon}\colon U_\Gamma \to \mathbb{R}$ is the signed distance function (also restricted to U_{Γ_ε}) for Γ_ε, i.e.,

$$d_{\Gamma_\varepsilon}(x) = \begin{cases} \operatorname{dist}(x, \Gamma_\varepsilon), & x \in U_{\Gamma_\varepsilon} \setminus \Omega_\varepsilon^{(2)} \\ -\operatorname{dist}(x, \Gamma_\varepsilon), & x \in U_{\Gamma_\varepsilon} \cap \Omega_\varepsilon^{(2)} \end{cases}.$$

[1]Incidentally, this implies $a^{(i)} < 1$.

Now, let $t \mapsto \Gamma_\varepsilon(t)$ and $t \mapsto \Omega_\varepsilon^{(i)}(t)$ denote the evolution of the interface and the domains, respectively, and set

$$Q_\varepsilon^{(i)} := \bigcup_{t\in S} \{t\} \times \Omega_\varepsilon^{(i)}(t), \qquad \Xi_\varepsilon := \bigcup_{t\in S} \{t\} \times \Gamma_\varepsilon(t).$$

In addition, let $V_{\Gamma_\varepsilon} \colon \Xi_\varepsilon \to \mathbb{R}$ be the normal velocity function of the interface.

For $k, l \in \mathbb{N}$, we introduce the Sobolev space

$$W^{(k,l),\infty}(S \times \Omega) = \left\{u \in L^\infty(S \times \Omega) : \partial_t^i u, D_x^j u \in L^\infty(S \times \Omega) \ (1 \leq i \leq k,\ 1 \leq j \leq l)\right\}$$

and note that $W^{(k,k),\infty}(S \times \Omega) = W^{k,\infty}(S \times \Omega)$.

Now, take $\theta_\varepsilon^{(i)} = \theta_\varepsilon^{(i)}(t,x)$ $(i = 1,2)$ to represent the temperature in the respective domains. In the following, we consider the moving boundary problem:

Moving boundary problem with prescribed normal velocity v_ε

$$\partial_t \theta_\varepsilon^{(1)} - \operatorname{div}(K^{(1)} \nabla \theta_\varepsilon^{(1)}) = f_\varepsilon^{(1)} \quad \text{in } Q_\varepsilon^{(1)}, \tag{6.3a}$$

$$\partial_t \theta_\varepsilon^{(2)} - \operatorname{div}(\varepsilon^2 K^{(2)} \nabla \theta_\varepsilon^{(2)}) = f_\varepsilon^{(2)} \quad \text{in } Q_\varepsilon^{(2)}, \tag{6.3b}$$

$$\theta_\varepsilon^{(1)} = \theta_\varepsilon^{(2)} \quad \text{on } \Xi_\varepsilon, \tag{6.3c}$$

$$-\left(K^{(1)} \nabla \theta_\varepsilon^{(1)} - \varepsilon^2 K^{(2)} \nabla \theta_\varepsilon^{(2)}\right) \cdot n_\varepsilon = L V_{\Gamma_\varepsilon} \quad \text{on } \Xi_\varepsilon, \tag{6.3d}$$

$$V_{\Gamma_\varepsilon} = \varepsilon v_\varepsilon \quad \text{on } \Xi_\varepsilon, \tag{6.3e}$$

$$-K^{(1)} \nabla \theta_\varepsilon^{(1)} \cdot \nu = 0 \quad \text{on } S \times \partial\Omega, \tag{6.3f}$$

$$\theta_\varepsilon^{(1)}(0) = \vartheta_\varepsilon^{(1)} \quad \text{in } \Omega_\varepsilon^{(1)}, \tag{6.3g}$$

$$\theta_\varepsilon^{(2)}(0) = \vartheta_\varepsilon^{(2)} \quad \text{in } \Omega_\varepsilon^{(2)}. \tag{6.3h}$$

Here, $K^{(i)}$ denote the constant heat conductivity coefficients and L denotes the constant of latent heat. The actual mathematical problem connected to this system is as follows: Given volume heat source densities $f_\varepsilon^{(i)} \colon Q_\varepsilon^{(i)} \to \mathbb{R}$, a function $v_\varepsilon \colon \Xi_\varepsilon \to \mathbb{R}$ describing the normal velocity of the moving interface, and initial values $\vartheta_\varepsilon^{(i)} \colon \Omega_\varepsilon^{(i)} \to \mathbb{R}$, find the corresponding evolution of the domains, i.e., find $\Omega_\varepsilon^{(i)}$ and $\Gamma_\varepsilon(t)$ for all $t \in S$, and the temperature functions $\theta_\varepsilon^{(i)} \colon Q_\varepsilon^{(i)} \to (0,\infty)$ such that all equations of the above system are satisfied.

Now, let $v_\varepsilon \in W^{(1,2),\infty}(S \times \Omega)$ be the outward normal velocity of our moving interface $\Gamma_\varepsilon(t)$. Let us assume that the motion of Γ_ε can be described via a regular C^1-motion over the interval $[0,\delta)$ for some $0 < \delta \leq T$. Then, there it exists a level set function $\varphi_\varepsilon \colon [0,\delta) \times \Omega \to \mathbb{R}$ such that

$$\Gamma_\varepsilon(t) = \{x \in \Omega \ : \ \varphi_\varepsilon(t,x) = 0\},$$
$$|\nabla \varphi_\varepsilon(t,x)| > 0 \quad \text{on } \Xi_\varepsilon,$$
$$\varphi_\varepsilon(t,x) < 0 \quad \text{on } \partial\Omega.$$

The normal velocity $\varepsilon v_\varepsilon$ and the level set function φ_ε are connected via

$$\partial_t \varphi_\varepsilon = \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon \quad \text{on } \Xi_\varepsilon.$$

We refer to [OF02, Section 4.1].

Based on these geometric considerations, we formulate the motion problem as a level set problem:

Motion problem via level set equation

Find $\varphi_\varepsilon \in C^1(S \times \Omega)$ such that

$$\partial_t \varphi_\varepsilon = \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon \quad \text{on } \Xi_\varepsilon, \tag{6.4a}$$

$$|\nabla \varphi_\varepsilon(t,x)| > 0 \quad \text{on } \Xi_\varepsilon, \tag{6.4b}$$

$$\frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} \in W^{(0,1),\infty}(S \times \Omega) \tag{6.4c}$$

$$\Gamma_\varepsilon(0) = \{x \in \Omega \ : \ \varphi_\varepsilon(0,x) = 0\}, \tag{6.4d}$$

$$\Omega_\varepsilon^{(1)}(0) = \{x \in \Omega \ : \ \varphi_\varepsilon(0,x) < 0\}. \tag{6.4e}$$

The family of sets $(\Gamma_\varepsilon(t))_{t \in S}$ defined via

$$\Gamma_\varepsilon(t) = \{x \in \Omega \ : \ \varphi_\varepsilon(t,x) = 0\} \tag{6.4f}$$

is called the solution of the motion problem.

Here, the condition $\frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} \in W^{(0,1),\infty}(S \times \Omega)$ is a shorthand for: the function $\frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} : (S \times \Omega) \setminus \Xi_\varepsilon \to \mathbb{R}$ can be extended to a function in $W^{(0,1),\infty}(S \times \Omega)$. Note that this condition is merely technical in that it is not needed for the level set function φ_ε to correspond to the motion of the interface; it is, however, needed in Lemma 6.10.

We also point out that uniqueness of a solution of the motion problem only asserts uniqueness of the the family of hypersurfaces $(\Gamma_\varepsilon(t))_{t \in S}$ but not uniqueness of the level set function φ_ε. Indeed, for every $\alpha > 0$, $\alpha \varphi_\varepsilon$ corresponds to the same motion problem.

6.3 Main results

In this section, we present the main results of this chapter. As some of the proofs are fairly long and technical, they are postponed to subsequent chapters; namely, Section 6.4 and Section 6.5 are devoted to the proofs of Theorem 6.1 and Theorem 6.2, respectively.

We start by formulating the requirements for the data (normal velocity, source densities, and initial values) that are needed to ensure the well-posedness of the microscopic problems as well as to facilitate the passage $\varepsilon \to 0$.

Assumption (A3). *Let $v_\varepsilon \in W^{(1,3),\infty}(S \times \Omega)$ with $\operatorname{supp}(v_\varepsilon) \subset U_{\Gamma_\varepsilon}$ such that*

$$l_v := \sup_{\varepsilon > 0} \left(\|v_\varepsilon\|_{W^{1,\infty}(S \times \Omega)} + \varepsilon \|D_x^2 v_\varepsilon\|_\infty + \varepsilon^2 \|D_x^3 v_\varepsilon\|_\infty \right) < \infty.$$

Assumption (A4). *For $i = 1, 2$, let $f_\varepsilon^{(i)} \in L^2(Q_\varepsilon^{(i)})$ and $\vartheta_\varepsilon^{(i)} \in L^2(\Omega_\varepsilon^{(i)})$ such that*

$$\sup_{\varepsilon>0}\left(\|f_\varepsilon^{(i)}\|_{L^2(Q_\varepsilon^{(i)})} + \|\vartheta_\varepsilon^{(i)}\|_{L^2(\Omega_\varepsilon^{(i)})}\right) < \infty.$$

Assumption (A5). *There is a function $v \in L^2(S \times \Omega; W^{1,2}_\#(Y))^3$ such that*

$$[v_\varepsilon]^\varepsilon \to v, \quad [Dv_\varepsilon]^\varepsilon \to D_y v, \quad \varepsilon\left[D^2 v_\varepsilon\right]^\varepsilon \to D_y^2 v \quad \text{in } L^2(S \times \Omega \times Y)^3.$$

Here, $[v_\varepsilon]^\varepsilon := \mathcal{T}_\varepsilon v_\varepsilon \colon S \times \Omega \times Y \to \mathbb{R}$ is the periodic unfolding of $v_\varepsilon \colon S \times \Omega \to \mathbb{R}$; we refer to Section 2.4.2. The regularity and the estimates postulated via Assumption (A3) are needed to ensure well-posedness of the motion problem given by equations (6.4a) to (6.4f) as well and to show corresponding a priori estimates, respectively. With Assumption (A4), these results can be used to tackle the heat problem (equations (6.3a) to (6.3h)). Finally, Assumption (A5) is necessary for the homogenization processs: without it, strong two-scale convergence of the functions related to the coordinate transform can not be expected.

The following two results, namely, Theorem 6.1 and Theorem 6.2, are the main results of this chapter.

Theorem 6.1. *Under Assumption (A3), there is $T > 0$, which is independent of ε and l_v, and a function $h_\varepsilon \colon [0, l_v^{-1}T] \times \Gamma_\varepsilon \to (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$ such that*

$$\Gamma_\varepsilon(t) = \{\gamma + h_\varepsilon(t,\gamma) n_{\Gamma_\varepsilon}(\gamma) \ : \ \gamma \in \Gamma_\varepsilon\} \quad (t \in [0, l_v^{-1}T]).$$

Furthermore, there is a corresponding regular C^1-motion $s_\varepsilon \colon [0, l_v^{-1}T] \times \overline{\Omega} \to \overline{\Omega}$ satisfying $s_\varepsilon(0) = \mathrm{id}$, $s_\varepsilon(t, \Omega_\varepsilon^{(i)}) = \Omega_\varepsilon^{(i)}(t)$ $(i = 1, 2)$, and

$$\|Ds_\varepsilon\|_\infty \leq 2, \quad \|\left(Ds_\varepsilon\right)^{-1}\|_\infty \leq 2.$$

Proof. This statement follows from Theorem 6.14,Lemma 6.15, and Lemma 6.16. The statements and proof of these results are given in Section 6.4. □

In the following, we take $S_v = l_v^{-1}T$.

Theorem 6.2. *Under Assumptions (A3) and (A5), there is $s \in L^\infty(S_v \times \Omega \times Y)$ with $\partial_t s$, $D_y s \in L^\infty(S_v \times \Omega \times Y)$ such that $Ds_\varepsilon \overset{2}{\to} D_y s$.*

Proof. The proof of this theorem is given in Section 6.5, see Lemma 6.25. We refer the reader also to Lemma 2.17. □

Using the results given in Theorems 6.1 and 6.2, it is possible to investigate the associated heat conduction problem.

Theorem 6.3. *Under Assumptions (A3) and (A4), there is a unique solution of the mathematical problem corresponding to the system given via equations* (6.3a) *to* (6.3h). *In addition, we find that*

$$\sup_{\varepsilon>0}\left(\|\theta_\varepsilon\|^2_{L^\infty(S_v;L^2(\Omega))}+\|\nabla\theta^{(1)}_\varepsilon\|^2_{L^2(S_v;H^1(\Omega^{(1)}_\varepsilon))}+\varepsilon^2\|\nabla\theta^{(2)}_\varepsilon\|^2_{L^2(S_v;H^1(\Omega^{(2)}_\varepsilon))}\right)<\infty$$

Proof. Using the transformation function s_ε (given via Theorem 6.1) to arrive at a fixed-domain formulation of the problem, we are almost exactly in the situation described in [EM17b]. We particularly point the attention to Corollary 4.17. As a consequence, the existence of a unique solution together with the corresponding ε-independent a priori estimates are available. □

Theorem 6.4. *Let Assumptions (A3) to (A5) hold. Then, there are functions $\theta\in L^2(S_v;W^{1,2}(\Omega))$ and $\theta^{(2)}\in L^2(Q_Y)$,*[2] *where $\theta^{(2)}(t,x,\cdot)\in W^{1,2}(Y^{(2)}(t,x))$ for almost all $(t,x)\in S_v\times\Omega$, such that*

$$\mathbb{1}_{\Omega^{(1)}_\varepsilon}\theta^{(1)}_\varepsilon\rightharpoonup|Y^{(1)}(t,x)|\theta,\quad \mathbb{1}_{\Omega^{(2)}_\varepsilon}\theta^{(2)}_\varepsilon\rightharpoonup\int_{Y^{(2)}(t,x)}\theta^{(2)}\,\mathrm{d}y\quad \textit{in } L^2(S\times\Omega).$$

Moreover, they solve the following homogenized distributed microstructure problem: The macroscopic temperature θ is governed by an effective heat conduction problem given via

$$\partial_t\theta-\operatorname{div}(K^h\nabla\theta)=f^h+f^h_\Gamma\quad \textit{in}\ \ S_v\times\Omega,\tag{6.5a}$$
$$-K^h\nabla\theta\cdot\nu=0\quad \textit{on}\ \ S_v\times\partial\Omega,\tag{6.5b}$$
$$\theta(0)=\vartheta^h\quad \textit{in}\ \Omega,\tag{6.5c}$$

which is coupled, via the Dirichlet boundary condition (6.5e), *to a micro heat problem with time dependent microstructures for $\theta^{(2)}$ in the form of*

$$\partial_t\theta^{(2)}-\operatorname{div}_y(K^{(2)}\nabla_y\theta^{(2)})=f^{(2)}\quad \textit{in } Y^{(2)}(t,x),\ t\in S_v,\ x\in\Omega,\tag{6.5d}$$
$$\theta=\theta^{(2)}\quad \textit{on } \Gamma(t,x),\ t\in S_v,\ x\in\Omega,\tag{6.5e}$$
$$\theta^{(2)}(0)=\vartheta^{(2)}\quad \textit{in } \Omega\times Y^{(2)}.\tag{6.5f}$$

Finally, the motion of the interface $\Gamma(t,x)$ in normal direction is given via

$$V_\Gamma=v\qquad \textit{on } \Gamma(t,x),\ t\in S_v,\ x\in\Omega.\tag{6.5g}$$

Here, the effective coefficients are

$$f^h=\sum_{i=1}^2\int_{Y^{(i)}(t,x)}f^{(i)}\,\mathrm{d}y,\qquad f_\Gamma=\int_{\Gamma(t,x)}L+K^{(2)}\nabla_y\theta^{(2)}\cdot n\,\mathrm{d}\sigma,$$
$$\vartheta^h=\int_{Y^{(1)}(t,x)}\vartheta^{(1)}\,\mathrm{d}y,\qquad \left(K^h\right)_{ij}=\min_{\tau\in W^{1,2}(Y^{(1)}(t,x))}\int_{Y^{(1)}(t,x)}K^{(1)}\left(\nabla_y\tau+e_j\right)\cdot e_i\,\mathrm{d}y,$$

and $f^{(i)}$, $\vartheta^{(1)}$, and v are the two-scale limits of their corresponding ε-counterparts.

[2] Here, we have set $Q_Y=\bigcup_{(t,x)\in S_v\times\Omega}\{(t,x)\}\times Y^{(2)}(t,x)$.

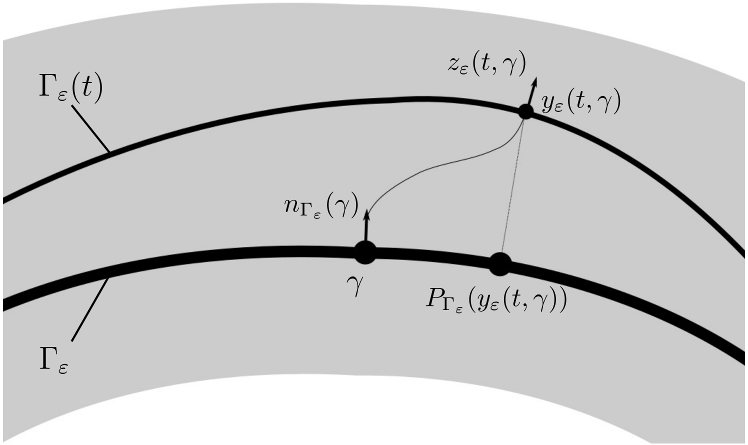


Figure 6.1: Part of the surface Γ_ε and its position at time t. The function y_ε characterizes the motion by tracking the paths of the material points. As an example, we see the path of y_ε for $\gamma = y_\varepsilon(0,\gamma)$ over the interval $(0,t)$. In addition, we see the change in the normal vector from $n_{\Gamma_e}(\gamma) = z_\varepsilon(0,\gamma)$ to $z_\varepsilon(\gamma)$. The goal is to find the corresponding height function h_ε that satisfies $h_\varepsilon(P_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma))) = d_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma))$. The gray area marks the tubular neighborhood U_{Γ_ε}.

Proof. Noting the strong convergence of the functions related to the transformation given by Lemma 6.25, this homogenization results follows via a standard two-scale limit procedure and is a special case of the homogenization of the thermoelasticity problem performed in Chapter 4. □

6.4 Interface motion (proof of Theorem 6.1)

This section is devoted to the proof of Theorem 6.1. As a short guideline, this proof follows the following strategy:

(*i*) We investigate a nonlinear, parametrized ODE-system – given by equations (6.6a) to (6.6d) – tracking the interface motion and establish a few important properties of its solution. This is done via Lemmas 6.6, 6.8, and 6.9.

(*ii*) Using these results, we then show that the motion problem via conditions (6.4a)-(6.4e) has a unique solutiom; see Lemma 6.10 and theorem 6.11

(*iii*) In Theorem 6.14, the existence of the height function h_ε is then deduced via the implicit function theorem.

(*iv*) Finally, we construct the C^1-diffeomorphism $s_\varepsilon\colon \overline{S\times\Omega} \to \overline{\Omega}$ and investigate its properties; see Lemma 6.16.

The first two steps can be found in Section 6.4.1, and steps (*iii*) and (*iv*) are the topic of Section 6.4.2.

Note that this section is structurally similar to [Che92, Section 3], where the main substantial differences are due to the parameter ε and its role in the context of homogenization. At some points, we transferred (instead of just referenced) whole arguments/proofs in order to be able to fully track the role of the parameter ε; wherever this is the case (as an example, take Lemma 6.10), this is explicitly indicated.

6.4.1 Interface motion problem

We consider the nonlinear ODE system:

ODE system describing the interface motion

Find $y_\varepsilon, z_\varepsilon \colon S \times U_{\Gamma_\varepsilon} \to \mathbb{R}^3$ such that

$$\partial_t y_\varepsilon(t,x) = -\varepsilon \frac{z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} v_\varepsilon(t, y_\varepsilon(t,x)) \quad \text{in } S \times U_{\Gamma_\varepsilon}, \tag{6.6a}$$

$$\partial_t z_\varepsilon(t,x) = \varepsilon |z_\varepsilon(t,x)| \nabla v_\varepsilon(t, y_\varepsilon(t,x)) \quad \text{in } S \times U_{\Gamma_\varepsilon}, \tag{6.6b}$$

$$y_\varepsilon(0,x) = x \quad \text{in } U_{\Gamma_\varepsilon}, \tag{6.6c}$$

$$z_\varepsilon(0,x) = -n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon} x) \quad \text{in } U_{\Gamma_\varepsilon}. \tag{6.6d}$$

Here, $P_{\Gamma_\varepsilon} \colon U_{\Gamma_\varepsilon} \to \Gamma$ is the projection operator. We extend every solution y_ε to all of Ω by setting $y_\varepsilon(t,x) = x$ for all $x \notin U_\Gamma$.

Remark 6.5. *We show later, Lemma 6.10, that the function y_ε characterizes the interface motion in the sense that $\Gamma_\varepsilon(t) = y_\varepsilon(t, \Gamma_\varepsilon)$; we also refer to Figure 6.1. The function z_ε describes the direction of the motion. Note that, if $\nabla v_\varepsilon \equiv 0$, the solution satisfies $y_\varepsilon(t,\gamma) = \gamma + d_{\Gamma_\varepsilon}(y_\varepsilon(t,\gamma)) n_{\Gamma_\varepsilon}(\gamma)$ for all $\gamma \in \Gamma$.*

For any solution $(y_\varepsilon, z_\varepsilon)$, we see that $y_\varepsilon(t, U_{\Gamma_\varepsilon}(1/2)) \subset U_{\Gamma_\varepsilon}(1) = U_{\Gamma_\varepsilon}$ as long as $-a^{(2)} \leq 2v_\varepsilon t \leq a^{(1)}$. This is due to $|\partial_t y_\varepsilon| \leq |v_\varepsilon|$. In addition, as $\operatorname{supp} v_\varepsilon \subset U_{\Gamma_\varepsilon}$, $y_\varepsilon(t,x) = x$ for all $x \in \Omega \setminus U_{\Gamma_\varepsilon}$.

We define functions

$$f_\varepsilon \colon S \times \left(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}\right) \to \mathbb{R}^3 \times \mathbb{R}^3, \quad f_\varepsilon(t,(y,z)) = \left(\frac{z}{|z|} v_\varepsilon(t,y), |z|\, \nabla v_\varepsilon(t,y)\right)^T,$$

$$g_\varepsilon \colon \Omega \to \mathbb{R}^3 \times \mathbb{R}^3, \quad g_\varepsilon(x) = \begin{pmatrix} x \\ -n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x)) \end{pmatrix}.$$

Setting $w_\varepsilon = (y_\varepsilon, z_\varepsilon)^T$, Equations (6.6a) to (6.6d) become

$$\partial_t w_\varepsilon(t,x) = \varepsilon f_\varepsilon\,(t, w_\varepsilon(t,x)) \quad \text{in } S \times \Omega, \tag{6.7a}$$

$$w_\varepsilon(0,x) = g_\varepsilon(x) \quad \text{in } \Omega. \tag{6.7b}$$

Lemma 6.6. *Let $v_\varepsilon \in W^{(1,3),\infty}(S \times \mathbb{R}^3)$. The ODE-system given via equations* (6.6a) *to* (6.6d) *admits a unique solution $(y_\varepsilon, z_\varepsilon) \in W^{(1,2),\infty}(S \times \Omega)^6$. Additionally, there exist*

$C_y, C_z > 0$ such that

$$\|D_x y_\varepsilon - \mathbb{I}\|_\infty + \|\partial_t D_x y_\varepsilon\|_\infty + \varepsilon\|D_x^2 y_\varepsilon\|_\infty \leq l_v C_y,$$
$$\varepsilon\|D_x z_\varepsilon\|_\infty + \varepsilon^2\|D_x^2 z_\varepsilon\|_\infty \leq C_z(l_v + 1).$$

Proof. (i) Existence and Uniqueness. Due to the embedding $W^{k,\infty}(\Omega) = C^{k-1,1}(\Omega)$ $(k \geq 1)$,[3] we have $v_\varepsilon, \partial_j v_\varepsilon \in C^{1,1}(S \times \Omega)$ $(j = 1, 2, 3)$ which then implies that $f_\varepsilon \in C^{1,1}(S \times (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}))$. Therefore, for every $x \in \Omega$, *Picard-Lindeloef's existence theorem*[4] guarantees the existence of a time $t_\varepsilon(x) > 0$ and a unique solution $w_\varepsilon(\cdot, x) = (y_\varepsilon(\cdot, x), z_\varepsilon(\cdot, x))^T \in C^{1,1}([0, t_\varepsilon(x)])^6$. Note that $|z_\varepsilon(0, x)| = 1$ independently of $x \in U_{\Gamma_\varepsilon}$. Taking a look at equation (6.6b), we see that

$$-\varepsilon t l_v \leq \int_0^t \frac{\partial_t(z_\varepsilon \cdot e_j)}{|z_\varepsilon|}\,\mathrm{d}\tau \leq \varepsilon t l_v \quad j = 1, 2, 3.$$

The norm of every solution z_ε is therefore bounded from below and above via

$$e^{-\varepsilon l_v t} \leq |z_\varepsilon(t, x)| \leq e^{\varepsilon l_v t}.$$

As a consequence, a blow up due to $|z_\varepsilon| \to 0$ is not possible in finite time; hence, we can extend to a unique solution $w_\varepsilon(\cdot, x) \in C^{1,1}(\overline{S})^6$ for $x \in \Omega$.

(ii) Regularity and Estimates. For any $x_1, x_2 \in \Omega$, we find that

$$w_\varepsilon(t, x_1) - w_\varepsilon(t, x_2) = g_\varepsilon(x_1) - g_\varepsilon(x_2) + \int_0^t f_\varepsilon(\tau, w_\varepsilon(\tau, x_1)) - f_\varepsilon(\tau, w_\varepsilon(\tau, x_1))\,\mathrm{d}\tau.$$

From $g_\varepsilon \in C^2(\Omega)$, the Lipschitz continuity of f_ε as well as Df_ε, and Gronwall's inequality, we can infer $w_\varepsilon(t, \cdot) \in W^{(1,2),\infty}(S \times \Omega)^6$.

In the following, let $\varepsilon > 0$ be sufficiently small such that $1/\sqrt{2} \leq \|z_\varepsilon\|_\infty \leq \sqrt{2}$. Differentiating the ODE with respect to $x \in \Omega$, we get

$$\partial_t D w_\varepsilon(t, x) = \varepsilon D_x \left(f_\varepsilon\left(t, w_\varepsilon(t, x)\right)\right). \tag{6.8}$$

We define $A_\varepsilon \colon S \times (\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}^{6\times 6}$ via

$$A_\varepsilon(t, (y, z)) := D_{(y,z)} f_\varepsilon\left(t, (y, z)\right) = \begin{pmatrix} \frac{z}{|z|} \otimes \nabla v_\varepsilon(t, y) & v_\varepsilon B(z) \\ |z| D^2 v_\varepsilon(t, y) & \nabla v_\varepsilon(t, y) \otimes \frac{z}{|z|} \end{pmatrix},$$

where $B \colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^{3\times 3}$ is given via

$$B(z) = D\left(z \mapsto \frac{z}{|z|}\right) = \frac{1}{|z|^3} \begin{pmatrix} z_2^2 + z_3^2 & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_1^2 + z_3^2 & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & z_1^2 + z_2^2 \end{pmatrix}. \tag{6.9}$$

[3] $W^{1,\infty}(\Omega) = C^{0,1}(\Omega)$ if and only if Ω is a *uniformly locally quasiconvex* domain Ω; see [HKT08, Theorem 7]. It is straightforward to show that this is the case for a rectilinear domain.

[4] For a statement of this result, we refer to [Zei86, Proposition 1.8].

Note that $\|B(z)\|_F \leq \sqrt{2}/|z|$. Equation (6.8) can be rewritten into

$$\partial_t Dw_\varepsilon(t,x) = \varepsilon A_\varepsilon(t, w_\varepsilon(t,x)) Dw_\varepsilon(t,x). \tag{6.10}$$

We estimate

$$|A_\varepsilon^{(11)}(t,(y,z))| = |A_\varepsilon^{(22)}(t,(y,z))| = \left| \frac{z}{|z|} \otimes \nabla v_\varepsilon(t,y) \right| \leq l_v, \tag{6.11a}$$

$$|A_\varepsilon^{(21)}(t,(y,z))| = \left| |z| D^2 v_\varepsilon(t,(y,z)) \right| \leq \frac{l_v}{\varepsilon}|z|, \tag{6.11b}$$

$$|A_\varepsilon^{(12)}(t,(y,z))| = \frac{\sqrt{2} l_v}{|z|}. \tag{6.11c}$$

For sufficiently small ε, this yields

$$\varepsilon |A_\varepsilon(t,(y_\varepsilon, z_\varepsilon)| \leq l_v(3\varepsilon + \sqrt{2}) \leq 2 l_v.$$

For the inital values of the *Jacobian* matrices, we have

$$\begin{aligned} &Dy_\varepsilon(0,x) = \mathbb{I}_3, \\ &Dz_\varepsilon(0,x) = D\left(n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right) = -L_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\left(\mathbb{I} - d_{\Gamma_\varepsilon(x)} L_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right)^{-1}. \end{aligned}$$

For the derivative of $n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))$, we refer to Lemma 2.3. Using the scaling properties of the involved operators, we estimate

$$|Dz_\varepsilon(0,x)| \leq \frac{C}{\varepsilon}.$$

We deduce that

$$\begin{aligned} |Dy_\varepsilon(t,x)| &\leq 1 + \varepsilon l_v \int_0^t |Dy_\varepsilon(\tau,x)| + 2|Dz_\varepsilon(\tau,x)| \,\mathrm{d}\tau, \\ |Dz_\varepsilon(t,x)| &\leq \frac{C_\Gamma}{\varepsilon} + \varepsilon l_v \int_0^t \sqrt{2}\varepsilon^{-1}|Dy_\varepsilon(\tau,x)| + |Dz_\varepsilon(\tau,x)| \end{aligned}$$

leading us, using *Gronwall's inequality*, to

$$\varepsilon\,|Dw_\varepsilon(t,x)| \leq C \exp(2 l_v T) =: C_w. \tag{6.12}$$

Specifically, for y_ε, we have

$$\begin{aligned} &Dy_\varepsilon(t,x) \\ &\quad = \mathbb{I}_3 + \varepsilon \int_0^t \left(A_\varepsilon^{(11)}(t, w_\varepsilon(\tau,x)) Dy_\varepsilon(\tau,x) + A_\varepsilon^{(12)}(t, w_\varepsilon(\tau,x)) Dz_\varepsilon(\tau,x) \right) \mathrm{d}\tau. \end{aligned} \tag{6.13}$$

Inserting the estimate given in inequality (6.12) into equation (6.13), we are led to

$$|Dy_\varepsilon(t,x)| \leq 1 + 3 l_v C_w T =: C_y.$$

Looking at equation (6.13) and using the estimates given via inequalities (6.11a) to (6.11c), we get (ε sufficiently small)

$$|\partial_t Dy_\varepsilon(t,x)| \leq \varepsilon l_v C_y + 2 l_v C_w \leq 3 l_v C_w.$$

Now, $\partial_k \partial_j y_\varepsilon(0,x) = 0$ and $\partial_k \partial_j z_\varepsilon(0,x) = -\partial_k \partial_j \left(n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\right)$. Since $n_{\Gamma_\varepsilon}(x) = [n]_\varepsilon(x)$, we have $\partial_k \partial_j n_{\Gamma_\varepsilon}(x) = \varepsilon^{-2}\left[\partial_k \partial_j (n \circ P_\Gamma)\right]_\varepsilon(x)$. As a next step, we calculate

$$\begin{aligned}
&\partial_t \partial_{x_j} \begin{pmatrix} Dy_\varepsilon(t,x) \\ Dz_\varepsilon(t,x) \end{pmatrix} \\
&\quad = -\varepsilon \begin{pmatrix} \partial_{x_j}\left(A_\varepsilon^{(11)}(t, w_\varepsilon(t,x))\right) Dy_\varepsilon(t,x) + A_\varepsilon^{(11)}(t, w_\varepsilon(t,x)) \partial_{x_j} Dy_\varepsilon(t,x) \\ \partial_{x_j}\left(A_\varepsilon^{(21)}(t, w_\varepsilon(t,x))\right) Dy_\varepsilon(t,x) + A_\varepsilon^{(21)}(t, w_\varepsilon(t,x)) \partial_{x_j} Dy_\varepsilon(t,x) \end{pmatrix} \\
&\quad\quad - \varepsilon \begin{pmatrix} \partial_{x_j}\left(A_\varepsilon^{(12)}(t, w_\varepsilon(t,x))\right) Dz_\varepsilon(t,x) + A_\varepsilon^{(12)}(t, w_\varepsilon(t,x)) \partial_{x_j} Dz_\varepsilon(t,x) \\ \partial_{x_j}\left(A_\varepsilon^{(22)}(t, w_\varepsilon(t,x))\right) Dz_\varepsilon(t,x) + A_\varepsilon^{(22)}(t, w_\varepsilon(t,x)) \partial_{x_j} Dz_\varepsilon(t,x) \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\partial_{x_j}\left(A_\varepsilon^{(11)}(t, w_\varepsilon(t,x))\right) \\
&\quad = B(z_\varepsilon(t,x)) \partial_{x_j} z_\varepsilon(t,x) \otimes \nabla v_\varepsilon(t, y_\varepsilon(t,x)) + \frac{z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} \otimes \partial_{x_j}\left(\nabla v_\varepsilon(t, y_\varepsilon(t,x))\right), \\
&\partial_{x_j}\left(A_\varepsilon^{(12)}(t, w_\varepsilon(t,x))\right) \\
&\quad = \sum_{j=1}^{3} \partial_{x_j} B(z_\varepsilon(t,x)) v_\varepsilon(t, y_\varepsilon(t,x)) \partial_{x_j} z_\varepsilon(t,x) + B(z_\varepsilon(t,x)) \partial_{x_j}(v_\varepsilon(t, y_\varepsilon(t,x))), \\
&\partial_{x_j}\left(A_\varepsilon^{(21)}(t, w_\varepsilon(t,x))\right) \\
&\quad = \frac{z_\varepsilon(t,x) \cdot \partial_{x_j} z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} D_y^2 v_\varepsilon(t, y_\varepsilon(t,x)) + |z_\varepsilon(t,x)| \partial \partial_{x_j}\left(D_y^2 v_\varepsilon(t, y_\varepsilon(t,x))\right), \\
&\partial_{x_j}\left(A_\varepsilon^{(22)}(t, w_\varepsilon(t,x))\right) \\
&\quad = \partial_{x_j}\left(\nabla v_\varepsilon(t, y_\varepsilon(t,x))\right) \otimes \frac{z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} + \nabla v_\varepsilon(t, y_\varepsilon(t,x)) \otimes B(z_\varepsilon(t,x)) \partial_{x_j} z_\varepsilon(t,x).
\end{aligned}$$

Let ε be small enough so that $\max_{j \in \{1,2,3\}} |\partial_j B(z_\varepsilon(t,x))| \leq 2$. We can estimate

$$\begin{aligned}
\left|\partial_{x_j}\left(A_\varepsilon^{(11)}(t, w_\varepsilon(t,x))\right)\right| &= \left|\partial_{x_i}\left(A_\varepsilon^{(22)}(t, w_\varepsilon(t,x))\right)\right| \leq \frac{l_v}{\varepsilon}\left(C_w + C_y\right), \\
\left|\partial_{x_j}\left(A_\varepsilon^{(12)}(t, w_\varepsilon(t,x))\right)\right| &\leq l_v \left(\frac{2C_w}{\varepsilon} + \sqrt{2} C_y\right), \\
\left|\partial_{x_j}\left(A_\varepsilon^{(21)}(t, w_\varepsilon(t,x))\right)\right| &\leq \frac{l_v}{\varepsilon^2}\left(C_w + \sqrt{2} C_y\right).
\end{aligned}$$

After integrating equation (6.10) over $(0,t)$, we get

$$\begin{aligned}
\partial_{x_i} \begin{pmatrix} Dy_\varepsilon(t,x) \\ Dz_\varepsilon(t,x) \end{pmatrix} &= \partial_{x_i} \begin{pmatrix} Dy_\varepsilon(0,x) \\ Dz_\varepsilon(0,x) \end{pmatrix} + \int_0^t \partial_t \partial_{x_i} \begin{pmatrix} Dy_\varepsilon(\tau,x) \\ Dz_\varepsilon(\tau,x) \end{pmatrix} \mathrm{d}\tau \\
&= \frac{1}{\varepsilon^2} \begin{pmatrix} 0 \\ \left[\partial_i \nabla (n \circ P_\Gamma)\right]_\varepsilon(x) \end{pmatrix} - \varepsilon \int_0^t \partial_{x_i}\left(A_\varepsilon(\tau, w_\varepsilon(\tau,x)) D w_\varepsilon(\tau,x)\right) \mathrm{d}\tau
\end{aligned}$$

which then leads to (ε being small enough)

$$|\partial_{x_i} Dy_\varepsilon(t,x)| \leq t l_v (C_y^2 + C_w^2 + 1) + \varepsilon l_v \int_0^t |\partial_{x_i} Dy_\varepsilon(t,x)| + 2\varepsilon |\partial_{x_i} Dz_\varepsilon(t,x)| \,\mathrm{d}\tau,$$

$$|\partial_{x_i} Dz_\varepsilon(t,x)| \leq \frac{C}{\varepsilon^2} + \frac{tl_v}{\varepsilon}\left(\left(C_w + \sqrt{2}C_y\right)C_y + \varepsilon(C_yC_w+1)\right) \\ + l_v \int_0^t \sqrt{2}|\partial_{x_i} Dy_\varepsilon(t,x)| + \varepsilon|\partial_{x_i} Dz_\varepsilon(t,x)|\,\mathrm{d}\tau.$$

Making use of Gronwall's inequality, we deduce that there is an ε-independent $C > 0$ such that

$$\varepsilon^2|\partial_j Dw_\varepsilon(t,x)| \leq C.$$

With this estimate, we can further bound $|\partial_j Dy_\varepsilon(t,x)|$ via

$$|\partial_{x_i} Dy_\varepsilon(t,x)| \leq tl_v\left(C_y^2 + C_w^2 + 1 + \frac{3C}{\varepsilon}\right).$$

□

The solution y_ε is assumed to track the movement of the interface in the sense that $y_\varepsilon(t,\Gamma) = \Gamma_\varepsilon(t)$ – a fact that we proof in Lemma 6.10. Note that at this point, however, it is not even clear that y_ε is actually a homeomorphism; a minimal requirement for it to correspond to a meaningful transformation.

Definiton 6.7. *Let* $(y_\varepsilon, z_\varepsilon)$ *be the solution of the ODE system given via equations* (6.6a) *to* (6.6d). *We set*

$$t_\varepsilon = \sup\big\{t \geq 0 \;:\; y_\varepsilon(\tau,\cdot)\colon U_{\Gamma_\varepsilon} \to y_\varepsilon(\tau, U_{\Gamma_\varepsilon}) \\ \textit{is a Lipschitz homeomorphism for all } \tau \in [0,t]\big\}. \quad (6.14)$$

Without any additional arguments, we only know that $t^\varepsilon_{max} \geq 0$. Note that $y_\varepsilon(0,\cdot) = \mathrm{Id}$. The following lemma shows that it is positive and uniformly bounded away from zero with respect to the parameter ε.

Lemma 6.8. *There is* $t_\delta > 0$*, which is independent of* ε *and* l_v*, such that* $t_\varepsilon \geq l_v^{-1} t_\delta$.

Proof. We recall the characterization of Dy_ε established in the proof of the preceding lemma, i.e., equation (6.13):

$$Dy_\varepsilon(t,x) = \mathbb{I}_3 + \varepsilon\int_0^t \left(A_\varepsilon^{(11)}(t, w_\varepsilon(\tau,x))Dy_\varepsilon(\tau,x) + A_\varepsilon^{(12)}(t, w_\varepsilon(\tau,x))Dz_\varepsilon(\tau,x)\right)\,\mathrm{d}\tau.$$

From here, we conclude that

$$\|Dy_\varepsilon(t,\cdot) - \mathbb{I}_3\|_\infty \leq 3tl_vC_w \quad \text{for all } t \in \overline{S}.$$

This shows[5]. that $y_\varepsilon(t,\cdot)\colon U_{\Gamma_\varepsilon} \to y_\varepsilon(U_{\Gamma_\varepsilon})$ is a Lipschitz homeomorphism for all $t \in [0, l_v^{-1}t_\delta)$ with $t_\delta = (4C_w)^{-1}$. Therefore, $t_\varepsilon \geq l_v^{-1}t_\delta > 0$. □

[5]This follows via the *Neumann* series, we refer to [Wer08, Satz II.1.11].

Now, for $t \in S$, let

$$y_{\varepsilon,t}^{-1} \colon y_\varepsilon(t, U_{\Gamma_\varepsilon}) \to U_{\Gamma_\varepsilon}$$

be the unique function that satisfies $y_{\varepsilon,t}^{-1}(y_\varepsilon(t,x)) = x$ for all $x \in U_{\Gamma_\varepsilon}$.

Lemma 6.9. *The function*

$$y_\varepsilon^{-1} \colon \bigcup_{t\in S} \big(\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon})\big) \to U_{\Gamma_\varepsilon}, \quad y_\varepsilon^{-1}(t,w) := y_{\varepsilon,t}^{-1}(w)$$

is Lipschitz continuous in $t \in S$.

Proof. It holds $y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) = x$ for all $(t,x) \in \bigcup_{t\in S}(\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon}))$. Implicit differentiation leads to

$$\partial_t \left(y_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right) = \partial_t y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) + D y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) \partial_t y_\varepsilon^{-1}(t,x) = 0$$

and, therefore,

$$\begin{aligned} \partial_t y_\varepsilon^{-1}(t,x) &= -\left(D y_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right)^{-1} \partial_t y_\varepsilon(t, y_\varepsilon^{-1}(t,x)) \\ &= \varepsilon \left(D y_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right)^{-1} \frac{z_\varepsilon(t, y_\varepsilon^{-1}(t,x))}{|z_\varepsilon(t, y_\varepsilon^{-1}(t,x))|} v_\varepsilon(t, y_\varepsilon(t, y_\varepsilon^{-1}(t,x))). \end{aligned} \tag{6.15}$$

As the right hand side is bounded by virtue of the estimates provided in Lemma 6.6, this implies Lipschitz continuity y_ε^{-1} with respect to $t \in S$. ☐

In the following lemma, we prove that any solution of the motion problem given by equations (6.4a) to (6.4e) can be characterized via y_ε and, in the subsequent theorem, we show that, indeed, there is a unique solution.

Lemma 6.10. *Let $\{\Gamma_\varepsilon(t)\}_{t\in[0,\delta]}$ be a solution of the free boundary problem given by equations* (6.4a) *to* (6.4e) *for some $\delta \in (0, l_v^{-1} t_\delta)$. Then, for all $t \in [0,\delta)$, $\Gamma_\varepsilon(t) = y_\varepsilon(t, \Gamma_\varepsilon)$.*

Proof. The following proof is taken from [Che92, Lemma 3.2]. Let $\varphi_\varepsilon \colon [0,\delta) \times \Omega \to \mathbb{R}$ be a corresponding level set function such that

$$r_\varepsilon := \frac{\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon}{\varphi_\varepsilon} \in W^{(0,1),\infty}([0,\delta) \times \Omega).$$

We have

$$\partial_t \varphi_\varepsilon - \varepsilon |\nabla \varphi_\varepsilon| v_\varepsilon = \varphi_\varepsilon r_\varepsilon \quad \text{in } [0,\delta) \times \Omega. \tag{6.16}$$

Now, let $\widetilde{y}_\varepsilon \colon [0,\delta) \times U_{\Gamma_\varepsilon} \to U_{\Gamma_\varepsilon}$ be the solution of the nonlinear ODE[6]

$$\begin{aligned} \partial_t \widetilde{y}_\varepsilon(t,x) &= -\varepsilon \frac{\nabla \varphi_\varepsilon(t, \widetilde{y}_\varepsilon(t,x))}{|\nabla \varphi_\varepsilon(t, \widetilde{y}_\varepsilon(t,x))|} v_\varepsilon(t, \widetilde{y}_\varepsilon(t,x)), \\ \widetilde{y}_\varepsilon(0,x) &= x. \end{aligned}$$

[6] That there is a unique solution is a consequence of $|\nabla \varphi_\varepsilon| > 0$ and the *Picard-Lindelöf* theorem; cf. Lemma 6.6.

Introducing $\psi_{\varepsilon,1}, \psi_{\varepsilon,2} \colon [0,\delta) \times \Omega \to \mathbb{R}^3$ via

$$\psi_{\varepsilon,1}(t,x) = \varphi_\varepsilon(t, \widetilde{y}_\varepsilon(t,x)), \quad \psi_{\varepsilon,2}(t,x) = \nabla\varphi_\varepsilon(t, \widetilde{y}_\varepsilon(t,x)),$$

the level set equation (6.16) leads us to

$$\begin{aligned} \partial_t \widetilde{y}_\varepsilon &= -\varepsilon \frac{\psi_{\varepsilon,2}}{|\psi_{\varepsilon,2}|} v_\varepsilon(\widetilde{y}_\varepsilon), \\ \partial_t \psi_{\varepsilon,1} &= \psi_{\varepsilon,1} r_\varepsilon(\widetilde{y}_\varepsilon), \\ \partial_t \psi_{\varepsilon,2} &= \psi_{\varepsilon,2} r_\varepsilon(\widetilde{y}_\varepsilon) + \psi_{\varepsilon,1} \nabla r_\varepsilon(\widetilde{y}_\varepsilon) + \varepsilon |\psi_{\varepsilon,2}| \nabla v_\varepsilon(\widetilde{y}_\varepsilon) \end{aligned}$$

together with the initial conditions

$$\widetilde{y}_\varepsilon(0,x) = x, \quad \psi_{\varepsilon,1}(0,x) = \varphi_\varepsilon(0,x), \quad \psi_{\varepsilon,2}(0,x) = \nabla\varphi_\varepsilon(0,x).$$

For $\psi_{\varepsilon,1}$, we get

$$\psi_{\varepsilon,1}(t,x) = \varphi_\varepsilon(0,x) \exp\left(\int_0^t r_\varepsilon(\tau, \widetilde{y}_\varepsilon(\tau,x)) \,\mathrm{d}\tau \right),$$

which implies that $\psi_{\varepsilon,1}(t,x) = 0$ if and only if $\varphi_\varepsilon(0,x) = 0$. Therefore,

$$\Gamma_\varepsilon(t) = \{x \in \Omega \ : \ \varphi_\varepsilon(t,x) = 0\} = \{\widetilde{y}_\varepsilon(t,x) \ : \ \psi_{\varepsilon,1}(t,x) = 0\} = \widetilde{y}_\varepsilon(t, \Gamma_\varepsilon(t)).$$

For $x \in \Gamma_\varepsilon$ and $t \in [0,\delta)$, we set

$$z_\varepsilon(t,x) = \frac{\psi_{\varepsilon,2}(t,x)}{|\psi_{\varepsilon,2}(0,x)|} \exp\left(- \int_0^t r_\varepsilon(\tau, \widetilde{y}_\varepsilon(\tau,x)) \,\mathrm{d}\tau \right).$$

Now, substituting $(\widetilde{y}_\varepsilon, z_\varepsilon)$ in the ODE system given by equations (6.6a) to (6.6d), we see that it is the solution and $\widetilde{y}_\varepsilon(t,x) = y_\varepsilon(t,x)$ for all $x \in \Gamma_\varepsilon$. □

Theorem 6.11. *There is a unique solution to the motion problem posed in the time interval* $[0, l_v^{-1} t_\delta)$.

Proof. This proof follows closely along the lines of [Che92, Theorem 3.1] adapting the ideas to our setting. As some parts of the proof are referenced at later points, the proof is recounted here. We introduce a Lipschitz continuous function $\widetilde{\varphi} \colon [0, l_v^{-1} t_\delta) \times \Omega \to [-\varepsilon a^{(2)}, \varepsilon a^{(1)}]$ via

$$\widetilde{\varphi}_\varepsilon(t,x) = \begin{cases} -\varepsilon a^{(1)}, & x \in \Omega_\varepsilon^{(1)} \setminus y_\varepsilon(t, U_{\Gamma_\varepsilon}) \\ -\varepsilon l, & x \in y_\varepsilon(t, \Gamma_\varepsilon^{(l)}) \text{ for some } l \in (-a^{(2)}, a^{(1)}) \\ \varepsilon a^{(2)}, & y \in \Omega_\varepsilon^{(2)} \setminus y_\varepsilon(t, U_{\Gamma_\varepsilon}) \end{cases}.$$

Here, we recall the definition (see equation (6.1))

$$\Gamma_\varepsilon^{(l)} := \{\Lambda_\varepsilon(\gamma, l) \ : \ \gamma \in \Gamma_\varepsilon\}.$$

We calculate

$$\begin{aligned}\partial_t\left(Dy_\varepsilon z_\varepsilon\right) &= Dy_\varepsilon \partial_t z_\varepsilon + D\partial_t y_\varepsilon z_\varepsilon \\ &= \varepsilon Dy_\varepsilon |z_\varepsilon| \nabla v_\varepsilon(y_\varepsilon) - \varepsilon D\left(\frac{z_\varepsilon}{|z_\varepsilon|} v_\varepsilon(y_\varepsilon)\right) z_\varepsilon \\ &= \varepsilon Dy_\varepsilon |z_\varepsilon| \nabla v_\varepsilon(y_\varepsilon) - \varepsilon D\left(\frac{z_\varepsilon}{|z_\varepsilon|}\right) v_\varepsilon(y_\varepsilon) z_\varepsilon - \varepsilon \frac{z_\varepsilon \cdot z_\varepsilon}{|z_\varepsilon|} Dy_\varepsilon \nabla v_\varepsilon(y_\varepsilon) \\ &= -\varepsilon v_\varepsilon(y_\varepsilon) D\left(\frac{z_\varepsilon}{|z_\varepsilon|}\right) z_\varepsilon.\end{aligned}$$

Calculating the spatial derivative of $\frac{z_\varepsilon}{|z_\varepsilon|}$, it follows that

$$\partial_t\left(Dy_\varepsilon z_\varepsilon\right) = -\varepsilon v_\varepsilon(y_\varepsilon) D\left(\frac{z_\varepsilon}{|z_\varepsilon|}\right) z_\varepsilon = 0.$$

Therefore, we are led to

$$Dy_\varepsilon(t,x) z_\varepsilon(t,x) = Dy_\varepsilon(0,x) z_\varepsilon(0,x) = -\mathbb{I}_3 n_{\Gamma_\varepsilon}(x) = -n_{\Gamma_\varepsilon}(x).$$

Furthermore, we have

$$\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) = \widetilde{\varphi}_\varepsilon(0,x) \tag{6.17}$$

for all $x \in U_\Gamma$ and, consequently,

$$D\left(\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x))\right) = Dy_\varepsilon(t,x) \nabla\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) = \nabla\widetilde{\varphi}_\varepsilon(0,x). \tag{6.18}$$

For $x \in U_{\Gamma_\varepsilon}$, it holds $\nabla\widetilde{\varphi}_\varepsilon(0,x) = -n_{\Gamma_\varepsilon}(x)$. This is leading to

$$\nabla\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) = z_\varepsilon(t,x) \quad \text{in } [0,\delta] \times U_{\Gamma_\varepsilon} \tag{6.19}$$

and

$$|\nabla\widetilde{\varphi}_\varepsilon(t,x)| \geq e^{-\varepsilon l_v t} \quad \text{for all } x \in y_\varepsilon(t, U_{\Gamma_\varepsilon}),\ t \in [0,\delta]. \tag{6.20}$$

We also have the identity

$$\begin{aligned} 0 &= \partial_t\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) + \nabla\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) \cdot \partial_t y_\varepsilon(t,x) \\ &= \partial_t\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) - \varepsilon\nabla\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) \frac{z_\varepsilon(t,x)}{|z_\varepsilon(t,x)|} v_\varepsilon(t, z_\varepsilon(t,x)) \\ &= \partial_t\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x)) - \varepsilon|\nabla\widetilde{\varphi}_\varepsilon(t, y_\varepsilon(t,x))| v_\varepsilon(t, z_\varepsilon(t,x)). \end{aligned}$$

Thus,

$$\partial_t\widetilde{\varphi}_\varepsilon(t,y) = \varepsilon|\nabla\widetilde{\varphi}_\varepsilon(t,y)| v_\varepsilon(t,y) \quad \text{in } \bigcup_{t\in[0,\delta]} \left(\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon})\right). \tag{6.21}$$

Due to the Lipschitz continuity of the involved derivatives, we get

$$\widetilde{\varphi}_\varepsilon \in W^{(2,2),\infty}\left(\bigcup_{t\in[0,\delta]} \left(\{t\} \times y_\varepsilon(t, U_{\Gamma_\varepsilon})\right)\right).$$

Now, let $g\colon \mathbb{R} \to [0,1]$ be a C^2-function such that $g(0) = 0$, $g'(0) = 1$, $g'(r) = 0$ if $r \notin (-{}^{a^{(2)}}\!/_2, {}^{a^{(1)}}\!/_2)$, $|g''_{|(-\infty,0)}| \le {}^2\!/_{a^{(2)}}$, and $|g''_{|[0,\infty)}| \le {}^2\!/_{a^{(1)}}$. We introduce $\varphi_\varepsilon = \varepsilon g \circ (\varepsilon^{-1}\widetilde{\varphi}_\varepsilon) \in W^{(2,2),\infty}([0,\delta] \times \Omega)$. Then, $\varphi_\varepsilon = 0$ if and only if $\widetilde{\varphi}_\varepsilon = 0$ which implies

$$\Gamma_\varepsilon = \{x \in \Omega \ : \ \varphi_\varepsilon(0,x) = 0\}.$$

and

$$\{x \in \Omega \ : \ \varphi_\varepsilon(t,x) = 0\} = \{x \in \Omega \ : \ \widetilde{\varphi}_\varepsilon(t,x) = 0\} = y_\varepsilon(t, \Gamma_\varepsilon).$$

In addition,

$$|\nabla \varphi_\varepsilon| = |\varepsilon g'(\varepsilon^{-1}\varphi_\varepsilon)\nabla \widetilde{\varphi}_\varepsilon| \ge |\nabla \widetilde{\varphi}_\varepsilon| > 0$$

and, for $x \in \partial\Omega$, it holds $\varphi_\varepsilon(t,x) = \varepsilon g(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t,x)) = \varepsilon g(a) < 0$. □

Lemma 6.12. *There is a constant $C_\varphi > 0$, which is independent of the parameters ε and l_v, such that*

$$\begin{aligned} \varepsilon^{-1}\|\partial_t \widetilde{\varphi}_\varepsilon\|_\infty + \|\partial_t \nabla \widetilde{\varphi}_\varepsilon\|_\infty &\le l_v C_\varphi, \\ \|\nabla \widetilde{\varphi}_\varepsilon\|_\infty + \varepsilon \|D^2 \widetilde{\varphi}_\varepsilon\|_\infty &\le (1 + l_v) C_\varphi, \end{aligned}$$

wherever the involved derivatives exist.

Proof. In this proof, we rely on the estimates provided in Lemma 6.6. Let $t \in [0, l_v^{-1} t_\delta]$ and $x \in y_\varepsilon(t, U_{\Gamma_\varepsilon})$. As shown in the preceding lemma in the form of equations (6.18) and (6.19), it holds

$$\nabla \widetilde{\varphi}_\varepsilon(t,x) = -\left(D y_\varepsilon(t, y_\varepsilon^{-1}(t,x))\right)^{-1} n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x)) = z_\varepsilon(t, y_\varepsilon^{-1}(t,x)).$$

As a consequence,

$$e^{-\varepsilon l_v t} \le |\nabla \widetilde{\varphi}_\varepsilon(t,x)| \le e^{\varepsilon l_v t}.$$

The second spatial derivative is given as

$$D^2 \widetilde{\varphi}_\varepsilon(t,x) = (D y_\varepsilon(t, y_\varepsilon^{-1}(t,x)))^{-1} D z_\varepsilon(t, y_\varepsilon^{-1}(t,x))$$

and can therefore be estimated via

$$\left|D^2 \widetilde{\varphi}_\varepsilon(t,x)\right| \le \frac{4}{\varepsilon}(1 + l_v) C_z.$$

Furthermore, since $\widetilde{\varphi}_\varepsilon$ satisfies equation (6.21), we can estimate

$$|\partial_t \widetilde{\varphi}_\varepsilon(t,x)| \le |\nabla \widetilde{\varphi}_\varepsilon(t,x)||v_\varepsilon(t,x)| \le \varepsilon e^{\varepsilon l_v t} l_v.$$

Taking the derivative with respect to $x \in y_\varepsilon(t, U_{\Gamma_\varepsilon})$ in equation (6.21),

$$\partial_t \nabla \widetilde{\varphi}_\varepsilon(t,x) = |\nabla \widetilde{\varphi}_\varepsilon(t,x)| \nabla v_\varepsilon(t,x) + D^2 \widetilde{\varphi}_\varepsilon(t,x) \frac{\nabla \widetilde{\varphi}_\varepsilon(t,x)}{|\nabla \widetilde{\varphi}_\varepsilon(t,x)|} |v_\varepsilon(t,x)|,$$

we finally find the upper bound

$$|\partial_t \nabla \widetilde{\varphi}_\varepsilon(t,x)| \le l_v \left(e^{\varepsilon l_v t} + 4 C_z\right)$$

□

6.4.2 Motion function

For $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$, we introduce the function $F_{\varepsilon,\gamma}\colon [0, t_\delta) \times (-\varepsilon a^{(1)}, \varepsilon a^{(2)}) \to \mathbb{R}$ via $F_{\varepsilon,\gamma}(t, r) = \varphi_\varepsilon(t, \Lambda_\varepsilon(\gamma, r))$. Then, $F_{\varepsilon,\gamma}(0, 0) = \varphi_\varepsilon(0, \Lambda_\varepsilon(\gamma, 0)) = 0$ for all $\gamma \in \Gamma_\varepsilon$.

Lemma 6.13. *For all $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$, it holds $\partial_2 F_{\varepsilon,\gamma}(0, 0) = -1$. Furthermore, there is $T_\delta > 0$ and $0 < R_\delta < a$ (independent of $\varepsilon > 0$, $\gamma \in \Gamma_\varepsilon$, and l_v) such that $\partial_2 F_{\varepsilon,\gamma}(t, r) \le -1/3$ for all $t \in [0, l_v^{-1} T_\delta]$ and $r \in [-\varepsilon R_\delta, \varepsilon R_\delta]$.*

Proof. We calculate

$$\partial_2 F_{\varepsilon,\gamma}(t, r) = g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)))\nabla\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)) \cdot n_{\Gamma_\varepsilon}(\gamma) \tag{6.22}$$

and see that

$$\partial_2 F_{\varepsilon,\gamma}(0, 0) = -1 < 0.$$

For arbitrary $t \in (0, l_v^{-1} t_\delta)$ and $r \in (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$, we have

$$\partial_2 F_{\varepsilon,\gamma}(t, r) = -1 + \int_0^r \partial_2^2 F_{\varepsilon,\gamma}(0, s)\,\mathrm{d}s + \int_0^t \partial_t \partial_2 F_{\varepsilon,\gamma}(\tau, r)\,\mathrm{d}\tau.$$

Starting off with the first integrand, $\partial_2^2 F_{\varepsilon,\gamma}$, we get

$$\begin{aligned}\partial_2^2 F_{\varepsilon,\gamma}(t, r) = \varepsilon^{-1} g''(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)))\left(\nabla\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)) \cdot n_{\Gamma_\varepsilon}(\gamma)\right)^2 \\ + D^2\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)) n_{\Gamma_\varepsilon}(\gamma) \cdot n_{\Gamma_\varepsilon}(\gamma).\end{aligned}$$

Using the estimates collected in Lemma 6.12, we can conclude that

$$\varepsilon\left|\partial_2^2 F_{\varepsilon,\gamma}(t, r)\right| \le \frac{3}{a} e^{2\varepsilon l_v t} + (1 + l_v) C_\varphi.$$

Here, and in the following, $a := \min\{a^{(1)}, a^{(2)}\}$. For the second integrand, $\partial_t \partial_2 F_{\varepsilon,\gamma}$, we calculate

$$\begin{aligned}\partial_t \partial_2 F_{\varepsilon,\gamma}(t, r) = \varepsilon^{-1} g''(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)))\partial_t\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r))\nabla\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)) \cdot n_{\Gamma_\varepsilon}(\gamma) \\ + g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)))\partial_t \nabla\widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, r)) \cdot n_{\Gamma_\varepsilon}(\gamma).\end{aligned}$$

We estimate

$$|\partial_t \partial_2 F_{\varepsilon,\gamma}(t, r)| \le l_v C_\varphi \left(\frac{3}{a} e^{\varepsilon l_v t} + 1\right)$$

and finally arrive at

$$\partial_2 F_{\varepsilon,\gamma}(t, r) \le -1 + \frac{r}{\varepsilon}\left(\frac{3}{a} e^{2\varepsilon l_v t} + (1 + l_v) C_\varphi\right) + t l_v C_\varphi \left(\frac{3}{a} e^{\varepsilon l_v t} + 1\right).$$

□

Theorem 6.14 (Height function)**.** *There is $T > 0$, which is independent of $\varepsilon > 0$ and l_v, and a height function $h_\varepsilon\colon \Gamma_\varepsilon \times [0, l_v^{-1} T] \to (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$ satisfying*

$$\Gamma_\varepsilon(t) = \{\Lambda_\varepsilon(\gamma, h_\varepsilon(t, \gamma)) \;:\; \gamma \in \Gamma_\varepsilon\}.$$

Proof. Note that $F_{\varepsilon,\gamma}(0,0) = 0$ and $\partial_2 F_{\varepsilon,\gamma}(0,0) = -1$ By the *Implicit Function Theorem*, we infer that, for every $\varepsilon > 0$ and for every $\gamma \in \Gamma_\varepsilon$, there is a time $\tau_{\varepsilon,\gamma} > 0$ and a differentiable function $h_{\varepsilon,\gamma} \colon [0, \tau_{\varepsilon,\gamma}] \to (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$ such that $F_{\varepsilon,\gamma}(t, h_{\varepsilon,\gamma}(t)) = 0$ for all $t \in [0, \tau_{\varepsilon,\gamma}]$. Let $\tau_{\varepsilon,\gamma} \in \overline{S}$ always be the maximal possible point in time for this to be true. We claim that

$$\inf\{\tau_{\varepsilon,\gamma} \ : \varepsilon > 0,\ \gamma \in \Gamma_\varepsilon\} \geq T,$$

where $T = l_v^{-1} \min\{T_\delta, R_\delta\}$. The values T_δ and R_δ are given by Lemma 6.13. Let us assume on the contrary that

$$\inf\{\tau_{\varepsilon,\gamma} \ :\ \varepsilon > 0,\ \gamma \in \Gamma_\varepsilon\} < T.$$

Then, we can choose $\varepsilon > 0$ and $\gamma \in \Gamma_\varepsilon$ such that $\tau_{\varepsilon,\gamma} < T$. Since

$$(i)\ F_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma}, h_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma})) = 0,$$
$$(ii)\ \partial_2 F_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma}, h_{\varepsilon,\gamma}(\tau_{\varepsilon,\gamma})) < -\frac{1}{3},$$

we can apply the *Implicit Function Theorem* again which contradicts the assumption that $\tau_{\varepsilon,\gamma}$ is maximal. Here, (ii) holds true by virtue of Lemma 6.13. As a result, we can define $h_\varepsilon \colon [0, l_v^{-1}T] \times \Gamma_\varepsilon \to (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$ via $h_\varepsilon(t,\gamma) := h_{\varepsilon,\gamma}(t)$. □

We introduce the positive part, $h^{(1)} := h^+ = \max\{0, h\}$, and the negative part, $h^{(2)} := h^- = \max\{0, -h\}$, of the height function $h_\varepsilon \colon \Gamma_\varepsilon \times [0, l_v^{-1}T] \to (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$.

Lemma 6.15. *There is $T_v > 0$, which is independent of ε and l_v, such that*

$$\sum_{i=1}^{2} \frac{5}{\varepsilon a^{(i)}} \|h_\varepsilon^{(i)}\|_{L^\infty((0,l_v^{-1}T_v)\times\Gamma_\varepsilon)} + 2\|\nabla_{\Gamma_\varepsilon} h_\varepsilon\|_{L^\infty((0,l_v^{-1}T_v)\times\Gamma_\varepsilon)} \leq \frac{1}{2}.$$

Moreover, $\|\partial_t h_\varepsilon\|_\infty \leq 3\varepsilon l_v C_\varphi$.

Proof. Due to the regularity of the involved functions, namely Λ_ε and φ_ε, we get $h_\varepsilon \in W^{2,\infty}((0,T) \times \Gamma_\varepsilon)$. It is clear that $\|h_\varepsilon\|_\infty \leq \|d_{\Gamma_\varepsilon}(y_\varepsilon)\|_\infty \leq \varepsilon t l_v$. For all $t \in [0,T]$ and $\gamma \in \Gamma_\varepsilon$, we have $F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma)) = 0$ implying vanishing derivatives with respect to time and space. Implicit differentiation with respect to time yields

$$\partial_t h_\varepsilon(t,\gamma) = -\frac{\partial_t F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))}{\partial_2 F_{\varepsilon,\gamma}(t, h_\varepsilon(t,\gamma))}. \tag{6.23}$$

Considering that $\|g'\|_\infty \leq 1$, we are therefore led to

$$|\partial_t h_\varepsilon(t,\gamma)| \leq 3\,|\partial_t \widetilde{\varphi}_\varepsilon(t, \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)))| \leq 3\varepsilon l_v C_\varphi.$$

Let us first observe that $\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma) = 0$ if and only if

$$n_{\Gamma_\varepsilon}(t, \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma))) = n_{\Gamma_\varepsilon}(\gamma).$$

The normal vector at $\gamma \in \Gamma_\varepsilon(t)$ is given as

$$n_{\Gamma_\varepsilon}(t,\gamma) = \frac{\nabla\varphi_\varepsilon(t,\gamma)}{|\nabla\varphi_\varepsilon(t,\gamma)|} = \frac{\nabla\widetilde{\varphi}_\varepsilon(t,\gamma)}{|\nabla\widetilde{\varphi}_\varepsilon(t,\gamma)|}.$$

For the surface gradient of h_ε, we can find the representation (we point to [PS16, Section 2.5])

$$\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma) = (\mathbb{I}_3 - h_\varepsilon(t,\gamma)L_{\Gamma_\varepsilon}(\gamma))\left(n_{\Gamma_\varepsilon}(\gamma) - \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\right), \qquad (6.24)$$

where we have set $\overline{\gamma}_t = y_\varepsilon(t,\gamma)$. Due to

$$n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) = n_{\Gamma_\varepsilon}(\gamma) + \int_0^t \underbrace{\frac{\partial_t \nabla\widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)|\nabla\widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)| - \nabla\widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)\partial_t|\nabla\widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)|}{|\nabla\widetilde{\varphi}_\varepsilon(t,\overline{\gamma}_t)|^2}}_{=:\Phi_\varepsilon(\tau,\overline{\gamma}_t)} \,\mathrm{d}\tau,$$

we estimate

$$|n_{\Gamma_\varepsilon}(\gamma) - n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)| \le \int_0^t |\Phi_\varepsilon(\tau,\overline{\gamma}_t)|\,\mathrm{d}\tau \le 2te^{3\varepsilon l_v t} l_v C_\varphi,$$

and (for small t)

$$0 < 1 - 2te^{3\varepsilon l_v t} l_v C_\varphi \le n_{\Gamma_\varepsilon}(\gamma)\cdot n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) \le 1.$$

Combining these estimates to bound the difference

$$\begin{aligned} n_{\Gamma_\varepsilon}(\gamma) &- \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) \\ &= n_{\Gamma_\varepsilon}(\gamma) - n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t) + \frac{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma) - 1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t), \end{aligned}$$

we are led to

$$\left| n_{\Gamma_\varepsilon}(\gamma) - \frac{1}{n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\cdot n_{\Gamma_\varepsilon}(\gamma)} n_{\Gamma_\varepsilon}(t,\overline{\gamma}_t)\right| \le 2te^{3\varepsilon l_v t} l_v C_\varphi + 2te^{3\varepsilon l_v t} l_v C_\varphi \sum_{k=0}^{\infty} \left(2te^{3\varepsilon l_v t} l_v C_\varphi\right)^k.$$

In summary, estimating from Equation (6.24), leads us to

$$|\nabla_{\Gamma_\varepsilon} h_\varepsilon(t,\gamma)| \le \left(1 + \frac{t l_v}{2a}\right)\left(2te^{3\varepsilon_0 l_v t} l_v C_\varphi \left(1 + \sum_{k=0}^{\infty}\left(2te^{3\varepsilon_0 l_v t} l_v C_\varphi\right)^k\right)\right).$$

□

Let $\chi \in \mathcal{D}(\mathbb{R}_{\ge}0)$ be a cut-off function such that

$$0 \le \chi \le 1, \qquad \chi(r) = 1 \text{ if } r < \frac{1}{3}, \qquad \chi(r) = 0 \text{ if } r > \frac{2}{3}.$$

In addition, let $\chi'(r) < 0$ if $1/3 < r < 2/3$ as well as $\|\chi'\|_\infty \le 4$.

We introduce the function $s_\varepsilon\colon [0, l_v^{-1}T_v] \times \overline{\Omega} \to \overline{\Omega}$ via

$$s_\varepsilon(t,x) = \begin{cases} x + h_\varepsilon(t, P_{\Gamma_\varepsilon}(x)) n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\chi\left(\frac{\mathrm{dist}(x,\Gamma_\varepsilon)}{\varepsilon a^{(1)}}\right), & x \in U_{\Gamma_\varepsilon}^{(1)} \cup \Gamma_\varepsilon \\ x + h_\varepsilon(t, P_{\Gamma_\varepsilon}(x)) n_{\Gamma_\varepsilon}(P_{\Gamma_\varepsilon}(x))\chi\left(\frac{\mathrm{dist}(x,\Gamma_\varepsilon)}{\varepsilon a^{(2)}}\right), & x \in U_{\Gamma_\varepsilon}^{(2)} \\ x, & x \notin U_{\Gamma_\varepsilon} \end{cases}.$$

Lemma 6.16. *The function* $s_\varepsilon\colon [0, l_v^{-1}T_v] \times \overline{\Omega} \to \overline{\Omega}$ *is a regular* C^1*-motion with* $\Gamma_\varepsilon(t) = s_\varepsilon(t, \Gamma_\varepsilon)$ *for all* $t \in [0, l_v^{-1}T_v]$.

Proof. As a consequence of the estimates provided in Lemma 6.15, we can conclude with Lemma 2.9 that $s_\varepsilon(t, \cdot)\colon \overline{\Omega} \to \overline{\Omega}$ is a regular C^1-deformation with $\Gamma_\varepsilon(t) = s_\varepsilon(t, \Gamma_\varepsilon)$ for all $t \in [0, l_v^{-1}T_v]$. The regularity with respect to time follows via $h_\varepsilon \in C^{1,1}([0, l_v^{-1}T_v] \times \overline{\Omega})$. □

6.5 Limit behavior (proof of Theorem 6.2

In this section, the limit behavior of the functions related to the Hanzawa transformation s_ε, in particular $F_\varepsilon = Ds_\varepsilon$ and $J_\varepsilon = \det F_\varepsilon$, are investigated. To be able to pass to the limit $\varepsilon \to 0$, strong two-scale convergence of these quantities has to be established. We start by introducing the folding and unfolding operators; similar (in spirit) considerations can be found, e.g., in [MCP08]. In the following section, we start by introducing the notions of folding, unfolding, and strong two-scale convergence (we also refer to Section 2.4) and by formulating a few technical lemmas needed in what follows.

In an effort to keep the notations for the estimations shorter, we introduce the functions

$$q_\varepsilon\colon S_v \times \Omega \to \Omega, \quad q_\varepsilon(t,x) := z_\varepsilon(t, y_\varepsilon^{-1}(t,x)), \tag{6.25a}$$
$$\eta_\varepsilon\colon S_v \times \Gamma \to \Omega, \quad \eta_\varepsilon(t,\gamma) := \Lambda_\varepsilon(\gamma, h_\varepsilon(t,\gamma)). \tag{6.25b}$$

6.5.1 Preliminaries and auxiliary lemmas

We recall that for $x \in \mathbb{R}^3$, $[x]$ is defined to be the unique $k \in \mathbb{Z}^3$ such that $\{x\} := x - [x] \in [0,1)^3$; see Figure 2.6. For functions $f\colon \Omega \to \mathbb{R}$ and $f_b\colon \Gamma_\varepsilon \to \mathbb{R}$, we denote the periodic unfolding via $[f]^\varepsilon := \mathcal{T}_\varepsilon f\colon \Omega \times Y \to \mathbb{R}$ and $[f_b]^\varepsilon := \mathcal{T}_\varepsilon f_b\colon \Omega \times \Gamma \to \mathbb{R}$, respectively; for details, we refer to Definition 2.15. In addition, for functions $g\colon \Omega \times Y \to \mathbb{R}$ and $g_b\colon \Omega \times \Gamma \to \mathbb{R}$ we set

$$[g]_\varepsilon\colon \Omega \to \mathbb{R}, \qquad [g]_\varepsilon(x) = g\left(x, \left\{\frac{x}{\varepsilon}\right\}\right),$$
$$[g_b]_\varepsilon\colon \Gamma_\varepsilon \to \mathbb{R}, \quad [g_b]_\varepsilon(x) = g_b\left(x, \left\{\frac{x}{\varepsilon}\right\}\right).$$

For these functions, we have the following integral identities (see Lemma 2.16)

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\Omega\times Y} [f]^\varepsilon(x,y)\,\mathrm{d}(x,y),$$
$$\int_{\Gamma_\varepsilon} f(x)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega\times\Gamma} [f]^\varepsilon(x,y)\,\mathrm{d}(x,y).$$

For Id: $\Omega \to \Omega$ and $y \in Y$, $x \in \Omega$, and $n, m \in \mathbb{N}$, it holds

$$|[\mathrm{id}]^{\varepsilon_n} - [\mathrm{id}]^{\varepsilon_m}| \leq \sqrt{2}\,(\varepsilon_n + \varepsilon_m)\,. \tag{6.26}$$

Moreover, for any function $f \in W^{1,2}(\Omega; W^{1,2}_{\#}(Y))$, we find that

$$\begin{aligned} &\left\| f - [[f]_\varepsilon]^\varepsilon \right\|^2_{L^2(\Omega\times Y)} \\ &\qquad = \int_{\Omega\times Y} \left| f(x,y) - f\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right], \left[y + \left[\frac{x}{\varepsilon}\right]\right]\right)\right|^2 \mathrm{d}(x,y) \to 0 \end{aligned} \tag{6.27}$$

since $\left(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon}\right], \left[y + \left[\frac{x}{\varepsilon}\right]\right]\right)$ converges uniformly to (x, y).

We start by formulating important relations for the unfolding of the operators related to the geometric properties of Γ_ε. These identities rely on in the periodicity of the initial configuration. For $x \in \Omega$, $y \in Y$, $\gamma \in \Gamma$, and $r \in (-\varepsilon a^{(2)}, \varepsilon a^{(1)})$, it holds

$$[n_\varepsilon]^\varepsilon (x,\gamma) = n(\gamma), \tag{6.28a}$$

$$[\Lambda_\varepsilon]^\varepsilon (x,\gamma,r) = \varepsilon\Lambda\left(\gamma, \frac{r}{\varepsilon}\right) + \varepsilon\left[\frac{x}{\varepsilon}\right], \tag{6.28b}$$

$$[L_{\Gamma_\varepsilon}]^\varepsilon (x,\gamma) = \varepsilon^{-1} L_\Gamma(\gamma), \tag{6.28c}$$

$$[P_{\Gamma_\varepsilon}]^\varepsilon (x,y) = \varepsilon P_\Gamma(y) + \varepsilon\left[\frac{x}{\varepsilon}\right], \tag{6.28d}$$

$$[DP_{\Gamma_\varepsilon}]^\varepsilon (x,y) = \left(\mathbb{I} - d_\Gamma(y) L_\Gamma(P_\Gamma(y))\right)^{-1} \left(\mathbb{I} - n(P_\Gamma(y)) \otimes n(P_\Gamma(y))\right). \tag{6.28e}$$

With these relations in mind, we are able to connect the limit behavior of η_ε and h_ε.

Lemma 6.17. *Let n, $m \in \mathbb{N}$. It holds*

$$\left|\varepsilon_n^{-1} [\eta_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \leq \left|\varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m}\right|$$

as well as

$$\begin{aligned} &\left| [D\eta_{\varepsilon_n}]^{\varepsilon_n} - [D\eta_{\varepsilon_m}]^{\varepsilon_m} \right| \\ &\qquad \leq \sum_{i=1}^{2} \frac{1}{2a^{(i)}} \left|\varepsilon_n^{-1} \left[h^{(i)}_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1} \left[h^{(i)}_{\varepsilon_n}\right]^{\varepsilon_n}\right| + \left| [\nabla h_{\varepsilon_n}]^{\varepsilon_n} - [\nabla h_{\varepsilon_m}]^{\varepsilon_m} \right|. \end{aligned}$$

Proof. Since Λ is contractive and equations (6.28a) and (6.28b) hold, we conclude

$$\begin{aligned} &\left|\varepsilon_n^{-1} [\eta_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [\eta_{\varepsilon_m}]^{\varepsilon_m}\right| \\ &\qquad = \left|\Lambda\left(\gamma, \varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n}\right) - \Lambda(\gamma, \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m})\right| \leq \left|\varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m}\right|. \end{aligned}$$

The spatial derivative of η_ε is given as

$$D_{\Gamma_\varepsilon}\eta_\varepsilon = \mathrm{Id} + \nabla_{\Gamma_\varepsilon} h_\varepsilon \otimes n_\varepsilon - h_\varepsilon L_{\Gamma_\varepsilon}.$$

Using equations (6.28a) to (6.28c), we estimate

$$\begin{aligned} &\left| [D_{\Gamma_\varepsilon}\eta_{\varepsilon_n}]^{\varepsilon_n} - [D_{\Gamma_\varepsilon}\eta_{\varepsilon_m}]^{\varepsilon_m} \right| \\ &\qquad \leq \left| \left[\nabla_{\Gamma_{\varepsilon_n}} h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[\nabla_{\Gamma_{\varepsilon_m}} h_{\varepsilon_m}\right]^{\varepsilon_m} \right| + \sum_{i=1}^{2} \frac{1}{2a^{(i)}} \left|\varepsilon_n^{-1} \left[h^{(i)}_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1} \left[h^{(i)}_{\varepsilon_n}\right]^{\varepsilon_n}\right|. \end{aligned}$$

□

In the next few lemmas, we establish some technical results regarding ε-independent estimates and the limit behavior of unfolded function that are needed to show the strong two-scale convergence of F_ε and J_ε.

Lemma 6.18. *Let $u_\varepsilon \in W^{1,2}(\Omega)$ and $u \in L^2(\Omega; W^{1,2}_\#(Y))$ such that $[u_\varepsilon]^\varepsilon \to u$ and $\varepsilon\,[\nabla u_\varepsilon]^\varepsilon \to \nabla_y u$ strongly in $L^2(\Omega \times Y)$. Then, $[u_\varepsilon]^\varepsilon \to u$ strongly in $L^2(\Omega \times \Gamma)$.*

Proof. We find that

$$\begin{aligned}\int_{\Omega\times\Gamma} &\left|[u_\varepsilon]^\varepsilon(x,\gamma) - u(x,\gamma)\right|^2 \mathrm{d}(x,\gamma)\\ &\le C_{tr}\Bigg(\int_{\Omega\times Y} \left|[u_\varepsilon]^\varepsilon(x,y) - u(x,y)\right|^2 \mathrm{d}(x,y)\\ &\qquad + \int_{\Omega\times Y} \left|\nabla_y [u_\varepsilon]^\varepsilon(x,y) - \nabla_y u(x,y)\right|^2 \mathrm{d}(x,y)\Bigg).\end{aligned}$$

Here, C_{tr} denotes the continuity constant for the trace embedding operator $W^{1,2}(Y) \hookrightarrow L^2(\Gamma)$. Both integrals converge to zero because $[u_\varepsilon]^\varepsilon \to u$, $\varepsilon\,[\nabla u_\varepsilon]^\varepsilon \to \nabla_y u$, and

$$\begin{aligned}\int_{\Omega\times Y} &\left|\nabla_y [u_\varepsilon]^\varepsilon(x,y) - \nabla_y u(x,y)\right|^2 \mathrm{d}(x,y)\\ &= \int_{\Omega\times Y} \left|\varepsilon_n [\nabla u_\varepsilon]^\varepsilon(x,y) - \nabla_y u(x,y)\right|^2 \mathrm{d}(x,y) \to 0\end{aligned}$$

□

The following technical lemma allows us to estimate interface terms.

Lemma 6.19. *For all $u \in W^{1,2}(\Omega)$, it holds that*

$$\varepsilon\|u\|^2_{L^2(\Gamma_\varepsilon(t))} \le 4C_{tr}\left(\|u\|^2 + \varepsilon^2\|\nabla u\|^2\right).$$

Proof. For $u \in W^{1,2}(\Omega)$ and $t \in [0, T_v]$, we have

$$\begin{aligned}\varepsilon\int_{\Gamma_\varepsilon(t)} |u(\gamma)|^2 \,\mathrm{d}\gamma &= \varepsilon\int_{\Gamma_\varepsilon} |u(y_\varepsilon(t,\gamma))|^2 |\det(D_{\Gamma_\varepsilon} y_\varepsilon(t,\gamma))| \,\mathrm{d}\gamma\\ &\le 2C_{tr}\left(\int_\Omega |u\circ y_\varepsilon(x)|^2 \,\mathrm{d}x + \varepsilon^2\int_\Omega |\nabla(u\circ y_\varepsilon)(x)|^2 \,\mathrm{d}x\right)\\ &\le 2C_{tr}\left(\int_\Omega |u\circ y_\varepsilon(x)|^2 \,\mathrm{d}x + 2\varepsilon^2\int_\Omega |(\nabla u)\circ y_\varepsilon(x)|^2 \,\mathrm{d}x\right).\end{aligned}$$

A time parametrized coordinate transformation $x \mapsto y_\varepsilon^{-1}(t,x)$ (note that $y_\varepsilon^{-1}(t,\Omega) = \Omega$) then leads to

$$\varepsilon\|u\|^2_{L^2(\Gamma_\varepsilon(t))} \le 4C_{tr}\left(\|u\|^2 + \varepsilon^2\|\nabla u\|^2\right)$$

□

Parts of the next analysis rely on the ability to estimate certain differences of some composites of functions involving y_ε. In the following lemma, we collect general results which are used at several points in the following analysis.

Lemma 6.20. *Let $(f_\varepsilon) \subset W^{1,\infty}(\Omega)$ and $n, m \in \mathbb{N}$ such that $n > m$.*

1. *Let $\|\nabla f_{\varepsilon_m}\|_\infty$ be bounded independently of the parameter ε and $[f_\varepsilon]^\varepsilon$ be a Cauchy sequence. Then, there are $C, C_m > 0$ such that*

$$\|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_m}]^{\varepsilon_m})\|_{L^2(\Omega\times Y)} \leq C_m + C\big\| [y_{\varepsilon_n}]^{\varepsilon_n} - [y_{\varepsilon_m}]^{\varepsilon_m} \big\|^2_{L^2(\Omega\times Y)}.$$

and such that $\lim_{m\to\infty} C_m = 0$.

2. *Let $f \in W^{1,\infty}(\Omega; W^{1,\infty}_\#(Y))$ such that $[f_\varepsilon]^\varepsilon \to f$. For $g_\varepsilon = y_\varepsilon$ or $g_\varepsilon = y_\varepsilon^{-1}$, we can estimate*

$$\begin{aligned}&\|f_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\|^2_{L^2(\Omega\times Y)}\\ &\leq C_m + C\left(\big\| [g_{\varepsilon_n}]^{\varepsilon_n} - [g_{\varepsilon_m}]^{\varepsilon_m} \big\|^2_{L^2(\Omega\times Y)} + \big\|\varepsilon_n^{-1} [g_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [g_{\varepsilon_m}]^{\varepsilon_m} \big\|^2_{L^2(\Omega\times Y)}\right)\end{aligned}$$

where $C, C_m > 0$ and $\lim_{m\to\infty} C_m = 0$.

3. *Let $f \in W^{1,\infty}(\Omega; W^{1,\infty}_\#(Y))$ such that $[f_\varepsilon]^\varepsilon \to f$ and $\varepsilon [\nabla f_\varepsilon]^\varepsilon \to \nabla_y f$. Then, we estimate*

$$\begin{aligned}&\|f_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\|^2_{L^2(\Omega\times \Gamma)}\\ &\leq C_m + C\left(\big\| [h_{\varepsilon_n}]^{\varepsilon_n} - [h_{\varepsilon_m}]^{\varepsilon_m} \big\|^2_{L^2(\Omega\times \Gamma)} + \big\|\varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m} \big\|^2_{L^2(\Omega\times \Gamma)}\right)\end{aligned}$$

where $C, C_m > 0$ and $\lim_{m\to\infty} C_m = 0$.

Proof. (Part 1). It holds

$$\big|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_m}]^{\varepsilon_m})\big| \leq \big|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_n}]^{\varepsilon_n})\big| + \|\nabla f_{\varepsilon_m}\|_\infty \big| [y_{\varepsilon_n}]^{\varepsilon_n} - [y_{\varepsilon_m}]^{\varepsilon_m} \big|.$$

The L^2 norm of the first term can be estimated (using the facts that y_ε is a diffeomorphism and $|\det Dy^{-1}_{\varepsilon_n}| \leq 4$) as

$$\begin{aligned}\int_{\Omega\times Y} \big|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_n}]^{\varepsilon_n})\big|^2 \,\mathrm{d}(x,y) &= \int_\Omega \big|f_{\varepsilon_n}(y_{\varepsilon_n}) - f_{\varepsilon_m}(y_{\varepsilon_n})\big|^2 \,\mathrm{d}x\\ &\leq 4\int_\Omega \big|f_{\varepsilon_n} - f_{\varepsilon_m}\big|^2 \,\mathrm{d}x = 4\int_{\Omega\times Y} \big| [f_{\varepsilon_n} - f_{\varepsilon_m}]^{\varepsilon_n} \big|^2 \,\mathrm{d}(x,y).\end{aligned}$$

Applying inequality (6.26), we finally see that

$$\begin{aligned}&\int_{\Omega\times Y} \big|f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_n}]^{\varepsilon_n})\big|^2 \,\mathrm{d}(x,y)\\ &\qquad\leq 4\int_{\Omega\times Y} \big| [f_{\varepsilon_n}]^{\varepsilon_n} - [f_{\varepsilon_m}]^{\varepsilon_m} \big|^2 + \big| [f_{\varepsilon_m}]^{\varepsilon_m} - [f_{\varepsilon_m}]^{\varepsilon_n} \big|^2 \,\mathrm{d}(x,y)\\ &\qquad\leq 4\int_{\Omega\times Y} \big| [f_{\varepsilon_n}]^{\varepsilon_n} - [f_{\varepsilon_m}]^{\varepsilon_m} \big|^2 + 32\|\nabla f_{\varepsilon_m}\|^2_\infty |\Omega| \varepsilon_m^2.\end{aligned}$$

(Part 2). With the use of the triangle inequality, we get

$$\begin{aligned}
&\int_{\Omega\times Y} \left|f_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\right|^2 \mathrm{d}(x,y) \\
&\qquad \le \int_{\Omega\times Y} \left|f_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n})\right|^2 + \left|f_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m}) - [f]_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\right|^2 \\
&\qquad\qquad + \left|\,[f]_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\right|^2 \mathrm{d}(x,y). \qquad (6.29)
\end{aligned}$$

For $i = n, m$, we estimate (note that $|\det Dg_\varepsilon^{-1}| \le 4$ for both choices of g_ε)

$$\begin{aligned}
&\int_{\Omega\times Y} \left|f_{\varepsilon_i}([g_{\varepsilon_i}]^{\varepsilon_i}) - [f]_{\varepsilon_i}([g_{\varepsilon_i}]^{\varepsilon_i})\right|^2 \mathrm{d}(x,y) \\
&\qquad\qquad \le 4\int_{\Omega\times Y} \left|\,[f_{\varepsilon_i}]^{\varepsilon_i} - f\right|^2 + \left|f - \left[[f]_{\varepsilon_i}\right]^{\varepsilon_i}\right|^2 \mathrm{d}(x,y).
\end{aligned}$$

Now, taking to the last term of equation (6.29) and using the y-periodicity of f, we have

$$\begin{aligned}
&\int_{\Omega\times Y} \left|\,[f]_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\right|^2 \mathrm{d}(x,y) \\
&\qquad = \int_{\Omega\times Y} \left|f\left([g_{\varepsilon_n}]^{\varepsilon_n}, \varepsilon_n^{-1}[g_{\varepsilon_n}]^{\varepsilon_n}\right) - f\left([g_{\varepsilon_m}]^{\varepsilon_m}, \varepsilon_m^{-1}[g_{\varepsilon_m}]^{\varepsilon_m}\right)\right|^2 \mathrm{d}(x,y).
\end{aligned}$$

As $f \in W^{1,\infty}(\Omega; W^{1,\infty}_\#(Y))$, this leads to

$$\begin{aligned}
&\left\|\,[f]_{\varepsilon_n}([g_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_m}([g_{\varepsilon_m}]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times Y)} \\
&\le \|\nabla_x f\|_\infty \left\|\,[g_{\varepsilon_n}]^{\varepsilon_n} - [g_{\varepsilon_m}]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} + \|\nabla_y f\|_\infty \left\|\varepsilon_n^{-1}[g_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[g_{\varepsilon_m}]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)}.
\end{aligned}$$

(Part 3). We calculate

$$\begin{aligned}
&\int_{\Omega\times\Gamma} \left|f_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m}))\right|^2 \mathrm{d}(x,\gamma) \\
&\qquad\qquad \le \int_{\Omega\times\Gamma} \left|\,[f]_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\right|^2 \\
&\qquad\qquad\qquad + \sum_{i=n,m} \left|f_{\varepsilon_i}([\eta_{\varepsilon_i}]^{\varepsilon_i}) - [f]_{\varepsilon_i}([\eta_{\varepsilon_i}]^{\varepsilon_i})\right|^2 \mathrm{d}(x,\gamma). \qquad (6.30)
\end{aligned}$$

Using the trace estimate from Lemma 6.19, the second part can be estimated via (note that $|\nabla_{\Gamma_\varepsilon} h_\varepsilon| + |h_\varepsilon L_{\Gamma_\varepsilon}| \le 1$)

$$\begin{aligned}
&\int_{\Omega\times\Gamma} \left|f_{\varepsilon_i}([\eta_{\varepsilon_i}]^{\varepsilon_i}) - [f]_{\varepsilon_i}([\eta_{\varepsilon_i}]^{\varepsilon_i})\right|^2 \mathrm{d}(x,\gamma) \\
&\qquad = \varepsilon_i \int_{\Gamma_{\varepsilon_i}} \left|f_{\varepsilon_i}(\eta_{\varepsilon_i}) - [f]_{\varepsilon_i}(\eta_{\varepsilon_i})\right|^2 \mathrm{d}\gamma \\
&\qquad \le 2\varepsilon_i \int_{\Gamma_{\varepsilon_i}(t)} \left|f_{\varepsilon_i}(\gamma) - [f]_{\varepsilon_i}(\gamma)\right|^2 \mathrm{d}\gamma \\
&\qquad \le 8C_{tr} \int_\Omega \left|f_{\varepsilon_i} - [f]_{\varepsilon_i}\right|^2 + \varepsilon_i^2 \left|\nabla f_{\varepsilon_i} - \nabla [f]_{\varepsilon_i}\right|^2 \mathrm{d}x \\
&\qquad = 8C_{tr} \int_{\Omega\times Y} \left|\,[f_{\varepsilon_i}]^{\varepsilon_i} - \left[[f]_{\varepsilon_i}\right]^{\varepsilon_i}\right|^2 + \varepsilon_i^2 \left|\,[\nabla f_{\varepsilon_i}]^{\varepsilon_i} - \left[\nabla [f]_{\varepsilon_i}\right]^{\varepsilon_i}\right|^2 \mathrm{d}x,
\end{aligned}$$

where, for the last term, it holds

$$\varepsilon_i \big| [\nabla f_{\varepsilon_i}]^{\varepsilon_i} - \left[\nabla [f]_{\varepsilon_i}\right]^{\varepsilon_i} \big| \leq \left|\varepsilon_i [\nabla f_{\varepsilon_i}]^{\varepsilon_i} - \nabla_y f\right| + \left|\nabla_y f - \nabla_y \left[[f]_{\varepsilon_i}\right]^{\varepsilon_i}\right| + \varepsilon_i \left|\nabla_x \left[[f]_{\varepsilon_i}\right]^{\varepsilon_i}\right|.$$

Now, for the first term on the right hand side of inequality (6.30), we use the ε-periodicity of $[f]_\varepsilon$ and estimate

$$\begin{aligned}
&\left\| [f]_{\varepsilon_n} ([\eta_{\varepsilon_n}]^{\varepsilon_n}) - [f]_{\varepsilon_m} ([\eta_{\varepsilon_m}]^{\varepsilon_m}) \right\|^2_{L^2(\Omega\times Y)} \\
&\leq \|\nabla_x f\|_\infty \left\| [h_{\varepsilon_n}]^{\varepsilon_n} - [h_{\varepsilon_m}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times\Gamma)} + \|\nabla_y f\|_\infty \left\|\varepsilon_n^{-1} [h_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [h_{\varepsilon_m}]^{\varepsilon_m} \right\|^2_{L^2(\Omega\times\Gamma)}.
\end{aligned}$$

□

6.5.2 Limit behavior

Based on Theorem 2.12 and on the estimates established via Lemma 6.6, it is clear that y_ε converges strongly to the identity operator and that both Dy_ε and z_ε have two-scale converging subsequences. This in itself, however, is not enough to guarantee strong convergence of their unfolded counterparts, which in consequence may also impede strong convergence of $[F_\varepsilon]^\varepsilon$ and $[J_\varepsilon]^\varepsilon$ – a property that is needed to make sure that passing to the limit $\varepsilon \to 0$ is justified.

In the following lemma, we investigate the limit behavior of the dilated functions $\varepsilon^{-1} [y_\varepsilon]^\varepsilon$ and $[z_\varepsilon]^\varepsilon$.

Lemma 6.21. *There exist functions $y, z \in L^2(S \times \Omega; H^1_\#(Y))^3$ such that*

$$\frac{1}{\varepsilon} [y_\varepsilon]^\varepsilon \to y, \quad [z_\varepsilon]^\varepsilon \to z, \quad [Dy_\varepsilon]^\varepsilon \to D_y y, \quad \varepsilon [Dz_\varepsilon]^\varepsilon \to D_y z.$$

Proof. Let $\delta > 0$ be given and let $n, m \in \mathbb{N}$, such that $n > m$ and such that $e^{\varepsilon_m l_v T_v} < 2$.[7] Taking a look at the ODE sytem given by equations (6.6a) to (6.6d) and its corresponding system that emerges by differentiation with respect to the spatial variable, we find that (in $S \times \Omega \times \Sigma$, $(i = n, m)$)

$$\varepsilon_i^{-1} \partial_t [y_{\varepsilon_i}]^{\varepsilon_i} = \frac{[z_{\varepsilon_i}]^{\varepsilon_i}}{\left| [z_{\varepsilon_i}]^{\varepsilon_i} \right|} v_{\varepsilon_i}([y_{\varepsilon_i}]^{\varepsilon_i}), \tag{6.31a}$$

$$\partial_t [z_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i \left| [z_{\varepsilon_i}]^{\varepsilon_i} \right| \nabla v_{\varepsilon_i}([y_{\varepsilon_i}]^{\varepsilon_i}), \tag{6.31b}$$

$$\partial_t [Dy_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i A^{(11)}_{\varepsilon_i} ([w_{\varepsilon_i}]^{\varepsilon_i}) [Dy_{\varepsilon_i}]^{\varepsilon_i} + \varepsilon_i A^{(11)}_{\varepsilon_i} ([w_{\varepsilon_i}]^{\varepsilon_i}) [Dz_{\varepsilon_i}]^{\varepsilon_i}, \tag{6.31c}$$

$$\varepsilon_i \partial_t [Dz_{\varepsilon_i}]^{\varepsilon_i} = \varepsilon_i^2 A^{(21)}_{\varepsilon_i} ([w_{\varepsilon_i}]^{\varepsilon_i}) [Dy_{\varepsilon_i}]^{\varepsilon_i} + \varepsilon_i^2 A^{(22)}_{\varepsilon_i} ([w_{\varepsilon_i}]^{\varepsilon_i}) [Dz_{\varepsilon_i}]^{\varepsilon_i}. \tag{6.31d}$$

[7] This is a mere technicality to allow for a more compact notation of the estimates. Here, we do not care about the details of the specific estimates, we only want to ensure convergence.

Now, subtracting these equations for $i = n$ and $i = m$ from one another, multiplying with the corresponding differences, and integrating over $\Omega \times Y$, we are led to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varepsilon_n^{-1}\left[y_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[y_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} &\\ \leq 2\int_{\Omega\times Y}\left|\frac{\left[z_{\varepsilon_n}\right]^{\varepsilon_n}}{\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\right|}v_{\varepsilon_n}(\left[y_{\varepsilon_n}\right]^{\varepsilon_n}) - \frac{\left[z_{\varepsilon_m}\right]^{\varepsilon_m}}{\left|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|}v_{\varepsilon_m}(\left[y_{\varepsilon_m}\right]^{\varepsilon_m})\right| &\\ \left|\varepsilon_n^{-1}\left[y_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^{-1}\left[y_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y), &\quad (6.32a)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} &\\ \leq 2\int_{\Omega\times Y}\left|\varepsilon_n\right|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\left|\nabla v_{\varepsilon_n}(\left[y_{\varepsilon_n}\right]^{\varepsilon_n}) - \varepsilon_m\right|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\left|\nabla v_{\varepsilon_m}(\left[y_{\varepsilon_m}\right]^{\varepsilon_m})\right| &\\ \left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y). &\quad (6.32b)\end{aligned}$$

Moreover, for the spatial derivatives, we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} &\\ \leq 2\int_{\Omega\times Y}\left|\varepsilon_n A^{(11)}_{\varepsilon_n}(\left[w_{\varepsilon_n}\right]^{\varepsilon_n})\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m A^{(11)}_{\varepsilon_m}(\left[w_{\varepsilon_m}\right]^{\varepsilon_m})\left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right| &\\ \left|\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y) &\\ + 2\int_{\Omega\times Y}\left|\varepsilon_n A^{(12)}_{\varepsilon_n}(\left[w_{\varepsilon_n}\right]^{\varepsilon_n})\left[Dz_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m A^{(12)}_{\varepsilon_m}(\left[w_{\varepsilon_m}\right]^{\varepsilon_m})\left[Dz_{\varepsilon_m}\right]^{\varepsilon_m}\right| &\\ \left|\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y) &\quad (6.32c)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\varepsilon_n\left[Dz_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m\left[Dz_{\varepsilon_m}\right]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times Y)} &\\ \leq 2\int_{\Omega\times Y}\left|\varepsilon_n^2 A^{(21)}_{\varepsilon_n}(\left[w_{\varepsilon_n}\right]^{\varepsilon_n})\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^2 A^{(21)}_{\varepsilon_m}(\left[w_{\varepsilon_m}\right]^{\varepsilon_m})\left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right| &\\ \left|\varepsilon_n\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m\left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y) &\\ + 2\int_{\Omega\times Y}\left|\varepsilon_n^2 A^{(22)}_{\varepsilon_n}(\left[w_{\varepsilon_n}\right]^{\varepsilon_n})\left[Dz_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^2 A^{(22)}_{\varepsilon_m}(\left[w_{\varepsilon_m}\right]^{\varepsilon_m})\left[Dz_{\varepsilon_m}\right]^{\varepsilon_m}\right| &\\ \left|\varepsilon_n\left[Dy_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m\left[Dy_{\varepsilon_m}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y). &\quad (6.32d)\end{aligned}$$

To proceed in showing that these sequences are Cauchy sequences, several independent estimates are needed to manage the right hand sides of inequalities (6.32a) to (6.32d). In the following, we heavily rely on the estimates established by Lemma 6.6. With the reverse triangle inequality, we get

$$\left|\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\right| - \left|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|\right| \leq \left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|, \quad (6.33a)$$

Since $e^{\varepsilon_m l_v T_v} < 2$, we also see that

$$\left|\frac{\left[z_{\varepsilon_n}\right]^{\varepsilon_n}}{\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n}\right|} - \frac{\left[z_{\varepsilon_m}\right]^{\varepsilon_m}}{\left|\left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|}\right| \leq 10\left|\left[z_{\varepsilon_n}\right]^{\varepsilon_n} - \left[z_{\varepsilon_m}\right]^{\varepsilon_m}\right|. \quad (6.33b)$$

Moreover, for $f_\varepsilon = v_\varepsilon, \varepsilon\nabla v_\varepsilon, \varepsilon_n^2 D^2 v_\varepsilon$, we can apply Lemma 6.20 to get

$$\begin{aligned}&\left\| f_{\varepsilon_n}([y_{\varepsilon_n}]^{\varepsilon_n}) - f_{\varepsilon_m}([y_{\varepsilon_n}]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times Y)} \\ &\quad \le 4\left\| \, [f_{\varepsilon_n}]^{\varepsilon_n} - [f_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} + l_v \left\| \, [y_{\varepsilon_n}]^{\varepsilon_n} - [y_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} + 32 l_v^2 |\Omega| \varepsilon_m^2, \end{aligned} \tag{6.33c}$$

The matrix valued function B, which is defined via equation (6.9), is Lipschitz continuous with Lipschitz constant 2, i.e.,

$$\left| B([z_{\varepsilon_n}]^{\varepsilon_n}) - B([z_{\varepsilon_m}]^{\varepsilon_m})\right| \le 2 \left| \, [z_{\varepsilon_n}]^{\varepsilon_n} - [z_{\varepsilon_m}]^{\varepsilon_m} \, \right|. \tag{6.33d}$$

Adding inequalities (6.32a) and (6.32b), using the estimates given by inequalities (6.33a) to (6.33c) as well as Assumption (A5), and applying Gronwall's inequality, we infer

$$\begin{aligned}&\left\| \varepsilon_n^{-1} \, [y_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} \, [y_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} + \left\| \, [z_{\varepsilon_n}]^{\varepsilon_n} - [z_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} \\ &\qquad \le C\left(\varepsilon_m^2 + \|[v_{\varepsilon_n}]^{\varepsilon_n} - [v_{\varepsilon_m}]^{\varepsilon_m}\|^2 + \left\| \varepsilon_n \, [\nabla v_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m \, [\nabla v_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2 \right) < \delta \end{aligned} \tag{6.34}$$

for all $n, m \in \mathbb{N}$ such that $n, m > N$ for sufficiently large $N \in \mathbb{N}$ (which is independent of ε and t). This implies

$$\frac{1}{\varepsilon} \, [y_\varepsilon]^\varepsilon \to y, \quad [z_\varepsilon]^\varepsilon \to z \quad \text{in} \ \ L^2(S\times\Omega\times Y)^3.$$

Similarly, adding inequalities (6.32c) and (6.32d) and using above estimates given by given by inequalities (6.33a) to (6.33d) and Assumption (A5), we get

$$\begin{aligned}&\left\| \, [Dy_{\varepsilon_n}]^{\varepsilon_n} - [Dy_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} + \left\| \varepsilon_n \, [Dz_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m \, [Dz_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2_{L^2(\Omega\times Y)} \\ &\qquad \le C\Big(\varepsilon_m^2 + \varepsilon_m^3 + \|[z_{\varepsilon_n}]^{\varepsilon_n} - [z_{\varepsilon_m}]^{\varepsilon_m}\|^2 + \left\| \, [y_{\varepsilon_n}]^{\varepsilon_n} - [y_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2 \\ &\qquad + \left\| \varepsilon_n \, [\nabla v_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m \, [\nabla v_{\varepsilon_m}]^{\varepsilon_m} \, \right\|^2 + \left\| \varepsilon_n^2 \left[D^2 \nabla^2 v_{\varepsilon_n}\right]^{\varepsilon_n} - \varepsilon_m^2 \left[D^2 v_{\varepsilon_m}\right]^{\varepsilon_m} \, \right\|^2 \Big) \le \delta\end{aligned}$$

for all $n, m \in \mathbb{N}$ such that $n, m > N$ for some sufficiently large $N \in \mathbb{N}$ (which is independent of ε and t). We conclude that

$$[Dy_\varepsilon]^\varepsilon \to D_y y, \quad \varepsilon \, [Dz_\varepsilon]^\varepsilon \to D_y z \quad \text{in} \ \ L^2(S\times\Omega\times Y)^{3\times 3}.$$

□

Remark 6.22. *As a consequence of Lemma 6.18, we also have*

$$\frac{1}{\varepsilon} \, [y_\varepsilon]^\varepsilon \to y, \quad [z_\varepsilon]^\varepsilon \to z \quad \text{in} \ \ L^2(S\times\Omega\times\Gamma)^3$$

in the sense of traces.

Lemma 6.23. *The following convergences hold:*

$$\frac{1}{\varepsilon} \left[y_\varepsilon^{-1}\right]^\varepsilon \to y^{-1}, \quad [q_\varepsilon]^\varepsilon \to z(y^{-1}), \quad \varepsilon^{-1}\widetilde{\varphi}_\varepsilon \to \widetilde{\varphi}, \quad \varepsilon\nabla q_\varepsilon \to \nabla_y q \quad \text{in} \ \ L^2(S\times\Omega\times Y).$$

Proof. We recall that y_ε^{-1} can be characterized by equation (6.15). This leads us to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n}-\varepsilon_m^{-1}\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m}\big\|^2_{L^2(\Omega\times Y)}\\ \leq\int_{\Omega\times Y}\Big|Dy_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})^{-1}\frac{z_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})}{|z_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n})|}v_{\varepsilon_n}(y_{\varepsilon_n}(\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n}))\\ -Dy_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})^{-1}\frac{z_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})}{|z_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m})|}v_{\varepsilon_m}(y_{\varepsilon_m}(\left[y_{\varepsilon_m}^{-1}\right]^{\varepsilon_m}))\Big|\\ \cdot\left|\varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_n}-\varepsilon_n^{-1}\left[y_{\varepsilon_n}^{-1}\right]^{\varepsilon_m}\right|\mathrm{d}(x,y).\end{aligned}$$

Taking into considerations the a-priori estimates available for the involved functions and the strong convergence results formulated in Lemma 6.21, as well as the estimates given in Lemma 6.20, it is possible (while cumbersome) to estimate the individual differences and see that $\varepsilon^{-1}\left[y_\varepsilon^{-1}\right]^\varepsilon$ is a Cauchy sequence. Similarly, it is also possible to show that $[q_\varepsilon]^\varepsilon=[z_\varepsilon(y_\varepsilon^{-1})]^\varepsilon$ also is a Cauchy sequence using Lemma 6.20 (2). Since $\partial_t\widetilde{\varphi}_\varepsilon$ is governed by equation (6.21) and because $\nabla\widetilde{\varphi}_\varepsilon=q_\varepsilon$ (see equation (6.19)), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\varepsilon_n^{-1}\widetilde{\varphi}_{\varepsilon_n}-\varepsilon_m^{-1}\widetilde{\varphi}_{\varepsilon_m}\big\|^2_{L^2(\Omega\times Y)}\leq\int_{\Omega\times Y}\big||q_{\varepsilon_n}|\,v_{\varepsilon_n}-|q_{\varepsilon_m}|\,v_{\varepsilon_m}\big|\,\mathrm{d}(x,y)$$

which shows that $\varepsilon^{-1}\widetilde{\varphi}_\varepsilon$ also converges strongly. Finally, as

$$\varepsilon\nabla q_\varepsilon=\varepsilon D^2\widetilde{\varphi}_\varepsilon=\varepsilon\big(Dy_\varepsilon(y_\varepsilon^{-1})\big)^{-1}Dz_\varepsilon(y_\varepsilon^{-1}),$$

we also get the strong convergence of $\varepsilon\left[\nabla q_\varepsilon\right]^\varepsilon$. □

Since the quantity $\varepsilon\|h_\varepsilon\|_\infty+\|\nabla_{\Gamma_\varepsilon}h_\varepsilon\|_\infty$ is bounded indepedently of the parameter ε, we can find a constant $C_h>0$ such that

$$\|h_\varepsilon\|_{L^2(S\times\Gamma_\varepsilon)}+\sqrt{\varepsilon}\|\nabla_{\Gamma_\varepsilon}h_\varepsilon\|_{L^2(S\times\Gamma_\varepsilon)^3}\leq C_h.$$

As a result, we conclude the existence of a function $h\in L^2(S,H^1(\Omega;H^1(\Gamma)))$ such that, up to a subsequence,

$$\frac{1}{\varepsilon}h_\varepsilon\overset{2}{\rightharpoonup}h,\quad\nabla_{\Gamma_\varepsilon}h_\varepsilon\overset{2}{\rightharpoonup}\nabla_\Gamma h$$

Furthermore, it is clear that $h\in L^\infty(S\times\Omega\times\Gamma)$ and that $[h_\varepsilon]^\varepsilon\in L^\infty(S\times\Omega\times\Gamma)$ is bounded independently of ε. As a consequence, there is a function $\tilde{h}\in L^\infty(S\times\Omega\times\Gamma)$ such that $[h_\varepsilon]^\varepsilon\rightharpoonup\tilde{h}$ in $L^2(S\times\Omega\times Y)$. In the following, we are concerned with the limit behavior of h_ε.

Lemma 6.24. *There is $h\in L^2(S;W^{1,2}(\Omega;H^1_\#(\Gamma)))$ such that $\varepsilon^{-1}[h_\varepsilon]^\varepsilon\to h$ and such that $[\nabla_{\Gamma_\varepsilon}h_\varepsilon]^\varepsilon\to\nabla_y h$ in $L^2(S\times\Omega\times\Gamma)$.*

Proof. Let $\delta>0$ and $n,m\in\mathbb{N}$, $n>m$. Using the representation of the height function h_ε in terms of $F_{\varepsilon,\gamma}$ as given by equation (6.23), we have

$$\partial_t h_\varepsilon(t,\gamma)=-\frac{\partial_t F_{\varepsilon,\gamma}(t,h_\varepsilon(t,\gamma))}{\partial_2 F_{\varepsilon,\gamma}(t,h_\varepsilon(t,\gamma))}\quad(t\in[0,T_v],\,\gamma\in\Gamma_\varepsilon).\tag{6.35}$$

Now, integrating over $\Omega\times\Gamma$ and testing with the difference $\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}-\varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_m}$ leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}-\varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\le 2\int_{\Omega\times\Gamma}\left|\varepsilon_n^{-1}\frac{\partial_t[F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})]^{\varepsilon_n}}{[\partial_2F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})]^{\varepsilon_n}}-\varepsilon_m^{-1}\frac{\partial_t[F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})]^{\varepsilon_m}}{[\partial_2F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})]^{\varepsilon_m}}\right|\\ &\qquad\left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}-\varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_m}\right|\mathrm{d}(x,\gamma).\end{aligned}$$

Using that $\partial_t\widetilde{\varphi}_\varepsilon$ is governed by equation (6.21) and the characterization of $\nabla\widetilde{\varphi}_\varepsilon$ given via equation (6.19), we get

$$\varepsilon^{-1}\partial_t[F_{\varepsilon,\gamma}(h_\varepsilon)]^\varepsilon=|q_\varepsilon([\eta_\varepsilon]^\varepsilon))|\,v_\varepsilon([\eta_\varepsilon]^\varepsilon). \tag{6.36}$$

Applying Lemma 6.20(3) to q_ε and v_ε, respectively, and using the strong convergence of $[v_\varepsilon]^\varepsilon$, $[\nabla v_\varepsilon]^\varepsilon$, $[q_\varepsilon]^\varepsilon$, and $\varepsilon[\nabla q_\varepsilon]^\varepsilon$, we are led to

$$\begin{aligned}&\left\|\varepsilon_n^{-1}\partial_t[F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})]^{\varepsilon_n}-\varepsilon_m^{-1}\partial_t[F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\le C(m)+C\left(\left\|[h_{\varepsilon_n}]^{\varepsilon_n}-[h_{\varepsilon_m}]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times\Gamma)}+\left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}-\varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}\right\|^2_{L^2(\Omega\times\Gamma)}\right)\end{aligned} \tag{6.37}$$

where $\lim_{m\to\infty}C(m)=0$. As a next step, we estimate the difference with respect to $\partial_2F_{\varepsilon,\gamma}$. In view of equations (6.19) and (6.22), we have

$$[\partial_2F_{\varepsilon,\gamma}(h_\varepsilon)]^\varepsilon=g'(\varepsilon^{-1}\widetilde{\varphi}_\varepsilon([\eta_\varepsilon]^\varepsilon))q_\varepsilon([\eta_\varepsilon]^\varepsilon)\cdot n \tag{6.38}$$

and, due to the strong convergence of $\varepsilon^{-1}[\widetilde{\varphi}_\varepsilon]^\varepsilon$, $[q_\varepsilon]^\varepsilon=[\nabla\widetilde{\varphi}_\varepsilon]^\varepsilon$, and $\varepsilon[\nabla q_\varepsilon]^\varepsilon$, we can infer (again applying Lemma 6.20(3))

$$\begin{aligned}&\left\|\varepsilon_n^{-1}[\partial_2F_{\varepsilon_n,\gamma}(h_{\varepsilon_n})]^{\varepsilon_n}-\varepsilon_m^{-1}[\partial_2F_{\varepsilon_m,\gamma}(h_{\varepsilon_m})]^{\varepsilon_m}\right\|^2_{L^2(\Omega\times\Gamma)}\\ &\le C_m+C\left(\left\|[h_{\varepsilon_n}]^{\varepsilon_n}-[h_{\varepsilon_m}]^{\varepsilon_m})\right\|^2_{L^2(\Omega\times\Gamma)}+\left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}-\varepsilon_m^{-1}[h_{\varepsilon_n}]^{\varepsilon_n}\right\|^2_{L^2(\Omega\times\Gamma)}\right)\end{aligned} \tag{6.39}$$

where $\lim_{m\to\infty}C_m\to 0$. Combining the estimates given by inequalities (6.37) and (6.39) and applying Gronewall's inequality, it is then easy to see that $\varepsilon^{-1}[h_\varepsilon]^\varepsilon$ is, in fact, Cauchy.

Using the representation of h_ε given in equation (6.24), we have

$$[\nabla_{\Gamma_\varepsilon}h_\varepsilon]^\varepsilon=\left(\mathbb{I}_3-\varepsilon^{-1}[h_\varepsilon]^\varepsilon L_\Gamma\right)\left(n-\frac{1}{n_{\Gamma_\varepsilon}([\eta_\varepsilon]^\varepsilon)\cdot n}n_{\Gamma_\varepsilon}([\eta_\varepsilon]^\varepsilon)\right).$$

Consequently, since $n_{\Gamma_\varepsilon}(\eta_\varepsilon)\cdot n_\varepsilon>1/2$ and $|\varepsilon^{-1}h_\varepsilon|\le a/10$ in $[0,T_v]\times\Gamma_\varepsilon$, we are led to

$$\begin{aligned}&\left\|[\nabla_{\Gamma_{\varepsilon_n}}h_{\varepsilon_n}]^{\varepsilon_n}-[\nabla_{\Gamma_{\varepsilon_m}}h_{\varepsilon_m}]^{\varepsilon_m}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\le\frac{3}{2a}\left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^\varepsilon-\varepsilon_m^{-1}[h_{\varepsilon_m}]^\varepsilon\right\|_{L^2(S\times\Gamma_\varepsilon)}+6\left\|\frac{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})\cdot n}-\frac{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\cdot n}\right\|_{L^2(S\times\Gamma_\varepsilon)}\end{aligned}$$

Now, due to $n_{\Gamma_\varepsilon}(\eta_\varepsilon) = \frac{\nabla\widetilde{\varphi}_\varepsilon(\eta_\varepsilon)}{|\nabla\widetilde{\varphi}_\varepsilon(\eta_\varepsilon)|} = \frac{q_\varepsilon(\eta_\varepsilon)}{|q_\varepsilon(\eta_\varepsilon)|}$, we further estimate

$$\begin{aligned}\left\|\frac{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{n_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})\cdot n} - \frac{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{n_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\cdot n}\right\|_{L^2(S\times\Gamma_\varepsilon)}&\\ \le 6\left\|\frac{q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})}{|q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n})|} - \frac{q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})}{|q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})|}\right\|_{L^2(S\times\Gamma_\varepsilon)}&\\ \le 36\left\|q_{\varepsilon_n}([\eta_{\varepsilon_n}]^{\varepsilon_n}) - q_{\varepsilon_m}([\eta_{\varepsilon_m}]^{\varepsilon_m})\right\|_{L^2(S\times\Gamma_\varepsilon)}.&\end{aligned}$$

As both $[q_\varepsilon]^\varepsilon$ and $\varepsilon\,[\nabla q_\varepsilon]^\varepsilon$ converge, we can apply Lemma 6.20(3) and conclude

$$\begin{aligned}&\left\|\left[\nabla_{\Gamma_{\varepsilon_n}} h_{\varepsilon_n}\right]^{\varepsilon_n} - \left[\nabla_{\Gamma_{\varepsilon_m}} h_{\varepsilon_m}\right]^{\varepsilon_m}\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad\le C_m + \frac{3}{2a}\left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^\varepsilon - \varepsilon_m^{-1}[h_{\varepsilon_m}]^\varepsilon\right\|_{L^2(S\times\Gamma_\varepsilon)}\\ &\qquad+ C\left(\|[h_{\varepsilon_n}]^\varepsilon - [h_{\varepsilon_m}]^\varepsilon\|_{L^2(S\times\Gamma_\varepsilon)} + \left\|\varepsilon_n^{-1}[h_{\varepsilon_n}]^\varepsilon - \varepsilon_m^{-1}[h_{\varepsilon_m}]^\varepsilon\right\|_{L^2(S\times\Gamma_\varepsilon)}\right),\end{aligned}$$

where, again, $\lim_{m\to\infty} C_m = 0$.

□

Lemma 6.25. *There is $\psi \in L^\infty(S; W^{1,\infty}(\Omega; W^{1,2}_\#(Y)))$ such that $\varepsilon^{-1}[\psi_\varepsilon]^\varepsilon \to \psi$ and such that $[\nabla\psi_\varepsilon]^\varepsilon \to \nabla_y\psi$ in $L^2(S\times\Omega\times Y)$.*

Proof. Let $n, m \in \mathbb{N}$ such that $m > n$ and set $\mu_\varepsilon(t,x) = h_\varepsilon(t, P_{\Gamma_\varepsilon}(x))$ as well as $\mu(t,x,y) = h(t,x,P_\Gamma(y))$. We calculate

$$\varepsilon_n^{-1}[\psi_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\psi_{\varepsilon_m}]^{\varepsilon_m} = \left(\varepsilon_n^{-1}[\mu_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\mu_{\varepsilon_m}]^{\varepsilon_m}\right)\chi\left(a^{-1}d_\Gamma\right)n(P_\Gamma).$$

As a consequence,

$$\begin{aligned}&\int_{\Omega\times U_\Gamma}\left|\varepsilon_n^{-1}[\psi_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1}[\psi_{\varepsilon_m}]^{\varepsilon_m}\right|^2\mathrm{d}(x,y)\\ &\qquad\le \int_{\Omega\times U_\Gamma}\left|\varepsilon_n^{-1}[\mu_{\varepsilon_n}]^{\varepsilon_n} - \mu\right|^2 + \left|\varepsilon_m^{-1}[\mu_{\varepsilon_m}]^{\varepsilon_m} - \mu\right|^2\mathrm{d}(x,y).\end{aligned}$$

Now, for fixed $x \in \Omega$, $[\mu_\varepsilon]^\varepsilon$ and μ are constant in the y variable in the direction of the normal vector. As a consequence,

$$\int_{\Omega\times U_\Gamma}\left|\varepsilon^{-1}[\mu_\varepsilon]^\varepsilon - \mu\right|^2\mathrm{d}(x,y) = 2a\int_{\Omega\times\Gamma}\left|\varepsilon_n^{-1}[h_{\varepsilon_n}]^{\varepsilon_n} - h\right|^2\mathrm{d}(x,y).$$

The unfolded deformation gradient (see equation (2.8)) is given via

$$\begin{aligned}[\nabla\psi_\varepsilon]^\varepsilon &= \left([\nabla\mu_\varepsilon]^\varepsilon\right)^T n(P_\Gamma)\chi\left(a^{-1}d_\Gamma\right)\\ &\quad+ \varepsilon^{-1}[\mu_\varepsilon]^\varepsilon\Big(L_\Gamma(P_\Gamma)\left(\mathbb{I} - d_\Gamma L_\Gamma(P_\Gamma)\right)^{-1}\left(\mathbb{I} - n(P_\Gamma)\otimes n(P_\Gamma)\right)\chi\left(a^{-1}d_\Gamma\right)\\ &\qquad\qquad + \chi'\left(a^{-1}d_\Gamma\right)n(P_\Gamma)\otimes n(P_\Gamma)\Big).\end{aligned}$$

which leads us to

$$\int_{\Omega\times U_\Gamma} |[\nabla\psi_{\varepsilon_n}]^{\varepsilon_n} - [\nabla\psi_{\varepsilon_m}]^{\varepsilon_m}|^2 \,\mathrm{d}(x,y)$$
$$\leq C \int_{\Omega\times U_\Gamma} \left|\varepsilon_n^{-1} [\mu_{\varepsilon_n}]^{\varepsilon_n} - \varepsilon_m^{-1} [\mu_{\varepsilon_m}]^{\varepsilon_m}\right|^2 + |[\nabla\mu_{\varepsilon_n}]^{\varepsilon_n} - [\nabla\mu_{\varepsilon_m}]^{\varepsilon_m}|^2 \,\mathrm{d}(x,y),$$

where $C > 0$ is independent of ε. Since

$$\nabla\mu_\varepsilon(t,x) = (DP_{\Gamma_\varepsilon}(x))^T \,\nabla_{\Gamma_\varepsilon} h_\varepsilon(t, P_{\Gamma_\varepsilon}(x))$$

and

$$\int_{\Omega\times U_\Gamma} |[\nabla_{\Gamma_\varepsilon} h_\varepsilon(P_{\Gamma_\varepsilon})]^\varepsilon (x,y) - \nabla_y h(t,x,P_\Gamma(y))|^2 \,\mathrm{d}(x,y)$$
$$= 2a \int_{\Omega\times\Gamma} |[\nabla_{\Gamma_\varepsilon} h_{\varepsilon_n}]^{\varepsilon_n} - \nabla_y h|^2 \,\mathrm{d}(x,y),$$

we can conclude $[\nabla\psi_\varepsilon]^\varepsilon \to \nabla_y\psi$. □

CHAPTER 7

Summary and outlook

7.1 Summary

In this thesis, we investigated the analysis and the mathematical homogenization of systems corresponding to thermoelasticity models for phase transformations in highly heterogeneous, periodic media: see equations (3.10a) to (3.10e).

In Chapter 4, we considered a simplified problem, where the changes in the geometry were assumed to be prescribed at the outset. Starting with a particular motion function describing these changes (see Section 4.2), we arrived at a fixed domain formulation of the problem. In Theorem 4.7 and Theorem 4.8, we then showed that the problem has a unique solution for which uniform (with respect to the parameter ε) energy estimates are available. Finally, we performed a homogenization limit procedure to arrive at a distributed microstructure model (see equations (4.30a) to (4.30d)) describing the effective behavior of the associated ε-problem. While similar transformation settings for homogenization problems were considered in [Mei08, Dob14], this is, to our knowledge, the first time, that such upscaling has be done in the context of thermoelasticity. Here, the main difficulties were due to the coupling between the mechanical part and the heat part as well as the time dependency of the involved operators.

Based on these results, we proved quantitative error/corrector estimates, i.e., convergence rates with respect to the parameter ε, under certain reasonable simplifications: (i) the *weakly coupled problem* where either the mechanical dissipation or the thermal stresses are assumed to negligible (see Theorem 5.9), (ii) the problem with *microscale coupling* where the mechanical dissipation and thermal stresses are only significant in the slow-conducting phase (see Theorem 5.10). We also argued why similar estimates for the fully-coupled problem are not obtainable via the same strategy. To our knowledge, this is the first time that quantitative corrector estimates have been proven in the context of thermoelasticity.

Finally, in Chapter 6, we considered the case where the changes in the geometry are not given at the outset but where they are the result of a prescribed function v_ε representing the normal velocity of the interface. We showed that, if v_ε satisfies certain regularity conditions and ε-uniform estimates, a *Hanzawa* transformation via a corresponding height function exists (Theorem 6.1). Relying on the notion of periodic unfolding, we then proved that strong-two scale convergence of the functions related to the transformation can be inferred from the strong two-scale convergence of v_ε (see Theorem 6.2). Combining these results with those from Chapter 4, we concluded that the thermoelasticity problem with moving boundary and prescribed normal velocity has a unique solution satisfying standard energy estimates (Theorem 6.3). Finally, we deduced the corresponding homogenization limit in Theorem 6.4.

Going back to the research question formulated in Chapter 1 – and in particular to the example of the formation of Bainite from Austenite steel – and taking a look at the homogenized system given via equations (4.30a) to (4.30e), we can trace two possible ways in which micro stresses manifest in the upscaled problem: (i) we have the volume force density H^h, which accounts for the surface stresses due to the curvature, and (ii) we have the Dirichlet coupling between the deformations u and $u^{(2)}$ (equation (4.30e)). We speculate that the volume force density H^h can be linked to the TRIP phenomenon, which is often interpreted as a force density due to micro effects; we refer to [Cha09, MSA$^+$15, WBDH08].

7.2 Outlook

We want to point out some possible further research directions linked to the work presented in this thesis.

Fully-coupled moving boundary problem. The most important and natural continuation is the treatment of the fully-coupled moving boundary problem, where the normal velocity is not prescribed, and where it might be dependent on the temperature function, the stress fields, and the geometry of the interface, e.g., $V_{\Gamma_\varepsilon} = \mathcal{V}_\varepsilon(\theta, e(u_\varepsilon), H_{\Gamma_\varepsilon})$. For simplicity, let us discuss the more instructive example $\mathcal{V}_\varepsilon(\theta, e(u_\varepsilon), H_{\Gamma_\varepsilon}) = \varepsilon(\theta_\varepsilon - \theta_{crit})$, where θ_{crit} denotes the critical temperature for the particular phase transformation.[1] With the results of this thesis, we established that:

(i) For every function $v_\varepsilon \in W^{(1,3),\infty}(S \times \Omega)$ satisfying the assumptions outlined in Section 6.3, we can obtain the *Hanzawa* transformation corresponding to the interface motion with normal velocity $V_{\Gamma_\varepsilon} = v_\varepsilon$.

(ii) Given a motion function characterizing the moving boundary, we can tackle the resulting two-phase thermoelasticity problem.

This situation already points to the construction of a possible fixed-point argument: Let us start with a normal velocity $v_\varepsilon \in W^{(1,3),\infty}(S \times \Omega)$ satisfying the assumptions outlined in Section 6.3 and solve the thermoelasticty problem with the corresponding

[1] Note that this corresponds to the law of *kinetic undercooling*.

moving boundary. This induces an operator $\mathcal{T}_\varepsilon\colon W^{(1,3),\infty}(S\times\Omega)\to L^2(S\times\Gamma)$ defined via $\mathcal{T}_\varepsilon(v_\varepsilon)=\theta_\varepsilon$. To tackle the fully-coupled moving boundary problem, we would have to show additionally that:

(*i*) There is a function $\widetilde{\theta_\varepsilon}\in W^{(1,3),\infty}(S\times\Omega)$ satisfying the assumptions outlined in Section 6.3 such that $\widetilde{\theta}_{\varepsilon|\Sigma}=\theta_\varepsilon$, i.e., there is an extension operator

$$\mathrm{Ext}\colon \mathcal{T}_\varepsilon(W^{(1,3),\infty}(S\times\Omega))\to W^{(1,3),\infty}(S\times\Omega).$$

(*ii*) The operator $(\mathrm{Ext}\circ\mathcal{T}_\varepsilon)\colon W^{(1,3),\infty}(S\times\Omega)\to W^{(1,3),\infty}(S\times\Omega)$ has a fixed point.

(*iii*) This fixed point satisfies the assumptions outlined in Section 6.3.

There are several important challenges in proving these steps but the most problematic is the need for uniform estimates: While the $W^{(1,3),\infty}$-regularity can possibly be established using methods borrowed from maximal parabolic regularity,[2], the necessary control with regards to ε is more problematic as we would need to, among other things, find a uniform bound of the form

$$\sup_{\varepsilon>0}\left(\|\theta_\varepsilon\|_{W^{1,\infty}(S\times\Gamma)}+\varepsilon\|D^2\theta_\varepsilon\|_{L^\infty(S\times\Gamma)}\right)\leq\infty.$$

Such uniform estimates are not the standard energy estimates and they are usually more difficult to obtain. There are a few results for higher estimates available but they are tailored to very specific situations, we refer to [Sch99, Yeh10]. Unfortunately, they are not applicable in our context. Using constructive methods for higher regularity estimates for PDEs as outlined in, e.g., [LSU95], uniform L^∞-estimates for θ_ε can be established. However, these methods fail for the gradient and the second derivative due to the two-phase structure of the problem. Nevertheless, we do expect these bounds to hold uniformly under reasonable assumptions.

Numerical analysis and simulations. As stated in Chapter 1, the general goal of homogenization procedures is to derive mathematical models that are both accurate and efficient in describing and predicting real world phenomena. In this thesis, we focus on the mathematical analysis of the problem and the natural next step is the numerical analysis of the upscaled models and corresponding multiscale simulations. With the convergence rates, i.e., corrector estimates, proved in this work, an important first step in establishing efficient numerical schemes like the *Multiscale FEM* is already taken; we refer to [AB05, HW97]. Using such schemes, simulations and first comparisons with real world data could be conducted.

Non-elastic effects. There are several ways to generalize the model – some of them already pointed out in Chapter 3 and Section 4.5. As we are primarily interested in the interplay of mechanical effects and phase transformations, allowing for non-elastic phenomena, like *plasticity* or *damage mechanics*, would be particularly interesting. Here,

[2]For similar problems where this has been achieved see [EPS03, DPZ08].

existing results, see, e.g., [Boe13], could likely be adapted to the resulting ε-dependent problem leaving the homogenization procedure as the main potential challenge.

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