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REGULARIZATION METHODS IN BANACH SPACES APPLIED TO INVERSE MEDIUM SCATTERING PROBLEMS

Doctoral Thesis

Regularization Methods in Banach Spaces Applied to Inverse Medium Scattering Problems

submitted by
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When I heard the learn'd astronomer,

When the proofs, the figures, were ranged in columns before me,
When I was shown the charts and diagrams, to add, divide, and measure them,
When I sitting heard the astronomer where he lectured with much applause in the lecture-room,
How soon unaccountable I became tired and sick,
Till rising and gliding out I wander'd off by myself,
In the mystical moist night-air, and from time to time,
Look'd up in perfect silence at the stars.

— Walt Whitman

ABSTRACT

This work handles inverse scattering problems for both acoustic and electromagnetic waves. That is to reconstruct the irradiated media from measurements of the scattered fields by regularization methods. As a particular feature, the contrasts of the scattering objects are assumed to be supported within a small region, hence called sparse. To apply sparsity regularization schemes it becomes crucial to model the problems in Banach spaces. Traditionally, they are given in a Hilbert space setting, such that reformulation in an L^p-sense becomes a key point. Contrasts are linked to the data by forward operators, basing on beforehand stated solution operators and their continuity properties. Thereby, appropriate regularization techniques providing sparsity are given.

As the case of scalar-valued contrast functions is already covered in the literature, mainly inverse scattering problems for anisotropic media are shown. In the case where electromagnetic waves are considered, a distinction is made between magnetic and non-magnetic media, since the latter is less complex. Finally, the case of inverse acoustic backscattering is handled, which is rarely seen in literature.

ZUSAMMENFASSUNG

Diese Arbeit behandelt inverse Streuprobleme für akustische und elektromagnetische Wellen, also die Objektrekonstruktion aus Messungen gestreuter Felder mit Hilfe sogenannter Regularisierungsmethoden. Dabei wird angenommen, dass die Kontraste der bestrahlten Medien einen verhältnismäßig kleinen Träger haben, d. h. sparse sind. Um sparsity-erhaltende Regularisierungsmethoden anzuwenden, ist es erforderlich das zugrundeliegende Problem in Banach Räumen zu modellieren. Üblicherweise sind diese in Hilbert Räumen gegeben, sodass der neuen Darstellung im L^p-Sinn eine Schlüsselrolle zukommt. Kontraste und Daten werden dann durch Vorwärtsoperatoren miteinander verknüpft, die auf Lösungsoperatoren und deren Stetigkeitseigenschaften basieren. Das erlaubt passende sparsity-erhaltende Regularisierungstechniken zu formulieren.

Da der Fall skalarwertiger Kontraste schon in der Literatur behandelt wurde, werden hauptsächlich inverse Streuprobleme an anisotropen Medien betrachtet. Im Fall elektromagnetischer Wellen wird dabei zwischen magnetischen und nicht-magnetischen Medien unterschieden, da der letzte Fall weniger technisch ist. Abschließend wird der in der Literatur schwer aufzufindende Fall akustischer Rückstreuung untersucht.

PUBLICATIONS

Note that the ideas and figures of Chapter 3, concerning regularization techniques for anisotropic acoustic scattering, have been submitted December 2015 and published previously in [LR17] (after revision in September 2016). Likewise, the concepts for electromagnetic scattering on non-magnetic anisotropic media, shown in Chapter 4, have been submitted for publication recently, see [Ren16].

[LR17] A. Lechleiter and M. Rennoch. "Non-linear Tikhonov regularization in Banach spaces for inverse scattering from anisotropic penetrable media." In: *Inverse Probl. Imaging* 11.1 (2017), pp. 151–176.

[Ren16] M. Rennoch. "Non-linear Tikhonov regularization in Banach spaces for inverse electromagnetic scattering from anisotropic penetrable non-magnetic media." Submitted. 2016.

How sad,
to think I will end
as only
a pale green mist
drifting the far fields.

— Ono no Komachi

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1 WHAT IS AT ISSUE?

What if a planet does not behave in the way it should do? Question the validity of observations? Disregard the underlying theory since it causes false predictions? As confusing this problem may be, as simple appears the solution: consider a lack of information.

At the beginning of the 19th century Kepler's laws of planetary motion, derived from Newton's law of universal gravitation, were used to calculate the orbit and the movement of Uranus, the farthermost known planet of the solar system. But surprisingly the mathematically derived forecast did not match the observation made. Put crudely, Uranus did not behave as expected, although disturbances caused by its neighbors Jupiter and Saturn were taken into account. As the assumption of existence of another planet would explain the discrepancy, the French mathematician U. Le Verrier accurately calculated its hypothetical position from given observations. By that, the predicted object, nowadays known as Neptune, was observed.

Yet we have seen a famous example of a so-called inverse problem, which means, roughly speaking, to determine the cause of an observed effect. More explicit, the best possible reconstruction of a missing information is used either to identify the source of observed effects (respectively their cause) or to determine the value of unknown model parameters. Throughout this thesis we are going to handle the latter, so-called parameter identification problems. Therefore, the abstract definition of a mathematical problem is given by a mapping

$$A \colon X \to Y,$$

for a forward operator A modeling the relation between a set of causes or parameters X and a set of observations or data Y. Thus, calculation of observable effects $A\,x \in Y$ from unknown values $x \in X$ is referred to a direct problem. However, the inverse problem is to determine $x \in X$ from given $y \in Y$, such that $A\,x = y$ holds.

Typically inverse problems are ill-posed in the sense of Hadamard. That is, either existence of solution of $A\,x = y$ is not given for all $y \in Y$, or the solution is not unique or the solution x does not depend continuously on the data y. Usually the latter causes the ill-posedness, since the other ones normally can be avoided by enlarging the solution space or by considering further properties of the model respectively. But if the solution is not stable due to perturbed data, additional information about the solution is necessary to achieve useful results (see, e.g., Kirsch [Kir11]).

HISTORICAL BACKGROUND Bearing in mind the little story about Neptune's discovery, told at the chapter's beginning, it may not surprise that the first result on mathematically rigorously described inverse problems was obtained by the Armenian mathematician and astronomer V. A. Ambartsumian. He devoted his early works to investigations of the theory of eigenvalues of differential equations and in 1929 published the first result for the inverse Sturm-Liouville problem, consisting in the reconstruction of a potential $q \in L^2(0,\pi)$ such that

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0,\pi], \qquad y'(0) = y'(\pi) = 0.$$

If q is identical to zero almost everywhere on $(0,\pi)$, the eigenvalues λ_n are of the form n^2, $n \geqslant 0$. But due to Ambartsumian's theorem the converse also holds (see, e.g., [Kab12, Theorem 6.2.1]). In 1946 the Swedish mathematician Borg showed that in general a single spectrum is not sufficient to determine the potential uniquely, but that uniqueness holds if an additional set of eigenvalues corresponding to a different boundary condition is given (see, e.g., [Kab12, Theorem 6.2.2]). We will see this kind of uniqueness result in Theorem 2.13 in Section 2.4. However, it was not until Levinson simplified Borg's proof in 1949, thus many years after the first results of Ambartsumian, that the importance of his discovery to scattering theory was realized (see, e.g., Sections 6.4 and 6.5 in the book of Kabanikhin [Kab12]).

From the applications' point of view the significance of inverse scattering problems was quickly realized during World War II, when radar and sonar were invented, trying to determine the distance of an object by the use of electromagnetic and acoustic waves. But whereas determining the location of a target is kind of straightforward, the problem of identification is ill-posed. As indicated above, that is, the solution of the modeled problem does not depend continuously on the measured data and is therefore way more difficult to handle. Thus, it was not until the Russian mathematician A. N. Tikhonov and his Western pendant K. Miller established their mathematical theories on solving ill-posed problems in the 1970s, that prepared the ground for the invention of synthetic aperture radar (SAR), marking the first successful application in object identification using electromagnetic waves [Col03]. Subsequently theoretical and numerical improvements consolidated the position of inverse scattering in physics and mechanics, like, e.g., acoustics, spectroscopy, geophysics, oceanography, and biomedical engineering. Therefore, the theory of inverse scattering has become an inherent part of daily life, for example electrical impedance tomography (EIT) or ultrasonics. Consequently, inverse scattering problems have become the most popular and well-studied amongst ill-posed problems (for a survey introduction see, e.g., [CK13]).

AT PRESENT This thesis is motivated by establishing reconstruction schemes for inverse scattering problems at so-called sparse media, roughly speaking relatively small objects that is. The required mathematical ideas are inspired by known techniques for solving inverse problems with sparsity constraints and for this reason base on regularization methods, which is primarily the minimization of Tikhonov functionals as briefly sketched in Section 2.1.

To acquaint us with the concept of sparsity, we have a more detailed glance at this terminology in Section 2.2. Hence, we see how regularization methods takes sparsity into account and why Banach spaces, especially L^p-spaces, play a major role. Initially, the presented ideas are influenced by linear inverse problems of image processing and became most famous by the work of Daubechies, Defrise, and De Mol [DDD04], inspiring a whole community in a short time. It consequently entailed a bunch of expanding results [BT09; LST11], as well as extensions to non-linearity [GHS08] and varied penalties [BB09; RZ09] as for discrepancies in Banach spaces [Bon+08; Sch+12]. Note that the mentioned references are just few examples out of a well-filled pool of publications, in which especially vast results relate to topics of image processing, e.g., [Sch+09; Str14]. Nonetheless the field of applications expands, see, e.g., [KSS09; Dah+12; JM12a]. For example an extensive survey on parameter identification problems concerning partial differential equations is provided by Jin and Maass [JM12b].

However, sparse regularization techniques in inverse scattering theory are highly underrepresented. Thus inspired, Lechleiter, Kazimierski, and Karamehmedović [LKK13] presented so-called Tikhonov and soft-shrinkage regularization methods for non-linear inverse medium scattering problems with sparsity-promoting penalty terms. The motivation of their work is justified by the assumption that the contrast of the medium is supported within a small region. As mentioned, modeling the problem in Banach spaces is a crucial basis for the use of sparsity regularization schemes. Since the analyzed scattering problems are traditionally given in a Hilbert space setting, reformulating them in an L^p-space setting becomes a key point. To achieve that, we intend in this thesis to transfer ideas of [LKK13] to different models of inverse scattering problems. That's why we see a sketched outline of their results in Section 2.3. In that context, the uniqueness result presented in Section 2.4 can be seen as an addendum, which overcomes a limitation in [LKK13], originating in the authors passion for simplicity.

In the subsequent chapters we have a glance at different scattering problems, which we are going to transfer in a Banach space setting. To begin with, we study the scattering of acoustic waves from anisotropic penetrable sparse media with matrix-valued material parameter in Chapter 3. In fact, these results are already published in [LR17].

In Chapter 4 we likewise handle inhomogeneous anisotropic objects with matrix-valued material parameter, but this time illuminated by electromagnetic waves. For this purpose we assume that the scatterer is non-magnetic, since this avoids technical complications occurring by direct transference of the beforehand derived methods. Once more these results are submitted for publication in [Ren16]. The mentioned difficulties arising from magnetic media are thereafter outlined in Chapter 5, where we show similarities and differences between the two cases. A slightly different idea is finally presented in Chapter 6, in which the problem consists of reconstructing a scalar-valued contrast function from observations, only measured in the incident direction. This model is commonly known as backscattering and enables us to rely more on the ideas of [LKK13] than the settings handled before.

NUMERICAL ANALYSIS Similar to the work of Lechleiter, Kazimierski, and Karamehmedović [LKK13] this thesis is heavily based on the theoretical aspects of the studied problems. Consequently, no proper numerical analysis of the reconstruction methods used in Section 3.7 is given.

Therefore, we refer to the work of Bürgel, Kazimierski, and Lechleiter [BKL17], in which the authors present a computational framework for the scalar inverse medium scattering problem, outlined in Section 2.3. Basically, they transfer the problem of reconstruction into a minimization problem for Tikhonov functionals, cf. (2.3). Additionally, different kinds of penalty terms are supported, like sparsity or total variation. Matching our principle approach, the used algorithm exploits Fréchet differentiability of the forward operator. Unfortunately, handling multiple penalties goes beyond the capabilities of iterated shrinkage schemes, used in [LKK13] as well as in Section 3.7. For that reason, the authors adjust the concept of primal-dual algorithms to their setting. Kindly, all numerical examples of that paper were made available in a kind of Matlab® toolbox, containing the solver for the direct scattering problem as well as the framework for both inversion methods, i.e., the primal-dual algorithm and the soft-shrinkage iteration.

Because of that, to put it crudely, the numerics shown in Section 3.7 rely on that framework, which we adapted to our problem. Therefore, in principle, two changes had to be made: first, the solver of the direct problem was extended as it is originally limited to the scalar problem of [LKK13]. Second, the adjoint of the forward operator's linearization needed to be calculated and implemented, see Section 3.8.

Thus, in Section 3.7 itself, there is a description of how the algorithms are used explicitly for the problem handled in Chapter 3. However, for a detailed portrayal of the discretized framework we refer to [BKL17].

NEW CONTRIBUTIONS As mentioned beforehand, the handled scattering problems are traditionally given in Hilbert space settings, whereas a Banach space setting would be appropriate. Hence, at first, we state the scattering models in an L^p-setting for the acoustic cases, or at least L^∞ for the electromagnetic cases. As this results in weak formulations, we gain corresponding solution operators for which we prove continuity and differentiability results. Afterwards, to model the inverse problem, we construct forward operators from the solution operators and transfer the beforehand shown properties. This allows us to show that assumptions of already existing Tikhonov regularization results are satisfied. Relying on those, we construct sparsity-promoting penalties via wavelets or total variation approaches to gain statements on sparsity regularization.

By and large, this procedure is done for anisotropic acoustic and electromagnetic scattering problems. Only the case of acoustic backscattering relies more on a volume potential approach as seen in [LKK13] due to the simpler Helmholtz model. However, the way of showing continuity and differentiability properties as well as regularization results orientates on the techniques established in the cases before.

Since numerical calculations exploit the computational framework for scalar acoustic scattering problems from [BKL17], at least adjustments of the solver of the direct problem as well as of the forward operator's linearization had to be made. By that, synthetic examples for scalar isotropic contrast functions complement the analysis of the anisotropic acoustic scattering problem. Note that as the framework does not handle electromagnetic problems properly, numerical examples for these cases remain.

In conclusion, by combining scattering theory with regularization techniques originally designed for image processing this thesis provides a first approach of how to reconstruct sparse media from noisy measurements concerning various scattering problems.

2 FUNDAMENTAL TERMS AND IDEAS

The mathematical ideas to establish regularization methods for inverse scattering problems used throughout this thesis are presented in this chapter. Readers familiar with regularization methods to solve inverse ill-posed problems and, probably, with the terminology of sparsity and how this affects the solution schemes, may skip Sections 2.1 and 2.2.

Since we are ultimately interested in non-linear inverse medium scattering problems, Section 2.3 provides an existing approach from Lechleiter, Kazimierski, and Karamehmedović [LKK13] handling a scalar acoustic scattering problem. Finally, results of their work is supplemented by showing a uniqueness result for a broader range of contrast functions in Section 2.4.

2.1 CONCEPTS OF SOLVING ILL-POSED PROBLEMS

By and large the theory of inversion methods provides diverse mathematical techniques to gain useful information of a problem from imperfect data, based on deductions from observations. In view of the impreciseness of the data, the techniques provide numerous solutions concerning the given uncertainties. Out of them the appropriate ones are chosen according to additional assumptions, like physical plausibility as for example in the algorithm of Bürgel, Kazimierski, and Lechleiter [BKL17]. The a-priori information used in this thesis will be clarified in Section 2.2.

One option to investigate inverse ill-posed problems bases on functional analysis, trying to make the problem well-posed by changing the space of the variables or its corresponding topology. Another approach takes the existing uncertainties into account by considering all variables to be random. Corresponding methods, for example Monte Carlo Methods, therefore aim for the related probability density function, see, e.g., the book of Tarantola [Tar05].

REGULARIZATION METHODS A third way to solve ill-posed problems follows the path paved by A. N. Tikhonov as well as his colleagues V. Badeva and V. A. Morozov. Throughout this thesis we will rely on these so-called regularization methods, which provide solutions approximating the exact solution (supposing it exists). Therefore, one constructs a framework concerning the additional a-priori information.

For a more detailed explanation we rely, to some extent, on the introduction of chapter 6 of the book on mathematical image processing of Bredies and Lorenz [BL11]. Therefore, remind that an inverse problem can be modeled by an operator equation, i.e., for given $y \in Y$ we seek the solution to $A\,x = y$ for a corresponding model $A\colon X \to Y$ and the Banach spaces X and Y. Note that A is often chosen to be an integral operator where its kernel is linked to the data. Because of perturbed measurements or miscellaneous errors in most applications we do not have access to exact data $y^\dagger$, since the given data is typically perturbed by some noise, which for example could be additive, multiplicative or Poisson (also called Salt-and-Pepper noise). Throughout this thesis we implicitly assume to have additive noise, i.e., we work with data $y^\delta = y^\dagger + \delta$, compromised by some noise level δ. Put crudely, the mathematical task of solving the inverse problem is to minimize the mismatch between noisy measurements y^δ and approximations of the exact solution $x^\dagger$ of $A\,x^\dagger = y^\dagger$. Thus, solving an inverse problem becomes a little bit tricky.

To obtain such "minimizing approximations" we would like to be able to distinguish between solutions and noise using their local properties. For instance, one assumes that the searched-for solution $x^\dagger$ (perhaps a picture in image processing tasks or a contrast function in an inverse scattering problem) is a structured object, i.e., one can derive local characteristics of $x^\dagger$. In contrast we suppose that the noise does not have such a structure. To quantify how well these assumptions fit into the underlying model, we rely on two functionals. The obvious one is the discrepancy Φ, depending on the noise and indicating for every solution $x \in X$ the mismatch between y^δ and $A\,x$. Intuitively, its minimization implies better approximations of the exact solution. However, solely minimizing the discrepancy causes numerical instabilities, as the model does not favor unconstrained output (see, e.g., [DDD04, Section 1.2]). Thus, one has to regularize the inverse problem, i.e., avoiding instable results by incorporate a-priori knowledge about the solution. Therefore, one adds a so-called penalty term $\mathcal{R}$, a stabilizing functional rating how well a solution x satisfies the a-priori stated properties.

Both functionals are designed to have a decreasing behavior if their arguments match well with the stated properties and vice versa. Consequently, a quantification of fulfillment of the stated requirements for every x, and therefore implicitly for every noise level $\delta = y^\delta - y^\dagger$, is given by

$$\mathcal{J}_{\alpha,\delta}(x) := \Phi(A\,x, y^\delta) + \alpha\,\mathcal{R}(x). \tag{2.1}$$

The additional weight $\alpha > 0$ emphasizes how much the a-priori known properties of the solution are taken into account. Due to the behaviors of Φ and $\mathcal{R}$, we prefer the solution which minimizes the functional $\mathcal{J}_{\alpha,\delta}$. The existence and even the uniqueness of such a solution under appropriate assumptions can be guaranteed by a result from convex analysis (see, e.g., [BL11, Satz 6.31]):

THEOREM 2.1. *Let* X *be a reflexive Banach space and* $\mathcal{J}\colon X \to \mathbb{R} \cup \{\infty\}$ *a convex, lower semicontinuous, and coercive functional. Then there exists a solution in* X *of the minimization*

$$\operatorname*{argmin}_{x \in X} \mathcal{J}(x).$$

If $\mathcal{J}$ *is strictly convex, then the solution is unique.*

FORWARD OPERATOR Note that the functional $\mathcal{J}_{\alpha,\delta}$ in (2.1) depends on the model of the analyzed problem, therein portrayed by a linear map A. Unfortunately, a lot of inverse problems are not covered by such a mapping, since either the forward operator is simply non-linear or corresponding solutions has to be found in domains which structure conflicts with the linear setting. Even though the underlying problem is linear, the forward operator can be non-linear. This typically holds for parameter identification problems concerning partial differential equations, on which the regularization techniques for inverse scattering problems presented in this work rely. For that reason we now see in a rough outline, following an approach provided by Jin and Maass [JM12b], how such a forward operator can be achieved.

As we have seen, mathematical problems which are subjects of discussion in this thesis can be modeled by an operator equation. Thus, let $A(x)\colon Y \to Z$ be a differential operator depending on a parameter x. Therefore, one aims to determine x from measurements of y, satisfying

$$A(x)y = z \quad \text{on a domain } \Omega \tag{2.2}$$

and some appropriate boundary conditions on $\partial\Omega$. Note that we incorporate the right-hand side of the boundary condition with z, such that by sloppy notation we write down the differential equation as in (2.2). This differential equation is then said to be solved by the solution operator $L(x, z) = y$. If $A(x)$ is linear, then $y = L(x, z) = A(x)^{-1}z$.

Since $A(x)$ for a fixed parameter x maps a function y on the right-hand side z, there is the operator valued mapping

$$A\colon X \to \{Y \to Z\}, \quad x \mapsto A(x).$$

We emphasize that the choices of the function spaces X, Y, and Z play a significant role in determining the analytic properties of $A(x)$ and of the associated parameter-to-state map, given for fixed z by

$$\mathcal{F} := F(\cdot)z\colon X \to Y, \quad x \mapsto L(x, z) = y.$$

Note that $\mathcal{F}$ is non-linear even for linear differential operators $A(x)$. Throughout this thesis $\mathcal{F}$, or rather $F(\cdot)z$, will mainly be called forward operator and is used to model the inverse problem.

Consequently, the analysis of the presented inverse problems rests on the analysis of the corresponding forward operators. Because of their construction, the analytic discussion of the solution operators will be the basis for all presented examples.

TIKHONOV REGULARIZATION Now we adjust the functional defined in (2.1), commonly known by the Russian mathematician A. N. Tikhonov, to the kind of beforehand derived non-linear forward operators

$$\mathcal{F}\colon \mathcal{D}(\mathcal{F}) \subseteq X \to Y$$

on infinite dimensional reflexive Banach spaces X and Y, such that $\mathcal{D}(\mathcal{F})$ is a convex and closed subset of X. Of course, also non-linear inverse problems can be ill-posed in a sense that is analogous to Hadamard's: violation of his third principle, solution's continuous dependence on data that is, transfers to so-called local ill-posedness, see, e.g., [Sch+12, Definition 3.15]. This means at a point $x_0 \in \mathcal{D}(\mathcal{F})$, such that $\mathcal{F}(x_0) = y$ for a y in the intersection of Y with the range of $\mathcal{F}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \overline{B}_\epsilon(x_0) \cap \mathcal{D}(\mathcal{F})$ for arbitrarily small radii $\epsilon > 0$, such that $\mathcal{F}(x_n) \to \mathcal{F}(x_0)$ in Y as $n \to \infty$, but $x_n \not\to x_0$ in X.

Note that we assume to have some data y^δ such that $\|y^\delta - y\|_Y \leqslant \delta$ for y in the range of $\mathcal{F}$ and a noise level δ. Because of that, we like to approximate the minimum of the set of solutions, that is, $\{x \in X,\, \mathcal{F}(x) = y\}$. Such minimum is subsequently referred to as penalty minimizing solution:

Definition 2.2 ($\mathcal{R}$-minimizing solution). We call $x^* \in \mathcal{D}(\mathcal{F}) \subseteq X$ a $\mathcal{R}$-minimizing solution if

$$\mathcal{F}(x^*) = y \quad \text{and} \quad \mathcal{R}(x^*) = \min\{\mathcal{R}(x),\ \mathcal{F}(x) = y,\ x \in \mathcal{D}(\mathcal{F})\}.$$

Further we assume that the discrepancy as well as the added convex penalty term are determined by Banach space norms, such that a regularized solution is given as the minimizer of

$$\operatorname*{argmin}_{x\in X} \mathcal{J}_{\alpha,\delta}(x) := \frac{1}{q}\|\mathcal{F}(x) - y^\delta\|_Y^q + \alpha\,\mathcal{R}(x), \tag{2.3}$$

where $\alpha > 0$ is called regularization parameter. If an estimate of the noise is known, α can be chosen appropriately [DDD04, Section 1.2]. That means it depends somehow on the noise level δ, such that the mismatch of approximations tends to zero for $\delta \to 0$.

One can show existence, stability, and convergence of regularized solutions of (2.3) under certain conditions, depending mostly on the forward operator, the function spaces, and the penalty term.

Theorem 2.3 (Existence). *Assume that the above introduced forward operator $\mathcal{F}\colon \mathcal{D}(\mathcal{F}) \subseteq X \to Y$ is continuous and weakly sequentially closed. Further assume that the penalty $\mathcal{R}\colon X \to \mathbb{R}_+$ is a proper convex functional* *with $\mathcal{D}(\mathcal{F}) \cap \mathcal{D}(\mathcal{R}) \neq \emptyset$ and is weakly lower semicontinuous with weakly sequentially pre-compact level sets. Then for all $\alpha > 0$ and $y^\delta \in Y$ there exists a minimizer of (2.3).*

THEOREM 2.4 (Stability). *Under the assumptions of Theorem 2.3, the minimizers of (2.3) are stable with respect to the data y^δ. That means for a data sequence $\{y_n^\delta\}_{n\in\mathbb{N}}$ converging to y^δ with respect to the norm-topology of Y, every corresponding sequence $\{x_n\}_{n\in\mathbb{N}}$ of minimizers of (2.3), has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges in the weak topology of X to a minimizer x of (2.3). In addition, $\mathcal{R}(x_{n_k}) \to \mathcal{R}(x)$ as $k \to \infty$.*

THEOREM 2.5 (Convergence). *Under the assumptions of Theorem 2.3, assume that a $\mathcal{R}$-minimizing solution exists. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a non-negative sequence of noise levels decreasing to zero and assume that $\{y^{\delta_n}\}_{n\in\mathbb{N}} \subset Y$ satisfies $\|y - y^{\delta_n}\| \leqslant \delta_n$. For an a-priori parameter choice $\alpha_n := \alpha(\delta_n) > 0$ such that*

$$\alpha_n \to 0 \quad \text{and} \quad \frac{\delta_n^q}{\alpha_n} \quad \text{as} \quad \delta_n \to 0.$$

Then any sequence of minimizers x_n of $\mathcal{J}_{\alpha_n,\delta_n}$ has a weakly convergent subsequence, converging to $\mathcal{R}$-minimizing solutions.

Remark 2.6. The same or similar results can be found proved in different textbooks or articles, see, e.g., Propositions 4.1 and 4.2 as well as Corollary 4.6 in the textbook of Schuster et al. [Sch+12], or rather Theorems 3.1 and 3.3 in the survey of Jin and Maass [JM12b]. Sometimes assumptions vary a little, for example the weakly sequentially closedness of $\mathcal{F}$ can be replaced by stronger conditions, e.g., by weak-weak continuity. Throughout this thesis we profit from that variability according to the actual problem. Thus, versions appropriate to treated problems are referenced at the corresponding parts.

2.2 TERMINOLOGY OF SPARSE SCATTERING OBJECTS

Relying on the minimization of a Tikhonov functional $\mathcal{J}$ as in (2.3) to solve a corresponding inverse problem, one classically assumes that the quantification of the noise δ can be measured in the L^2-norm (see, e.g., [BDV78]). Thus, if δ is a function on the domain Ω, one has $Y = L^2(\Omega)$. In addition, X is typically supposed to be a subspace of a Hilbert space. Hence, $\mathcal{R}$ is chosen to be the pth power of the Hilbert space norm. Obviously, one is tempted to exploit the provided scalar products and, consequently, chooses both norms to be squared, that is $p = q = 2$ [Sch+09, Section 3.1]. Of course this is not always an appropriate choice, since such penalties result in overly smoothed minimizers. For example, methods in image processing fall back on squared H^1-seminorms, using weak gradients to penalize the pointwise noise (see, e.g., [BL11, Beispiel 6.1]). Unfortunately, denoising is then coupled with a loss of sharpness as this minimization scheme is similar to a linear filter. Because of that another approach "takes away" the squares [Osh+05; CKP98; AV94]. This results in the seminorm corresponding to the space of functions with bounded variation, simply known as total variation; we refer to Definition 4.17 for a rigorous definition. For a proper introduction to so-called TV-regularization methods see, e.g., the survey of Burger and Osher [BO13].

SPARSITY In fact there are a lot of applications preferring highly localized approximations of their solution. Therefore, for example, techniques of inverse parameter identification problems in material mechanics are proposed for non-destructive material testing. Since this includes characterizing the intactness of a machine-made component, it is, roughly speaking, about detection of non-obvious cracks, bumps or the like. It is further about identification of singularities and physical material parameters.

Consequently, one can legitimately assume that the samples tested contain objects, e.g., the cracks, which are very small compared to its surrounding media, in fact the manufactured item. Mathematically, the contrast of such an object is described by few non-zero coefficients for a chosen basis and is commonly called sparse, see, e.g., [DDD04; JM12b]: Assume that $\{\varphi_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis in X. If the searched-for parameter $x^\dagger$ is supposed to be sparse, then it can be represented by a finite number of basis functions $\{\varphi_i\}_{i\in I}$ with finite index set I. One distinguishes then between exact sparsity, that is $x^\dagger = \sum_{i\in I} x_i\varphi_i$, and approximated sparsity, which means for a predefined tolerance that

$$\|x^\dagger - \sum_{i\in I} x_i\varphi_i\|_X \leqslant \text{TOL}.$$

Throughout this thesis, our numerical examples rely on synthetic objects, which are supported within a small subdomain of a known search domain. Examples of such constructed sparse scatterers are shown in Figure 2.1. We emphasize that although the examples only show objects with the same properties at every point, it is admissible and highly reasonable to suppose structures with irregularities, i.e., inhomogeneous media.

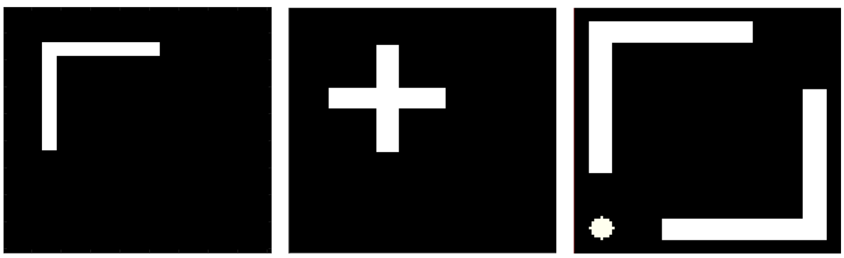

Figure 2.1: Examples of synthetic localized structures.

ADAPTING PENALTIES In this thesis we rely on the concept of Tikhonov regularization, which means minimizing a functional consisting of the discrepancy and an added penalty term, as seen in Section 2.1. For our purpose the penalty is supposed to incorporate a sparse representation of the solution. The perhaps most famous results for solving (linear) inverse problems with sparsity constraints were given by Daubechies, Defrise, and De Mol [DDD04].

As mentioned, the concept of sparsity, and, therefore, the choice of the penalty term, is linked to the chosen orthonormal basis. For example the object is assumed to be sparse in the Fourier domain, i.e., $x^\dagger$ has only a few nonzero Fourier components. Consequently, one would choose a Fourier basis. Another example requires singular value decomposition (SVD) expansions, e.g., if operators can be diagonalized in an analytic way or fairly small-scale problems are to be solved. Sometimes significant aspects of $x^\dagger$ correspond to the smallest singular values. This can be problematic since the penalization in linear regularization methods discards those values. In such cases it may be useful to replace basis functions by singular vectors [DDD04].

In some instances objects are sparse in the original domain, which also holds for the scattering problems handled in this thesis. Such objects are modeled in a pixel basis, the characteristic functions of pixels that is. Inverse problems discretized in a pixel space then are regularized using L^p-norms (or rather ℓ^p-norms in a finite-dimensional situation) as penalty, $1 \leqslant p \leqslant 2$. In general one uses weighted ℓ^p-norms. More precisely, for an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ and a sequence of strictly positive weights $\omega = \{\omega_n\}_{n\in\mathbb{N}}$, the penalty term is defined as

$$\mathcal{R}_p(x) \coloneqq \sum_{n\in\mathbb{N}} \omega_n |\langle x, \varphi_n\rangle|^p. \tag{2.4}$$

Note that this corresponds to the penalty in (2.3) for a Banach space X equipped with the weighted ℓ^p-norm.

Now let the weights equal a fixed constant and observe the penalties $\mathcal{R}_p$ with p decreasing from 2 to 1. Notice that while the exponent decreases, parameters x with “small” coefficients for which $|\langle x, \varphi_n\rangle| < 1$ are more penalized than parameters with “large coefficients”, i.e., with $|\langle x, \varphi_n\rangle| > 1$. In other words, in a range of small values the quadratic function is smaller than the absolute value function (Figure 2.2 provides a plot of both functions). So to say, for ℓ^1 there is less penalty on functions with large but few components (with respect to the chosen basis), whereas ℓ^2 favors sums of many small components. Further remember that results on Tikhonov regularization in Section 2.1 base on convex functionals $\mathcal{J}$. Since ℓ^p is strictly convex for $1 < p < \infty$ and at least convex for $p = 1$, one restricts themselves mostly to $p \geqslant 1$, although some applications rely on exponents p between zero and one. Therefore, $p \in [1, 2)$ promotes sparsity properties of $x^\dagger$ with respect to the basis functions.

Finally we emphasize that Tikhonov regularization with ℓ^1-penalty yields sparse minimizers, although the true solution is not sparse, see, e.g., [Sch+12, Section 1.5]. Thus, a-priori knowledge of the searched-for object becomes crucial.

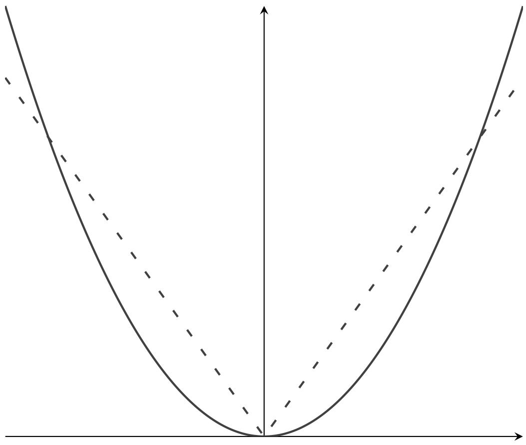

Figure 2.2: Plot of a quadratic function (solid line) as well as the absolute value function (dashed line).

2.3 REGULARIZATION TECHNIQUES PROVIDING SPARSITY FOR INVERSE MEDIUM SCATTERING

As mentioned before, Tikhonov regularization schemes with sparsity-promoting penalties found their way into the theory of inverse problems due to image processing tasks. After the community drew its attention for those techniques to parameter identification problems, Lechleiter, Kazimierski, and Karamehmedović aimed to shift the focus on inverse medium scattering problems, that is to reconstruct the (sparse) contrast of the medium from measurements of scattered waves.

SCATTERING PROBLEM Roughly speaking, in [LKK13] Lechleiter, Kazimierski, and Karamehmedović assume to have an incident wave u^i with time-dependence $\exp(-i\omega t)$ for the frequency ω. Such time-harmonic field is a solution to the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ for the positive wave number $k = \omega / c_0$, where c_0 denotes the speed of sound. Now they consider that u^i illuminates an inhomogeneous medium, given by a bounded and open set $D \subset \mathbb{R}^d$, $d = 2, 3$, which physical properties are modeled by a contrast function $q \colon \mathbb{R}^d \to \mathbb{C}$, supported in $\overline{D}$. The thereby arising scattered field u^s is described by Sommerfeld's radiation condition,

$$\lim_{|x| \to \infty} |x|^{(d-1)/2} \left(\frac{\partial}{\partial |x|} - ik \right) u^s(x) = 0$$

uniformly in all directions $\hat{x} = x/|x|$, and is therefore called radiating. Consequently, the total field $u = u^i + u^s$ satisfies

$$\Delta u + k^2(1+q)u = 0 \quad \text{in } \mathbb{R}^d.$$

Likewise, the scattered field also solves an Helmholtz equation. For a schematic illustration see Figure 2.3.

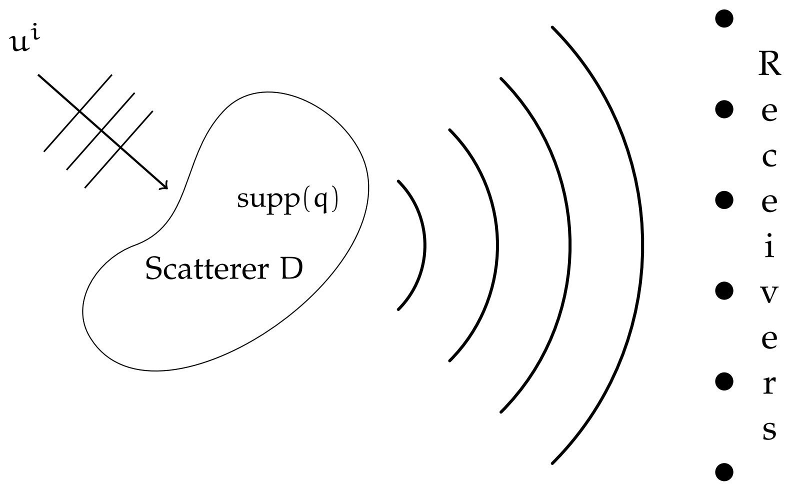

Figure 2.3: Schematic illustration of the scattering model.

However, the authors reformulate u^s as a solution to the so-called Lippmann-Schwinger integral equation

$$u^s - k^2 V(qu^s) = k^2 V(qu^i) \quad \text{in } D,$$

via radiating volume potentials $V(f)(x) := \int_D \Phi(x-y)f(y)\,dy$ for $x \in \mathbb{R}^d$. The herein used function Φ denotes the radiating fundamental solution of the Helmholtz equation, given by

$$\Phi(x) := \begin{cases} \frac{i}{4} H_0^{(1)}(k|x|), & \text{if } d = 2, \\ \frac{\exp(ik|x|)}{4\pi|x|}, & \text{if } d = 3, \end{cases} \qquad x \neq 0. \tag{2.5}$$

Remember that in Section 2.2 we have seen that penalties based on L^p-norms for small p are appropriate to respect the sparsity property, as, roughly speaking, small values are strongly penalized. Consequently, Lechleiter, Kazimierski, and Karamehmedović base their choice of penalties on the analysis of scattering theory for time-harmonic waves with scalar valued refractive indices in L^p, $p \in (1, \infty)$. Note that the restriction of p distinct from one or infinity guarantees to work with reflexive function spaces. Because of that they derive compactness properties for $u \mapsto V(qu)$. We remark that Chapter 6 heavily relies on these properties. Although the way of working presented in the other parts of this thesis differs in this aspect, we will use volume potential representations several times. Afterwards they show uniqueness of L^p-solutions to the inverse problem under appropriate assumptions.

INVERSE PROBLEM To model the inverse problem as an operator equation, the authors construct the forward operator $\mathcal{F}$ as a so-called contrast-to-measurement operator. If $q^\dagger$ is the searched-for exact contrast, the inverse problem hence becomes to determine q such that

$$\mathcal{F}(q) = F^\delta_{\text{meas}}, \tag{2.6}$$

for noisy measurements F^δ_{meas}, that is $\|\mathcal{F}(q^\dagger) - F^\delta_{\text{meas}}\| \leqslant \delta$. According to the outlined properties of inverse problems, this equation is locally ill-posed about any $q^\dagger$. Note that the results in [LKK13] use near field data, but that there is no restriction for far field measurements. To apply standard Tikhonov regularization results, see Section 2.1, they show continuity, compactness and weakly lower semicontinuity in appropriate function spaces for the contrast-to-measurement operator. Due to the volume potential-based solution theory, this is done via the collective compactness of a certain family of integral operators, which seems impossible for most of our approaches, except the one outlined in Chapter 6. Finally they have all the proper ingredients to show that approximative solutions of the regularization scheme converge to the unique exact solution of the stated operator equation (2.6).

EXPLANATORY NOTE The statement of uniqueness of solution to the operator equation (2.6) given in [LKK13, Theorem 11], is in the case of dimension three restricted to contrast functions in L^p, $p > 3$. This limitation arises from the uniqueness result for the contrast function the authors use. They admit that this result is not optimal, but was chosen for its elementary proof. Sharing their opinion that an elementary argumentation optimized in p has its own interest, we provide such a proof in the following Section 2.4. Therefore, the statement also holds for functions in L^p, $p > 3/2$.

We remark that throughout this thesis, we will consider complex-valued material parameters and at least in the case of anisotropic acoustic scattering we also consider Banach spaces for the image space of the contrast-to-measurement operator. Although we handle slightly different problem settings, we thereby try to cover outstanding issues of [LKK13].

In closing we like to remark that their paper due to its immense appendix may also be attractive for readers interested in results on Sobolev embeddings, collectively compact operator theory, Hilbert-Schmidt operators, and non-linear Tikhonov regularization.

2.4 UNIQUE DETERMINATION OF CONTRASTS

In Section 2.3 we have already outlined the work of Lechleiter, Kazimierski, and Karamehmedović [LKK13]. Therein, in detail, the support of the contrast function q modeling the scattering object is assumed to be in a ball of radius R, i.e., $\overline{D} \coloneqq \operatorname{supp}(q) \subset B_R$, and, consequently, volume potentials $V(f)(x) = \int_{B_R} \Phi(x-y)f(y)\,dy$ for $x \in \mathbb{R}^d$ and the fundamental solution Φ of (2.5) play a major role. This is because the arising scattered field can be found as a solution to the Lippmann-Schwinger integral equation

$$\left(\mathrm{Id} - k^2 V(q\,\cdot)\right) u^s = k^2 V(qu^i) \quad \text{in } B_R. \tag{2.7}$$

In an additional note we explained that in dimension three the originally given statement relating to uniqueness of solution of the operator equation (2.6) can be extended by concerning a uniqueness result for contrast functions $q \in L^p(B_R)$ for $p > 3/2$ instead of $p > 3$. To prove this statement, we reproduce in detail the appropriate assumptions for which uniqueness of solution to the Lippmann-Schwinger equation, respectively the Helmholtz equation for the scattered field, was shown in [LKK13].

Assumption 2.7. Let $p > 3/2 \geqslant 1$ and choose

$$t > \max\left\{\frac{p}{p-1}, \frac{6}{5}\right\},$$

which in fact guarantees $t > 1$. Compactness results, e.g., shown in [LKK13, Proposition 2], thus imply that the Lippmann-Schwinger integral equation (2.7) is well-defined in $L^t(B_R)$. Defining additionally a number $\overline{p}$ such that

$$\overline{p} \coloneqq \frac{tp}{t+p} > \frac{6}{5},$$

enables to apply a unique continuation property of L^p-solutions to the Helmholtz equation, see, e.g., [LKK13, Lemma 3].

The authors showed that under this assumptions the Lippmann-Schwinger equation has a unique solution in $L^t(B_R)$, which norm is bounded by the norm of the (arbitrary) right-hand side. Thus, for an incident field $u^i \in L^t(B_R)$, the Lippmann-Schwinger equation (2.7) defines a radiating solution $u^s \in W^{2,\overline{p}}_{\mathrm{loc}}(\mathbb{R}^3)$ of the Helmholtz equation

$$\Delta u^s + k^2(1+q)u^s = -k^2 q u^i \quad \text{in } L^t_{\mathrm{loc}}(\mathbb{R}^3). \tag{2.8}$$

Since u^s can be found as a solution to (2.7), we denote for convenience

$$T_q \coloneqq \left(\mathrm{Id} - k^2 V(q\cdot)\right)^{-1}, \quad q \in L^p_{\mathrm{Im}\geqslant 0}(B_R),$$

which is a bounded operator on $L^t(B_R)$ due to Assumption 2.7.

Remark 2.8. For $t \geqslant 2$ the operator $T_q \colon L^t(B_R) \to L^t(B_R) \hookrightarrow L^2(B_R)$ is continuous, such that also its adjoint T_q^* is a continuous function from $L^2(B_R)$ to $L^{t'}(B_R)$ for $t' \leqslant 2$.

SOBOLEV EMBEDDINGS Under Assumption 2.7 the authors have shown that the scattered field is a radiating solution of the Helmholtz equation (2.8). We now specify $u_q^s(\cdot,\theta) \in W^{2,\overline{p}}_{loc}(\mathbb{R}^3)$ to be the unique weak solution of the corresponding scattering problem for the incident field $u^i(\cdot,\theta)$ from directions $\theta \in \mathbb{S}^2$. Therefore, we are going to state some embedding results which indicate that the integrals arising from equations given in the weak sense are well defined.

Denote by p' the conjugated exponent of p, such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then in regard to Remark 2.8 it seems beneficial to aim for $t \geqslant 2$ as $t > \max\{p', 6/5\}$ by Assumption 2.7. Thus, $t = 2p'$ is a reasonable choice, since then

$$\overline{p} = \frac{tp}{t+p} = \left(\frac{1}{p} + \frac{1}{2p'}\right)^{-1} = \left(1 - \frac{1}{2p'}\right)^{-1} = \frac{2p'}{2p'-1} = (2p')'.$$

By that, one shows that $\overline{p}' = 2p'$, where $\overline{p}'$ denotes the conjugated exponent of $\overline{p}$, that is $\frac{1}{\overline{p}} + \frac{1}{\overline{p}'} = 1$. Now $p > 3/2$ by Assumption 2.7 implies that $p' < 3$, which is equivalent to $2p' < 6$ such that $(2p')' = \overline{p} > 6/5$. Consequently, $\overline{p}, \overline{p}' \in (6/5, 6)$.

Assume that $\overline{p}' \geqslant 3$, which implies $\overline{p} > 3/2$. Then Sobolev embedding results, see, e.g., [AF03, Theorem 6.3], state that $W^{2,\overline{p}}(B_R) \hookrightarrow L^r(B_R)$ for all $r \in [1,\infty)$. Otherwise if $\overline{p}' < 3$, then $W^{2,\overline{p}}(B_R) \hookrightarrow L^r(B_R)$ for all $r \in [1, \frac{3\overline{p}}{3-2\overline{p}})$. Note that we can choose $r = \overline{p}'$ if $\frac{3\overline{p}}{3-2\overline{p}} > \overline{p}'$. Fortunately this holds if and only if $3\overline{p}^2 - 3\overline{p} > 3\overline{p} - 2\overline{p}^2$, that is $\overline{p} > 6/5$, which is true as we have seen. However, we can always find a compact embedding of $W^{2,\overline{p}}(B_R)$ into $L^{\overline{p}'}(B_R)$.

Further Sobolev embeddings also show that $W^{1,\overline{p}}(B_R) \hookrightarrow L^r(B_R)$ for all $r \in [1, \frac{3\overline{p}}{3-\overline{p}})$. Since $\overline{p} \mapsto \frac{3\overline{p}}{3-\overline{p}}$ is strictly monotonically increasing for $\overline{p} \in (6/5, 3)$, we have that

$$\frac{3\overline{p}}{3-\overline{p}} > \frac{3\frac{6}{5}}{3-\frac{6}{5}} = \frac{18/5}{(15-6)/5} = \frac{18}{9} = 2,$$

which means that $W^{1,\overline{p}}(B_R) \hookrightarrow L^2(B_R)$.

PREPARATORY WORK The aim of this chapter is to show that the knowledge of the far field pattern $u_q^\infty(\hat{x},\theta)$, corresponding to an incident field from direction $\theta \in \mathbb{S}^2$ and measured in direction $\hat{x} \in \mathbb{S}^2$, provides sufficient information to determine the contrast function uniquely. Therefore we assume to have two functions $q_1, q_2 \in L^p(B_R)$, $p > 3/2$, such that their corresponding far field patterns coincide. Then we show that the contrast functions are equal to each other. Note that for this approach we heavily rely on an analogous proof for the refractive index $n = \sqrt{1+q} \in L^\infty(\mathbb{R}^3)$ given by Kirsch [Kir11, Section 6.4], which originally rest upon different works from A. Ramm, R. Novikov, and A. Nachman from the 1980s. The proof is subdivided into three lemmata which we enroll in the following.

At first we show, roughly speaking, that for a fixed contrast function the span of all total fields corresponding to scattering problems arising from incident plane waves are dense in the space of solutions to the Helmholtz equation.

LEMMA 2.9 (Denseness). *Let* $q \in L^p(B_R)$ *such that* $q(x) = 0$ *for all* $x \notin B_R$ *and let* $u_q(\cdot, \theta) \in W^{2,\overline{p}}_{loc}(\mathbb{R}^3)$ *be the total field corresponding to the incident field* $u^i(x, \theta) = e^{ik\theta\cdot x}$. *Then the linear span*

$$U := \operatorname{span}\left\{u_q(\cdot,\theta)\colon B_R \to \mathbb{C},\ u_q(\cdot,\theta) = u^i(\cdot,\theta) + u^s_q(\cdot,\theta) \text{ for } \theta \in \mathbb{S}^2\right\},$$

$U \subset W^{2,\overline{p}}(B_R)$, *is dense in the* $L^{\overline{p}'}(B_R)$*-closure of*

$$H := \left\{v \in W^{2,\overline{p}}(B_R)\,, \int_{B_R} \left[\nabla v \cdot \nabla \psi - k^2(1+q)v\psi\right] dx = 0 \right.$$
$$\left. \text{for all } \psi \in W^{2,\overline{p}'}(B_R)\right\} \subset L^{\overline{p}'}(B_R). \tag{2.9}$$

Proof. To handle functions of $L^{\overline{p}}(B_R)$ and $L^{\overline{p}'}(B_R)$, we extend the $L^2(B_R)$-scalar product, $(\cdot,\cdot)_{L^2}$, to the dual product between $L^p(B_R)$ and $L^{p'}(B_R)$ for $1 < p < \infty$. We thus assume that the closure of U in $L^{\overline{p}'}$ is a proper subset of H, i.e., that there exists a function $f \in L^{\overline{p}}(B_R)$ such that

$$(f, u(\cdot,\theta)) = \int_{B_R} f(x)\overline{u(x,\theta)}\, dx = 0 \quad \text{for all } \theta \in \mathbb{S}^2$$

and further there is another function $\tilde{v} \in H \setminus U$ such that $(f, \tilde{v}) > 0$.

By definition of $T_q = (\operatorname{Id} - k^2 V(q\,\cdot\,))^{-1}$ its adjoint is

$$T_q^* = (\operatorname{Id} - [k^2 V(q\cdot)]^*)^{-1} = (\operatorname{Id} - k^2\overline{q}\overline{V(\overline{\cdot})})^{-1}.$$

Due to that one claims, that

$$0 = (f, u(\cdot,\theta)) = (f, T_q(u^i(\cdot,\theta))) = (T_q^*(f), u^i(\cdot,\theta)).$$

Note that the potential $V(\overline{f}) \in W^{2,\overline{p}}(B_R)$ solves $(\Delta + k^2)\, V(\overline{f}) = -\overline{f}$ in $L^{\overline{p}}(B_R)$, i.e.,

$$\int_{B_R} \left[\nabla V(\overline{f}) \cdot \nabla\overline{\psi} - k^2 V(\overline{f})\overline{\psi}\right] dx = \int_{B_R} \overline{f\psi}\, dx \quad \text{for all } \psi \in W^{2,\overline{p}'}_0(B_R). \tag{2.10}$$

Consequently, $w := T_q^*(f) \in L^{\overline{p}}(B_R)$ satisfies

$$f(x) = w(x) - k^2\overline{q}(x) \int_{B_R} \overline{\Phi(x,y)} w(y)\, dy, \quad x \in B_R.$$

Note that $g := \overline{V(\overline{w})} = \int_{B_R} \overline{\Phi(x,y)} w(y) dy$ in $W^{2,\overline{p}}(B_R)$ with $L^{\overline{p}}$-density w. Therefore, $\overline{g}$ is a weak radiating solution of the Helmholtz equation $(\Delta + k^2)\overline{g} = -w$ in $L^{\overline{p}}(B_R)$. Observe that its far field pattern vanishes:

$$\begin{aligned}\overline{g}^{\infty}(-\theta) &= \frac{1}{4\pi}\int_{B_R} e^{ik\theta\cdot y}\overline{w}(y)\,dy \\ &= \frac{1}{4\pi}\int_{B_R}\overline{w(y)e^{-ik\theta\cdot y}}\,dy = \frac{1}{4\pi}\overline{(w, u^i(\cdot,\theta))} = 0.\end{aligned}$$

Now Rellich's lemma, see, e.g., [CK13, Lemma 2.12], implies that $g(x) = 0$ for all $x \notin B_R$, that means $\overline{g}, g \in W_0^{2,\overline{p}}(B_R)$.

Due to the definition of H, we have for all $v \in H$ that

$$\int_{B_R} \left[\nabla v \cdot \nabla\overline{\psi} - k^2 v\overline{\psi}\right] dx = k^2 \int_{B_R} qv\overline{\psi}\,dx \quad \text{for all } \psi \in W_0^{2,\overline{p}'}(B_R).$$

Choosing $\psi = g \in W_0^{2,\overline{p}}(B_R) \hookrightarrow L^{\overline{p}'}(B_R)$ we have also that

$$\int_{B_R} \left[\nabla v \cdot \nabla\overline{g} - k^2 v\overline{g}\right] dx = k^2 \int_{B_R} qv\overline{g}\,dx. \tag{2.11}$$

In addition the weak formulation corresponding to the Helmholtz equation solved by g, implies for $\psi = v \in H$ after complex conjugation that

$$\int_{B_R} \left[\nabla\overline{g} \cdot \nabla v - k^2\overline{g}v\right] dx = \int_{B_R} \overline{w}v\,dx. \tag{2.12}$$

Finally equations (2.11) and (2.12) show that for all $v \in H$ it holds that

$$0 = \int_{B_R} v(\overline{w} - k^2 q\overline{g})\,dx = \int_{B_R} \overline{v(w - k^2\overline{q}\overline{V(\overline{w})})}\,dx = \int_{B_R} v\overline{f}\,dx = \overline{(f,v)}.$$

In fact, this contradicts the assumption of existence of $v \in H$ such that $(f, v) > 0$. □

In the next lemma we prove some kind of orthogonality relations between solutions of the Helmholtz equation corresponding to different contrast functions q_1 and q_2.

LEMMA 2.10 (Orthogonality relations). *Let* $q_1, q_2 \in L^p(B_R)$, $p > 3/2$, *be two contrast functions such that* $q_1(x) = q_2(x) = 0$ *for all* $x \notin B_R$ *and assume that* $u^{\infty}_{q_1} = u^{\infty}_{q_2}$ *with respect to all directions. Then*

$$\int_{B_R} v_1 v_2 (q_1 - q_2)\,dx = 0$$

for all weak solutions $v_{1,2} \in W^{2,\overline{p}}(B_R)$ *of the Helmholtz equation* $\Delta v_{1,2} + k^2(1 + q_{1,2})v_{1,2} = 0$ *in* B_R.

Proof. Fix $v_1 \in W^{2,\overline{p}}(B_R)$ as a weak solution of $\Delta v_1 + k^2(1+q_1)v_1 = 0$ in B_R. Now we have a glance at the difference of the total fields $u := u_{q_1}(\cdot,\theta) - u_{q_2}(\cdot,\theta)$, which corresponds to the difference of the scattered fields $u^s_{q_1}$ and $u^s_{q_2}$ causing the far field patterns $u^\infty_{q_1}$ and $u^\infty_{q_2}$. Since $u^\infty_{q_1}(\cdot,\theta) = u^\infty_{q_2}(\cdot,\theta)$ by assumptions, Rellich's lemma, see, e.g., [CK13, Lemma 2.12], implies that u vanishes outside of B_R, that is $u \in W^{2,\overline{p}}_0(B_R)$. Further u satisfies the Helmholtz equation $\Delta u + k^2((1+q_1)u_{q_1} - (1+q_2)u_{q_2}) = 0$, i.e.,

$$\Delta u + k^2(1+q_1)u = k^2(q_2-q_1)u_{q_2} \quad \text{in } B_R$$

in the weak sense, that means for all $\psi \in W^{2,\overline{p}'}(B_R)$ it holds that

$$\int_{B_R} \left[\nabla u \cdot \nabla\overline{\psi} - k^2(1+q_1)u\overline{\psi}\right] dx = -k^2 \int_{B_R} (q_2-q_1)u_{q_2}\overline{\psi}\, dx.$$

Choosing $\overline{\psi} = v_1$ as test functions this yields

$$k^2 \int_{B_R} v_1 u_{q_2}(q_1-q_2)\, dx = \int_{B_R} \left[\nabla u \cdot \nabla v_1 - k^2(1+q_1)uv_1\right] dx,$$

where the right-hand side vanishes, since v_1 weakly solves

$$(\Delta + k^2(1+q_1))v_1 = 0 \quad \text{in } B_R,$$

as u is compactly supported in B_R, i.e., $u \in W^{2,\overline{p}}_0(B_R) \hookrightarrow L^{\overline{p}'}(B_R)$. Finally by the denseness result of Lemma 2.9 u_{q_2} approximates $v_2 \in W^{2,\overline{p}}(B_R)$ as it is a weak solution of the Helmholtz equation. □

Following the techniques of Serov [Ser12, Lemma 3], for contrast functions $q \in L^p(B_R)$ we construct distributional solutions $u_z \in L^{\overline{p}}(B_R)$ of the form $u_z(x) := u(x,z) = e^{ix\cdot z}(1+v_z(x))$, solving the Helmholtz equation $\Delta u_z + k^2(1+q)u_z = 0$ in $\mathbb{R}^3$ for $z \in \mathbb{C}^3$ such that $z \cdot z = 0$.

Lemma 2.11 (Distributional solutions). *For all $z \in \mathbb{C}^3$ such that $z \cdot z = 0$ and $|z|$ large enough, there exists a unique solution $v_z \in W^{2,\overline{p}}(B_R)$ of*

$$(\Delta + 2iz\cdot\nabla)v_z + k^2(1+q)(1+v_z) = 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.13}$$

Further v_z satisfies for $c > 0$ the inequality

$$\|v_z\|_{L^{\overline{p}'}(B_R)} \leqslant \frac{c}{|z|^\gamma},$$

where $\gamma := (1-2s)\theta$ for $\theta = 1 - \frac{3}{2p} > 0$ and $0 < s < \frac{1}{2}$. Consequently, there exists a solution $u_z \in W^{2,\overline{p}}(B_R)$ of the form $u_z(x) = e^{ix\cdot z}(1+v_z(x))$ solving

$$\Delta u_z + k^2(1+q)u_z = 0 \text{ in } \mathbb{R}^3.$$

Proof. We claim that for $z \in \mathbb{C}^3$ such that $z \cdot z = 0$ and for $v_z(x) := v(x,z)$, the searched-for distributional solutions $u_z(x) := u(x,z) = e^{ix\cdot z}(1 + v_z(x))$ solve the Helmholtz equation $\Delta u_z + k^2(1+q)u_z = 0$ in $\mathbb{R}^3$. Thus applying the general product rule for the Laplace operator yields

$$\begin{aligned}
0 &= \Delta u_z(x) + k^2(1+q(x))u_z(x) \\
&= \Delta\left(e^{ix\cdot z}(1+v_z(x))\right) + k^2(1+q(x))e^{ix\cdot z}(1+v_z(x)) \\
&= e^{ix\cdot z}\Delta(1+v_z(x)) + 2\nabla e^{ix\cdot z}\cdot\nabla(1+v_z(x)) + (1+v_z(x))\Delta e^{ix\cdot z} \\
&\qquad + k^2(1+q(x))e^{ix\cdot z}(1+v_z(x)) \\
&= e^{ix\cdot z}\Delta v_z(x) + 2ie^{ix\cdot z}z\cdot\nabla v_z(x) - z\cdot z e^{ix\cdot z}(1+v_z(x)) \\
&\qquad + k^2(1+q(x))e^{ix\cdot z}(1+v_z(x)).
\end{aligned}$$

This implies that v_z solves equation (2.13).

To show that v_z satisfies the stated inequality, we rely on the integral equation

$$\begin{aligned}
v(x) &= \int_{B_R} g_z(x-y)k^2(1+q(y))(1+v(y))\,dy \\
&= \left(g_z * k^2(1+q)(1+v)\right)(x),
\end{aligned} \tag{2.14}$$

for $q \in L^p(B_R)$ and the Green-Faddeev function $g_z(x)$, which is the fundamental solution of the following operator with constant coefficients, see, e.g., [Ser12, p.3]:

$$(\Delta + 2iz\cdot\nabla)\,g_z(x) = -\delta(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.15}$$

(Here, $\mathcal{D}'(\mathbb{R}^3)$ is the dual space of the space of test functions $\mathcal{D}(\mathbb{R}^3)$ on $\mathbb{R}^3$ and denotes the space of distributions on $\mathbb{R}^3$.) We further denote by $G_z(f)$ the convolution of g_z with f. Then using the embedding $v \in W^{2,\overline{p}}(B_R) \hookrightarrow L^{\overline{p}'}(B_R)$, we can rewrite equation (2.14) as an operator equation $A_z(v) = v$ in $L^{\overline{p}'}(B_R)$ with integral operator

$$A_z\colon f \mapsto G_z(k^2(1+q)(1+f)).$$

For $q \in L^p(B_R)$ the general Hölder inequality states that

$$\|qf\|_{\overline{p}} \leqslant \|q\|_p\|f\|_{\overline{p}'}.$$

Due to that, the convolution operator $G_z\colon L^{\overline{p}}(B_R) \to L^{\overline{p}'}(B_R)$ extends to $L^{\overline{p}'}(B_R)$ such that application of [LPS08, equation (15)] yields

$$\|G_z(q\cdot)\|_{L^{\overline{p}'}(B_R)\to L^{\overline{p}'}(B_R)} \leqslant c\|q\|_{L^p(B_R)}/|z|^\gamma.$$

If we now assume that $v \in B_\rho(0) = \{u \mid \|u\|_{L^{\overline{p}'}(B_R)} < \rho\} \subset L^{\overline{p}'}(\mathbb{R}^3)$ for arbitrary $\rho > 0$, a solution v of $A_z(v) = v$ satisfies

$$\|v\|_{L^{\overline{p}'}(B_R)} \leqslant \frac{ck^2}{|z|^\gamma}(1+\rho)\|1+q\|_{L^p(B_R)}. \tag{2.16}$$

Let further $v, w \in W^{2,\overline{p}}(B_R) \hookrightarrow L^{\overline{p}'}(B_R)$ and $|z|$ be large enough such that

$$\frac{ck^2}{|z|^\gamma}\|1+q\|_{L^p(B_R)} < 1.$$

Consequently, A is contractive, i.e.,

$$\|A(v) - A(w)\|_{L^{\overline{p}'}(B_R)} \leqslant \frac{ck^2}{|z|^\gamma}\|1+q\|_{L^p(B_R)}\|v-w\|_{L^{\overline{p}'}(B_R)},$$

such that there exists a unique solution v_z of $A_z(v) = v$ by the Banach fixed-point theorem. Note that $v_z(x) = v(x,z)$ satisfy the estimate (2.16) and solves the integral equation (2.14) as well as equation (2.13), since $g_z(x)$ is the fundamental solution of (2.15). □

UNIQUENESS OF INVERSE PROBLEM We now show that the set of products of two solutions of the Helmholtz equation is dense in $L^{p'}(B_R)$. This becomes the key factor of the aimed-for statement to uniquely determine the contrast function from given far field measurements.

THEOREM 2.12. *Let $q_1, q_2 \in L^p(B_R)$ be two contrast functions compactly supported in B_R. Then the span of the set of products of weak solutions of the Helmholtz equation, i.e.,*

$$\operatorname{span}\{u_1u_2,\ u_{1,2} \in W^{2,\overline{p}}(B_R) \textit{ weakly solve } \Delta u_{1,2}+k^2(1+q_{1,2})u_{1,2}=0\},$$

is dense in $L^{p'}(B_R)$.

Proof. Let $g \in L^p(B_R)$ such that $\int_{B_R} gu_1u_2\, dx = 0$ for all weak solutions $u_{1,2} \in W^{2,\overline{p}}(B_R)$ of the Helmholtz equation. Further fix an arbitrary vector $l \in \mathbb{R}^3 \setminus \{0\}$ such that for a $\hat{c} \in \mathbb{S}^2$ it holds that

$$\hat{c} \cdot l = 0.$$

Likewise choose $b \in \mathbb{R}^3$ as well as a positive number $\tau > 0$, such that

$$b \cdot l = 0, \quad b \cdot \hat{c} = 0 \ \text{ and } \ |b|^2 = |l|^2 + \tau^2.$$

Thus, $\{l, \hat{c}, b\}$ forms an orthogonal system in $\mathbb{R}^3$. By that, define $z_{1,2} \in \mathbb{C}^3$ in the sense that

$$z_1 := -\frac{l}{2} + \frac{\tau\hat{c}}{2} + \frac{i}{2}b, \quad z_2 := -\frac{l}{2} - \frac{\tau\hat{c}}{2} - \frac{i}{2}b.$$

Note that it holds that $z_{1,2} \cdot z_{1,2} = 0$, $|z_{1,2}| \geqslant \tau/2$ and $z_1 + z_2 = -l$. By Lemma 2.11 there exists $T > 0$ and $C > 0$ such that there are solutions $u_{z_{1,2}} \in W^{2,\overline{p}}(B_R)$ of the homogeneous Helmholtz equation in the form $u_{z_{1,2}} = e^{ix\cdot z_{1,2}}(1 + v_{z_{1,2}})$, where $\|v_{z_{1,2}}\|_{L^{\overline{p}'}(B_R)} \leqslant C/|z_{1,2}|^\gamma \leqslant 2C/\tau$ for all $\tau > T$.

Due to the choice of g, we thus have that

$$0 = \int_{B_R} g(x)u_{z_1}(x)u_{z_2}(x)\,dx = \int_{B_R} e^{i(z_1+z_2)\cdot x}(1+v_{z_1}(x))(1+v_{z_2}(x))g(x)\,dx.$$

As $\overline{p}' > 2 > \overline{p} > 1$, the product of the integrands is in $L^1(B_R)$ and the L^1-norm is uniformly bounded in $\tau > T$. Consequently, $v_{z_{1,2}}$ vanish for $\tau \to \infty$ after interchanging limit and integral. Thus it holds that

$$0 = \int_{B_R} g(x)e^{-il\cdot x}\,dx = \hat{g}(l),$$

where $\hat{g}$ denotes the Fourier transform of g. By the inversion theorem, see, e.g., [Rud91, 7.7(c)], the inverse Fourier transform

$$g_0(x) = \int_{B_R} \hat{g}(l)e^{il\cdot x}\,dl = 0, \quad x \in B_R$$

is identical to $g(x)$ for almost every $x \in B_R$. Therefore, also g vanishes almost everywhere. □

Now by Lemma 2.10 we have, almost as a corollary, the aimed-for uniqueness result.

THEOREM 2.13. *Let $q_1, q_2 \in L^p(B_R)$, $p > 3/2$, be two contrast functions such that $q_1(x) = q_2(x) = 0$ for all $x \notin B_R$. Further denote by $u^\infty_{q_1}$ and $u^\infty_{q_2}$ the corresponding far field patterns. If both patterns coincide, that is, $u^\infty_1(\hat{x};\theta) = u^\infty_2(\hat{x};\theta)$ for all $\hat{x}, \theta \in \mathbb{S}^2$, then $q_1 = q_2$.*

Proof. Lemma 2.10 states that for such $q_1, q_2 \in L^p(B_R)$ and all weak solutions of the Helmholtz equation it holds that

$$\int_{B_R} v_1 v_2 (q_1 - q_2)\,dx = 0.$$

Since we know from Theorem 2.12 that the set of products of such solutions is dense in $L^{p'}(B_R)$ and, consequently, $q_1 = q_2$. □

3 INVERSE ACOUSTIC SCATTERING FROM ANISOTROPIC PENETRABLE MEDIA

In this chapter we consider non-linear Tikhonov regularization and sparsity-promoting techniques in Banach spaces for inverse scattering from penetrable anisotropic media. Therefore, we recall in Section 3.2 well-known weak solution theory for the anisotropic Helmholtz equation (3.1) in the ball B_{2R} for an admissible set of material parameters with the L^p-topology. While the material parameters perform the searched-for contrast, we assume that its support is strictly included in B_R. Although this is not crucial, it avoids technicalities and enables us to directly rely on a specific bound in Meyers' seminal work [Mey63].

Using Meyers' gradient estimate we analyze the dependence of scattered fields and their Fréchet derivatives on the material parameter in Sections 3.3 and 3.4. Section 3.5 extends these results to a forward operator mapping the contrast to its far field operator. This allows to show convergence of a non-linear Tikhonov regularization against a minimum-norm solution to the inverse problem in Section 3.6, but also to set up sparsity-promoting versions of that regularization method. Note, for both approaches, the discrepancy is defined via a q-Schatten norm or an L^q-norm with $1 < q < \infty$. Several numerical examples presented in Section 3.7 indicate the reconstruction quality of the method, as well as the qualitative dependence of the reconstructions on q.

Finally, as the adjoint operator of the forward operator's linearization is a crucial ingredient for the reconstruction scheme, an explicit and computable representation of this adjoint can be found in Section 3.8. All results presented in this chapter are already published in [LR17].

3.1 PRESENTATION OF THE PROBLEM

We consider direct and inverse scattering of time harmonic waves from a penetrable and anisotropic inhomogeneous medium with density described by a matrix-valued material contrast parameter $Q \in \mathbb{C}^{d\times d}$

$$\operatorname{div}((\mathrm{Id}_d + Q)\nabla u) + k^2 u = 0 \qquad \text{in } \mathbb{R}^d,\ d = 2,3. \tag{3.1}$$

To this end, we set up weak solution theory for the scattering problem in Lebesgue spaces L^t with $t \geqslant 2$ to be able to treat contrast functions in L^p with $p < \infty$ in some admissible set. The analytic results allow to prove convergence of a sparsity-promoting version of

Tikhonov regularization in Banach spaces for a specifically designed penalty term towards, roughly speaking, a minimum-norm solution. Numerical examples of contrast reconstructions in two dimensions show feasibility of the proposed algorithm.

Regarding incident waves $u^i(x,\theta) = \exp(ik\theta \cdot x)$ from direction $\theta \in \mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d \mid |x| = 1\right\}$ we seek solutions $u(\cdot,\theta)$ to (3.1) such that the scattered field $u^s(\cdot,\theta) = u(\cdot,\theta) - u^i(\cdot,\theta)$ additionally satisfies Sommerfeld's radiation condition,

$$\lim_{r\to\infty} r^{(d-1)/2} \left(\frac{\partial u^s}{\partial r}(r\hat{x},\theta) - iku^s(r\hat{x},\theta) \right) = 0 \tag{3.2}$$

uniformly in all directions $\hat{x} \in \mathbb{S}^{d-1}$. By construction, the scattered field in particular solves the Helmholtz equation $\Delta u + k^2 u = 0$ outside some ball $B_R = \{|x| < R\}$ containing $\overline{D}$; such solutions are called radiating in the sequel. It is well-known that radiating solutions to the Helmholtz equation have the asymptotic behavior

$$u^s(r\hat{x},\theta) = \gamma_d \frac{e^{ikr}}{r^{(d-1)/2}} u^\infty(\hat{x},\theta) + \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty,$$

where $\gamma_2 = \exp(i\pi/4)/\sqrt{8\pi k}$, $\gamma_3 = 1/(4\pi)$, and $u^\infty \colon \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{C}$ is the so-called far field pattern of the scattered field. This function is analytic in both variables and defines the far field operator $F = F_Q \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$,

$$\left(F_Q g\right)(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x},\theta) g(\theta) \, dS(\theta), \qquad \hat{x} \in \mathbb{S}^{d-1}.$$

The inverse scattering problem we are interested in is to stably approximate the contrast function Q_{exa} from noisy measurements of the far field pattern u^∞, that is, from a noisy version F^δ_{meas} such that $\|F_{\text{exa}} - F^\delta_{\text{meas}}\| \leqslant \delta$ for some noise-level δ. To this end, we show that different variants of Tikhonov regularization can be employed for this task and in particular suggest a sparsity-promoting variant of that technique. The latter variant hence provides a solution $Q^\dagger$ such that $F_{Q^\dagger} = F_{\text{exa}}$ in the limit as the noise level δ tends to zero, such that moreover $Q^\dagger$ minimizes the sparsity promoting penalty term defining the Tikhonov functional.

Note that the convergence analysis of minimization-based regularization methods requires Banach spaces with some smoothness properties that rule out L^∞ as a suitable space for contrasts. Following the ideas of Jin and Maass in [JM12a], we use Meyers' gradient estimates for weak solutions to elliptic equations to obtain that gradients of weak solutions to (3.1) actually belong to L^t-spaces with $t > 2$. This in turn allows to prove various analytic properties for the solution to (3.1) as Lipschitz continuity or directional Gâteaux differentiability that only require the contrast Q to be measured in some L^p-norm with $2 < p < \infty$. While [JM12a] applies sparsity-promoting regularization methods based on Tikhonov regularization to determine a conductivity distribution in electrical impedance tomography (EIT), the only known investigation of corresponding techniques in inverse scattering was done by Lechleiter, Kazimierski, and Karamehmedović in [LKK13]. As mentioned in Chapter 2.3, their work tackled the simpler Helmholtz equation $\Delta u + k^2 n^2 u = 0$.

A specificity of our approach compared to the ones of Lechleiter, Kazimierski, and Karamehmedović as well as of Jin and Maass is that we do not only incorporate penalty terms that are linked to Hilbert spaces but also measure the discrepancy, that is, the difference between the measured far field data and the computed approximation, in a Banach-space: We consider either the full range of Schatten classes $\mathcal{S}_q$ on the space of linear operators on $L^2(\mathbb{S}^{d-1})$ for $1 < q < \infty$, or define a norm on the space of measurement operators by considering the norm of their integral kernels in $L^q(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$. For $q = 2$, both notions coincide. The choice of q significantly influences both reconstruction time and quality, as we demonstrate numerically.

On the very technical level, the ellipticity of the conductivity equation tackled in [JM12a] generally makes uniform estimates for solutions to the governing differential equation with different conductivities arguably easier than for the indefinite Helmholtz equation treated in this case. (This problem did not occur in [LKK13] due to the much easier solution theory in L^t handling the simpler Helmholtz equation.)

Note also that nonlinear Tikhonov regularization requires performant minimization routines; we rely on the classical shrinked Landweber iteration from Daubechies, Defrise, and De Mol mentioned in Section 2.2 and also the Chambolle-Pock algorithm from [CP11], which is connected to trust region algorithms. Since we not attempt to improve reconstruction algorithms, we leave further algorithmic tools aside. A proper numerical analysis of these techniques can be found in [BKL17].

We finally remark that estimates use a generic constant C that might change its value from one occurrence to the other.

3.2 THE SCATTERING PROBLEM

In this section, we recall weak solution theory in Sobolev spaces $W^{1,t}$ with $t \geqslant 2$ for the anisotropic Helmholtz equation (3.1) subject to the radiation condition (3.2) for the scattered field. The latter equation is understood in the distributional sense. After recalling conditions for solvability of that problem in H^1, we provide L^t-theory using Meyers' gradient estimates. As it leads to somewhat shorter expressions, we actually tackle the scattering problem for the corresponding scattered fields, which are required to be locally in H^1 and to satisfy the differential equation

$$\operatorname{div}((\mathrm{Id}_d + Q)\nabla u^s) + k^2 u^s = -\operatorname{div}(Q\nabla u^i) \qquad \text{in } \mathbb{R}^d \tag{3.3}$$

weakly in the sense of $L^2(\mathbb{R}^d)$, subject to the radiation condition (3.2).

Remark 3.1. If the material parameter $A = \mathrm{Id}_d + Q$ is piecewise differentiable, then any weak solution u and its co-normal derivative $\partial u/\partial \nu_A := \nu^\top A \nabla u$ are continuous over interfaces Γ where A jumps: $[u]_\Gamma = 0$ and $[\nu^\top A \nabla u]_\Gamma = 0$, where ν denotes a unit normal to Γ and $[v]_\Gamma$ denotes the jump of the function v across Γ.

ADMISSIBLE CONTRASTS Before recalling the standard L^2-based solution approach via Riesz-Fredholm theory, we introduce a set of contrasts that depends on a fixed parameter $\lambda \in (0,1)$,

$$\mathcal{Q} := \Big\{ Q \in L^\infty(B_{2R}, \mathbb{C}^{d\times d}) \;\Big|\; \lambda \leqslant 1 + \mathrm{Re}(\bar{z}^\top Q z) \leqslant \lambda^{-1}, \\ -\lambda^{-1} \leqslant \mathrm{Im}(\bar{z}^\top Q z) \leqslant 0 \text{ for all } z \in \mathbb{C}^d \text{ with } |z| = 1 \\ \text{and such that } \mathrm{supp}(Q) \Subset B_R \Big\}. \tag{3.4}$$

Thus, λ determines the class of possible material parameters $A = \mathrm{Id}_d + Q$ such that $\lambda \leqslant \mathrm{Re}\, \bar{z}^\top A(x) z$ and $|A(x)| \leqslant \lambda^{-1}$ for all $z \in \mathbb{C}^d$ with $|z| = 1$ and almost every $x \in \mathbb{R}^d$. (We always implicitly extend $Q \in \mathcal{Q}$ by zero from B_{2R} to all of $\mathbb{R}^d$.) We further endow $\mathcal{Q}$ with the $L^p(B_{2R})^{d\times d}$-norm for $1 \leqslant p \leqslant \infty$. Note first that for $p < \infty$, the set $\mathcal{Q}$ then has no interior points for the L^p-topology, since for any $Q \in \mathcal{Q}$ and any $\epsilon > 0$, the open L^p-ball $\{Q' \in L^p(B_{2R})^{d\times d} \mid \|Q - Q'\|_{L^p(B_{2R})^{d\times d}} < \epsilon\}$ is not completely contained in $\mathcal{Q}$. Second, any $Q \in \mathcal{Q}$ obviously belongs to all spaces $L^p(B_{2R})^{d\times d}$ for $1 \leqslant p \leqslant \infty$.

Remark 3.2. We consider contrasts Q on B_{2R} supported in $\overline{B_R}$ since this straightforwardly allows to directly rely on a specific Meyers' estimate from [Mey63], avoiding technicalities. (B_{2R} could be replaced by any bounded domain that strictly contains B_R.)

SOLUTION THEORY VIA RIEZ-FREDHOLM Seeking to solve for the scattered field instead of the total one, we rewrite equation (3.3) for all test functions $\psi \in C_0^\infty(\mathbb{R}^d)$ in the weak sense by multiplying that equation with ψ, integrating over B_{2R}, and integrating by parts the divergence term. Thus,

$$\int_{B_{2R}} \left[(\mathrm{Id}_d + Q) \nabla u^s \cdot \nabla \bar{\psi} - k^2 u^s \bar{\psi} \right] \, dx - \int_{\partial B_{2R}} \frac{\partial u^s}{\partial \nu} \bar{\psi} \, dS \\ = - \int_{B_R} Q \nabla u^i \cdot \nabla \bar{\psi} \, dx, \tag{3.5}$$

and density of smooth functions in $H^1(B_{2R})$ implies that the latter equation holds for all $\psi \in H^1(B_{2R})$. As the trace operator $\gamma(u) = u|_{\partial B_{2R}}$ has a unique continuation to a linear operator from $H^1(B_{2R})$ into $H^{1/2}(\partial B_{2R})$, see [McL00, Lemma 3.35], such that $u^s|_{\partial B_{2R}}$ belongs to $H^{1/2}(\partial B_{2R})$, see [Mon03, Theorem 3.24]. Further, $(\mathrm{Id}_d + Q)\nabla u^s$ belongs to $H(\mathrm{div}, B_{2R})$ since u^s solves (3.3) in $L^2(\mathbb{R}^d)$, such that the trace theorem in $H(\operatorname{div}, B_{2R})$ shows that $\partial u^s/\partial \nu = \nu \cdot \nabla u^s = \nu \cdot (\mathrm{Id}_d + Q)\nabla u^s$ belongs to $H^{-1/2}(\partial B_{2R})$, see [Mon03, Theorem 3.24]. Thus, the boundary integral in (3.5) is well-defined as a duality pairing between $H^{\pm 1/2}(\partial B_{2R})$.

Now, we denote by $\Lambda_{2R} \colon H^{1/2}(\partial B_{2R}) \to H^{-1/2}(\partial B_{2R})$ the exterior Dirichlet-to-Neumann operator, see [Néd01], which maps Dirichlet boundary values ϕ on ∂B_{2R} to the normal derivative $\partial v/\partial \nu$ of the unique radiating solution v to the exterior Dirichlet scattering boundary problem. More precisely, $v \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{B_{2R}})$ is the unique radiating solution to $\Delta v + k^2 v = 0$ in $\mathbb{R}^d \setminus \overline{B}_{2R}$, and can be written down explicitly in series form using Hankel functions, see Remark 3.5. As u^s is a radiating solution to the Helmholtz equation, $\Lambda_{2R}(\gamma(u^s))$ equals $\partial u^s/\partial \nu$, such that the left-hand side of (3.5) becomes

$$\int_{B_{2R}} \left[(\mathrm{Id}_d + Q)\nabla u^s \cdot \nabla \bar{\psi} - k^2 u^s \bar{\psi}\right] \mathrm{d}x - \int_{\partial B_{2R}} \Lambda_{2R}(\gamma(u^s))\gamma(\bar{\psi})\, \mathrm{d}S \tag{3.6}$$

for all $\psi \in H^1(B_{2R})$, equal to the right-hand side $\Psi(\psi) = -\int_{B_R} Q\nabla u^i \cdot \nabla\bar{\psi}\, \mathrm{d}x$. (We omit the trace operator γ from now on if a restriction to the boundary is obvious.)

To prove existence of solution of (3.6), we follow Hähner [Hä00], see also the proof of Theorem 5.7 in [CK13], and rely on an additional Dirichlet-to-Neumann map $\Lambda_{\Delta,2R}$ that maps Dirichlet data on ∂B_{2R} to Neumann data of the solution to an exterior Dirichlet boundary problem for the Laplace equation. Note that $-\Lambda_{\Delta,2R}$ is coercive, that is, $-\int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\psi)\bar{\psi}\, \mathrm{d}S \geqslant c\|\psi\|^2_{H^{1/2}(\partial B_{2R})}$ for all $\psi \in H^{1/2}(\partial B_{2R})$, see, e.g., [CK13, p. 131]. The sesquilinear forms

$$s(\varphi,\psi) \coloneqq \int_{B_{2R}} \left[(\mathrm{Id}_d + Q)\nabla\varphi \cdot \nabla\bar{\psi} + \varphi\bar{\psi}\right] \mathrm{d}x - \int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\varphi)\bar{\psi}\, \mathrm{d}S,$$
$$s_1(\varphi,\psi) \coloneqq (k^2+1)\int_{B_{2R}} \varphi\bar{\psi}\, \mathrm{d}x + \int_{\partial B_{2R}} (\Lambda_{2R} - \Lambda_{\Delta,2R})(\varphi)\bar{\psi}\, \mathrm{d}S,$$

allow to reformulate the variational form (3.6) as

$$s(v,\psi) - s_1(v,\psi) = -\int_{B_R} Q\nabla u^i \cdot \nabla\bar{\psi}\, \mathrm{d}x \quad \text{for all } \psi \in H^1(B_{2R}). \tag{3.7}$$

Both Λ_{2R} and $\Lambda_{\Delta,2R}$ are bounded from $H^{1/2}(\partial B_{2R})$ into $H^{-1/2}(\partial B_{2R})$, such that s and s_1 are bounded sesquilinear forms. The coercivity of $\Lambda_{\Delta,2R}$ implies that s is coercive,

$$s(\varphi,\varphi) \geqslant \|\varphi\|^2_{H^1(B_{2R})} - \int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\varphi)\bar{\varphi}\, \mathrm{d}S \geqslant C\|\varphi\|^2_{H^1(B_{2R})},$$

for all $\varphi \in H^1(B_{2R})$. Moreover, compactness of $\Lambda_{2R} - \Lambda_{\Delta,2R}$, see, e.g., [CK13, p. 131], and the compact embedding of $H^1(B_{2R})$ in $L^2(B_{2R})$ imply that s_1 is a compact sesquilinear form.

By Riesz' representation theorem there exists a bounded operator $S \colon H^1(B_{2R}) \to H^1(B_{2R})$ and a compact operator S_1 such that

$s(\varphi,\psi) = (S\,\varphi,\psi)_{H^1(B_{2R})}$ and $s_1(\varphi,\psi) = (S_1\,\varphi,\psi)_{H^1(B_{2R})}$ for all $\varphi,\,\psi \in H^1(B_{2R})$. By Lax-Milgram's lemma, S is further boundedly invertible. Further introducing $r \in H^1(B_{2R})$ such that $-\int_{B_R} Q\nabla u^i \cdot \nabla\bar{\psi}\,dx = (r,\psi)_{H^1(B_{2R})}$ for all $\psi \in H^1(B_{2R})$, the variational formulation (3.7) can be equivalently rewritten as $S\nu - S_1\,\nu = r$ in $H^1(B_{2R})$. Multiplying with the inverse S^{-1} yields $\nu - K\nu = S^{-1}\,r$ with a compact operator $K := S^{-1}\,S_1$. Thus, Riesz-Fredholm theory implies that uniqueness of solution to the latter equation implies existence of solution for all right-hand sides.

Lemma 3.3. *If the only solution to the homogeneous problem corresponding to (3.6) is the trivial solution, then that variational problem possesses a unique solution for all bounded anti-linear functionals $\Psi \in H^1(B_{2R})^*$ and there is C_Q independent of Ψ such that*

$$\|\nu\|_{H^1(B_{2R})} \leqslant C_Q\|\Psi\|_{H^1(B_{2R})^*}\left[= C_Q\|Q\|_{L^\infty(B_R)}\|u^i\|_{H^1(B_R)} \quad \textit{if}\ \Psi(\psi) = -\int_{B_R} Q\nabla u^i\cdot\nabla\psi\,dx\right]. \tag{3.8}$$

Uniqueness of solution for the variational problem (3.6) is strongly linked to the unique continuation property for solutions to that equation. In [Häh00], Hähner shows uniqueness of solution for contrasts Q that are C^1-smooth and supported in domains of class C^2; this result can be generalized to contrasts that are piecewise differentiable on a decomposition of B_{2R} into finitely many Lipschitz domains.

Corollary 3.4. *If there is a decomposition of $\overline{B_{2R}} = \bigcup_{j=1}^n \overline{\Omega_j}$ of B_{2R} into finitely many Lipschitz domains Ω_j such that $Q \in \mathcal{Q}$ belongs to $C^1(\overline{\Omega_j})$ for $j = 1,\dots,n$, then the variational formulation (3.6) possesses a unique solution ν for all bounded anti-linear functionals $\Psi \in H^1(B_{2R})^*$.*

Remark 3.5. It is well-known that solutions ν to (3.6) can be uniquely extended to radiating solutions in $H^1_{loc}(\mathbb{R}^d)$ of the Helmholtz equation in $H^1_{loc}(\mathbb{R}^d)$, see [Néd01]: For the spherical Hankel function $h_n^{(1)}$ and the spherical harmonics Y_n^m this extension is given by

$$\tilde{\nu}(x) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \nu_n^m \frac{h_n^{(1)}(2k|x|)}{h_n^{(1)}(2kR)} Y_n^m\left(\frac{x}{|x|}\right), \quad |x| > R, \tag{3.9}$$

where $\nu_n^m = \frac{1}{R^2}\int_{\partial B_{2R}} \nu\overline{Y}_n^m\,dS$. Thereby one defines extended functions as $w = \{\nu \text{ in } B_R,\ \tilde{\nu} \text{ in } \mathbb{R}^3 \setminus B_{2R}\}$ as radiating solutions of (3.3). By abuse of notation, we denote such functions for simplicity by ν again.

regularity estimate To be able to handle derivatives of scattered fields in L^p-spaces, we give a version of Meyers' well-known gradient estimate from [Mey63].

THEOREM 3.6. *For the bounded Lipschitz domain* $B_{2R} \subset \mathbb{R}^d$ *and for* $Q \in \mathcal{Q}$ *and* $f \in L^t(B_{2R})^d$ *let* $v \in H^1(B_{2R})$ *be a weak solution to*

$$\operatorname{div}((\mathrm{Id}_d + Q)\nabla v) + k^2 v = -\operatorname{div} f \quad \text{in } B_{2R},$$

i.e. v *solves the variational formulation* (3.6) *for all* $\psi \in H_0^1(B_{2R})$. *Then there exists a constant* $T_\lambda \in (2, \infty)$ *depending on* λ *and* d *such that for all* $t \in (2, T_\lambda)$ *the gradient* ∇v *belongs to* $L^t(B_{2R})^d$ *and satisfies*

$$\|\nabla v\|_{L^t(B_R)^d} \leqslant C\left((1+k^2)\|v\|_{L^2(B_{2R})} + \|f\|_{L^t(B_{2R})^d}\right), \tag{3.10}$$

where $C = C(\lambda, d, t, R)$. *As* $\lambda \to 0$ *(or* $\lambda \to 1$*) the constant* T_λ *tends to* 2 *(or* ∞*).*

Proof. In [Mey63, Theorem 2] the original statement is shown more generally for $f \in L^t(B_{2R})^d$ and $h \in L^r(B_{2R})$ with $r^* \geqslant t > 2$, such that $\nabla u \in L^2(B_{2R})$ weakly solves

$$\operatorname{div}(A\nabla u) = \operatorname{div} f + h \quad \text{in } B_{2R}.$$

The number r^* is defined by $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{d}$ if $r < d$ or as any number in $(1, \infty)$ else. Since in our case $h = k^2 u$ for $u \in H^1(B_{2R})$, the choice $r = 2$ is natural. If $d = 2$, we can hence choose an arbitrary $r^* \in (1, \infty)$ such that $r^* \geqslant t > 2$. In three dimensions, the analogous condition for r^* is fulfilled: $r^* = 2d/(d-2) = 6 \geqslant t > 2$.

The necessary condition $t > 2$ enforces $t > 2d/(d+2)$, which allows to use estimate (49) of [Mey63, Theorem 2],

$$\|\nabla u\|_{L^t(B_R)^d} \leqslant C\Big(R^{d(\frac{1}{t}-\frac{1}{2})-1}\|u\|_{L^2(B_{2R})} + \|f\|_{L^t(B_{2R})^d} + R^{d(\frac{1}{t}-\frac{1}{2})+1}\|h\|_{L^2(B_{2R})}\Big),$$

and gives the stated result for $v = u$, $h = k^2 u$, and $r = 2$. □

3.3 THE SOLUTION OPERATOR

To investigate the solution operator mapping the contrast Q and the incident field u^i to the weak solution of the scattering problem (3.6), we define a sesquilinear form for $Q \in \mathcal{Q}$ for all $\varphi, \psi \in H^1(B_{2R})$ by

$$a_Q(\varphi, \psi) := \int_{B_{2R}} \left[(\mathrm{Id}_d + Q)\nabla\varphi \cdot \nabla\bar{\psi} - k^2 \varphi\bar{\psi}\right] dx - \int_{\partial B_{2R}} \Lambda_{2R}(\varphi)\bar{\psi}\, dS.$$

For $Q \in \mathcal{Q}$, we assume that the forward problem (3.6) is solvable for all right-hand sides and denote by $L(Q, \cdot)\colon H^1(B_{2R}) \to H^1(B_{2R})$ the solution operator mapping $f \in H^1(B_{2R})$ to the solution of the variational problem $a_Q(v, \psi) = -\int_{B_{2R}} Q\nabla f \cdot \nabla\bar{\psi}\, dx$ for all $\psi \in H^1(B_{2R})$. Choosing $f = u^i$ as a solution to the Helmholtz equation in $\mathbb{R}^d$,

$v_Q = L(Q, u^i)$ is hence the weak solution to the variational formulation (3.6), i.e.,

$$a_Q(v_Q, \psi) = -\int_{B_R} Q\nabla u^i \cdot \nabla\bar{\psi}\, dx \qquad \text{for all } \psi \in H^1(B_{2R}) \qquad (3.11)$$

and thus the radiating extension of v_Q to $\mathbb{R}^d$, see (3.9), weakly solves $\operatorname{div}((\operatorname{Id}_d + Q)\nabla v) + k^2 v = -\operatorname{div}(Q\nabla u^i)$ in $\mathbb{R}^d$.

PERTURBED CONTRASTS To state a perturbation result for $L(Q, \cdot)$, note that boundedness of the solution operator $L(Q, \cdot)$ implies by Riesz' representation theorem the existence of a boundedly invertible operator $A_Q : H^1(B_{2R}) \to H^1(B_{2R})$ such that

$$(A_Q\, v_Q, \psi)_{H^1(B_{2R})} = a_Q(v_Q, \psi) \qquad \text{for all } \psi \in H^1(B_{2R}). \qquad (3.12)$$

LEMMA 3.7. *Assume that for $Q \in \mathcal{Q}$ the forward problem (3.6) is uniquely solvable and let Q' be a perturbation of Q, small enough such that*

$$\|Q'\|_{L^\infty(B_{2R})^{d\times d}} \leqslant \frac{1}{2}\|\, A_Q^{-1}\,\|^{-1}_{H^1(B_{2R})\to H^1(B_{2R})}. \qquad (3.13)$$

Then for all $\Psi \in H^1(B_{2R})^*$ there is a unique $v \in H^1(B_{2R})$ solving

$$a_{Q+Q'}(v, \psi) = \Psi(\psi) \qquad \textit{for all } \psi \in H^1(B_{2R}). \qquad (3.14)$$

Thus, the solution operator $L(Q + Q', \cdot)$ exists for all Q' that satisfy (3.13) and is uniformly bounded by $\|\, L(Q + Q', u^i)\|_{H^1(B_{2R})} \leqslant C\|u^i\|_{H^1(B_R)}$, with $C = C(Q)$ independent of Q'.

Proof. Let Q, $\widetilde{Q} \in \mathcal{Q}$ such that $\|Q - \widetilde{Q}\|_{L^\infty(B_{2R})^{d\times d}} \leqslant \delta$ for $\|\, A_Q^{-1}\,\| < \delta^{-1}$. (We omit to explicitly denote operator norms in this proof.) Define $A_{\widetilde{Q}}$ as the Riesz representation A_Q of $a_{\widetilde{Q}}$ in (3.12) and note that

$$\left|((A_Q - A_{\widetilde{Q}})v, \psi)_{H^1(B_{2R})}\right| = \left|\int_{B_{2R}} (Q - \widetilde{Q})\nabla v \cdot \nabla\psi\, dx\right|$$
$$\leqslant \|Q - \widetilde{Q}\|_{L^\infty(B_{2R})}\|\nabla v\|_{L^2(B_{2R})}\|\nabla\psi\|_{L^2(B_{2R})},$$

such that $\|\, A_Q - A_{\widetilde{Q}}\,\| \leqslant \|Q - \widetilde{Q}\|_{L^\infty(B_{2R})} \leqslant \delta$. Due to the choice of δ there holds that $\|\operatorname{Id} - A_Q^{-1} A_{\widetilde{Q}}\,\| \leqslant \|\, A_Q^{-1}\,\|\,\|\, A_Q - A_{\widetilde{Q}}\,\| < 1$, such that the Neumann series

$$\sum_{j=0}^{\infty}\left(\operatorname{Id} - A_Q^{-1} A_{\widetilde{Q}}\right)^j = \left(\operatorname{Id} - (\operatorname{Id} - A_Q^{-1} A_{\widetilde{Q}})\right)^{-1} = A_{\widetilde{Q}}^{-1} A_Q,$$

converges and defines $A_{\widetilde{Q}}^{-1} A_Q$. Thus, $A_{\widetilde{Q}}^{-1}$ exists as a bounded operator on $H^1(B_{2R})$ and its operator norm is bounded by $\|\, A_{\widetilde{Q}}^{-1}\,\| \leqslant \|\, A_Q^{-1}\,\|/(1 - \delta\|\, A_Q^{-1}\,\|)$. Setting $Q' = \widetilde{Q} - Q$ and $\delta = 1/2$, the claim follows from the equivalence of (3.14) and the equation

$$(A_Q\, v_Q, \psi)_{H^1(B_{2R})} = \Psi(\psi)_{L^2(B_{2R})}$$

holds for all $\psi \in H^1(B_{2R})$. □

Remark 3.8. A. It is well-known that the technique of the proof of Lemma 3.7 allows to show solvability for all contrasts of the form $Q + Q'$ with $Q' \in L^\infty(B_R)^{d\times d}$ such that $\|Q'\|_{L^\infty(B_R)^{d\times d}} < \| A_Q^{-1} \|^{-1}_{H^1(B_{2R})\to H^1(B_{2R})}$.

B. Combining the last Lemma 3.7 with Corollary 3.4, one can show that the forward problem (3.6) is solvable in the union of open L^∞-balls around, roughly speaking, all piecewise continuously differentiable contrasts Q with radius $\| A_Q^{-1} \|^{-1}_{H^1(B_{2R})\to H^1(B_{2R})}$.

SOLUTION OPERATOR'S CONTINUITY Recall now that the constant $T_\lambda > 2$ has been defined in Theorem 3.6 and that u^i denotes a generic solution to the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^d$.

THEOREM 3.9. *For* $p > 2T_\lambda/(T_\lambda - 2)$ *and* $Q, Q + Q' \in \mathcal{Q}$ *such that the solution operator* $L(Q, \cdot)$ *exists and* Q' *satisfies* (3.13), *there holds that*

$$\| L(Q + Q', u^i) - L(Q, u^i)\|_{H^1(B_{2R})} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}, \tag{3.15}$$

with a constant C *that only depends on* Q *but not on* Q' *or on* u^i.

Proof. Due to Lemma 3.7, the assumptions on Q and Q' imply that both solution operators $L(Q, \cdot)$ and $L(Q + Q', \cdot)$ are bounded operators on $H^1(B_{2R})$. For the same incident field u^i we set $v_{Q+Q'} = L(Q + Q', u^i)$ and $v_Q = L(Q, u^i)$ and denote the radiating extensions (see (3.9)) of these functions to $\mathbb{R}^d$ again by $v_{Q+Q'}$ and the corresponding total fields by $u_{Q+Q'} = u^i + v_{Q+Q'}$ and $u_Q = u^i + v_Q$. The difference $v_{Q+Q'} - v_Q = u_{Q+Q'} - u_Q$ is the weak, radiating solution to

$$\begin{aligned} \operatorname{div}((\mathrm{Id}_d + Q)\nabla(u_{Q+Q'} - u_Q)) + k^2(u_{Q+Q'} - u_Q) \\ = -\operatorname{div}(Q'\nabla u_{Q+Q'}) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Thus, the boundedness of the solution operator $L(Q, \cdot)$ hence shows that $\|u_{Q+Q'} - u_Q\|_{H^1(B_{2R})} \leqslant C_Q\|Q'\nabla u_{Q+Q'}\|_{L^2(B_R)^d}$. Choosing p and t such that $1/p + 1/t = 1/2$, the generalized Hölder inequality yields

$$\|Q'\nabla u_{Q+Q'}\|_{L^2(B_R)^d} \leqslant \|Q'\|_{L^p(B_R)^{d\times d}}\|\nabla u_{Q+Q'}\|_{L^t(B_R)^d}. \tag{3.16}$$

The choices of p and t imply $p = 2t/(t - 2)$ and since by assumption $p > 2T_\lambda/(T_\lambda - 2)$ we have $2T_\lambda/(T_\lambda - 2) < 2t/(t - 2)$. The strict monotonicity of $s \mapsto 2s/(s - 2)$ on $(2, \infty)$ consequently implies that $t < T_\lambda$, which allows to conclude by Meyers' estimate (3.10) that

$$\|\nabla u_{Q+Q'}\|_{L^t(B_R)^d} \leqslant C\|u_{Q+Q'}\|_{L^2(B_{2R})}. \tag{3.17}$$

Due to the assumption on Q' and Lemma 3.7, the right-hand side of the latter estimate can be bounded by

$$\begin{aligned} \|u_{Q+Q'}\|_{L^2(B_{2R})} &\leqslant \|u^i\|_{H^1(B_{2R})} + \|v_{Q+Q'}\|_{H^1(B_{2R})} \\ &\leqslant (1 + 2C_Q)\|u^i\|_{H^1(B_{2R})}, \end{aligned}$$

where C_Q is the constant from (3.8). Together, the last four estimates yield the claim. □

Remark 3.10. The case $p = \infty$ is not covered by Meyers' estimate, but could be treated directly by standard L^2-theory from Lemma 3.3 and Lemma 3.7. This also holds for all further statements.

Theorem 3.9 requires assumption (3.13) for Q' merely to bound the norm of $u_{Q+Q'}$ independently of Q'. If one knows a-priori that the norm of solutions to the scattering problem is uniformly bounded, then assumption (3.13) can obviously be dropped.

Corollary 3.11. *Assume that $\| L(Q,\cdot)\|_{H^1(B_{2R})\to H^1(B_{2R})}$ is uniformly bounded for all Q in a subset $\mathcal{Q}'$ of $\mathcal{Q}$. Then for all $p > 2T_\lambda/(T_\lambda - 2)$ there is $C_* > 0$ depending on $\mathcal{Q}'$ and p such that (3.15) holds for all Q, $Q + Q' \in \mathcal{Q}'$ with C replaced by C_*.*

3.4 DIFFERENTIABILITY OF THE SOLUTION OPERATOR

We now have a glance at the differentiability of the solution operator and, therefore, fix the incident field u^i in this entire section. We further fix $Q \in \mathcal{Q}$ such that the solution operator $L(Q, \cdot)$ is bounded on $H^1(B_{2R})$ and denote the derivative of L with respect to Q in direction $Q' \in L^p(B_{2R})^{d\times d}$ by $v' := L'(Q,u^i)[Q']$, defined by

$$a_Q(v',\psi) = -\int_{B_R} Q'\nabla\left[L(Q,u^i) + u^i\right]\cdot\nabla\bar{\psi}\,dx \quad \text{for all } \psi \in H^1(B_{2R}). \tag{3.18}$$

We show in Theorem 3.15 that v' can be interpreted as a Gâteaux derivative of $L(Q,u^i)$ in direction Q' (see also Remark 3.16).

CONTINUITY PROPERTIES

Lemma 3.12. *For every $Q \in \mathcal{Q}$ the mapping $Q' \mapsto L'(Q,u^i)[Q']$, which is in $\mathcal{L}(L^p(B_R)^{d\times d}, H^1(B_{2R}))$, has the following continuity properties:*

(i) For $p > 2T_\lambda/(T_\lambda - 2)$, there is $C = C(Q) > 0$ such that

$$\| L'(Q,u^i)[Q']\|_{H^1(B_{2R})} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

(ii) For every $p > 2T_\lambda/(T_\lambda - 2)$ there is $t \in (2, T_\lambda)$ and $C = C(Q) > 0$ such that

$$\|\nabla L'(Q,u^i)[Q']\|_{L^t(B_R)^d} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

Proof. (i) Lemma 3.3 implies that

$$\| L'(Q,u^i)[Q']\|_{H^1(B_{2R})} \leqslant C\|Q'\nabla u_Q\|_{L^2(B_R)^d},$$

where $u_Q = L(Q,u^i) + u^i$ represents the total field whose radiating extension to $\mathbb{R}^d$, see (3.9), satisfies the anisotropic Helmholtz equation $\operatorname{div}((\mathrm{Id}_d + Q)\nabla u_Q) + k^2 u = 0$ weakly in $\mathbb{R}^d$. Choosing p and t such that $1/p + 1/t = 1/2$, the generalized Hölder inequality further implies that $\|Q'\nabla u_Q\|_{L^2(B_R)^d} \leqslant \|Q'\|_{L^p(B_R)^{d\times d}} \|\nabla u_Q\|_{L^t(B_R)^d}$. Again, $p = 2t/(t-2)$ and, as in the proof of Theorem 3.9, Meyers' estimate (3.10) yields

$$\|\nabla u_Q\|_{L^t(B_R)^d} \leqslant C\|u_Q\|_{L^2(B_{2R})}.$$

Next, we exploit Lemma 3.3 to estimate

$$\begin{aligned}\|u_Q\|_{L^2(B_{2R})} &\leqslant \|v_Q\|_{H^1(B_{2R})} + \|u^i\|_{H^1(B_{2R})} \\ &\leqslant C\|Q\nabla u^i\|_{L^2(B_R)^d} + \|u^i\|_{H^1(B_{2R})},\end{aligned}$$

such that $\|u_Q\|_{L^2(B_{2R})} \leqslant \left[1 + C\|Q\|_{L^\infty(B_R)^{d\times d}}\right]\|u^i\|_{H^1(B_{2R})}$, and we conclude that

$$\| L'(Q,u^i)[Q']\|_{H^1(B_{2R})} \leqslant C\|Q\|_{L^\infty(B_R)^{d\times d}} \|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

(ii) For $t \in (2, T_\lambda)$ and $p > 2T_\lambda/(T_\lambda - 2)$, Meyers' estimate (3.10) yields as in the proof of part (i) that

$$\begin{aligned}\|\nabla L'(Q,u^i)[Q']\|_{L^t(B_R)^d} \leqslant C \Big[&\| L'(Q,u^i)[Q']\|_{L^2(B_{2R})} \\ &+\|Q'\nabla u_Q\|_{L^t(B_{2R})^d}\Big],\end{aligned} \tag{3.19}$$

where the radiating extension of the total field $u_Q = L(Q,u^i) + u^i$ to $\mathbb{R}^d$ solves $\operatorname{div}((\mathrm{Id}_d + Q)\nabla u_Q) + k^2 u = 0$ weakly in $\mathbb{R}^d$ (see (3.9)). The first term in (3.19) is bounded by $C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}$ due to (i), such that it remains to bound the second term: For arbitrary $\epsilon > 0$ such that $t' = t + \epsilon \in (t, T_\lambda)$, we set $p = t't/(t'-t)$ and compute that

$$\begin{aligned}\|Q'\nabla u_Q\|^t_{L^t(B_{2R})^d} &= \int_{B_R} |Q'|_2^t |\nabla u_Q|^t \,dx \\ &\leqslant \left(\int_{B_R} |Q'|^{\frac{tt'}{t'-t}} dx\right)^{1-\frac{t}{t'}} \|\nabla u_Q\|^t_{L^{t'}(B_R)^d} \\ &\overset{(3.10)}{\leqslant} C\|u_Q\|^t_{L^2(B_{2R})}\left(\int_{B_R} |Q'|^p dx\right)^{1-\frac{t}{t'}}.\end{aligned}$$

Since $p(t'-t)/t't = 1$ and as $\|u_Q\|_{L^2(B_{2R})} \leqslant C(Q)\|u^i\|_{H^1(B_{2R})}$ as seen in the proof of (i), we have that

$$\begin{aligned}\|Q'\nabla u_Q\|_{L^t(B_{2R})^d} &\leqslant C\|Q'\|^{\frac{t'-t}{t't}p}_{L^p(B_R)^{d\times d}} \|u_Q\|_{L^2(B_{2R})} \\ &\leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})},\end{aligned}$$

which shows the claimed estimate for $\|\nabla L'(Q,u^i)[Q']\|_{L^t(B_R)^d}$. □

LEMMA 3.13. *For $p > 2T_\lambda/(T_\lambda - 2)$ and for every $Q, Q + Q' \in \mathcal{Q}$ such that Q' satisfies (3.13), there exists $t \in (2, T_\lambda)$ and $C > 0$ independent of Q' and u^i such that*

$$\|\nabla L(Q+Q',u^i) - \nabla L(Q,u^i)\|_{L^t(B_R)^d} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

Proof. As in Theorem 3.9, we exploit that the difference of $u_{Q+Q'} = u^i + L(Q+Q', u^i)$ and $u_Q = u^i + L(Q, u^i)$ can be extended to a radiating function in $\mathbb{R}^d$ that solves

$$\begin{aligned} \operatorname{div}((\mathrm{Id}_d + Q)\nabla(u_{Q+Q'} - u_Q)) + k^2(u_{Q+Q'} - u_Q) \\ = -\operatorname{div}(Q'\nabla u_{Q+Q'}) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Consequently, Meyers estimate (3.10) implies that

$$\begin{aligned} &\|\nabla[L(Q+Q', u^i) - L(Q, u^i)]\|_{L^t(B_R)^d} \\ &= \|\nabla[L(Q+Q', u^i) + u^i - L(Q, u^i) - u^i]\|_{L^t(B_R)^d} \\ &\leqslant C\left(\| L(Q+Q', u^i) - L(Q, u^i)\|_{L^2(B_{2R})} + \|Q'\nabla u_{Q+Q'}\|_{L^t(B_{2R})^d}\right). \end{aligned}$$

The first term of the right hand side is bounded by Theorem 3.9, whereas the second one estimates as in the proof of Lemma 3.12 part (ii),

$$\|\nabla[L(Q+Q', u^i) - L(Q, u^i)]\|_{L^t(B_R)^d} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

□

THEOREM 3.14. *Under the assumptions of Lemma 3.13, the map $Q \mapsto L'(Q, u^i)$ is locally Lipschitz continuous: There is $C > 0$ independent of Q' and u^i such that for all $P \in L^p(B_R)^{d\times d}$ there holds*

$$\begin{aligned} &\| L'(Q+Q', u^i)[P] - L'(Q, u^i)[P]\|_{H^1(B_{2R})} \\ &\leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|P\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})} \end{aligned}$$

Proof. For $P \in L^p(B_R)^{d\times d}$, $w_{Q+Q'} = L'(Q+Q', u^i)[P]$ and $w_Q = L'(Q, u^i)[P]$ satisfy by (3.18) the variational formulations

$$\begin{aligned} a_{Q+Q'}(w_{Q+Q'}, \psi) &= -\int_{B_R} P\nabla\left[L(Q+Q', u^i) + u^i\right]\cdot\nabla\bar{\psi}\,dx \qquad \text{and} \\ a_Q(w_Q, \psi) &= -\int_{B_R} P\nabla\left[L(Q, u^i) + u^i\right]\cdot\nabla\bar{\psi}\,dx \end{aligned}$$

for all $\psi \in H^1(B_{2R})$. Thus, $w := w_{Q+Q'} - w_Q$ satisfies

$$\begin{aligned} a_Q(w, \psi) = &-\int_{B_R} P\nabla\left[L(Q+Q', u^i) - L(Q, u^i)\right]\cdot\nabla\bar{\psi}\,dx \\ &-\int_{B_R} Q'\nabla L'(Q+Q', u^i)[P]\cdot\nabla\bar{\psi}\,dx. \end{aligned}$$

Lemma 3.7 and the generalized Hölder inequality with Lebesgue indices p and t, such that $1/p + 1/t = 1/2$, then imply that

$$\begin{aligned} &\|w\|_{H^1(B_{2R})} \\ &\leqslant C_Q\,\|P\nabla[L(Q+Q', u^i) - L(Q, u^i)] + Q'\nabla L'(Q+Q', u^i)[P]\|_{L^2(B_R)^d} \\ &\leqslant C_Q\left[\|P\|_{L^p(B_R)^{d\times d}}\|\nabla[L(Q+Q', u^i) - L(Q, u^i)]\|_{L^t(B_R)^d}\right. \\ &\left. + C_Q\,\|Q'\|_{L^p(B_R)^{d\times d}}\|\nabla L'(Q+Q', u^i)[P]\|_{L^t(B_R)^d}\right]. \end{aligned}$$

Herein the L^t-norms are bounded by Lemma 3.13

$$\|\nabla[L(Q+Q',u^i)-L(Q,u^i)]\|_{L^t(B_R)^d} \leqslant C(Q)\|Q'\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}$$

as well as by Lemma 3.12

$$\|\nabla L'(Q+Q',u^i)[P]\|_{L^t(B_R)^d} \leqslant C_Q\|P\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

Combining these bounds with the above estimate for w shows the claim, as

$$\|w\|_{H^1(B_{2R})} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}}\|P\|_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}.$$

□

GÂTEAUX DERIVATIVE

THEOREM 3.15. *For $p > 2T_\lambda/(T_\lambda-2)$, the solution operator L is differentiable in the sense that for every $Q, Q+Q' \in \mathcal{Q}$ such that Q' satisfies (3.13), it holds that*

$$\|L(Q+Q',u^i)-L(Q,u^i)-L'(Q,u^i)[Q']\|_{H^1(B_{2R})}$$

is bounded by $\|Q'\|^2_{L^p(B_R)^{d\times d}}\|u^i\|_{H^1(B_{2R})}$, scaled by a constant independent of Q' and u^i. Thus, if $\{Q'_n\}_{n\in\mathbb{N}}$ satisfies (3.13) for all $n \in \mathbb{N}$ as well as $\|Q'_n\|_{L^p(B_R)^{d\times d}} \to 0$ as $n \to \infty$, then

$$\frac{\|L(Q+Q'_n,u^i)-L(Q,u^i)-L'(Q,u^i)[Q'_n]\|_{H^1(B_{2R})}}{\|Q'_n\|_{L^p(B_R)^{d\times d}}} \to 0 \quad \text{as } n \to \infty.$$

Proof. For $w := L(Q+Q',u^i)-L(Q,u^i)-L'(Q,u^i)[Q']$ we first consider the variational formulations defining all three terms,

$$\begin{aligned} a_{Q+Q'}(L(Q+Q',u^i),\psi) &= -\int_{B_R}(Q+Q')\nabla u^i\cdot\nabla\bar{\psi}\,dx,\\ a_Q(L(Q,u^i),\psi) &= -\int_{B_R} Q\nabla u^i\cdot\nabla\bar{\psi}\,dx,\\ a_Q(L'(Q,u^i)[Q'],\psi) &= -\int_{B_R} Q'\nabla[L(Q,u^i)+u^i]\cdot\nabla\overline{\psi}\,dx \end{aligned}$$

for all $\psi \in H^1(B_{2R})$. Thus, for all $\psi \in H^1(B_{2R})$ there holds

$$\begin{aligned} &a_{Q+Q'}(w,\psi)\\ &= a_{Q+Q'}(L(Q+Q',u^i),\psi) - a_Q(L(Q,u^i),\psi) - a_Q(L'(Q,u^i)[Q'],\psi)\\ &\quad -\int_{B_{2R}} Q'\nabla L(Q,u^i)\cdot\nabla\overline{\psi}\,dx - \int_{B_{2R}} Q'\nabla L'(Q,u^i)[Q']\cdot\nabla\overline{\psi}\,dx\\ &= \int_{B_R} Q\nabla u^i\cdot\nabla\bar{\psi}\,dx - \int_{B_R}(Q+Q')\nabla u^i\cdot\nabla\bar{\psi}\,dx - \int_{B_R} Q'\nabla L(Q,u^i)\cdot\nabla\overline{\psi}\,dx\\ &\quad + \int_{B_R} Q'\nabla\left[L(Q,u^i)+u^i\right]\cdot\nabla\overline{\psi}\,dx - \int_{B_R} Q'\nabla L'(Q,u^i)[Q']\cdot\nabla\overline{\psi}\,dx\\ &= -\int_{B_R} Q'\nabla L'(Q,u^i)[Q']\cdot\nabla\overline{\psi}\,dx. \end{aligned}$$

Generalized Hölder's inequality for p and t such that $1/t + 1/p = 1/2$ and Lemma 3.7 imply that

$$\begin{aligned}\|w\|_{H^1(B_{2R})} &\leqslant 2C_Q \|Q'\nabla L'(Q,u^i)[Q']\|_{L^2(B_R)^d} \\ &\leqslant C_Q \|Q'\|_{L^p(B_R)^{d\times d}} \|\nabla L'(Q,u^i)[Q']\|_{L^t(B_R)^d},\end{aligned}$$

where by Lemma 3.12 part (ii)

$$\|\nabla L'(Q,u^i)[Q']\|_{L^t(B_R)^d} \leqslant C\|Q'\|_{L^p(B_R)^{d\times d}} \|u^i\|_{H^1(B_{2R})},$$

which finally implies the claimed estimate. □

Remark 3.16. For $Q' \in L^\infty(B_{2R})^{d\times d}$ that satisfies (3.13), the last theorem shows that $Q' \mapsto L'(Q,u^i)[Q']$ is the Gâteaux derivative of $Q \mapsto L(Q,u^i)$ at Q in direction Q'. As $Q \mapsto L(Q,u^i)$ is, however, not defined on an open set in $L^p(B_{2R})^{d\times d}$, see the discussion below (3.4), $Q \mapsto L(Q,u^i)$ is not Gâteaux differentiable (or Fréchet differentiable) in the L^p-sense and $L'(Q,u^i)$ is not a Gâteaux differential.

3.5 THE FORWARD OPERATOR

In this section, we define the forward operator corresponding to the inverse scattering problem we are ultimately interested in. This operator maps a contrast function to the corresponding far field operator.

POTENTIAL REPRESENTATION Therefore, we reformulate the anisotropic Helmholtz equation under investigation for a source $f \in L^2(B_{2R})$, extended by zero to all of $\mathbb{R}^d$, as $\operatorname{div}((\mathrm{Id}_d + Q)\nabla v) + k^2 v = f$ in $\mathbb{R}^d$ for a radiating weak solution $v \in H^1_{loc}(\mathbb{R}^d)$. By [Kir08, Lemma 2.1], v can be represented as a volume potential defined via

$$\Phi_k(x) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x|), & \text{if } d = 2, \\ \frac{\exp(ik|x|)}{4\pi|x|}, & \text{if } d = 3, \end{cases} \qquad x \neq 0,$$

which is the radiating fundamental solution to the Helmholtz equation. More precisely,

$$v = \operatorname{div} \int_{B_R} \Phi_k(\cdot - y)[Q\nabla v + f](y)\, dy \quad \text{in } \mathbb{R}^d. \tag{3.20}$$

Note that the choice of $f = \nabla u^i$ corresponds to the radiating extension of $v = L(Q,u^i)$ to $\mathbb{R}^d$. In analogy to this potential representation, the radiating extension of the derivative $v' = L'(Q,u^i)[Q']$ to $\mathbb{R}^d$ satisfies

$$v' = \operatorname{div} \int_{B_R} \Phi_k(\cdot - y) \left[Q\nabla v' + Q'\nabla \left(L(Q,u^i) + u^i\right)\right](y)\, dy \tag{3.21}$$

in $\mathbb{R}^d$, since v' solves, by definition, the variational formulation (3.18).

Further we assume from now on that the solution operator $L(Q, u^i)$ is well-defined and bounded on $H^1(B_{2R})$ for all $Q \in \mathcal{Q}$. Due to Lemma 3.7, this can always be guaranteed by choosing the parameter $\lambda \in (0, 1)$ defining $\mathcal{Q}$ small enough.

ASSUMPTION 3.17. The solution operator $L(Q, u^i)$ exists for all $Q \in \mathcal{Q}$ as a bounded operator on $H^1(B_{2R})$.

Due to the potential representation (3.20) of $v = L(Q, u^i)$, the far field pattern of $v^\infty(\hat{x})$ for a direction $\hat{x} \in \mathbb{S}^{d-1}$ hence equals

$$\begin{aligned} v^\infty(\hat{x}) &= \left(\operatorname{div} \int_{B_R} \Phi_k(\cdot - y)\; Q(y) \nabla [v(y) + u^i(y)] \, dy \right)^\infty (\hat{x}) \\ &= - \int_{B_R} \left(\nabla_y \Phi_k(\cdot - y) \right)^\infty (\hat{x}) \cdot Q(y) \nabla [v(y) + u^i(y)] \, dy \\ &= - \int_{B_R} \nabla_y e^{-ik\hat{x}\cdot y} \cdot Q(y) \nabla [v(y) + u^i(y)] \, dy \\ &= ik\hat{x} \cdot \int_{B_R} Q(y) \nabla [v(y) + u^i(y)] e^{-ik\hat{x}\cdot y} \, dy, \quad \hat{x} \in \mathbb{S}^{d-1}. \end{aligned} \tag{3.22}$$

CONSTRUCTING THE FORWARD OPERATOR As the latter integral expression is an analytic function in $\hat{x}$, the far field v^∞ is analytic as well. Let us now introduce, for brevity, the integral operator

$$V: L^2(B_{2R})^d \to H^2(B_{2R})^d, \qquad Vf = \int_{B_R} \Phi_k(\cdot - y) f(y) \, dy.$$

(See [CK13, Theorem 8.2] for the mapping properties of V.) The total field $v + u^i$ restricted to B_{2R} satisfies in $H^1(B_{2R})$:

$$\begin{aligned} v + u^i &= [\mathrm{Id} - \operatorname{div} V(Q\nabla(\cdot))]^{-1} \operatorname{div} V(Q\nabla u^i) + u^i \\ &= [\mathrm{Id} - \operatorname{div} V(Q\nabla(\cdot))]^{-1} u^i. \end{aligned}$$

Thus, we abbreviate the (bounded) inverse of $\mathrm{Id} - \operatorname{div} V(Q\nabla(\cdot))$ on $H^1(B_{2R})$ by S_Q and represent the far field $v^\infty = L(Q, u^i)^\infty$, computed in (3.22), as

$$v^\infty(\hat{x}) = ik \int_{B_R} \hat{x} \cdot Q(y) \nabla \big(S_Q\, u^i\big)(y)\, e^{-ik\hat{x}\cdot y} \, dy, \quad \hat{x} \in \mathbb{S}^{d-1}.$$

If we further introduce the integral operator

$$Z: L^t(B_R)^d \to L^2(\mathbb{S}^{d-1}), \qquad f \mapsto ik \int_{B_R} \hat{x} \cdot f(y)\, e^{-ik\hat{x}\cdot y} \, dy, \tag{3.23}$$

then there holds that

$$L(Q, u^i)^\infty = Z \circ \left[Q\nabla S_Q\big(u^i\big) \right].$$

As $Q \in \mathcal{Q} \subset L^\infty(B_{2R})^{d\times d}$ and $S_Q\big(u^i\big) \in L^2(B_{2R})$, the following lemma shows that the composition on the right is well-defined and bounded. This is basically due to the smoothing property of Z, which is a trace class operator, see [Gro55].

Lemma 3.18. *Choose* $m \in \mathbb{N}$, $1 < t < \infty$, *and* $f \in L^t(B_{2R})^d$.

(i) There is $C = C(m,t)$ *such that* $\|Zf\|_{C^m(\mathbb{S}^{d-1})} \leqslant C(m,t)\|f\|_{L^t(B_{2R})^d}$.

(ii) The operator Z *is of trace class from* $L^t(B_{2R})^d$ *into* $L^2(\mathbb{S}^{d-1})$.

Proof. (i) The kernel of the integral operator Z is smooth in $\hat{x}$ and y, such that one easily shows the claimed bound by partial integration and the Hölder inequality.

(ii) Due to the bounds shown in part (i), Z is bounded from $L^t(B_{2R})^d$ into any Hilbert space $H^m(\mathbb{S}^{d-1})$. Choosing m large enough then implies that the embedding of $H^m(\mathbb{S}^{d-1})$ in $L^2(\mathbb{S}^{d-1})$ is a trace class operator, see [Gra68]. As those operators form an ideal, Z is also a trace class operator that maps $L^t(B_{2R})^d$ into $L^2(\mathbb{S}^{d-1})$. □

We are now ready to rigorously introduce the forward operator that, by definition, maps contrasts to far field operators. To this end, we consider incident fields in form of Herglotz wave functions,

$$v_g(x) = \int_{\mathbb{S}^{d-1}} e^{ikx\cdot\theta} g(\theta)\, dS(\theta), \quad \text{for } x \in \mathbb{R}^d \text{ and } g \in L^2(\mathbb{S}^{d-1}), \tag{3.24}$$

that are well-known entire solutions to the Helmholtz equation in $\mathbb{R}^d$. It is moreover well-known that $g \mapsto v_g|_{B_{2R}}$ is a bounded operation from $L^2(\mathbb{S}^{d-1})$ into $H^1(B_{2R})$, see [CK13, Section 3.3]. Using v_g as an incident field then defines a far field operator $F_Q \in L^2(\mathbb{S}^{d-1})$ by $F_Q g = (L(Q, v_g))^\infty$ for $g \in L^2(\mathbb{S}^{d-1})$. As the integral kernel $u^\infty = u_Q^\infty \colon \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{C}$ of $F(Q) = F_Q$ is analytic in both variables, F_Q is compact and even belongs to the set $\mathcal{S}_1$ of trace class operators on $L^2(\mathbb{S}^{d-1})$, since its singular values $s_j(F_Q)$ are summable, i.e., $\|F_Q\|_{\mathcal{S}_1} = \sum_{j\in\mathbb{N}} |s_j(F_Q)| < \infty$. The embedding $\ell^p \subset \ell^q$ for $1 \leqslant p < q \leqslant \infty$ of the sequence spaces ℓ^p further implies that trace class operators belong to the qth Schatten class $\mathcal{S}_q$ for all $q \in [1, \infty)$, a Banach space of all compact operators on $L^2(\mathbb{S}^{d-1})$ with q-summable singular values $s_j(F_Q)$, equipped with the norm defined by

$$\|F_Q\|_{\mathcal{S}_q}^q = \sum_{j\in\mathbb{N}} |s_j(F_Q)|^q, \quad \text{for } q \geqslant 1.$$

This allows to define the contrast-to-far field mapping,

$$F\colon \mathcal{Q} \to \mathcal{S}_q, \qquad F(Q)g = Z \circ \big(Q\nabla S_Q(v_g)\big) \text{ for } g \in L^2(\mathbb{S}^{d-1}),\ q \geqslant 1, \tag{3.25}$$

as an operator from $\mathcal{Q}$ into the qth Schatten class $\mathcal{S}_q$.

Properties of the forward operator

Remark 3.19. A. Due to Lemma 3.18 with $t = 2$ and the continuity properties of the solution operator L, the composition $Z \circ \big(Q\nabla S_Q(v_g)\big)$ is well-defined in $L^2(\mathbb{S}^{d-1})$.

B. Since trace class operators form an ideal in the space of all bounded operators, and as $F(Q)g = Z(L(Q, \nu_g))$ with a trace class operator Z, the forward operator is a trace class operator as well, and, hence, belongs to all spaces $\mathcal{S}_q$ for $q \geqslant 1$.

C. An alternative to the $\mathcal{S}_q$-norms are L^q-norms for integral operators on the sphere: Since $F(Q)g = \int_{\mathbb{S}^{d-1}} u^\infty(\cdot, \theta) g(\theta)\, dS(\theta)$ is represented by the far field pattern $u^\infty(\cdot, \theta)$ of the scattered fields $u^s = L(Q, \nu_g)$, the L^q-Norm of u^∞ defines an operator norm for $F(Q)$ by $\|F(Q)\|_q := \|u^\infty\|_{L^q(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})}$, $1 < q < \infty$. The contrast-to-far field map $Q \mapsto F(Q)$ as defined in (3.25) is then continuous from $L^{q'}(\mathbb{S}^{d-1})$ into $L^q(\mathbb{S}^{d-1})$ with $q' = q/(q-1)$, because $g \mapsto \nu_g|_D$ is continuous from $L^{q'}(\mathbb{S}^{d-1})$ into $C^1(D)$ for all $q \in (1, \infty)$. For $q = 2$, it is well-known that $\|\cdot\|_{\mathcal{S}_2} = \|\cdot\|_2$. The advantage of the L^q-norms with respect to the implementation of inversion algorithms is that the computation of adjoints or subdifferentials is straightforward for these spaces. Since the subsequent theoretic results do not depend on the choice of the discrepancy norm, we continue to work with the Schatten norms $\|\cdot\|_{\mathcal{S}_q}$, noting that all results holds as well for the $\|\cdot\|_q$-norms.

The link between the solution operator L and the non-linear forward operator F enables us to show various properties of F via those of L. To this end, note first that the far field of the radiating extension of $L(Q, \nu_g)$ depends boundedly and linearly on $L(Q, \nu_g)$, such that the derivative $Q' \mapsto F'(Q)[Q'] \in \mathcal{L}(L^p(B_{2R})^{d \times d}, \mathcal{S}_q)$ with respect to $Q \in \mathcal{Q}$ of F equals by the product rule in Banach spaces, see [Zei86],

$$F'(Q)[Q']g = Z \circ \left[Q \nabla L'(Q, \nu_g)[Q'] + Q' \nabla S_Q(\nu_g) \right], \qquad (3.26)$$

since $L(Q, \nu_g) = S_Q(\nu_g) - \nu_g$. This allows to transfer the results of Theorems 3.9, 3.14, and 3.15 from L to F.

COROLLARY 3.20. *Choose* $Q \in \mathcal{Q}$ *and* $Q + Q' \in \mathcal{Q}$ *such that* Q' *satisfies* (3.13), *and* $q \geqslant 1$.

(i) If $p > 2T_\lambda/(T_\lambda - 2)$ *then there is* $C = C(Q)$ *such that*

$$\|F(Q + Q') - F(Q)\|_{\mathcal{S}_q} \leqslant C \|Q'\|_{L^p(B_R)^{d \times d}}. \qquad (3.27)$$

(ii) The operator $F'(Q)$ *is locally Lipschitz continuous with respect to* $L^p(B_R)^{d \times d}$*: There is* $C = C(Q)$ *such that*

$$\|F'(Q + Q') - F'(Q)\|_{\mathcal{L}(L^p(B_{2R})^{d \times d}, \mathcal{S}_q)} \leqslant C \|Q'\|_{L^p(B_R)^{d \times d}}.$$

(iii) If $p > 4T_\lambda/(T_\lambda - 2)$ *the far field operator* $F(Q)$ *is differentiable in the sense that*

$$\|F(Q + Q') - F(Q) - F'(Q)[Q']\|_{\mathcal{S}_q} \leqslant C \|Q'\|^2_{L^p(B_R)^{d \times d}}$$

for $C = C(Q)$*. Is there further a sequence* $\{Q'_n\}_{n\in\mathbb{N}}$ *satisfying* (3.13) *for all* $n \in \mathbb{N}$ *and if* $\|Q'_n\|_{L^p(B_R)^{d\times d}} \to 0$ *as* $n \to \infty$*, then*

$$\|F(Q+Q'_n) - F(Q) - F'(Q)[Q'_n]\|_{\mathcal{S}_q} / \|Q'_n\|_{L^p(B_R)^{d\times d}} \to 0.$$

Proof. The basic ingredient of the proof is the smoothing property of the far field map Z defined in (3.23), which is a trace class operator from $L^2(B_{2R})^d$ into $L^2(\mathbb{S}^{d-1})$. Choosing the incident field u^i as a Herglotz wave function v_g for some $g \in L^2(\mathbb{S}^{d-1})$,

$$\begin{aligned}
&\|F(Q+Q') - F(Q)\|_{\mathcal{S}_q} \\
&= \left\| g \mapsto Z\left[(Q+Q')\nabla S_{Q+Q'}(v_g) - Q\nabla S_Q(v_g)\right] \right\|_{\mathcal{S}_q} \\
&\leqslant \left\| g \mapsto Z\left[(Q+Q')\nabla S_{Q+Q'}(v_g) - Q\nabla S_Q(v_g)\right] \right\|_{\mathcal{S}_1} \\
&\overset{(*)}{\leqslant} C \left\| g \mapsto \left[(Q+Q')\nabla S_{Q+Q'}(v_g) - Q\nabla S_Q(v_g)\right] \right\|_{\mathcal{L}(L^2(\mathbb{S}^{d-1}), L^2(B_R)^d)} \\
&\leqslant C \sup_{\|g\|_{L^2}=1} \Big[\left\| Q'\nabla S_{Q+Q'}(v_g) \right\|_{L^2(B_R)^d} \\
&\qquad\qquad + \left\| Q\nabla [S_{Q+Q'}(v_g) - S_Q(v_g)] \right\|_{L^2(B_R)^d} \Big],
\end{aligned}$$

where inequality $(*)$ follows from Lemma 3.18(ii) and the fact that the composition of the trace class operator Z with a bounded and linear operator is of trace class as well. Now we use again the technique from the proof of Theorem 3.9, see (3.16) and (3.17), to obtain the bound

$$\|Q'\nabla S_{Q+Q'}(v_g)\|_{L^2(B_R)^d} \leqslant \|Q'\|_{L^p(B_R)^{d\times d}} \|S_{Q+Q'}(v_g)\|_{H^1(B_R)}$$

where due to Lemma 3.7 the total wave field satisfies the estimates

$$\|S_{Q+Q'}(v_g)\|_{H^1(B_R)} \leqslant C\, \|v_g\|_{H^1(B_{2R})} \leqslant C\, \|g\|_{L^2(\mathbb{S}^{d-1})} = C,$$

with a constant $C = C(Q)$ independent of Q'. As the difference of the total fields $S_{Q+Q'}(v_g) - S_Q(v_g)$ equals the difference of the scattered fields $L(Q+Q', v_g) - L(Q, v_g)$ for the fixed incident field v_g, Theorem 3.9 further shows that

$$\begin{aligned}
&\left\| Q\nabla [S_{Q+Q'}(v_g) - S_Q(v_g)] \right\|_{L^2(B_R)^d} \\
&\qquad \leqslant C\, \|Q\|_{L^\infty(B_R)^{d\times d}} \|Q'\|_{L^p(B_R)^{d\times d}} \|v_g\|_{H^1(B_{2R})}
\end{aligned} \tag{3.28}$$

such that by plugging the last estimates together we conclude that $\|F(Q+Q') - F(Q)\|_{\mathcal{S}_q} \leqslant C(Q)\|Q'\|_{L^p(B_R)^{d\times d}}$. The bounds in (ii) and (iii) are shown analogously, using Theorems 3.14 and 3.15 instead of Theorem 3.9. □

As for Theorem 3.9, the last corollary's assumption that (3.13) holds for Q' can be replaced by uniformly bounded solution operators, see Corollary 3.11.

COROLLARY 3.21. *Assume that* $\| L(Q, \cdot)\|_{H^1(B_{2R})\to H^1(B_{2R})}$ *is uniformly bounded for all* Q *in a subset* $\mathcal{Q}'$ *of* $\mathcal{Q}$*. Then for all* $p > 2T_\lambda/(T_\lambda - 2)$ *there is* $C_* > 0$ *depending on* $\mathcal{Q}'$ *and* p *such that* (3.27) *holds for all* Q, $Q + Q' \in \mathcal{Q}'$ *with* C *replaced by* C_*.

Proof. Instead of Theorem 3.9, use Corollary 3.11 to obtain the bound in (3.28) of the last proof. □

3.6 NON-LINEAR TIKHONOV AND SPARSITY REGULARIZATION

The inverse problem we consider is to stably approximate a contrast Q_{exa} from perturbed measurements of its far field operator $F(Q_{exa})$. More precisely, for noisy measurements F^δ_{meas} with noise level $\delta > 0$ such that $\|F(Q_{exa}) - F^\delta_{meas}\|_{\mathcal{S}_q} \leqslant \delta$, we seek to approximate Q by non-linear Tikhonov regularization. Thus, for some penalty term $\mathcal{R}$ we consider to minimize the Tikhonov functional

$$\mathcal{J}_{\alpha,\delta}(Q) := \frac{1}{q}\|F(Q) - F^\delta_{meas}\|^q_{\mathcal{S}_q} + \alpha\,\mathcal{R}(Q), \quad q \in [1, \infty), \tag{3.29}$$

over some appropriate admissible set of contrasts included in $\mathcal{Q}$. As the functional $\mathcal{J}_{\alpha,\delta}$ requires $F(Q)$ to be well-defined, we suppose for the rest of the chapter that the variational formulation (3.6) of the forward problem is uniquely solvable for all $Q \in \mathcal{Q}$ and all incident fields u^i that solve the Helmholtz equation in $\mathbb{R}^d$.

ASSUMPTION 3.22. The variational formulation (3.6) is uniquely solvable for all $Q \in \mathcal{Q}$ and all incident fields u^i that solve the Helmholtz equation in $\mathbb{R}^d$, and the norm of the solution operator $L(Q, \cdot)$ on $H^1(B_{2R})$ is uniformly bounded for $Q \in \mathcal{Q}$.

Due to Lemma 3.7, the first part of the latter assumption can always be guaranteed by choosing the parameter $\lambda \in (0, 1)$ that defines the set $\mathcal{Q}$ in (3.4) close enough to one, as (3.6) is uniquely solvable if Q is the identity matrix. The second part can be guaranteed by merely considering contrasts in $\mathcal{Q} \cap X$ for some space $X \subset L^\infty(B_{2R})^{d\times d}$ that embeds compactly into $L^\infty(B_{2R})^{d\times d}$.

Before presenting the Tikhonov regularization framework in detail, we first show Lipschitz continuity of the discrepancy $\mathcal{E}(Q) := \|F(Q) - F^\delta_{meas}\|_{\mathcal{S}_q}$.

THEOREM 3.23. *If Assumption* 3.22 *holds, then* $|\mathcal{E}(Q) - \mathcal{E}(Q + Q')| \leqslant C\,\|Q'\|_{L^p(B_R)^{d\times d}}$ *for* $p > 2T_\lambda/(T_\lambda - 2)$ *for all elements* Q *and* $Q + Q'$ *of* $\mathcal{Q}$.

Proof. For all contrasts Q and $Q + Q'$ in $\mathcal{Q}$ the reverse triangle inequality for norms implies that $|\mathcal{E}(Q) - \mathcal{E}(Q + Q')| \leqslant \|F(Q) - F(Q + Q')\|_{\mathcal{S}_q}$. By Assumption 3.22, Corollary 3.21 bounds the last right-hand side uniformly in Q and Q' by $\|F(Q) - F(Q + Q')\|_{\mathcal{S}_q} \leqslant C\,\|Q'\|_{L^p(B_R)^{d\times d}}$ for any choice of $p > 2T_\lambda/(T_\lambda - 2)$. Thus, $\mathcal{E}$ is Lipschitz continuous on $\mathcal{Q}$ with respect to $L^p(B_R)^{d\times d}$ for $p > 2T_\lambda/(T_\lambda - 2)$. □

TIKHONOV REGULARIZATION Fixing $p > 2T_\lambda/(T_\lambda - 2)$, let us now set $p_* = dp/(p+d)$, such that $1 < p_* < d$. Sobolev's embedding theorem then implies that $W^{1,p_*}(B_{2R})^{d\times d}$ embeds compactly into $L^p(B_{2R})^{d\times d}$; moreover, the Sobolev inequality

$$\|Q\|_{L^p(B_{2R})^{d\times d}} \leqslant C\,\|Q\|_{W^{1,p_*}(B_{2R})^{d\times d}}$$

holds for all $Q \in W^{1,p_*}(B_{2R})^{d\times d}$. Note that functions in $W^{1,p_*}(B_{2R})^{d\times d}$ are in general discontinuous, since an embedding into Hölder spaces would require $p_* > d$. In the sequel we consider $W_0^{1,p_*}(B_{2R})^{d\times d}$ to be the space of functions that vanish on ∂B_R and extend those by zero to all of $\mathbb{R}^d$, such that the intersection of $W_0^{1,p_*}(B_R)^{d\times d}$ with $\mathcal{Q}$ is well-defined. (By abuse of notation, we do not denote this extension explicitly.)

Non-linear Tikhonov regularization is classically based on the assumption that the penalty term $\mathcal{R}$ is coercive in the space of interest $L^p(B_R)^{d\times d}$, such that weak convergence results can be obtained for a minimizing sequence. If $\mathcal{R}$ is even coercive in a space compactly embedded in $L^p(B_R)^{d\times d}$, then one directly obtains strong convergence of the minimizing sequence.

THEOREM 3.24 (Tikhonov regularization). *If we choose the penalty term $\mathcal{R}$ of the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$ as $\mathcal{R}(Q) = \|Q\|^{p_*}_{W^{1,p_*}(B_R)^{d\times d}}$, then $\mathcal{J}_{\alpha,\delta}$ possesses a minimizer in $\mathcal{Q}\cap W^{1,p_*}(B_R)^{d\times d}$. If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weakly converges to a solution $Q^\dagger \in W^{1,p_*}(B_R)^{d\times d}\cap\mathcal{Q}$ such that $F(Q^\dagger) = F(Q_{exa})$ holds in $\mathcal{S}_q$ and $Q^\dagger$ minimizes the $W^{1,p_*}(B_R)^{d\times d}$-norm amongst all solution to the latter equation.*

SPARSITY REGULARIZATION As the proof of Theorem 3.24 is well-known, see, e.g., [Sch+12] or the proof of Theorem 3.26 below, we directly present a sparsity-promoting alternative based on a wavelet basis of W^{1,p_*}, see [Tri06]. Assume that $\psi_{Mo} \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, is a compactly supported (mother) wavelet with scaling function ψ_{Fa}, associated to a multi-resolution analysis, such that the corresponding one-dimensional wavelets

$$\psi^j_m(x_r) = \begin{cases} \psi_{Fa}(x_r - m) & \text{if } j = 0,\ m \in \mathbb{Z}, \\ 2^{(j-1)/2}\psi_{Mo}(2^{j-1}x_r - m) & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}, \end{cases}$$

form a wavelet basis for $x = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$. By that define d-dimensional n-wavelets as usual by setting $\Psi_m(x) = \prod_{r=1}^d \psi_{Fa}(x_r - m_r)$ for $m \in \mathbb{Z}^d$. For $\{Fa, Mo\}^{d*} = \{G \in \{Fa, Mo\}^d$: at least one component of G equals $Mo\}$ and $x \in \mathbb{R}^d$, we further set

$$\Psi^G_m(x) = \prod_{r=1}^d \psi_{G_r}(x_r - m_r), \quad m \in \mathbb{Z}^d,\ G = (G_r)_{r=1}^d \in \{Fa, Mo\}^{d*},$$

introduce $G^0 = \{(\mathrm{Fa})^d\} = \{(\mathrm{Fa}, \dots, \mathrm{Fa})\}$ and $G^j = \{\mathrm{Fa}, \mathrm{Mo}\}^{d*}$ for $j \in \mathbb{N}$, and for $m \in \mathbb{Z}^d$ define n-wavelets on $\mathbb{R}^d$ by

$$\Psi_m^{j,G}(x) = \begin{cases} \Psi_m(x) & \text{for } j = 0,\ G \in G^0, \\ 2^{(j-1)d/2}\Psi_m^G(2^{j-1}x) & \text{for } j \in \mathbb{N},\ G \in G^j. \end{cases}$$

We finally define wavelet coefficients of functions $Q \in L^1(\mathbb{R}^d)^{d\times d}$ by

$$Q_m^{j,G} = \left(\int_{\mathbb{R}^d} Q_{\ell,\ell'} \Psi_m^{j,G}\, dx \right)_{\ell,\ell'=1}^{d} \in \mathbb{C}^{d\times d}. \tag{3.30}$$

Examples for suitable wavelets include the well-known Daubechies wavelets, see [Dau88; Dau92]; however, the following result holds as well for differently constructed Meyer wavelets, see Chapter 3.1.5 in [Tri06], in particular Theorem 3.12.

THEOREM 3.25 (See [Tri06, Theorem 3.5]). *For $1 \leqslant p_* < \infty$ and the above-defined n-wavelets $\Psi_m^{j,G}$ with*

$$n \in \mathbb{N} \quad \textit{such that} \quad n > \max(1, 2\tfrac{d}{p_*} + \tfrac{d}{2} - 1),$$

there holds that the set of functions $\{\Psi_m^{j,G}\}$ is an unconditional basis in $W^{1,p_*}(\mathbb{R}^d)^{d\times d}$. Further, there are constants $A, B > 0$ such that for all $Q \in W^{1,p_*}(\mathbb{R}^d)^{d\times d}$ there holds

$$A\|Q\|_{W^{1,p_*}(\mathbb{R}^d)^{d\times d}} \leqslant \left(\sum_{j\in\mathbb{N}_0} 2^{j(p_*-d)} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^d} 2^{jdp_*/2} |Q_m^{j,G}|^{p_*} \right)^{1/p_*} \leqslant B\|Q\|_{W^{1,p_*}(\mathbb{R}^d)^{d\times d}}. \tag{3.31}$$

In the following, we use the representation of the W^{1,p_*}-norm in (3.31) for contrasts $Q \in W_0^{1,p_*}(B_R)^{d\times d}$ that are extended by zero to all of $\mathbb{R}^d$ and, to this end, abbreviate the series in (3.31) by $\sum_{j,G,m}$. For all numbers $1 \leqslant r \leqslant p_*$ and all sequences $(a_j)_{j\in\mathbb{N}}$ in $\ell^{p_*}(\mathbb{N})$ there holds that $\left(\sum_{j=1}^\infty |a_j|^{p_*}\right)^{1/p_*} \leqslant \left(\sum_{j=1}^\infty |a_j|^r\right)^{1/r}$. Fix such $r \in [1, p_*]$ and choose any sequence of weights $(\omega_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}^\infty$ such that $\omega_j \geqslant 2^{j(1-d/p_*+d/2)r}$. Then the functional

$$\begin{aligned} \mathcal{R}_r(Q) := \frac{1}{r} \sum_{j,G,m} \omega_j |Q_m^{j,G}|^r &\geqslant \frac{1}{r} \sum_{j,G,m} 2^{j(1-d/p_*+d/2)r} |Q_m^{j,G}|^r \\ &\geqslant \frac{1}{r} \left(\sum_{j,G,m} 2^{j(p_*-d+dp_*/2)} |Q_m^{j,G}|^{p_*} \right)^{r/p_*} \geqslant \frac{A^r}{r} \|Q\|^r_{W^{1,p_*}(B_R)^{d\times d}} \end{aligned} \tag{3.32}$$

bounds the rth power of the W^{1,p_*}-norm of $Q \in W_0^{1,p_*}(B_R)^{d\times d}$ from above.

Recall now that $F(Q_{\mathrm{exa}})$ and $F^\delta_{\mathrm{meas}} \in \mathcal{S}_q$ model exact and noisy measurements, respectively, with noise level $\|F(Q_{\mathrm{exa}}) - F^\delta_{\mathrm{meas}}\|_{\mathcal{S}_q} \leqslant \delta$.

Due to the the above setting, the following result follows straightforwardly from standard non-linear regularization theory, see, e.g., [Sch+09; Sch+12].

THEOREM 3.26 (Sparsity regularization). *For $1 \leqslant r \leqslant p_* = dp/(p+d)$, the functional $\mathcal{J}_{\alpha,\delta}$ with $\mathcal{R} = \mathcal{R}_r$ from (3.32) possesses a minimizer in $\mathcal{Q} \cap W^{1,p_*}(B_R)^{d\times d}$. If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weakly converges to an $\mathcal{R}_r$-minimizing solution $Q^\dagger \in W^{1,p_*}(B_R)^{d\times d} \cap \mathcal{Q}$ of the equation $F(Q) = F(Q_{exa})$ in $\mathcal{S}_q$.*

Recall that $Q^\dagger$ is a $\mathcal{R}_r$-minimizing solution to $F(Q^\dagger) = F(Q_{\text{exa}})$ if

$$\mathcal{R}_r(Q^\dagger) = \min\left\{\mathcal{R}_r(Q),\ Q \in \mathcal{Q} \cap W^{1,p_*}(B_R)^{d\times d},\ F(Q) = F_{\text{exa}}\right\}.$$

Proof. We repeat the proof for the existence of a minimizer of $\mathcal{J}_{\alpha,\delta}$. For an arbitrary minimizing sequence $\{Q^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ the penalty $\{\mathcal{R}_r(Q^{(n)})\}$ is uniformly bounded, such that

$$\begin{aligned}\|Q^{(n)}\|^r_{L^p(B_R)^{d\times d}} &\leqslant C\|Q^{(n)}\|^r_{W^{1,p_*}(B_R)^{d\times d}} \\ &\leqslant \frac{Cr}{A^r}\sum_{j,G,m}\omega_j\left|(Q^{(n)})^{j,G}_m\right|^r = \frac{Cr}{A^r}\mathcal{R}_r(Q^{(n)})\end{aligned}$$

is uniformly bounded as well. As $W^{1,p_*}(B_R)^{d\times d}$ is a reflexive Banach space, the sequence $\{Q^{(n)}\}_{n\in\mathbb{N}}$ contains a weakly convergent subsequence that converges in $L^p(B_R)^{d\times d}$ due to the compact embedding of $W^{1,p_*}(B_R)^{d\times d}$ in $L^p(B_R)^{d\times d}$, say, to $Q \in W^{1,p_*}(B_R)^{d\times d}$. Since $\mathcal{Q}$ is a convex set, the limit Q belongs to $\mathcal{Q}$ and Lipschitz continuity of the discrepancy term $\mathcal{E}$ with respect to $L^p(B_R)^{d\times d}$ implies that $\mathcal{E}(Q_n) \to \mathcal{E}(Q)$. Lower semi-continuity of the penalty term $\mathcal{R}_r$ with respect to $W^{1,p_*}(B_R)$ shows that Q is a minimizer of $\mathcal{J}_{\alpha,\delta}$. Consistency of the minimizers for vanishing noise level under the given choice of the regularization parameter α can be shown as in, e.g., [Sch+12]. □

We omit here to show well-known source conditions that imply convergence rates of the minimizers, as these are classic and can be straightforwardly transferred from either abstract results in, e.g., [Sch+12], or from [JM12a], to our setting. All required analytic properties of the forward operator F can be derived from Corollary 3.20.

As it is well-known that a solution Q to the inverse problem $F(Q) = F^\delta_{\text{meas}}$ is only unique up to a change of variables, Theorem 3.26 shows that all we can hope for is to determine an $\mathcal{R}_r$-minimizing solution. Even if we restrict ourselves to an (scalar) isotropic contrast of the form $Q = q^{\text{sc}}\,\text{Id}_d$, it is unclear to us whether the far field operator $F_{q^{\text{sc}}}$ corresponding to q^{sc} uniquely determines the isotropic contrast q^{sc}.

3.7 NUMERICAL EXAMPLES

After elaborating a theoretic framework that guarantees convergence of the Tikhonov iterates against a minimum-norm solution, we now present a couple of numerical experiments for contrasts, that are sparse in a wavelet basis. Referring to Section 2.2, that means that few wavelet coefficients of the isotropic contrast function are non-zero. To this end, we minimize the Tikhonov functional in (3.29) for the sparsity-promoting penalty (3.32) in a simplified setting: we model an isotropic test media by merely a scalar material parameter q^{sc} with wavelet coefficients $Wq^{sc} = \{(q^{sc})_m^{j,G}\}$ defined as in (3.30).

Starting with an initial guess q_0^{sc} (that will always be chosen as zero), we consider a Tikhonov functional for the linearization of the forward operator at the current iterate q_ℓ^{sc} and seek a minimizer h_ℓ of

$$h \mapsto \frac{1}{q}\|F(q_\ell^{sc}) + F'(q_\ell^{sc})[h] - F_{\text{meas}}^{\delta}\|_{\mathcal{S}_q}^{q} + \alpha \mathcal{R}_r(q_\ell^{sc} + h), \qquad (3.33)$$

where $\mathcal{R}_r$ is defined in (3.32). We tackle the latter minimization problem numerically by either a shrinked Landweber iteration as proposed by Daubechies, Defrise, and De Mol [DDD04], or, alternatively, by a primal-dual algorithm as proposed Chambolle and Pock [CP11], see also [HH14]. Whilst the first algorithm is simpler to implement and essentially equals the one used for the numerical experiments in [LKK13], its disadvantage, to some extend, is that it requires (squared) Hilbert space norms defining the discrepancy.

COMPUTATIONAL FRAMEWORK All examples moreover rely on simulated scattering data for 32 incident and scattering directions. Computation of synthetic data and the evaluation of the forward operator, as well as the adjoint of its derivative, require to numerically approximate solutions to the scattering problem (3.1–3.2) or to corresponding adjoint problems. To this end, we discretize the volumetric integral equation (3.20) by a collocation approach using trigonometric polynomials as in [Vai00], see also [LN14] for the analysis of a corresponding Galerkin method applied to a periodic variant of (3.1). The advantage of the resulting method is that the integral operator can be rapidly evaluated by the fast Fourier transform, which makes the solution of the discrete system by an iterative solver attractive. (We use the generalized minimal residual method (GMRES) with an accuracy of 10^{-6} as linear solver.) Moreover, the uniform grid of the domain-of-interest can remain fixed during the iteration.

All synthetic far field data are computed on a uniform grid of size 2048×2048 of $[-0.4, 0.4)^2$, which leads in the examples to a relative numerical error of less than one percent. (Wave lengths equal either $\pi/70 \approx 0.044$ and $\pi/50 \approx 0.0628$.) In the inversion schemes, we approximate solutions to scattering problems on a grid of size 512×512 on the same domain; the contrast function itself is resolved on a grid of size 128×128. We did not attempt to speed up the forward solver, such that most of the inversion time is due to solving (adjoint) forward problems (speeding up the forward solver hence yields a corresponding speed-up for the inverse solver). All computations are coded in Matlab® and indicated computation times are measured on an eight-core Intel® Core™i7-3770 CPU@3.40 GHz with 32 GB RAM.

When adding artificial noise to the synthetic data, we scale a matrix containing independent and normally distributed random numbers with mean zero and standard deviation one such that the sum of the synthetic data and the random matrix has a prescribed relative error, equal to $\delta = 0.01$, 0.05, or 0.1. Note that instead of discretizing the adjoint of the derivative of the forward operator, we rely on the adjoint of the discretization of the integral operator, to obtain exact adjunction up to the precision of the iterative solver. Figure 3.1 shows plots of the two contrasts $q^{sc\,(1,2)}$ of the isotropic test media we consider for inversion (for the complex-valued $q^{sc\,(1)}$ we plot real and imaginary part).

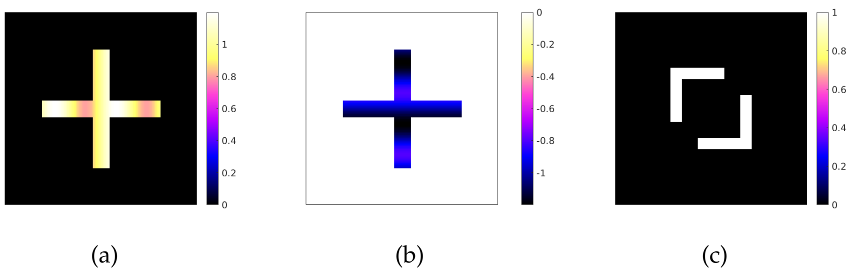

Figure 3.1: Contrasts plotted in $[-0.4, 0.4)^2$. (a) Real part of $q^{sc\,(1)}$ (b) Imaginary part of $q^{sc\,(1)}$ (c) Real-valued contrast $q^{sc\,(2)}$.

SHRINKED LANDWEBER ITERATION In detail the shrinked Landweber iteration determines $q^{sc}_{\ell+1} = q^{sc}_\ell + h_\ell$ by computing the ℓth step h_ℓ by resolving the first-order optimality conditions for a (scalar-valued) minimum of the non-linear functional $\mathcal{J}_{\alpha,\delta}$ in (3.29),

$$q^{sc}_\ell + \alpha\mu \sum_{j,G,m} \omega_j \operatorname{sign}((q^{sc}_\ell)^{j,G}_m) \left|(q^{sc}_\ell)^{j,G}_m\right|^{r-1} \psi^{j,G}_m = q^{sc}_\ell - \mu F'(q^{sc}_\ell)^* (F(q^{sc}_\ell) - F^\delta_{\text{meas}})$$

for all $\mu > 0$, which motivates to compute the update by a traditional fixed point iteration (see, e.g., [DDD04])

$$h_\ell = W^{-1} \circ \mathcal{S} \circ W\Big(q^{sc}_\ell - \mu_\ell F'(q^{sc}_\ell)^* [F(q^{sc}_\ell) - F^\delta_{\text{meas}}] \Big),$$

with a step-size parameter $\mu_\ell > 0$ determined by Armijo's rule, see, e.g., [Arm66], and W and W^{-1} as the forward and the inverse wavelet transform. Further, $\mathcal{S} = \mathcal{S}_{\alpha\mu_n\omega,r}$ is the so-called soft-thresholding operator: For $\tilde{\omega} = (\tilde{\omega}_j)_j$ with positive entries there holds $\mathcal{S}_{\tilde{\omega},r}(f_j) = (S_{\tilde{\omega}_j,r}(f_j))_j$ with scalar functions $S_{\alpha,r}$ defined as inverse function to $t \mapsto t + \alpha\,\mathrm{sign}(t)|t|^{r-1}$ for $r > 1$, see [DDD04]. (For $r = 1$ there holds $S_{\alpha,r}(t) = \mathrm{sign}(t)\,\max\{|t| - \alpha, 0\}$.)

Independent of how the update h_ℓ is computed, we stop the iteration for the q_ℓ^{sc} if the discrepancy is less than a fixed tolerance $\tau = 1.5$ times the (relative) noise level. All examples are computed using the Cohen-Daubechies-Feauveau 9/7 wavelets.

When choosing the weights $\omega = (\omega_j)_{j\in\mathbb{N}}$ such that the penalty term $\mathcal{R}_r$ with $r = 1$ is for a fixed wavelet discretization numerically equivalent to the $W^{1,3/2}(B_R)$-norm, the resulting reconstructions both for the shrinked Landweber iteration and the primal-dual algorithm did neither substantially differ in the visual norm nor regarding the resulting reconstruction errors from reconstructions for the constant sequence where $\omega_j \equiv 1$ (such that $\mathcal{R}_1$ is the ℓ^1-norm of the wavelet coefficients). As, additionally, the shrinked Landweber iteration required considerably more iteration steps, all results shown below are computed with for constant weights $\omega_j \equiv 1$ and $r = 1$.

In the following first set of examples we used the shrinked Landweber iteration sketched above for artificial noise levels δ equal to 0.01, 0.05, and 0.1 and regularization parameter $\alpha = \delta$. The wave number equals $k = 140$, such that the wave length is about 0.045. The iteration is stopped by the discrepancy principle if the (relative) discrepancy is less than $1.5\,\delta$. Figure 3.2 shows that the shape of the cross in Figures 3.1a and 3.1b is well reconstructed and that the magnitude of the reconstruction is roughly matched, at least for small noise level. However, the small variations of the contrast inside the cross-shape are not well resolved but tend either to thicken or to thin the width of the cross. For $\delta = 0.01$, the relative discrepancy does not reach the prescribed value of 0.015 in 500 iterations, which might be due to the numerical noise level of the synthetic data. We hence plot the 500th iterate; after the 400th iteration, the first two digits of the reconstruction do no longer change, such that this is, arguably, justified. Reconstruction times notably become tremendous for so many iteration steps.

Figure 3.3 shows the corresponding results in the same reconstruction setting for the real-valued double L-shape from Figure 3.1c. The inversion scheme converges somewhat faster; again, for $\delta = 0.01$ the reconstructions do not reach a relative discrepancy of $1.5\,\delta$ until the sequence of reconstructions becomes stationary at about the 200th iterate. Notably, the imaginary part of the reconstruction remains small during the iteration without imposing it to vanish by the algorithm. Again, the reconstruction times are rather high, which is a well-known disadvantage of soft-shrinking techniques. Generally speaking, the inversion problem is to our impression somewhat harder to

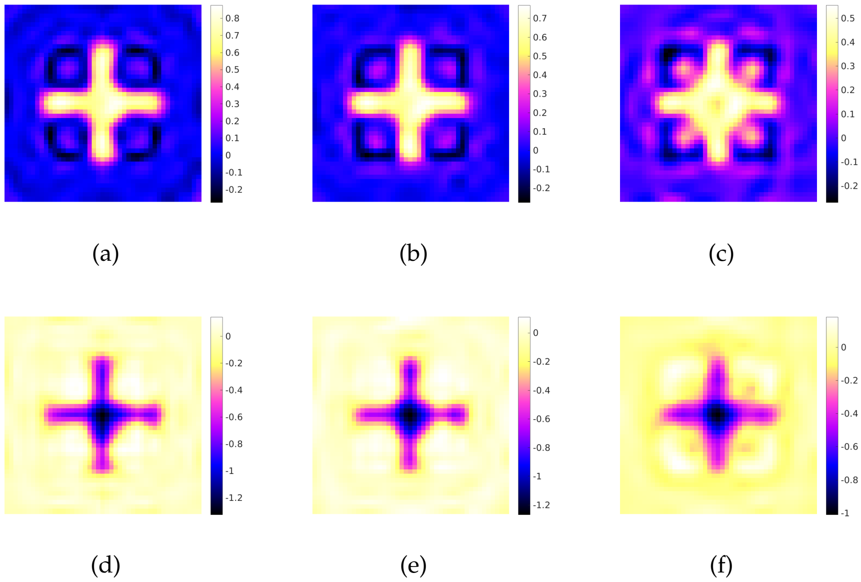

Figure 3.2: Reconstructions of $q^{sc\,(1)}$ by shrinked Landweber method, plotted in $[-0.4, 0.4)^2$ (real parts in top row, imaginary parts in bottom row). (a/d) $\delta = 0.01$, 500 iter., 2145 min., rel. error is 0.533 (b/e) $\delta = 0.05$, 300 iter., 748 min., rel. error is 0.565 (c/f) $\delta = 0.1$, 57 iter., 126 min., rel. error is 0.677.

tackle numerically by the shrinked Landweber iteration compared to the Helmholtz equation $\Delta u + k^2(1+q)u = 0$ considered in [LKK13].

Since due to their parallel edges the above shown examples are well suited for a tensor product basis, Figure 3.4 shows results of the less restrictive case of non-axis-parallel edges. Therefore, we reconstruct a rotation by 25° of above real-valued double L-shape from Figure 3.1c. As seen before, for $\delta = 0.01$ the reconstructions do not even reach a relative discrepancy of $1.7\,\delta$ until the sequence of reconstructions starts to get stationary at about the 300th iterate. Although considerably more iterations are needed, the reconstructions are slightly inferior to those in Figure 3.3.

PRIMAL-DUAL ALGORITHM To cope with the two most obvious disadvantages of the shrinked Landweber iteration, we finally consider the primal-dual algorithm by Chambolle and Pock. This allows first to consider different norms for the discrepancy term of the Tikhonov functional (we choose discretized L^p-norms for functions on $\mathbb{S}^1 \times \mathbb{S}^1$-norms as explained in Remark 3.19) and second yields smaller computation times. This algorithm computes the minimizer of the Tikhonov functional in (3.33) by explicitly considering the resolvents of the subdifferentials of the convex functionals $F \mapsto \|F - F^{\delta}_{\text{meas}} + F(q^{sc}_{\ell})\|^q_{\mathcal{S}_q}$ and $h \mapsto \alpha\,\mathcal{R}_r(q^{sc}_{\ell} + h)$. More precisely, let us consider general proper, convex, and lower semicontinuous func-

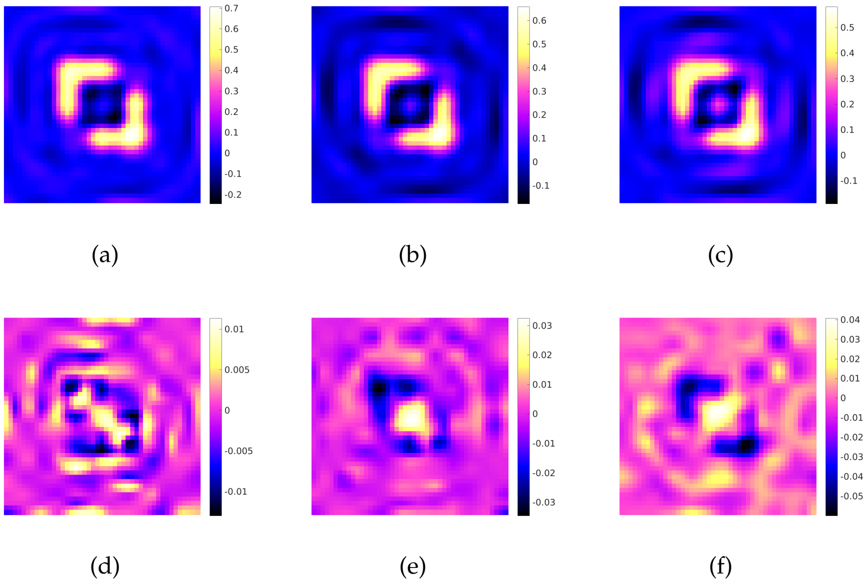

Figure 3.3: Reconstructions of $q^{sc\,(2)}$ by shrinked Landweber method, plotted in $[-0.4, 0.4]^2$ (real parts in top row, imaginary parts in bottom row). (a/d) $\delta = 0.01$, 200 iter., 390 min., rel. error is 0.653 (b/e) $\delta = 0.05$, 48 iter., 87 min., rel. error is 0.665 (c/f) $\delta = 0.1$, 20 iter., 38 min., rel. error is 0.703.

tionals $\mathcal{E} : \mathcal{S}_q \to [0, \infty]$ and $\mathcal{P} : \mathcal{Q} \to [0, \infty]$, as long as the resolvents $(I + \sigma\partial\mathcal{E}^*)^{-1}$ and $(I + \eta\partial\mathcal{P})^{-1}$ of the subgradients of the Fenchel conjugate $\mathcal{E}^*$ of $\mathcal{E}$ and of $\mathcal{P}$ are explicitly computable for $\eta, \sigma > 0$. The primal-dual algorithm then computes the minimizer of

$$h \mapsto \mathcal{E}(F'(q_\ell^{sc})[h] - (F_{meas}^\delta - F(q_\ell^{sc}))) + \alpha\,\mathcal{P}(q_\ell^{sc} + h) \tag{3.34}$$

via these resolvents. As already mentioned, $\mathcal{E} = \|\cdot\|_q^q/q$ for $1 < q < \infty$, see Remark 3.19. We further define $\mathcal{P}(\cdot)$ as sum of $\mathcal{R}_r$ from (3.32) and a convex functional $\mathbb{1}_b$ that ensures that the reconstructed contrast respects a-priori known pointwise bounds: $\mathbb{1}_b(q^{sc}) = 0$ if $-1 \leqslant \operatorname{Re} q^{sc}(x) \leqslant 3$ and $0 \leqslant \operatorname{Im} q^{sc}(x) \leqslant 3$ in $[-0.4, 0.4]^2$; and $\mathbb{1}_b(q^{sc}) = \infty$ otherwise. For both functionals, the subgradients can be computed using basic rules of convex analysis, see, e.g., [Roc97], and both resolvents can be computed explicitly.

For the remaining example, we invert $q^{sc\,(2)}$ for scattering data for $k = 100$, such that $2\pi/k \approx 0.063$. The artificial noise level δ is set to 0.01, the regularization parameter α in (3.34) equals 0.01 and the remaining parameters η and σ to $(5/4\|F'(q_\ell^{sc})\|)^{1/2}$. We stop the primal-dual algorithm applied to the linearized functional in (3.34) when the relative residuum of the linear equation is less than 0.05 (which typically yields less than ten steps and is finished in less than a minute). The largest part of the computation time of the primal-dual algorithm is due to the computation of the entire (factorized)

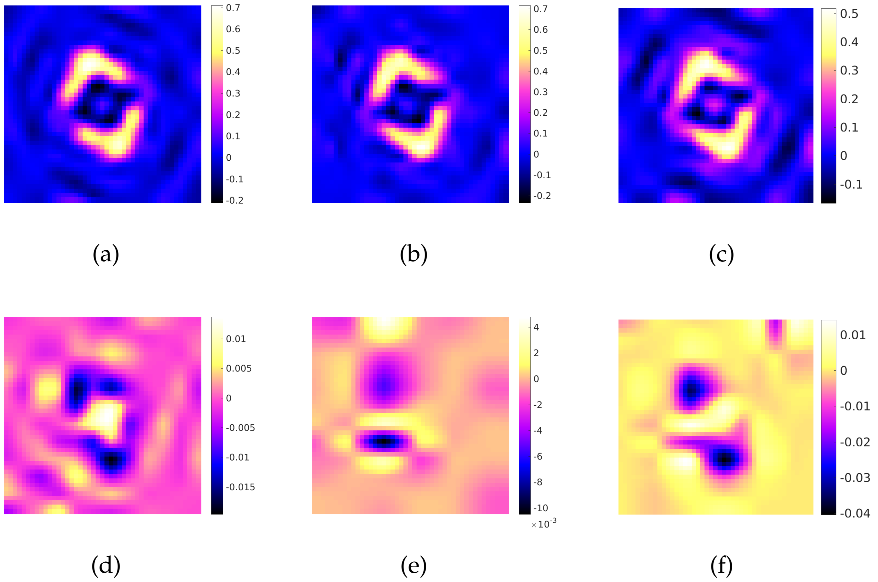

Figure 3.4: Reconstructions of $q^{sc\,(2)}$ rotated by 25° by shrinked Landweber method, plotted in $[-0.4, 0.4)^2$ (real parts in top row, imaginary parts in bottom row). (a/d) $\delta = 0.01$, 300 iter., rel. error is 0.668 (b/e) $\delta = 0.05$, 445 iter., rel. error is 0.669 (c/f) $\delta = 0.1$, 99 iter., rel. error is 0.734.

matrix representing the derivative of the forward operator at the current iterate q^{sc}_ℓ. (This typically takes less than two minutes for the examples below.) The numerical computation of the (matrix) norm of $F'(q^{sc}_\ell)$ takes about one minute and executing the algorithm for one linearized problem typically less than three minutes. Figure 3.5 shows the effect of changing the parameter $q \in (1, \infty)$ of the discrepancy term $\| \cdot \|_q^q / q$ by plotting reconstructions for $q = 1.6$, 2, and 3. (We simply plot the iterate with the smallest error.) Let us first note that for all reconstructions, the computation times are much smaller than for the shrinked Landweber iteration. Generally, choosing q larger/smaller results in smaller/larger iteration numbers to reach to optimal reconstruction in the entire range in between $q = 1$ and $q = 5$. On the other hand, the reconstruction quality is best for $q = 2$, where the accuracy roughly matches that of the shrinked Landweber iteration. (Arguably, this might be due to the Gaussian distribution of the additive noise.) Choosing q larger or smaller than 2 yields increasingly worse reconstructions; in particular, the contrasts do not reach the true values anymore. Thus, the choice of the discrepancy norm has obviously a significant influence on the inversion result.

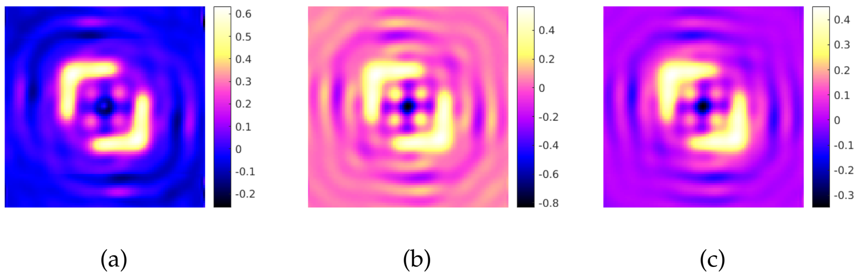

Figure 3.5: Real part of reconstructions of $q^{sc\,(2)}$ by primal-dual algorithm for different discrepancy norms $\|\cdot\|_q^q/q$ (see Remark 3.19) and fixed artificial noise level $\delta = 0.01$, plotted on $[-0.4, 0.4)^2$. (a) $q = 2$, 5 iter., 12 min., rel. error is 0.658 (b) $q = 3$, 2 iter., 4 min., rel. error is 0.738 (c) $q = 1.6$, 41 iter., 82 min., rel. error is 0.763.

3.8 ADJOINT OF THE FORWARD OPERATOR'S LINEARIZATION

The adjoint operator of the linearization F' is a crucial ingredient for most gradient-based schemes tackling the inverse scattering problem to stably solve the non-linear equation $F(Q) = F_{meas}$ for some given $F_{meas} \in \mathcal{S}_q$. This is our main motivation to give an explicit and computable representation of this adjoint. We fix $Q \in \mathcal{Q}$, consider $F'(Q)\colon L^p(B_R)^{d\times d} \to \mathcal{S}_q$ and aim to determine $F'(Q)^*\colon \mathcal{S}_{q'} \to L^{p'}(B_R)^{d\times d}$ such that for all $P \in L^p(B_{2R})^{d\times d}$, $K \in \mathcal{S}_{q'}$ it holds

$$(F'(Q)[P], K)_{\mathcal{S}_2} \stackrel{!}{=} (P, F'(Q)^* K)_{L^2(B_R)^{d\times d}}. \tag{3.35}$$

Here, p' and q' are the conjugate Lebesgue indices to p and q, respectively, such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, and $(\cdot,\cdot)_{L^2(B_R)^{d\times d}}$ is the usual scalar product in $L^2(B_R)^{d\times d}$,

$$(A, B)_{L^2(B_R)^{d\times d}} = \int_{B_R} A : B \,dx = \int_{B_R} \sum_{i,j=1}^{d} \overline{A}_{ij} B_{ij} \,dx,$$

extended to the anti-linear dual product between $L^p(B_R)^{d\times d}$ and $L^{p'}(B_R)^{d\times d}$. Further, $(\cdot,\cdot)_{\mathcal{S}_2}$ is the scalar product in the Hilbert space of Hilbert-Schmidt operators,

$$(F, K)_{\mathcal{S}_2} = \sum_{j\in\mathbb{N}} s_j(F)\overline{s_j(K)} = \sum_{j=1}^{\infty} (F g_j, K g_j)_{L^2(\mathbb{S}^{d-1})}$$

for an arbitrary orthonormal basis $(g_j)_{j\in\mathbb{N}}$ of $L^2(\mathbb{S}^{d-1})$. Consequently, (3.35) becomes

$$\sum_{j=1}^{\infty} (F'(Q)[P] g_j, K g_j)_{L^2(\mathbb{S}^{d-1})} \stackrel{!}{=} (P, F'(Q)^* K)_{L^2(B_R)^{d\times d}}$$

for all $P \in L^p(B_{2R})^{d\times d}$ and $K \in \mathcal{S}_{q'}$. Thus, we consider at first a single L^2-scalar product for fixed $Q \in \mathcal{Q}$ and $g \in L^2(\mathbb{S}^{d-1})$ and seek for $A : L^2(\mathbb{S}^{d-1}) \to L^{p'}(B_{2R})^{d\times d}$ such that for all $P \in L^p(B_{2R})^{d\times d}$ and $h \in L^2(\mathbb{S}^{d-1})$ it holds, that

$$(F'(Q)[P]g, h)_{L^2(\mathbb{S}^{d-1})} \overset{!}{=} (P, Ah)_{L^2(B_R)^{d\times d}}.$$

Recall from (3.21) that $L'(Q, v_g)[P] = v' \in H^1(B_{2R})$, a function whose radiating extension satisfies

$$v' = S_Q\left[\operatorname{div} V\big(P\nabla[L(Q, v_g) + v_g]\big)\right] \text{ in } H^1(B_{2R}),$$

where $S_Q = [\mathrm{Id} - \operatorname{div} V(Q\nabla(\cdot))]^{-1}$. Since the derivative F', see (3.26), involves the far field of L', we note that

$$\begin{aligned} F'(Q)[P]g &= Z \circ \left[Q\nabla v' + P\nabla S_Q(v_g)\right] \\ &= Z \circ \left[Q\nabla S_Q\left[\operatorname{div} V\big(P\nabla S_Q(v_g)\big)\right] + P\nabla S_Q(v_g)\right]. \end{aligned}$$

Consequently, we compute that

$$\begin{aligned} &(F'(Q)[P]g,\, h)_{L^2(\mathbb{S}^{d-1})} \\ &\quad= \Big(Q\nabla S_Q\left[\operatorname{div} V\big(P\nabla S_Q(v_g)\big)\right] + P\nabla S_Q(v_g),\, Z^*h\Big)_{L^2(B_R)^d} \\ &\quad= \Big(P\nabla S_Q(v_g),\, [Q\nabla S_Q \circ (\operatorname{div} V)]^* \circ Z^*h\Big)_{L^2(B_R)^d} \\ &\qquad+ \Big(P\nabla S_Q(v_g),\, Z^*h\Big)_{L^2(B_R)^d} \\ &\quad= \Big(P,\, \big(\big[[Q\nabla S_Q \circ (\operatorname{div} V)]^* + \mathrm{Id}\big] \circ Z^*h\big) \otimes \nabla\overline{S_Q(v_g)}\Big)_{L^2(B_R)^{d\times d}} \end{aligned}$$

where the last matrix-valued function is defined by $(a \otimes b)_{i,j} = a_i b_j$ for $1 \leqslant i, j \leqslant d$ and $[Q\nabla S_Q \operatorname{div} V]^*$ denotes the L^2-adjoint of the bounded operator $w \mapsto Q\nabla S_Q \circ (\operatorname{div} V(w))$ on $L^2(B_R)$. (If Q is a twice continuously differentiable function, then the latter adjoint can be represented by $f \mapsto V^*(\nabla S_Q^*(\operatorname{div}(\overline{Q}^\top f)))$ for all $f \in H^2(B_R)$.)

Lemma 3.27. *For $Q \in \mathcal{Q}$ and $g \in L^2(\mathbb{S}^{d-1})$, the adjoint of the mapping $P \mapsto F'(Q)[P](g)$ with respect to the L^2-inner product maps $L^2(\mathbb{S}^{d-1})$ into $L^{p/(p-1)}(B_{2R})^{d\times d}$ for $p > 2T_\lambda(T_\lambda - 2)$ and is represented by*

$$g \mapsto \Big(\big[[Q\nabla S_Q \circ (\operatorname{div} V)]^* + \mathrm{Id}\big] \circ Z^*g\Big) \otimes \nabla\overline{S_Q(v_g)}.$$

For all orthonormal bases $\{g_j\}_{j\in\mathbb{N}}$ of $L^2(\mathbb{S}^{d-1})$ and all $K \in \mathcal{S}_{q'}$, the bounded operator $F'(Q)^* : \mathcal{S}_{q'} \to L^{p/(p-1)}(B_{2R})^{d\times d}$ is represented by

$$F'(Q)^*(K) = \sum_{j=1}^{\infty} \Big(\big[[Q\nabla S_Q \circ (\operatorname{div} V)]^* + \mathrm{Id}\big] \circ Z^*\big(K g_j\big)\Big) \otimes \nabla\overline{S_Q[v_{g_j}]}. \tag{3.36}$$

Note. If $Q = q^{sc}\,\mathrm{Id}_d$ is represented by a scalar function q^{sc} and $h \mapsto F'(q)[h]$ maps $L^p(B_R)$ into $\mathcal{S}_q$, then $F'(Q)^*(K) = F'(q^{sc})^*(K)$ in (3.36) becomes scalar, i.e.,

$$F'(q^{sc})^*(K) = \sum_{j=1}^{\infty} \Big(\big[[q^{sc}\nabla S_{q^{sc}} \circ (\operatorname{div} V)]^* + \mathrm{Id}\big] \circ Z^*\big(K g_j\big)\Big) \cdot \nabla\overline{S_{q^{sc}}[v_{g_j}]}$$

is a function in $L^{p/(p-1)}(B_R)$.

4 INVERSE ELECTROMAGNETIC SCATTERING FROM ANISOTROPIC NON-MAGNETIC MEDIA

Whereas both the works of Section 2.3 and Chapter 3 deal with acoustic waves, we now explore non-linear Tikhonov regularization and sparsity-promoting techniques for inverse electromagnetic scattering from penetrable linear inhomogeneous non-magnetic anisotropic media, like aluminum-copper alloys.

Thus, we expose the scattering problem in more details in Section 4.2. For that purpose, as in the previous chapter, we work with material parameters of an admissible set, equipped with the L^∞-topology. In Section 4.3 we then construct the associated solution operator, which maps material parameters to scattered fields. Further we establish H^1-regularity estimates, provided by Saranen [Sar82], to analyze the dependence of scattered fields and their derivatives on the material parameter, see Section 4.4 as well. We extend these results in Section 4.5 to a parameter-to-far field mapping, called the forward operator. Due to that, we show in Section 4.6 convergence of a non-linear Tikhonov regularization against a minimum-norm solution to the inverse problem, which seems to be impossible when one works in an $H(\mathrm{curl})$-setting. Hence, we establish sparsity-promoting Tikhonov regularization results in wavelet bases as well as for functions of bounded variation. Finally, the adjoint of the forward operator's linearization is calculated in Section 4.7.

Note that in contrast to Section 3.7 numerical results are not provided, since even computational implementation of Maxwell's equations with material parameter in the first order term into the used framework of [BKL17] is still an open task.

NOTATION By $\mathbb{S}^2 = \{x \in \mathbb{R}^3,\ |x| = 1\}$ we denote the unit sphere in $\mathbb{R}^3$ and $B_R(x)$ is the ball of radius R about $x \in \mathbb{R}^3$. For any bounded Lipschitz domain $B \subset \mathbb{R}^3$ we denote the Sobolev space $W^{1,2}(B, \mathbb{C}^3)$ by $H^1(B, \mathbb{C}^3)$. Hence, we define the Hilbert space $H(\mathrm{curl}, B) := \{u \in L^2(B, \mathbb{C}^3),\ \mathrm{curl}\, u \in L^2(B, \mathbb{C}^3)\}$, with inner product $(u, w)_{H(\mathrm{curl},B)} := (u, w)_{L^2(B)} + (\mathrm{curl}\, u, \mathrm{curl}\, w)_{L^2(B)}$. The closure of $C_0^\infty(B, \mathbb{C}^3)$ in the norm of $H(\mathrm{curl}, B)$ is called $H_0(\mathrm{curl}, B) = \{u \in H(\mathrm{curl}, B),\ \nu \times u = 0 \text{ on } \partial B\}$. Further,

$$H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3) := \left\{u\colon \mathbb{R}^3 \to \mathbb{C}^3,\ u|_B \in H(\mathrm{curl}, B) \text{ for all balls } B \subset \mathbb{R}^3\right\}$$

and $H_t^{-1/2}(\partial B) := \{u \in H^{-1/2}(\partial B, \mathbb{C}^3),\ u \cdot \nu = 0$ a.e. on $\partial B\}$ in which ν denotes the unit outward normal to B. By that one defines the trace space of $H(\mathrm{curl}, B)$ with respect to the trace $u \mapsto \nu \times u$,

$$H^{-1/2}(\mathrm{Div}, \partial B) := \left\{u \in H_t^{-1/2}(\partial B),\ \nabla_{\partial B} \cdot u \in H^{-1/2}(\partial B)\right\},$$

where $\nabla_{\partial B}\cdot$ denotes the surface divergence. Its dual space is given by $H^{-1/2}(\mathrm{Curl}, \partial B) := \{u \in H_t^{-1/2}(\partial B),\ \nabla_{\partial B} \times u \in H^{-1/2}(\partial B)\}$, within use of the surface scalar curl $\nabla_{\partial B}\times$ (for details see, e.g, Monk [Mon03, Section 3.4]). By abuse of notation, a duality pairing between the trace space of $H(\mathrm{curl}, B)$ and its dual (see, e.g., [Mon03, Section 3.5.3], or [BCS02], [BH03]) will for simplicity always be written as a boundary integral over ∂B. Analogously we define the Hilbert space $H(\mathrm{div}, B) := \{u \in L^2(B, \mathbb{C}^3),\ \mathrm{div}\, u \in L^2(B, \mathbb{C}^3)\}$ with inner product $(u, w)_{H(\mathrm{div},B)} := (u, w)_{L^2(B)} + (\mathrm{div}\, u, \mathrm{div}\, w)_{L^2(B)}$. To improve readability, we use a generic constant C in our estimates, maybe changing its value from one occurrence to the other.

4.1 MAXWELL'S EQUATIONS

Recall that in general the propagation of time-harmonic electromagnetic waves in three dimensions is governed by Maxwell's equations for the electric and magnetic fields E and H. Given a circular frequency $\omega > 0$ and a medium with electric permittivity ε, magnetic permeability μ, and conductivity σ, linear and time-harmonic electromagnetic waves are governed by the differential equations

$$\begin{aligned} \mathrm{curl}\, E - i\omega\mu H &= 0\,, \\ \mathrm{curl}\, H + i\omega\varepsilon E &= \sigma E \end{aligned} \quad \text{in } \mathbb{R}^3. \tag{4.1}$$

We assume the tangential components of E and H to be continuous on interfaces, where σ, ε and μ are discontinuous. (Instead, the normal components might jump across the material boundary.) Denoting the constant background permittivity and permeability by ε_0 and μ_0, we introduce the anisotropic relative permittivity ε_r and relative permeability μ_r

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0} + i\frac{\sigma(x)}{\omega\varepsilon_0}, \quad \mu_r(x) = \frac{\mu(x)}{\mu_0}.$$

In the following we assume that $\varepsilon \equiv \varepsilon_0$, $\mu \equiv \mu_0$, and $\sigma \equiv 0$ outside some bounded domain. Further the scattered fields satisfy the Silver-Müller radiation condition

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty,$$

uniformly with respect to $x/|x| \in \mathbb{S}^2$. As mentioned above, we intend to handle the important case of non-magnetic media, that is the magnetic permeability μ is constant and equal to the permeability μ_0

of vacuum such that $\mu_r \equiv 1$. An example for that case can be seen during the solidification of an aluminum-copper alloy with a non-magnetic Al_2CU phase. Herein strong magnetic fields are used to control the crystal growth and the solid-liquid interface morphology to align the phase and, therefore, avoiding the embrittlement of the material. Hence, we will work from now on with the magnetic field H only, which is divergence free in case of non-magnetic media. Thus, the system (4.1) can be reduced to the second-order Maxwell system

$$\operatorname{curl}\left(\varepsilon_r^{-1}\operatorname{curl} H\right) - k^2 H = 0 \quad \text{in } \mathbb{R}^3 \tag{4.2}$$

for the positive wave number $k := \omega\sqrt{\varepsilon_0\mu_0} \in \mathbb{C}\setminus\{0\}$, such that $\operatorname{Re} k > 0$ and $\operatorname{Im} k \geqslant 0$. Accordingly, the electric field is determined by $E = i \operatorname{curl} H/(\omega\varepsilon_0\varepsilon_r)$.

4.2 SCATTERING FROM NON-MAGNETIC MEDIA

We consider the time-harmonic Maxwell's equations to model scattering of an incident electromagnetic wave from a non-magnetic medium modeled by space-dependent relative electric permittivity ε_r. As the material parameter $\varepsilon_r \in L^\infty(D, \mathrm{Sym}(3))$ takes values in the complex-valued symmetric 3×3 matrices $\mathrm{Sym}(3) \subset \mathbb{C}^{3\times 3}$, its real part correlates physically to the electric permittivity, whereas its imaginary part is proportional to the electric conductivity σ. We assume that there exists a positive constant $\lambda > 0$ such that $\lambda|\xi|^2 \leqslant \operatorname{Re}(\overline{\xi}^\top \varepsilon_r\, \xi)$ for all $\xi \in \mathbb{C}^3$ and for almost all x on the bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{D}$. Since in particular we have that also $\varepsilon_r^{-1} \in L^\infty(D, \mathrm{Sym}(3))$, we suppose that the closure of D equals the support of $I_3 - \varepsilon_r^{-1}$ and, moreover, that the imaginary part of ε_r^{-1} is bounded from above, that is, $\operatorname{Im}(\overline{\xi}^\top \varepsilon_r^{-1}\xi) \leqslant 0$ for $\xi \in \mathbb{C}^3$. To generalize notation we thus abbreviate the material parameter as an element $\rho := \varepsilon_r^{-1}$ of the bounded subset $\mathcal{P}$ of $L^\infty(D, \mathrm{Sym}(3))$, which is equipped with the L^∞-topology and defined for a $\lambda > 0$ as

$$\mathcal{P} := \Big\{\rho \in L^\infty(D, \mathrm{Sym}(3)),\ \lambda|\xi|^2 \leqslant \operatorname{Re}(\overline{\xi}^\top \rho^{-1}\xi), \\ \operatorname{Im}(\overline{\xi}^\top \rho\, \xi) \leqslant 0,\ \text{a.e. in } D \text{ and for all } \xi \in \mathbb{C}^3\Big\}.$$

THE SCATTERED FIELD Remember that we have already derived in the introduction that the total magnetic field solves

$$\operatorname{curl}(\rho \operatorname{curl} H) - k^2 H = 0 \quad \text{in } \mathbb{R}^3. \tag{4.3}$$

On interfaces where ρ^{-1} is discontinuous, the tangential components of the magnetic field H and of $\rho \operatorname{curl} H$ are continuous across the interface. In particular, if ρ^{-1} is discontinuous across ∂D, then

$$\nu \times [H]_{\partial D} = 0 \quad \text{and} \quad \nu \times [\rho \operatorname{curl} H]_{\partial D} = 0, \tag{4.4}$$

where $[\cdot]_{\partial D}$ denotes the jump of a function across ∂D. Assume that a time-harmonic incident plane wave

$$H^i(x, d; p) := p\, e^{ikx\cdot d}, \quad x \in \mathbb{R}^3, \quad \text{where } d \in \mathbb{S}^2,\ p \in \mathbb{C}^3, \text{ and } p \cdot d = 0,$$

with direction d and polarization p propagates through the inhomogeneity D. Due to the different material parameters inside D there arises a scattered wave H^s, solving

$$\operatorname{curl}(\rho \operatorname{curl} H^s) - k^2 H^s = \operatorname{curl}\left((I_3 - \rho)\operatorname{curl} H^i\right) \quad \text{in } \mathbb{R}^3. \tag{4.5}$$

Since H^i solves $\operatorname{curl}^2 H^i - k^2 H^i = 0$ in $\mathbb{R}^3$, the total field $H = H^i + H^s$ is still a solution to (4.3). Furthermore H^s is radiating, i.e., it satisfies the Silver-Müller radiation condition

$$\operatorname{curl} H^s(x) \times \hat{x} - ikH^s(x) = \mathcal{O}\left(|x|^{-2}\right) \quad \text{as } |x| \to \infty, \tag{4.6}$$

uniformly with respect to $\hat{x} := x/|x| \in \mathbb{S}^2$.

THE FAR FIELD Since H^s is a radiating solution to 4.5, it has the asymptotic behavior

$$H^s(x) = \frac{\exp(ik|x|)}{4\pi|x|} H^\infty(\hat{x}, d; p) + \mathcal{O}\left(|x|^{-2}\right), \quad \text{as } |x| \to \infty,$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. Here H^∞ is called the far field pattern of H^s, which (see, e.g., [CK13, Theorem 6.9]) is an analytic and tangential vector field on the unit sphere, i.e.,

$$H^\infty(\hat{x}, d; p) \cdot \hat{x} = 0 \quad \text{for all } \hat{x} \in \mathbb{S}^2 \text{ and all } d \in \mathbb{S}^2, p \in \mathbb{C}^3 \text{ with } p \cdot d = 0.$$

In particular, H^∞ belongs to the space of square-integrable tangential vector fields

$$L^2_t(\mathbb{S}^2) := \left\{g \in L^2(\mathbb{S}^2, \mathbb{C}^3),\ g(\hat{x}) \cdot \hat{x} = 0 \text{ for a.e. } \hat{x} \in \mathbb{S}^2\right\} \subset L^2(\mathbb{S}^2, \mathbb{C}^3).$$

The far field patterns H^∞ define the far field operator $F\colon L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} H^\infty(\hat{x}, d; g(d))\, dS(d) \quad \text{for } \hat{x} \in \mathbb{S}^2, \tag{4.7}$$

which is linear since H^∞ depends linearly on p, i.e., $H^\infty(\hat{x}, d; p) = \hat{H}^\infty(\hat{x}, d)p$ for all $p \in \mathbb{C}^3$ with $p \cdot d = 0$ and $\hat{H}^\infty(\hat{x}, d) \in \mathbb{C}^{3\times 3}$. Due to reciprocity relations, H^∞ is moreover a smooth function in both variables $\hat{x}$ and d which implies that F is a compact operator on $L^2_t(\mathbb{S}^2)$. Note that Fg with $g \in L^2_t(\mathbb{S}^2)$ is the far field pattern of the magnetic field corresponding to an incident Herglotz wave function,

$$v_g(x) = \int_{\mathbb{S}^2} H^i(x, d; g(d))\, dS(d) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d)\, dS(d), \quad x \in \mathbb{R}^3, \tag{4.8}$$

in $H(\operatorname{curl}, B_R)$.

Regarding general source terms $f \in C^\infty(D, \mathbb{C}^3)$ on the right-hand side of (4.5), we seek for weak radiating solutions $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ to

$$\operatorname{curl}(\rho \operatorname{curl} v) - k^2 v = \operatorname{curl}\big((I_3 - \rho) f\big) \quad \text{in } \mathbb{R}^3, \tag{4.9}$$

$$\left.\begin{aligned} \nu \times v|_- &= \nu \times v|_+ , \\ \nu \times \rho \operatorname{curl} v|_- - \nu \times \operatorname{curl} v|_+ &= \nu \times (I_3 - \rho) f \end{aligned}\right\} \quad \text{on } \partial D. \tag{4.10}$$

Note that we always implicitly use natural, homogeneous transmission conditions (4.10) on ∂D in the rest of this chapter and that setting $f = \operatorname{curl} H^i$ yields the original problem (4.5). The weak radiating solution $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ thus needs to satisfy

$$\int_{\mathbb{R}^3} \left[\rho \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right] dx = \int_{\mathbb{R}^3} (I_3 - \rho) f \cdot \operatorname{curl} \overline{\psi}\, dx \tag{4.11}$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support.

Remark 4.1. A. Choosing $\psi = \nabla \varphi$ to be a gradient field, the equation $\operatorname{curl} \nabla \varphi = 0$ implies that $\int_{\mathbb{R}^3} v \cdot \nabla \overline{\varphi}\, dx = 0$ for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support, i.e., $\operatorname{div} v = 0$ in $\mathbb{R}^3$. Thus, the solution v is divergence free.

B. The Silver-Müller radiation condition is well-defined for any weak solution v to (4.11):
Outside D the solution v solves $\operatorname{curl}^2 v - k^2 v = 0$ together with $\operatorname{div} v = 0$; thus, the identity $\Delta = \nabla \operatorname{div} - \operatorname{curl}^2$ implies that $\Delta v + k^2 v = 0$ and elliptic regularity results imply that v is a smooth function in $\mathbb{R}^3 \setminus \overline{D}$.

4.3 THE SOLUTION OPERATOR

Now we transform the weak formulation (4.11) into a variational equation on a bounded domain. Therefore, we denote by B_R a ball, containing the support $\overline{D}$ of $I_3 - \rho$ in its interior and the tangential trace mapping $\gamma_t \colon H(\operatorname{curl}, B_R) \to H^{-1/2}(\operatorname{Div}, \partial B_R)$ by $\gamma_t(u) = \nu \times u|_{\partial B_R}$ for the outward unit normal vector $\nu = \nu(x)$ at $x \in \partial B_R$. Further the "dual" tangential trace $\gamma_T \colon H(\operatorname{curl}, B_R) \to H^{-1/2}(\operatorname{Curl}, \partial B_R)$ is given by $\gamma_T(u) = (\nu \times u)|_{\partial B_R} \times \nu$, see [Mon03, Theorem 3.31] or [BCS02] for a generalization to Lipschitz domains. If $v \in H_{loc}(\operatorname{curl}, \mathbb{R}^3)$ solves (4.11), then v solves also

$$\begin{aligned} \int_{B_R} \left[\rho \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \gamma_t(\operatorname{curl} v) \cdot \gamma_T(\overline{\psi})\, dS \\ = \int_D (I_3 - \rho) f \cdot \operatorname{curl} \overline{\psi}\, dx, \end{aligned} \tag{4.12}$$

for all test functions $\psi \in H(\operatorname{curl}, B_R)$ with compact support included in B_R, because $\rho \equiv 1$ on ∂B_R. Regarding both the transmission conditions (4.10) and the relation between the magnetic and electric fields (see below (4.2)), we denote the exterior magnetic-to-electric Calderon operator by

$$\Lambda\colon H^{-1/2}(\mathrm{Div}, \partial B_R) \to H^{-1/2}(\mathrm{Div}, \partial B_R),$$

mapping $\varphi \in H^{-1/2}(\mathrm{Div}, \partial B_R)$ into $\left(\nu \times \frac{i}{\omega\varepsilon_0} \operatorname{curl} u\right)\Big|_{\partial B_R}$, where u satisfies

$$\operatorname{curl}^2 u - k^2 u = 0 \text{ in } \mathbb{R}^3 \setminus \overline{B_R}, \quad \gamma_t(u) = \nu \times u = -i\omega\varepsilon_0\varphi \text{ on } \partial B_R$$

and the Silver-Müller radiation condition (4.6). Exploiting that, we can rewrite (4.12) as

$$\begin{aligned}\int_{B_R} \left[\rho \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \Lambda(\gamma_t(v)) \cdot \gamma_T(\overline{\psi})\, dS \\ = \int_D (I_3 - \rho)\, f \cdot \operatorname{curl} \overline{\psi}\, dx\end{aligned} \tag{4.13}$$

for all $\psi \in H(\operatorname{curl}, B_R)$. (We omit the trace operators γ_t and γ_T from now on if a tangential restriction to the boundary is obvious.)

Remark 4.2. If $v \in H(\operatorname{curl}, B_R)$ solves (4.13) then v can be extended into the exterior of B_R: For the spherical Hankel functions $h_n^{(1)}$ and the spherical harmonics Y_n^m there exist unique $\alpha_n^m, \beta_n^m \in \mathbb{C}$, such that the extension is given by

$$\begin{aligned}\tilde{v}(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \Bigg\{ \alpha_n^m \operatorname{curl}\operatorname{curl} \left(x\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right)\right) \\ - \beta_n^m \operatorname{curl} \left(x\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right)\right)\Bigg\}, \quad |x| > R.\end{aligned}$$

For details on the coefficients, see Section 9.3.3 of [Mon03], especially Theorem 9.17 and Remark 9.18, as well as Section 2.7 of [KH15], Corollary 2.47 and Theorem 2.50 in particular. Such defined extended functions $w = \{v \text{ in } B_R, \tilde{v} \text{ in } \mathbb{R}^3 \setminus B_R\}$ solve (4.11) and for simplicity are denoted by v again.

We now define a sesquilinear form for $\rho \in \mathcal{P}$ and for all $\varphi, \psi \in H(\operatorname{curl}, B_R)$ by

$$a_\rho(\varphi, \psi) := \int_{B_R} \left[\rho \operatorname{curl} \varphi \cdot \operatorname{curl} \overline{\psi} - k^2 \varphi \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \Lambda(\varphi) \cdot \overline{\psi}\, dS,$$

and the solution operator $L\colon \mathcal{P} \times H(\operatorname{curl}, B_R) \to H(\operatorname{curl}, B_R)$, which maps material parameters ρ and incident fields u^i to the solution of the variational problem

$$a_\rho(L(\rho, u^i), \psi) = \int_D (I_3 - \rho) \operatorname{curl} u^i \cdot \operatorname{curl} \overline{\psi}\, dx \quad \text{for all } \psi \in H(\operatorname{curl}, B_R). \tag{4.14}$$

Thus, $L(\rho, u^i) = v$ is still the weak solution to the variational formulation (4.13) for $f = \operatorname{curl} u^i$ and the radiating extension of v to $\mathbb{R}^3$ (see Remark 4.2) weakly solves

$$\operatorname{curl}(\rho \operatorname{curl} v) - k^2 v = \operatorname{curl}\left((I_3 - \rho) \operatorname{curl} u^i\right) \quad \text{in } \mathbb{R}^3.$$

SOLUTION THEORY VIA RIESZ-FREDHOLM Using either this variational formulation involving the exterior Calderon operator, as done by Monk [Mon03] or a volume integral approach seen by Kirsch [Kir07], it is possible to show that the underlying problem (4.11) can be reduced to a Fredholm problem of index zero (see, e.g., [Kir07, Lemma 2.4]). Therefore, uniqueness implies existence of solution (see, e.g., [Kir07, Theorem 2.5], [Mon03, Theorem 10.2]):

LEMMA 4.3. *The scattering problem* (4.9) *and* (4.10) *with radiation condition* (4.6) *satisfies the Fredholm alternative, i.e., there exists a unique radiating solution* $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ *of* (4.11) *for every* $f \in L^2(D, \mathbb{C}^3)$*, provided uniqueness holds for all* $\rho \in \mathcal{P}$*. If uniqueness holds, then there exists a constant* $C > 0$ *(depending on* B_R, k, ρ *only) such that*

$$\|v\|_{H(\text{curl},B_R)} \leqslant C\|(I_3 - \rho)f\|_{L^2(D,\mathbb{C}^3)} \tag{4.15}$$

for the right-hand side of (4.11)*. Further, the restriction* $v|_D$ *is the unique solution of the integro-differential representation* (4.24) *in* $H(\text{curl}, B_R)$*.*

ASSUMPTION 4.4. We assume in the following that for the connected, convex set $\mathcal{P}$ any solution to (4.11) for $f \in L^2(D, \mathbb{C}^3)$ is unique, such that existence and continuous dependence of this solution follow from uniqueness. For example, in the case of dielectric media (i.e., $\sigma \equiv 0$), this assumption is always satisfied if $\rho \in \mathcal{P}$ is globally Hölder continuous and differentiable, except at one point of Coulomb-type singularity, since, under this smoothness assumption, unique continuation results for Maxwell's equations are applicable, see [Ō02],[Vog91].

Thus, the solution operator $L(\rho, \cdot)$ exists for all $\rho \in \mathcal{P}$, together with a constant $C = C(\mathcal{P})$ such that $\| L(\rho, u^i)\|_{H(\text{curl},B_R)} \leqslant C \|u^i\|_{H(\text{curl},D)}$.

REGULARITY ESTIMATE To handle derivatives of L in L^p-spaces, we use an H^1-regularity estimate stated by Saranen [Sar82]. Since we will work with a couple of solutions to equations with slightly different right-hand sides, we state the following result for a broader range of parameters on the right-hand side.

THEOREM 4.5. *Let* $\rho \in \mathcal{P}$, $f \in L^2(D, \mathbb{C}^3)$ *and the support* $\overline{D}$ *of* $A \in L^\infty(B_R, \text{Sym}(3))$ *be a subset of the ball* B_R*. If* v *in* $H(\text{curl}, B_R)$ *is a weak solution of*

$$\text{curl}\,(\rho\,\text{curl}\,v) - k^2 v = \text{curl}\,(A\,f) \quad \text{in } \mathbb{R}^3, \tag{4.16}$$

then $v \in H^1(B_R, \mathbb{C}^3)$ *and there holds that*

$$\|v\|_{H^1(B_R,\mathbb{C}^3)} \leqslant C\,\|Af\|_{L^2(B_R,\mathbb{C}^3)}, \tag{4.17}$$

for some constant C *depending on* B_R, k, *and* ρ *only.*

Proof. To apply results of Saranen [Sar82], we have to ensure that the solution is part of an appropriate function space. Therefore, we choose a cut-off function $\chi \in C_c^\infty(\mathbb{R}^3)$, such that $\chi \equiv 1$ in $B_R \supseteq$

$\overline{D} = \operatorname{supp} A$ and vanishes outside of the convex domain $B_{2R} \supseteq B_R$. Then the function $w := \chi v$ satisfies $\nu \times w = 0$ on ∂B_{2R}, such that $w \in H_0(\operatorname{curl}, B_{2R})$. We further have $\operatorname{div} w \in L^2(B_{2R}, \mathbb{C}^3)$—as we will see in the next lines—and, thus, $w \in H_0(\operatorname{curl}, B_{2R}) \cap H(\operatorname{div}, B_{2R})$. Due to Theorem 4.2 of [Sar82], the field w then satisfies

$$\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leqslant C\left(\|\operatorname{curl} w\|_{L^2(B_{2R},\mathbb{C}^3)} + \|\operatorname{div} w\|_{L^2(B_{2R},\mathbb{C}^3)}\right).$$

Considering that $w = \chi v$, we can rewrite the norms on the right-hand side by applying the product rules of the rotation, respectively of the divergence, as

$$\|\operatorname{curl} w\|_{L^2(B_{2R},\mathbb{C}^3)} = \|\chi \operatorname{curl} v + \nabla\chi \times v\|_{L^2(B_{2R},\mathbb{C}^3)} < \infty,$$
$$\|\operatorname{div} w\|_{L^2(B_{2R},\mathbb{C}^3)} = \|\chi \operatorname{div} v + \nabla\chi \cdot v\|_{L^2(B_{2R},\mathbb{C}^3)} < \infty.$$

Further, for two vectors a and b, the identities

$$|a \times b| = |a||b| \sin \sphericalangle(a, b) \quad \text{and} \quad a \cdot b = |a||b| \cos \sphericalangle(a, b),$$

where the absolute values of sin and cos are bounded by one, provide estimates for the cross product and dot product respectively. By that, after applying the triangle-inequality, we gain that

$$\|\chi \operatorname{curl} v + \nabla\chi \times v\|_{L^2(B_{2R},\mathbb{C}^3)}$$
$$\leqslant \|\chi\|_{L^\infty(B_{2R})}\|\operatorname{curl} v\|_{L^2(B_{2R},\mathbb{C}^3)} + \|\chi\|_{C^1(B_{2R})}\|v\|_{L^2(B_{2R},\mathbb{C}^3)}$$

and, respecting that $\operatorname{div} v = 0$, further

$$\|\chi \operatorname{div} v + \nabla\chi \cdot v\|_{L^2(B_{2R},\mathbb{C}^3)} \leqslant \|\chi\|_{C^1(B_{2R})}\|v\|_{L^2(B_{2R},\mathbb{C}^3)}.$$

So far, we have shown that $\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leqslant C(\chi)\|v\|_{H(\operatorname{curl},B_{2R})}$. Using Lemma 4.3 and bearing in mind that $\operatorname{supp} A$ is strictly contained in B_R only, we have that

$$\|w\|_{H^1(B_{2R},\mathbb{C}^3)} \leqslant C(\chi, \rho)\|Af\|_{L^2(B_R,\mathbb{C}^3)}.$$

Finally, regarding that the H^1-norm of $w = \chi v$ over B_{2R} is bounded from below by the H^1-norm of v over B_R, finishes the proof. □

COROLLARY 4.6. *Let $\rho \in \mathcal{P}$ and $\operatorname{supp}(I_3 - \rho) = \overline{D} \subset B_R \subset \mathbb{R}^3$. If v in $H(\operatorname{curl}, B_R)$ is a weak solution of (4.9), then $v \in H^1(B_R, \mathbb{C}^3)$ and there holds that*

$$\|v\|_{H^1(B_R,\mathbb{C}^3)} \leqslant C\,\|(I_3 - \rho)\, f\|_{L^2(B_R,\mathbb{C}^3)}, \tag{4.18}$$

for some constant C depending on B_R, k, and ρ only.

Note. Due to Remark 4.1 A, solutions $u \in H(\operatorname{curl}, B_R)$ of (4.11) are divergence free. Since the tangential trace of u is in $H^{-1/2}(\operatorname{Div}, \partial B_R)$,

$\nu \times u$ is also an element of $H^{-1/2}(\partial B_R, \mathbb{C}^3)$, which can be characterized as the completion of $L^2(\partial B_R, \mathbb{C}^3)$ [McLoo, p.98], and further satisfies $\nu \cdot (\nu \times u) = 0$ a.e. on ∂B_R. We thus have that $\nu \times u \in L^2_t(\partial B_R)$, such that u is a function of

$$W_N = \{u \in H(\mathrm{curl}, B_R) \cap H(\mathrm{div}, B_R), \\ \mathrm{div}\, u = 0 \text{ in } B_R,\ \nu \times u \in L^2_t(\partial B_R)\},$$

which is compactly embedded in $L^2(B_R, \mathbb{C}^3)$ [Mon03, Corollary 3.49]. Thus, one can show Theorem 4.5 alternatively via Riesz-Fredholm theory.

SOLUTION OPERATOR'S CONTINUITY We now show that L is Lipschitz continuous:

THEOREM 4.7. *Let Assumption 4.4 hold and $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ be a perturbation of $\rho \in \mathcal{P}$, such that $\rho + \rho' \in \mathcal{P}$, then*

$$\| L(\rho+\rho', u^i) - L(\rho, u^i)\|_{H^1(B_R,\mathbb{C}^3)} \leqslant C\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u^i\|_{H(\mathrm{curl},B_R)},$$

where $C > 0$ depends on B_R, k and ρ, but is independent of ρ' and u^i.

Proof. For a fixed incident field u^i we set $v_{\rho+\rho'} = L(\rho + \rho', u^i)$ and $v = L(\rho, u^i)$ and denote the radiating extensions (see Remark 4.2) of these functions to $\mathbb{R}^3$ again by $v_{\rho+\rho'}$, v and the corresponding total fields by $u_{\rho+\rho'} = u^i + v_{\rho+\rho'}$ and $u = u^i + v$. The difference $v_{\rho+\rho'} - v = u_{\rho+\rho'} - u$ is the weak, radiating solution in $\mathbb{R}^3$ to

$$\mathrm{curl}\left(\rho\, \mathrm{curl}(u_{\rho+\rho'} - u)\right) - k^2(u_{\rho+\rho'} - u) = -\,\mathrm{curl}\left(\rho'\, \mathrm{curl}\, u_{\rho+\rho'}\right).$$

Now applying Theorem 4.5, yields

$$\begin{aligned}\|u_{\rho+\rho'} - u\|_{H^1(B_R,\mathbb{C}^3)} &\leqslant C(\rho)\,\|\rho'\,\mathrm{curl}\, u_{\rho+\rho'}\|_{L^2(B_R,\mathbb{C}^3)} \\ &\leqslant C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u_{\rho+\rho'}\|_{H(\mathrm{curl},B_R)}.\end{aligned}$$

By triangle-inequality $\|u_{\rho+\rho'}\| \leqslant \|u^i\| + \|v_{\rho+\rho'}\|$ in $H(\mathrm{curl})$-norms, we get rid of the total field, where due to Assumption 4.4

$$\|v_{\rho+\rho'}\|_{H(\mathrm{curl},B_R)} \leqslant C(B_R, \rho)\|u^i\|_{H(\mathrm{curl},B_R)}.$$

Note that the underlying inequality originally gives an upper bound in D, but we simply increased the norm by enlarging the domain. This will be done implicitly during further estimates. □

4.4 DIFFERENTIABILITY OF THE SOLUTION OPERATOR

To have a glance at the differentiability of the solution operator, we fix the incident field and the parameter $\rho \in \mathcal{P}$ in this section, such that the solution operator $L(\rho, \cdot)$ is bounded on $H(\mathrm{curl}, B_R)$. Further we introduce the function $v' \in H(\mathrm{curl}, B_R)$ by

$$a_\rho(v', \psi) = -\int_D \theta \,\mathrm{curl}\left(L(\rho, u^i) + u^i\right) \cdot \mathrm{curl}\, \overline{\psi}\, dx, \tag{4.19}$$

for all $\psi \in H(\mathrm{curl}, B_R)$. In Theorem 4.10 we will show that v' is indeed the derivative $L'(\rho, u^i)[\theta]$ of L with respect to $\rho \in \mathcal{P}$ in direction $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$.

CONTINUITY PROPERTIES

LEMMA 4.8. *For every $\rho \in \mathcal{P}$ the linear mapping $\theta \mapsto L'(\rho, u^i)[\theta]$ from $L^\infty(B_R, \mathrm{Sym}(3))$ to $H(\mathrm{curl}, B_R)$ has the following continuity property:*

$$\| L'(\rho, u^i)[\theta]\|_{H^1(B_R, \mathbb{C}^3)} \leqslant C \|\theta\|_{L^\infty(B_R, \mathrm{Sym}(3))} \|u^i\|_{H(\mathrm{curl}, B_R)},$$

where $C > 0$ depends on B_R, k and ρ only.

Proof. In the following we denote by $u = L(\rho, u^i) + u^i$ the total field, such that due to (4.19) applying the H^1-estimate of Theorem 4.5 gains

$$\begin{aligned} \| L'(\rho, u^i)[\theta]\|_{H^1(B_R, \mathbb{C}^3)} &\leqslant C(\rho)\, \|\theta \,\mathrm{curl}\, u\|_{L^2(B_R, \mathbb{C}^3)} \\ &\leqslant C\, \|\theta\|_{L^\infty(B_R, \mathrm{Sym}(3))} \left(\|u^i\|_{H(\mathrm{curl}, B_R)} + \| L(\rho, u^i)\|_{H(\mathrm{curl}, B_R)} \right), \end{aligned}$$

where for the last step we estimated the L^2-norm by $H(\mathrm{curl})$-norm and separated the total field by triangle inequality. Herein Lemma 4.3 states that

$$\begin{aligned} \| L(\rho, u^i)\|_{H(\mathrm{curl}, B_R)} &\leqslant C\, \| I_3 - \rho\|_{L^\infty(B_R, \mathrm{Sym}(3))} \| \mathrm{curl}\, u^i\|_{L^2(B_R, \mathbb{C}^3)} \\ &\leqslant C\, \|u^i\|_{H(\mathrm{curl}, B_R)}, \end{aligned} \tag{4.20}$$

due to the boundedness of the L^∞-term. □

THEOREM 4.9. *Under the assumptions of Theorem 4.7, the mapping $\rho \mapsto L'(\rho, u^i)$ is locally Lipschitz continuous: There is a $C > 0$ independent of ρ' and u^i such that for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$ there holds*

$$\begin{aligned} &\| L'(\rho + \rho', u^i)[\theta] - L'(\rho, u^i)[\theta]\|_{H^1(B_R, \mathbb{C}^3)} \\ &\qquad \leqslant C\, \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))} \|\theta\|_{L^\infty(B_R, \mathrm{Sym}(3))} \|u^i\|_{H(\mathrm{curl}, B_R)}. \end{aligned}$$

where $C > 0$ depends on B_R, k and ρ only.

Proof. For $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$, the functions $L'(\rho + \rho', u^i)[\theta]$ and $L'(\rho, u^i)[\theta]$ satisfy by (4.19) the variational formulations

$$\begin{aligned} a_{\rho+\rho'}(L'(\rho + \rho', u^i)[\theta], \psi) &= -\int_D \theta \,\mathrm{curl}\, u_{\rho+\rho'} \cdot \mathrm{curl}\, \overline{\psi}\, dx, \\ a_\rho(L'(\rho, u^i)[\theta], \psi) &= -\int_D \theta \,\mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\psi}\, dx, \end{aligned}$$

where the perturbed total field $u_{\rho+\rho'}$ consists of the perturbed scattered field $L(\rho+\rho', u^i)$ and the incident field u^i, analogous $u = L(\rho, u^i) + u^i$. Thus, $w := L'(\rho+\rho', u^i)[\theta] - L'(\rho, u^i)[\theta]$ satisfies

$$a_\rho(w, \psi) = -\int_D \theta \operatorname{curl}(L(\rho+\rho', u^i) - L(\rho, u^i)) \cdot \operatorname{curl}\overline{\psi}\, dx - \int_D \rho' \operatorname{curl} L'(\rho+\rho', u^i)[\theta] \cdot \operatorname{curl}\overline{\psi}\, dx.$$

Therefore, Theorem 4.5 states

$$\begin{aligned}\|w\|_{H^1(B_R,\mathbb{C}^3)} &\leqslant C\Big(\|\theta \operatorname{curl}(L(\rho+\rho', u^i) - L(\rho, u^i))\|_{L^2(B_R,\mathbb{C}^3)} \\ &\qquad + \|\rho' \operatorname{curl} L'(\rho+\rho', u^i)[\theta]\|_{L^2(B_R,\mathbb{C}^3)}\Big) \\ &\leqslant C\Big(\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))} \| L(\rho+\rho', u^i) - L(\rho, u^i)\|_{H(\mathrm{curl},B_R)} \\ &\qquad + \|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))} \| L'(\rho+\rho', u^i)[\theta]\|_{H(\mathrm{curl},B_R)}\Big).\end{aligned}$$

Here, each $H(\mathrm{curl})$-norm occurring in the last estimate is bounded by a constantly scaled $H(\mathrm{curl})$-norm of u^i times a factor; this is for the first one $\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}$ due to Theorem 4.7 and $\|\theta\|_{L^\infty(B_R,\mathrm{Sym}(3))}$ for the second one due to Lemma 4.8. □

GÂTEAUX DERIVATIVE

THEOREM 4.10. *Let Assumption 4.4 hold, then the solution operator L is differentiable in the sense that for every perturbation $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ of $\rho \in \mathcal{P}$ such that $\rho + \rho' \in \mathcal{P}$, it holds that*

$$\| L(\rho+\rho', u^i) - L(\rho, u^i) - L'(\rho, u^i)[\rho']\|_{H^1(B_R,\mathbb{C}^3)} \leqslant C \|\rho'\|^2_{L^\infty(B_R,\mathrm{Sym}(3))} \|u^i\|_{H(\mathrm{curl},B_R)},$$

where $C > 0$ depends on B_R, k and ρ only.
Thus, if $\{\rho'_n\}_{n\in\mathbb{N}} \subset L^\infty(B_R, \mathrm{Sym}(3))$ such that $\rho + \rho'_n \in \mathcal{P}$ for all $n \in \mathbb{N}$ as well as $\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))} \to 0$ as $n \to \infty$, then

$$\frac{\| L(\rho+\rho'_n, u^i) - L(\rho, u^i) - L'(\rho, u^i)[\rho'_n]\|_{H^1(B_R,\mathbb{C}^3)}}{\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))}} \to 0 \quad \text{as } n \to \infty.$$

Proof. For $w := L(\rho+\rho', u^i) - L(\rho, u^i) - L'(\rho, u^i)[\rho']$ we first consider the variational formulations defining all three terms,

$$\begin{aligned} a_{\rho+\rho'}(L(\rho+\rho', u^i), \psi) &= \int_D (I_3 - \rho - \rho') \operatorname{curl} u^i \cdot \operatorname{curl}\overline{\psi}\, dx, \\ a_\rho(L(\rho, u^i), \psi) &= \int_D (I_3 - \rho) \operatorname{curl} u^i \cdot \operatorname{curl}\overline{\psi}\, dx, \\ a_\rho(L'(\rho, u^i)[\rho'], \psi) &= -\int_D \rho' \operatorname{curl}(L(\rho, u^i) + u^i) \cdot \operatorname{curl}\overline{\psi}\, dx,\end{aligned}$$

for all $\psi \in H(\mathrm{curl}, B_R)$. Thus, for all $\psi \in H(\mathrm{curl}, B_R)$ there holds

$$\begin{aligned} &a_{\rho+\rho'}(w,\psi) \qquad (4.21)\\ &= a_{\rho+\rho'}(L(\rho+\rho',u^i),\psi) - a_\rho(L(\rho,u^i)\psi) - a_\rho(L'(\rho,u^i)[\rho'],\psi)\\ &\quad - \int_{B_R} \rho' \,\mathrm{curl}\, L(\rho,u^i)\cdot \mathrm{curl}\,\overline{\psi}\,dx - \int_{B_R} \rho'\,\mathrm{curl}\, L'(\rho,u^i)[\rho']\cdot\mathrm{curl}\,\overline{\psi}\,dx\\ &= \int_D (I_3-\rho-\rho')\,\mathrm{curl}\, u^i\cdot\mathrm{curl}\,\overline{\psi}\,dx - \int_D \rho'\,\mathrm{curl}\, L(\rho,u^i)\cdot\mathrm{curl}\,\overline{\psi}\,dx\\ &\quad - \int_D (I_3-\rho)\,\mathrm{curl}\, u^i\cdot\mathrm{curl}\,\overline{\psi}\,dx + \int_D \rho'\,\mathrm{curl}(L(\rho,u^i)+u^i)\cdot\mathrm{curl}\,\overline{\psi}\,dx\\ &\quad - \int_D \rho'\,\mathrm{curl}\, L'(\rho,u^i)[\rho']\cdot\mathrm{curl}\,\overline{\psi}\,dx\\ &= -\int_D \rho'\,\mathrm{curl}\, L'(\rho,u^i)[\rho']\cdot\mathrm{curl}\,\overline{\psi}\,dx. \end{aligned} \qquad (4.22)$$

Now the H^1-estimate of Theorem 4.5 implies that

$$\begin{aligned} \|w\|_{H^1(B_R,\mathbb{C}^3)} &\leqslant C(\rho)\,\|\rho'\,\mathrm{curl}\, L'(\rho,u^i)[\rho']\|_{L^2(B_R,\mathbb{C}^3)}\\ &\leqslant C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\, L'(\rho,u^i)[\rho']\|_{H(\mathrm{curl},B_R)}. \end{aligned}$$

Due to the equation (4.22), we gain by Lemma 4.3 that

$$\begin{aligned} \|\, L'(\rho,u^i)[\rho']\|_{H(\mathrm{curl},B_R)} &\leqslant C\,\|\rho'\;\mathrm{curl}\, u\|_{L^2(B_R,\mathbb{C}^3)}\\ &\leqslant C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|u\|_{H(\mathrm{curl},B_R)}, \end{aligned}$$

for the total field $u = L(\rho,u^i)+u^i$. After separation into incident and scattered fields, again applying Lemma 4.3 like in (4.20) finally results in the stated estimate. □

4.5 THE FORWARD OPERATOR

In this section we define the so called forward operator which maps material parameters to their corresponding far field operators, such that the forward operator corresponds to the inverse scattering problem we are actually interested in.

POTENTIAL REPRESENTATION Therefore, we follow the volume integral approach mentioned in Section 4.3, by which one can show, see Kirsch [Kir07, Theorem 2.3], that the scattering problem (4.9), (4.10), and (4.6) is equivalent to an integro-differential equation defined via the radiating fundamental solution to the Helmholtz equation:

$$\Phi_k(x) = \frac{1}{4\pi|x|}\,e^{ik|x|}, \quad x \neq 0. \qquad (4.23)$$

More precisely, $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is a radiating solution to (4.11) if and only if v solves in $H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$:

$$v = \mathrm{curl}\int_{B_R} \Phi_k(\cdot - y)(I_3-\rho)(y)\,\mathrm{curl}\left[v(y)+u^i(y)\right]dy. \qquad (4.24)$$

Analogously the radiating extension of the function $v' = L'(\rho, u^i)[\theta]$ to $\mathbb{R}^3$ satisfies

$$v' = \operatorname{curl} \int_{B_R} \Phi_k(\cdot - y) \left[(I_3 - \rho) \operatorname{curl} v' - \theta \operatorname{curl}(L(\rho, u^i) + u^i) \right] (y) \, dy \tag{4.25}$$

in $H(\operatorname{curl}, B_R)$, because v' solves, by definition, the variational formulation (4.19). Now for a direction $\hat{x} \in \mathbb{S}^2$, the far field pattern of $v^\infty(\hat{x})$ hence equals (see, e.g., [LR15, Proposition 3])

$$\begin{aligned} v^\infty(\hat{x}) &= \left(\operatorname{curl} \int_{B_R} \Phi_k(\cdot - y)(I_3 - \rho)(y) \operatorname{curl} \left[v(y) + u^i(y) \right] dy \right)^\infty (\hat{x}) \\ &= \int_{B_R} [\operatorname{curl} e^{-ik\hat{x}\cdot y}](I_3 - \rho)(y) \operatorname{curl} \left[v(y) + u^i(y) \right] dy \qquad (4.26) \\ &= ik\hat{x} \times \int_{B_R} e^{-ik\hat{x}\cdot y}(I_3 - \rho)(y) \operatorname{curl} \left[v(y) + u^i(y) \right] dy, \quad \hat{x} \in \mathbb{S}^2. \end{aligned}$$

This shows that the far field v^∞ is an analytic function, since the latter integral expression is analytic in $\hat{x}$.

CONSTRUCTING THE FORWARD OPERATOR To keep notation simple, we introduce the integral operator

$$V \colon L^2(B_R, \mathbb{C}^3) \to H^2(B_R, \mathbb{C}^3), \quad Vf = \int_{B_R} \Phi_k(\cdot - y) f(y) \, dy.$$

(See [CK13, Theorem 8.2] for the mapping properties of V.) The scattered field restricted to B_R satisfies

$$v = \left\{ I_3 - \operatorname{curl} V \left((I_3 - \rho) \operatorname{curl}(\cdot) \right) \right\}^{-1} \left[\operatorname{curl} V((I_3 - \rho) \operatorname{curl} u^i) \right],$$

such that the total field $v + u^i$ equals $S_\rho u^i$, in particular

$$S_\rho(u^i) := \left[I_3 - \operatorname{curl} V \left((I_3 - \rho) \operatorname{curl}(\cdot) \right) \right]^{-1} (u^i) = v + u^i.$$

Thus, we represent the far field pattern $v^\infty = L(\rho, u^i)^\infty$, computed for direction $\hat{x} \in \mathbb{S}^2$ in (4.26), as

$$v^\infty(\hat{x}) = ik \int_{B_R} \hat{x} \times (I_3 - \rho)(y) \operatorname{curl}(S_\rho u^i)(y) \, e^{-ik\hat{x}\cdot y} \, dy.$$

If we further introduce the integral operator

$$Z \colon L^r(B_R, \mathbb{C}^3) \to L^2_t(\mathbb{S}^2), \quad f \mapsto ik \int_{B_R} \hat{x} \times f(y) \, e^{-ik\hat{x}\cdot y} \, dy, \tag{4.27}$$

then there holds that

$$L(\rho, u^i)^\infty = Z \circ [(I_3 - \rho) \operatorname{curl} S_\rho(u^i)].$$

Using smoothing properties of Z, which again as in Chapter 3, is a trace class operator such that by analogy with Lemma 3.18 the composition on the right is well-defined and bounded, since $I_3 - \rho \in L^\infty(B_R, \operatorname{Sym}(3))$ and $\operatorname{curl} S_\rho u^i \in L^2(B_R, \mathbb{C}^3)$.

LEMMA 4.11. *Choose* $m \in \mathbb{N}$, $1 < r < \infty$, *and* $f \in L^r(B_R, \mathbb{C}^3)$.

(i) There is $C = C(m,r)$ *such that* $\|Zf\|_{C^m(\mathbb{S}^2)} \leqslant C(m,r)\|f\|_{L^r(B_R,\mathbb{C}^3)}$.

(ii) The operator Z *is of trace class from* $L^r(B_R, \mathbb{C}^3)$ *into* $L^2_t(\mathbb{S}^2)$.

Now we are able to introduce the forward operator, which maps material parameters to associated far field operators. As mentioned in Section 4.2, from now on we assume to have Herglotz wave functions v_g for $g \in L^2_t(\mathbb{S}^2)$, see (3.24), as incident fields. Thus, note that $g \mapsto v_g|_{B_R}$ is a bounded mapping from $L^2_t(\mathbb{S}^2)$ into $H^1(B_R)$, see Colton and Kress [CK13, Section 3.3]. For the incident field v_g a far field operator $F_\rho \colon L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defines by $F_\rho g = (L(\rho, v_g))^\infty$ for $g \in L^2_t(\mathbb{S}^2)$. We mention that F_ρ is compact, since the integral kernel $u^\infty = u^\infty_\rho \colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{C}$ of $F(\rho) = F_\rho$ is analytic in both variables. Because of the summability of its singular values $s_j(F_\rho)$, i.e., $\|F_\rho\|_{\mathcal{S}_1} = \sum_{j\in\mathbb{N}} |s_j(F_\rho)| < \infty$, it even belongs to the set $\mathcal{S}_1$ of trace class operators on $L^2_t(\mathbb{S}^2)$. The embedding $\ell^p \subset \ell^q$ for $1 \leqslant p < q \leqslant \infty$ of the sequence spaces ℓ^p further implies that trace class operators belong to the qth Schatten class $\mathcal{S}_q$ for all $q \in [1, \infty)$, a Banach space of all compact operators on $L^2_t(\mathbb{S}^2)$ with q-summable singular values $s_j(F_\rho)$, equipped with the norm defined by

$$\|F_\rho\|^q_{\mathcal{S}_q} = \sum_{j\in\mathbb{N}} |s_j(F_\rho)|^q, \quad \text{for } q \geqslant 1.$$

Because of that, the contrast-to-far field mapping defines as an operator from $\mathcal{P}$ into the qth Schatten class $\mathcal{S}_q$ for $g \in L^2_t(\mathbb{S}^2)$, $q \geqslant 1$:

$$F(\cdot)g \colon \mathcal{P} \to \mathcal{S}_q, \quad F(\rho)g = Z \circ [(I_3 - \rho) \operatorname{curl} S_\rho(v_g)]. \tag{4.28}$$

PROPERTIES OF THE FORWARD OPERATOR We emphasize that the properties mentioned in Remark 3.19 hold analogously for F. Further, be aware that the far field of the radiating extension of $L(\rho, v_g)$ depends boundedly and linearly on $L(\rho, v_g)$. Thus, since $L(\rho, v_g) = S_\rho(v_g) - v_g$, the derivative $\theta \mapsto F'(\rho)[\theta] \in \mathcal{L}(L^\infty(B_R, \mathrm{Sym}(3)), \mathcal{S}_q)$ of F with respect to $\rho \in \mathcal{P}$ in direction $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$ equals, by the product rule in Banach spaces, see [Zei86],

$$F'(\rho)[\theta]g = Z \circ [(I_3 - \rho) \operatorname{curl}(L'(\rho, v_g)[\theta]) + \theta \operatorname{curl}(S_\rho(v_g))]. \tag{4.29}$$

Since the non-linear forward operator F is linked to the solution operator L, we are able to transfer the results of Theorem 4.7, 4.9, and 4.10 from L to F.

COROLLARY 4.12. *Let Assumption 4.4 hold and* $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ *be a small perturbation of* $\rho \in \mathcal{P}$, *such that* $\rho + \rho' \in \mathcal{P}$ *and let* $q \geqslant 1$.

(i) There is a constant $C = C(\rho, B_R, k)$ *such that*

$$\|F(\rho + \rho') - F(\rho)\|_{\mathcal{S}_q} \leqslant C\, \|\rho'\|_{L^\infty(B_R, \mathrm{Sym}(3))}. \tag{4.30}$$

(ii) The operator $F'(\rho)$ is locally Lipschitz continuous, i.e., there is a constant $C = C(\rho, B_R, k)$ such that

$$\|F'(\rho+\rho') - F'(\rho)\|_{\mathcal{L}(L^\infty(B_R,\mathrm{Sym}(3)),\mathcal{S}_q)} \leqslant C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}.$$

(iii) The far field operator $F(\rho)$ is differentiable in the sense that

$$\|F(\rho+\rho') - F(\rho) - F'(\rho)[\rho']\|_{\mathcal{S}_q} \leqslant C\,\|\rho'\|^2_{L^\infty(B_R,\mathrm{Sym}(3))}$$

for a constant C depending on B_R, k and ρ.
If $\{\rho'_n\}_{n\in\mathbb{N}} \subset L^\infty(B_R,\mathrm{Sym}(3))$ such that $\rho+\rho'_n \in \mathcal{P}$ for all $n \in \mathbb{N}$ as well as $\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))} \to 0$ as $n \to \infty$, then $\|F(\rho+\rho'_n) - F(\rho) - F'(\rho)[\rho'_n]\|_{\mathcal{S}_q}/\|\rho'_n\|_{L^\infty(B_R,\mathrm{Sym}(3))} \to 0$.

Proof. The basic ingredient of the proof is the smoothing property of the far field mapping Z defined in (4.27), which is a trace class operator from $L^2(B_R,\mathbb{C}^3)$ into $L^2_t(\mathbb{S}^2)$. Since the incident field u^i is chosen to be a Herglotz wave function v_g for some $g \in L^2(\mathbb{S}^2)$, we have

$$\begin{aligned}
&\|F(\rho+\rho') - F(\rho)\|_{\mathcal{S}_q}\\
&= \|g \mapsto Z\left[(I_3-(\rho+\rho'))\operatorname{curl} S_{\rho+\rho'}(v_g) - (I_3-\rho)\operatorname{curl} S_\rho(v_g)\right]\|_{\mathcal{S}_q}\\
&\leqslant C\,\|g \mapsto Z\left[(I_3-(\rho+\rho'))\operatorname{curl} S_{\rho+\rho'}(v_g) - (I_3-\rho)\operatorname{curl} S_\rho(v_g)\right]\|_{\mathcal{S}_1}\\
&\overset{(*)}{\leqslant} C\,\|g \mapsto \left[(I_3-(\rho+\rho'))\operatorname{curl} S_{\rho+\rho'}(v_g)\right.\\
&\qquad\qquad \left.- (I_3-\rho)\operatorname{curl} S_\rho(v_g)\right]\|_{\mathcal{L}(L^2_t(\mathbb{S}^2),L^2(B_R,\mathbb{C}^3))}\\
&\leqslant C \sup_{\|g\|_{L^2}=1} \Big[\|\rho' \operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R,\mathbb{C}^3)}\\
&\qquad\qquad + \|(I_3-\rho)\operatorname{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)]\|_{L^2(B_R,\mathbb{C}^3)}\Big],
\end{aligned}$$

where inequality $(*)$ follows from Lemma 4.11. Now we obtain the bound

$$\begin{aligned}
\|\rho' \operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R,\mathbb{C}^3)}
&\leqslant \|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|\operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R,\mathbb{C}^3)}\\
&\leqslant \|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|S_{\rho+\rho'}(v_g)\|_{H(\operatorname{curl},B_R)},
\end{aligned}$$

where for the total field $\|S_{\rho+\rho'}(v_g)\|_{H(\operatorname{curl},B_R)} \leqslant C\|v_g\|_{H(\operatorname{curl},B_R)} \leqslant C\|g\|_{L^2(\mathbb{S}^2)} = C$ with a constant $C = C(\rho)$ independent of ρ', due to Assumption 4.4. The same technique yields

$$\begin{aligned}
&\|(I_3-\rho)\operatorname{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)]\|_{L^2(B_R,\mathbb{C}^3)}\\
&\qquad \leqslant \|I_3-\rho\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\operatorname{curl},B_R)}.
\end{aligned}$$

As $S_{\rho+\rho'}(v_g) - S_\rho(v_g) = L(\rho+\rho', v_g) - L(\rho, v_g)$, Theorem 4.7 further shows that

$$\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\operatorname{curl},B_R)} \leqslant C\,\|\rho'\|_{L^\infty(B_R,\mathrm{Sym}(3))}\|v_g\|_{H(\operatorname{curl},B_R)},$$

such that by plugging the last estimates together we deduce the statement. The bounds in (ii) and (iii) are shown analogously, using Theorems 4.9 and 4.10 instead of Theorem 4.7. □

4.6 NON-LINEAR TIKHONOV AND SPARSITY REGULARIZATION

We observe the stable approximation of ρ_{exa} from perturbed measurements of its far field operator $F(\rho_{\mathrm{exa}})$. This will be referred to as our inverse problem. In detail, we seek to approximate ρ by non-linear Tikhonov regularization, for noisy measurements $F^{\delta}_{\mathrm{meas}}$ with noise level $\delta > 0$ such that $\|F(\rho_{\mathrm{exa}}) - F^{\delta}_{\mathrm{meas}}\|_{\mathcal{S}_q} \leqslant \delta$. Thus, for a convex regularization functional $\mathcal{R}$ we consider to minimize the Tikhonov functional

$$\mathcal{J}_{\alpha,\delta}(\rho) := \tfrac{1}{2}\|F(\rho) - F^{\delta}_{\mathrm{meas}}\|^2_{\mathcal{S}_q} + \alpha\,\mathcal{R}(\rho), \tag{4.31}$$

over some appropriate admissible parameter set included in $\mathcal{P}$. Note that under Assumption 4.4 the operator $F(\rho)$ is well-defined.

THEOREM 4.13 (Tikhonov regularization). *If $\mathcal{D}(F)$ is a closed subset of a Banach space, equipped with the weak*-topology such that additionally $\mathcal{D}(F)$ is weak*-closed, and if the imagespace of F is also a Banach space for which any (norm-)bounded subset is weakly precompact, and if F is a (norm-norm)-continuous map, whose graph is (weak*,weak)-closed, then for any weak*-lower semicontinuous $\mathcal{R}$ with weak*-precompact level sets, such that $\mathcal{R}(\mathcal{D}(F)) \cap \mathbb{R} \neq \emptyset$, there exists a minimizer for the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$, defined in (4.31).*

If further $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weak-converges to a solution $\rho^\dagger$ such that $F(\rho^\dagger) = F(\rho_{exa})$ holds in the imagespace of F and $\rho^\dagger$ minimizes $\mathcal{R}$.*

Proof. Definition 5.2.1 and Theorems 5.2.2 to 5.2.4 of [Res12] provide a more general version of this theorem for a broader range of topologies. But due to [Fol84, Chapter 5, Ex. 51], the weak*-topology is suitable. □

We thus apply this result to our setting by following the techniques of Ressel [Res12]. Therefore, be aware that the domain of definition $\mathcal{P}$ of F, equipped with the weak*-topology, is a closed and bounded subset of the Banach space $L^\infty(B_R, \mathrm{Sym}(3))$. Alaoglu's theorem then states that closed balls are weak*-compact, in particular $\mathcal{P}$ is weak*-closed. Recalling (4.28), the forward operator F can be written as

$$F(\cdot)g\colon \mathcal{P} \to \mathcal{S}_q, \quad F(\rho)g = Z \circ \left[(I_3 - \rho)\,\mathrm{curl}(L(\rho, \nu_g) + \nu_g)\right] \in L^2_t(\mathbb{S}^2),$$

for all $g \in L^2(\mathbb{S}^2)$ and for the Banach space $\mathcal{S}_q$ of qth Schatten-class operators, $q \geqslant 1$. Additionally, we like to quote the following statement from [Res12, Corollary 8.3.7], combining results for weak*-convergent sequences:

LEMMA 4.14. *Let* $\{f_n\} \in L^q(X)$ *a weak*-convergent sequence, where* (X, λ) *is a finite measure space and* $q \in (1, \infty]$. *Then we find a subsequence* $\{f_{n_k}\}$, *converging in the* L^r*-norm for all* $1 \leqslant r < q$.

We now show the (weak*,weak)-closedness of the graph of F, i.e., F is sequentially closed from ($\mathcal{P}$,weak*) to $\mathcal{S}_q$ with its weak topology. Therefore, one assumes to have a sequence of parameters $\{\rho_n\}_{n\in\mathbb{N}}$ from $\mathcal{P}$, such that $\rho_n \rightharpoonup^* \rho_0$ and $F(\rho_n) \rightharpoonup F_{\rho_0}$ in $\mathcal{S}_q$ for $n \to \infty$. Then one has to show that this implies that $F(\rho_0) = F_{\rho_0}$ (since $\rho_0 \in \mathcal{P}$ due to the weak*-closedness). Hence, we define $v_n := L(\rho_n, v_g)$, i.e., $F(\rho_n)g = Z((I_3 - \rho_n)\operatorname{curl}(v_n + v_g))$ for all $g \in L^2(\mathbb{S}^2)$.

LEMMA 4.15. F *is sequentially closed from (*$\mathcal{P}$*,weak*) to* $\mathcal{S}_q$ *equipped with its weak topology.*

Proof. As in Banach spaces weak*-convergent sequences are bounded (see e.g., [Alt12]), the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is norm-bounded and due to Theorem 4.7 L is Lipschitz continuous. Hence, the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in H^1-norm. (Alternatively, Theorem 4.5 states the same.) Thus, there exists a subsequence $\{v_{n_m}\}_{m\in\mathbb{N}}$, weakly converging to a $v \in H^1(B_R, \mathbb{C}^3)$ by Alaoglu's theorem. Because of that, $\operatorname{curl} v_{n_m} \rightharpoonup \operatorname{curl} v$ in $L^2(B_R, \mathbb{C}^3)$ and due to the compact embedding of $H^1(B_R, \mathbb{C}^3)$ in $L^q(B_R, \mathbb{C}^3)$ for $1 \leqslant q < 6$, we have also that $v_{n_m} \to v$ in $L^2(B_R, \mathbb{C}^3)$ for $m \to \infty$. (Note that throughout this proof we always mean convergence as $m \to \infty$, although it is not mentioned for improved readability.)

Further, by assumption, $\rho_n \rightharpoonup^* \rho_0$ in $L^\infty(B_R, \mathrm{Sym}(3))$, for the finite measure space (B_R, λ) with λ denoting the Lebesgue-measure, such that Lemma 4.14 implies the existence of a subsequence $\{\rho_{n_m}\}_{m\in\mathbb{N}}$ which converges in $L^r(B_R, \mathrm{Sym}(3))$ for all $1 \leqslant r < \infty$; hence, $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$.

We first reformulate

$$Z \circ [(I_3 - \rho_{n_m})\operatorname{curl}(v_{n_m} + v_g)] \to Z \circ [(I_3 - \rho_0)\operatorname{curl}(v + v_g)] \quad \text{in } L_t^2(\mathbb{S}^2),$$

as

$$ik \int_{B_R} \hat{x} \times e^{-ik\hat{x}\cdot y} \big[\operatorname{curl}(v_{n_m} - v) - \operatorname{curl} v_g(\rho_0 - \rho_{n_m}) \\ - \rho_{n_m} \operatorname{curl} v_{n_m} + \rho_0 \operatorname{curl} v\big](y)\, dy \to 0.$$

Note that the absolute value of the integral is bounded by

$$\|\operatorname{curl} e^{-ik\hat{x}\cdot y}\|_\infty \|v_{n_m} - v\|_{H^1(B_R, \mathbb{C}^3)} \\ + \|\operatorname{curl} e^{-ik\hat{x}\cdot y}\|_\infty \|\operatorname{curl} v_g\|_\infty \|\rho_0 - \rho_{n_m}\|_{L^1(B_R, \mathrm{Sym}(3))} \\ + \|\operatorname{curl} e^{-ik\hat{x}\cdot y}\|_\infty \|\operatorname{curl}(v_{n_m} - v)\|_{L^2(B_R, \mathbb{C}^3)} \|\rho_0 - \rho_{n_m}\|_{L^2(B_R, \mathrm{Sym}(3))}.$$

The terms vanish due to the boundedness of the maximum norms and to the above discussion. Since we have by assumption that

$$Z((I_3 - \rho_n)\operatorname{curl}(v_n + v_g)) = F(\rho_n)g \rightharpoonup F_{\rho_0} g \quad \text{in } L_t^2(\mathbb{S}^2)$$

for all $g \in L^2(\mathbb{S}^2)$, we deduce that

$$F_{\rho_0} g = Z((I_3 - \rho_0) \operatorname{curl}(v + v_g)).$$

Now we show that $v = L(\rho_0, v_g)$, because this implies $F_{\rho_0} g = Z((I_3 - \rho_0) \operatorname{curl}(v + v_g)) = Z((I_3 - \rho_0) \operatorname{curl}(L(\rho_0, v_g) + v_g)) = F(\rho_0)g$. Therefore remember that, according to (4.14), v_{n_m} solves

$$a_{\rho_{n_m}}(v_{n_m}, \psi) = \int_D (I_3 - \rho_{n_m}) \operatorname{curl} v_g \cdot \operatorname{curl} \overline{\psi} \, dx \quad \text{for all } \psi \in C^1(\overline{B_R})$$

for

$$a_{\rho_{n_m}}(v_{n_m}, \psi) = \int_{B_R} [\rho_{n_m} \operatorname{curl} v_{n_m} \cdot \operatorname{curl} \overline{\psi} - k^2 v_{n_m} \overline{\psi}] \, dx + \int_{\partial B_R} \Lambda(\nu \times v_{n_m}) \cdot \gamma_T(\overline{\psi}) \, dS.$$

Note that the equation was originally stated for test functions in $H(\operatorname{curl}, B_R)$, but since $C^1(\overline{B_R}, \mathbb{C}^3)$-functions are dense in $H(\operatorname{curl}, B_R)$, we switch to those test functions to profit from their boundedness in the maximum norm.

Now, instead of showing that $a_{\rho_{n_m}}(v_{n_m}, \psi) - a_{\rho_0}(v, \psi) \to 0$ we rewrite the difference into $a_{\rho_{n_m}}(v, \psi) - a_{\rho_0}(v, \psi) + a_{\rho_{n_m}}(v_{n_m} - v, \psi)$.

To show convergence of the difference of the first terms, note that both the boundary integrals and the integrals which do not contain any parameter ρ cancel themselves out, such that we only have to have a glance at

$$\left| \int_{B_R} (\rho_{n_m} - \rho_0) \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} \, dx \right| \leqslant \|\rho_{n_m} - \rho_0\|_{L^2(B_R, \mathrm{Sym}(3))} \| \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} \|_{L^2(B_R, \mathbb{C}^3)}.$$

As discussed above, we know that $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$. Since the other term is bounded, the integral tends to zero.

To show that

$$a_{\rho_{n_m}}(v_{n_m} - v, \psi) = \int_{B_R} [\rho_{n_m} \operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl} \overline{\psi} - k^2 (v_{n_m} - v) \overline{\psi}] \, dx + \int_{\partial B_R} \Lambda(\nu \times (v_{n_m} - v)) \cdot \gamma_T(\overline{\psi}) \, dS$$

converges to zero, we first have a glance at the integral over B_R without material parameter. Recall that $v_{n_m} \to v$ in $L^2(B_R, \mathbb{C}^3)$ and, thus,

$$\left| k^2 \int_{B_R} (v_{n_m} - v) \overline{\psi} \, dx \right| \leqslant k^2 \|v_{n_m} - v\|_{L^2(B_R, \mathbb{C}^3)} \|\psi\|_{L^2(B_R, \mathbb{C}^3)} \to 0.$$

To show convergence of the integral containing the material parameter, recall that $\operatorname{curl} v_{n_m} \rightharpoonup \operatorname{curl} v$ in $L^2(B_R, \mathbb{C}^3)$ and by the same arguments as above we deduce again that $\rho_{n_m} \to \rho_0$ in $L^2(B_R, \mathrm{Sym}(3))$. Respecting the a.e. boundedness of ψ and $\operatorname{curl}\psi$, this implies

$$\begin{aligned}
\left|\int_{B_R} \rho_{n_m} \operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl}\overline{\psi}\, dx\right| \\
&\leqslant \left|\int_{B_R} (\rho_{n_m} - \rho_0) \operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl}\overline{\psi}\, dx\right| \\
&\quad + \left|\int_{B_R} \rho_0 \operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl}\overline{\psi}\, dx\right| \\
&\leqslant \|\rho_{n_m} - \rho_0\|_{L^2(B_R, \mathrm{Sym}(3))} \|\operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl}\overline{\psi}\|_{L^2(B_R, \mathbb{C}^3)} \\
&\quad + \left|\int_{B_R} \rho_0 \operatorname{curl}(v_{n_m} - v) \cdot \operatorname{curl}\overline{\psi}\, dx\right|.
\end{aligned}$$

As in Banach spaces also weakly convergent sequences are bounded (see e.g., [Alt12]), $\operatorname{curl}(v_{n_m} - v)$ is a bounded term and while $\|\rho_{n_m} - \rho_0\|_{L^2(B_R, \mathrm{Sym}(3))} \to 0$, the first term vanishes. The last one converges since $\operatorname{curl}(v_{n_m} - v) \rightharpoonup 0$ in $L^2(B_R, \mathbb{C}^3)$.

At least, to see the convergence of the boundary integral, we have to be aware, that the solution is a smooth function on a neighborhood S of ∂B_R, not containing $\overline{D}$, such that $\overline{S} \cap \overline{D} = \emptyset$. To see this, we choose a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(\chi) \subset S$, such that $\chi \equiv 1$ in a neighborhood of ∂B_R but vanishes elsewhere. Therefore, $v|_S := \chi v$ is a smooth function outside D, see Remark 4.1 B, and, thus, for $j \geqslant 1$ one derives, using the integro-differential form (4.24) of the solution, the estimate

$$\begin{aligned}
\| v|_S \|_{C^j(\overline{S})} &\leqslant C(S, j) \|(I_3 - \rho)\, [\operatorname{curl} v + f]\|_{L^2(B_R, \mathbb{C}^3)} \\
&\leqslant C(S, j) \| I_3 - \rho\|_{L^\infty(B_R, \mathbb{C}^3)} \left(\|v\|_{H(\operatorname{curl}, B_R)} + \|f\|_{L^2(B_R, \mathbb{C}^3)}\right),
\end{aligned}$$

where in fact $f = \operatorname{curl} v_g$. That shows $v_{n_m}|_S, v|_S \in C^\infty(\overline{S})$ for all $m \in \mathbb{N}$, implying $v_{n_m} \to v$ in $H(\operatorname{curl}, S)$ due to the density of $C^\infty(S)$ in $H(\operatorname{curl}, S)$, from where the tangential trace mapping γ_t and the exterior Calderon operator Λ maps $v_{n_m} - v$ into $H^{-1/2}(\operatorname{Div}, \partial B_R)$, the dual space of the range $H^{-1/2}(\operatorname{Curl}, \partial B_R)$ of the trace γ_T. Thus,

$$\int_{\partial B_R} \Lambda(\gamma_t(v_{n_m} - v)) \cdot \gamma_T(\overline{\psi})\, dS \to 0.$$

Hence, we have shown that

$$\begin{aligned}
&\int_{B_R} [\rho_{n_m} \operatorname{curl} v_{n_m} \cdot \operatorname{curl}\overline{\psi} - k^2 v_{n_m} \overline{\psi}]\, dx + \int_{\partial B_R} \Lambda(\nu \times v_{n_m}) \cdot \gamma_T(\overline{\psi})\, dS \\
&\quad \to \int_{B_R} [\rho_0 \operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v \overline{\psi}]\, dx + \int_{\partial B_R} \Lambda(\nu \times v) \cdot \gamma_T(\overline{\psi})\, dS,
\end{aligned}$$

i.e., $a_{\rho_{n_m}}(v_{n_m}, \psi) \to a_{\rho_0}(v, \psi)$ for all $\psi \in C^1(\overline{B}_R)$. Since due to Assumption 4.4 the problem is uniquely solvable, this implies that $v = L(\rho_0, v_g)$. □

SPARSITY REGULARIZATION To gain sparse reconstruction techniques, we follow an approach with respect to basis functions of $\mathcal{D}(F)$. Note that, since B_R is of finite measure, it holds that $\mathcal{P} \subseteq L^\infty(B_R, \mathrm{Sym}(3)) \subseteq L^2(B_R, \mathrm{Sym}(3))$ with continuous embedding. We thus fix a biorthogonal wavelet Riesz basis $\{\psi_i\}_i$, $\{\widetilde{\psi}_i\}_i$, assuming that each ψ_i is also a function in $L^\infty(B_R, \mathrm{Sym}(3))$. Further, due to Hölder interpolation it holds that $(L^p(B_R, \mathrm{Sym}(3)) \cap \mathcal{P}) \subseteq L^2(B_R, \mathrm{Sym}(3))$ for $p \in (1, 2]$, such that we define our penalty term as some weighted ℓ^p-norm, i.e.,

$$\mathcal{R}_p(\rho) := \frac{1}{p} \sum_{i \in \mathbb{N}} \omega_i |\langle \rho, \widetilde{\psi}_i \rangle|^p, \quad \rho \in \mathcal{P},\ p \in (1, 2], \tag{4.32}$$

with non-negative weights $(\omega_i)_i$, satisfying $\|\rho\|_{L^\infty} \leqslant \mathcal{R}_p(\rho)$. Note that such weights exist, since one can achieve a norm equivalence for an appropriate Besov space (see, e.g., [Coh03, Theorem 3.7.7]), such that Sobolev/Besov embedding theorems yield an L^∞-embedding.

THEOREM 4.16 (Sparsity regularization I). *For $p \in (1, 2]$, the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$, defined in (3.29), with $\mathcal{R} = \mathcal{R}_p$, defined in (3.32), possesses a minimizer in $\mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3))$.*

If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that converges $\mathcal{P}$-weakly to a $\mathcal{R}_p$-minimizing solution $\rho^\dagger \in \mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3))$ of the equation $F(\rho) = F(\rho_{exa})$ in $\mathcal{S}_q$.

Recall that $\rho^\dagger$ is a $\mathcal{R}_p$-minimizing solution to $F(\rho^\dagger) = F(\rho_{\mathrm{exa}})$ if

$$\mathcal{R}_p(\rho^\dagger) = \min\{\mathcal{R}_p(\rho),\ \rho \in \mathcal{P} \cap L^p(B_R, \mathrm{Sym}(3)),\ F(\rho) = F_{\mathrm{exa}}\}.$$

Proof. As carried out above, the choices of $\mathcal{P}$ and $\mathcal{S}_q$ satisfy the conditions for the Tikhonov regularization of Theorem 4.13, as well as the sequentially closedness of F, shown in Lemma 4.15. Further note that $|\langle \cdot, \widetilde{\psi}_i \rangle|^p$ is L^2-weakly lower semicontinuous and any $\mathcal{P}$-weak* convergent sequence is also L^2-weakly convergent due to continuous embedding of $\mathcal{P}$ into $L^2(B_R, \mathrm{Sym}(3))$. Since scalar multiplication does not impact lower semicontinuity properties as well as summation of lower semicontinuous functions (see, e.g., [BL11, Lemma 6.14]), the penalty term $\mathcal{R}_p$ is weak*-lower semicontinuous, for $p \in (1, 2]$. Finally, it has weak*-precompact sublevel sets, since $\mathcal{P}$ itself is weak*-compact by Alaoglu's theorem. □

To avoid Hölder continuous spaces we now give a second approach by adapting techniques of image processing, where the gradient is used to highlight edges of objects, whereas homogeneous regions stay as more connected areas. Traditionally this leads to Sobolev penalty terms $\|D^m w\|_{L^p(\Omega)}$ for all $w \in W^{m,p}(\Omega)$ with $p \in (1, \infty)$ and $m \geqslant 1$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. It can be shown that $p \to 1$ yields better reconstructions. Since the $W^{1,1}$-seminorm is not lower semicontinuous [BL11, Satz 6.101], the boundedness of a sequence does not imply the existence of a weak-convergent subsequence, such that the Sobolev penalty term for $p = 1$ can not be used directly.

In fact, this holds in general for $p = 1$, since $L^1(\Omega)$ without a σ-finite measure is not a reflexive dual space of $L^\infty(\Omega)$. Thus, one generalizes the integral $\int_\Omega |Dw| dx$ by regarding $L^1(\Omega)$ as a subset of the space $\mathfrak{M}(\Omega, \mathbb{R}^3)$ of vector valued, finite Radon measures $\mu \colon \mathfrak{B}(\Omega) \to \mathbb{R}^3$, which is equipped with the norm $\|\mu\|_{\mathfrak{M}(\Omega,\mathbb{R}^3)} = |\mu|(\Omega)$, called the total variation measure. Therefore, $\mathfrak{M}(\Omega, \mathbb{R}^3)$ is a Banach space and isometrically isomorphic to the dual space $C_0(\Omega, \mathbb{R}^3)^*$ by Riesz-Markow's representation theorem. Note that the characterization as a dual space implies a weak*-convergence.

Due to that, one then defines the distributional gradient by the representation of such a vector-valued finite Radon measure:

Definition 4.17. For a domain $\Omega \subset \mathbb{R}^3$, $\mu \in \mathfrak{M}(\Omega, \mathbb{R}^3)$ is the distributional gradient of $w \in L^1_{loc}(\Omega)$, if for every $\psi \in C_0^\infty(\Omega, \mathbb{R}^3)$ it holds, that

$$\int_\Omega w \operatorname{div} \psi \, dx = -\int_\Omega \psi \, d\mu.$$

The norm of such a measure μ is called total variation of w and we write

$$\mathrm{TV}_\Omega(w) := \begin{cases} \|Dw\|_{\mathfrak{M}(\Omega,\mathbb{R}^3)} & \text{if the measure } \mu =: Dw \text{ exists,} \\ \infty & \text{else.} \end{cases}$$

If the distributional derivative of w can be written as a finite Radon measure, we say the function w has bounded total variation. Therefore, one often also writes

$$\mathrm{TV}_\Omega(w) = \sup\left\{ \int_\Omega w \operatorname{div} \psi \, dx, \ \psi \in C_0^\infty(\Omega), \ \|\psi\|_{L^\infty(\Omega)} \leqslant 1 \right\}. \tag{4.33}$$

The space of all functions with bounded total variation is thus defined by

$$\mathrm{BV}(\Omega) := \left\{ w \in L^1(\Omega), \ \mathrm{TV}_\Omega(w) < \infty \right\}.$$

Roughly speaking, $\mathrm{BV}(\Omega)$ contains functions in $L^1(\Omega)$, whose distributional gradients are finite Radon measures, and is a Banach space equipped with the norm $\|w\|_{\mathrm{BV}(\Omega)} := \|w\|_{L^1(\Omega)} + \mathrm{TV}_\Omega(w)$, whereas $\mathrm{TV}_\Omega(w)$ is the BV-seminorm (obviously $\|w\|_{W^{1,1}(\Omega)} = \|w\|_{\mathrm{BV}(\Omega)}$ for $w \in W^{1,1}(\Omega)$). Note that compared to Sobolev spaces, the BV-space also contains piecewise smooth functions, such that by total variation as penalty, one can handle functions with discontinuities.

Since BV is the dual space of the separable space L^1 on which bounded sets are pre-compact (see, e.g., [AFP00, Theorem 3.23] and [AVCM04, Remark B.7]) a weak*-convergence can be defined which yields the general weak*-topology, i.e., DEFINITION 4.18. Let $w, w_n \in BV(\Omega)$, then $\{w_n\}_{n\in\mathbb{N}}$ is said to be weakly*-convergent to w in $BV(\Omega)$, if $w_n \to w$ in $L^1(\Omega)$ and $Dw_n \xrightarrow{\mathfrak{M}} Dw$ in Ω, i.e.,

$$\lim_{n\to\infty} \int_\Omega \psi \, dDw_n = \int_\Omega \psi \, dDw \quad \text{for all } \psi \in C_0(\Omega).$$

We now restrict our set $\mathcal{P}$ and further operate on the set of material parameters, called

$$\mathcal{P}_{TV} := \left\{ \rho \in \mathcal{P}, \; TV_{(B_R, \mathrm{Sym}(3))}(\rho) < \infty \right\}.$$

Although Schuster et al. [Sch+12] suggest to use the BV-seminorm (4.33) as penalty, i.e., $\mathcal{R} = TV_\Omega$, Bachmayr and Burger [BB09] remark that using only the BV-seminorm as penalty, does not guarantee the possibility to make the to-be-solved variational problem locally convex, such that the global minimum can be computed by local descent methods. Because of that, they add a multiple of the squared L^2-norm to gain sufficient compactness properties of the functional. Even though the full BV-norm as penalty would provide the same compactness properties, they state that numerical minimization can be handled easier by adding L^2-norm instead of L^1-norm. However, since Bürgel, Kazimierski, and Lechleiter [BKL17] provide promising reconstructions for the full BV-norm (taking into account an additional term respecting some physical constraints), we suppose to use for $\rho \in \mathcal{P}$ the penalty

$$\mathcal{R}_{BV}(\rho) := \|\rho\|_{BV(B_R, \mathrm{Sym}(3))} = \|\rho\|_{L^1(B_R, \mathrm{Sym}(3))} + TV_{(B_R, \mathrm{Sym}(3))}(\rho). \tag{4.34}$$

THEOREM 4.19 (Sparsity regularization II). *The Tikhonov functional* $\mathcal{J}_{\alpha,\delta}$, *defined in* (4.31), *with* $\mathcal{R} = \mathcal{R}_{BV}$, *defined in* (4.34), *possesses a minimizer in* $\mathcal{P}_{TV} \cap BV(B_R, \mathrm{Sym}(3))$.

If $\delta_n \to 0$ *as* $n \to \infty$ *and if one chooses* $\alpha_n = \alpha_n(\delta_n)$ *such that* $0 < \alpha_n \to 0$ *and* $0 < \delta_n^2/\alpha_n \to 0$, *then every sequence of minimizers of* $\mathcal{J}_{\alpha_n,\delta_n}$ *contains a subsequence that converges* $\mathcal{P}_{TV}$*-weakly to an* $\mathcal{R}_{BV}$*-minimizing solution* $\rho^\dagger \in \mathcal{P}_{TV} \cap BV(B_R, \mathrm{Sym}(3))$ *of the equation* $F(\rho) = F(\rho_{exa})$ *in* $\mathcal{S}_q$.

Proof. As in Theorem 4.16, $\mathcal{S}_q$ and especially $\mathcal{P}_{TV}$ satisfy the conditions for the Tikhonov regularization of Theorem 4.13, since $\mathcal{P}_{TV} \subset \mathcal{P}$. Further, Lemma 4.15, i.e., the sequentially closedness of F, also holds for $\mathcal{P}_{TV}$. Since the total variation TV is weak*-lower semicontinuous (see, e.g., [BO13, Proposition 3.7]) as well as the L^1-norm, the penalty $\mathcal{R}_{BV}$ is weak*-lower semicontinuous as well. Again, $\mathcal{R}$ has weak*-precompact sublevel sets, since $\mathcal{P}_{TV}$ is again weak*-compact by Alaoglu's theorem. □

4.7 ADJOINT OF THE FORWARD OPERATOR'S LINEARIZATION

Most gradient-based schemes, which are used to solve the inverse scattering problem, i.e., stably solving the non-linear equation $F(\rho) = F_{meas}$ for some given data $F_{meas} \in \mathcal{S}_q$, like iterated shrinkage algorithm, rely on the adjoint operator of the linearization F'. This is why we give an explicit and computable representation, following Section 3.8. Therefore, we fix $\rho \in \mathcal{P}$, consider $F'(\rho)\colon L^\infty(B_R, \mathrm{Sym}(3)) \to \mathcal{S}_q$ and aim to determine $F'(\rho)^*\colon \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))$ such that

$$(F'(\rho)[\theta], K)_{\mathcal{S}_2} \stackrel{!}{=} (\theta, F'(\rho)^* K)_{L^2} \quad \text{for all } \theta \in L^\infty(B_R, \mathrm{Sym}(3)),\ K \in \mathcal{S}_{q'}. \tag{4.35}$$

Here, q' denotes the conjugate Lebesgue index to q, such that $1/q + 1/q' = 1$, and $(\cdot,\cdot)_{L^2}$ is the usual scalar product in $L^2(B_R, \mathrm{Sym}(3))$,

$$(A, B)_{L^2} = \int_{B_R} A : B \, dx = \int_{B_R} \sum_{i,j=1}^{d} \overline{A}_{ij} B_{ij} \, dx.$$

extended to the anti-linear dual product between $L^\infty(B_R, \mathrm{Sym}(3))$ and $L^1(B_R, \mathrm{Sym}(3))$. Further, $(\cdot,\cdot)_{\mathcal{S}_2}$ is the scalar product in the Hilbert space of Hilbert-Schmidt operators,

$$(F, K)_{\mathcal{S}_2} = \sum_{j \in \mathbb{N}} s_j(F) \overline{s_j(K)} = \sum_{j=1}^{\infty} (F g_j, K g_j)_{L^2_t(S^2)}$$

for an arbitrary orthonormal basis $(g_j)_{j \in \mathbb{N}}$ of $L^2_t(S^2)$. Consequently, (4.35) becomes

$$\sum_{j=1}^{\infty} (F'(\rho)[\theta] g_j, K g_j)_{L^2_t(S^2)} \stackrel{!}{=} (\theta, F'(\rho)^* K)_{L^2}$$

for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$, $K \in \mathcal{S}_{q'}$. Thus, we consider at first a single L^2-scalar product and for fixed $\rho \in \mathcal{P}$ and $g \in L^2_t(S^2)$ we seek for $A\colon L^2_t(S^2) \to L^1(B_R, \mathrm{Sym}(3))$, such that for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))$ and $f \in L^2_t(S^2)$ it holds that

$$(F'(\rho)[\theta] g, f)_{L^2_t(S^2)} \stackrel{!}{=} (\theta, Af)_{L^2}.$$

Recall from (4.25) that $L'(\rho, v_g)[\theta] = v' \in H(\mathrm{curl}, B_R)$, a function whose radiating extension satisfies

$$v' = -S_\rho \left[\mathrm{curl}\, V(\theta \, \mathrm{curl}[L(\rho, v_g) + v_g])\right] \ \text{ in } H(\mathrm{curl}, B_R),$$

where $S_\rho = [I_3 - \mathrm{curl}\, V((I_3 - \rho)\, \mathrm{curl})]^{-1}$. Since F' involves the far field of L', see (4.29), we note that

$$\begin{aligned} F'(\rho)[\theta] g &= Z \circ \left[(I_3 - \rho)\, \mathrm{curl}\, v' + \theta \, \mathrm{curl}\, S_\rho(v_g)\right] \\ &= Z \circ \left[\theta \, \mathrm{curl}\, S_\rho(v_g) - (I_3 - \rho)\, \mathrm{curl}\, S_\rho \left[\mathrm{curl}\, V(\theta \, \mathrm{curl}\, S_\rho(v_g))\right]\right]. \end{aligned}$$

Consequently, we compute that

$$
\begin{aligned}
&(F'(\rho)[\theta]g,\, f)_{L^2_t(\mathbb{S}^2)} \\
&= \big(\theta\,\mathrm{curl}\, S_\rho(\nu_g) - (I_3 - \rho)\,\mathrm{curl}\, S_\rho[\mathrm{curl}\, V(\theta\,\mathrm{curl}\, S_\rho(\nu_g))]\,,\, Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} \\
&= \big(\theta\,\mathrm{curl}\, S_\rho(\nu_g),\, Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} \\
&\quad - \big(\theta\,\mathrm{curl}\, S_\rho(\nu_g),\, [\mathrm{curl}\, V]^* \circ [(I_3 - \rho)\,\mathrm{curl}\, S_\rho]^* \circ Z^* f\big)_{L^2(B_R,\mathbb{C}^3)} \\
&= \big(\theta, \big[\big(I_3 - [(I_3 - \rho)\,\mathrm{curl}\, S_\rho \circ \mathrm{curl}\, V]^*\big) \circ Z^* f\big] \otimes \mathrm{curl}\, \overline{S_\rho(\nu_g)}\big)_{L^2}
\end{aligned}
$$

where the last matrix-valued function is defined by $(a \otimes b)_{i,j} = a_i b_j$ for $1 \leqslant i, j \leqslant 3$.

LEMMA 4.20. *For $\rho \in \mathcal{P}$ and $g \in L^2_t(\mathbb{S}^2)$, the adjoint of $\theta \mapsto F'(\rho)[\theta](g)$ with respect to the L^2-inner product maps $L^2_t(\mathbb{S}^2)$ into $L^1(B_R, \mathrm{Sym}(3))$ and is represented by*

$$
g \mapsto \Big(\big[I_3 - [(I_3 - \rho)\,\mathrm{curl}\, S_\rho \circ \mathrm{curl}\, V]^*\big] \circ Z^* g\Big) \otimes \mathrm{curl}\, \overline{S_\rho(\nu_g)}.
$$

For all orthonormal bases $\{g_j\}_{j \in \mathbb{N}}$ of $L^2_t(\mathbb{S}^2)$ and all $K \in \mathcal{S}_{q'}$, the bounded operator $F'(\rho)^* : \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))$ is represented by

$$
F'(\rho)^*(K) = \sum_{j=1}^{\infty} \Big[\big(I_3 - [(I_3 - \rho)\,\mathrm{curl}\, S_\rho \circ \mathrm{curl}\, V]^*\big) \circ Z^*\big(K g_j\big)\Big] \otimes \mathrm{curl}\, \overline{S_\rho[\nu_{g_j}]}. \tag{4.36}
$$

5 ANISOTROPIC MAGNETIC MEDIA

In Chapter 4 we derived some non-linear Tikhonov regularization and sparsity-promoting schemes for inverse electromagnetic scattering from anisotropic inhomogeneities which are non-magnetic. Complementarily, we now set aside this limitation and take also magnetic media into account, see Section 5.1. As pointed out in Section 5.2, this results into significant complexity regarding the techniques we use for our analysis. Hence, Sections 5.3 and 5.4 provide an analysis related to the one given in the chapter before, pointing out the main differences and where additional assumptions are required. Due to a short discussion in Section 5.5, we finally rely on the regularization techniques derived for the non-magnetic case and, again, close with the calculation of the adjoint of the forward operator's linearization, see Section 5.6.

Consider Maxwell's equations governing the propagation of time-harmonic electric and magnetic fields E and H in $\mathbb{R}^3$. Remember that the anisotropic relative permittivity ε_r and relative permeability μ_r are defined as

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0} + i\frac{\sigma(x)}{\omega\varepsilon_0}, \qquad \mu_r(x) = \frac{\mu(x)}{\mu_0}$$

and assumed to equal the constant background permittivity ε_0 and permeability μ_0 outside some bounded domain, i.e., $\varepsilon \equiv \varepsilon_0$ and $\mu \equiv \mu_0$. Working with the magnetic field only, the problem can be reduced to a second-order Maxwell system once more. For the positive wave number $k := \omega\sqrt{\varepsilon_0\mu_0}$ this reads

$$\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} H\right) - k^2\mu_r H = 0 \qquad \text{in } \mathbb{R}^3. \tag{5.1}$$

Accordingly, the electric field is determined by $E = i \operatorname{curl} H/(\omega\varepsilon_0\varepsilon_r)$.

Remark 5.1. Alternatively, one could also represent the magnetic field H by $\operatorname{curl} E/(i\omega\mu_0\mu_r)$ and derive

$$\operatorname{curl}\left(\mu_r^{-1} \operatorname{curl} E\right) - k^2\varepsilon_r E = 0 \qquad \text{in } \mathbb{R}^3.$$

Interchanging the roles of ε_r and μ_r in the following, hence, yields analogous results for the electric instead of the magnetic field.

5.1 MEDIUM SCATTERING

In contrast to Chapter 4 we now assume that the irradiated medium is modeled by relative electric permittivity $\varepsilon_r \in L^\infty(D, \mathrm{Sym}(3))$ as well as relative magnetic permeability $\mu_r \in W^{1,3+\delta}_{\mathrm{loc}}(\mathbb{R}^3, \mathrm{Sym}(3))$ for some $\delta > 0$. The material parameters are supposed to take values in the complex-valued symmetric 3×3 matrices $\mathrm{Sym}(3) \subset \mathbb{C}^{3\times 3}$ with uniformly positive definite real parts and symmetric imaginary parts, i.e., there exist a positive constant $\lambda > 0$ such that

$$2\lambda|\xi|^2 \leqslant \overline{\xi}^\top \mathrm{Re}(\varepsilon_r)\xi, \quad 2\lambda|\xi|^2 \leqslant \overline{\xi}^\top \mathrm{Re}(\mu_r)\xi, \quad \text{and } |\mu_r| + |\varepsilon_r| \leqslant \lambda^{-1}, \tag{5.2}$$

for almost all $x \in D$ and $\xi \in \mathbb{C}^3$. Thus, the real part of ε_r is the physical electric permittivity, whereas its imaginary part is proportional to the electric conductivity σ. Further the imaginary part of the complex magnetic permeability μ_r, modeling, e.g., magnetic dissipation or lag time, shall be bounded from below, i.e., $\overline{\xi}^\top \mathrm{Im}(\mu_r)\xi \geqslant 0$ for $\xi \in \mathbb{C}^3$. In particular we have that also $\varepsilon_r^{-1} \in L^\infty(D, \mathrm{Sym}(3))$, such that the imaginary part of ε_r^{-1} is bounded from above, that is to say $\overline{\xi}^\top \mathrm{Im}(\varepsilon_r^{-1})\xi \leqslant 0$ for $\xi \in \mathbb{C}^3$. Moreover, we suppose that the support of $I_3 - \varepsilon_r^{-1}$ and $\mu_r - I_3$ is the closure of a bounded and connected open set $D \subset \mathbb{R}^3$ with $C^{1,1}$-boundary and connected complement $\mathbb{R}^3 \setminus \overline{D}$. Suppose also that $\mathrm{supp}(\mu_r - \mu_0) \subset B_R$, where $\overline{D} \subset B_R$.

To keep notation simple and related to the one of Chapter 4, we assume the parameters $\rho := (\varepsilon_r^{-1}, \mu_r)$ to be in $\mathcal{P} = \mathcal{P}_{\varepsilon_r} \times \mathcal{P}_{\mu_r}$, for

$$\mathcal{P}_{\varepsilon_r} = \Big\{ \varepsilon_r \in L^\infty(D, \mathrm{Sym}(3)),\ \lambda|\xi|^2 \leqslant \mathrm{Re}(\overline{\xi}^\top \varepsilon_r\, \xi), \\ \mathrm{Im}(\overline{\xi}^\top \varepsilon_r^{-1}\, \xi) \leqslant 0,\ \text{a.e. in } D \text{ and for all } \xi \in \mathbb{C}^3 \Big\}.$$

and

$$\mathcal{P}_{\mu_r} = \Big\{ \mu_r \in W^{1,3+\delta}_{\mathrm{loc}}(\mathbb{R}^3, \mathrm{Sym}(3)),\ \delta > 0,\ \lambda|\xi|^2 \leqslant \mathrm{Re}(\overline{\xi}^\top \mu_r\, \xi), \\ \mathrm{Im}(\overline{\xi}^\top \mu_r\, \xi) \geqslant 0,\ \text{a.e. in } D \text{ and for all } \xi \in \mathbb{C}^3 \Big\}.$$

Both parameter sets could be equipped with the topology induced by L^q-norms, $2 \leqslant q \leqslant \infty$. But note that the sets will not contain any interior point if $q < \infty$. However, as we need $q = \infty$ for regularization results, we follow the approach of Chapter 4 and directly assume that the parameter sets are equipped with the topology induced by the L^∞-norm. $\mathcal{P}$ is endowed with the product norm, i.e., $\|\cdot\|_{\mathcal{P}} = \|\cdot\|_{\mathcal{P}_{\varepsilon_r}} + \|\cdot\|_{\mathcal{P}_{\mu_r}}$.

SCATTERING PROBLEM Remember that the scattering problem in Chapter 4 was given by (4.9) and (4.10), where its solution satisfies the Silver-Müller radiation condition

$$\mathrm{curl}\, v(x) \times \hat{x} - ikv(x) = \mathcal{O}\left(|x|^{-2}\right), \quad \text{as } |x| \to \infty, \tag{5.3}$$

uniformly with respect to $\hat{x} \in \mathbb{S}^2$. In the following, we are going to evolve an analogous analysis for the problem, which is to seek for $f, g \in C^\infty(D, \mathbb{C}^3)$ a weak radiating solution $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ to

$$\mathrm{curl}\left(\varepsilon_r^{-1}\,\mathrm{curl}\, v\right) - k^2\mu_r v = k^2(\mu_r - I_3)\, g + \mathrm{curl}\left((I_3 - \varepsilon_r^{-1})\, f\right) \quad \text{in } \mathbb{R}^3, \tag{5.4}$$

$$\begin{aligned} \nu \times v|_- &= \nu \times v|_+ , \\ \nu \times \varepsilon_r^{-1} \operatorname{curl} v|_- - \nu \times v|_+ &= \nu \times (I_3 - \varepsilon_r^{-1})\, f \end{aligned} \quad \text{on } \partial D. \tag{5.5}$$

Thus, the weak solution $v \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ needs to satisfy

$$\begin{aligned} &\int_{\mathbb{R}^3} \left[\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 \mu_r v \cdot \overline{\psi}\right] dx \\ &\qquad = \int_{\mathbb{R}^3} \left[k^2 (\mu_r - I_3)\, g \cdot \overline{\psi} + (I_3 - \varepsilon_r^{-1})\, f \cdot \operatorname{curl} \overline{\psi}\right] dx \end{aligned} \tag{5.6}$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support and, additionally, the Silver-Müller radiation condition (5.3).

Remark 5.2. Note that choosing $\psi = \nabla \varphi$ as a gradient field, the equation $\operatorname{curl} \nabla \varphi = 0$ implies that $\int_{\mathbb{R}^3} \mu_r v \cdot \nabla \overline{\varphi}\, dx = -\int_{\mathbb{R}^3} Ph \cdot \nabla \overline{\phi}\, dx$ for all $\varphi \in H^1(\mathbb{R}^3)$ with compact support, i.e., $\operatorname{div}(\mu_r v) = -\operatorname{div}(Ph)$ in $\mathbb{R}^3$. In contrast to the non-magnetic case, the solution v is not necessarily divergence free anymore.

5.2 THE SOLUTION OPERATOR

Following Section 4.3 we transform the weak formulation (5.6) into a variational equation on a bounded domain. Therefore, we integrate over a ball B_R, containing the supports of $I_3 - \varepsilon_r^{-1}$ and $\mu_r - I_3$ in its interior, against test functions $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support, and integrating by parts the rotation term. Since $\varepsilon_r^{-1} \equiv 1$ on ∂B_R, it holds that

$$\begin{aligned} &\int_{B_R} \left[\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 \mu_r v \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \gamma_t(\operatorname{curl} v) \cdot \gamma_T(\overline{\psi})\, dS \\ &\qquad = \int_D \left[k^2 (\mu_r - I_3)\, g \cdot \overline{\psi} + (I_3 - \varepsilon_r^{-1})\, f \cdot \operatorname{curl} \overline{\psi}\right] dx, \end{aligned} \tag{5.7}$$

where γ_t and γ_T denote the tangential trace and the "dual" tangential trace mapping defined in Section 4.3. Again, bringing the exterior Calderon operator Λ into play, motivates to define for $\rho \in \mathcal{P}$ and for all $\varphi, \psi \in H(\operatorname{curl}, B_R)$ the sesquilinear form

$$a_\rho(\varphi, \psi) := \int_{B_R} \left[\varepsilon_r^{-1} \operatorname{curl} \varphi \cdot \operatorname{curl} \overline{\psi} - k^2 \mu_r \varphi \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \Lambda(\varphi) \cdot \overline{\psi}\, dS.$$

Due to that, we define the solution operator

$$L \colon \mathcal{P} \times H(\operatorname{curl}, B_R) \to H(\operatorname{curl}, B_R),$$

mapping the parameters $\rho = (\varepsilon_r^{-1}, \mu_r)$ and the incident field u^i to the weak solution of the scattering problem (5.7). Choosing $f = \operatorname{curl} u^i$ and $g = u^i$ in (5.7), $L(\rho, u^i)$ becomes hence the weak solution to

$$a_\rho(L(\rho, u^i), \psi) = \int_D \left[(I_3 - \varepsilon_r^{-1}) \operatorname{curl} u^i \cdot \operatorname{curl} \overline{\psi} + k^2 (\mu_r - I_3)\, u^i \cdot \overline{\psi}\right] dx \tag{5.8}$$

for all $\psi \in H(\text{curl}, B_R)$, such that the radiating extension ν of $L(\rho, u^i)$ to $\mathbb{R}^3$ weakly solves

$$\text{curl}(\varepsilon_r^{-1} \,\text{curl}\, \nu) - k^2 \mu_r \nu = \text{curl}\left((I_3 - \varepsilon_r^{-1})\,\text{curl}\, u^i\right) + k^2(\mu_r - I_3)\, u^i.$$

Remark 5.3. Note that for non-magnetic media we have $\mu \equiv \mu_0$ and, therefore, the material parameter ρ solely equals ε_r^{-1}, such that the sesquilinear form was defined as

$$a_\rho(\varphi, \psi) := \int_{B_R} \left[\varepsilon_r^{-1} \,\text{curl}\, \varphi \cdot \text{curl}\, \overline{\psi} - k^2 \varphi \cdot \overline{\psi}\right] dx + \int_{\partial B_R} \Lambda(\varphi) \cdot \overline{\psi}\, dS.$$

Thus, the solution operator was for all $\psi \in H(\text{curl}, B_R)$ the solution of

$$a_\rho(L(\rho, u^i), \psi) = \int_D (I_3 - \varepsilon_r^{-1})\,\text{curl}\, u^i \cdot \text{curl}\, \overline{\psi}\, dx.$$

Again, existence of solution can be provided by uniqueness such that Lemma 4.3 holds analogously according to the right-hand side of (5.6).

Assumption 5.4. We assume in the following that for the set $\mathcal{P}$ any solution to (5.6) for $f, g \in L^2(D, \mathbb{C}^3)$ is unique, such that existence and continuous dependence of this solution follow from uniqueness. For example, conductors (i.e., complex-valued ε_r) satisfy this assumption if $\varepsilon_r \in C^{1,\alpha}(D)$ and $\mu_r \in C^{2,\alpha}(D)$, see [Kir07, Theorem 2.5]. Further uniqueness for Maxwell's equations also holds, if $\varepsilon_r, \mu_r \in L^\infty(\mathbb{R}^3)^{3\times 3}$ satisfy the ellipticity conditions (5.2) and if $|k|$ is sufficiently small, or if for a fixed k the domain is of small measure and bounded diameter, see [BCTX12].

Therefore, the solution operator $L(\rho, \cdot)$ exists for all $\rho \in \mathcal{P}$, with a constant $C = C(\mathcal{P}) > 0$ such that $\| L(\rho, u^i)\|_{H(\text{curl}, B_R)} \leqslant C \|u^i\|_{H(\text{curl}, B_R)}$.

NOTATION To improve readability, we abbreviate from now on notation of norms. Due to its frequent occurrence, we simply denote the norm of $L^\infty(B_R, \text{Sym}(3))$ by $\|\cdot\|$. Further we indicate the norm of $H(\text{div}, B_R)$ by $H(\text{div})$, if the domain does not differ from B_R. Analogously, $H(\text{curl})$ abbreviates $H(\text{curl}, B_R)$. Finally, we rely on notation used in [Mon03] and write $L^2(B_R)^3$ instead of $L^2(B_R, \mathbb{C}^3)$ and $H^1(B_R)^3$ for $H^1(B_R, \mathbb{C}^3)$ respectively. Otherwise, deviations of this notation is indicated.

ELLIPTIC REGULARITY Like in Chapter 4 we are going to handle derivatives of scattered fields in L^p-spaces and, hence, use H^1-estimates stated for Maxwell's equations by Alberti and Capdeboscq [AC14, Theorem 2].

Theorem 5.5. *For $\rho \in \mathcal{P}$ let $\nu \in H(\text{curl}, B_R)$ be a weak solution of*

$$\text{curl}\left(\varepsilon_r^{-1} \,\text{curl}\, \nu\right) - k^2 \mu_r \nu = \text{curl}\,(A\, f_1) + k^2 B\, f_2 \quad \text{in } \mathbb{R}^3, \tag{5.9}$$

where $A, B \in L^\infty(B_R, \mathrm{Sym}(3))$ *denote arbitrary coefficients and* f_1, f_2 *are functions in* $L^2(B_R)^3 \times H(\mathrm{div}, B_R)$.

Then $v \in H^1(B_R)^3$ *and there holds that*

$$\|v\|_{H^1(B_R)^3} \leqslant C\Big\{\big[1 + \|B\| + \|A\|\big]\,\|v\|_{H(\mathrm{curl})} + \|B\|\|f_2\|_{L^2(B_R)^3} + \big[\|A\| + \|\varepsilon_r A\|\big]\,\|f_1\|_{L^2(B_R)^3} + |k|\,\|B\, f_2\|_{H(\mathrm{div})}\Big\}. \tag{5.10}$$

for a constant C *depending on* $D \subset B_R$, λ, k, *and* $\|\mu_r\|_{W^{1,3+\delta}(B_R,\mathrm{Sym}(3))}$ *only.*

Proof. Due to Kirsch [Kir07, Theorem 2.3] v can be written in the form of an integro-differential equation

$$v(x) = (k^2 + \nabla\,\mathrm{div}) \int_{B_R} B\,[v(y) + f_2(y)]\,\Phi_k(x,y)\,dy + \mathrm{curl} \int_{B_R} A\,[\mathrm{curl}\, v(y) + f_1(y)]\,\Phi_k(x,y)\,dy,$$

with densities in $L^2(B_R)^3$ and kernel in $C^\infty(B_R)^3$ for $x \in S$. Here S denotes a neighborhood of ∂B_R such that S does not contain $\overline{D}$. Therefore, v is a smooth function outside D, see Remark 4.1 B, and one derives for $j \geqslant 1$ the estimate

$$\begin{aligned}\| v|_S \|_{C^j(B_R)} &\leqslant C(S,j)\Big[\|B\,[v + f_2]\|_{L^2(B_R)^3} + \|A\,[\mathrm{curl}\, v + f_1]\|_{L^2(B_R)^3}\Big] \\ &\leqslant C(S,j)\Big[\|B\|(\|v\|_{H(\mathrm{curl})} + \|f_2\|_{L^2(B_R)^3}) \\ &\qquad + \|A\|(\|v\|_{H(\mathrm{curl})} + \|f_1\|_{L^2(B_R)^3})\Big].\end{aligned}$$

Thus, $v|_S \in C^\infty(B_R)$ yields a boundary condition which is bounded from above in the $H^1(B_R)$-norm.

Now we rewrite the second-order form (5.9) as $\mathrm{curl}\, w + ik\mu_r v = -ikB\, f_2$, where $ikw = \varepsilon_r^{-1}\,\mathrm{curl}\, v - A\, f_1$. Consequently, (5.9) turns into the first-order system

$$\begin{aligned}\mathrm{curl}\, v - ik\varepsilon_r w &= \varepsilon_r A\, f_1\,, \\ \mathrm{curl}\, w + ik\mu_r v &= -ikB\, f_2,\end{aligned} \quad \text{in } B_R, \qquad v \times \nu = v|_S \times \nu \quad \text{on } \partial B_R. \tag{5.11}$$

Note that the right-hand side of its first equation is in $L^2(B_R)^3$, since $f_1 \in L^2(B_R)^3$ and $A \in L^\infty(B_R, \mathrm{Sym}(3))$. Likewise we have the same for the right-hand side of the second equation as $f_2 \in H(\mathrm{div}, B_R)$ and $B \in L^\infty(B_R, \mathrm{Sym}(3))$.

Thus, the first-order system is equivalent to the Maxwell's equations (2) of [AC14] with interchanged roles of the electric and magnetic fields. Since the ball $B_R \subset \mathbb{R}^3$ is a bounded and connected domain with $C^{1,1}$-boundary, Theorem 1 of Alberti and Capdeboscq [AC14] for the case $p = 2$ therefore provides the estimate

$$\|v\|_{H^1(B_R)^3} \leqslant C\left(\|v\|_{H(\mathrm{curl})} + \| v|_S \|_{H^1(B_R)} + \|\varepsilon_r A\, f_1\|_{L^2(B_R)^3} + |k|\|B\, f_2\|_{H(\mathrm{div})}\right).$$

Using the estimate of the boundary term as above, one derives the statement (5.10). □

COROLLARY 5.6. *For* $\rho \in \mathcal{P}$ *let* $v \in H(\mathrm{curl}, B_R)$ *be a weak solution of* (5.4), *then* $v \in H^1(B_R)^3$ *and there holds that*

$$\|v\|_{H^1(B_R)^3} \leqslant C\Big\{ \left[1 + \|(\mu_r - I_3)\| + \|(I_3 - \varepsilon_r^{-1})\|\right] \|v\|_{H(\mathrm{curl})} + \left[\| I_3 - \varepsilon_r^{-1}\| + \|(\varepsilon_r - I_3)\|\right] \|f\|_{L^2(B_R)^3} + \|(\mu_r - I_3)\| \|g\|_{L^2(B_R)^3} + |k| \, \|(\mu_r - I_3)\, g\|_{H(\mathrm{div})}\Big\}, \tag{5.12}$$

where C *is a constant depending on* D, λ, k, *and* $\|\mu_r\|_{W^{1,3+\delta}(B_R,\mathrm{Sym}(3))}$ *only.*

Proof. Remember that v solves equation (4.9), means v is a solution to

$$\mathrm{curl}\left(\varepsilon_r^{-1}\,\mathrm{curl}\, v\right) - k^2\mu_r v = k^2(\mu_r - I_3)\, g + \mathrm{curl}\left((I_3 - \varepsilon_r^{-1})\, f\right) \quad \text{in } \mathbb{R}^3$$

and, thus, is of the form (5.9) for $A = I_3 - \varepsilon_r^{-1}$ and $B = \mu_r - I_3$ and $(f_1, f_2) = (f, g)$. Thus, Theorem 5.5 yields (5.12). □

Remark 5.7. In comparison to the non-magnetic scattering problem handled in Chapter 4, the regularity estimate of Theorem 4.5 and its resulting Corollary 4.6 respectively, provides a similar although less complex statement. Having a glance at the magnetic case, first of all we observe that the constant C, scaling the upper bound of the solution, depends on the norm of the relative magnetic permeability μ_r, but which in this case can be compactly embedded into appropriate differentiability classes C^k by general Sobolev inequalities, see, e.g., [Eva02, Theorem 6]. Further, the arising extra terms cost us slightly more effort in the ongoing analysis of the solution operator's properties. Most problematic is the $H(\mathrm{div}, B_R)$-norm containing the part of the right-hand side of (5.4), linked to the relative magnetic permeability μ_r. To provide analogous properties to the ones derived in Sections 4.3 and 4.4 this term has somehow to be bounded by the $H(\mathrm{curl}, B_R)$-norm of the incident field. At this point we are not aware how to get naturally such a bound without stating massive smoothness assumptions for μ_r. However, during the analysis of the solution operator we drag this term along, such that it becomes apparent where such a bound would be necessary.

SOLUTION OPERATOR'S CONTINUITY For the ongoing part we assume to have a perturbation $\rho' = (\rho_1', \rho_2')$ of the parameter $\rho \in \mathcal{P}$, small enough such that ρ' is bounded in the $L^\infty(B_R, \mathrm{Sym}(3))$-norm and $\rho + \rho' \in \mathcal{P}$.

THEOREM 5.8. *Let* $\rho, \rho + \rho' \in \mathcal{P}$ *satisfying Assumption* 5.4. *Then there holds that*

$$\begin{aligned}\| L(\rho+\rho', u^i) - L(\rho, u^i)\|_{H^1(B_R)^3} \leqslant C \Big\{ & \big[(\|\rho_1'\| + \|\rho_2'\|)^3 \\ & +(2+\|\varepsilon_r\|_\infty)(\|\rho_1'\| + \|\rho_2'\|)^2\big] \, \|u^i\|_{H(\mathrm{curl})} \\ & +|k| \|\rho_2' \, [L(\rho+\rho', u^i) + u^i]\|_{H(\mathrm{div})} \Big\},\end{aligned}$$

where $C > 0$ *depends on* B_R, k, ρ, $\| I_3 - \varepsilon_r^{-1} \|$ *and* $\|\mu_r - I_3\|$ *only.*

Proof. For the same incident field u^i we set $v_{\rho+\rho'} = L(\rho+\rho', u^i)$ and $v = L(\rho, u^i)$ and denote the radiating extensions of these functions to $\mathbb{R}^3$ again by $v_{\rho+\rho'}$, v and the corresponding total fields by $u_{\rho+\rho'} = u^i + v_{\rho+\rho'}$ and $u = u^i + v$. The difference $v_{\rho+\rho'} - v = u_{\rho+\rho'} - u$ is the weak, radiating solution to

$$\begin{aligned}\mathrm{curl}\left(\varepsilon_r^{-1}\,\mathrm{curl}(u_{\rho+\rho'} - u)\right) - k^2\mu_r(u_{\rho+\rho'} - u) & \\ = k^2\rho_2' u_{\rho+\rho'} - \mathrm{curl}\left(\rho_1'\,\mathrm{curl}\, u_{\rho+\rho'}\right) & \quad \text{in } \mathbb{R}^3.\end{aligned}$$

Now applying Theorem 5.5, yields

$$\begin{aligned}\|u_{\rho+\rho'} - u\|_{H^1(B_R)^3} \leqslant C \Big\{ & \big[1 + \|\rho_2'\| + \|\rho_1'\|\big] \, \|u_{\rho+\rho'} - u\|_{H(\mathrm{curl})} \\ & + \big[\|\rho_1'\| + \|\varepsilon_r \rho_1'\|\big] \, \|\mathrm{curl}\, u_{\rho+\rho'}\|_{L^2(B_R)^3} \\ & + \|\rho_2'\| \|u_{\rho+\rho'}\|_{L^2(B_R)^3} + |k| \, \|\rho_2' \, u_{\rho+\rho'}\|_{H(\mathrm{div})} \Big\},\end{aligned}$$

where, in fact, ε_r is bounded to its maximum norm, such that for a large enough constant the second line simplifies to

$$[1 + C_*]\|\rho_1'\| \|\,\mathrm{curl}\, u_{\rho+\rho'}\|_{L^2(B_R)^3}.$$

Due to analogous version of Lemma 4.3 we further have the bound

$$\begin{aligned}\|u_{\rho+\rho'} - u\|_{H(\mathrm{curl})} \leqslant C \Big[& \|\rho_2'\|_{L^\infty(D,\mathrm{Sym}(3))} \|u_{\rho+\rho'}\|_{L^2(D)^3} \\ & + \|\rho_1'\|_{L^\infty(D,\mathrm{Sym}(3))} \|\,\mathrm{curl}\, u_{\rho+\rho'}\|_{L^2(D)^3} \Big],\end{aligned}$$

note that here we dropped down in the domain; we therefore increase the norm by enlarging the domain again. We will do this implicitly during further estimates without mentioning. Estimating norms in L^2 by norms in $H(\mathrm{curl})$, we thus yield—not regarding the $H(\mathrm{div})$-term—inside the curly brackets

$$\big[\|\rho_1'\|^2 + (2 + C_*)\|\rho_1'\| + 2\|\rho_1'\|\|\rho_2'\| + 2\|\rho_2'\| + \|\rho_2'\|^2\big] \, \|u_{\rho+\rho'}\|_{H(\mathrm{curl})}.$$

Afterwards we use triangle-inequality $\|u_{\rho+\rho'}\| \leqslant \|u^i\| + \|v_{\rho+\rho'}\|$ in $H(\mathrm{curl})$-norm to get rid of the total field, where the analogous of Lemma 4.3 again implies

$$\begin{aligned}\|v_{\rho+\rho'}\|_{H(\mathrm{curl})} \leqslant C \Big\{ & \| I_3 - \varepsilon_r^{-1} - \rho_1' \| \|\,\mathrm{curl}\, u^i\|_{L^2(B_R)} \\ & + \|\mu_r + \rho_2' - I_3\| \|u^i\|_{L^2(B_R)} \Big\}.\end{aligned}$$

Herein triangle-inequality again yields

$$\| I_3 - \varepsilon_r^{-1} - \rho_1' \| \leqslant \| I_3 - \varepsilon_r^{-1} \| + \| \rho_1' \|,$$

where the first term is bounded by a constant C. Do the same procedure for $\|\mu_r - I_3 + \rho_2'\|$ to get another constant C. Thus, we have that

$$\|v_{\rho+\rho'}\|_{H(\mathrm{curl})} \leqslant C(\|\rho_1'\| + \|\rho_2'\|)\|u^i\|_{H(\mathrm{curl})}.$$

Putting this altogether yields the inequality

$$\|u_{\rho+\rho'} - u\|_{H^1(B_R)^3} \leqslant C \left\{ T(\rho') \|u^i\|_{H(\mathrm{curl})} + |k| \|\rho_2' u_{\rho+\rho'}\|_{H(\mathrm{div})} \right\},$$

such that $T(\rho')$ denotes a term, depending on the $L^\infty(B_R, \mathrm{Sym}(3))$-norm of the perturbation ρ', defined by

$$\begin{aligned}
&(\|\rho_1'\| + \|\rho_2'\|)^3 + \big((2 + C_*)\|\rho_1'\|^2 + (4 + C_*)\|\rho_1'\|\|\rho_2'\| + 2\|\rho_2'\|^2\big) \\
&\leqslant (\|\rho_1'\| + \|\rho_2'\|)^3 + \big((2 + C_*)\|\rho_1'\|^2 + (4 + C_*)\|\rho_1'\|\|\rho_2'\| + (2 + C_*)\|\rho_2'\|^2\big) \\
&= (\|\rho_1'\| + \|\rho_2'\|)^3 + (2 + C_*)(\|\rho_1'\| + \|\rho_2'\|)^2. \qquad (5.13)
\end{aligned}$$

Thus, we have T bounded by the last two terms and, therefore, yield the statement. □

Note. As mentioned in Remark 5.7 the just derived estimate is slightly more complex compared to the one of Theorem 4.7. In particular, therein, the term concerning the L^∞-norm of the perturbation ρ' extends to (5.13). Further note that the H(div)-term has to be bounded by the H(curl)-norm of the incident field to provide Lipschitz continuity for the solution operator L.

5.3 DIFFERENTIABILITY OF THE SOLUTION OPERATOR

To have a glance at the differentiability of the solution operator, we fix the incident field in this section and the parameters $\rho \in \mathcal{P}$, such that the solution operator $L(\rho, \cdot)$ is bounded on $H(\mathrm{curl}, B_R)$. Then we denote the derivative of L with respect to ρ in direction $\rho' \in L^\infty(B_R, \mathrm{Sym}(3))$ by $v' := L'(\rho, u^i)[\rho']$, defined by

$$\begin{aligned}
a_\rho(v', \psi) = -\int_D \big[\rho_1' \,\mathrm{curl}\, \big(L(\rho, u^i) + u^i\big) \cdot \mathrm{curl}\, \overline{\psi} \\
- k^2 \rho_2' \big(L(\rho, u^i) + u^i\big) \cdot \overline{\psi}\big] dx.
\end{aligned} \qquad (5.14)$$

CONTINUITY PROPERTIES

LEMMA 5.9. *For every $\rho \in \mathcal{P}$ the linear mapping $\rho' \mapsto L'(\rho, u^i)[\rho']$ from $L^\infty(B_R, \mathrm{Sym}(3))$ to $H(\mathrm{curl}, B_R)$) has the following continuity property:*

$$\begin{aligned}
\| L'(\rho, u^i)[\rho'] \|_{H^1(B_R)^3} \leqslant C \Big\{ \Big[&(\|\rho_1'\| + \|\rho_2'\|)^2 \\
&+ (2 + \|\varepsilon_r\|_\infty)\|\rho_1'\| + 2\|\rho_2'\| \Big] \|u^i\|_{H(\mathrm{curl})} \\
&+ |k| \|\rho_2' [L(\rho, u^i) + u^i]\|_{H(\mathrm{div})} \Big\},
\end{aligned}$$

where $C > 0$ depends on B_R, k, ρ, $\| I_3 - \varepsilon_r^{-1} \|$ and $\|\mu_r - I_3\|$ only.

Proof. In the following we denote by $u = L(\rho, u^i) + u^i$ the total field and, thus, apply the H^1-estimate of Theorem 5.5 to gain

$$\begin{aligned}\| L'(\rho, u^i)[\rho']\|_{H^1(B_R)^3} \leqslant C \Big\{ &[1 + \|\rho_1'\| + \|\rho_2'\|] \, \| L'(\rho, u^i)[\rho']\|_{H(\mathrm{curl})} \\ &+ [1 + C_*]\|\rho_1'\| \|\operatorname{curl} u\|_{L^2(B_R)^3} \\ &+ \|\rho_2'\| \|u\|_{L^2(B_R)^3} + |k| \, \|\rho_2' \, u\|_{H(\mathrm{div})} \Big\},\end{aligned}$$

using again that the maximum norm $\|\varepsilon_r\|_\infty$ is bounded by C_*. Herein Lemma 4.3 states that

$$\begin{aligned}\| L'(\rho, u^i)[\rho']\|_{H(\mathrm{curl})} &\leqslant C \left[\|\rho_2' \, u\|_{L^2(B_R)^3} + \|\rho_1' \, \operatorname{curl} u\|_{L^2(B_R)^3}\right] \\ &\leqslant C \left[\|\rho_2'\| \|u\|_{L^2(B_R)^3} + \|\rho_1'\| \|\operatorname{curl} u\|_{L^2(B_R)^3}\right],\end{aligned}$$

(In the last line we could also have used Hölders inequality for some indices p, t such that $1/p + 1/t = 1/2$ if there would be an equivalent to Meyers gradient estimate for the curl-operator.) Now estimating both $\|\operatorname{curl} u\|_{L^2}$ and $\|u\|_{L^2}$ by $\|u\|_{H(\mathrm{curl})}$ we have

$$\begin{aligned}\| L'(\rho, u^i)[\rho']\|_{H^1(B_R)^3} \leqslant C \Big\{ \Big[&\|\rho_1'\|^2 + (2 + C_*)\|\rho_1'\| + 2\|\rho_1'\|\|\rho_2'\| \\ &+2\|\rho_2'\| + \|\rho_2'\|^2\Big] \|u\|_{H(\mathrm{curl})} + |k| \, \|\rho_2' \, u\|_{H(\mathrm{div})} \Big\},\end{aligned}$$

Again triangle-inequality estimates the total field u by the incident field u^i and the scattered field $L(\rho, u^i)$. Further, the analogous of Lemma 4.3 yields

$$\begin{aligned}\| L(\rho, u^i)\|_{H(\mathrm{curl})} \leqslant C\Big\{&\| I_3 - \varepsilon_r^{-1} \|\|\operatorname{curl} u^i\|_{L^2(B_R)^3} \\ &+ \|\mu_r - I_3 \|\|u^i\|_{L^2(B_R)^3}\Big\},\end{aligned}$$

where the L^∞-terms are bounded by constants as beforehand, such that

$$\| L(\rho, u^i)\|_{H(\mathrm{curl})} \leqslant C\sqrt{2}\|u^i\|_{H(\mathrm{curl})},$$

where we used that $\|u\|_{L^2(B_R)^3} + \|\operatorname{curl} u\|_{L^2(B_R)^3} \leqslant \sqrt{2}\|u\|_{H(\mathrm{curl})}$. Summing everything up, one finally yields

$$\begin{aligned}\| L'(\rho, u^i)[\rho']\|_{H^1(B_R)^3} \leqslant C\Big\{ \Big[&(\|\rho_1'\| + \|\rho_2'\|)^2 + (2 + C_*)\|\rho_1'\| \\ &+2\|\rho_2'\|\Big] 2\sqrt{2}C\|u^i\|_{H(\mathrm{curl})} + |k|\|\rho_2' \, [L(\rho, u^i) + u^i]\|_{H(\mathrm{div})}\Big\}.\end{aligned}$$

□

THEOREM 5.10. *The map $\rho \mapsto L'(\rho, u^i)$ is locally Lipschitz continuous: There is a $C > 0$ independent of ρ' and u^i, such that it holds for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))^2$ that*

$$\begin{aligned}\| L'(\rho + \rho', u^i)[\theta] - L'(\rho, u^i)[\theta]\|_{H^1(B_R)^3} \leqslant C\Big\{2\,\hat{T}(\rho', \theta) \, \|u^i\|_{H(\mathrm{curl})} \\ +|k| \left[\|\theta_2(L(\rho + \rho', u^i) - L(\rho, u^i))\|_{H(\mathrm{div})} + \|\rho_2' \, L'(\rho + \rho', u^i)[\theta]\|_{H(\mathrm{div})}\right]\Big\}.\end{aligned}$$

where $\hat{T}(\rho', \theta)$ *is given by* (5.15) *and* $C > 0$ *depends on* B_R, k, ρ, $\| I_3 - \varepsilon_r^{-1} \|$ *and* $\|\mu_r - I_3\|$ *only.*

Proof. For $\theta \in L^\infty(B_R, \mathrm{Sym}(3))^2$, $w_{\rho+\rho'} = L'(\rho + \rho', u^i)[\theta]$ and $w_\rho = L'(\rho, u^i)[\theta]$ satisfy by (5.14) the variational formulations

$$a_{\rho+\rho'}(w_{\rho+\rho'}, \psi) = -\int_D \left[\theta_1 \operatorname{curl} u_{\rho+\rho'} \cdot \operatorname{curl} \overline{\psi} - k^2 \theta_2 u_{\rho+\rho'} \cdot \overline{\psi}\right] dx,$$
$$a_\rho(w_\rho, \psi) = -\int_D \left[\theta_1 \operatorname{curl} u \cdot \operatorname{curl} \overline{\psi} - k^2 \theta_2 u \cdot \overline{\psi}\right] dx,$$

where the perturbed total field $u_{\rho+\rho'}$ consists of the perturbed scattered field $v_{\rho+\rho'} = L(\rho + \rho', u^i)$ and the incident field u^i. Thus, $w := w_{\rho+\rho'} - w_\rho$ satisfies

$$\begin{aligned} a_\rho(w, \psi) = &-\int_D \left[\theta_1 \operatorname{curl}(v_{\rho+\rho'} - v) \cdot \operatorname{curl} \overline{\psi} - k^2 \theta_2 (v_{\rho+\rho'} - v) \cdot \overline{\psi}\right] dx \\ &- \int_D \left[\rho_1' \operatorname{curl} w_{\rho+\rho'} \cdot \operatorname{curl} \overline{\psi} - k^2 \rho_2' w_{\rho+\rho'} \cdot \overline{\psi}\right] dx. \end{aligned}$$

Therefore, Theorem 5.5 states

$$\begin{aligned} \|w\|_{H^1(B_R)^3} \leqslant C\Big\{ &\left[1 + \|\theta_1\| + \|\theta_2\| + \|\rho_1'\| + \|\rho_2'\|\right] \|w\|_{H(\mathrm{curl})} \\ &+ \left[\|\theta_1\| + \|\varepsilon_r \theta_1\|\right] \|\operatorname{curl}(v_{\rho+\rho'} - v)\|_{L^2(B_R)^3} \\ &+ \left[\|\rho_1'\| + \|\varepsilon_r \rho_1'\|\right] \|\operatorname{curl} w_{\rho+\rho'}\|_{L^2(B_R)^3} \\ &+ \|\theta_2\| \|v_{\rho+\rho'} - v\|_{L^2(B_R)^3} + \|\rho_2'\| \|w_{\rho+\rho'}\|_{L^2(B_R)^3} \\ &+ |k| \|\theta_2 (v_{\rho+\rho'} - v)\|_{H(\mathrm{div})} + |k| \|\rho_2' w_{\rho+\rho'}\|_{H(\mathrm{div})} \Big\}. \end{aligned}$$

Further, use that the maximum norm $\|\varepsilon_r\|_\infty$ is bounded by a generic C_* and that the analogous of Lemma 4.3 states that

$$\begin{aligned} \|w\|_{H(\mathrm{curl})} \leqslant C\Big\{ &\|\theta_1\| \|\operatorname{curl}(v_{\rho+\rho'} - v)\|_{L^2(B_R)^3} + \|\rho_2'\| \|w_{\rho+\rho'}\|_{L^2(B_R)^3} \\ &+ \|\theta_2\| \|v_{\rho+\rho'} - v\|_{L^2(B_R)^3} + \|\rho_1'\| \|\operatorname{curl} w_{\rho+\rho'}\|_{L^2(B_R)^3} \Big\}. \end{aligned}$$

We thus gain that

$$\begin{aligned} \|w\|_{H^1(B_R)^3} \leqslant C\Big\{ &\Big[(2 + C_*)\|\theta_1\| + \|\theta_1\|^2 + \|\theta_1\| \|\rho_1'\| + \|\theta_1\| \|\theta_2\| \\ &+ \|\theta_1\| \|\rho_2'\|\Big] \|\operatorname{curl}(v_{\rho+\rho'} - v)\|_{L^2(B_R)^3} + \Big[2\|\theta_2\| + \|\theta_2\|^2 + \|\theta_2\| \|\rho_2'\| \\ &+ \|\theta_1\| \|\theta_2\| + \|\theta_2\| \|\rho_1'\|\Big] \|v_{\rho+\rho'} - v\|_{L^2(B_R)^3} + \Big[(2 + C_*)\|\rho_1'\| + \|\rho_1'\|^2 \\ &+ \|\theta_1\| \|\rho_1'\| + \|\rho_1'\| \|\rho_2'\| + \|\theta_2\| \|\rho_1'\|\Big] \|\operatorname{curl} w_{\rho+\rho'}\|_{L^2(B_R)^3} \\ &+ \Big[2\|\rho_2'\| + \|\rho_2'\|^2 + \|\theta_2\| \|\rho_2'\| + \|\rho_1'\| \|\rho_2'\| + \|\theta_1\| \|\rho_2'\|\Big] \|w_{\rho+\rho'}\|_{L^2(B_R)^3} \\ &+ |k| \|\theta_2 (v_{\rho+\rho'} - v)\|_{H(\mathrm{div})} + |k| \|\rho_2' w_{\rho+\rho'}\|_{H(\mathrm{div})} \Big\}. \end{aligned}$$

Now we can estimate $\|w_{\rho+\rho'}\|_{L^2(B_R)^3}$ and $\|\operatorname{curl} w_{\rho+\rho'}\|_{L^2(B_R)^3}$ by $\|w_{\rho+\rho'}\|_{H(\mathrm{curl})}$; analogously for the norms of $v_{\rho+\rho'} - v$. Therefore, again in analogy to Lemma 4.3 it holds that

$$\|w_{\rho+\rho'}\|_{H(\mathrm{curl})} \leqslant C\Big\{\|\theta_1\|\|\operatorname{curl} u_{\rho+\rho'}\|_{L^2(B_R)^3} + \|\theta_2\|\|u_{\rho+\rho'}\|_{L^2(B_R)^3}\Big\}$$
$$\leqslant C(\|\theta_1\| + \|\theta_2\|)\|u_{\rho+\rho'}\|_{H(\mathrm{curl})}.$$

Since $\|v_{\rho+\rho'}\|_{H(\mathrm{curl})} \leqslant C(\|\rho_1'\| + \|\rho_2'\|)\|u^i\|_{H(\mathrm{curl})}$ for a constant depending also on the L^∞-norms of $\mu_r - I_3$ and $I_3 - \varepsilon_r$, triangle inequality implies that

$$\|w_{\rho+\rho'}\|_{H(\mathrm{curl})} \leqslant C\Big\{(\|\theta_1\| + \|\theta_2\|)\|u^i\|_{H(\mathrm{curl})}$$
$$+(\|\theta_1\| + \|\theta_2\|)(\|\rho_1'\| + \|\rho_2'\|)\|u^i\|_{H(\mathrm{curl})}\Big\}.$$

By the same way we also gain that

$$\|v_{\rho+\rho'} - v\|_{H(\mathrm{curl})} \leqslant C\Big\{(\|\rho_1'\| + \|\rho_2'\|)\|u^i\|_{H(\mathrm{curl})}$$
$$+(\|\rho_1'\| + \|\rho_2'\|)^2\|u^i\|_{H(\mathrm{curl})}\Big\}.$$

Plugging these in and sorting the terms, we finally gain

$$\|w\|_{H^1(B_R)^3} \leqslant C\Big\{2\,\hat{T}(\rho',\theta)\,\|u^i\|_{H(\mathrm{curl})}$$
$$+|k|\Big[\|\theta_2(v_{\rho+\rho'} - v)\|_{H(\mathrm{div})} + \|\rho_2' w_{\rho+\rho'}\|_{H(\mathrm{div})}\Big]\Big\},$$

where $\hat{T}(\rho',\theta)$, depending on the $L^\infty(B_R, \mathrm{Sym}(3))$-norms of θ and ρ', is defined by

$$\hat{T}(\rho',\theta) := (\|\rho_1'\| + \|\rho_2'\|)^3(\|\theta_1\| + \|\theta_2\|)$$
$$+(\|\rho_1'\| + \|\rho_2'\|)^2(\|\theta_1\| + \|\theta_2\|)^2 + (\|\rho_1'\| + \|\rho_2'\|)(\|\theta_1\| + \|\theta_2\|)^2$$
$$+\big[(3+\|\varepsilon_r\|_\infty)(\|\rho_1'\|+\|\rho_2'\|)+(2+\|\varepsilon_r\|_\infty)\big]\,(\|\rho_1'\|+\|\rho_2'\|)(\|\theta_1\|+\|\theta_2\|) \tag{5.15}$$

□

Gâteaux derivative

Theorem 5.11. *The solution operator* L *is differentiable in the sense that for every* $\rho, \rho + \rho' \in \mathcal{P}$*, satisfying Assumption 5.4, it holds that*

$$\|L(\rho+\rho', u^i) - L(\rho, u^i) - L'(\rho, u^i)[\rho']\|_{H^1(B_R)^3} \leqslant C\Big\{\Big[(\|\rho_1'\| + \|\rho_2'\|)^3$$
$$+(2 + \|\varepsilon_r\|_\infty)(\|\rho_1'\| + \|\rho_2'\|)^2\Big]\|u^i\|_{H(\mathrm{curl})} + |k|\|\rho_2'\ L'(\rho, u^i)[\rho']\|_{H(\mathrm{div})}\Big\},$$

where $C > 0$ *depends on* B_R*,* k*,* ρ*,* $\|I_3 - \varepsilon_r^{-1}\|$ *and* $\|\mu_r - I_3\|$ *only.*

Proof. For $w := L(\rho+\rho', u^i) - L(\rho, u^i) - L'(\rho, u^i)[\rho']$ we first consider the variational formulations defining all three terms,

$$\begin{aligned} a_{\rho+\rho'}(L(\rho+\rho', u^i), \psi) &= \int_D \Big[(I_3 - \varepsilon_r^{-1} - \rho_1') \operatorname{curl} u^i \cdot \operatorname{curl} \overline{\psi} \\ &\qquad + k^2(\mu_r - I_3 + \rho_2') u^i \cdot \overline{\psi}\Big] dx, \\ a_\rho(L(\rho, u^i), \psi) &= \int_D \Big[(I_3 - \varepsilon_r^{-1}) \operatorname{curl} u^i \cdot \operatorname{curl} \overline{\psi} \\ &\qquad + k^2(\mu_r - I_3) u^i \cdot \overline{\psi}\Big] dx, \\ a_\rho(L'(\rho, u^i)[\rho'], \psi) &= -\int_D \Big[\rho_1' \operatorname{curl}(L(\rho, u^i) + u^i) \cdot \operatorname{curl} \overline{\psi} \\ &\qquad - k^2 \rho_2' (L(\rho, u^i) + u^i) \cdot \overline{\psi}\Big] dx, \end{aligned}$$

for all $\psi \in H(\operatorname{curl}, B_R)$. A short calculation analogously to the one seen for example in the proof of Theorem 4.10, shows that for all $\psi \in H(\operatorname{curl}, B_R)$ there holds

$$\begin{aligned} a_{\rho+\rho'}(w, \psi) &= \int_D k^2 \rho_2' L'(\rho, u^i)[\rho'] \cdot \overline{\psi} \, dx \\ &\quad - \int_D \rho_1' \operatorname{curl} L'(\rho, u^i)[\rho'] \cdot \operatorname{curl} \overline{\psi} \, dx. \end{aligned}$$

Now the H^1-estimate of Theorem 5.5 implies that

$$\begin{aligned} \|w\|_{H^1(B_R)^3} \leqslant C\Big\{&[1 + \|\rho_1'\|]\|w\|_{H(\operatorname{curl})} + [1 + C_*]\|\rho_1'\|\|\operatorname{curl} v'\|_{L^2(B_R)^3} \\ &+ \|\rho_2'\|\|w\|_{H(\operatorname{curl})} + \|\rho_2'\|\|v'\|_{L^2} + |k|\|\rho_2' v'\|_{H(\operatorname{div})}\Big\} \end{aligned}$$

using again that $\|\varepsilon_r\|_\infty$ is bounded to its maximum norm by C_*. Due to the above shown equation, the analogous of Lemma 4.3 implies that

$$\begin{aligned} \|w\|_{H(\operatorname{curl})} &\leqslant C\left[\|\rho_2' v'\|_{L^2(B_R)^3} + \|\rho_1' \operatorname{curl} v'\|_{L^2(B_R)^3}\right] \\ &\leqslant C\left[\|\rho_2'\|\|v'\|_{L^2(B_R)^3} + \|\rho_1'\|\|\operatorname{curl} v'\|_{L^2(B_R)^3}\right], \end{aligned}$$

Further estimating $\|v'\|_{L^2(B_R)^3}$ and $\|\operatorname{curl} v'\|_{L^2(B_R)^3}$ by $\|v'\|_{H(\operatorname{curl})}$ and, again, the analogous of Lemma 4.3 states

$$\|w\|_{H^1(B_R)^3} \leqslant C\left\{T(\rho')\,\|u^i\|_{H(\operatorname{curl})} + |k|\|\rho_2' u_{\rho'}\|_{H(\operatorname{div})}\right\},$$

where T is bounded as in (5.13). □

5.4 THE FORWARD OPERATOR

In this section, we define the forward operator corresponding to the inverse scattering problem we are ultimately interested in. This operator maps a contrast function to the corresponding far field operator. We recall that, to shorten notation, a parameter $\rho \in \mathcal{P}$ was set to be $\rho = (\rho_1, \rho_2) = (\varepsilon_r^{-1}, \mu_r)$.

POTENTIAL REPRESENTATION As in Section 4.5, we rely on a volume integral approach, see [Kir07, Theorem 2.3]. Thus, the scattering problem (5.4), (5.5), and (5.3) is equivalent to an integro-differential equation defined via the radiating fundamental solution $\Phi_k(x)$ to the Helmholtz equation in $\mathbb{R}^3$, see (4.23). In detail, $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ is a radiating solution to (5.6) if and only if v satisfies

$$v = (k^2 + \nabla \operatorname{div}) \int_{B_R} \Phi_k(\cdot - y)(\rho_2 - I_3)(y) \left[v(y) + u^i(y)\right] dy \\ + \operatorname{curl} \int_{B_R} \Phi_k(\cdot - y)(I_3 - \rho_1)(y) \operatorname{curl} \left[v(y) + u^i(y)\right] dy \qquad \text{in } \mathbb{R}^3.$$

In analogy, the radiating extension of $v' = L'(\rho, u^i)[\rho']$ to $\mathbb{R}^3$ satisfies

$$v' = (k^2 + \nabla \operatorname{div}) \int_{B_R} \Phi_k(\cdot - y)\left[(\rho_2 - I_3)v'(y) + \rho_2'(L(\rho, u^i) + u^i)(y)\right] dy \\ + \operatorname{curl} \int_{B_R} \Phi_k(\cdot - y)\left[(I_3 - \rho_1) \operatorname{curl} v'(y) - \rho_1' \operatorname{curl}(L(\rho, u^i) + u^i)(y)\right] dy, \tag{5.16}$$

because v' solves, by definition, the variational formulation (5.14).

FAR FIELD PATTERN Technically, the potential representation of a radiating solution consists of the two potentials

$$u_1 = \operatorname{curl} \int_{B_R} \Phi_x(\cdot - y) f_1(y)\, dy, \tag{5.17}$$

$$u_2 = (k^2 + \nabla \operatorname{div}) \int_{B_R} \Phi_k(\cdot - y) f_2(y)\, dy, \tag{5.18}$$

for appropriate densities $f_{1,2}$. Note that in both cases the tangential components are continuous on the boundary and $u_{1,2}$ satisfy the Silver-Müller radiation condition uniformly with respect to directions $\hat{x} \in \mathbb{S}^2$.

As seen in the non-magnetic case, see also, e.g., [LR15, Proposition 3], the corresponding far field pattern to radiated waves described by (5.17) is given by

$$u^\infty(\hat{x}) = \frac{ik\hat{x}}{4\pi} \times \int_{B_R} e^{-ik\hat{x}\cdot y} f_1(y)\, dy, \quad \hat{x} \in \mathbb{S}^2.$$

Further, the far field pattern for radiated waves of the form (5.18) is given by

$$u^\infty(\hat{x}) = \frac{k^2}{4\pi} (1 - \hat{X}) \int_{B_R} e^{-ik\hat{x}\cdot y} f_2(y)\, dy, \quad \hat{x} \in \mathbb{S}^2,$$

where the matrix $\hat{X}$ is defined as in (5.19). This can be seen as follows: the first part of the sum simply deduces from [CK13, eq. (5.94)] where $(\Phi_k(\cdot - y))^\infty(\hat{x})$ is claimed to be $\exp(-ik\hat{x} \cdot y)$, while the other part can be seen as the adjoint of the operator $H\colon L^2(\mathbb{S}^2)^3 \to L^2(B_R)^3$ defined by

$$(H\,p)(y) \coloneqq \nabla_y \operatorname{div}_y \int_{\mathbb{S}^2} p(\hat{x}) e^{ik\hat{x}\cdot y}\, dS(\hat{x}), \quad y \in B_R.$$

For its calculation we first remind the following identities for scalar valued functions $\lambda\colon \mathbb{R}^3 \to \mathbb{C}$ and vector valued functions $F, G\colon \mathbb{R}^3 \to \mathbb{C}^3$ [KH15, (6.6)]

$$\operatorname{div}(\lambda F) = F \cdot \nabla\lambda + \lambda \operatorname{div} F,$$

so that $\operatorname{div}_y(e^{ik\hat{x}\cdot y} p(\hat{x})) = p(\hat{x}) \cdot \nabla_y e^{ik\hat{x}\cdot y}$. Now [KH15, (6.8)] states

$$\nabla_y (F \cdot G) = (F')^\top G + (G')^\top F,$$

where $F'(y), G'(y) \in \mathbb{C}^{3\times 3}$ are the Jacobian matrices of F and G at y, i.e. $F'_{i,j} = \partial F_i / \partial y_j$. Thus, we have $\nabla_y (p(\hat{x}) \cdot \nabla_y e^{ik\hat{x}\cdot y}) = (E')^\top p(\hat{x})$, where E' denotes the Jacobian matrix of $\nabla_y e^{ik\hat{x}\cdot y}$ and is given by

$$E' \coloneqq -k^2 e^{ik\hat{x}\cdot y} \hat{X} = -k^2 e^{ik\hat{x}\cdot y} \begin{pmatrix} \hat{x}_1^2 & \hat{x}_1\hat{x}_2 & \hat{x}_1\hat{x}_3 \\ \hat{x}_1\hat{x}_2 & \hat{x}_2^2 & \hat{x}_2\hat{x}_3 \\ \hat{x}_1\hat{x}_3 & \hat{x}_2\hat{x}_3 & \hat{x}_3^2 \end{pmatrix}. \tag{5.19}$$

Note. Since $\hat{X}$ is symmetric, we have that $\hat{X}^\top = \hat{X}$ and, therefore, $(E')^\top = E'$.

Hence, we obtain

$$\begin{aligned}
(H\,p, f)_{L^2(B_R)^3} &= \int_{B_R} \left(\nabla_y \operatorname{div}_y \int_{\mathbb{S}^2} p(\hat{x}) e^{ik\hat{x}\cdot y}\, dS(\hat{x}) \right) \overline{f}(y)\, dy \\
&= \int_{B_R} \int_{\mathbb{S}^2} \nabla_y \operatorname{div}_y \left(p(\hat{x}) e^{ik\hat{x}\cdot y} \right) dS(\hat{x})\, \overline{f}(y)\, dy \\
&= \int_{B_R} \int_{\mathbb{S}^2} E' p(\hat{x})\, dS(\hat{x})\, \overline{f}(y)\, dy \\
&= \int_{\mathbb{S}^2} \overline{(-k^2) \int_{B_R} e^{-ik\hat{x}\cdot y} f(y)\, dy\, \hat{X}} p(\hat{x})\, dS(\hat{x}) \\
&= \overline{(H^* f, p)}_{L^2(\mathbb{S}^2)^3}.
\end{aligned}$$

Combined with the first part yields the stated far field pattern.

Consequently, for a direction $\hat{x} \in \mathbb{S}^2$, the far field pattern of $v^\infty(\hat{x})$ hence equals,

$$\begin{aligned}
v^\infty(\hat{x}) &= \left((k^2 + \nabla \operatorname{div}) \int_{B_R} \Phi_k(\cdot - y)(\rho_2 - I_3)(y) \left[v(y) + u^i(y) \right] dy \right)(\hat{x}) \\
&\quad + \left(\operatorname{curl} \int_{B_R} \Phi_k(\cdot - y)(I_3 - \rho_1)(y) \operatorname{curl} \left[v(y) + u^i(y) \right] dy \right)(\hat{x}) \\
&= \int_{B_R} [(k^2 + \nabla \operatorname{div}) e^{-ik\hat{x}\cdot y}](\rho_2 - I_3)(y) \left[v(y) + u^i(y) \right] dy \\
&\quad + \int_{B_R} [\operatorname{curl} e^{-ik\hat{x}\cdot y}](I_3 - \rho_1)(y) \operatorname{curl} \left[v(y) + u^i(y) \right] dy
\end{aligned} \tag{5.20}$$

$$= k^2(1-\hat{X})\int_{B_R} e^{-ik\hat{x}\cdot y}(\rho_2 - I_3)(y)\,\left[v(y)+u^i(y)\right]dy$$
$$+ ik\hat{x}\times\int_{B_R} e^{-ik\hat{x}\cdot y}(I_3-\rho_1)(y)\,\mathrm{curl}\left[v(y)+u^i(y)\right]dy,$$

where the matrix $\hat{X}$ is given in (5.19). As the latter integral expression is an analytic function in $\hat{x}$, the far field v^∞ is analytic as well.

CONSTRUCTING THE FORWARD OPERATOR Let us now introduce, for brevity, the integral operator

$$V\colon L^2(B_R)^3 \to H^2(B_R)^3, \qquad Vf = \int_{B_R} \Phi_k(\cdot - y)f(y)\,dy.$$

(See [CK13] for the mapping properties of V.) Now abbreviate

$$S_\rho := \left\{I_3 - [(k^2+\nabla\,\mathrm{div})V((\rho_2-I_3)\cdot) + \mathrm{curl}\,V((I_3-\rho_1)\,\mathrm{curl}\,\cdot)]\right\}^{-1}$$

as (bounded) linear inverse. Thereby, the scattered field restricted to B_R satisfies

$$v = S_\rho\left[(k^2+\nabla\,\mathrm{div})V((\rho_2-I_3)u^i) + \mathrm{curl}\,V((I_3-\rho_1)\,\mathrm{curl}\,u^i)\right].$$

Consequently, the total field $v+u^i$ equals $S_\rho u^i$. According to that we represent the far field $v^\infty = L(\rho,u^i)^\infty$, computed into direction $\hat{x}\in\mathbb{S}^2$ in (5.20), as

$$v^\infty(\hat{x}) = k^2(1-\hat{X})\int_{B_R}(\rho_2-I_3)(y)(S_\rho u^i)(y)\,e^{-ik\hat{x}\cdot y}\,dy$$
$$+ ik\int_{B_R}\hat{x}\times(I_3-\rho_1)(y)\,\mathrm{curl}(S_\rho u^i)(y)\,e^{-ik\hat{x}\cdot y}\,dy.$$

If we further introduce the integral operators

$$Z_{\rho_1}\colon L^2(B_R)^3\to L^2_t(\mathbb{S}^2), \quad f\mapsto ik\int_{B_R}\hat{x}\times f(y)\,e^{-ik\hat{x}\cdot y}\,dy, \quad (5.21)$$
$$Z_{\rho_2}\colon L^2(B_R)^3\to L^2_t(\mathbb{S}^2), \quad f\mapsto k^2(1-\hat{X})\int_{B_R} f(y)\,e^{-ik\hat{x}\cdot y}\,dy, \quad (5.22)$$

then there holds that

$$L(\rho,u^i)^\infty = Z_{\rho_1}\circ[(I_3-\rho_1)\,\mathrm{curl}\,S_\rho(u^i)] + Z_{\rho_2}\circ[(\rho_2-I_3)S_\rho(u^i)].$$

As $\rho\in\mathcal{P}\subset L^\infty(B_R,\mathrm{Sym}(3))^2$ and $S_\rho u^i\in L^2(B_R)^3$, the smoothing properties of Lemma 4.11 hold for Z_{ρ_1,ρ_2} as well. Consequently, the forward operator defines analogous to Section 4.5, that is, the contrast-to-far field mapping $F(\cdot)g\colon\mathcal{P}\to\mathcal{S}_q$, defined by

$$F(\rho)g = Z_{\rho_2}\circ[(\rho_2-I_3)S_\rho(v_g)] + Z_{\rho_1}\circ[(I_3-\rho_1)\,\mathrm{curl}\,S_\rho(v_g)] \quad (5.23)$$

for $g\in L^2_t(\mathbb{S}^2)$, $q\geqslant 1$, is an operator from $\mathcal{P}$ into the qth Schatten class $\mathcal{S}_q$.

PROPERTIES OF THE FORWARD OPERATOR Again, we refer to Remark 3.19, which still holds for this case. Further, the link between the solution operator L and the non-linear forward operator F enables us to show various properties of F via those of L, too. To this end, note first that the far field of the radiating extension of $L(\rho, \nu_g)$ depends boundedly and linearly on $L(\rho, \nu_g)$. Thus, since $L(\rho, \nu_g) = S_\rho(\nu_g) - \nu_g$, the derivative $\rho' \mapsto F'(\rho)[\rho']$ with respect to $\rho \in \mathcal{P}$ of F equals, by the product rule in Banach spaces, see [Zei86],

$$\begin{aligned} F'(\rho)[\rho']g = Z_{\rho_1} \circ [(I_3 - \rho_1) \operatorname{curl}(L'(\rho, \nu_g)[\rho']) + \rho_1' \operatorname{curl}(S_\rho(\nu_g))] \\ + Z_{\rho_2} \circ [(\rho_2 - I_3) L'(\rho, \nu_g)[\rho'] + \rho_2' S_\rho(\nu_g)] \end{aligned} \tag{5.24}$$

This allows to transfer the results of Theorem 5.8, 5.10, and 5.11 from L to F.

COROLLARY 5.12. *Choose $\rho, \rho + \rho' \in \mathcal{P}$ such that Assumption 5.4 holds and $q \geqslant 1$.*

(i) There is $C = C(\rho)$ such that

$$\begin{aligned} \|F(\rho + \rho') - F(\rho)\|_{\mathcal{S}_q} \leqslant C\Big\{ (\|\rho_1'\| + \|\rho_2'\|)^3 + (\|\rho_1'\| + \|\rho_2'\|) \\ + (2 + \|\varepsilon_r\|_\infty)(\|\rho_1'\| + \|\rho_2'\|)^2 + |k| \|\rho_2' [L(\rho + \rho', \nu_g) + \nu_g]\|_{H(\operatorname{div})} \Big\}. \end{aligned} \tag{5.25}$$

(ii) If one bounds $\hat{T}$ of (5.15) by $\|\rho'\|\|\theta\|$, scaled by a term depending on the $L^\infty(B_R, \operatorname{Sym}(3))$-norms of ρ' and θ, then the operator $F'(\rho)$ is locally Lipschitz continuous with respect to $L^\infty(B_R, \operatorname{Sym}(3))^2$: There is $C = C(\rho)$ such that

$$\begin{aligned} \|F'(\rho + \rho') - F'(\rho)\| \leqslant C\|\rho'\| + \Big[\|\theta_2(\nu_{\rho+\rho'} - \nu)\|_{H(\operatorname{div})} \\ + \|\rho_2' w_{\rho+\rho'}\|_{H(\operatorname{div})} \Big]. \end{aligned}$$

(iii) The far field operator $F(\rho)$ is differentiable in the sense that

$$\begin{aligned} \|F(\rho + \rho') - F(\rho) - F'(\rho)[\rho']\| \leqslant C\Big\{ \Big[(\|\rho_1'\| + \|\rho_2'\|)^3 \\ + (2 + \|\varepsilon_r\|_\infty)(\|\rho_1'\| + \|\rho_2'\|)^2 \Big] \|u^i\|_{H(\operatorname{curl})} \\ + |k| \|\rho_2' L'(\rho, u^i)[\rho']\|_{H(\operatorname{div})} \Big\} \end{aligned}$$

for $C = C(\rho)$. By abuse of notation denote the right-hand side by a scaling factor, depending on the $L^\infty(B_R, \operatorname{Sym}(3))$-norm of ρ', of $\|\rho'\|^2$. Hence, if $\{\rho_n'\}_{n \in \mathbb{N}} \subset L^\infty(B_R, \operatorname{Sym}(3))^2$, such that $\rho + \rho_n' \in \mathcal{P}$ satisfies Assumption 5.4 for all $n \in \mathbb{N}$ as well as $\|\rho_n'\| \to 0$ as $n \to \infty$, then $\|F(\rho + \rho_n') - F(\rho) - F'(\rho)[\rho_n']\|_{\mathcal{S}_q} / \|\rho_n'\| \to 0$.

Proof. The basic ingredient of the proof is the smoothing property of the far field mappings Z_{ρ_1} and Z_{ρ_2} definied in (5.21),(5.22), which are trace class operators from $L^2(B_R)^3$ into $L^2_t(\mathbb{S}^2)$. Choosing the incident field u^i as a Herglotz wave function v_g for some $g \in L^2_t(\mathbb{S}^2)$,

$$\begin{aligned}
&\|F(\rho+\rho') - F(\rho)\|_{\mathcal{S}_q} \\
&= \|g \mapsto Z_{\rho_2}\left[(\rho_2+\rho_2'-I_3)S_{\rho+\rho'}(v_g) - (\rho_2 - I_3)S_\rho(v_g)\right] \\
&\quad + Z_{\rho_1}\left[(I_3 - (\rho_1+\rho_1'))\operatorname{curl} S_{\rho+\rho'}(v_g) - (I_3-\rho_1)\operatorname{curl} S_\rho(v_g)\right]\|_{\mathcal{S}_q} \\
&\overset{(*)}{\leqslant} C\|g \mapsto \left[(I_3 - (\rho_1+\rho_1'))\operatorname{curl} S_{\rho+\rho'}(v_g) - (I_3-\rho_1)\operatorname{curl} S_\rho(v_g)\right. \\
&\quad \left. + (\rho_2+\rho_2'-I_3)S_{\rho+\rho'}(v_g) - (\rho_2-I_3)S_\rho(v_g)\right]\|_{\mathcal{L}(L^2_t(\mathbb{S}^2), L^2(B_R)^3)} \\
&\leqslant C \sup_{\|g\|_{L^2}=1}\Big[\|\rho_2' S_{\rho+\rho'}(v_g) - \rho_1' \operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R)^3} \\
&\quad + \|(I_3-\rho_1)\operatorname{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)] + (\rho_2 - I_3)[S_{\rho+\rho'}(v_g)-S_\rho(v_g)]\|_{L^2(B_R)^3}\Big],
\end{aligned}$$

where inequality $(*)$ follows from Lemma 3.18. Now we use triangle inequality to obtain the bound

$$\begin{aligned}
&\|\rho_2' S_{\rho+\rho'}(v_g) - \rho_1' \operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R)^3} \\
&\leqslant \|\rho_2'\|\|S_{\rho+\rho'}(v_g)\|_{L^2(B_R)^3} + \|\rho_1'\|\|\operatorname{curl} S_{\rho+\rho'}(v_g)\|_{L^2(B_R)^3} \\
&\leqslant (\|\rho_1'\| + \|\rho_2'\|)\|S_{\rho+\rho'}(v_g)\|_{H(\operatorname{curl})}
\end{aligned}$$

together with the estimate

$$\|S_{\rho+\rho'}(v_g)\|_{H(\operatorname{curl})} \leqslant C\|v_g\|_{H(\operatorname{curl})} \leqslant C\|g\|_{L^2_t(\mathbb{S}^2)} = C$$

for the total wave field, with a constant $C = C(\rho)$ independent of ρ'. The same technique yields

$$\begin{aligned}
&\|(I_3-\rho_1)\operatorname{curl}[S_{\rho+\rho'}(v_g)-S_\rho(v_g)] + (\rho_2 - I_3)[S_{\rho+\rho'}(v_g)-S_\rho(v_g)]\|_{L^2(B_R)^3} \\
&\leqslant \|I_3-\rho_1\|\|\operatorname{curl}[S_{\rho+\rho'}(v_g) - S_\rho(v_g)]\|_{L^2(B_R)^3} \\
&\quad + \|\rho_2 - I_3\|\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{L^2(B_R)^3} \\
&\leqslant C(\|I_3-\rho_1\|, \|\rho_2 - I_3\|)\sqrt{2}\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\operatorname{curl})}.
\end{aligned}$$

As $S_{\rho+\rho'}(v-g) - S_\rho(v_g) = L(\rho+\rho', v_g) - L(\rho, v_g)$, Theorem 5.8 further shows that

$$\begin{aligned}
\|S_{\rho+\rho'}(v_g) - S_\rho(v_g)\|_{H(\operatorname{curl})} \leqslant C\Big\{\Big[&|k|\|\rho_2'[L(\rho+\rho', v_g) + v_g]\|_{H(\operatorname{div})} \\
&+ (\|\rho_1'\| + \|\rho_2'\|)^3 + (2 + \|\varepsilon_r\|_\infty)(\|\rho_1'\| + \|\rho_2'\|)^2\Big]\|v_g\|_{H(\operatorname{curl})}\Big\},
\end{aligned}$$

such that by plugging the last estimates together we deduce the statement. The bounds in (ii) and (iii) are shown analogously, using Theorems 5.10 and 5.11 instead of Theorem 5.8. □

Remark 5.13. Note that, as mentioned in Remark 5.7, the properties of F, stated in the last corollary, just hold, in fact, if the H(div)-terms are bounded appropriately, that is, the H(curl)-norm of the incident field scaled by the $L^\infty(B_R, \mathrm{Sym}(3))$-norm of the contained perturbations.

5.5 NON-LINEAR TIKHONOV REGULARIZATION

In analogy to the non-magnetic case of Chapter 4, we want to stably approximate ρ_{exa} from perturbed measurements of its far field operator $F(\rho_{exa})$. As we seek approximations ρ by non-linear Tikhonov regularization, where the given data F^{δ}_{meas} is again perturbed with noise level $\delta > 0$ such that $\|F(\rho_{exa}) - F^{\delta}_{meas}\|_{\mathcal{S}_q} \leqslant \delta$, we consider to minimize the Tikhonov functional

$$\mathcal{J}_{\alpha,\delta}(\rho) := \tfrac{1}{2}\|F(\rho) - F^{\delta}_{meas}\|^2_{\mathcal{S}_q} + \alpha\,\mathcal{R}(\rho).$$

Note that from viewpoint of notation this equals the functional (4.31) of the non-magnetic case, but now the admissible set of parameters—over which the minimization is assumed—includes in $\mathcal{P} = \mathcal{P}_{\varepsilon_r} \times \mathcal{P}_{\mu_r}$. Further, the Tikhonov regularization results of Theorem 4.13 still holds for a convex functional $\mathcal{R}$.

To apply these regularization results we again rely on the techniques of Ressel [Res12]. Note that therefore we already fixed the topologies of the admissible parameter sets to be induced by the L^∞-norm. Thus, we gain closed, bounded subsets of Banach spaces, which we equip with the weak*-topology. Due to Alaoglu's theorem closed balls are weak*-compact, such that the sets are closed with respect to their topology. Thus, the tuple of sets $\mathcal{P}$ is itself a closed and bounded subset of a Banach space and weak*-closed. We further assume, that for all the beforehand mentioned $H(\mathrm{div})$-norms there exists a bound, depending on the $H(\mathrm{curl})$-norm of the incident field, such that Corollary 5.12 holds as mentioned in Remark 5.13 (as well as all related Theorems). Consequently, the forward operator F becomes for fixed $g \in L^2_t(\mathbb{S}^2)$ a mapping from $\mathcal{P}$ to $\mathcal{S}_q(L^2_t(\mathbb{S}^2), L^2_t(\mathbb{S}^2))$. One then shows analogously to the proof of Lemma 4.15, that F is sequentially closed from ($\mathcal{P}$,weak*) to $\mathcal{S}_q$ with its weak topology.

5.6 ADJOINT OF THE FORWARD OPERATOR'S LINEARIZATION

Again, as in Sections 3.8 and 4.7, we are going to calculate the adjoint operator of the linearization F'. Recall that $\rho = (\rho_1, \rho_2) = (\varepsilon_r^{-1}, \mu_r)$, then we fix $\rho \in \mathcal{P}$, consider $F'(\rho)\colon L^\infty(B_R, \mathrm{Sym}(3))^2 \to \mathcal{S}_q$ and aim to determine $F'(\rho)^*\colon \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))^2$ such that

$$(F'(\rho)[\theta], K)_{\mathcal{S}_2} \stackrel{!}{=} (\theta, F'(\rho)^*K)_{L^2} \quad \text{for all } \theta \in L^\infty(B_R, \mathrm{Sym}(3)),\ K \in \mathcal{S}_{q'}. \tag{5.26}$$

Here, q' is the conjugate Lebesgue index to q, such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Further $(\cdot,\cdot)_{L^2}$ denotes the usual scalar product in $L^2(B_R, \mathrm{Sym}(3))$,

$$(A, B)_{L^2} = \int_{B_R} A : B\,dx = \int_{B_R} \sum_{i,j=1}^{d} \overline{A}_{ij} B_{ij}\,dx.$$

extended to the anti-linear dual product between $L^\infty(B_R, \mathrm{Sym}(3))$ and $L^1(B_R, \mathrm{Sym}(3))$. Further, $(\cdot,\cdot)_{\mathcal{S}_2}$ is the scalar product in the Hilbert space of Hilbert-Schmidt operators,

$$(F, K)_{\mathcal{S}_2} = \sum_{j\in\mathbb{N}} s_j(F)\overline{s_j(K)} = \sum_{j=1}^{\infty} (F g_j,\, K g_j)_{L^2_t(\mathbb{S}^2)}$$

for an arbitrary orthonormal basis $(g_j)_{j\in\mathbb{N}}$ of $L^2_t(\mathbb{S}^2)$. Consequently, (5.26) becomes

$$\sum_{j=1}^{\infty} (F'(\rho)[\theta]g_j, K\, g_j)_{L^2_t(\mathbb{S}^2)} \overset{!}{=} (\theta, F'(\rho)^* K)_{L^2}$$

for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))^2$, $K \in \mathcal{S}_{q'}$. Thus, we consider at first a single L^2-scalar product for fixed $\rho \in \mathcal{P}$ and $g \in L^2_t(\mathbb{S}^2)$ we seek for $A\colon L^2_t(\mathbb{S}^2) \to L^1(B_R, \mathrm{Sym}(3))^2$, such that

$$(F'(\rho)[\theta]g, f)_{L^2_t(\mathbb{S}^2)} \overset{!}{=} (\theta, Af)_{L^2}$$

for all $\theta \in L^\infty(B_R, \mathrm{Sym}(3))^2$ and $f \in L^2_t(\mathbb{S}^2)$. Recall from (5.16) that $L'(\rho, v_g)[\theta] = v' \in H(\mathrm{curl}, B_R)$, a function whose radiating extension satisfies

$$v' = S_\rho\left[(k^2 + \nabla\,\mathrm{div})V(\theta_2[L(\rho, v_g) + v_g]) - \mathrm{curl}\,V(\theta_1\,\mathrm{curl}[L(\rho, v_g) + v_g])\right]$$

in $H(\mathrm{curl}, B_R)$, for the bounded linear inverse

$$S_\rho = [I_3 - ((k^2 + \nabla\,\mathrm{div})V((\rho_2 - I_3)\cdot) + \mathrm{curl}\,V((I_3 - \rho_1)\,\mathrm{curl}\,\cdot))]^{-1}.$$

Since F' involves the far field of L', see (5.24), we note that

$$\begin{aligned}
F'(\rho)[\theta]g &= Z_{\rho_2} \circ \left[(\rho_2 - I_3)v' + \theta_2 S_\rho(v_g)\right] \\
&\quad + Z_{\rho_1} \circ \left[(I_3 - \rho_1)\,\mathrm{curl}\, v' + \theta_1\,\mathrm{curl}\, S_\rho(v_g)\right] \\
&= Z_{\rho_2} \circ \Big[(\rho_2 - I_3)S_\rho\left[(k^2 + \nabla\,\mathrm{div})V(\theta_2 S_\rho(v_g))\right. \\
&\qquad\qquad \left. - \mathrm{curl}\,V(\theta_1\,\mathrm{curl}\,S_\rho(v_g))\right] + \theta_2 S_\rho(v_g)\Big] \\
&\quad + Z_{\rho_1} \circ \Big[(I_3 - \rho_1)\,\mathrm{curl}\,S_\rho\left[(k^2 + \nabla\,\mathrm{div})V(\theta_2 S_\rho(v_g))\right. \\
&\qquad\qquad \left. - \mathrm{curl}\,V(\theta_1\,\mathrm{curl}\,S_\rho(v_g))\right] + \theta_1\,\mathrm{curl}\,S_\rho(v_g)\Big].
\end{aligned}$$

Consequently, one computes that

$$\begin{aligned}
(F'(\rho)[\theta]g, f)_{L^2_t(\mathbb{S}^2)} = &\Big(\theta_1, \Big([I_3 - [(I_3 - \rho_1)\,\mathrm{curl}\,S_\rho \circ \mathrm{curl}\,V]^*] \circ Z^*_{\rho_1} f \\
&- [(\rho_2 - I_3)S_\rho \circ \mathrm{curl}\,V]^* \circ Z^*_{\rho_2} f\Big) \otimes \mathrm{curl}\,\overline{S_\rho(v_g)}\Big)_{L^2} \\
&+ \Big(\theta_2, \Big([(I_3 - \rho_1)\,\mathrm{curl}\,S_\rho \circ (k^2 + \nabla\,\mathrm{div})V]^* \circ Z^*_{\rho_1} f \\
&+ [[(\rho_2 - I_3)S_\rho \circ (k^2 + \nabla\,\mathrm{div})V]^* + I_3] \circ Z^*_{\rho_2} f\Big) \otimes \overline{S_\rho(v_g)}\Big)_{L^2},
\end{aligned}$$

where the last matrix-valued function is defined by $(a \otimes b)_{i,j} = a_i b_j$ for $1 \leqslant i, j \leqslant 3$.

Lemma 5.14. *For* $\rho \in \mathcal{P}$ *and* $g \in L^2_t(\mathbb{S}^2)$*, the adjoint of* $\theta \mapsto F'(\rho)[\theta](g)$ *with respect to the* L^2*-inner product maps* $L^2_t(\mathbb{S}^2)$ *into* $L^1(B_R, \mathrm{Sym}(3))$ *and is represented by*

$$\begin{aligned} g \mapsto \Big(\big[I_3 - [(I_3 - \rho_1) \operatorname{curl} S_\rho \circ \operatorname{curl} V]^* \big] \circ Z^*_{\rho_1} g \\ - [(\rho_2 - I_3) S_\rho \circ \operatorname{curl} V]^* \circ Z^*_{\rho_2} g \Big) \otimes \operatorname{curl} \overline{S_\rho(\nu_g)} \\ + \Big([(I_3 - \rho_1) \operatorname{curl} S_\rho \circ (k^2 + \nabla \operatorname{div}) V]^* \circ Z^*_{\rho_1} g \\ + \big[[(\rho_2 - I_3) S_\rho \circ (k^2 + \nabla \operatorname{div}) V]^* + I_3 \big] \circ Z^*_{\rho_2} g \Big) \otimes \overline{S_\rho(\nu_g)}. \end{aligned}$$

For all orthonormal bases $\{g_j\}_{j\in\mathbb{N}}$ *of* $L^2_t(\mathbb{S}^2)$ *and all* $K \in \mathcal{S}_{q'}$*, the bounded operator* $F'(\rho)^* : \mathcal{S}_{q'} \to L^1(B_R, \mathrm{Sym}(3))$ *is represented by*

$$\begin{aligned} F'(\rho)^*(K) = \sum_{j=1}^{\infty} \Big(\big[I_3 - [(I_3 - \rho_1) \operatorname{curl} S_\rho \circ \operatorname{curl} V]^* \big] \circ Z^*_{\rho_1} \big(K g_j\big) \\ - [(\rho_2 - I_3) S_\rho \circ \operatorname{curl} V]^* \circ Z^*_{\rho_2} \big(K g_j\big) \Big) \otimes \operatorname{curl} \overline{S_\rho[\nu_{g_j}]} \\ + \sum_{j=1}^{\infty} \Big([(I_3 - \rho_1) \operatorname{curl} S_\rho \circ (k^2 + \nabla \operatorname{div}) V]^* \circ Z^*_{\rho_1} \big(K g_j\big) \\ + \big[[(\rho_2 - I_3) S_\rho \circ (k^2 + \nabla \operatorname{div}) V]^* + I_3 \big] \circ Z^*_{\rho_2} \big(K g_j\big) \Big) \otimes \overline{S_\rho[\nu_{g_j}]}. \end{aligned} \tag{5.27}$$

6 INVERSE ACOUSTIC BACKSCATTERING

In this chapter we want to recover informations of an object by sending in acoustic plane waves of a certain direction and measuring the far field of the corresponding scattered field in the opposite direction, see Figure 6.1. This concept is known as inverse backscattering problem and differs from the settings of the previous chapters, where measurements from different angles, perhaps distinct from the incident ones, were considered.

In contrast to the scattering problems treated before, the setting of backscattering is remarkably underrepresented in the literature. An in-depth investigation of the mapping from a potential $q(x)$ on $\mathbb{R}^3$ to the backscattering amplitude associated with the Hamiltonian $-\Delta + q(x)$ is given by Eskin and Ralston [ER89], as well as an addendum for two dimensions, see [ER91].

In 1990, Stefanov [Ste90] derived partial results on uniqueness of the backscattering problem in three dimensions under certain conditions on potentials, considering both wave equation and Schrödinger equation. Some time later Rakesh and Uhlmann [RU14] stated some further uniqueness results for the three dimensional backscattering problem, yet also limited to potentials with certain conditions (see also [RU15]). Consequently, global uniqueness of the backscattering problem remains unsolved.

Additionally, uniqueness of reconstruction from the inverse problem is not necessarily guaranteed, as we only have, e.g., for the two dimensional problem, scalar valued measurements. Therefore, it is not clear, how to uniquely reconstruct a multi dimensional object from lower dimensional data.

However, we provide some regularization techniques in analogy to the concepts seen before, although uniqueness of solution is not guaranteed. To begin with we introduce the acoustic scattering problem in Section 6.1 to provide an appropriate setting. We emphasize, that in contrast to previous techniques, we rely directly on a volume potential approach to gain from regularity results of [LKK13]. In line with the course of action established in the chapters before, we thereby show properties of the solutions of the problem, see Section 6.2, and define a forward operator corresponding to our model in Section 6.3. Finally, Section 6.4 contains results on regularized solutions of the inverse problem.

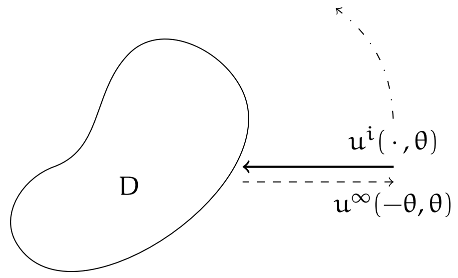

Figure 6.1: Illumination of a medium D from direction θ (solid arrow) and measurement of the corresponding far field pattern u^∞ in direction $-\theta$ (dashed arrow). Directions θ of incident field u^i vary over $\mathbb{S}^{d-1}$ (dashdotted arrow).

6.1 THE SCATTERING PROBLEM

For our scattering problem in $\mathbb{R}^d$, $d = 2, 3$, we illuminate an inhomogeneous medium D in direction θ by an incident plane wave $u^i(\cdot, \theta)$, which solves the homogeneous Helmholtz equation. Assuming $\overline{D}$ to be the support of a contrast function $q \in L^\infty(\overline{D})$ with $\mathrm{Im}(q) \geqslant 0$ (such that uniqueness holds) the therefore present total field u then solves

$$\Delta u + k^2(1+q)u = 0 \qquad \text{in } \mathbb{R}^d,\ d = 2, 3. \tag{6.1}$$

The herein covered scattered field $u^s = u - u^i$ satisfies Sommerfeld's radiation condition,

$$\lim_{r\to\infty} r^{(d-1)/2} \left(\frac{\partial u^s}{\partial r}(r\hat{x}, \theta) - iku^s(r\hat{x}, \theta) \right) = 0 \tag{6.2}$$

uniformly in all directions $\hat{x} \in \mathbb{S}^{d-1}$. We call such solutions radiating in the sequel. The radiation condition implies an asymptotic behavior of the scattered field at infinity:

$$u^s(r\hat{x}, \theta) = \gamma_d \frac{e^{ikr}}{r^{(d-1)/2}} u^\infty(\hat{x}, \theta) + \mathcal{O}(r^{-1}) \qquad \text{as } r \to \infty,$$

for $\gamma_2 = \exp(i\pi/4)/\sqrt{8\pi k}$, $\gamma_3 = 1/(4\pi)$. Note that the so-called far field pattern $u^\infty \colon \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{C}$ is an analytic function in both variables and is uniquely determined by the scattered field.

SOLUTION THEORY VIA RIESZ-FREDHOLM Integrating the Helmholtz equation (6.1) against test functions $\psi \in C_0^\infty(\mathbb{R}^d)$ over the ball $B_{2R} \subset \mathbb{R}^d$, which contains $\overline{D} \subseteq B_R$, and using Green's theorem yields

$$\int_{B_{2R}} \left[\nabla u \cdot \nabla \bar{\psi} - k^2(1+q)u\bar{\psi} \right] \,dx = \int_{\partial B_{2R}} \frac{\partial u}{\partial \nu} \bar{\psi} \,dS. \tag{6.3}$$

Since smooth functions are dense in $H^1(B_{2R})$, the latter equation holds for all $\psi \in H^1(B_{2R})$. As the trace operator $\gamma(u) = u|_{\partial B_{2R}}$

has a unique continuation to a linear operator from $H^1(B_{2R})$ into $H^{1/2}(\partial B_{2R})$, see [McL00, Lemma 3.35], $u|_{\partial B_{2R}}$ belongs to $H^{1/2}(\partial B_{2R})$. Further, ∇u belongs to $H^1(B_{2R})$ and in particular to $H(\mathrm{div}, B_{2R})$ since u solves (6.1) in $L^2(\mathbb{R}^d)$, such that the trace theorem in $H(\mathrm{div}, B_{2R})$ shows that $\partial u/\partial \nu = \nu \cdot \nabla u$ belongs to $H^{-1/2}(\partial B_{2R})$, see [Mon03, Theorem 3.24]. Thus, the boundary integral in (6.3) is well-defined as a duality pairing between $H^{\pm 1/2}(\partial B_{2R})$.

Denote by $\Lambda_{2R} \colon H^{1/2}(\partial B_{2R}) \to H^{-1/2}(\partial B_{2R})$ the exterior Dirichlet-to-Neumann operator, see [Néd01], which maps Dirichlet boundary values ϕ on ∂B_{2R} to the normal derivative $\partial v/\partial \nu$ of the unique radiating solution v to the exterior Dirichlet scattering boundary problem. More precisely, $v \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{B_{2R}})$ is the unique radiating solution to $\Delta v + k^2 v = 0$ in $\mathbb{R}^d \setminus \overline{B}_{2R}$, and can be written down explicitly in series form using Hankel functions. As u is a radiating solution to the Helmholtz equation, $\Lambda_{2R}(\gamma(u))$ equals $\partial u/\partial \nu$, such that (6.3) becomes

$$\int_{B_{2R}} \left[\nabla u \cdot \nabla \bar{\psi} - k^2(1+q)u\bar{\psi}\right] \mathrm{d}x - \int_{\partial B_{2R}} \Lambda_{2R}(\gamma(u))\gamma(\bar{\psi}) \,\mathrm{d}S = \Psi(\psi) \tag{6.4}$$

for all $\psi \in H^1(B_{2R})$, with right-hand side

$$\Psi(\psi) = \int_{\partial B_R} \left[\frac{\partial u^i}{\partial \nu} - \Lambda_{2R}(\gamma(u^i))\right] \gamma(\bar{\psi}) \,\mathrm{d}S.$$

(We omit the trace operator γ from now on if a restriction to the boundary is obvious.)

To prove the existence of a solution of (6.4), we follow the ideas of Kirsch [Kir93], also presented in the book of Colton and Kress [CK13, Proof of Theorem 5.7], and rely on an additional Dirichlet-to-Neumann map $\Lambda_{\Delta,2R}$ that maps Dirichlet data on ∂B_{2R} to Neumann data of the solution to an exterior Dirichlet boundary problem for the Laplace equation. Note that $-\Lambda_{\Delta,R}$ is coercive, that is

$$-\int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\psi)\bar{\psi} \,\mathrm{d}S \geqslant c\|\psi\|^2_{H^{1/2}(\partial B_{2R})}, \quad \text{for all } \psi \in H^{1/2}(\partial B_{2R}).$$

The sesquilinear forms

$$s(\varphi, \psi) := \int_{B_{2R}} \left[\nabla \varphi \cdot \nabla \bar{\psi} + \varphi \bar{\psi}\right] \mathrm{d}x - \int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\varphi)\bar{\psi} \,\mathrm{d}S,$$
$$s_1(\varphi, \psi) := \int_{B_{2R}} [k^2(1+q)+1]\varphi\bar{\psi} \,\mathrm{d}x + \int_{\partial B_{2R}} (\Lambda_{2R} - \Lambda_{\Delta,2R})(\varphi)\bar{\psi} \,\mathrm{d}S,$$

allow to reformulate the variational form (6.4) for all $\psi \in H^1(B_{2R})$ as

$$s(u, \psi) - s_1(u, \psi) = \int_{\partial B_R} \left[\frac{\partial u^i}{\partial \nu} - \Lambda_{2R}(\gamma(u^i))\right] \bar{\psi} \,\mathrm{d}S. \tag{6.5}$$

Both Λ_{2R} and $\Lambda_{\Delta,2R}$ are bounded from $H^{1/2}(\partial B_{2R})$ into $H^{-1/2}(\partial B_{2R})$, such that s and s_1 are bounded sesquilinear forms. The coercivity of $\Lambda_{\Delta,2R}$ implies that s is coercive,

$$s(\varphi,\varphi) = \|\varphi\|^2_{H^1(B_{2R})} - \int_{\partial B_{2R}} \Lambda_{\Delta,2R}(\varphi)\bar{\varphi}\, dS \geqslant C\|\varphi\|^2_{H^1(B_{2R})},$$

for all $\varphi \in H^1(B_{2R})$. Moreover, compactness of $\Lambda_{2R} - \Lambda_{\Delta,2R}$, see, e.g., [CK13, p. 131] and the compact embedding of $H^1(B_{2R})$ in $L^2(B_{2R})$ imply that s_1 is a compact sesquilinear form.

By the representation theorem of Riesz there exists a bounded operator $S\colon H^1(B_{2R}) \to H^1(B_{2R})$ and a compact operator S_1 such that $s(\varphi,\psi) = (S\,\varphi,\psi)_{H^1(B_{2R})}$ and $s_1(\varphi,\psi) = (S_1\,\varphi,\psi)_{H^1(B_{2R})}$ for all $\varphi, \psi \in H^1(B_{2R})$. By Lax-Milgram's lemma, S is further boundedly invertible. Further introducing $r \in H^1(B_{2R})$ such that $\int_{\partial B_R}[\frac{\partial u^i}{\partial \nu} - \Lambda_{2R}(\gamma(u^i))]\gamma(\bar{\psi})\, dS = (r,\psi)_{H^1(B_{2R})}$ for all $\psi \in H^1(B_{2R})$, the variational formulation (6.5) can be equivalently rewritten as $S\,u - S_1\,u = r$ in $H^1(B_{2R})$. Multiplying with the inverse S^{-1} yields $u - K\,u = S^{-1}\,r$ with a compact operator $K := S^{-1}\,S_1$. Thus, Riesz-Fredholm theory implies that uniqueness of solution to the latter equation implies existence of solution for all right-hand sides.

Lemma 6.1. *If the only solution to the homogeneous problem corresponding to (6.4) is the trivial solution, then that variational problem possesses a unique solution for all bounded anti-linear functionals $\Psi \in H^1(B_{2R})^*$ and there is C_q independent of Ψ such that*

$$\|u\|_{H^1(B_{2R})} \leqslant C_q\|\Psi\|_{H^1(B_{2R})^*}. \tag{6.6}$$

Corollary 6.2 (See [Kir11, Theorem 6.5]). *For $q \in L^\infty(B_R)$ such that $\mathrm{Im}(q) \geqslant 0$, the scattering problem (6.1) and (6.2) has at most one solution, that is, if u is a solution corresponding to an incident wave $u^i = 0$ then $u = 0$.*

Proof. Using Sommerfeld's radiation condition and Green's first theorem, one can exploit the non-negativity of the contrast's imaginary part. Owing to that, one shows that the integral of the solution u to the Helmholtz equation vanishes, such that u itself vanishes outside B_R. Finally u equals zero in the whole space due to a unique continuation principle, see, e.g., [Kir11, Theorem 6.4]. □

INTEGRAL EQUATIONS We now use an integral equation approach to profit from regularity results presented in the work of Lechleiter, Kazimierski, and Karamehmedović [LKK13], already discussed in Section 2.3. Therefore, we reformulate the scattering problem (6.1) and (6.2) into the equivalent problem of solving the integral equation

$$u(x) = u^i + k^2\int_{\mathbb{R}^d} \Phi(x-y)q(y)u(y)\, dy, \quad x \in \mathbb{R}^d,\ d = 2,3, \tag{6.7}$$

where Φ denotes the radiating fundamental solution of the Helmholtz equation, given by

$$\Phi_k(x) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x|), & \text{if } d = 2, \\ \frac{\exp(ik|x|)}{4\pi|x|}, & \text{if } d = 3, \end{cases} \qquad x \neq 0.$$

Note that (6.7) is known as Lippmann-Schwinger equation (for its derivation see, e.g., [CK13, Section 8.2]). Due to [LKK13, Lemma 1] the radiating volume potential

$$(Vf)(x) = \int_{B_R} \Phi(x-y) f(y) \, dy, \quad x \in \mathbb{R}^d, \ d = 2, 3,$$

extends to a bounded operator from $L^s(B_R)$ into $W^{2,s}(B_R)$ for all $s \in (1, \infty)$.

ASSUMPTION 6.3. From now on we fix the contrast q to be a function in $L^p(B_R)$ for $p > d/(d-1) \geqslant d/2$. Additionally, we choose

$$t > \max\left\{\frac{p}{p-1}, \frac{2d}{d+2}\right\}$$

large enough, such that

$$\frac{tp}{t+p} \geqslant \frac{d}{2}.$$

Note that the assumptions on p and t ensure that, first, the Lippmann-Schwinger integral equation is well-defined in $L^t(B_R)$ and, second, that a unique continuation principle holds to its solutions for $q \in L^p(B_R)$. To abbreviate Assumption 6.3, we define a set of contrasts, which also satisfy appropriate properties emphasized beforehand, as

$$\mathcal{Q} := \left\{q \in L^p(B_R), \, p > d/2, \, \mathrm{Im}(q) \geqslant 0\right\} \subset L^p_{\mathrm{Im} \geqslant 0}(B_R). \tag{6.8}$$

Therefore, compactness of $v \mapsto V(qv)$ on $L^t(B_R)$ as well as uniqueness of solution holds:

LEMMA 6.4 (See [LKK13, Theorem 6]). *Let $q \in \mathcal{Q}$ and choose $t > 1$ according to Assumption 6.3. Then the Lippmann-Schwinger equation*

$$v - k^2 V(qv) = f,$$

has a unique solution $v \in L^t(B_R)$ such that $\|v\|_{L^t(B_R)} \leqslant C\|f\|_{L^t(B_R)}$.

If $f = k^2 V(qu^i)$ for some incident field $u^i \in L^t(B_R)$, then (6.7) defines a radiating solution $u^s \in W^{2,tp/(t+p)}_{\mathrm{loc}}(\mathbb{R}^d)$ to the Helmholtz equation $\Delta u^s + k^2(1+q)u^s = -k^2 q u^i$ in $L^t_{\mathrm{loc}}(\mathbb{R}^d)$.

We now collect a couple of techniques, which will be used frequently in the ongoing analysis. Thus, assume to have a scattered field u^s satisfying (6.7). By Lemma 6.4, the radiating solution is in $W^{2,r}(B_R)$ for $r := tp/(t+p)$, such that

$$\|u^s\|_{W^{2,r}(B_R)} = k^2\|V(q(u^s + u^i))\|_{W^{2,r}(B_R)}.$$

As mentioned above, the volume potential V is a bounded operator from $L^r(B_R)$ into $W^{2,r}(B_R)$ and, hence,

$$k^2\|V(q(u^s + u^i))\|_{W^{2,r}(B_R)} \leqslant C(k)\,\|q(u^s + u^i)\|_{L^r(B_R)}.$$

Applying the generalized Hölder inequality for $r = tp/(t+p)$, see, e.g., [LKK13, Appendix (A.1)], and triangle inequality, the right-hand side is further bounded by

$$\|q(u^s + u^i)\|_{L^r(B_R)} \leqslant \|q\|_{L^p(B_R)}\left(\|u^s\|_{L^t(B_R)} + \|u^i\|_{L^t(B_R)}\right).$$

Now exploit that the scattered field is bounded due to Lemma 6.4, that is

$$\|u^s\|_{L^t(B_R)} \leqslant C(k)\,\|V(qu^i)\|_{L^t(B_R)}.$$

Therefore, compactness results for V(see, e.g., [LKK13, Proposition 2]), basing, in fact, on Sobolev embedding theorems, imply that

$$\|V(qu^i)\|_{L^t(B_R)} \leqslant C\|V(qu^i)\|_{W^{2,r}(B_R)} \leqslant C(k,q)\,\|u^i\|_{L^t(B_R)}, \qquad (6.9)$$

with $C(k,q) = C(k, \|q\|_{L^p(B_R)})$ depending continuously on $\|q\|_{L^p(B_R)}$. Plugging all estimates together finally yields, that

$$\|u^s\|_{W^{2,r}(B_R)} \leqslant C(k,q)\,\|q\|_{L^p(B_R)}\|u^i\|_{L^t(B_R)}.$$

6.2 PROPERTIES OF THE TOTAL FIELD

As in the case of anisotropic acoustic and electromagnetic scattering, we collect necessary properties of the solutions. Therefore, in the following we indicate the dependency of the total field on the contrast q by writing $u_q = u(\cdot, \theta)$, corresponding to an incident plane wave $u^i(\cdot, \theta)$ from direction θ. Note that we still abbreviate $r = tp/(t+p)$. At first show Lipschitz continuity of the total field.

THEOREM 6.5. *Let $q + q' \in \mathcal{Q}$ be a perturbed contrast, such that Assumption 6.3 holds. Then*

$$\|u_{q+q'} - u_q\|_{W^{2,tp/(t+p)}(B_R)} \leqslant C\,\|q'\|_{L^p(B_R)}\|u^i\|_{L^t(B_R)},$$

where the constant C depends on k and the L^p-norms of q and $q + q'$ but not on q' itself.

Proof. Regarding that u_q solves (6.1), $u_{q+q'}$ for the perturbed contrast $q + q'$ consequently solves

$$\Delta u_{q+q'} + k^2(1 + q + q')u_{q+q'} = 0.$$

Because of that, the difference of the total fields is a solution to

$$\Delta(u_{q+q'} - u_q) + k^2(1+q)(u_{q+q'} - u_q) = -k^2 q' u_{q+q'}.$$

As mentioned beforehand, this is equivalent to solving a Lippmann-Schwinger equation in the form of (6.7), such that Lemma 6.4 states for $r := tp/(t+p)$ that

$$\|u_{q+q'} - u_q\|_{W^{2,r}(B_R)} = k^2 \|V(q(u^s_{q+q'} - u^s_q) + q' u_{q+q'})\|_{W^{2,r}(B_R)}.$$

(Note that $\|u_{q+q'} - u_q\| = \|u^s_{q+q'} - u^s_q\|$ for fixed incident fields.) Exploiting boundedness of V and triangle inequality, we have that

$$\begin{aligned}\|V(q(u^s_{q+q'} - u^s_q) + q' u_{q+q'}))\|_{W^{2,r}(B_R)} \leqslant C_k \, \|q(u^s_{q+q'} - u^s_q)\|_{L^r(B_R)} \\ + C_k \, \|q' u_{q+q'}\|_{L^r(B_R)}.\end{aligned}$$

Now apply generalized Hölder inequality to each term. Then the second norm is bounded by $\|q'\|_{L^p(B_R)} \|u_{q+q'}\|_{L^t(B_R)}$ times some constant, which is fine as we incorporate that later just as it is. However, for the first term we obtain the following inequality:

$$\begin{aligned}\|q(u^s_{q+q'} - u^s_q)\|_{L^r(B_R)} &\leqslant C \, \|q\|_{L^p(B_R)} \|u^s_{q+q'} - u^s_q\|_{L^t(B_R)} \\ &\leqslant C \, \|q\|_{L^p(B_R)} \|V(q' u_{q+q'})\|_{L^t(B_R)},\end{aligned}$$

where we made use of the boundedness of solution by its right-hand side, see Lemma 6.4. Now, an analogous sleight of hand as in (6.9), that is, relying on compactness results of V, and subsequently exploiting boundedness of the operator V as well as generalized Hölder inequality, consequently yields in order of appearance:

$$\begin{aligned}\|q\|_{L^p(B_R)} \|V(q' u_{q+q'})\|_{L^t(B_R)} &\leqslant \|q\|_{L^p(B_R)} C_{q+q'} \|V(q' u_{q+q'})\|_{W^{2,r}(B_R)} \\ &\leqslant C_{q,q+q'} \, \|q' u_{q+q'}\|_{L^r(B_R)} \\ &\leqslant C_{q,q+q'} \, \|q'\|_{L^p(B_R)} \|u_{q+q'}\|_{L^t(B_R)},\end{aligned}$$

where C_q indicates a constant depending only on the L^p-norm of $q \in \mathcal{Q}$. At long last, we have shown that

$$\|u_{q+q'} - u_q\|_{W^{2,r}(B_R)} \leqslant C \, \|q'\|_{L^p(B_R)} \|u_{q+q'}\|_{L^t(B_R)}$$

for a constant $C = C(k, \|q\|_{L^p(B_R)}, \|q+q'\|_{L^p(B_R)})$, which does not depend on q' itself.

In a final step one shows that the L^t-norm of $u_{q+q'}$ is bounded by the L^t-norm of the incident field. Separating the total field into its scattered and incident parts, this can be achieved using the same techniques as before. In detail, that is, triangle inequality, boundedness of solution by Lemma 6.4, compactness of V as in (6.9), boundedness of V, and generalized Hölder inequality. Putting all estimates together finally yields the claimed statement. □

DERIVATIVE OF TOTAL FIELDS For convenience we now define the sequilinear form

$$a_q(\varphi,\psi) := \int_{B_{2R}} \left[\nabla\varphi\cdot\nabla\bar{\psi} - k^2(1+q)\varphi\bar{\psi}\right]\,dx \\ - \int_{\partial B_{2R}} \Lambda_{2R}(\gamma(\varphi))\gamma(\bar{\psi})\,dS$$

for all $\varphi,\psi \in H^1(B_{2R})$. Note that this equals the difference $s(\varphi,\psi) - s_1(\varphi,\psi)$ of sesquilinear forms in (6.5), such that (6.4) can be written as $a_q(u_q,\psi) = \Psi(\psi)$. Due to that, we define a function u' that is for all $\psi \in H^1(B_{2R})$ a solution to

$$a_q(u',\psi) = k^2\int_{B_R} h u_q \bar{\psi}\,dx. \tag{6.10}$$

We foreclose that $u' = u'_q[h]$ is somewhat the derivative of u_q with respect to q in direction $h \in L^p(B_R)$ as we show in Theorem 6.7.

THEOREM 6.6. *For $h \in L^p(B_R)$ and a perturbed contrast $q+q' \in \mathcal{Q}$, such that Assumption6.3 holds, the map $q \mapsto u'_q$ is Lipschitz continuous:*

$$\|u'_{q+q'}[h] - u'_q[h]\|_{W^{2,tp/(t+p)}(B_R)} \leqslant C\,\|q'\|_{L^p(B_R)}\|h\|_{L^p(B_R)}\|u^i\|_{L^t(B_R)},$$

where C depends on k and the L^p-norms of q and $q+q'$ but not on q' itself.

Proof. Relying on (6.10), the functions $u'_{q+q'}[h]$ and $u'_q[h]$ satisfy for $h \in L^p(B_R)$ and all test functions $\psi \in H^1(B_R)$, the formulations

$$a_{q+q'}(u'_{q+q'}[h],\psi) = k^2\int_{B_R} h u_{q+q'}\bar{\psi}\,dx$$

and

$$a_q(u'_q[h],\psi) = k^2\int_{B_R} h u_q\bar{\psi}\,dx.$$

Consequently, $w := u'_{q+q'}[h] - u'_q[h]$ satisfies

$$a_q(w,\psi) = k^2\int_D \left(h(u_{q+q'} - u_q) - q'u'_{q+q'}[h]\right)\bar{\psi}\,dx.$$

For the proof we rely on the techniques used beforehand in the proof of Theorem 6.5, which is somewhat technical, but does not contain new ideas. That's why we just present the overall picture but not every detailed step. To start with, for $r := tp/(t+p)$, Lemma 6.4 implies that

$$\|w\|_{W^{2,r}(B_R)} = k^2\|V(qw + h(u_{q+q'} - u_q) - q'u'_{q+q'}[h])\|_{W^{2,r}(B_R)}.$$

Again, benefit from the boundedness of V and triangle inequality, such that

$$\|w\|_{W^{2,r}(B_R)} \leqslant C_k\left(\|qw\|_{L^r(B_R)} + \|h(u_{q+q'} - u_q)\|_{L^r(B_R)} \\ + \|q'u'_{q+q'}[h])\|_{L^r(B_R)}\right).$$

Initially, note that the first term $\|qw\|_{L^r(B_R)}$—due to generalized Hölder inequality, Lemma 6.4, compactness as well as boundedness of V—is bounded by

$$C(k,q)\ \left(\|h(u_{q+q'}-u_q)\|_{L^r(B_R)}+\|q'u'_{q+q'}[h])\|_{L^r(B_R)}\right),$$

where the constant C only depends on the L^p-norms of q and $q+q'$ but not on q' itself. We emphasize that this equals the other terms from the last inequality, such that it is sufficient to estimate these terms appropriately. To make a long story short, utilizing the prominently used techniques one derives the claimed estimate. □

THEOREM 6.7. *For* $q' \in L^p(B_{2R})$ *such that* $q+q' \in \mathcal{Q}$*, it holds that*

$$\|u_{q+q'}-u_q-u'_q[q']\|_{W^{2,tp/(t+p)}(B_R)} \leqslant C\,\|q'\|^2_{L^p(B_R)}\|u^i\|_{L^t(B_R)}.$$

for a constant C *depending on* k *and* L^p*-norms of* q *and* $q+q'$ *but not on* q' *itself.*

Proof. Be aware that

$$\begin{aligned} a_{q+q'}(u_{q+q'},\psi) &= \int_{\partial B_R}\left[\frac{\partial u^i}{\partial \nu}-\Lambda_{2R}(\gamma(u^i))\right]\gamma(\bar{\psi})\,dS,\\ a_q(u_q,\psi) &= \int_{\partial B_R}\left[\frac{\partial u^i}{\partial \nu}-\Lambda_{2R}(\gamma(u^i))\right]\gamma(\bar{\psi})\,dS,\\ a_q(u'_q[q'],\psi) &= k^2\int_{B_R} q'u_q\bar{\psi}\,dx, \end{aligned}$$

holds for all test functions $\psi \in H^1(B_R)$. Thus, $w := u_{q+q'}-u_q-u'_q[q']$ is a solution to

$$a_q(w,\psi)=k^2\int_{B_R} q'(u_{q+q'}-u_q)\overline{\psi}\,dx.$$

In a nutshell, exploiting the overall set of techniques, one can show that for $C=C(k,p)$ it holds that

$$\|w\|_{W^{2,tp/(t+p)}(B_R)} \leqslant C\left(\|w\|_{L^t(B_R)}+\|q'\|_{L^p(B_R)}\|u_{q+q'}-u_q\|_{L^t(B_R)}\right).$$

Note that, as seen in the proof of Theorem 6.6, the first term $\|w\|_{L^t(B_R)}$ is bounded by the second addend, such that only $\|q'\|_{L^p(B_R)}\|u_{q+q'}-u_q\|_{L^t(B_R)}$ remains. During the proof of Theorem 6.5 we have seen, that the L^t-term is bounded by a constant times $\|q'\|_{L^p(B_R)}\|u^i\|_{L^t(B_R)}$. Consequently, the statement holds. □

Remark 6.8. Note that the solution theory exposed in Section 6.1 also holds for the scattered field $u^s_q=u_q-u^i$. Therefore, the statement of Theorem 6.5 is also true for u^s_q, because $\|u_{q+q'}-u_q\|=\|u^s_{q+q'}-u^s_q\|$ assuming a fixed incident field u^i. Thus, defining $u'_q[h]$ to be the derivative of u^s_q which still solves equation (6.10), a corresponding version of Theorem 6.7 can be shown analogously for the scattered field. Because of that, the properties of the derivative of the total field shown in this section hold true in the same manner for the scattered field.

6.3 THE FORWARD OPERATOR

As in Chapters 3 and 4 concerning the anisotropic acoustic and electromagnetic scattering problem, we aim to gain information of the contrast q from measurements of the far field pattern. Consequently, therein all regularization results solving arising inverse problems rely on the far field operator. However, the actual setting of backscattering limits the angle of measurement to the direction $-\theta$ for the corresponding scattered field $u^s(\cdot,\theta)$, arising from incident plane waves $u^i(\cdot,\theta)$ from direction θ. Obviously the superposition of far fields that forms the far field operator $g \mapsto \int_{\mathbb{S}^{d-1}} u^\infty(\cdot,\theta)g(\theta)\,dS(\theta)$, then simplifies to the far field $u^\infty(-\theta,\theta)$.

To transfer the properties of the total field, shown in Theorems 6.5 to 6.7, to the far field pattern, we define the total-to-far field mapping from $L^r(B_R)$ into $L^2(\mathbb{S}^{d-1})$:

$$Z_\theta : f \mapsto \gamma_d \int_{B_R} e^{-ik\theta\cdot y} f(y)\,dy. \tag{6.11}$$

Due to [CK13, eq. (8.28)], the function $\theta \mapsto u^\infty(-\theta,\theta)$ is then represented by

$$u^\infty(-\theta,\theta) = Z_{-\theta} \circ [qu(\cdot,\theta)] = \gamma_d \int_{B_R} e^{ik\theta\cdot y} q(y)u(y,\theta)\,dy. \tag{6.12}$$

The composition is well-defined due to the smoothing properties of the total-to-far field mapping Z_θ, which is a trace class operator, see Lemma 3.18.

To make notation clear and provide a consistent concept, we define a forward operator modeling our problem, by

$$F\colon \mathcal{Q} \to L^2(\mathbb{S}^{d-1}), \quad F(q)\theta = u_q^\infty(-\theta,\theta) = Z_{-\theta} \circ [qu_q(\cdot,\theta)], \quad \theta \in \mathbb{S}^{d-1}. \tag{6.13}$$

Regarding (6.12), we thus find the derivative of $h \mapsto F'(q)[h]$ in $\mathcal{L}(L^p(B_R), L^2(\mathbb{S}^{d-1}))$ with respect to q of $F(q)$ by the product rule, as

$$F'(q)[h]\theta = u_q^{\infty\,\prime}(-\theta,\theta)[h] = Z_{-\theta} \circ \left[qu_q'(\cdot,\theta)[h] + hu_q(\cdot,\theta)\right]. \tag{6.14}$$

PROPERTIES OF THE FORWARD OPERATOR Again, since the forward operator F is linked to the total-to-far field map, we can show various properties of F via Theorems 6.5, 6.6, and 6.7.

COROLLARY 6.9. *Let Assumption 6.3 hold and assume to have an incident plane wave such that* $\|u^i\|_{L^t(B_{2R})} \leqslant C$ *for* $C > 0$.

(i) *Let* q' *be a perturbation such that* $q + q' \in \mathcal{Q}$, *then there is* $C = C(k,q)$ *such that*

$$\|F(q) - F(q+q')\|_{L^2(\mathbb{S}^{d-1})} \leqslant C\,\|q'\|_{L^p(B_R)}.$$

(ii) *The operator* $F'(q)$ *is Lipschitz continuous with respect to* $L^p(B_R)$*, i.e., there is a constant* $C = C(k, q)$ *such that*

$$\|F'(q+q') - F'(q)\|_{\mathcal{L}(L^p(B_R), L^2(S^{d-1}))} \leqslant C\, \|q'\|_{L^p(B_R)}.$$

(iii) *The forward operator is differentiable in the sense that*

$$\|F(q+q') - F(q) - F'(q)[q']\|_{L^2(S^{d-1})} \leqslant C\, \|q'\|^2_{L^p(B_R)},$$

for $C = C(k, q)$ *and* $q + q' \in \mathcal{Q}$. *If there is* $\{q'_n\}_{n\in\mathbb{N}}$ *such that* $q + q'_n \in \mathcal{Q}$ *for every* $n \in \mathbb{N}$ *and* $\|q'_n\|_{L^p(B_R)} \to 0$ *as* $n \to \infty$*, then* $\|F(q+q'_n) - F(q) - F'(q)[q'_n]\|_{L^2(S^{d-1})} / \|q'_n\|_{L^p(B_R)} \to 0$.

Proof. Due to the smoothing properties of Z, see Lemma 3.18, we have that

$$\begin{aligned} \|F(q) - F(q+q')\|_{L^2(S^{d-1})} &= \|Z \circ \left[q u_q - (q+q') u_{q+q'}\right] \|_{L^2(S^{d-1})} \\ &\leqslant C \left[\|q \left(u_q - u_{q+q'}\right) \|_{L^r(S^{d-1})} + \|q' u_{q+q'}\|_{L^r(S^{d-1})} \right], \end{aligned}$$

for $r = tp/(t+p)$. As shown in the proof of Theorem 6.5, both terms are bounded by a constant C, depending on the L^p-norm of q and $q + q'$ only, times $\|q'\|_{L^p(B_R)} \|u^i\|_{L^t(B_{2R})}$. Thus, for an incident wave satisfying the given assumption we have the stated inequality. Statements (ii) and (iii) can be shown analogously by exploiting Theorems 6.6 and 6.7. □

6.4 NON-LINEAR TIKHONOV AND SPARSITY REGULARIZATION

As said before, the inverse problem is to stably approximate a contrast q_{exa} from perturbed measurements of its far field pattern $F(q_{exa})$. Once more, for noisy measurements F^δ_{meas} with noise level $\delta > 0$ such that $\|F(q_{exa}) - F^\delta_{meas}\|_{L^2(S^{d-1})} \leqslant \delta$, we seek approximations of q from non-linear Tikhonov regularization. Thus, we consider to minimize the Tikhonov functional

$$\mathcal{J}_{\alpha,\delta}(q) := \tfrac{1}{2}\|F(q) - F^\delta_{meas}\|^2_{L^2(S^{d-1})} + \alpha\, \mathcal{R}(q), \tag{6.15}$$

over some appropriate admissible set of contrasts included in $\mathcal{Q}$, a subset of the closed and convex set $L^p_{\mathrm{Im}\geqslant 0}(B_R)$, and for some penalty term $\mathcal{R}$. Hence, we emphasize that the graph of F is weak-to-strong closed, due to [LKK13, Lemma 7], that is, for a weakly convergent sequence $q_n \in \mathcal{Q}$ to q implies $q \in \mathcal{Q}$ and $u_{q_n} \to u_q$ in $L^t(B_R)$ pointwise, as $n \to \infty$.

In analogy to Section 3.6, recall that $p > d/(d-1) \geqslant d/2$ due to Assumption 6.3 and set $p_* = dp/(p+d)$, such that $1 < p_* < d$. Again, Sobolev's embedding states that $W^{1,p_*}(B_R)$ embeds compactly into $L^p(B_R)$, such that

$$\|q\|_{L^p(B_R)} \leqslant C \|q\|_{W^{1,p_*}(B_R)} \quad \text{for all } q \in W^{1,p_*}(B_R).$$

Note that an embedding into Hölder spaces would require $p_* > d$. As in the case of anisotropic contrasts, we are going to consider an extension by zero to all of $\mathbb{R}^d$ of functions of $W_0^{1,p_*}(B_R)$, the space of functions that vanish on ∂B_R that is, without denoting it explicitly. Therefore, we derive standard Tikhonov regularization statements, analogous to Theorem 3.24:

THEOREM 6.10 (Tikhonov regularization). *Let the penalty of the Tikhonov functional $\mathcal{J}_{\alpha,\delta}$ be $\mathcal{R}(q) = \|q\|^{p_*}_{W^{1,p_*}(B_R)}$, then $\mathcal{J}_{\alpha,\delta}$ possesses a minimizer in $\mathcal{Q} \cap W^{1,p_*}(B_R)$. If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weakly converges to a solution $q^\dagger \in W^{1,p_*}(B_R) \cap \mathcal{Q}$ such that $F(q^\dagger) = F(q_{exa})$ holds in $L^2(\mathbb{S}^{d-1})$ and $q^\dagger$ minimizes the $W^{1,p_*}(B_R)$-norm amongst all solution to the latter equation.*

SPARSITY REGULARIZATION A sparsity-promoting alternative yet again bases on a wavelet basis of W^{1,p_*} as seen in Section 3.6. Therefore, we assume that $\psi_{Mo} \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, is a compactly supported mother wavelet with scaling function ψ_{Fa}, the so-called father wavelet. Consequently, an associated multi-resolution analysis provides one-dimensional wavelets

$$\psi^j_m(\cdot) = \begin{cases} \psi_{Fa}(\cdot - m) & \text{if } j = 0,\ m \in \mathbb{Z}, \\ 2^{(j-1)/2}\psi_{Mo}(2^{j-1}\cdot - m) & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}. \end{cases}$$

As shown in Section 3.6, those define d-dimensional n-wavelets $\Psi^{j,G}_m$. Then for all numbers $1 \leqslant r \leqslant p_*$ choose weights $\{\omega_j\}_{j\in\mathbb{N}_0}$ such that $\omega_j \geqslant 2^{j(1-d/p_*+d/2)r}$, which is a normalizing factor coming from Theorem 3.25. Thus, the functional

$$\mathcal{R}_r(q) := \frac{1}{r}\sum_{j,G,m} \omega_j \left|q^{j,G}_m\right|^r, \tag{6.16}$$

defines an appropriate penalty term as described by (3.32). One more time, this yields a result for non-linear Tikhonov regularization which provides sparsity, analogous to Theorem 3.26.

THEOREM 6.11 (Sparsity regularization). *For $1 \leqslant r \leqslant p_* = dp/(p+d)$, the functional $\mathcal{J}_{\alpha,\delta}$ with $\mathcal{R} = \mathcal{R}_r$ from (6.16) possesses a minimizer in $\mathcal{Q} \cap W^{1,p_*}(B_R)$. If $\delta_n \to 0$ as $n \to \infty$ and if one chooses $\alpha_n = \alpha_n(\delta_n)$ such that $0 < \alpha_n \to 0$ and $0 < \delta_n^2/\alpha_n \to 0$, then every sequence of minimizers of $\mathcal{J}_{\alpha_n,\delta_n}$ contains a subsequence that weakly converges to an $\mathcal{R}_r$-minimizing solution $q^\dagger \in W^{1,p_*}(B_R) \cap \mathcal{Q}$ of the equation $F(q) = F(q_{exa})$ in $L^2(\mathbb{S}^{d-1})$.*

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DECLARATION

I, Marcel Rennoch, declare that this thesis titled, "Regularization Methods in Banach Spaces applied to Inverse Medium Scattering Problems" and the work presented in it are my own. I confirm that:

- This work was done while in candidature for a doctoral degree at the University of Bremen.
- Where any part of this thesis has previously been submitted, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

Bremen, May, 2017

Marcel Rennoch

COLOPHON

As mathematics is generally done with pen, pencil, or chalk, a handwritten rather than a mechanical style seems more pleasant to read. That's why the *AMS Euler* math font was designed by the German typographer Hermann Zapf with the assistance of the American computer scientist Donald Knuth (widely known as the creator of TEX). The design tries "to capture the flavor of mathematics as it might be written by a mathematician with excellent handwriting" [Knu89]—that is actually the opposite of my personal handwriting—and is therefore used as math font in this thesis.

Whilst the Euler design blends very badly with the default TEX font Computer Modern, it goes very well with other typefaces of Zapf. Hence, the serif font used is *Palatino*, provided by Diego Puga and typeset in 11/13.6 × 28 (resulting in a "double square text block" with 1:2 ratio). The typewriter font used is *Bera Mono*, originally developed by Bitstream, Inc. as Bitstream Vera.

Vector graphics are set up with PGF/Ti*k*Z, whereas plots are provided by Matlab®. The typesetting of the document bases on the typographical look-and-feel `classicthesis` developed by André Miede. Its style is inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*". `classicthesis` is available for both LATEX and LYX:

`https://bitbucket.org/amiede/classicthesis/`