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The Inside-Outside Duality in Inverse Scattering Theory

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ABSTRACT

In this thesis we investigate a connection between far field data that arises from time-harmonic scattering problems and interior eigenvalues of corresponding scattering objects. This connection has been used to develop the so-called "inside-outside duality" method, which can be used to detect the interior eigenvalues from far field data. In this method a particular focus lies on the behavior of certain eigenvalues of the far field operator, which characterizes the interior eigenvalues. This thesis is separated into two parts. In the first part, we consider acoustic, time-harmonic scattering from impenetrable and penetrable scattering objects. We start by considering acoustic scattering from impenetrable objects and subsequently outline the principle arguments for the derivation of the inside-outside duality. In this context we also show how to work with near field data instead of far field data. In the remainder of the first part, the arguments are then adapted to scattering from penetrable scattering objects that may contain cavities. For all scattering scenarios under investigation, numerical examples for the verification of the theoretical results are provided.

In the second part of this thesis we consider elastic and electromagnetic scattering problems. In the case of elastic scattering, we assume an isotropic background medium in which either a rigid or a penetrable scattering object is embedded. For electromagnetic scattering, we consider penetrable objects that may contain cavities. The main challenge in this part lies in adapting the preceding arguments for the different scattering equations. Therefore we focus on theoretical results, which can potentially be used to detect interior eigenvalues from corresponding far field data.

PREFACE

The topic of this thesis is a connection between interior eigenvalues of scattering objects and far field data that arises from a corresponding scattering problem. Originally this connection has been examined in [EP95] for impenetrable Dirichlet scatterers and in [KL13] for penetrable scatterers. Building upon this basis, we extend and modify the ideas from these articles for different scattering scenarios. In the first part of this thesis we consider acoustic scattering. In Chapter 2 and in Chapter 3 we review and extend the ideas from the above-mentioned articles. In Chapter 4 we then consider more complex acoustic scattering scenarios, involving anisotropic densities. In the second part of this thesis we consider different scattering models. In Chapter 6 we consider elastic scattering and in Chapter 7 we consider electromagnetic scattering and summarize and extend the ideas from [LR15].

Note that most parts of Chapter 2 have already been published in [LP14, LP15b], where the principle ideas for this thesis have been worked out by the author of this thesis in collaboration with his supervisor Armin Lechleiter. Additionally parts of Chapter 4, in particular Section 4.3 and Section 4.6, have been published in [LP15a] by the same authors. The extensions in Section 3.3 and Section 3.5 have been published in [PK16] by the author of this thesis and Andreas Kleefeld, who kindly also provided the numerical data for the verification of the inside-outside duality in Section 3.5. Finally most parts of Chapter 6 have been submitted for publication in [Pet16].

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Part I.

The Inside-Outside Duality for Acoustic Scattering

CHAPTER 1

INTRODUCTION

In applied mathematics and physics interior eigenvalues of objects play a crucial role. For example the eigenvalues of the Laplacian with Dirichlet boundary conditions represent the fundamental modes of vibrations of an idealized drum. In this context the famous question "Can one hear the shape of a drum" was posed [Kac66], which raises the question if the knowledge of all modes of vibration of the drum enables one to gain information about its shape. Since the Laplacian is also employed in several other applications, for example in the description of small waves on the surface of an idealized pool or modes of an idealized optical fiber in the paraxial approximation [AK96], its properties are well-studied. In particular the self-adjointness of the corresponding eigenvalue problem grants easy access to important properties. For example it is well-known that if we assume either Dirichlet or Neumann boundary conditions, the eigenvalues of the Laplacian are non-negative, discrete and tend to infinity [GN13].

In direct and inverse time-harmonic acoustic scattering theory interior eigenvalues appear naturally since the propagation of acoustic waves is described by the Helmholtz equation, in which the Laplacian is an integral part. From a physical point of view, interior eigenvalues relate to non-scattering waves, i.e. in the presence of interior eigenvalues there is an incoming wave that produces no scattered field such that the scattering object is invisible. The specifics of eigenvalues that are important depend on the scattering problem under investigation. If we consider for example scattering by impenetrable scattering objects then interior Dirichlet, Neumann, or Robin eigenvalues of the negative Laplacian play an important role, depending on the boundary conditions we assume. In direct scattering theory, the presence of interior eigenvalues might lead to failure in the application of integral equation methods for the solution of exterior scattering problems, see, e.g. [CK13, SS13]. In inverse scattering theory on the other hand, well-known object reconstruction and shape identification techniques like the linear sampling method or the factorization method can fail at interior eigenvalues, [CK13, KG08].

When we consider acoustic scattering by a penetrable, inhomogeneous scattering object a different eigenvalue problem appears, the so-called interior transmission eigenvalue problem. It was first investigated in [CM88, CM89]. Unlike the eigenvalue problem of the Laplacian, this eigenvalue problem is more difficult to analyze since it is no longer self-adjoint. Indeed it was unclear if interior transmission eigenvalues even exist until 2008, when it was proven in [PS08] that, if the index of refraction, describing the inhomogeneity, is large enough, a finite number of transmission eigenvalues exist. In a number of subsequent papers, this result was extended to different scattering scenarios [CH09, Kir09]. The research on the existence and discreteness of transmission eigenvalues culminated in the work [CHG10], where the existence of an infinite number of discrete interior transmission eigenvalues was shown under very general conditions. In particular the discreteness of interior transmission eigenvalues is an important result, since, as in the case of impenetrable scattering objects, reconstruction techniques can fail at interior transmission eigenvalues [CC06, KG08]. In inverse scattering theory interior transmission eigenvalues have a further important application, since they provide bounds for the index of refraction of the scattering object [CH13a], thereby granting information about the unknown scattering object.

Our objective in the first part of this thesis is to determine interior eigenvalues from acoustic far field data by a relation that is known as the "inside-outside duality" [EP95]. The name is motivated by the fact that we can use information obtained in the exterior of the scattering object, the far field data, to gain information about properties of the interior of the scattering object, the interior eigenvalues. An essential part of the inside-outside duality technique is the construction of far field operators from far field data for many wavenumbers. The properties of the far field operator then bridge the gap between exterior scattering data and interior eigenvalues. For our analysis we require the far field operator to have certain properties. More precisely, we require compactness and normality and rely on the special structure of its eigenvalues, which lie on a circle in the upper half in the complex plane. To guarantee these properties, we choose the material parameters of the scattering model under consideration accordingly. The inside-outside duality then examines how the behavior of certain eigenvalues of the far field operator with varying wavenumber relates to interior eigenvalues of the scattering object. For the derivation of this relation we need a factorization of the far field operator, which will help us link interior eigenvalues to the far field data. As we will see, the particular form of the factorization determines the quality of the inside-outside duality.

The remainder of the first part of this thesis is structured as follows. In Chapter 2 we consider acoustic, time-harmonic scattering from impenetrable scattering objects and show how corresponding interior eigenvalues of the negative Laplacian can be determined by the inside-outside duality from near field data and far field data. Then we will focus on scattering by penetrable scattering objects whose material properties are described by different material parameters and determine interior transmission eigenvalues of the corresponding interior transmission eigenvalue problems. In Chapter 3 we will focus on scattering objects whose properties are described by a scalar function, the index of refraction. In this context we will also discuss the influence of the presence of cavities in scattering objects. In Chapter 4 we will continue the discussion on penetrable scattering objects by considering scattering equations that include anisotropic densities. For all scattering models under consideration, we will also show how the theoretical analysis of the inside-outside duality can be turned into a working algorithm that can be used to numerically detect interior eigenvalues.

CHAPTER 2

SCATTERING FROM IMPENETRABLE OBJECTS

2.1. Introduction

In this chapter, we want to use the inside-outside duality technique to determine interior eigenvalues of the negative Laplacian from far field data and from near field data. The results in this section are based on the articles [EP95, LP14] for the derivation of the inside-outside duality for far field data and on the article [LP15b] for the derivation for near field data. Let us first specify how interior eigenvalues are defined. Our scattering object is represented by a bounded Lipschitz domain $D \subset \mathbb{R}^3$ with connected complement and $k > 0$ represents the wavenumber. Then the number k^2 is a Dirichlet, Neumann, or Robin eigenvalue if there is a non-trivial solution of the problem

$$\Delta u + k^2 u = 0 \quad \text{in } D, \qquad \mathcal{B}u = 0 \quad \text{on } \partial D,$$

where ∂D denotes the boundary of D and $\mathcal{B}$ represents either Dirichlet, Neumann or Robin boundary conditions, i.e.

$$\mathcal{B}u = u \qquad \text{or} \qquad \mathcal{B}u = \frac{\partial u}{\partial \nu} + \tau u,$$

where ν is the outward normal to ∂D and $\tau \in L^\infty(\partial D)$ is assumed to be real-valued. Note that the Neumann boundary condition $\mathcal{B}u = \partial u/\partial \nu$ is implied in the Robin boundary condition, since we can choose $\tau = 0$. The eigenvalue problems have to be understood in a weak sense, i.e. k^2 is a Dirichlet eigenvalue if there is a non-trivial function $u \in H_0^1(D)$ such that

$$\int_D \left(\nabla u \cdot \nabla \psi - k^2 u \psi\right) \,\mathrm{d}x = 0 \qquad \forall \psi \in H_0^1(D),$$

and a Robin eigenvalue if there is a non-trivial function $u \in H^1(D)$ such that

$$\int_D \left(\nabla u \cdot \nabla \psi - k^2 u \psi\right) \,\mathrm{d}x = \int_{\partial D} \tau u \psi \;\mathrm{d}s \qquad \forall \psi \in H^1(D).$$

For each of these eigenvalues problems, we consider a corresponding exterior time-harmonic scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \mathcal{B}u = 0 \quad \text{on } \partial D, \tag{2.1}$$

where the total field $u = u^i + u^s$ is the sum of an incident field u^i and a scattered field u^s that satisfies Sommerfeld's radiation condition

$$\left(\frac{\partial u^s}{\partial |x|} - \mathrm{i}ku^s\right) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \text{ as } |x| \to \infty, \qquad \text{uniformly in } \hat{x} = \frac{x}{|x|} \in \mathbb{S}_1, \tag{2.2}$$

where $\mathbb{S}_1$ is the unit sphere, or more generally $\mathbb{S}_R := \{x \in \mathbb{R}^3 : \ |x| = R\}$ for a Radius $R > 0$. In this thesis we will call solutions to the Helmholtz equation that satisfy Sommerfeld's radiation condition (2.2), or a similar radiation condition, radiating solutions. As a consequence of this radiation condition, the scattered wave behaves like an outgoing spherical wave,

$$u^s(x) = \frac{e^{\mathrm{i}k|x|}}{4\pi|x|} u^\infty(\hat{x}) + \mathcal{O}(|x|^{-2}), \qquad \hat{x} \in \mathbb{S}_1, \tag{2.3}$$

with a far field pattern $u^\infty \in L^2(\mathbb{S}_1)$. As incident fields, we consider in the following either incident plane waves $u^i(x,\theta) = \exp(\mathrm{i}k\,\theta \cdot x)$ with direction $\theta \in \mathbb{S}_1$ or radiating point-sources

$$u^i(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, \quad x \neq y \in \mathbb{R}^3, \qquad \text{for source points } y \notin \overline{D}. \tag{2.4}$$

If the incident field u^i is a plane wave with direction $\theta \in \mathbb{S}_1$, we indicate the dependence of the far field pattern u^∞ on the incident direction by writing $u^\infty(\hat{x},\theta)$ for $\hat{x},\theta \in \mathbb{S}_1$. Then the far field operator is defined by

$$F : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1), \qquad Fg(\hat{x}) = \int_{\mathbb{S}_1} u^\infty(\hat{x},\theta) g(\theta) \,\mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}_1. \tag{2.5}$$

Later we will come across far field operators that correspond to different scattering problems. However there are some properties that all far field operators we are going to consider share and which are essential for the derivation of the inside-outside duality. One of those properties is compactness, which is due to the smoothness of its kernel and another is normality, see [KG08]. In the case of Robin boundary condition normality of the far field operator is due to the fact that τ is real-valued. Note that the required normality of the far field operator will influence our choice of material parameters in scattering scenarios we discuss later, mostly assuming those parameters to be real-valued. Normality of the far field operator is also essential to show another important property, which is the particular structure of its eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. They lie on a circle of radius $8\pi^2/k$ with center $8\pi^2\mathrm{i}/k$ in the complex plane. This special structure can be seen in Figure 2.1, which also reveals another characteristic of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. For Dirichlet boundary conditions they converge to zero from the left side and for Robin boundary conditions they converge to zero from the right side, that is, $\operatorname{Re}\lambda_j \lessgtr 0$ for $j \in \mathbb{N}$ large enough. Writing the eigenvalues in polar coordinates

$$\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j), \qquad \vartheta_j \in [0,\pi],$$

each eigenvalue λ_j corresponds to a phase ϑ_j. For completeness, we define the phase of the eigenvalue $\lambda_j = 0$ as $\vartheta_j = \pi$ if the eigenvalues converge to zero from the left and as $\vartheta_j = 0$ if they converge to zero from the right. Note that this case is not important in the context of this thesis. Due to their special structure, we can conclude that there is one eigenvalue λ_* with a smallest phase $\vartheta_* := \min_{j\in\mathbb{N}} \vartheta_j$ in the case of Dirichlet boundary conditions and one eigenvalue λ^* with a largest phase $\vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j$ in the case of Robin boundary conditions. Since the far field operator $F = F_k$ depends on the wavenumber k, its eigenvalues and in particular their phases also depend on the wavenumber. Depending on the boundary conditions we consider, the inside-outside duality for far field data now states the following: The value k_0^2 is an interior eigenvalue of $-\Delta$ if and only

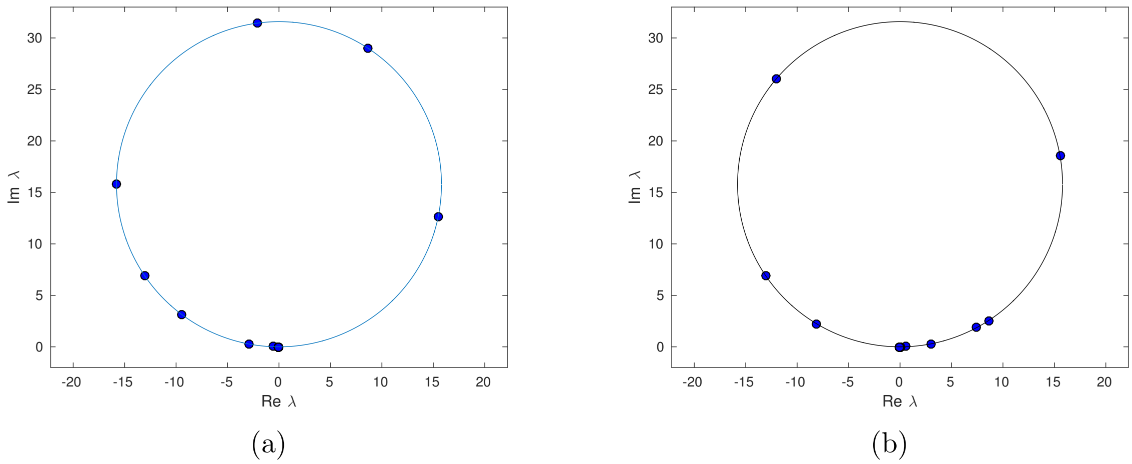


Figure 2.1.: The eigenvalues λ_j of the far field operator F with $k = 5$ on a circle in the complex plane for (a) Dirichlet boundary conditions (b) Neumann boundary conditions.

if the smallest phase $\vartheta_*(k)$ converges to zero, or the largest phase $\vartheta^*(k)$ converges to π for $k \to k_0$. For a precise statement see Theorem 2.8 and Theorem 2.9 for Dirichlet scattering and Theorem 2.16 and Theorem 2.17 for Robin scattering.

An essential ingredient for the derivation of the inside-outside duality is a factorization of the far field operator. For all three boundary conditions considered here, the far field operator satisfies a factorization of the form $F = GTG^*$ with a solution operator G mapping boundary data ψ on ∂D to the far field of the radiating solution of the following scattering problem,

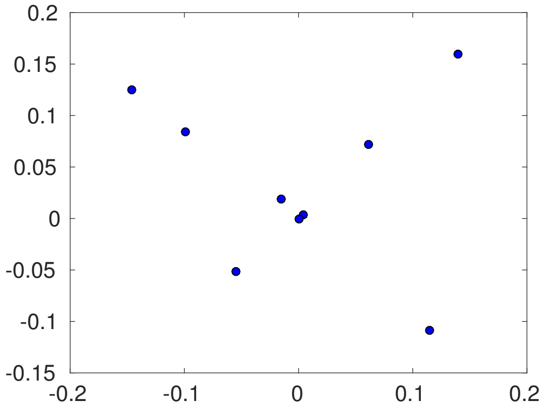
$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \mathcal{B}(v) = \psi \quad \text{on } \partial D. \tag{2.6}$$

The precise form of T, in particular the correct space for ψ, depends on the boundary conditions implemented in $\mathcal{B}$. In the subsequent analysis, the idea is the following: We will use the properties of the middle operator T to derive the behavior of the eigenvalue with the smallest or largest phase, stated by the inside-outside duality. In this context the question appears if we can neglect the outer operators in a sensible way without influencing our analysis. As we will see later, this is possible in scattering problems with impenetrable objects, since the outer operator G^* has dense range in its image space.

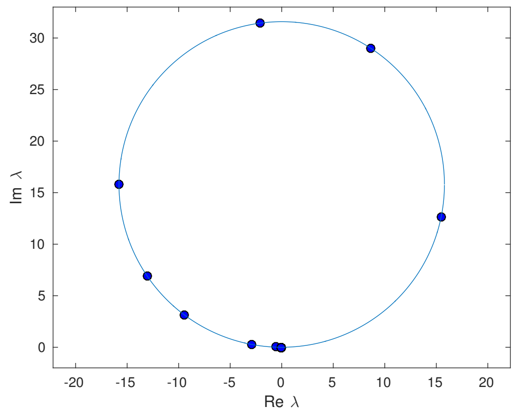
Next we will introduce the near field operator and contrast its properties against those of the far field operator. If the incident field is a point source at $y \in \mathbb{R}^3 \setminus \overline{D}$, see (2.4), we denote the scattered field of the solution to (2.1) by $u^s(\cdot, y)$. Let R be chosen large enough such that $\overline{D} \subset B_R := \{x \in \mathbb{R}^3 : \; |x| \leq R\}$. Then we define the near field operator $N_R : L^2(\mathbb{S}_R) \to L^2(\mathbb{S}_R)$ corresponding to incident point sources on $\mathbb{S}_R$ and near field wave measurements on the same surface $\mathbb{S}_R$ by

$$N_R g(x) = \int_{\mathbb{S}_R} u^s(x, y) g(y) \, \mathrm{d}S(y), \qquad x \in \mathbb{S}_R. \tag{2.7}$$

Following the approach from above for far field data, a naive attempt to characterize interior eigenvalues would be to examine the behavior of eigenvalues of the near field operator. Although the near field operator shares some properties with the far field operator like compactness and denseness of range, other important properties are lost. Most importantly the eigenvalues of the near field operator do not lie on any circle in the complex plane and do not show any specific structure, which makes it impossible to define phases of those eigenvalues in any sensible way, see Figure 2.2. The missing structure of the eigenvalues is related to other missing properties of the near field operator.



(a)



(b)

Figure 2.2.: Scattering from a unit ball with Dirichlet boundary conditions. (a) Eigenvalues of the near field operator N_R in the complex plane for wavenumber $k = 5$ and radius $R = 2$. (b) In comparison the eigenvalues of the far field operator F for $k = 5$.

For example, this operator is in general not normal and there is also no sensible factorization with outer operators that are adjoint to each other.

Therefore we need to change our approach and follow an ansatz from [HYZZ14], where the near field operator is modified by a unitary operator such that the modification shows some important properties which we can use to derive the inside-outside duality. Most importantly there is a factorization of this modified operator that is similar to the factorization of the far field operator and its eigenvalues show a structure which allows us to define phases of these eigenvalues in a sensible way. However in contrast to the derivation of the inside-outside duality for far field data a full characterization of interior eigenvalues is still not possible by merely relying on the phases of these eigenvalues. Therefore we introduce a new concept, the numerical range of an operator. In our main result in Corollary 2.33 we then show that interior Dirichlet eigenvalues can be fully characterized by the behavior of the element in the numerical range of the modified near field operator with the smallest phase.

Before we proceed, we introduce some important tools for our analysis. The fundamental radiating solution Φ to the Helmholtz equation is given by

$$\Phi(x, y) := \frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|}, \quad x \neq y.$$

Now the single layer and double layer potential are defined by

$$\mathrm{SL}\varphi(x) := \int_{\partial D} \Phi(x, y)\varphi(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{2.8}$$

$$\mathrm{DL}\,\psi(x) := \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\psi(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^3 \setminus \partial D. \tag{2.9}$$

Here and later on, ν denotes the outwards pointing unit normal vector field to D. It is well-known [McL00] that SL and DL are bounded from $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$ into $H^1(B_R)$ and $H^1(B_R \setminus \partial D)$ for any ball B_R centered in the origin with radius $R > 0$, respectively. Both potentials are smooth solutions to the Helmholtz equation in $\mathbb{R}^3 \setminus \partial D$ and radiating in $\mathbb{R}^3 \setminus \overline{D}$. Let us denote the exterior and interior trace operator on ∂D by $[\cdot]|^+$ and $[\cdot]|^-$, respectively. Then it is also well-known that the traces $\mathrm{SL}\varphi\big|^{\pm}, \partial\mathrm{SL}\varphi/\partial\nu\big|^{\pm}$ and $\mathrm{DL}\,\varphi\big|^{\pm}, \partial\,\mathrm{DL}\,\varphi/\partial\nu\big|^{\pm}$ are given by

$$\mathrm{SL}\varphi\big|^{\pm} = S\varphi \qquad \text{in } H^{1/2}(\partial D), \tag{2.10}$$

$$\mathrm{DL}\,\psi\big|^{\pm} = \pm\frac{1}{2}\psi + K\psi \qquad \text{in } H^{1/2}(\partial D), \tag{2.11}$$

$$\frac{\partial\mathrm{SL}\varphi}{\partial\nu}\Big|^{\pm} = \mp\frac{1}{2}\varphi + K'\varphi \qquad \text{in } H^{-1/2}(\partial D), \tag{2.12}$$

$$\frac{\partial\,\mathrm{DL}\,\psi}{\partial\nu}\Big|^{\pm} = N\psi \qquad \text{in } H^{-1/2}(\partial D), \tag{2.13}$$

where the boundary integral operator S is bounded from $H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$, K is bounded on $H^{1/2}(\partial D)$, K' is bounded on $H^{-1/2}(\partial D)$ and N in bounded from $H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$.

To simplify notation, let us in this chapter denote both the duality pairing between $H^{\pm 1/2}(\partial D)$ that extends the $L^2(\partial D)$-inner product and the inner product itself by $(\cdot,\cdot)$ or $(\cdot,\cdot)_{L^2(\partial D)}$. The inner product on $L^2(\mathbb{S}_1)$ is denoted by $(\cdot,\cdot)_{L^2(\mathbb{S}_1)}$ or by $(\cdot,\cdot)$ if there is no danger of confusion. As mentioned above, the open ball of radius R centered in the origin is denoted by B_R.

The remainder of this Chapter is structured as follows: In Section 2.2 we will derive the inside-outside duality for scattering from impenetrable objects with Dirichlet boundary conditions. In Section 2.3 we will consider Robin boundary conditions. In Section 2.4 we will show how we can use the results we have attained for far field data in order to base the inside-outside duality for near field data upon these results. The last two sections are dedicated to show that the theory can be turned into a working algorithm to numerically detect interior eigenvalues. In Section 2.5 we will derive an algorithm that we can use numerically to detect interior eigenvalues from discrete far field data. Finally in Section 2.6 we will use these results to numerically detect interior eigenvalues from discrete near field data.

2.2. Characterizing Dirichlet Eigenvalues from Far Field Data

This section and the next section are based on the work in [LP14]. Recall that $D \subset \mathbb{R}^3$ is a bounded Lipschitz domain with connected complement. In this section we examine the exterior Dirichlet scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D, \tag{2.14}$$

that is, $\mathcal{B}(u) = u$. We denote again by $u^s(\cdot,\theta)$ the radiating scattered field for an incident plane wave with direction θ, by $u^\infty(\cdot,\theta) \in L^2(\mathbb{S}_1)$ its far field pattern, and by F the far field operator, see (2.5). Note that a solution to the exterior Dirichlet scattering problem is understood in a variational sense, i.e. we seek a radiating scattered field $u^s(\cdot,\theta) \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ such that

$$\int_{\mathbb{R}^3\setminus\overline{D}} \left(\nabla u^s \cdot \nabla\overline{\psi} - k^2 u^s \overline{\psi}\right) \, \mathrm{d}x = 0$$

for all test functions $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ with compact support in $\mathbb{R}^3 \setminus \overline{D}$ such that $u^s = -u^i$ on the boundary ∂D. The first essential part of our analysis is a factorization of the far field operator. In [KG08, Theorem 1.15], it was shown the F can be written as

$$F = -GS^*G^* \tag{2.15}$$

where the single-layer operator S on ∂D has been defined in (2.10) and $G : H^{1/2}(\partial D) \to L^2(\mathbb{S}_1)$ is a solution operator mapping ψ to the far field pattern v^∞ of the unique radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ to

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad v = \psi \quad \text{on } \partial D. \tag{2.16}$$

We can use the properties of the middle operator S of the factorization to relate interior Dirichlet eigenvalues to the far field data. The link is provided by the following lemma, which summarizes the properties of the operator S, see [LP14, Lemma 1] and [KG08, Lemma 1.15] for a proof.

Lemma 2.1. *For $k > 0$ the following holds:*
(a) If k^2 is no Dirichlet eigenvalue, then S is an isomorphism from the Sobolev space $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$.
(b) For all $\varphi \in H^{-1/2}(\partial D)$ it holds that $\mathrm{Im}\,(\varphi, S\varphi) \leq 0$.
(c) If $\mathrm{Im}\,(\varphi, S\varphi) = 0$ for a non-trivial $\varphi \in H^{-1/2}(\partial D)$, then k^2 is a Dirichlet eigenvalue and the restriction of $w = \mathrm{SL}\varphi$ to D is a corresponding eigenfunction.
(d) If k^2 is a Dirichlet eigenvalue with eigenfunction $w \in H_0^1(D)$, then it holds that $\varphi = \partial w/\partial\nu|^- \in H^{-1/2}(\partial D) \neq 0$ satisfies $\mathrm{Im}\,(\varphi, S\varphi) = 0$.
(e) Denote by S_i the single layer boundary operator (2.8) for the wavenumber $k = \mathrm{i}$. Then S_i is self-adjoint and coercive as a map from $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$, i.e. there is a constant $c_0 > 0$ so that

$$(\varphi, S_\mathrm{i}\varphi) \geq c_0 \|\varphi\|^2_{H^{-1/2}(\partial D)} \qquad \forall \varphi \in H^{-1/2}(\partial D).$$

(f) The difference $S - S_\mathrm{i}$ is compact from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$.

From this lemma we conclude that the dimension of the eigenspace of the Dirichlet eigenvalue k^2 equals the dimension of the kernel of $\varphi \to \mathrm{Im}\,(\varphi, S\varphi)$. This property will also show later in our numerical experiments. Now recall from the introduction that the eigenvalues λ_j of F all lie on the circle $\{z \in \mathbb{C} : |z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$ and they converge to 0 as $j \to \infty$ since F is compact. We will now prove that they converge to zero from the left side. The principle arguments of the proof can also be found in the proof of [KL13, Lemma 4.1].

Lemma 2.2. *Assume that k^2 is no interior Dirichlet eigenvalue of $-\Delta$ in D. Then the eigenvalues λ_j of F converge to zero from the left, i.e., $\mathrm{Re}\,\lambda_j < 0$ for $j \in \mathbb{N}$ large enough.*

Proof. Let $g_j \in L^2(\mathbb{S}_1)$ be the eigenfunction corresponding to the eigenvalue λ_j of the far field operator F. Due to the normality and compactness of F, the eigenfunctions $(g_j)_{j\in\mathbb{N}}$ form a complete orthonormal system in $L^2(\mathbb{S}_1)$. We define $\psi_j = G^* g_j/\sqrt{|\lambda_j|}$. Then

$$\begin{aligned}(\psi_j,\, S\psi_\ell)_{L^2(\partial D)} &= \frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(G^* g_j,\, SG^* g_\ell)_{L^2(\partial D)} = \frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(GS^*G^* g_j,\, g_\ell)_{L^2(\mathbb{S}_1)} \\ &= -\frac{1}{\sqrt{|\lambda_j|\,|\lambda_\ell|}}(Fg_j,\, g_\ell)_{L^2(\mathbb{S}_1)} = -\frac{\lambda_j}{|\lambda_\ell|}\delta_{j,\ell} = -s_j\delta_{j,\ell}\end{aligned}$$

where $s_j := \lambda_j/|\lambda_j|$. By construction, $|s_j| = 1$ and $\mathrm{Im}\,(s_j) > 0$. Since λ_j converges to zero, the only possible accumulation point of s_j is either 1 oder -1. In the remainder of this proof we will show that the accumulation point is -1, which implies the statement of the lemma.

Lemma 2.1 implies a representation $S = S_\mathrm{i} + C$ where S_i is self-adjoint and coercive and C is a compact operator. Therefore

$$-s_j = (\psi_j,\, S_\mathrm{i}\psi_j)_{L^2(\partial D)} + (\psi_j,\, C\psi_j)_{L^2(\partial D)}, \qquad j \in \mathbb{N}. \tag{2.17}$$

This implies in particular that $-\mathrm{Re}\,(s_j) \geq c_0\|\psi_j\|^2_{H^{1/2}(\partial D)} + \mathrm{Re}\,(\psi_j,\, C\psi_j)_{L^2(\partial D)}$. Next, we show that the sequence ψ_j is bounded using a contradiction argument: Assume that there is a subsequence, also denoted by ψ_j, such that $\|\psi_j\|_{H^{1/2}(\partial D)} \to \infty$ as $j \to \infty$. Then $\psi'_j := \psi_j/\|\psi_j\|_{H^{1/2}(\partial D)}$ satisfies

$$c_0 + \mathrm{Re}\,(\psi'_j,\, C\psi'_j)_{L^2(\partial D)} \leq -\frac{\mathrm{Re}\,(s_j)}{\|\psi_j\|^2_{H^{1/2}(\partial D)}} \to 0 \quad \text{as } j \to \infty. \tag{2.18}$$

Since the sequence ψ_j' is bounded, we can extract a weakly convergent subsequence, again denoted by ψ_j' such that $\psi_j' \rightharpoonup \psi'$ as $j \to \infty$. Since C is compact, the image sequence $C\psi_j'$ converges strongly in $H^{1/2}(\partial D)$ and $(\psi_j', C\psi_j')_{L^2(\partial D)} \to (\psi', C\psi')_{L^2(\partial D)}$. Now (2.18) allows to conclude that

$$c_0 + \lim_{j\to\infty} \operatorname{Re}(\psi_j', C\psi_j')_{L^2(\partial D)} = c_0 + \operatorname{Re}(\psi', C\psi')_{L^2(\partial D)} \le 0. \tag{2.19}$$

Since $c_0 > 0$, this means that $\operatorname{Re}(\psi', C\psi')_{L^2(\partial D)} < 0$. Similar arguments applied to the imaginary part of (2.17) yield

$$0 = -\lim_{j\to\infty} \frac{\operatorname{Im}(s_j)}{\|\psi_j\|^2_{H^{1/2}(\partial D)}} = \lim_{j\to\infty} \operatorname{Im}(\psi_j', S\psi_j')_{L^2(\partial D)} = \operatorname{Im}(\psi', S\psi')_{L^2(\partial D)}.$$

Our assumption that k^2 is no interior eigenvalue together with Lemma 2.1 now implies that $\psi' = 0$. This contradicts the fact that $\operatorname{Re}(\psi', C\psi')_{L^2(\partial D)} < 0$ and finally shows that $\{\psi_j\}_{j\in\mathbb{N}}$ is bounded. To conclude the proof, consider again the imaginary part of (2.17) and use that the expression $(\psi_j, S_{\mathrm{i}}\psi_j)_{L^2(\partial D)}$ is real-valued together with $\operatorname{Im} s_j \to 0$ to deduce that

$$\operatorname{Im}(\psi_j, C\psi_j)_{L^2(\partial D)} \to \operatorname{Im}(\psi, C\psi) = 0 \qquad \text{as } j \to \infty.$$

This shows that $\operatorname{Im}(\psi, S\psi) = \operatorname{Im}(\psi, C\psi) = 0$. Since k^2 is no interior eigenvalue, implies that $\psi = 0$. Hence, $(\psi_j, C\psi_j) \to 0$ and $-\operatorname{Re}(s_j) \ge c_0\|\psi_j\|^2 \ge 0$ as $j \to \infty$, such that the accumulation point of s_j has to be -1. ■

Remark 2.3. In this remark we summarize the essential ingredients of the last proof. In order to show that the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of a normal far field operator converge to zero from one specific side, it is sufficient to find a factorization of the far field operator, in which the middle operator is either strictly positive or negative in the absence of interior eigenvalues and can be written as a compact perturbation of a coercive operator.

To proceed we represent the eigenvalues λ_j of the far field operator F in polar coordinates, such that

$$\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j), \qquad \vartheta_j \in [0, \pi].$$

For completeness, we define $\vartheta_j = \pi$ whenever $\lambda_j = 0$. Since $\operatorname{Re}\lambda_j < 0$ by Lemma 2.2 for large $j \in \mathbb{N}$, the phases ϑ_j converge to π as $j \to \infty$ and there is hence a smallest phase

$$\vartheta_* = \vartheta_{j_*} = \min_{j\in\mathbb{N}} \vartheta_j$$

among all phases ϑ_j. The eigenvalue λ_{j_*} with smallest phase is from now on denoted by λ_*. Since we are interested in the behavior of ϑ_* with varying wavenumber, we prove the following characterization of the cotangent of the smallest phase.

Theorem 2.4. *If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, then*

$$\cot\vartheta_* = \max_{g\in L^2(\mathbb{S}_1)} \frac{\operatorname{Re}(Fg, g)_{L^2(\mathbb{S}_1)}}{\operatorname{Im}(Fg, g)_{L^2(\mathbb{S}_1)}}. \tag{2.20}$$

The maximum is attained at any eigenvector g_* to the eigenvalue λ_* of F with smallest phase.

To simplify notation, we neglected to exclude the zero vector from the maximum in (2.20). Note also that the denominator $\operatorname{Im}(Fg, g)_{L^2(\mathbb{S}_1)}$ is positive if k^2 is not an interior eigenvalue due to Lemma 2.1 and the factorization $F = -GS^*G^*$: $\operatorname{Im}(Fg, g) = -\operatorname{Im}(G^*g, SG^*g) > 0$ for $g \neq 0$ since

G^* is injective. Since this characterization of the smallest phase is an essential part of the inside-outside duality, which we will also need when we consider other scattering problems, we will include the proof of this characterization from [LP14], which uses the special structure of the eigenvalues of F to apply a monotonicity argument (see Lemma 2.5) and relies on the normality and compactness of the far field operator to guarantee the existence of an orthonormal basis of eigenfunctions.

Lemma 2.5. *Assume that f, g are continuous functions on $I := (0, \beta) \subset \mathbb{R}$ such that g takes positive values and that $\alpha \mapsto f(\alpha)/g(\alpha)$ is strictly monotonically decreasing on I. Assume further that $(\alpha_j)_{j\in\mathbb{N}} \subset I$ is a sequence such that $\alpha_j \geq \alpha_* > 0$ for all $j \in \mathbb{N}$. Further let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-negative numbers. If both series $\sum_{j\in\mathbb{N}} c_j f(\alpha_j)$ and $\sum_{j\in\mathbb{N}} c_j g(\alpha_j)$ are unconditionally convergent, then*

$$\frac{\sum_{j\in\mathbb{N}} c_j f(\alpha_j)}{\sum_{j\in\mathbb{N}} c_j g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)}.$$

Equality holds if and only if $c_j = 0$ whenever $\alpha_j \neq \alpha_*$ and if there is at least one α_j that equals α_*.

Proof. Due to the monotonicity of $\alpha \mapsto f(\alpha)/g(\alpha)$,

$$\frac{f(\alpha_j)}{g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)} \tag{2.21}$$

for all $j \in \mathbb{N}$. In particular, since $g(\alpha_j)$ is positive, $f(\alpha_j) \leq f(\alpha_*)\, g(\alpha_j)/g(\alpha_*)$ for all $j \in \mathbb{N}$, that is,

$$\sum_{j\in\mathbb{N}} c_j f(\alpha_j) \leq \sum_{j\in\mathbb{N}} c_j \frac{f(\alpha_*)}{g(\alpha_*)} g(\alpha_j) = \frac{f(\alpha_*)}{g(\alpha_*)} \sum_{j\in\mathbb{N}} c_j g(\alpha_j).$$

Since $\sum_{j\in\mathbb{N}} c_j g(\alpha_j)$ is a positive number, the latter inequality implies that

$$\frac{\sum_{j\in\mathbb{N}} c_j f(\alpha_j)}{\sum_{j\in\mathbb{N}} c_j g(\alpha_j)} \leq \frac{f(\alpha_*)}{g(\alpha_*)}. \tag{2.22}$$

The strict monotonicity of $\alpha \mapsto f(\alpha)/g(\alpha)$ yields that equality in (2.21) holds if and only if $\alpha_j = \alpha_*$. Thus, equality in (2.22) holds if and only if $c_j = 0$ whenever $\alpha_j \neq \alpha_*$ and if there is at least one α_j that equals $\alpha_* > 0$. ∎

Proof of Theorem 2.4. We exploit that the eigenvectors $g_j \in L^2(\mathbb{S}_1)$ of F form a complete orthonormal basis of $L^2(\mathbb{S}_1)$ to represent $g \in L^2(\mathbb{S}_1)$ as $g = \sum_{j\in\mathbb{N}} (g, g_j) g_j$. Since $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$ this shows in particular that

$$(Fg, g) = \sum_{j\in\mathbb{N}} \lambda_j |(g, g_j)|^2. \tag{2.23}$$

Now we set $r_j = |\lambda_j|$. Since $\mathrm{Re}\,(\lambda_j) = r_j \cos(\vartheta_j)$ and $\mathrm{Im}\,(\lambda_j) = r_j \sin(\vartheta_j)$ we want to apply Lemma 2.5 to $f(\alpha) = \cos(\alpha)$ and $g(\alpha) = \sin(\alpha)$ on $(0, \pi)$ and need to check the monotonicity of $h(\alpha) := f(\alpha)/g(\alpha) = \cot(\alpha)$. We find that $h'(\alpha) = 2/(\cos(2\alpha) - 1) < 0$ in $(0, \pi)$, that is, h is strictly monotonically decreasing. Setting $\alpha_j = \vartheta_j$, $\alpha_* = \vartheta_* \leq \vartheta_j$ and $c_j = r_j |(g, g_j)|^2$ for arbitrary $g \in L^2(\mathbb{S}_1)$, Lemma 2.5 implies that

$$\frac{\sum_{j\in\mathbb{N}} \mathrm{Re}\,(\lambda_j) |(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \mathrm{Im}\,(\lambda_j) |(g, g_j)|^2} = \frac{\sum_{j\in\mathbb{N}} \cos(\vartheta_j) r_j |(g, g_j)|^2}{\sum_{j\in\mathbb{N}} \sin(\vartheta_j) r_j |(g, g_j)|^2} \leq \frac{\cos(\vartheta_*)}{\sin(\vartheta_*)} = \cot(\vartheta_*).$$

Due to the orthonormality of the eigenfunctions g_j and since $r_j > 0$ for all $j \in \mathbb{N}$ since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, equality holds if and only if g is chosen as an eigenfunction for the eigenvalue $\lambda_* = \lambda_{j_*}$ with the smallest phase among all eigenvalues of F. ∎

To progress with the analysis of the behavior of the smallest phase ϑ_*, we want to use the properties of the middle operator S of our factorization $F = -GS^*G^*$. At this point the question arises if we can neglect the outer operators of the factorization without influencing the results. As we will see in the next remark this is possible if the far field operator has a factorization with outer operators that have dense range in their image space, independent of the wavenumber k.

Remark 2.6. Due to the factorization $F = -GS^*G^*$ and the denseness of the range of G^* in $H^{-1/2}(\partial D)$, see [KG08], (2.20) can also be expressed using the single-layer operator S: Indeed, $(Fg,\, g)_{L^2(\mathbb{S}_1)} = -(S^*G^*g,\, G^*g)_{L^2(\partial D)} = -(\varphi,\, S\varphi)_{L^2(\partial D)}$ for $\varphi = G^*g \in H^{-1/2}(\partial D)$; in particular,

$$\cot \vartheta_* = \max_{\varphi \in H^{-1/2}(\partial D)} \frac{\operatorname{Re}(\varphi,\, S\varphi)_{L^2(\partial D)}}{\operatorname{Im}(\varphi,\, S\varphi)_{L^2(\partial D)}}, \tag{2.24}$$

where the maximum is attained at $\varphi = G^*g_*$.

This possibility to reformulate the characterization of the smallest phase in terms of the middle operator is the main reason why the inside-outside duality yields particularly good results for scattering from impenetrable scattering objects. For scattering from penetrable scattering objects, there is no sensible factorization where the outer operators have dense range in their image space for all wavenumbers. Therefore the analysis becomes more complicated as we will see in later chapters.

At this point it becomes crucial to consider the dependence of all the involved quantities on the wavenumber $k > 0$: We write $\vartheta_* = \vartheta_*(k)$, $S = S_k$ and $\mathrm{SL} = \mathrm{SL}_k$ to indicate this dependence. Further, we write $k \nearrow k_0$ to indicate that the positive number k tends to $k_0 > 0$ from below, i.e., $k_0 > k \to k_0$. Before we give a precise formulation of the first part of the inside-outside duality, we prove a final auxiliary result: The derivative of the middle operator S_k with respect to k is positive if k^2 is a Dirichlet eigenvalue.

Lemma 2.7. *Assume that k_0^2 is a Dirichlet eigenvalue of $-\Delta$ in D. Then S_{k_0} has a non-trivial kernel and for all elements φ_0 in this kernel it holds that $(\varphi_0,\, S_{k_0}\varphi_0)_{L^2(\partial D)} = 0$. The mapping $k \mapsto (\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)}$ is differentiable at k_0 and*

$$\alpha := \frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)}\bigg|_{k=k_0} = 2k_0 \int\limits_D |u_{k_0}|^2 \,\mathrm{d}x, \qquad \textit{where } u_{k_0} = \mathrm{SL}_{k_0}\varphi_0.$$

Proof. We already saw in Lemma 2.1 that $\operatorname{Im}(\varphi,\, S_k\varphi)_{L^2(\partial D)}$ vanishes for a non-zero φ if and only if $S_k\varphi = 0$, that is, if and only if k^2 is a Dirichlet eigenvalue of $-\Delta$ in D. Set $u_k = \mathrm{SL}_k\varphi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, in particular, $u_{k_0} = \mathrm{SL}_{k_0}\varphi_0$. Since the fundamental solution Φ is weakly singular, we compute that

$$\frac{\mathrm{d}}{\mathrm{d}k}u_k(x) = \frac{\mathrm{d}}{\mathrm{d}k}\int_{\partial D} \Phi(x,y)\varphi_0(y)\,\mathrm{d}S(y) = \int_{\partial D} \frac{\mathrm{d}}{\mathrm{d}k}\Phi(x,y)\varphi_0(y)\,\mathrm{d}S(y) = \int_{\partial D} \frac{\mathrm{i}}{4\pi} e^{\mathrm{i}k|x-y|}\varphi_0(y)\,\mathrm{d}S(y),$$

for $x \in \mathbb{R}$. The derivative of u_k with respect to k is hence well-defined in, e.g., $H^1_{\mathrm{loc}}(\mathbb{R}^3)$. In particular, the chain rule implies that

$$\Delta u_k' + k^2 u_k' + 2k u_k = 0, \qquad \text{where } u_k' := \frac{\mathrm{d}}{\mathrm{d}k}u_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{2.25}$$

Now we compute the derivative with respect to k of $k \mapsto (\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)}$,

$$\frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)} = \left(\varphi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}S_k\varphi_0\right) = \left(\varphi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}u_k\right) = \left(\frac{\partial u_k}{\partial \nu}\bigg|^- - \frac{\partial u_k}{\partial \nu}\bigg|^+,\, \frac{\mathrm{d}}{\mathrm{d}k}u_k\right)_{L^2(\partial D)}.$$

Note that the normal derivative $(\partial u_{k_0}/\partial\nu)|^+$ taken from the exterior vanishes, since the radiating solution $u_{k_0} = \mathrm{SL}_{k_0}\varphi_0$ to the Helmholtz equation vanishes by construction on ∂D and hence by the unique solvability of the exterior Dirichlet scattering problem everywhere in $\mathbb{R}^3 \setminus \overline{D}$. Now we use Green's first identity for $u_{k_0} \in H_0^1(D)$ and u'_{k_0} and exploit (2.25) to get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)}\bigg|_{k=k_0} &= \left(\frac{\partial u_{k_0}}{\partial\nu}\bigg|^-,\, \frac{\mathrm{d}u_{k_0}}{\mathrm{d}k}\right)_{L^2(\partial D)} = \int_D \left[\Delta u_{k_0}\overline{u'_{k_0}} + \nabla u_{k_0}\nabla\overline{u'_{k_0}}\right]\,\mathrm{d}x \\ &= \int_D \left[-k_0^2 u_{k_0}\overline{u'_{k_0}} - u_{k_0}\Delta\overline{u'_{k_0}}\right]\,\mathrm{d}x \\ &= \int_D \left[-k_0^2 u_{k_0}\overline{u'_{k_0}} + k_0^2\overline{u'_{k_0}}u_{k_0} + 2k_0\,|u_{k_0}|^2\right]\,\mathrm{d}x = 2k_0\int_D |u_{k_0}|^2\,\mathrm{d}x.\end{aligned}$$

■

The next theorem states the first part of the inside-outside duality, which characterizes Dirichlet eigenvalues by the behavior of the smallest phase $\vartheta_*(k)$ with varying wavenumber k. The proof relies on the fact the the auxiliary derivative α from the last Lemma is real-valued and does not vanish. For other scattering scenarios with penetrable scattering objects, we will also derive real-valued auxiliary derivatives. Unlike the scattering from impenetrable objects however, it is in those cases not clear if the derivative does not vanish. This is due to the fact that for scattering from impenetrable scattering objects we can make special use of the fact that the smallest phase ϑ_* is characterized only in terms of the middle operator S in (2.24).

Theorem 2.8 (Inside-Outside Duality - Part 1)**.** *Let k_0^2 be an interior Dirichlet eigenvalue of $-\Delta$. Then it holds that* $\lim_{k\nearrow k_0}\vartheta_*(k) = 0$.

Proof. Since k_0^2 is an interior eigenvalue, we know from Lemma 2.1 that there exists a non-trivial $\varphi_0 \in H^{-1/2}(\partial D)$ such that $(\varphi_0,\, S_{k_0}\varphi_0)_{L^2(\partial D)} = 0$. Assume that $I = (k_0 - \varepsilon, k_0 + \varepsilon)$ is an interval that does not contain other Dirichlet eigenvalues. We have shown that

$$\cot\vartheta_*(k) = \max_{\varphi\in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S_k\varphi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(\varphi,\, S_k\varphi)_{L^2(\mathbb{S}_1)}} \qquad \text{for } k \in I\setminus\{k_0\},$$

see (2.24). Define $f(k) = (\varphi_0,\, S_k\varphi_0)_{L^2(\partial D)}$ for $k \in I$ and note that the last Lemma 2.7 states that this function is differentiable at k_0. Taylor's theorem states that

$$f(k) = f(k_0) + \alpha(k-k_0) + r(k),$$

where $f(k_0) = 0$ by construction and the remainder $r(k)$ satisfies $r(k) = o(|k-k_0|)$ as $k \to k_0$. Further, note that $\mathrm{Im}\,(r(k)) \le 0$ due to Lemma 2.1, because the derivative $\alpha = \mathrm{d}f/\,\mathrm{d}k\; f(k)$ at k_0 is real-valued and $\mathrm{Im}\,f(k) \le 0$. Hence,

$$\cot\vartheta_*(k) = \max_{\varphi\in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S_k\varphi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(\varphi,\, S_k\varphi)_{L^2(\mathbb{S}_1)}} \overset{\varphi=\varphi_0}{\ge} \frac{\alpha(k-k_0) + \mathrm{Re}\,(r(k))}{\mathrm{Im}\,(r(k))} \to \infty \quad \text{as } k\nearrow k_0. \tag{2.26}$$

Indeed, since α is positive, $k \nearrow k_0$ implies that $\alpha(k-k_0) \le 0$ tends slower to zero than $0 > \mathrm{Im}\,(r(k)) = o(|k-k_0|)$, that is, $[\alpha(k-k_0) + \mathrm{Re}\,(r(k))]/\mathrm{Im}\,(r(k)) \to \infty$. Obviously, $\cot\vartheta_*(k) \to \infty$ for $\vartheta_*(k) \in (0,\pi)$ implies that $\vartheta_*(k) \to 0$. ■

The final result in this section is the second part of the inside-outside duality, which together with Theorem 2.8 gives a full characterization of interior Dirichlet eigenvalues by the behavior of the phase ϑ_*. Roughly speaking, this second part states that interior eigenvalues k_0^2 are characterized by the fact that the phase $\vartheta_*(k) \in (0, \pi)$ of $\lambda_*(k)$ tends to 0 as $k \nearrow k_0$. Note that while the first part of the inside-outside duality only holds conditionally in later scattering scenarios, the second part holds true more universally. As we will see later, this is due to the fact the proof does not need to rely on the property of the outer operators in the factorization of F having dense range in its image space.

Theorem 2.9 (Inside-Outside Duality - Part 2). *Assume that $k_0 > 0$ and that $I = (k_0 - \varepsilon, k_0)$ contains no k such that k^2 is a Dirichlet eigenvalue of $-\Delta$ in D. If $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$, then k_0^2 is a Dirichlet eigenvalue of $-\Delta$ in D.*

Proof. To prove that $\lim_{k \nearrow k_0} \vartheta_*(k) = 0$ implies that k_0^2 is a Dirichlet eigenvalue we argue by contradiction: Assume that this limit relation holds but that k_0^2 is not a Dirichlet eigenvalue. Due to equation (2.24), $\vartheta_*(k) \to 0$ as $k \nearrow k_0$ implies that

$$\max_{\varphi \in H^{-1/2}(\partial D)} \frac{\mathrm{Re}\,(\varphi,\, S_k \varphi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(\varphi,\, S_k \varphi)_{L^2(\mathbb{S}_1)}} \to \infty \qquad \text{as } k \nearrow k_0.$$

Hence, there exist sequences $k_j \in I$ such that $k_j \nearrow k_0$ and $\varphi_j \in H^{-1/2}(\partial D)$ with $\|\varphi_j\|_{H^{-1/2}(\partial D)} = 1$ such that $0 > \mathrm{Im}\,(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} \to 0$ as $j \to \infty$ and $\mathrm{Re}\,(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} < 0$ for $j \in \mathbb{N}$ large enough. Since the sequence φ_j is bounded, there exists a weakly convergent subsequence that we also denote by φ_j, such that $\varphi_j \rightharpoonup \varphi_0$ for some $\varphi_0 \in H^{-1/2}(\partial D)$. Define $v_j = \mathrm{SL}_{k_j} \varphi_j$. Note that Green's first identity, the jump relation (2.12), and the Sommerfeld radiation condition imply that

$$\begin{aligned}(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} &= \int_{\partial D} \left[\frac{\partial v_j}{\partial \nu}\bigg|^- - \frac{\partial v_j}{\partial \nu}\bigg|^+ \right] \overline{v_j} \,\mathrm{d}S = \int_{B_R \setminus \partial D} [|\nabla v_j|^2 - k_j^2 |v_j|^2] \,\mathrm{d}x - \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j} \,\mathrm{d}S \\ &= \int_{B_R \setminus \partial D} [|\nabla v_j|^2 - k_j^2 |v_j|^2] \,\mathrm{d}x - \mathrm{i} k_j \int_{\partial B_R} |v_j|^2 \,\mathrm{d}S + \mathcal{O}(1/R) \quad \text{as } R \to \infty, \end{aligned} \tag{2.27}$$

such that the far field v_j^∞ of v_j satisfies

$$\mathrm{Im}\,(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} = -\frac{k_j}{4\pi^2} \|v_j^\infty\|_{L^2(\mathbb{S}_1)}^2, \qquad j \in \mathbb{N}. \tag{2.28}$$

The operator mapping φ_j to v_j^∞ is compact and hence the far fields v_j^∞ converge strongly in $L^2(\mathbb{S}_1)$. This strong limit equals the weak limit which is $v_0^\infty \in L^2(\mathbb{S}_1)$, the far field of $v_0 := \mathrm{SL}_{k_0} \varphi_0$. Note now that the right-hand side in (2.28) tends to zero, that is, v_0^∞ must vanish. Rellich's lemma then implies that v_0 vanishes in the exterior of D. However, since we assumed that k_0^2 is no interior Dirichlet eigenvalue, v_0 must vanish inside of D, too, and the jump relations for the single-layer potential imply that φ_0 must also vanish, that is, $\varphi_j \rightharpoonup 0$. Since the single-layer operator SL is bounded from $H^{-1/2}(\partial D)$ into $H^1(B_R)$ for all $R > 0$ it is also a compact operator into $L^2(B_R)$. Hence, $v_j \to 0$ strongly in $L^2(B_R)$. Due to elliptic regularity results, SL is also bounded from $H^{-1/2}(\partial D)$ into $H^2(B_{2R} \setminus B_{R/2})$ for $R > 0$ large enough. Since $\varphi_j \rightharpoonup 0$ this mapping property implies that $\int_{\partial B_R} (\partial v_j / \partial \nu) \overline{v_j} \,\mathrm{d}S$ tends strongly to zero as $j \to \infty$. Note that we already found above that $\mathrm{Re}\,(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} \leq 0$. This motivates to take the real part of (2.27),

$$0 \geq \mathrm{Re}\,(\varphi_j,\, S_{k_j} \varphi_j)_{L^2(\partial D)} = \int_{B_R} [|\nabla v_j|^2 - k^2 |v|^2] \,\mathrm{d}x - \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j} \,\mathrm{d}S,$$

to obtain that

$$\int_{B_R} |\nabla v_j|^2 \,\mathrm{d}x \leq \int_{B_R} |v_j|^2 \,\mathrm{d}x + \int_{\partial B_R} \frac{\partial v_j}{\partial \nu} \overline{v_j} \,\mathrm{d}S \to 0 \qquad \text{as } j \to \infty.$$

In particular, v_j converges strongly to zero in $H^1(B_R)$, as well as its trace $v_j|_{\partial D} = S_{k_j}\varphi_j$ tends strongly to zero in $H^{1/2}(\partial D)$. Since, by assumption k_0^2 is not a Dirichlet eigenvalue, the single-layer operator S_{k_0} is an isomorphism. This allows to conclude that $\varphi_j \to 0$ strongly in $H^{-1/2}(\partial D)$, which contradicts our initial assumption that $\|\varphi_j\|_{H^{-1/2}(\partial D)} = 1$ for all $j \in \mathbb{N}$. ∎

Remark 2.10. One can also prove that the number M of eigenvalue curves $k \mapsto \lambda_j(k)$ that tend to 0 from the right as $k \nearrow k_0$ equals the dimension N of the eigenspace of the interior Dirichlet eigenvalue k_0^2. The proof of Lemma 2.8 together with Lemma 2.1 implies that N linear independent eigenfunctions create N eigenvalues.

2.3. Characterizing Neumann and Robin Eigenvalues from Far Field Data

In this section we want to characterize interior Robin and Neumann eigenvalues from far field data. In this context the main challenge lies in adapting the arguments that have been used in the last section to derive the inside-outside duality for the present case. Although we will only consider Robin boundary conditions in this section, note that Neumann boundary conditions are implied in the derivation. From the arguments in the last section it is also clear that the far field operator under consideration needs to be normal to allow for an eigenvalue decomposition. This is why we only consider non-absorbing boundary conditions to guarantee this property.

Let $D \subset \mathbb{R}^3$ again be a bounded Lipschitz domain with connected complement and let the boundary operator $\mathcal{B}$ take the form $\mathcal{B}(u) = \partial u/\partial\nu + \tau u$ on ∂D for a real-valued function $\tau \in L^\infty(\partial D)$. Then we can state the exterior Robin scattering problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \frac{\partial u}{\partial \nu} + \tau u = 0 \quad \text{on } \partial D. \tag{2.29}$$

In the variational formulation of this problem, we seek a radiating scattered field $u^s(\cdot,\theta) \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})$ corresponding to an incoming plane wave $u^i(\cdot,\theta) = e^{ik\theta\cdot}$ with direction $\theta \in \mathbb{S}_1$, such that $u = u^s + u^i$ and the scattered field solves

$$\int_{\mathbb{R}^3\setminus\overline{D}} \left(\nabla u^s \cdot \nabla\overline{\psi} - k^2 u^s \overline{\psi}\right) \,\mathrm{d}x - \int_{\partial D} \tau u^s \overline{\psi} \,\mathrm{d}s = \int_{\partial D} \left(\frac{\partial u^i}{\partial \nu} + \tau u^i\right) \overline{\psi} \,\mathrm{d}s$$

for all $\psi \in H^1(\mathbb{R}^3 \setminus \overline{D})$ with compact support. Note that since we do not exclude the special case $\tau = 0$, all succeeding arguments also hold true for the Neumann case $B(u) = \partial u/\partial\nu$. Recall the definition of the far field operator F in (2.5). Since τ is real-valued, the far field operator F is a compact and normal operator [CK95]. We denote its eigensystem again as $(\lambda_j, g_j)_{j\in\mathbb{N}}$, that is, $Fg = \sum_{j\in\mathbb{N}} \lambda_j (g, g_j) g_j$ and note that the eigenvalues λ_j again lie on the circle $\{z \in \mathbb{C},\ |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$. In what follows we will provide the framework for the techniques from the last section for the derivation of the inside-outside duality. We start by stating a factorization of the far field operator F corresponding to the above-introduced Robin boundary conditions,

$$F = -GT^*G^*. \tag{2.30}$$

Here, $G : H^{-1/2}(\partial D) \to L^2(\mathbb{S}_1)$ is the compact and injective solution operator, mapping a Robin boundary datum ψ to the far field v^∞ of the unique radiating solution to the exterior Robin boundary value problem,

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \frac{\partial v}{\partial \nu} + \tau v = \psi \quad \text{on } \partial D. \tag{2.31}$$

Moreover, the operator $T:\, H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is given by

$$T\psi = N\psi + K'(\tau\psi) + \tau K\psi + \tau S(\tau\psi), \tag{2.32}$$

where N, K', K and S are the boundary integral operators defined in (2.10)–(2.13). For the proof of this factorization we refer to [KG08, Theorem 2.6]. The following lemma, which collects the properties of the operator T, is the equivalent to Lemma 2.1 for Robin boundary conditions.

Lemma 2.11. *For $k > 0$ the following holds:*
(a) If k^2 is no Robin eigenvalue, then T is an isomorphism from $H^{1/2}(\partial D)$ onto $H^{-1/2}(\partial D)$.
(b) For all $\psi \in H^{1/2}(\partial D)$ it holds that $\operatorname{Im}(T\psi, \psi)_{L^2(\partial D)} \geq 0$.
(c) There is a non-trivial $\psi \in H^{1/2}(\partial D)$ such that $\operatorname{Im}(T\psi, \psi) = 0$ *if and only if k^2 is a Robin eigenvalue.*
(d) T can be represented as $T = N(0)+C$ where $N(0)$ is the hypersingular boundary integral operator N from (2.13) *for wavenumber $k = 0$ and C is a compact operator. The operator $-N(0)$ is strictly positive and self-adjoint,*

$$-\big(N(0)\psi,\, \psi\big) \geq c_0 \|\psi\|^2_{H^{1/2}(\partial D)} \qquad \text{for all } \psi \in H^{1/2}(\partial D). \tag{2.33}$$

Proof. (a) See [KG08, Theorem 2.6] for a proof.
(b) Note first that the imaginary part of the far field operator $\operatorname{Im} F$ is positive, since for any $g \in L^2(\mathbb{S}_1)$

$$\operatorname{Im}(Fg,\, g) = \frac{k}{16\pi^2}\|Fg\|^2_{L^2(\mathbb{S}_1)} = \frac{k}{16\pi^2}\|F^*g\|^2_{L^2(\mathbb{S}_1)} \geq 0 \qquad \text{for all } g \in L^2(\mathbb{S}_1). \tag{2.34}$$

The equalities in the equation above are a direct consequence of [KG08, Theorem 2.5]. Now we can use the factorization of F to calculate,

$$0 \leq \frac{k}{16\pi^2}\|Fg\|^2_{L^2(\mathbb{S}_1)} = \operatorname{Im}(Fg,\, g)_{L^2(\mathbb{S}_1)} = -\operatorname{Im}(T^*G^*g,\, G^*g)_{L^2(\partial D)} = \operatorname{Im}(TG^*g,\, G^*g)_{L^2(\partial D)} \tag{2.35}$$

for $g \in L^2(\mathbb{S}_1)$. From the denseness of the range of G^* in $H^{1/2}(\partial D)$, see [KG08], it now follows that $\operatorname{Im}(\psi, T\psi)_{L^2(\partial D)} \geq 0$ for all $\psi \in H^{1/2}(\partial D)$.
(c) Assume now that $\operatorname{Im}(T\psi,\, \psi) = 0$ for a $0 \neq \psi \in H^{1/2}(\partial D)$. Since the range of G^* is dense in $H^{1/2}(\partial D)$, there exists $\{g_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{S}_1)$ such that $G^*g_j \to \psi$ as $j \to \infty$. Due to (2.35),

$$0 \leq \frac{k}{16\pi^2}\|Fg_j\|^2_{L^2(\mathbb{S}_1)} = \operatorname{Im}(TG^*g_j,\, G^*g_j)_{L^2(\mathbb{S}_1)} \to \operatorname{Im}(T\psi,\, \psi)_{L^2(\mathbb{S}_1)} = 0 \quad \text{as } j \to \infty.$$

We conclude that $Fg_j \to 0$ as $j \to \infty$ and (2.34) shows that $F^*g_j \to 0$ as well. For arbitrary $g \in L^2(\mathbb{S}_1)$ this implies that $-(G^*g,\, TG^*g_j)_{L^2(\partial D)} = (g,\, F^*g_j)_{L^2(\mathbb{S}_1)} \to 0$ as $j \to \infty$. Since $G^*g_j \to \psi$ as $j \to \infty$, it follows that $(G^*g, T\psi) = 0$ for all $g \in L^2(\mathbb{S}_1)$ and the denseness of the range of G^* shows that $T\psi = 0$.

Let now k^2 be an interior Robin eigenvalue of $-\Delta$ in D and $w \in H^1(D)$ a corresponding eigenfunction. Due to the representation theorem, w can be written as

$$w = \mathrm{SL}\bigg(\frac{\partial w}{\partial \nu}\bigg|^-\bigg) - \mathrm{DL}(w|^-) \qquad \text{in } H^1(D).$$

Since $\partial w/\partial\nu = -\tau w$ on ∂D, we find that $w = -\mathrm{SL}(\tau\, w|^-) - \mathrm{DL}(w|^-)$. Setting $\psi = w|^-$ and exploiting the jump relations (2.10)–(2.13) we obtain that

$$w|^{-} = -S(\tau\psi) + \frac{1}{2}\psi - K\psi \quad \text{in } H^{1/2}(\partial D), \qquad \frac{\partial w}{\partial\nu}\bigg|^{-} = -\frac{1}{2}\tau\psi - K'(\tau\psi) - Nw \quad \text{in } H^{-1/2}(\partial D).$$

Using these equations, we deduce that

$$\frac{\partial w}{\partial\nu}\bigg|^{-} + \tau w|^{-} = -\big[\tau S(\tau\psi) + \tau K\psi + K'(\tau\psi) + N\psi\big] = -T\psi.$$

Since w satisfies homogeneous Robin boundary conditions we obtain that $T\psi = 0$. The representation $w = -\mathrm{SL}(\tau\psi) - \mathrm{DL}\,\psi$ on the other hand implies that $\psi \neq 0$, since otherwise w would vanish in D, contradicting the assumption that w is an eigenfunction. Hence, the kernel of T is non-trivial, which implies the assertion. Note that if we assume that $T\psi = 0$ in $H^{-1/2}(\partial D)$ for some $0 \neq \psi \in H^{1/2}(\partial D)$, then the same arguments show that $w = -\mathrm{SL}(\tau\psi) - \mathrm{DL}\,\psi$ defines a Robin eigenfunction of $-\Delta$ in D.
(d) We refer to [LP14, Lemma 10] for a proof. ■

Now we want to proof that the eigenvalues λ_j of the far field operator converge to zero from one specific side. Note that the operator T fulfills all the requirements that have been mentioned in Remark 2.3. Therefore we can easily adapt the arguments from the proof of Lemma 2.2 to show such a characteristic.

Lemma 2.12. *Assume that k^2 is no interior Robin eigenvalue of $-\Delta$ in D. Then the eigenvalues λ_j of F converge to zero from the right, i.e., $\mathrm{Re}\,\lambda_j > 0$ for $j \in \mathbb{N}$ large enough.*

Note that contrary to the case with Dirichlet boundary conditions, the eigenvalues converge to zero from the opposite side. This is due to the fact the imaginary part of T is positive, whereas the imaginary part of the operator S in the previous section was negative. Let us again represent the eigenvalues λ_j of F in polar coordinates,

$$\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j), \qquad \vartheta_j \in [0, \pi].$$

Since $\mathrm{Re}\,\lambda_j > 0$ for large $j \in \mathbb{N}$, the phases ϑ_j converge to 0 as $j \to \infty$ and therefore we can define the largest phase

$$\vartheta^* = \vartheta_{j^*} = \max_{j\in\mathbb{N}} \vartheta_j$$

among all phases ϑ_j. We denote the eigenvalue corresponding to the largest phase ϑ^* as λ^*. Adapting the arguments of Theorem 2.4 and Lemma 2.5 to the different phase behavior for the Robin boundary conditions, we obtain the following characterization of the largest phase ϑ^*.

Theorem 2.13. *If k^2 is not a Robin eigenvalue of $-\Delta$ in D, then*

$$\cot\vartheta^* = \min_{g\in L^2(\mathbb{S}_1)} \frac{\mathrm{Re}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}, \tag{2.36}$$

where the minimum is attained at any eigenvector g^* corresponding to the eigenvalue λ^* of F with smallest phase.

Remark 2.14. Inserting the factorization (2.30) of the far field operator and using the denseness of the range of G^*, the equality in (2.36) can equivalently be expressed as

$$\cot\vartheta^* = \min_{\psi\in H^{1/2}(\partial D)} \frac{\mathrm{Re}\,(\psi,\, T\psi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(\psi,\, T\psi)_{L^2(\partial D)}}. \tag{2.37}$$

where the minimum is attained at $\psi = G^* g^*$.

To indicate the dependency of the relevant quantities on the wavenumber k, we write from now on again $\vartheta^* = \vartheta^*(k)$, $\mathrm{SL} = \mathrm{SL}_k$, $\mathrm{DL} = \mathrm{DL}_k$ as well as $T = T_k$. Further, we write $k \searrow k_0$ to indicate that the positive wavenumber k tends to k_0 from above, that is, $k_0 < k \to k_0$.

As the equivalent to Lemma 2.7 we will show that the derivative of T_k with respect to k is positive when it is restricted to the kernel of T_k.

Lemma 2.15. *Assume that k_0^2 is a Robin eigenvalue of $-\Delta$ in D. Then T_{k_0} has a non-trivial kernel and for all elements $\psi_0 \in H^{1/2}(\partial D)$ in this kernel it holds that $(\psi_0,\, T_{k_0}\psi_0)_{L^2(\partial D)} = 0$. The mapping $k \mapsto (\psi_0,\, T_k\psi_0)_{L^2(\partial D)}$ is differentiable at k_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}k}(\psi_0,\, T_k\psi_0)_{L^2(\partial D)}\bigg|_{k=k_0} = 2k_0 \int\limits_D |u_{k_0}|^2 \,\mathrm{d}x, \qquad \text{where } u_{k_0} = \mathrm{SL}_{k_0}(\tau\psi_0) + \mathrm{DL}_{k_0}\psi_0.$$

Proof. We have already proven in Lemma 2.11 that $\mathrm{Im}\,(\psi,\, T_k\psi)_{L^2(\partial D)} = 0$ for a non-trivial $\psi \in L^2(\partial D)$ implies that k^2 is an interior Robin eigenvalue. Define $u_k := \mathrm{SL}_k(\tau\psi_0) + \mathrm{DL}_k\,\psi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \partial D)$. In Lemma 2.7 we have shown that the single layer potential SL_k is differentiable in k. A similar calculation for the double layer potential DL_k shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{DL}_k(x) &= \frac{\mathrm{d}}{\mathrm{d}k}\int_{\partial D}\frac{\partial}{\partial\nu}\Phi(x,y)\psi_0(y)\,\mathrm{d}S(y) = \int_{\partial D}\frac{\partial}{\partial\nu}\frac{\mathrm{d}}{\mathrm{d}k}\Phi(x,y)\psi_0(y)\,\mathrm{d}S(y)\\ &= \int_{\partial D}\frac{\mathrm{i}}{4\pi}\frac{\partial}{\partial\nu}\exp(\mathrm{i}k|x-y|)\psi_0(y)\,\mathrm{d}S(y), \qquad x \in \mathbb{R}^3,\end{aligned}$$

implying that the derivative of u_k with respect to k is also well-defined in, e.g. $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \partial D)$. In particular, $u_k' := \mathrm{d}u_k/\,\mathrm{d}k \in H^1(D)$ and we can use the chain rule to obtain

$$\Delta u_k' + k^2 u_k' + 2k u_k = 0 \quad \text{in } D. \tag{2.38}$$

Since $u_k = \mathrm{SL}_k(\tau\psi_0) + \mathrm{DL}_k\,\psi_0$ one easily verifies the jump relation

$$u_k|^- - u_k|^+ = \psi_0. \tag{2.39}$$

Moreover, we have already computed in the proof of Lemma 2.11 that

$$T_k\psi_0 = \frac{\partial u_k}{\partial\nu}\bigg|^- + \tau\, u_k|^- .$$

These two relations allow to compute the derivative with respect to k of $k \mapsto (\psi_0,\, T_k\psi_0)_{L^2(\partial D)}$,

$$\frac{\mathrm{d}}{\mathrm{d}k}\big(\psi_0,\, T_k\psi_0\big)_{L^2(\partial D)} = \left(\psi_0,\, \frac{\mathrm{d}}{\mathrm{d}k}T_k\psi_0\right) = \left(\, u_k|^- - u_k|^+\,,\ \frac{\mathrm{d}}{\mathrm{d}k}\frac{\partial u_k}{\partial\nu}\bigg|^- + \tau\frac{\mathrm{d}}{\mathrm{d}k}\, u_k|^- \right)_{L^2(\partial D)}.$$

For $k = k_0$ the trace $u_{k_0}|^+$ taken from the exterior of D vanishes because k_0^2 is an interior eigenvalue. Indeed, the radiating solution u_{k_0} to the homogeneous Robin boundary value problem (2.29) vanishes outside of D and hence its trace vanishes on ∂D. Now we can apply Green's first identity for

$u_{k_0} \in H^1_0(D)$, use (2.38) and the boundary condition $\partial u_{k_0}/\partial\nu = -\tau u_{k_0}$ to compute that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(\psi_0,\, T_k\psi_0)_{L^2(\partial D)}\Big|_{k=k_0} &= \left(u_{k_0}|^- ,\, \frac{\mathrm{d}}{\mathrm{d}k}\frac{\partial u_{k_0}}{\partial\nu}\Big|^- + \tau\frac{\mathrm{d}}{\mathrm{d}k}\, u_{k_0}|^- \right)_{L^2(\partial D)} \\ &= -\int_D \left[\Delta\overline{u'_{k_0}}u_{k_0} + \nabla u_{k_0}\nabla\overline{u'_{k_0}}\right]\mathrm{d}x - \int_{\partial D}\tau\overline{u_{k_0}}'u_{k_0}|^-\,\mathrm{d}S \\ &= -\int_D \left[\Delta\overline{u'_{k_0}}u_{k_0} - \Delta u_{k_0}\overline{u'_{k_0}}\right]\mathrm{d}x - \int_{\partial D}\frac{\partial u_{k_0}}{\partial\nu}\Big|^-\overline{u'_{k_0}}\,\mathrm{d}S + \int_{\partial D}\frac{\partial u_{k_0}}{\partial\nu}\Big|^-\overline{u'_{k_0}}\,\mathrm{d}S \\ &= \int_D\left[2k_0\overline{u_{k_0}}u_{k_0} + k^2\overline{u'_{k_0}}u_{k_0} - k^2\overline{u'_{k_0}}u_{k_0}\right]\mathrm{d}x = 2k_0\int_D|u_{k_0}|^2\mathrm{d}x.\end{aligned}$$

■

Now we can state the first part of the inside-outside duality for scattering with Robin boundary conditions. As in the previous section we rely on the fact the auxiliary derivative in the last lemma is positive. Using the same arguments as in the proof of Theorem 2.8, it is easy to show that the following characterization of interior Robin eigenvalues holds.

Theorem 2.16 (Inside-Outside Duality - Part 1)**.** *Let k_0^2 be an interior Robin eigenvalue. Then it holds that* $\lim_{k\searrow k_0}\vartheta^*(k) = \pi$.

Note that unlike in the previous section, where the smallest phase converges to zero, here the largest phase ϑ^* converges to π. This is due to the fact that the imaginary part of the middle operator T is positive and the phase behavior of ϑ^* is described by a minimum instead of a maximum, taken over the expression in (2.36). The next theorem completes the inside-outside duality for scattering with Robin boundary conditions. While the first part of the inside-outside duality provides a necessary conditions for k_0^2 being an interior Robin eigenvalue, the second part states that the behavior of the largest phase ϑ^* with varying wavenumber k provides also a sufficient condition to characterize interior Robin eigenvalues.

Theorem 2.17 (Inside-Outside Duality - Part 2)**.** *Assume that $k_0 > 0$ and that $I = (k_0, k_0+\varepsilon)$ contains no k such that k^2 is a Robin eigenvalue of $-\Delta$ in D. If $\lim_{k\searrow k_0}\vartheta^*(k) = \pi$, then k_0^2 is a Robin eigenvalue of $-\Delta$ in D.*

Proof. If k_0^2 is a Robin eigenvalue, $\lim_{k\searrow k_0}\vartheta^*(k) = \pi$ follows directly from Lemma 2.16.

Assume now that $\lim_{k\searrow k_0}\vartheta^*(k) = \pi$ but that k_0^2 is no Robin eigenvalue. From Lemma 2.13 it follows that

$$\min_{\psi\in H^{1/2}(\partial D)}\frac{\mathrm{Re}\,(\psi,\, T_k\psi)_{L^2(\partial D)}}{\mathrm{Im}\,(\psi,\, T_k\psi)_{L^2(\partial D)}} \to -\infty \qquad \text{as } k\searrow k_0.$$

Hence, there is a sequence $\{k_j\}_{j\in\mathbb{N}} \subset I$ with $k_j \searrow k_0$ as $j\to\infty$ and functions $\psi_j \in H^{1/2}(\partial D)$ with $\|\psi_j\|_{H^{1/2}(\partial D)} = 1$ such that

$$0 > \mathrm{Im}\,(\psi_j,\, T_{k_j}\psi_j)_{L^2(\partial D)} \to 0 \quad \text{as } j\to\infty, \tag{2.40}$$

and such that $\mathrm{Re}\,(\psi_j,\, T_{k_j}\psi_j)_{L^2(\partial D)} > 0$ for j large enough. Since the range of G^* is dense in $H^{1/2}(\partial D)$, there exist sequences $\{g_{j,\ell}\}_{\ell\in\mathbb{N}} \subset L^2(\mathbb{S}_1)$ such that $\psi_j = \lim_{\ell\to\infty} G^*_{k_j} g_{j,\ell}$. Since the sequence $\{\psi_j\}_{j\in\mathbb{N}}$ is bounded in $H^{1/2}(\partial D)$ we can extract a weakly convergent subsequence, still denoted by ψ_j, such that $\psi_j \rightharpoonup \psi_0 \in H^{1/2}(\partial D)$. Define

$$v_j = \mathrm{DL}_{k_j}\,\psi_j + \mathrm{SL}_{k_j}(\tau\psi_j), \qquad j\in\mathbb{N}_0. \tag{2.41}$$

Since DL_{k_j} and SL_{k_j} form sequences of uniformly bounded linear operators, v_j converges weakly in $H^1(B_R \setminus \partial D)$ to $v_0 = \mathrm{DL}_{k_0}\, \psi_0 + \mathrm{SL}_{k_0}(\tau\psi_0) \in H^1(B_R \setminus \partial D)$ for $R > 0$ large enough such that $\overline{D} \subset B_R$. Due to the jump relations (2.10)–(2.13) it holds that $\partial v_j/\partial\nu|^+ + \tau\, v_j|^+ = T_{k_j}\psi_j$. Thus, the far fields of the radiating solutions v_j to the Helmholtz equation are given by

$$v_j^\infty = G_{k_j} T_{k_j} \psi_j = \lim_{\ell\to\infty} G_{k_j} T_{k_j} G_{k_j}^* g_{j,\ell} = - \lim_{\ell\to\infty} F_{k_j}^* g_{j,\ell}. \tag{2.42}$$

Since T is an isomorphism and G is compact, the mapping $\psi_j \mapsto v_j^\infty$ is compact and $v_j^\infty \to v_0^\infty \in L^2(\mathbb{S}_1)$ strongly in $L^2(\mathbb{S}_1)$. According to (2.34) we have

$$\begin{aligned} 0 < & \frac{k_j}{16\pi^2} \|F_{k_j}^* g_{j,\ell}\|_{L^2(\mathbb{S}_1)}^2 \overset{(2.34)}{=} \operatorname{Im}\left(F(k_j) g_{j,\ell},\, g_{j,\ell}\right)_{L^2(\partial D)} \overset{\ell\to\infty}{\longrightarrow} -\operatorname{Im}\left(T_{k_j}^* \psi_j,\, \psi_j\right)_{L^2(\partial D)} \\ & = -\operatorname{Im}\left(\psi_j,\, T_{k_j}\psi_j\right)_{L^2(\partial D)} \to 0 \quad \text{as } j \to \infty \text{ due to (2.40)}. \end{aligned}$$

Hence, $\lim_{\ell\to\infty} F_{k_j}^* g_{j,\ell} = v_j^\infty$ tends to zero in $L^2(\mathbb{S}_1)$ as $j \to \infty$, that is, $v_0^\infty = 0$. Rellich's lemma implies that v_0 vanishes in $\mathbb{R}^3 \setminus \overline{D}$. Moreover, k_0^2 is no Robin eigenvalue, that is, v_0 vanishes everywhere. The jump relations (2.10)–(2.13) imply that $\psi_0 = 0$ must vanish, too, that is, $\psi_j \rightharpoonup 0$ in $H^{1/2}(\partial D)$.

We will now show that v_j converges strongly to zero in $H^1(B_R \setminus \partial D)$. First we note that, up to extraction of a subsequence, $\tau\psi_j$ converges weakly to zero in $L^2(\partial D)$ and therefore strongly to zero in $H^{-1/2}(\partial D)$. Thus, $\mathrm{SL}_{k_j}(\tau\psi_j)$ also converges strongly to zero in $H^1(B_R \setminus \partial D)$. Second, we show that $\mathrm{DL}_{k_j}\, \psi_j$ converges strongly to zero in $H^1(B_R \setminus \partial D)$, too (the weak convergence to zero is clear). To this end, let us recall from the proof of Lemma 2.11 that T_{k_j} can be written as $T_{k_j} = N_{k_j} + C_{k_j}$ with a compact operator C_{k_j}. Thus,

$$\operatorname{Re}\left(\psi_j,\, T_{k_j}\psi_j\right)_{L^2(\partial D)} = \operatorname{Re}\left(\psi_j,\, N_{k_j}\psi_j\right)_{L^2(\partial D)} + \operatorname{Re}\left(\psi_j,\, C_{k_j}\psi_j\right)_{L^2(\partial D)}.$$

Since $\psi_j \rightharpoonup 0$ in $H^{1/2}(\partial D)$, the sequence $C_{k_j}\psi_j$ converges strongly in $H^{-1/2}(\partial D)$ to $C(k_0)\psi_0 = 0$. Setting $v_j' = \mathrm{DL}_{k_j}\, \psi_j$, Green's first identity shows that

$$\begin{aligned} \operatorname{Re}\left(\psi_j,\, T_{k_j}\psi_j\right)_{L^2(\partial D)} = & - \int_{B_R\setminus\partial D} \left[|\nabla v_j'|^2 - k_j^2 |v_j'|^2 \right] \mathrm{d}x \\ & + \operatorname{Re}\left(\psi_j,\, C_{k_j}\psi_j\right)_{L^2(\partial D)} + \operatorname{Re} \int_{\partial B_R} \frac{\partial v_j'}{\partial\nu} \overline{v_j'}\, \mathrm{d}S. \end{aligned}$$

The last surface integral tends to zero as $j \to \infty$ since $\psi_j \rightharpoonup 0$ and since both mappings $\psi_j \mapsto v_j'|_{\partial B_R}$ and $\psi_j \mapsto \partial v_j'/\partial\nu|_{\partial B_R}$ are compact due to elliptic regularity results. Exploiting the positivity of $\operatorname{Re}\left(\psi_j,\, T_{k_j}\psi_j\right)_{L^2(\partial D)} > 0$ for $j \in \mathbb{N}$ large enough yields that

$$\int_{B_R\setminus\partial D} |\nabla v_j'|^2\, \mathrm{d}x \leq \int_{B_R\setminus\partial D} |v_j'|^2\, \mathrm{d}x \qquad \text{for } j \in \mathbb{N} \text{ large enough.}$$

Since $v_j' = \mathrm{DL}_{k_j}\, \psi_j$ converges weakly to zero in $H^1(B_R \setminus \partial D)$, this series of functions converges strongly to zero in $L^2(B_R \setminus \partial D)$ and from the last inequality we get that $v_j' = \mathrm{DL}_{k_j}\, \psi_j$ converges even strongly in $H^1(B_R \setminus \partial D)$. Now it follows that $v_j = \mathrm{DL}_{k_j}\, \psi_j + \mathrm{SL}_{k_j}(\tau\psi_j)$, defined in (2.41), converges strongly to $0 = v_0 = \mathrm{DL}_{k_0}\, \psi_0 + \mathrm{SL}_{k_0}(\tau\psi_0)$ in $H^1(B_R \setminus \partial D)$. The jump relation (2.39) for the combined single- and double-layer potential implies that $\psi_0 = v_0|^- - v_0|^+ = 0$. Hence, $\psi_j \to 0$ strongly in $H^{1/2}(\partial D)$ as $j \to \infty$. This, however, contradicts our assumption $\|\psi_j\|_{H^{1/2}(\partial D)} = 1$. ■

2.4. Characterizing Dirichlet Eigenvalues from Near Field Data

In this section we derive the inside-outside duality for near field data that arises from the exterior Dirichlet scattering problem. In the last sections, we worked with the properties of the far field operator. Naturally our main focus in this section therefore lies on the properties of the near field operator N_R from (2.7). Recall its definition

$$N_R : L^2(\mathbb{S}_R) \to L^2(\mathbb{S}_R), \qquad N_R g(x) = \int_{\mathbb{S}_R} u^s(x,y) g(y) \,\mathrm{d}S(y), \qquad x \in \mathbb{S}_R,$$

where $u^s(\cdot, y)$ is the scattered field that arises from the solution of the exterior Dirichlet problem (2.14) when the incident plane wave is the point source $\Phi(\cdot, y)$ at $y \in \mathbb{R}^3 \setminus \overline{D}$, see (2.4).

Remark 2.18. The near field operator could also be defined on more general surfaces. For example if $\Gamma \subset \mathbb{R}^3 \setminus \overline{D}$ denotes the boundary of an arbitrary Lipschitz domain $\Omega_\Gamma \Supset D$ with connected complement, then by replacing the sphere $\mathbb{S}_R$ by surface Γ in the definition of N in (2.7), we obtain a more general form of the near field operator. However it has been shown in [LP15b, Section 3] that the generalized near field operator can be related to the near field operator defined in (2.7), such that it suffices to consider the latter one.

We start by discussing the properties of the near field operator, some of which have already been mentioned in the introduction to this chapter. Standard regularity results for elliptic differential equations as in [McL00] show that the scattered field is smooth in the exterior of the scattering object, i.e. $u^s(\cdot, y) \in C^\infty(\mathbb{R}^3 \setminus \overline{D})$ and since the reciprocity relation $u^s(x,y) = u^s(y,x)$ holds for $x \neq y \in \mathbb{R}^3 \setminus \overline{D}$, we have that the kernel $u^s(x,y)$ of N_R belongs to $C^\infty(\mathbb{S}_R \times \mathbb{S}_R)$. This implies in particular, that the near field operator is a compact linear operator on $L^2(\mathbb{S}_R)$. However, our numerical experiments indicate that the near field operator is not normal and therefore it is unclear if it possesses any eigenvalues at all. If the scattering object D is the unit ball B_1, analytically calculating the eigenvalues of the near field operator also shows that the eigenvalues do not show any particular structure at all, even if they exist, see Figure 2.2.

Moreover there is no factorization of the near field operator that has the necessary attributes for the inside-outside duality. For example, one could factorize the near field operator in terms of the single layer boundary operator and the corresponding potential by

$$N_R = \mathrm{SL}_{\partial D}|_{\mathbb{S}_R} S^{-1} \mathrm{SL}_{\mathbb{S}_R}|_{\partial D}.$$

However the outer operators in the factorization are not adjoint to each other, which is why we cannot use this factorization to prove the inside-outside duality as we have done in the previous sections. To deal with all these problems, we will not directly work with the near field operator but use the ansatz from [HYZZ14] and modify the near field operator by adding a unitary operator, such that the modification has a useful factorization. For the modification of the near field operator, we use that it is defined for spherical measurements in order to define a unitary operator T_R such that the composition $T_R N_R$ possesses a factorization that is suitable to derive a version of the inside-outside duality. First we will keep the wavenumber fixed and indicate the dependency of quantities on the radius R whenever necessary.

From now on we proceed in the following way. First we will derive the modifying unitary operator T_R, show that the operator $T_R N_R$ possesses infinitely many eigenvalues and obtain a factorization for this operator in (2.50). Then we will use this factorization to derive a relation between the far field operator F and the modified near field operator $T_R N_R$ in Theorem 2.28. In a next step we use this relation to obtain the first part of the inside-outside duality in Corollary 2.30. Finally we will use the concept of the numerical range to obtain a full characterization of interior Dirichlet eigenvalues in Corollary 2.33 and thereby completing the inside-outside duality for near field data. We start by deriving the modifying operator T_R.

Since the near field operator N_R in (2.7) takes functions as arguments whose domain is a sphere with radius R, we want to represent those functions in terms of their basis functions. We recall that the spherical harmonics $\{Y_n^m :\ n \in \mathbb{N}_0,\ -n \le m \le n\}$ form a complete orthogonal basis of the space $L^2(\mathbb{S}_R)$ of square-integrable functions on the sphere $\mathbb{S}_R$ for arbitrary $R > 0$, that is, every function $g \in L^2(\mathbb{S}_R)$ expands as

$$g(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_n^m Y_n^m(\hat{x}), \quad \text{where} \quad g_n^m = \frac{1}{R^2} \int_{\mathbb{S}_R} g(x) \overline{Y_n^m(\hat{x})} \, \mathrm{d}S \quad \text{and} \quad \hat{x} = \frac{x}{|x|}. \tag{2.43}$$

Using this expansion we define $\mathcal{P}_R :\ L^2(\mathbb{S}_R) \to \ell^2$ by

$$\mathcal{P}_R(g) = \boldsymbol{g}, \qquad \boldsymbol{g} = \{g_n^m :\ n \in \mathbb{N}_0,\ |m| \le n\} \in \ell^2. \tag{2.44}$$

(For simplicity, we do not explicitly introduce the corresponding index set of the sequence space ℓ^2.) Its inverse $\mathcal{P}_R^{-1} :\ \ell^2 \to L^2(\mathbb{S}_R)$ is then given by $\mathcal{P}_R^{-1}(\boldsymbol{g}) = \sum_{n,m} g_n^m Y_n^m$. Writing I_{ℓ^2} and $I_{L^2(\mathbb{S}_R)}$ for the identity operators on ℓ^2 and $L^2(\mathbb{S}_R)$, respectively, it is easy to compute that

$$\mathcal{P}_R \mathcal{P}_R^{-1} = I_{\ell^2}, \quad \mathcal{P}_R^{-1} \mathcal{P}_R = I_{L^2(\mathbb{S}_R)}, \quad \mathcal{P}_R^* = \frac{1}{R^2} \mathcal{P}_R^{-1}, \quad \text{and} \quad (\mathcal{P}_R^{-1})^* = R^2 \mathcal{P}_R.$$

We use $\mathcal{P}_R$ to transform both the far field operator F and the near field operator N_R into operators acting on the sequence space ℓ^2 by defining

$$\mathcal{F} = \mathcal{P}_1 F \mathcal{P}_1^{-1} \qquad \text{and} \qquad \mathcal{N}_R = \mathcal{P}_R N_R \mathcal{P}_R^{-1}. \tag{2.45}$$

Thus, both $\mathcal{F}$ and $\mathcal{N}_R$ are compact operators on ℓ^2 representing F and N_R in the orthogonal basis of spherical harmonics. Since any solution u to the exterior Dirichlet scattering problem (2.16) with boundary datum f can be expressed in terms of the spherical Hankel functions $h_n^{(1)}$ on any sphere $\mathbb{S}_\rho$ such that $D \Subset B_\rho$,

$$u(x)|_{\mathbb{S}_\rho} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m(f) h_n^{(1)}(k\rho) Y_n^m(\hat{x}) \qquad \text{with coefficients } b_n^m(f) \in \mathbb{C}, \tag{2.46}$$

the asymptotic expansion of the Hankel function $h_n^{(1)}$ for large arguments shows that the corresponding far field pattern is given by

$$u^\infty(\hat{x}) = \frac{1}{k} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{i^{n+1}} b_n^m(f) Y_n^m(\hat{x}).$$

The lifting $\mathcal{N}_R$ of N_R into ℓ^2 now allows to modify the latter operator such that it possesses a factorization where the outer operators are adjoint to each other: Following the trick from [HYZZ14], we define the unitary operator $\mathcal{T}_R :\ \ell^2 \to \ell^2$ by

$$\mathcal{T}_R \boldsymbol{g} = \left\{ -\frac{\overline{h_n^{(1)}(kR)}}{h_n^{(1)}(kR)} g_n^m :\ n \in \mathbb{N}_0,\ |m| \le n \right\}, \tag{2.47}$$

and the compact and linear operator $\mathcal{G}_R:\ H^{1/2}(\partial D)\to\ell^2$ by

$$\mathcal{G}_R(f)=\left\{b_n^m(f)h_n^{(1)}(kR):\ n\in\mathbb{N}_0,\ |m|\le n\right\},\tag{2.48}$$

where the coefficients $b_n^m(f)$ are defined in (2.46). The operator $\mathcal{T}_R$ is well-defined since the spherical Hankel function cannot vanish for positive arguments and $\mathcal{G}_R$ is compact, injective and has dense range in ℓ^2 (see [HYZZ14, Lemma 3.5]). Note that $\mathcal{P}_R^{-1}\mathcal{G}_R f$ is the evaluation of the solution to (2.16) on $\mathbb{S}_R$. Moreover, [HYZZ14, Equation (3.7)] shows that the modified near field operator $\mathcal{T}_R\mathcal{N}_R$ can be factorized as

$$\mathcal{T}_R\mathcal{N}_R=-R^2\,(\mathcal{T}_R\mathcal{G})S^*(\mathcal{T}_R\mathcal{G}_R)^*,\qquad \text{i.e.,}\qquad \mathcal{T}_R\mathcal{P}_R N_R\mathcal{P}_R^{-1}=-R^2(\mathcal{T}_R\mathcal{G}_R)S^*(\mathcal{T}_R\mathcal{G}_R)^*.\tag{2.49}$$

Lifting $\mathcal{T}_R$ back into the space $L^2(\mathbb{S}_R)$ yields $T_R=\mathcal{P}_R^{-1}\mathcal{T}_R\mathcal{P}_R$, a unitary operator on $L^2(\mathbb{S}_R)$ and the factorization in (2.49) directly shows that $T_RN_R:\ L^2(\mathbb{S}_R)\to L^2(\mathbb{S}_R)$ factorizes into

$$T_RN_R=-G_RS^*G_R^*,\qquad \text{where } G_R=\mathcal{P}_R^{-1}\mathcal{T}_R\mathcal{G}_R.\tag{2.50}$$

This factorization hence features adjoint outer operators due to the replacement of $\mathcal{P}_R^{-1}\mathcal{G}_R f$, evaluating the solution to (2.16) on $\mathbb{S}_R$, by $G_R=\mathcal{P}_R^{-1}\mathcal{T}_R\mathcal{G}_R$, which conjugates the spherical Hankel functions in (2.48) before evaluation. The above-mentioned properties of $\mathcal{G}_R$ clearly imply that $G_R:\ H^{1/2}(\partial D)\to L^2(\mathbb{S}_R)$ is compact, injective and has dense range in $L^2(\mathbb{S}_R)$.

We will later on use the factorization (2.50) to examine the structure of T_RN_R more closely. Prior to that, we show that the latter operator has infinitely many eigenvalues, following a technique from [CK95].

Lemma 2.19. *The operator T_RN_R has an infinite number of eigenvalues tending to zero.*

Proof. We restrict T_RN_R to an operator mapping the orthogonal complement $\ker(N_R)^\perp\subset L^2(\mathbb{S}_R)$ of its kernel $\ker(N_R)$ into the closure of its range $\overline{\mathrm{Rg}(T_RN_R)}\subset L^2(\mathbb{S}_R)$ by defining $A:\ \ker(N_R)^\perp\to\overline{\mathrm{Rg}(T_RN_R)}\subset L^2(\mathbb{S}_R)$ by $Ag=T_RN_Rg$. As T_R is unitary, A is hence injective and has dense range. Moreover, the factorization (2.50) implies that T_RN_R is compact since G_R is compact, such that A is compact, too. We next show that $\ker(N_R)$ is finite-dimensional, to conclude that the range of A is infinite-dimensional, too, due to injectivity of A:

If $N_Rg=0$ for some $g\neq 0$, then the radiating solution u to (2.16) for $f=\mathrm{SL}_{\mathbb{S}_R}g|_{\partial D}$ vanishes on $\mathbb{S}_R$, and hence entirely in $\mathbb{R}^3\setminus\overline{D}$, due to the radiation condition and Rellich's lemma. Thus, the single-layer potential $\mathrm{SL}_{\mathbb{S}_R}g$ vanishes on ∂D, such that $v=\mathrm{SL}_{\mathbb{S}_R}g|_D\in H_0^1(D)$ defines an Dirichlet eigenfunction of the (negative) Laplacian for the eigenvalue k^2. As the corresponding eigenspace is finite-dimensional due to Fredholm theory, there can at most exist a finite number of linearly independent g generating such eigenfunctions; consequently, $\ker(N_R)$ is finite-dimensional.

We next define the subspace of principle functions of A by

$$P(A)=\mathrm{span}\left\{g\in\ker(N)^\perp:\ (\mu\,\mathrm{Id}-A)^ng=0\text{ for some }n\in\mathbb{N}\text{ and }\mu\in\mathbb{C}\right\}\subset L^2(\mathbb{S}_R).$$

Assume for a moment that A is a trace class operator and that $\operatorname{Im}A\ge 0$, i.e., that the non-selfadjoint part of A is non-negative. Due to [Rin71, Theorem 3.5.1], these two properties imply that $\overline{\mathrm{Rg}(A)}=\overline{P(A)}$. We showed above that $\mathrm{Rg}(A)$ has infinite dimension and conclude that there exist infinitely many linearly independent principle functions. As for each principle function there exists an associated eigenvalue and an eigenfunction due to Riesz theory, see [Kre99], the infinitely many linearly independent principle functions guarantee the existence of infinitely many eigenfunctions of A. By definition of A, any eigenpair (μ,g) satisfies $\mu g=Ag=T_RN_Rg$ and hence also T_RN_R possesses infinitely many eigenvalues. Since T_RN_R is compact, these eigenvalues tend to zero.

It remains to show that A is a trace class operator and that $\mathrm{Im}\,(A) \geq 0$. The second property follows immediately from the factorization (2.50), since for any $g \in L^2(\mathbb{S}_R)$ it holds that

$$\mathrm{Im}\,(Ag,\, g)_{L^2(\mathbb{S}_R)} = \mathrm{Im}\,(T_R N_R g,\, g)_{L^2(\mathbb{S}_R)} = -\mathrm{Im}\,(S^* G_R^* g,\, G_R^* g)_{L^2(\partial D)} \geq 0,$$

where we exploited that the non-selfadjoint part $\mathrm{Im}\,S$ of the single layer operator S is non-negative, see [KG08, Lemma 1.14]. Since T_R is unitary, it is further sufficient to show that N_R is a trace class operator to prove this property for $T_R N_R$. For N_R, this is essentially due to the smoothness of its kernel $(x, y) \mapsto u^s(x, y) \in C^\infty(\mathbb{S}_R \times \mathbb{S}_R)$, since this smoothness implies that N_R is a bounded linear operator from $L^2(\mathbb{S}_R)$ into any Sobolev space $H^s(\mathbb{S}_R)$ for arbitrary $s \in \mathbb{R}$. Choosing $s > 2$ implies that the embedding of $H^s(\mathbb{S}_R)$ in $L^2(\mathbb{S}_R)$ is a trace class operator, see [Gra68], and finally proves that N_R itself is a trace class operator on $L^2(\mathbb{S}_R)$. ∎

The following corollary shows that any eigenvalue of $T_R N_R$ is contained in the upper half of the complex plane.

Corollary 2.20. *If k^2 is no Dirichlet eigenvalue of $-\Delta$ in D, then all eigenvalues of $T_R N_R$ are contained in the upper half $\{z \in \mathbb{C} :\ \mathrm{Im}\,(z) > 0\}$ of the complex plane; if k^2 is a Dirichlet eigenvalue, they are contained in $\{z \in \mathbb{C} :\ \mathrm{Im}\,(z) > 0\} \cup \{0\}$.*

Proof. If μ is an eigenvalue corresponding to a normalized eigenfunction g, then we compute as in the proof of Lemma 2.19 that

$$\mathrm{Im}\,(\mu) = \mathrm{Im}\,(T_R N_R g,\, g)_{L^2(\mathbb{S}_R)} = -\mathrm{Im}\,(S^* G_R^* g,\, G_R^* g)_{L^2(\partial D)} \geq 0$$

due to the factorization (2.50) of $T_R N_R$ and the property $\mathrm{Im}\,(S^* f,\, f)_{L^2(\partial S)} \leq 0$ for any $f \in H^{-1/2}(\partial D)$ of the single-layer operator S, see Lemma 2.1. The latter expression can only vanish if either $f = 0$ or else if k^2 is a Dirichlet eigenvalue of D and f is the normal derivative of a Dirichlet eigenfunction. ∎

For the remainder of this work, we fix a radius R and henceforth neglect the subscript R for better readability, such that, e.g., N_R and $T_R N_R$ become N and TN, respectively.

Since our goal is to prove an inside-outside duality for near field data relying on a corresponding duality for far field data, we derive a connection between the far field operator F and the modified near field operator TN. For this purpose we introduce a mapping $\mathcal{Z}$, which is later on used to relate far fields to near fields. For $\boldsymbol{g} = \{g_n^m :\ n \in \mathbb{N}_0,\, |m| \leq n\} \in \ell^2$, let

$$\mathcal{Z}\boldsymbol{g} = \Big\{ -k\mathrm{i}^{n+1}\overline{h_n^{(1)}(kR)}g_n^m :\ n \in \mathbb{N}_0,\, |m| \leq n \Big\}.$$

This map is unbounded on ℓ^2 since $n \mapsto |h_n^{(1)}(kR)|$ is an unbounded sequence, such that we restrict $\mathcal{Z}$ to its domain

$$\mathrm{dom}(\mathcal{Z}) = \{\boldsymbol{g} \in \ell^2 :\ \|\mathcal{Z}\boldsymbol{g}\|_{\ell^2} < \infty\}.$$

Then $\mathcal{Z} : \ell^2 \supset \mathrm{dom}(\mathcal{Z}) \to \ell^2$ is a well-defined unbounded linear operator.

Remark 2.21. The domain $\mathrm{dom}(\mathcal{Z})$ contains precisely those sequences $\boldsymbol{g} = (g_n^m)$ such that

$$v(x) = k \sum_{n \in \mathbb{N}_0} \mathrm{i}^{n+1} \sum_{m=-n}^{n} g_n^m h_n^{(1)}(k|x|) Y_n^m(\hat{x}), \qquad |x| > R,$$

is a radiating solution to the Helmholtz equation with trace in $L^2(\mathbb{S}_R)$, see [CK13, Theorem 2.17].

Lemma 2.22. *The domain $\mathrm{dom}(\mathcal{Z})$ is dense in ℓ^2, that is, $\overline{\mathrm{dom}(\mathcal{Z})} = \ell^2$.*

Proof. To show that the space $\operatorname{dom}(\mathcal{Z})$ is dense in ℓ^2, we choose an arbitrary $\boldsymbol{g} \in \ell^2$ and define

$$\boldsymbol{g}_M = \begin{cases} g_n^m & \text{for } n \leq M,\ |m| \leq n, \\ 0 & \text{else.} \end{cases}$$

Clearly, $\boldsymbol{g}_M \in \operatorname{dom}(\mathcal{Z})$ for all $M \in \mathbb{N}$. Furthermore, for every $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that $\|\boldsymbol{g} - \boldsymbol{g}_M\|_{\ell^2} < \varepsilon$. This concludes the proof. ■

The last lemma implies that the operator $\mathcal{Z} : \ell^2 \supset \operatorname{dom}(\mathcal{Z}) \to \ell^2$ is densely defined in ℓ^2. We next prove further properties of $\mathcal{Z}$ and its adjoint $\mathcal{Z}^* : \ell^2 \supset \operatorname{dom}(\mathcal{Z}^*) \to \ell^2$, before we exploit these operators in Theorem 2.25 to establish a connection between the lifted far- and near field operator $\mathcal{F}$ and $\mathcal{TN}$, defined in (2.45).

Lemma 2.23. *The operator $\mathcal{Z} : \ell^2 \supset \operatorname{dom}(\mathcal{Z}) \to \ell^2$ and its adjoint $\mathcal{Z}^* : \ell^2 \supset \operatorname{dom}(\mathcal{Z}^*) \to \ell^2$ are one-to-one and onto and* $\operatorname{dom}(\mathcal{Z}) = \operatorname{dom}(\mathcal{Z}^*)$. *Their inverse operators $\mathcal{Z}^{-1} : \ell^2 \to \ell^2$ and $(\mathcal{Z}^*)^{-1} : \ell^2 \to \ell^2$ are bounded and even compact on ℓ^2 with ranges* $\operatorname{Rg}(\mathcal{Z}^{-1}) = \operatorname{Rg}((\mathcal{Z}^*)^{-1}) = \operatorname{dom}(\mathcal{Z})$.

Proof. The domain of $\mathcal{Z}^*$ consists of those $\boldsymbol{f} = (f_n^m) \in \ell^2$ for which there is a $\boldsymbol{f}^* \in \ell^2$ such that

$$(\mathcal{Z}\boldsymbol{g}, \boldsymbol{f})_{\ell^2} = (\boldsymbol{g}, \boldsymbol{f}^*)_{\ell^2} \qquad \text{for all } \boldsymbol{g} \in \operatorname{dom}(\mathcal{Z}),$$

or, equivalently, such that

$$-k \sum_{n \in \mathbb{N}_0} \sum_{m=-n}^{n} \mathrm{i}^{n+1} \overline{h_n^{(1)}(kR)} g_n^m \overline{f_n^m} = (\boldsymbol{g}, \boldsymbol{f}^*)_{\ell^2} \qquad \text{for all } \boldsymbol{g} \in \operatorname{dom}(\mathcal{Z}),$$

which implies that $\boldsymbol{f}^* = \{k\,(-\mathrm{i})^{n+1} h_n^{(1)}(kR) f_n^m :\ n \in \mathbb{N}_0,\ |m| \leq n\}$. In particular, $\boldsymbol{f}^*$ exists in ℓ^2 if and only if $\boldsymbol{f} \in \operatorname{dom}(\mathcal{Z})$ and the adjoint adjoint $\mathcal{Z}^* : \operatorname{dom}(\mathcal{Z}^*) \to \ell^2$, defined by $\mathcal{Z}^*\boldsymbol{f} = \boldsymbol{f}^*$, has the same domain as $\mathcal{Z}$. To show that $\mathcal{Z}$ is onto, let $\boldsymbol{f} \in \ell^2$ be arbitrary and set

$$\boldsymbol{g} = \left\{ -\frac{1}{k \mathrm{i}^{n+1} \overline{h_n^{(1)}(kR)}} f_n^m :\ n \in \mathbb{N}_0,\ |m| \leq n \right\}.$$

Clearly $\boldsymbol{g} \in \ell^2$ and $\mathcal{Z}\boldsymbol{g} = \boldsymbol{f}$. For injectivity, we simply note that $\mathcal{Z}$ is a diagonal operator with non-trivial entries. The inverse operator $\mathcal{Z}^{-1} : \ell^2 \to \operatorname{dom}(\mathcal{Z}) \subset \ell^2$ is given by

$$\mathcal{Z}^{-1}\boldsymbol{g} = \left\{ -\frac{1}{k \mathrm{i}^{n+1} \overline{h_n^{(1)}(kR)}} g_n^m :\ n \in \mathbb{N}_0,\ |m| \leq n \right\}.$$

This operator is bounded, since for any $\boldsymbol{g} \in \ell^2$ it holds that

$$\|\mathcal{Z}^{-1}\boldsymbol{g}\|_{\ell^2}^2 = \frac{1}{k^2} \sum_{n \in \mathbb{N}} \sum_{m=-n}^{n} |h_n^{(1)}(kR)|^{-2} |g_n^m|^2 \leq c \sum_{n \in \mathbb{N}} \sum_{m=-n}^{n} |g_n^m|^2 = c\|\boldsymbol{g}\|_{\ell^2}^2,$$

because $|h_n^{(1)}(kR)|^{-2} \to 0$ for $n \to \infty$. As $\mathcal{Z}^{-1}$ is a diagonal operator with entries converging to zero, compactness of $\mathcal{Z}^{-1}$ follows from Cantor's diagonal argument. Bijectivity and compactness of $(\mathcal{Z}^*)^{-1}$ follow analogously. ■

Lemma 2.24. *Assume that $\mathcal{A} : H^{1/2}(\partial D) \to \ell^2$ is a bounded linear operator such that* $\operatorname{Rg}(\mathcal{A}) \subset \operatorname{dom}(\mathcal{Z})$ *and such that $\mathcal{ZA} : H^{1/2}(\partial D) \to \ell^2$ is also a bounded operator. Then it holds that* $\operatorname{dom}((\mathcal{ZA})^*) \supset \operatorname{dom}(\mathcal{A}^*\mathcal{Z}^*)$ *and $(\mathcal{ZA})^*\boldsymbol{g} = \mathcal{A}^*\mathcal{Z}^*\boldsymbol{g}$ for all $\boldsymbol{g} \in \operatorname{dom}(\mathcal{Z}^*)$.*

Proof. Since $\mathrm{dom}(\mathcal{A}^*\mathcal{Z}^*) = \mathrm{dom}(\mathcal{Z}^*) = \mathrm{dom}(\mathcal{Z})$ and $\mathrm{dom}\left((\mathcal{Z}\mathcal{A})^*\right) = \ell^2$, it follows that $\mathrm{dom}(\mathcal{A}^*\mathcal{Z}^*) \subset \mathrm{dom}\left((\mathcal{Z}\mathcal{A})^*\right)$. If $\boldsymbol{g} \in \mathrm{dom}(\mathcal{Z}^*)$, then for all $f \in H^{1/2}(\partial D)$ we have that

$$((\mathcal{Z}\mathcal{A})^*\boldsymbol{g}, f)_{L^2(\partial D)} = (\boldsymbol{g}, \mathcal{Z}\mathcal{A}f)_{\ell^2} = (\mathcal{Z}^*\boldsymbol{g}, \mathcal{A}f)_{\ell^2} = (\mathcal{A}^*\mathcal{Z}^*\boldsymbol{g}, f)_{L^2(\partial D)},$$

which proves the assertion. ■

Now we link the lifted far- and near field operators $\mathcal{F}$ and $\mathcal{N}$ with each other.

Theorem 2.25. *For all $\boldsymbol{g} \in \mathrm{dom}(\mathcal{Z}^*)$ it holds that $\mathcal{T}\mathcal{N}\boldsymbol{g} = R^2\, \mathcal{Z}\mathcal{F}\mathcal{Z}^*\boldsymbol{g}$ and for all $\boldsymbol{g} \in \ell^2$ it holds that $\mathcal{F}\boldsymbol{g} = R^{-2}\, \mathcal{Z}^{-1}\mathcal{T}\mathcal{N}(\mathcal{Z}^{-1})^*\boldsymbol{g}$.*

Proof. Recall the factorization $F = -G_\infty S^* G_\infty^*$ in (2.15), where we changed notation for this section. In a first step, we lift the operators from this factorization to the sequence space. To this end, we define an operator $\mathcal{G}_\infty : H^{1/2}(\partial D) \to \ell^2$ by

$$\mathcal{G}_\infty(f) = \left\{ \frac{1}{k\mathrm{i}^{n+1}} b_n^m(f) :\ n \in \mathbb{N}_0,\ |m| \le n \right\}, \tag{2.51}$$

where $b_n^m(f)$ are the coefficients from the expansion (2.46). Then $G_\infty f = \mathcal{P}_1^{-1}\mathcal{G}_\infty(f)$ holds for all $f \in H^{1/2}(\partial D)$ and $G_\infty^* = \mathcal{G}_\infty^*(\mathcal{P}_1^{-1})^* = \mathcal{G}_\infty^*\mathcal{P}_1$. Thus, the far field operator can be written as

$$F = -G_\infty S^* G_\infty^* = -\mathcal{P}_1^{-1}\mathcal{G}_\infty S^* \mathcal{G}_\infty^* \mathcal{P}_1$$

and, in particular,

$$\mathcal{F} = -\mathcal{G}_\infty S^* \mathcal{G}_\infty^*. \tag{2.52}$$

Next recall the factorization from (2.49),

$$\mathcal{T}\mathcal{N} = -R^2\ (\mathcal{T}\mathcal{G})S^*(\mathcal{T}\mathcal{G})^*, \tag{2.53}$$

where

$$\mathcal{T}\mathcal{G}(f) = \left\{ -b_n^m(f)\overline{h_n^{(1)}(kR)} :\ n \in \mathbb{N}_0,\ |m| \le n \right\}. \tag{2.54}$$

Comparing this to (2.51) yields $\mathcal{Z}\mathcal{G}_\infty = \mathcal{T}\mathcal{G}$ and by Lemma 2.24 we get $(\mathcal{T}\mathcal{G})^*\boldsymbol{g} = \mathcal{G}_\infty^*\mathcal{Z}^*\boldsymbol{g}$ for all $\boldsymbol{g} \in \mathrm{dom}(\mathcal{Z}^*)$. Inserting this equation into (2.53), we obtain

$$\mathcal{T}\mathcal{N}\boldsymbol{g} = -R^2\ \mathcal{Z}\mathcal{G}_\infty S^*\mathcal{G}_\infty^*\mathcal{Z}^*\boldsymbol{g} = R^2\ \mathcal{Z}\mathcal{F}\mathcal{Z}^*\boldsymbol{g}.$$

Finally setting $\mathcal{G}_\infty = \mathcal{Z}^{-1}\mathcal{T}\mathcal{G}$ and substituting $\mathcal{G}^\infty$ into (2.52) yields the second factorization of the theorem. ■

To establish a connection between TN and F, we first lift the operator $\mathcal{Z}$ into $L^2(\mathbb{S}_R)$,

$$Z :\ L^2(\mathbb{S}_1) \supset \mathrm{dom}(Z) = \left\{\mathcal{P}_1^{-1}\boldsymbol{g} :\ \boldsymbol{g} \in \mathrm{dom}(\mathcal{Z})\right\} \to L^2(\mathbb{S}_R), \qquad Z = \mathcal{P}_R^{-1}\mathcal{Z}\mathcal{P}_1.$$

The adjoint Z^* of Z is characterized as follows,

$$Z^* :\ L^2(\mathbb{S}_R) \supset \mathrm{dom}(Z^*) = \left\{\mathcal{P}_R^{-1}\boldsymbol{g} :\ \boldsymbol{g} \in \mathrm{dom}(\mathcal{Z}^*)\right\} \to L^2(\mathbb{S}_1), \qquad Z^* = \mathcal{P}_1^{-1}\mathcal{Z}\mathcal{P}_R.$$

Since $\mathcal{P}_1$ and $\mathcal{P}_R^{-1}$ are isomorphisms, we obtain the following corollaries from Lemmas 2.22 and 2.23 and Theorem 2.25.

Corollary 2.26. *It holds that $\overline{\mathrm{dom}(Z)} = L^2(\mathbb{S}_1)$ and $\overline{\mathrm{dom}(Z^*)} = L^2(\mathbb{S}_R)$.*

Corollary 2.27. *The operators* $Z : L^2(\mathbb{S}_1) \supset \mathrm{dom}(Z) \to L^2(\mathbb{S}_R)$ *and its adjoint* $Z^* : L^2(\mathbb{S}_R) \supset \mathrm{dom}(Z^*) \to L^2(\mathbb{S}_1)$ *are one-to-one and onto with bounded and compact inverse* $Z^{-1} : L^2(\mathbb{S}_R) \to L^2(\mathbb{S}_1)$ *and* $(Z^*)^{-1} : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_R)$*, respectively. The ranges of their inverses are* $\mathrm{Rg}(Z^{-1}) = \mathrm{dom}(Z)$ *and* $\mathrm{Rg}((Z^*)^{-1}) = \mathrm{dom}(Z^*)$.

Theorem 2.28. *For all* $g \in \mathrm{dom}(Z^*)$ *it holds that* $TNg = R^2\ ZFZ^*g$*, whereas for all* $g \in L^2(\mathbb{S}_1)$ *it holds that* $Fg = R^{-2}\ Z^{-1}TN(Z^{-1})^*g$.

Proof. One easily computes that

$$TNg = \mathcal{P}_R^{-1}\mathcal{T}\mathcal{N}\mathcal{P}_R g = \mathcal{P}_R^{-1}\mathcal{T}\mathcal{N}\boldsymbol{g} = R^2\ \mathcal{P}_R^{-1}\mathcal{Z}\mathcal{F}_1\mathcal{Z}^*\boldsymbol{g} = R^2\ \mathcal{P}_R^{-1}\mathcal{Z}\mathcal{F}_1\mathcal{Z}^*\boldsymbol{g} = R^2\ ZFZ^*g,$$

and a similar calculation yields the representation of Fg. ■

After these preliminary considerations, we will now state and prove the main result of this section on the characterization of interior Dirichlet eigenvalues of the scatterer D via the smallest phase in the numerical range of TN, thus proving an inside-outside duality for near field data. We have already shown in Corollary 2.20 that all eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ of TN lie in the upper half of the complex plane. Recall the representation of the eigenvalues $\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j)$ of the far field operator F in polar coordinates, such that there exists an eigenvalue λ_* with a smallest phase $\vartheta_* = \min_{j\in\mathbb{N}} \vartheta_j$. In this section we sort these eigenvalues in descending order according to their magnitude, i.e., $|\lambda_j| \geq |\lambda_{j+1}|$ for $j \in \mathbb{N}$. We further introduce the phases $\delta_n \in [0, \pi]$ of the eigenvalues μ_n of TN via polar coordinates, too, writing

$$\mu_n = |\mu_n| e^{\mathrm{i}\delta_n},$$

where again we set $\delta_n = \pi$ if $\mu_n = 0$. We also sort these eigenvalues by magnitude is descending order, i.e. $|\mu_n| \geq |\mu_{n+1}|$ for all $n \in \mathbb{N}$. Although we have no further information about the structure of these eigenvalues, we can prove that all phases $(\delta_n)_{n\in\mathbb{N}}$ are larger than or equal to the smallest phase ϑ_*.

Lemma 2.29. *Let* k^2 *be no Dirichlet eigenvalues of* $-\Delta$*. Let* ϑ_* *be the smallest phase among all the phases of the eigenvalues of the far field operator* F *and let* $(\delta_n)_{n\in\mathbb{N}}$ *be the phases of the eigenvalues* $(\mu_n)_{n\in\mathbb{N}}$ *of* TN*. Then it holds that* $\delta_n \geq \vartheta_* > 0$ *for all* $n \in \mathbb{N}$.

Proof. Let μ_n be any eigenvalue of TN with eigenfunction f_n and phase δ_n. Then we use the characterization of ϑ_* from Lemma 2.4 and the factorization of TN from Theorem 2.28 to get

$$\begin{aligned}\cot(\vartheta_*) &= \max_{g\in L^2(\mathbb{S}_1)} \frac{\mathrm{Re}\,(Fg, g)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(Fg, g)_{L^2(\mathbb{S}_1)}} = \max_{g\in L^2(\mathbb{S}_1)} \frac{\mathrm{Re}\,(TN(Z^{-1})^*g, (Z^{-1})^*g)_{L^2(\mathbb{S}_R)}}{\mathrm{Im}\,(TN(Z^{-1})^*g, (Z^{-1})^*g)_{L^2(\mathbb{S}_R)}} \\ &= \max_{f\in L^2(\mathbb{S}_R)} \frac{\mathrm{Re}\,(TNf, f)_{L^2(\mathbb{S}_R)}}{\mathrm{Im}\,(TNf, f)_{L^2(\mathbb{S}_R)}} \geq \frac{\mathrm{Re}\,(TNf_n, f_n)_{L^2(\mathbb{S}_R)}}{\mathrm{Im}\,(TNf_n, f_n)_{L^2(\mathbb{S}_R)}} = \cot(\delta_n)\end{aligned}$$

where we used the denseness of the image of $(Z^{-1})^*$ in $L^2(\mathbb{S}_R)$. Note that all expressions in the last chain of equations are well-defined since $\mathrm{Im}\,(Fg, g)$ and $\mathrm{Im}\,(Tf, f)$ do not vanish. The assertion now follows from the strictly monotonic decrease of the cotangent. ■

From now on the dependency of all quantities on the wavenumber $k > 0$ becomes important, which we will indicate by writing, e.g., $\vartheta_* = \vartheta_*(k)$, $\delta_n = \delta_n(k)$ and $g_* = g_*(k)$ for numbers and vectors, respectively, and by $TN = T_kN_k$ and $F = F_k$ for operators.

We can now use the result in Lemma 2.29 in combination with the inside-outside duality for far field data in Theorem 2.9 to formulate a first partial result for near field data.

Corollary 2.30. *Assume that $k_0 > 0$ and that $I = (k_0 - \varepsilon, k_0)$ contains no wavenumber k such that k^2 is a Dirichlet eigenvalue of $-\Delta$ in D and consider, for $k \in I$, the phase $\delta_n(k) \in (0, \pi)$ of an arbitrary eigenvalue $\mu_n(k)$ of $T_k N_k$. If $\delta_n(k) \to 0$ as k tends to k_0 from below, then k_0^2 is a Dirichlet eigenvalue of $-\Delta$ in D.*

Proof. As $\delta_n(k) \to 0$ it follows that $\vartheta_*(k) \to 0$ for $k \to k_0$ by Lemma 2.29, which proves the claim due to Theorem 2.9. ■

The latter corollary merely states a sufficient condition for k_0^2 being a Dirichlet eigenvalue of $-\Delta$ in D. To prove a necessary condition, and thus to arrive at a complete duality statement, we rely on the numerical range of an operator as further technical tool. If $\mathcal{H}$ is a Hilbert space, then the numerical range $W(B)$ of a bounded linear operator $B : \mathcal{H} \to \mathcal{H}$ is a subset of the complex plane given by

$$W(B) = \{(Bg, g)_{\mathcal{H}} : g \in \mathcal{H}, \|g\|_{\mathcal{H}} = 1\}.$$

In Lemma 2.31 we gather some important, well-known results about the numerical range of the operator B, which can be found in [Gus70, dBGS72, Lan75, Hil66]. Let us recall before that the boundary of $W(B)$ has infinite curvature at one of its points $\beta \in \partial W(B)$ if there is no closed disc contained in $W(B)$ that contains β. (As an illustrative example, any corner of a polygon hence has infinite curvature.)

Lemma 2.31. *(a) The numerical range of B is convex.*
(b) If $\beta \in W(B)$ is a boundary point at which $\partial W(B)$ has infinite curvature, then β is an eigenvalue of B.
(c) The spectrum of B is contained in the closure of the numerical range of B.
(d) If B is compact and normal, then the numerical range is the convex hull of its eigenvalues.

Due to the factorization of $T_k N_k$ in (2.50), it is clear that its numerical range $W(T_k N_k)$ is contained in the upper half of the complex plane. The factorizations shown in Theorem 2.28 will even allow to characterize the smallest phase of all elements of $W(T_k N_k)$ in Theorem 2.32 below. To this end, we will compare the numerical ranges of $T_k N_k$ and F_k, given by

$$W(T_k N_k) = \left\{(T_k N_k f, f)_{L^2(\mathbb{S}_R)} : f \in L^2(\mathbb{S}_R), \|f\|_{L^2(\mathbb{S}_R)} = 1\right\} \tag{2.55}$$

$$W(F_k) = \left\{(F_k g, g)_{L^2(\mathbb{S}_1)} : g \in L^2(\mathbb{S}_1), \|g\|_{L^2(\mathbb{S}_1)} = 1\right\}.. \tag{2.56}$$

For the subsequent theorem, recall that $\lambda_*(k)$ is the eigenvalue of F_k possessing the smallest phase $\vartheta_*(k)$ among the phases of all eigenvalues of F_k (the phase of the origin equals π, by definition).

Theorem 2.32. *If $0 \notin W(T_k N_k)$ then the union of the phases of all elements of $W(T_k N_k)$ is the interval $[\vartheta_*(k), \pi)$. If $0 \in W(T_k N_k)$ then the union of the phases of all elements of $W(T_k N_k)$ is the interval $[\vartheta_*(k), \pi]$.*

Proof. Assume first that $0 \notin W(T_k N_k)$. Let us introduce the set

$$W_{Z,k} = \left\{(T_k N_k f, f)_{L^2(\mathbb{S}_R)} : f \in \mathrm{dom}(Z_k^*), \|f\|_{L^2(\mathbb{S}_R)} = 1\right\} \subset \mathbb{C}$$

and note that $W_{Z,k}$ is dense in $W(T_k N_k)$ due to the denseness of $\mathrm{dom}(Z_k^*)$ in $L^2(\mathbb{S}_R)$ and the continuity of both $T_k N_k$ and the inner product of $L^2(\mathbb{S}_1)$. Now we use the factorization of $T_k N_k$

from Theorem 2.28,

$$\begin{aligned} W_{Z,k} &= \left\{ (T_k N_k f, f)_{L^2(\mathbb{S}_R)} \,:\, f \in \mathrm{dom}(Z_k^*),\, \|f\|_{L^2(\mathbb{S}_R)} = 1 \right\} \\ &= \left\{ R^2\, (F_k Z_k^* f, Z_k^* f)_{L^2(\mathbb{S}_1)} \,:\, f \in \mathrm{dom}(Z_k^*),\, \|f\|_{L^2(\mathbb{S}_R)} = 1 \right\} \\ &= \left\{ R^2\, \frac{(F_k g, g)_{L^2(\mathbb{S}_1)}}{\|f\|^2_{L^2(\mathbb{S}_R)}} \,:\, g = Z_k^* f,\, f \in \mathrm{dom}(Z_k^*),\, \|f\|_{L^2(\mathbb{S}_R)} = 1 \right\} \\ &= \left\{ R^2\, \frac{(F_k g, g)_{L^2(\mathbb{S}_1)}}{\|(Z_k^*)^{-1} g\|^2_{L^2(\mathbb{S}_R)}} \,:\, g \in L^2(\mathbb{S}_1),\, \|g\|_{L^2(\mathbb{S}_1)} = 1 \right\}, \end{aligned} \tag{2.57}$$

where we exploited that Z_k^* is one-to-one and onto from $\mathrm{dom}(Z_k^*)$ into $L^2(\mathbb{S}_R)$ to obtain the last equality.

Note that since $0 \notin W(T_k N_k)$, it follows that $0 \notin W_{Z,k}$ and therefore no eigenvalue of F vanishes, due to equation (2.57). By Lemma 2.31(d), the numerical range $W(F_k)$ is the convex hull of the eigenvalues $(\lambda_n(k))_{n\in\mathbb{N}}$ of F_k. Since the eigenvalues of F_k have phases in the interval $[\vartheta_*(k), \pi)$ and tend to the origin from the left, we conclude that for any phase in $[\vartheta_*(k), \pi)$ there is an element of $W(F_k)$ possessing that phase. Now we compare (2.57) and (2.56) and note that to each element $\gamma = (T_k N_k f, f)_{L^2(\mathbb{S}_R)}$ in $W_{Z,k}$ there corresponds an element $(F_k g, g)_{L^2(\mathbb{S}_1)}$ in $W(F_k)$ that possesses the same phase, and vice versa. In particular, the union $[\vartheta_*(k), \pi)$ of the phases of all elements in $W(F_k)$ equals the union of the phases of all the elements in $W_{Z,k}$.

Denote now by $g_*(k) \in L^2(\mathbb{S}_1)$ an eigenfunction for the eigenvalue $\lambda_*(k)$ of F_k with the smallest phase $\vartheta_*(k)$. Since $\vartheta_*(k)$, which is also the phase of, e.g., the element

$$\gamma_*(k) = \frac{(F_k g_*(k), g_*(k))_{L^2(\mathbb{S}_1)}}{\|(Z_k^*)^{-1} g_*(k)\|^2_{L^2(\mathbb{S}_R)}} \in W_{Z,k}\,,$$

is a distinct lower bound of the phases of the elements of $W_{Z,k}$, it follows from the density of $W_{Z,k}$ in $W(T_k N_k)$ that $\vartheta_*(k)$ is also a lower bound of the phases of the elements of $W(T_k N_k)$. Since $0 \notin W(T_k N_k)$ the union of all phases of this set is indeed $[\vartheta_*(k), \pi)$.
If $0 \in W(T_k N_k)$, the phase π is included in the set of phases, so that by the same arguments, the set of phases is $[\vartheta_*(k), \pi]$. ■

Finally, we formulate an inside-outside that establishes a relation between interior Dirichlet eigenvalues of the Laplacian and the smallest phase of the numerical range of the near field operator.

Corollary 2.33 (Inside-Outside Duality). *Assume that $k_0 > 0$ and that $I = (k_0 - \varepsilon, k_0)$ contains no k such that k^2 is a Dirichlet eigenvalue of $-\Delta$ in D and denote by $[\delta_*(k), \pi)$ the union of phases of elements from $W(T_k N_k)$. Then it holds that k_0^2 is a Dirichlet eigenvalues of $-\Delta$ if and only if $\delta_*(k)$ converges to zero as k approaches k_0 from below.*

Proof. We have shown in Theorem 2.32 that the union of phases of elements from $W(T_k N_k)$ is the half-open interval $[\vartheta_*(k), \pi)$, such that $\delta_*(k)$ equals the smallest phase of the eigenvalues $(\lambda_n(k))_{n\in\mathbb{N}}$ of F_k. The assertion now follows directly from the inside-outside duality for far field data in Theorem 2.8 and Theorem 2.9. ■

2.5. Numerically Detecting Interior Eigenvalues from Far Field Data

In this section we want to show that the theoretical derivation of the inside-outside duality for far field data from Section 2.2 and Section 2.3 can be turned into a working algorithm that enables one to detect interior eigenvalues from far field data. While we focus in this section on the detection of interior eigenvalues in a domain D of $-\Delta$ for Dirichlet and Neumann boundary conditions, the principle idea of the algorithm also applies for the other scattering scenarios we are going to consider, since we only require the knowledge of discretized far field operators for many wavenumber that correspond to the scattering problem under consideration. While the algorithm we present was in its simplest form successfully applied already in [LP14], a rigorous theoretical derivation was done in [JL15]. In this section we proceed in the following way. Let $\{\Gamma_n^{(j)}, 1 \leq j \leq N\} \subset \mathbb{S}_1$ be disjoint and relatively open subsets of $\mathbb{S}_1$ with Lipschitz boundary, such that the union of their closures is dense in $\mathbb{S}_1$ and

$$h_N := \sup_{j=1,..,N} \left\{ |\hat{x} - \hat{y}|, \hat{x}, \hat{y} \in \Gamma_N^{(j)} \right\} \to 0 \qquad \text{as } N \to \infty.$$

Let $\Theta_n := \{\theta_N^{(j)}\}_{j=1}^N \subset \mathbb{S}_1$ contain pairwise different directions such that $\theta_N^{(j)} \in \Theta_N$ belongs to $\Gamma_N^{(j)}$ for $j = 1, .., N$. Assume now that we have a set of discrete far field data

$$\mathbb{F}_N^\delta := u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(l)})_{j,l=1}^N \in \mathbb{C}^{N \times N}, \tag{2.58}$$

with noise level δ, i.e. $\|\mathbb{F}_N^\delta - \mathbb{F}_N\|_2 \leq \delta$, where $\mathbb{F}_N := u^\infty(\theta_N^{(j)}, \theta_N^{(l)})_{j,l=1}^N$ contains the exact far field data. We want to establish a connection between the eigenvalues λ_j of the far field operator F and the eigenvalues λ_j^N of its discrete, noisy counterpart $\mathbb{F}_N^\delta$. To establish this connection, we introduce an intermediary operator F_N^δ in (2.62) and establish first a connection between the eigenvalues of $\mathbb{F}_N^\delta$ and F_N^δ in Theorem 2.34. In a second step we then establish a connection between the eigenvalues of F_N^δ and F in Theorem 2.35 and Theorem 2.36. Recall that the statement of the inside-outside duality in the last sections is formulated in terms of the extremal phases $\vartheta_*(k)$ or $\vartheta^*(k)$. Therefore we are also interested in how the error δ in the discrete far field data influences the accuracy of the phases for the numerically computed eigenvalues. As a conclusion to this section, we are going to do numerical experiments for both Dirichlet and Neumann scattering objects and show how the inside-outside duality algorithm fares in practice.

Let the indicator function of the j-th surface patch be denoted by $\mathbf{1}_{\Gamma_N^j}$. We define a discrete interpolation operator $Q_N : \mathbb{C}^N \to L^2(\mathbb{S}_1)$ by

$$Q_N g_N = \sum_{j=1}^N w_N(j) g_{N,j} \mathbf{1}_{\Gamma_N^j}, \tag{2.59}$$

where $g_{N,j}$ is the j-th component of $g_N \in \mathbb{C}^N$ and the weights $w_N(j) > 0$ for $1 \leq j \leq N \in \mathbb{N}$ are given by

$$w_N(j) := \sigma(\Gamma_N^j), \tag{2.60}$$

where σ is the measure of the area of Γ_N^j. Obviously, Q_N is bounded since $\|Q_N g_N\|^2_{L^2(\mathbb{S}_1)} \leq 4\pi \|g_N\|^2_{\mathbb{C}^N}$. The adjoint $Q_N^* : L^2(\mathbb{S}_1) \to \mathbb{C}^N$ is then given by

$$Q_N^* g = \left[w_N(j) \mathbf{1}_{\Gamma_N^j} g(\theta_N^{(j)}) \right]_{j=1}^N. \tag{2.61}$$

Now we will use these interpolation operators to establish a connection between the eigenvalues of the far field operator F and the eigenvalues of the discrete counter part $\mathbb{F}_N^\delta$. As a first step we define a finite-dimensional approximation $F_N^\delta : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1)$ to the exact far field operator F by

$$F_N^\delta g := Q_N \mathbb{F}_N^\delta Q_N^* g = \sum_{j=1}^N \mathbf{1}_{\Gamma_N^j} \sum_{l=1}^N w_N^2(l) u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(l)}) g(\theta_N^{(l)}) \qquad \text{for } g \in L^2(\mathbb{S}_1) \tag{2.62}$$

and analogously $F_N : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1)$, where we just replace $\mathbb{F}_N^\delta$ by $\mathbb{F}_N$. Then we know from Lemma [JL15, Lemma 5,Lemma 6] that F_N converges to F in the operator norm, more precisely $\|F - F_N\|_{L^2(\mathbb{S}_1)\to L^2(\mathbb{S}_1)} \le Ch_N^2 \to 0$ for $N \to \infty$. Furthermore from $\|\mathbb{F}_N^\delta - \mathbb{F}_N\|_2 \le \delta =: \delta_N$ it follows that $\|F_N - F_N^\delta\|_{L^2(\mathbb{S}_1)\to L^2(\mathbb{S}_1)} \le 4\pi\delta_N$. These two results imply an error bound for the difference of F and F_N^δ. By [JL15, Theorem 7] it holds that

$$\|F - F_N^\delta\|_{L^2(\mathbb{S}_1)\to L^2(\mathbb{S}_1)} \le Ch_N^2 + 4\pi\delta_N, \qquad N \in \mathbb{N}$$

for a constant C independent of N. Now we need to provide a link between the eigenvalues of the matrix of our discrete far field data $\mathbb{F}_N^\delta$ and the eigenvalues of the finite-dimensional approximation F_N of the far field operator. Defining $\mathbb{W} = \operatorname{diag}(w_N(j)_{j=1}^N) \in \mathbb{R}^{N\times N}$, we can now provide this link in the following theorem, see [JL15, Theorem 8].

Theorem 2.34. *All eigenvalues of $\mathbb{W}_N\mathbb{F}_N\mathbb{W}_N$ and $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ are eigenvalues of F_N and F_N^δ, respectively, and any additional eigenvalue of F_N and F_N^δ must vanish.*

Now that we have established a connection between the eigenvalues of the discrete far field data $\mathbb{F}_N^\delta$ and the eigenvalues of the finite-dimensional approximation F_N to the far field operator, we will now establish a second connection between the eigenvalues of the far field operator F and the approximation F_N. For this purpose, we use [JL15, Corollary 14] to obtain

Theorem 2.35. *For all eigenvalues λ_l^N of F_N^δ it holds that*

$$\min_{j\in\mathbb{N}} |\lambda_l^N - \lambda_j| \le \|F_N^\delta - F\|.$$

Therefore all the eigenvalues λ_l^N of F_N^δ have a distance to the spectrum $\sigma(F)$ of the far field operator of at most $\|F - F_N^\delta\|$. In the inside-outside duality we work with the eigenvalue λ_* or λ^* of F with the smallest or largest phase. The last theorem does not guarantee that there is a discrete eigenvalue of F_N^δ that is close to the eigenvalues λ_* or λ^* with the extremal phase. This is guaranteed by [LP14, Lemma 15], which implies the following theorem.

Theorem 2.36. *Let $\|F - F_N^\delta\| < \varepsilon$ and λ_l be an eigenvalue of F such that*

$$\min_{j\in\mathbb{N}} |\lambda_j - \lambda_l| > 2\varepsilon.$$

Then there exists an eigenvalue λ_j^N of F_N^δ such that $|\lambda_j - \lambda_j^N| < \varepsilon$.

In other words, for every eigenvalue of the far field operator that is not too close to zero, there exists a corresponding eigenvalue of the finite-dimensional approximation F_N. Now recall that F_N^δ and $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ share the same set of eigenvalues. Summarizing the results, we know that for every non-zero eigenvalue λ_j of F there is one eigenvalue $\lambda_{l(j)}^N$ of $\mathbb{W}_N\mathbb{F}_N^\delta\mathbb{W}_N$ such that

$$|\lambda_{l(j)}^N - \lambda_j| \le \|F_N^\delta - F\| \le C(h_N^2 + 4\pi\delta_N). \tag{2.63}$$

If we assume that additionally $\delta_N \to 0$ for $N \to \infty$, then

$$|\lambda_{l(j)}^N - \lambda_j| \to 0 \qquad \text{for } N \to \infty.$$

For our algorithm we need to work with the phases of the eigenvalues of the far field operator. Note that small perturbations of eigenvalues close to zero can imply large errors in the phases. More precisely, if an eigenvalue λ_j of F has magnitude smaller than $\|F_N^\delta - F\|$, then it can be perturbed into an eigenvalue λ_j^N with arbitrary phase. Therefore we will later exclude such eigenvalues from our considerations and only work with phases of eigenvalues that are not too close to zero. To be more precise, define $\varepsilon_N := \|F_N^\delta - F\|$ and assume that λ_l^N is an eigenvalue of $\mathbb{W}_N \mathbb{F}_N^\delta \mathbb{W}_N$ such that the following two estimates

$$|\lambda_l^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N, \qquad \varepsilon_N^{1/2} < 4(\pi/k)^{1/2}, \tag{2.64}$$

hold. Then we know from [JL15, Lemma 18] that the phase of λ_l^N belongs to $(0,\pi)$ and that there is an eigenvalue λ_j of F such that $|\lambda_l^N - \lambda_j| \le \varepsilon_N$ and the phase difference $|\vartheta_l^N - \vartheta_j|$ is bounded by

$$|\vartheta_l^N - \vartheta_j| \le \frac{\pi}{2}\frac{\varepsilon_N}{r_j} \le \frac{1}{8}(k\varepsilon_N)^{1/2}.$$

Taking these considerations into account, it would make sense to try characterize interior Dirichlet or Robin eigenvalues in terms of the smallest phase or largest phase respectively among all phases of eigenvalues λ_j^N of $\mathbb{F}_N^\delta$ such that $|\lambda_j^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N$, i.e. to consider where this phase converges to zero or π. Therefore we introduce

$$\begin{aligned} \vartheta_\natural(k,N) &= \min\left\{\vartheta_j^N, \lambda_j^N \in \sigma(\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k)\mathbb{W}_N), |\lambda_j^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N\right\}, \\ \vartheta^\natural(k,N) &= \max\left\{\vartheta_j^N, \lambda_j^N \in \sigma(\mathbb{W}_N \mathbb{F}_N^{\delta_N}(k)\mathbb{W}_N), |\lambda_j^N| > 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N\right\}. \end{aligned} \tag{2.65}$$

It has been shown in [LP15b, Theorem 20] that instead of characterizing interior eigenvalues by the smallest or largest phases $\vartheta_*(k)$ or $\vartheta^*(k)$, it is from a numerical point of view sufficient to consider the behavior of $\vartheta_\natural(k,N)$ and $\vartheta^\natural(k,N)$ to characterize the interior eigenvalues. In practice however it turns out that the bound for ε_N is too cautious. We will later suggest better bounds, depending on the noise level we consider.

The theory in this section is as yet independent of the specific scattering model, since we have so far only used the knowledge of (noisy) far field data to approximate interior eigenvalues. For the subsequent numerical experiments we will now explain how we obtain the far field data for scattering from impenetrable objects with Dirichlet or Robin boundary conditions. To compute the numerical approximation to a scattered field we use boundary integral equations and we briefly sketch here which equations we solved numerically. For the exterior Dirichlet problem, any radiating solution u^s to

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u^s|_{\partial D}^+ = \psi \in H^{1/2}(\partial D)$$

can be represented as a single layer potential $\mathrm{SL}\varphi$ if k^2 is not an interior Dirichlet eigenvalue. Indeed, under this assumption, the boundary integral equation of the first kind

$$S\varphi = \psi \qquad \text{in } H^{1/2}(\partial D) \tag{2.66}$$

is always uniquely solvable for $\psi \in H^{1/2}(\partial D)$. For all computations, we opted to use integral equations of the first kind since the resulting eigenvalue approximations showed in our experiments to be always more accurate than those computed via equations of the second kind. Except for values of k^2 closer than about $1e-4$ to an interior eigenvalue we did not observe stability problems of equations of the first kind at interior eigenvalues. (For the case of the cube, we used the normality error of $\|F_N^* F_N - F_N F_N^*\| / \|F_N^* F_N\|$ as error and stability indicator.) To illustrate that the accuracy of the eigenvalue computations does not depend on the choice of a direct or an indirect method, we use an integral equation of the first kind coming from a direct method to solve for radiating solutions to the exterior Neumann problem

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial D}^+ = \phi \in H^{-1/2}(\partial D),$$

more precisely,

$$-N\psi = \frac{1}{2}\operatorname{Id}\phi + K'\phi \qquad \text{in } H^{-1/2}(\partial D), \tag{2.67}$$

which is uniquely solvable in $H^{1/2}(\partial D)$ if k^2 is not an interior Neumann eigenvalue.

We solved the boundary integral equations (2.66) and (2.67) using the software package BEM++ (see [SBA+15]). BEM++ discretizes (2.66) and (2.67) using a Galerkin discretization and solves the linear system using H-matrix compression and preconditioning techniques. The far-field pattern at

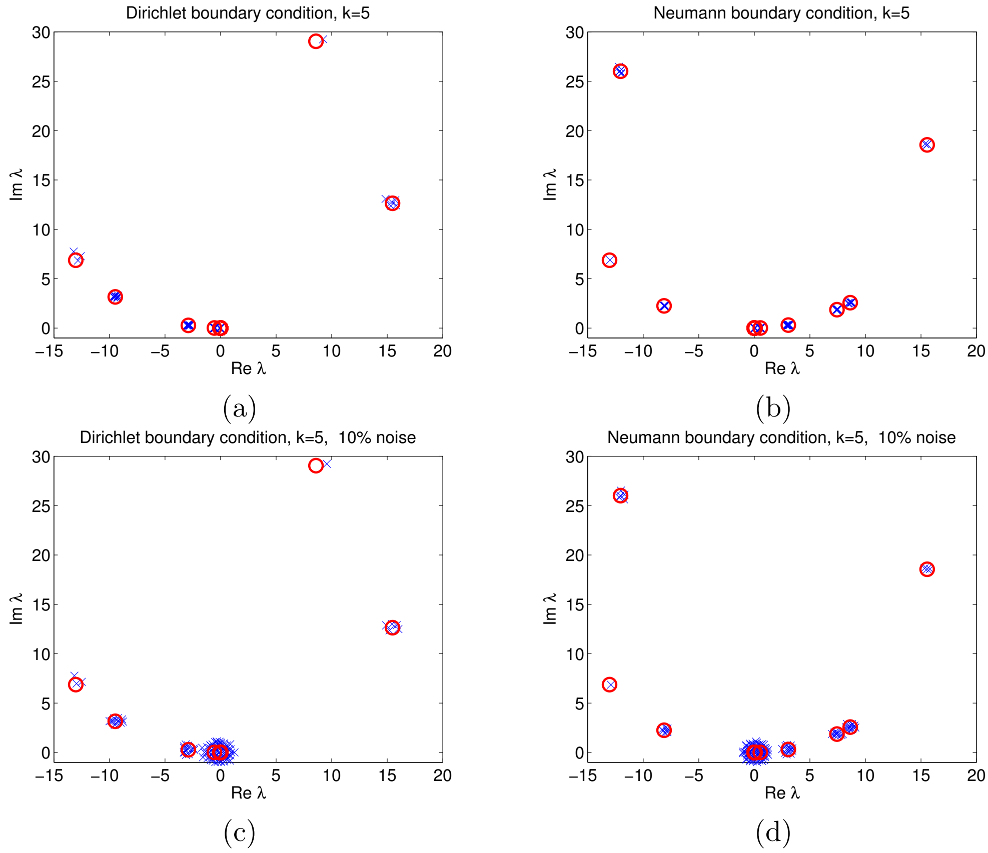


Figure 2.3.: Eigenvalues of the far field operator $F(k)$ and of $\mathbb{F}_N^{\delta_c}(k)$ for $k = 5$, $N = 120$, and $D = B$ (the unit ball). Red circles and blue crosses mark analytically computed eigenvalues of $F(k)$ and numerically computed eigenvalues of $F_N(k)$, respectively. For (c) and (d) we perturbed $\mathbb{F}_N^{\delta_c}(k)$ by adding artificial noise with a relative noise level of 10%. (a) Dirichlet boundary conditions, no artificial noise. (b) Neumann boundary conditions, no artificial noise. (c) Dirichlet boundary conditions, relative noise level of 10%. (d) Neumann boundary conditions, relative noise level of 10%.

points $\theta_j \subset \mathbb{S}_1$ of the numerical solution can directly be computed in BEM++ using its potential representation and yields the data $(u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$ we require to construct $\mathbb{F}_N^\delta$ as in (2.58), where δ is the noise that is produced by computational error. In the following examples, we always choose the same surface mesh of $\mathbb{S}_1$ from [Ces96, Section II.2.3.2.1] to obtain a partition of $\mathbb{S}_1$ into $N = 120$ quadrangles Γ_N of equal area. The incident and far field direction $\{\theta_j\}_{j=1}^N \subset \mathbb{S}_1$ then are the centers of the quadrangles.

Since all quadrangles have the same area, the weight matrix $\mathbb{W}_N^2 = w_N \operatorname{Id}_N$ is the scalar $w_N = 4\pi/N = 4\pi/120$ and therefore $\mathbb{W}_N \mathbb{F}_N^\delta(k) \mathbb{W}_N = (4\pi/N)\mathbb{F}_N^\delta(k) =: \mathbb{F}_N^{\delta_c}(k)$. To indicate the good accuracy of the resulting eigenvalues of $\mathbb{F}_N^{\delta_c}$, we plot in Figures 2.3(a) and (b) the analytically computed eigenvalues of $F(k)$ when the scatterer D is the open unit ball B, together with the N

largest (that is, non-zero) eigenvalues of $\mathbb{F}_N^{\delta_c}$ for $k = 5$. Since later on we will investigate the stability of the eigenvalue computations with respect to synthetic noise, we also indicate in Figures 2.3(c) and (d) how the numerically computed eigenvalues behave under artificial noise. To this end, we perturb the numerically computed data $(u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$ by adding a random matrix of size 120×120 containing normally distributed entries with mean zero such that the relative noise level in the spectral matrix norm equals 10%. These figures indicate the problem to attain accurate phase information of perturbed eigenvalues that are close to zero. To reduce the influence of noise, we will later work with the smallest or largest regularized phases, which have been defined in (2.65).

To verify the main assertions of this section from Theorem 2.8, Theorem 2.9 and the corresponding Theorem 2.16 and 2.17 for Robin boundary condition we compute the eigenvalues $\lambda_j^N(k)$, $j = 1, \dots, N$, of $\mathbb{F}_N^{\delta_c}(k)$ for several k and examine how their phases depend on the wavenumber.

Theorem 2.8 and Theorem 2.9 state, roughly speaking, that k_0^2 is an interior Dirichlet eigenvalue if and only if the eigenvalue $\lambda_*(k)$ of $F(k)$ with smallest phase converges to zero as k tends to k_0 from below. To verify this statement, we convert the positions of the eigenvalues in polar coordinates and plot the resulting phases. For eigenvalues close to zero, small position errors produce large phase errors. Therefore by omitting all eigenvalues $\lambda_j^N(k)$ such that

$$\lambda_j^N(k) \in R_+(\varepsilon(k)) := \{z \in \mathbb{C}, \, |z| \le \varepsilon(k), \, \operatorname{Re} z \ge 0\} \subset \mathbb{C}$$

counteracts the influence of noise. Note that if we choose $\varepsilon(k) = 4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N$, then the smallest phase of the remaining eigenvalues is equal to the smallest regularized phase $\vartheta_\natural(k_i, N)$ and hence from [LP15b, Theorem 20] and the discussion above, it follows that it is sufficient to examine the behavior of this phase. In practice however, cutting of all eigenvalues $\lambda_j^N(k)$ that have absolute value smaller than $4\pi(\varepsilon_N/k)^{1/2} + \varepsilon_N$ appears to be too cautious. Below we discuss a better choice of $\varepsilon(k)$ depending on the noise level.

To further stabilize the phase computations, we exploit the a-priori knowledge that the exact eigenvalues $\lambda_j(k)$ lie on the circle $\{z \in \mathbb{C}, \, |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane and project the eigenvalues $\lambda_j^N(k)$ outside $R_+(\varepsilon(k))$ orthogonally onto this circle, using the mapping

$$\mathcal{Q}: \, \lambda \mapsto \frac{8\pi^2 \mathrm{i}}{k} + \frac{8\pi^2}{k} \frac{\lambda - 8\pi^2 \mathrm{i}/k}{|\lambda - 8\pi^2 \mathrm{i}/k|}. \tag{2.68}$$

Although this projection might theoretically increase the phase error for certain eigenvalues λ_j^N, it has a stabilizing effect upon our computations and leads to data that is easier to interpret. Geometric considerations as in [JL15] also show that the projection operator leads to better error bounds for the phase error, in particular for eigenvalues close to zero.

Finally, we compute the phases of the projected eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ such that $\lambda_j^N(k) \notin R_+(\varepsilon(k))$. Following Theorem 2.9, interior eigenvalues are characterized by the fact that the exact eigenvalue $\lambda_*(k)$ with smallest phase tends to zero from the right. To be able to compare the resulting values of k in our computations with the true interior eigenvalues, we choose the scatterer to be either the unit ball B or the cube $C = (0,1)^3$, such that the interior Dirichlet eigenvalues are known exactly: For the unit ball B, the eigenvalues are given as positive roots of spherical Bessel functions and the first five eigenvalues appear at wavenumbers

$$k_B^{(1)} = \pi, \quad k_B^{(2)} \approx 4.49, \quad k_B^{(3)} \approx 5.76, \quad k_B^{(4)} \approx 6.28, \quad k_B^{(5)} \approx 6.99.$$

For the cube $C = (0,1)^3$ the wavenumbers k_C at which k_C^2 is an interior Dirichlet eigenvalue are given by $k_C = \sqrt{k_1 + k_2 + k_3}$ where $k_{1,2,3}$ is one of the numbers $\pi^2(n+1)^2$, $n \in \mathbb{N}_0$. Hence, the first

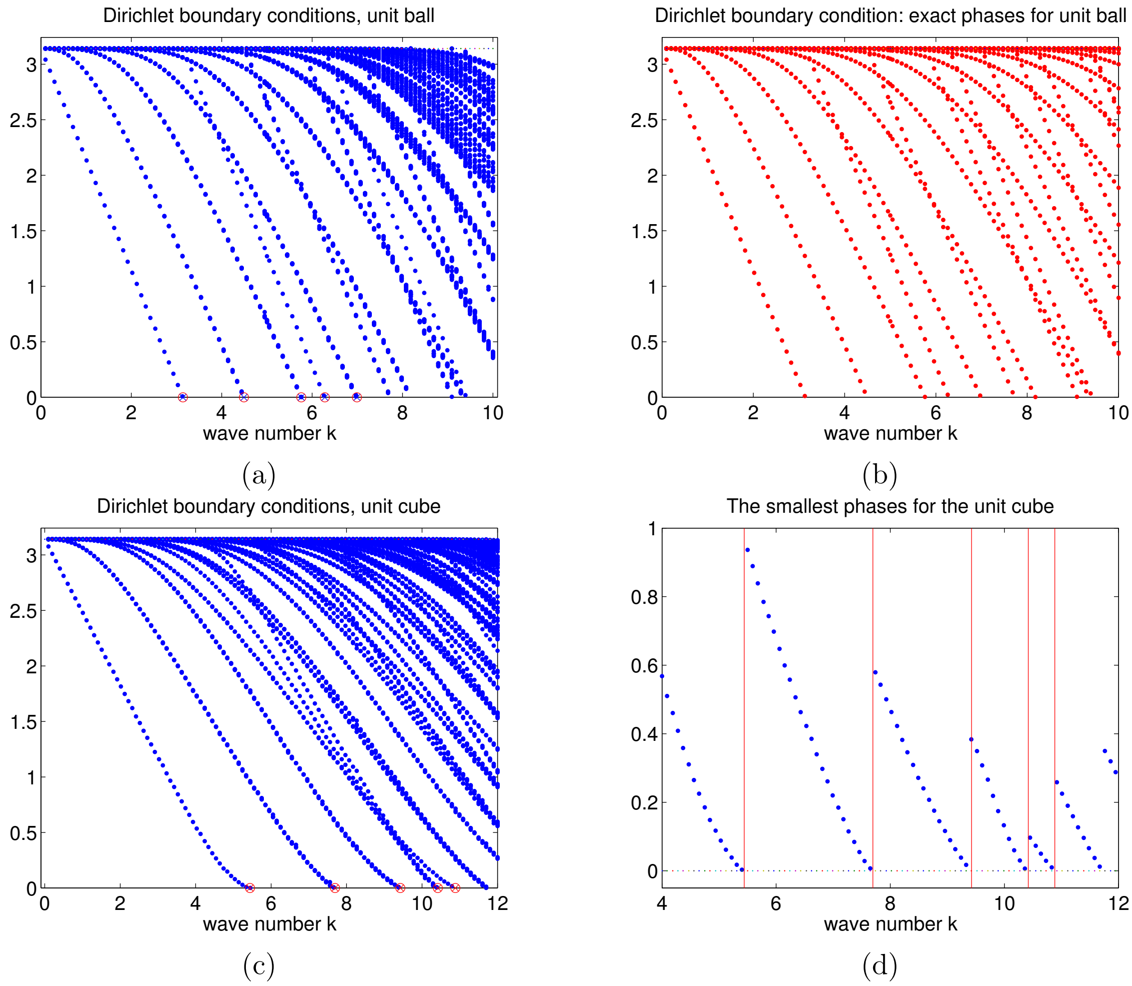


Figure 2.4.: Blue dots mark the phases of the projected numerical eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_+(\varepsilon(k))$ for Dirichlet boundary conditions, $N = 120$. Red dots mark the exact phases ϑ_j. Red circles on the k-axis mark the exact positions of the smallest five interior Dirichlet eigenvalues. (a) Phases of the projected numerical eigenvalues for the unit ball B. (b) Phases of the analytically known eigenvalues of F for the unit ball B. (c) Phases of the projected numerical eigenvalues for the unit cube. (d) Only the smallest phase from (c) was plotted. Vertical red lines mark the smallest five interior Dirichlet eigenvalues.

five Dirichlet eigenvalues arise at the wavenumbers

$$k_C^{(1)} = \sqrt{3}\pi, \quad k_C^{(2)} = \sqrt{6}\pi, \quad k_C^{(3)} = 3\pi, \quad k_C^{(4)} = \sqrt{11}\pi, \quad k_C^{(5)} = \sqrt{12}\pi.$$

Figure 2.4 shows plots of the phases of the projected eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ such that $\lambda_j^N(k) \notin R_+(\varepsilon(k))$ against the wavenumber k. In these computations, the value of $\varepsilon(k)$ has been set to $10^{-4} \cdot 16\pi^2/k$. The phases of the projected eigenvalues plotted in Figure 2.4(a) for wavenumbers in between 0 and, roughly speaking, 6 cannot be distinguished visually from the exact ones plotted in Figure 2.4(b). Further, for wavenumbers larger than 8 it is obvious that the numerical accuracy is not sufficient anymore to yield correct phases for eigenvalues lying in the left complex half-plane, that is, where the eigenvalues accumulate. However, Figures 2.4(a) and (c) show that the smallest phase tends to zero when k tends to an eigenvalue from below. Figure 2.4(d) shows that the location of the jumps in the curve of the smallest phase (that might, e.g., be found numerically using discrete derivatives) yield enclosures of the exact eigenvalues.

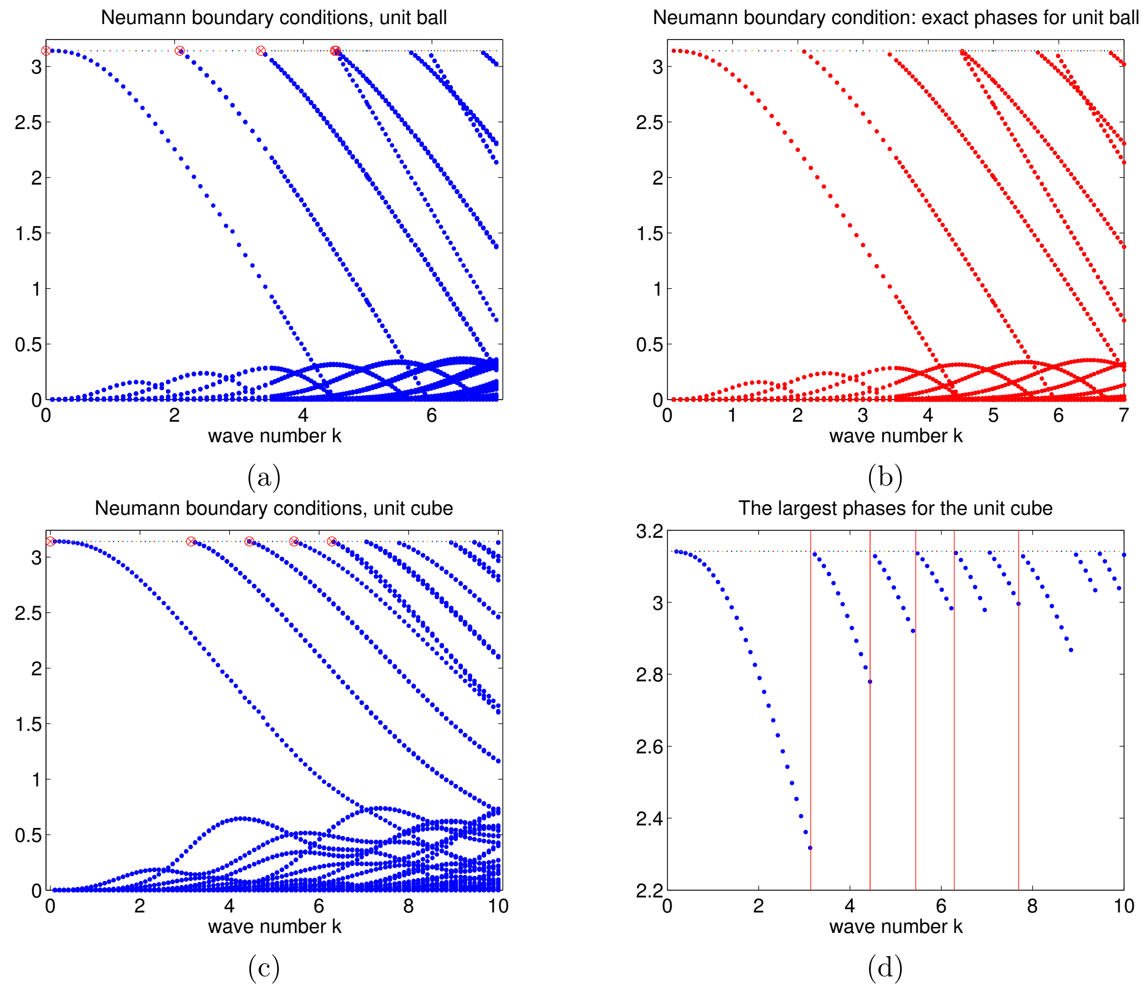


Figure 2.5.: Blue dots mark the phases of the projected numerical eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_-(\varepsilon(k))$ for Neumann boundary conditions, $N = 120$. Red dots make the exact phases ϑ_j. Red circles on the k-axis mark the exact positions of the smallest five interior Neumann eigenvalues. (a) Phases of the projected numerical eigenvalues for the unit ball B. (b) Phases of the analytically known eigenvalues of F for the unit ball B. (c) Phases of the projected numerical eigenvalues for the unit cube. (d) Only the smallest phase from (c) was plotted. Vertical red lines mark the exact positions of the smallest five non-zero interior Neumann eigenvalues.

In the case of Neumann boundary conditions on ∂D, Theorem 2.16 and Theorem 2.17 state that the phase $\vartheta^*(k)$ of the eigenvalue $\lambda^*(k)$ of the far field operator with largest phase converges to π if and only if k tends to an interior Neumann eigenvalue from above. In Figure 2.5 we show plots of the phases of the projected eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ for

$$\lambda_j^N(k) \notin R_-(\varepsilon(k)) := \{z \in \mathbb{C},\ |z| \leq \varepsilon(k),\ \mathrm{Re}\, z \leq 0\} \subset \mathbb{C}$$

for Neumann boundary conditions against the wavenumber k, again for the unit ball B and the cube C. As in the Dirichlet case, the simplicity of the domain allows to compute the interior Neumann eigenvalues explicitly. For the unit ball, the wavenumbers k at which interior eigenvalues arise are given by the roots of the derivative of the spherical Hankel function. The first few of those wavenumbers are

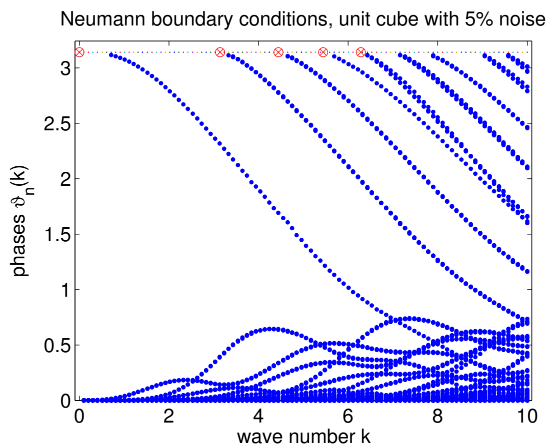
$$k_B^{(1)} = 0, \quad k_B^{(2)} \approx 2.08, \quad k_B^{(3)} \approx 3.34, \quad k_B^{(4)} \approx 4.49, \quad k_B^{(5)} \approx 4.51.$$

For the cube C, the wavenumbers k_C at which k_C^2 is an interior Neumann eigenvalue are given by $k_C = \sqrt{k_1 + k_2 + k_3}$ where $k_{1,2,3}$ is one of the numbers $\pi^2 n^2$ for $n \in \mathbb{N}_0$. Therefore the first few Neumann eigenvalues arise at the wavenumbers

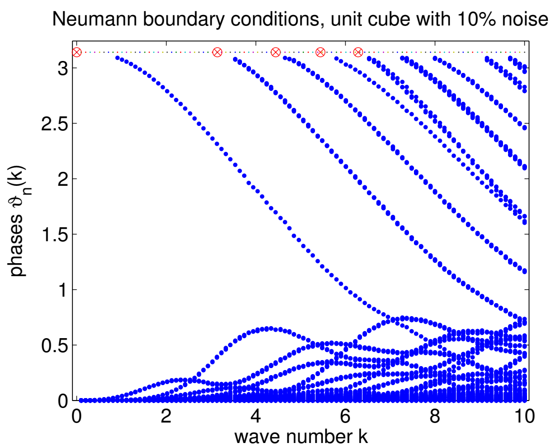
$$k_C^{(1)} = 0, \quad k_C^{(2)} = \pi, \quad k_C^{(3)} = \sqrt{2}\pi, \quad k_C^{(4)} = \sqrt{3}\pi, \quad k_C^{(5)} = 2\pi.$$

Figure 2.5 shows that both for the unit ball B and the cube C these values correspond to the wavenumbers for which the largest phase tends to π. Again, the jumps in the curve of the largest phase shown in Figure 2.5(d) can be used to derive enclosures of the exact interior eigenvalues.

Finally we want to test the stability of the computation of interior eigenvalues via the behavior of the smallest or largest phase when adding artificial noise to the data $(u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$. As a test case we choose the unit cube with Neumann boundary conditions as a test object. To obtain two instances of noisy data from the numerically computed data $(u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$, we added a matrix with random numbers following a normal distribution with mean zero and variance such that the relative error in the spectral matrix norm equals once 5% and once 10%. For the phase computations, we applied the same stabilization technique used above: We first omitted the eigenvalues $\lambda_j^N(k)$ in $R_-(\varepsilon(k)) := \{|z| \le \varepsilon(k), \operatorname{Re} z \le 0\}$ and then projected the remaining eigenvalues onto the circle $\{|z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ using the projection $\mathcal{Q}$ from (2.68). The number $\varepsilon(k)$ was set to $0.025 \cdot 16\pi^2/k$ and $0.05 \cdot 16\pi^2/k$. The results can be seen in Figure 2.6. Of course, the interior Neumann eigenvalues are not as precisely identifiable as in Figure 2.5(c). However, by, e.g., choosing the jump of the largest phase as an approximation to the exact interior eigenvalues yields an acceptable absolute error of less than 0.1 and 0.2 for $\lambda_C^{(j)}$, $j = 2, \ldots, 5$, for the two noise levels of 5% and 10%, respectively.



(a)



(b)

Figure 2.6.: Computed phase curves after adding synthetic noise to the numerically computed far field data for the cube C with Neumann boundary conditions, $N = 120$. Blue dots mark the phases of the projected numerical eigenvalues $Q[\lambda_j^N(k)]$ with $\lambda_j^N(k) \notin R_-(\varepsilon(k))$. Red circles on the k-axis mark the exact positions of the smallest five interior Neumann eigenvalues. (a) Relative noise level 5%. (b) Relative noise level 10%.

2.6. Numerically Detecting Interior Eigenvalues from Near Field Data

In this section we provide numerical examples to verify the theoretical results from Section 2.4. In particular, we show that it is possible to numerically compute the Dirichlet eigenvalues of the negative Laplacian in a domain D from the modified near field operators $T_k N_k$ from (2.50) in a given spherical setting, for a sufficiently dense grid of wavenumbers k. For simplicity, we assume that sources and measurements are done on the sphere $\mathbb{S}_R$ and drop the index R from now on; the index k will be dropped whenever this causes no confusion. In the last section, we used the surface mesh of $\mathbb{S}_1$ from [Ces96, Section II.2.3.2.1] to obtain a partition of $\mathbb{S}_1$ into $M_1 = 120$ quadrangles Γ_{M_1} of equal area. We will use this mesh also for the sphere $\mathbb{S}_R$ by projection each mesh point orthogonally onto $\mathbb{S}_R$. Then the incident fields are caused by point sources located at the points $\{y_j\}_{j=1}^{M_1} \subset \mathbb{S}_R$, which are again the centers of the quadrangles, and the near field data is measured at the same points. Following the structure of the previous section, we assume now that we have a set of discrete near field data

$$\mathbb{N}_{M_1} := u^s_\delta(y_j, y_l)_{j,l=1}^{M_1} \in \mathbb{C}^{M_1 \times M_1}, \tag{2.69}$$

with noise level δ, i.e. $\|\mathbb{N}^\delta_{M_1} - \mathbb{N}_{M_1}\|_2 \leq \delta$, where $\mathbb{N}_{M_1} := u^s(y_j, y_l)_{j,l=1}^{M_1}$ contains the exact near field data. To bridge the gap between the near field operator N and its discrete representation $\mathbb{N}^\delta_{M_1}$, let $\mathbf{1}_{\Gamma_j} : \mathbb{S}_R \to \mathbb{C}$ be the indicator function of the patch Γ_j and let the discrete interpolation operator $Q_{M_1} : \mathbb{C}^{M_1} \to L^2(\mathbb{S}_R)$ and its adjoint $Q^*_{M_1} : L^2(\mathbb{S}_R) \to \mathbb{C}^{M_1}$ be defined as in (2.59) and (2.61). Then we define the finite-dimensional approximation to the near field operator as

$$N_{M_1} g := Q_{M_1} \mathbb{N}^\delta_{M_1} Q^*_M g = \sum_{j=1}^{M} \mathbf{1}_{\Gamma^j_M} \sum_{l=1}^{M} w^2_M(l) u^s_\delta(y_j, y_l) g(y_l) \qquad \text{for } g \in L^2(\mathbb{S}_R),$$

where the weights w_M have been defined in (2.60). Note that all the results from the previous section about the approximation of the far field operator holds. For example, N_{M_1} converges to N is the operator norm and shares eigenvalues with the discrete approximation $\mathbb{N}_{M_1}$. Unlike in the previous section, however, we do not work directly with the eigenvalues of the near field operator N but instead with the properties of the modified operator TN. Therefore we will first introduce a finite dimensional approximation to the operator TN. For an element $g \in L^2(\mathbb{S}_R)$ we develop $N_{M_1} g$ into its coefficients $(N_{M_1} g)^m_n$ for the orthogonal basis of spherical harmonics by numerical integration on $\mathbb{S}_R$ and truncate the series expression defining T, see [HYZZ14, Equation (3.12)], at $M_2 \in \mathbb{N}$, such that

$$(T_{M_2} N_{M_1}) g(x) = \sum_{n=0}^{M_2} \sum_{m=-n}^{n} \left(\frac{\overline{h^{(1)}_n(kR)}}{h^{(1)}_n(kR)} (N_{M_1} g)^m_n \right) Y^m_n(\hat{x}), \quad x \in \mathbb{S}_R. \tag{2.70}$$

yields an approximation of TN. This approximation is then discretized by evaluating it for all indicator functions 1_{Γ_j}, $j = 1, \ldots, M_1$, at the source points $\{y_i,\, i = 1, \ldots, M_1\}$ to obtain the $M_1 \times M_1$ matrix $\mathbb{T}_{M_2} \mathbb{N}_{M_1}$. Is all our numerical experiments, we choose $M_2 = 12$ and $M_1 = 120$. Therefore we will neglect the indices from now on and simply write $\mathbb{TN}$. Note that unlike in the previous section, we will not show that it is sufficient to work with the properties of the discrete approximation $\mathbb{TN}$ instead of working with the properties of TN. However our numerical experiments show that the approach also works reasonably well.

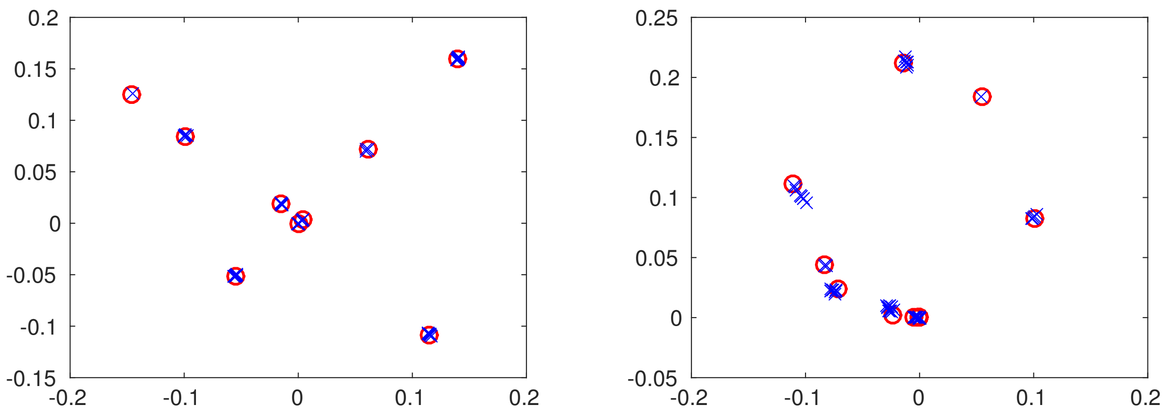
In order to construct the discrete approximation operator, we need measurements $(u^s(y_i, y_j))_{i,j=1}^{M_1}$

of scattered fields $u^s(\cdot, y_j)$ that are radiating solutions of the exterior Dirichlet scattering problem

$$\Delta u^s(\cdot, y_j) + k^2 u^s(\cdot, y_j) = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u^s(\cdot, y_j) = -u^i(\cdot, y_j) \quad \text{on } \partial D,$$

see (2.4). As in the previous section, we use the boundary element software package BEM++ to simulate this data set by computing numerical approximations $u_\delta(\cdot, y_j)$ to the solution $u(\cdot, y_j)$ of this problem for the M_1 source points $y_j \in \mathbb{S}_R$.

If the scattering object D is the unit ball $B_1(0)$, then the operators N and TN are diagonalizable in the basis of the spherical harmonics and their eigenvalues can be explicitly calculated. In Figure 2.7(a) we computed these eigenvalues for measurements on $\mathbb{S}_2$, i.e., for $R = 2$, and compared them to the numerically computed eigenvalues of the approximated near field operator $\mathbb{N}_{M_1}$. We note that the numerically computed eigenvalues to N are sufficiently accurate in the visible norm; however, they do not share any visibly apparent structure. In Figure 2.7(b) we computed the analytic eigenvalues of TN_k for the same setting and compared them to the eigenvalues of the matrix representation $\mathbb{TN}_k$. Although the the addition of the operator $\mathbb{T}$ visibly increases the inaccuracy of the approximated eigenvalues, one can see that they accumulate at zero from the left, corresponding to the eigenvalues of the far field operator, and that they lie approximately on a contour in the upper half of the complex plane. Expanding on this point, we note that the eigenvalues μ_n of TN and λ_n of F are



(a) (b)

Figure 2.7.: Eigenvalues in the complex plane for wavenumber $k = 5$ and radius $R = 2$. Red circles mark analytically calculated eigenvalues and blue crosses numerically computed eigenvalues of discretizations. (a) Eigenvalues of the near field operator N and its discretization $\mathbb{N}_{M_1}$. (b) Eigenvalues of the modified near field operator TN and its discretization $\mathbb{TN}$.

given by

$$\mu_n = ikR^2 |h_n^{(1)}(kR)|^2 \frac{j_n(k)}{h_n^{(1)}(k)} \quad \text{and} \quad \lambda_n = \frac{(4\pi)^2 i}{k} \frac{j_n(k)}{h_n^{(1)}(k)}, \qquad n \in \mathbb{N},$$

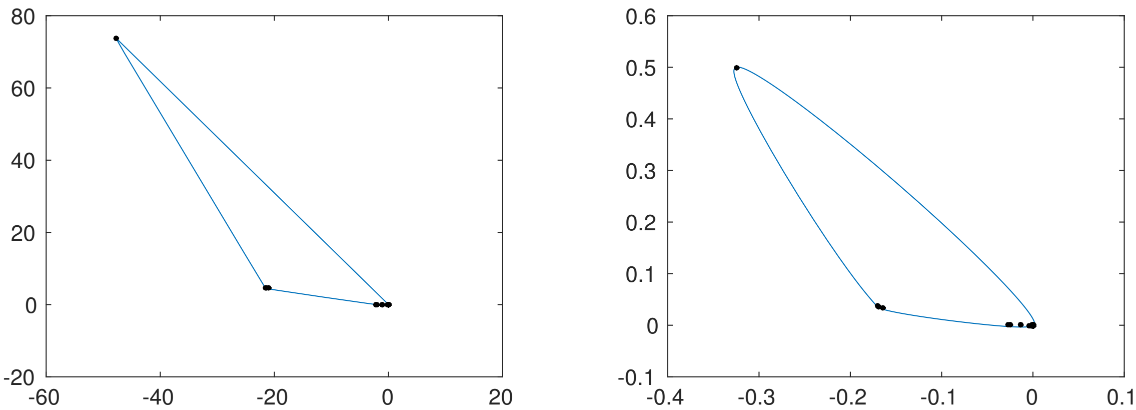
respectively, see [HYZZ14, Section 3.3] and [KG08, Section 1.5]. Comparing both expressions, we find that scaling the radii of the eigenvalues λ_n by $k^2\,|h_n^{(1)}(kR)|^2/(4\pi)^2$ precisely yields the eigenvalues μ_n. Note that this factor could also be derived from Theorem 2.25, since for $\boldsymbol{g} \in \mathrm{dom}(\mathcal{Z})$ we have that $\mathcal{Z}\mathcal{Z}^*\boldsymbol{g} = \{k^2 |h_n^{(1)}(kR)|^2 g_n^m,\ n \in \mathbb{N}_0,\ |m| \le n\}$. Obviously, the scaling factor does not change the phases of the eigenvalues. We would further like to point out that even for the other scatterers considered below the eigenvalues of $\mathbb{TN}$ retain the same phases as the eigenvalues of the discretization $\mathbb{F} := \mathbb{F}_{M_1}$ of the far field operator F, that was introduced in (2.58). In particular, the smallest phase among all eigenvalues of $\mathbb{TN}$ and $\mathbb{F}$ larger than the noise level of these discretization always agreed roughly up to discretization error.

Finally, we numerically verify the inside-outside duality for near field data. For that purpose we need to calculate the smallest phase of all the elements of the numerical range of $\mathbb{TN}$, given by $W(\mathbb{TN}) = \{(\mathbb{TN}v, v) : v \in \mathbb{C}^{M_1}, \|v\| = 1\}$. The algorithm we use to compute this numerical range follows [CH95]. The essential idea is to first rotate $\mathbb{TN}$ by multiplying a factor $\exp(-\mathrm{i}t)$ to $\mathbb{TN}$ and second to decompose the rotation $\exp(-\mathrm{i}t)\mathbb{TN}$ into its real and imaginary part, i.e. $\exp(-\mathrm{i}t)\mathbb{TN} = H_t + \mathrm{i}K_t$, with self-adjoint operators

$$H_t = \frac{\exp(-\mathrm{i}t)\mathbb{TN} + (\exp(-\mathrm{i}t)\mathbb{TN})^*}{2}, \qquad K_t = \frac{\exp(-\mathrm{i}t)\mathbb{TN} - (\exp(-\mathrm{i}t)\mathbb{TN})^*}{2\mathrm{i}}.$$

We denote by $\mu_{\max}(t)$ the largest eigenvalue of H_t and by P_t the orthogonal projection from $\mathbb{C}^{M_1}$ onto the eigenspace $\{v \in \mathbb{C}^{M_1} : H_t v = \mu_{\max}(t)v\}$ and calculate (not necessarily different) eigenvectors v_t^+ and v_t^- corresponding to $\mu_{\max}(t)$, which are also eigenvectors of the (not necessarily different) smallest and largest eigenvalue of $P_t K_t P_t$. For $t \in [0, 2\pi]$, the numbers $(\mathbb{TN}v_t^+, v_t^+)$ and $(\mathbb{TN}v_t^-, v_t^-)$ then belong to the boundary of the numerical range of $\mathbb{TN}$, and $W(\mathbb{TN})$ is the convex hull of all these numbers, see [CH95, Theorem 3].

Due to numerical inaccuracies, finding the smallest phase in this set is not an obvious task, as becomes apparent when comparing the numerical ranges of $\mathbb{TN}$ and $\mathbb{F}$. As a scattering object, we choose the unit cube $[0,1]^3$ and plot the the boundaries of these two numerical ranges in Figure 2.8(a) and (b). While the boundary of the numerical range of $\mathbb{F}$ in Figure 2.8(a) shows that the numerical



(a) (b)

Figure 2.8.: The numerical ranges of $\mathbb{F}$ and $\mathbb{TN}$ for the unit cube as obstacle, wavenumber $k = 1.5$ and measurements taken on the sphere with radius $R = 2$. (a) Boundary of the numerical range of $\mathbb{F}$. Black dots mark the numerically computed eigenvalues of $\mathbb{F}$. (b) Boundary of the numerical range of $\mathbb{TN}$. Black dots mark the numerically computed eigenvalues of $\mathbb{TN}$.

range is indeed the convex hull of the eigenvalues of $\mathbb{F}$, see Lemma 2.31(d), the inaccuracies in the computation of the numerical approximation of the operator $\mathbb{T}$ show up in the plot of the numerical range of $\mathbb{TN}$ in Figure 2.8(b). In particular, the boundary of the numerical range of $\mathbb{TN}$ between 0 and the corner with smallest phase fails to be straight, such that it is not obvious how to stably determine the element in $W(\mathbb{TN})$ possessing the smallest phase.

For this reason, we opted for the simple idea to use that eigenvalue of $\mathbb{TN}$ as an indicator for interior eigenvalues that possesses the smallest phase among all eigenvalues of $\mathbb{TN}$ larger than the discretization error. (The discretization error is estimated via the absolute value of the smallest negative imaginary part of these eigenvalues.) In all our computations, this eigenvalue coincided with that boundary point of the numerical range of $\mathbb{TN}$ possessing the smallest phase among all corner-like boundary points where boundary curvature peaks. This not surprising, since points in the boundary of the numerical range with infinite curvature are eigenvalues of the corresponding operator by Lemma 2.31(b).

The subsequent Figure 2.9 indicates that replacing the smallest phase of the numerical range by smallest phase of the eigenvalues of $\mathbb{TN}$ yields simple-to-compute and accurate indicator for Dirichlet eigenvalues of the unit sphere and the unit ball. For these particular scattering objects,

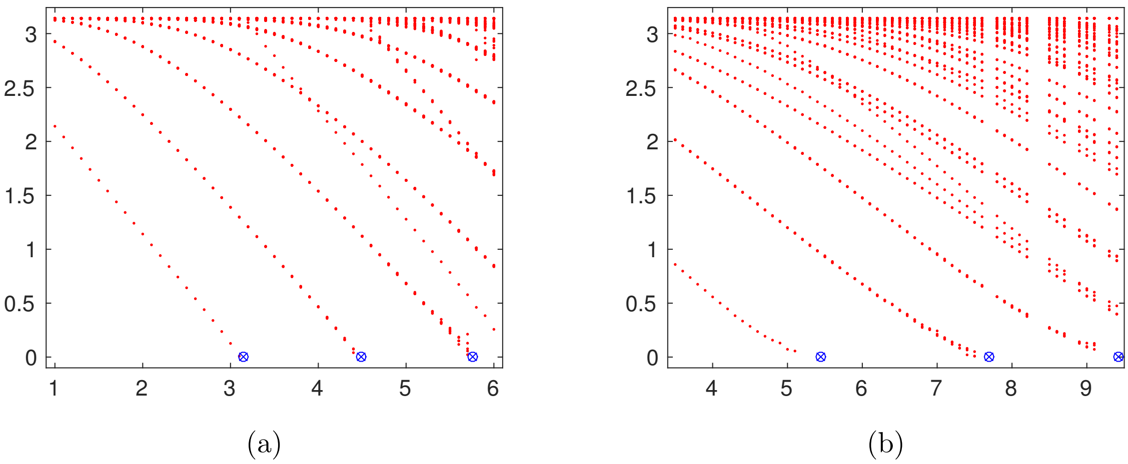


Figure 2.9.: Phases of the eigenvalues of $\mathbb{TN}_k$ for varying wavenumber k. Near field data is measured on the sphere $\mathbb{S}_2$ of radius two. (a) Scattering from the unit ball $B_1(0)$. Blue circles mark position of the exact square roots k_B to Dirichlet eigenvalues. (b) Scattering from the unit cube $[0, 1]^3$. Blue circles mark position of the exact square roots k_C to Dirichlet eigenvalues.

we have already analytically calculated the Dirichlet eigenvalues in the last section. The first three eigenvalues for the ball appear at the wavenumbers

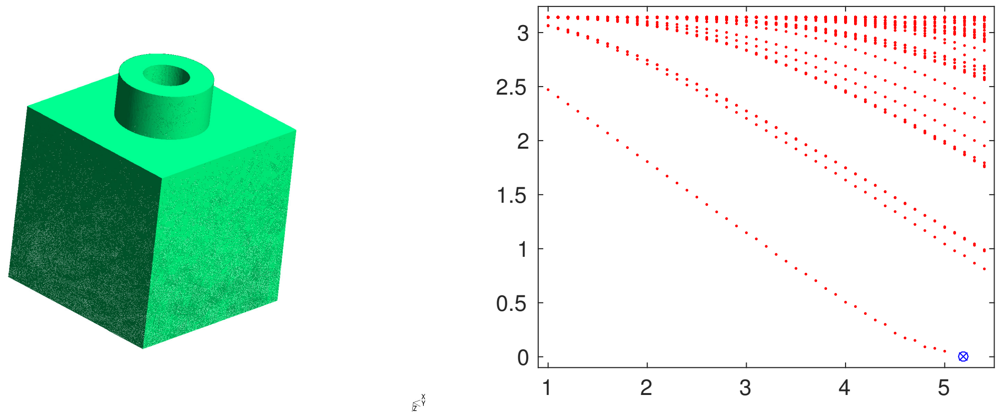
$$k_B^{(1)} = \pi, \qquad k_B^{(2)} \approx 4.49, \qquad k_B^{(3)} \approx 5.76.$$

The first three Dirichlet eigenvalues for the cube appear at the wavenumbers

$$k_C^{(1)} = \sqrt{3}\pi \approx 5.44, \qquad k_C^{(2)} = \sqrt{6}\pi \approx 7.70, \qquad k_C^{(3)} = 3\pi \approx 9.42.$$

Indeed, one can see in Figure 2.9 that the smallest phase converges to zero if and only if k_0^2 is a Dirichlet eigenvalue.

Finally, we provide an example for a non-convex scatterer D for which the Dirichlet eigenvalues of $-\Delta$ are not known analytically; the object is plotted in Figure 2.10(a) and, roughly speaking, consists of the unit square with a smaller cylinder on top. Due to numerical inaccuracies at larger wavenumbers, we only aim to approximate the smallest wavenumber k_0 such that k_0^2 is a Dirichlet eigenvalue. For this purpose, we take the last two smallest phases before the first phase jump at about 5.25, see Figure 2.10(b), and linearly extrapolate the line through these points with the 0-axis. This technique, which showed to yield stable results in [LP15a], then provides the approximation $k_0 \approx 5.19$ for the square root of the smallest Dirichlet eigenvalues of the plotted domain.



(a) (b)

Figure 2.10.: (a) The non-convex scattering object. (b) Phases of the eigenvalues of $\mathbb{TN}_k$ for varying wavenumber k. The blue circle on the 0-axis marks the extrapolated position of the square root of the Dirichlet eigenvalue.

CHAPTER 3

SCATTERING FROM PENETRABLE OBJECTS

3.1. Introduction

In this chapter we will derive the inside-outside duality for scattering by penetrable, inhomogeneous objects that are described by a scalar-valued contrast function and may or may not contain cavities. In this context we determine interior transmission eigenvalues of a corresponding transmission eigenvalue problem from far field data. This chapter is based on the work in [KL13] and [PK16] if cavities are present inside the scattering object. While the principle arguments from the last section can also be applied here, there is one major drawback in this case when compared to scattering by impenetrable scattering objects: The far field operator that arises from the scattering problem does no longer have a factorization in which the outer operators have dense range in a suitable image space independent of the wavenumber. This leads to a more complex derivation since the reduction of the analysis to the middle operator of the factorization that was done in Remark 2.6 and Remark 2.14 is no longer possible and one needs to find a way to work around this problem. As a consequence the price one pays is that the first part of the inside-outside duality only holds under certain conditions to the contrast function describing the scattering objects. Before we discuss this in more detail, we introduce the scattering problem and the corresponding transmission eigenvalue problem.

Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected complement that represents a penetrable scattering object. The properties of the scattering object are described by the real-valued refractive index $n \in L^\infty(D)$, which is equal to one outside the scattering object and bounded away from zero inside of D, i.e. there is a constant $c_0 > 0$ such that $n \geq c_0$ almost everywhere in D. To contrast the inhomogeneity against the background, we define the contrast function $q = n - 1$, which is also real-valued. We start by examining scattering objects that contain no cavities. Therefore we additionally assume that q is either strictly positive or negative almost everywhere in D and bounded away from zero, i.e. $|q| \geq c > 0$ for a constant c. In Section 3.3 we will relax this assumption to allow cavities inside the scattering object. Under this assumptions, the propagation of time harmonic acoustic waves is described by the equation

$$\Delta u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3.$$

Recall that ν and $[\cdot]_{\partial D}$ denote the exterior normal to D and the jump of a function across the boundary ∂D. We require our solution and its normal derivative not to jump across the boundary of D, i.e.

$$[u]_{\partial D} = 0 \qquad \text{and} \qquad \left[\frac{\partial u}{\partial \nu}\right]_{\partial D} = 0.$$

As in the previous sections, the total field $u = u^i + u^s$ is the sum of an incident plane wave $u^i(\cdot, \theta) = e^{ikx\cdot\theta}$ with direction $\theta \in \mathbb{S}_1$ and a scattered field $u^s(\cdot, \theta)$ that satisfies Sommerfeld's radiation condition (2.2). In the variational formulation for the scattered field, we seek a radiating solution $u^s \in H^1_{\text{loc}}(\mathbb{R}^3)$ that solves

$$\int_{\mathbb{R}^3} \left(\nabla u^s \cdot \nabla \overline{\psi} - k^2 n v \overline{\psi}\right) \mathrm{d}x = \int_D k^2 q u^i \overline{\psi} \, \mathrm{d}x \tag{3.1}$$

for all test functions $\psi \in H^1_{\text{loc}}(\mathbb{R}^3)$ with compact support, where we extended q by zero outside of D. The existence and uniqueness of a solution has been established in [CK13]. Recall that the radiating solution $u^s(\cdot, \theta)$ to the Helmholtz equation (3.1) can be expressed in terms of its far fields $u^\infty(\cdot, \theta)$, see (2.3), and the far field operator $F : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1)$ is then defined as in (2.5), i.e.

$$Fg(\hat{x}) = \int_{\mathbb{S}_1} u^\infty(\hat{x}, \theta) g(\theta) \, \mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}_1. \tag{3.2}$$

The far field operator is compact and since we choose the contrast function q to be real-valued, it retains the properties we already discussed in the previous chapter: It is normal and its eigenvalues lie on the circle $\{z \in \mathbb{C} : |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane, see [CK13].

The scattering problem is closely linked to an interior transmission eigenvalue problem, which we will introduce now. A squared wavenumber k^2 is called an interior transmission eigenvalue if there are non-trivial functions $u, w \in L^2(D)$, $u - w \in H^2_0(D)$, which solve the following interior transmission eigenvalue problem

$$\begin{aligned} \Delta u + k^2 n(x) u = 0 \quad \text{in } D, \qquad \Delta w + k^2 w = 0 \qquad \text{in } D, \\ u = w \quad \text{on } \partial D, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \qquad \text{on } \partial D, \end{aligned} \tag{3.3}$$

in a distributional sense, i.e.

$$\int_D w(\Delta\phi + k^2\phi) \, \mathrm{d}x = 0, \quad \int_D u(\Delta\phi + k^2(1+q)\phi) \, \mathrm{d}x = 0 \quad \forall \phi \in C_0^\infty(D)$$

and

$$\int_D w(\Delta\phi + k^2\phi) \, \mathrm{d}x = \int_D u(\Delta\phi + k^2(1+q)\phi) \, \mathrm{d}x \quad \forall \phi \in C^\infty(D).$$

The first and obvious link between the transmission eigenvalue problem and the scattering problem is the injectivity of the far field operator F, see [KG08]. More precisely, F is injective if k^2 is no interior transmission eigenvalue, or conversely, if F is not injective, then k^2 must be an interior transmission eigenvalue. We want to provide a second link via the inside-outside duality method by characterizing the interior transmission eigenvalues by the behavior of the eigenvalues of the far field operator. As we will show in Lemma 3.3, the eigenvalues λ_j of the far field operator converge to zero from one specific side, depending on the sign of the contrast q. More precisely, if q is positive or negative, then either $\operatorname{Re}\lambda_j > 0$ or $\operatorname{Re}\lambda_j < 0$ respectively for large $j \in \mathbb{N}$. To simplify notation, we follow [KL13] and define

$$\sigma := \begin{cases} 1 & \text{if } q > 0 \text{ in } D, \\ -1 & \text{if } q < 0 \text{ in } D. \end{cases}$$

To indicate the main result of this chapter, we rewrite the eigenvalues λ_j of the far field operator F

in polar coordinates, such that

$$\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j), \quad \vartheta_j \in [0, \pi] \tag{3.4}$$

and each eigenvalue λ_j corresponds to its phase ϑ_j. The convergence characteristic of the eigenvalues λ_j allows the definition of a smallest and a largest phase, i.e. if

$$\sigma = 1, \qquad \vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j \qquad\qquad \text{and if} \qquad\qquad \sigma = -1, \qquad \vartheta_* := \min_{j\in\mathbb{N}} \vartheta_j. \tag{3.5}$$

The inside-outside duality now states that interior transmission eigenvalues k_0^2 are characterized by the behavior of the smallest phase $\vartheta_*(k)$ or the largest phase $\vartheta^*(k)$, depending on the sign of q. More precisely, the first part of the inside-outside duality states that for interior transmission eigenvalues k_0^2, for which the expression $\alpha(k_0)$ in (3.14) does not vanish, the smallest phase $\vartheta_*(k)$ converges to zero if the contrast q is negative or the largest phase $\vartheta^*(k)$ converges to π if q is positive for $k \to k_0$, see Theorem 3.7. On the other hand, if one of the extremal phases converges to zero or to π respectively for $k \to k_0$, then k_0^2 is an interior transmission eigenvalue, see Theorem 3.8. A similar statement holds in the presence of cavities, see Theorem 3.16 and Theorem 3.17. Note that unlike in the previous chapter, where we obtained an unconditional characterization of interior eigenvalues, the first part of the inside-outside duality now only holds under the condition that the expression $\alpha(k_0)$ does not vanish.

The remainder of this chapter in structured as follows. In the next Section we derive the inside-outside duality for penetrable scattering objects that contain no cavities. In Section 3.3 we expand these results by allowing cavities inside the scattering object. In Section 3.4 we will then derive material parameter that allow for a full characterization of interior transmission eigenvalues. Finally in Section 3.5 we will show that we can use the analytical results to compute interior transmission eigenvalues for scattering objects with or without cavities by applying the algorithm we have introduced previously. In this context we will also discuss the advantages and disadvantages of the inside-outside duality algorithm from a numerical point of view.

3.2. Characterizing Interior Transmission Eigenvalues from Far Field Data

The derivation of the inside-outside duality for penetrable scattering objects follows along the same lines as the derivation of the inside-outside duality for impenetrable scattering objects. In this section we will therefore proceed in the following way. First we state a factorization of the far field operator F and examine its properties in Lemma 3.2, which will help us to establish a link between interior transmission eigenvalues and the far field operator. Then we will use these results in order to show in Theorem 3.3 that its eigenvalues λ_j converge to zero from one specific side. From this point on, relying on the same phase characterization that we have already used in Remark 3.4, we will calculate an the auxiliary derivative α in (3.14) and use it to finally state the inside-outside duality in Theorem 3.16 and Theorem 3.8.

To make the link between interior transmission eigenvalues and the properties of the far field operator more explicit, we will now introduce a factorization of the far field operator, that has been derived in [KL13, Theorem 2.5] in a slightly adapted form. First we introduce the Herglotz operator $H : L^2(\mathbb{S}_1) \to L^2(D)$ by

$$(H\psi)(x) = \int_{\mathbb{S}_1} \psi(\theta) e^{ikx\cdot\theta} \,\mathrm{d}s(\theta), \quad x \in D.$$

Its adjoint $H^* : L^2(D) \to L^2(\mathbb{S}_1)$ is then given by

$$H^*(\psi)(x) = \int_D \psi(\theta) e^{-ikx\cdot\theta} \, \mathrm{d}s(\theta), \quad x \in \mathbb{S}_1,$$

which is the far field w^∞ of the volume potential

$$w(x) = \int_D \psi(y)\Phi(x,y) \, \mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Due to the properties of the fundamental solution $\Phi(x,y) = e^{ikx\cdot y}/|x-y|, x \neq y$, it holds that w is a radiating solution to $\Delta w + k^2 w = -\psi$ in $\mathbb{R}^3$. Finally we introduce the operator $T : L^2(D) \to L^2(D)$ by $Tf = k^2 q(f + v|_D)$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the radiating weak solution to

$$\Delta v + k^2(1+q)v = -k^2 q f \quad \text{in } \mathbb{R}^3, \tag{3.6}$$

i.e.

$$\int_{\mathbb{R}^3} \left(\nabla v \cdot \nabla \overline{\psi} - k^2(1+q)v\overline{\psi}\right) \mathrm{d}x = \int_D k^2 q f \overline{\psi} \, \mathrm{d}x \tag{3.7}$$

for all $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Uniqueness and existence of the solution to (3.7) has also been established in [KG08]. The following lemma states a factorization of the far field operator, which we will use to examine the link between far field data and interior transmission eigenvalues more closely.

Lemma 3.1. *The far field operator can be factorized as $F = H^*TH$.*

Proof. For a proof see [KL13, Theorem 2.5] or the proof of Theorem 4.2, where we prove a factorization for a more complex scattering equation. The arguments easily transfer to this case. ■

It is essential to examine the properties of the middle operator T of the factorization for the derivation of the inside-outside duality. Our objective is to establish a statement similar to Lemma 2.1 in the previous section. To this end, we first give a characterization of the image of the Herglotz wave operator. When considering scattering by penetrable scattering objects, a characterization of the range of the outer operator of the factorization is essential, since its range is no longer dense in its image space and therefore the transformation in Remark 2.6 is no longer possible. The closure of the range of the Herglotz wave operator is given by the L^2-solutions of the Helmholtz equation in D, i.e. if we define

$$X = \left\{ w \in L^2(D) : \int_D w\left(\Delta\psi + k^2\psi\right) \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D) \right\}, \tag{3.8}$$

and denote by $\overline{\mathcal{R}(H)}$ the closure of the range of H in $L^2(D)$, we have that $X = \overline{\mathcal{R}(H)}$. Using this characterization, we can prove all the necessary properties of T that we will need for our analysis.

Lemma 3.2. *For $k > 0$ the following holds:*
(a) $\sigma\mathrm{Im}\,(Tf,f)_{L^2(D)} \geq 0$ for all $f \in L^2(D)$.
(b) If k^2 is an interior transmission eigenvalue with eigenpair (u,w), then $(Tw,w)_{L^2(D)} = 0$.
(c) If for a non-trivial $w \in X$ it holds that $\mathrm{Im}\,(Tw,w)_{L^2(D)} = 0$, then there exists a function $u \in L^2(D)$ such that k^2 is an interior transmission eigenvalue with corresponding eigenpair (u,w) and $u - w \in H^2_0(D)$ does not vanish.
(d) It holds that $T = k^2q(I + C)$, where I is the identity operator and $C : L^2(D) \to L^2(D)$ is a compact operator.

Proof. For a proof we refer to the proof of [KL13, Theorem 2.5, Theorem 3.1] or to Lemma 4.3, where we show these properties for a more complex version of the Helmholtz equation, which does involve two different material parameters. The arguments can easily be simplified to the present case. ■

Recall that the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of F lie on the circle $\{z \in \mathbb{C} : |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane. Now the factorization and the properties of the middle operator T from the last lemma can be used to show that the eigenvalues of the far field operator converge to zero from one distinct side. Since the proof uses essentially the same arguments as the proof of Lemma 2.2, we omit it here. For a full proof, see [KL13, Lemma 4.1].

Lemma 3.3. *Let k^2 be no interior transmission eigenvalue. Then the eigenvalues λ_j converge to zero from the right if $q > 0$ in D or from the left if $q < 0$ in D, i.e. $\sigma \mathrm{Re}\,\lambda_j > 0$ if j is large enough.*

Recall the representation of the eigenvalues λ_j of the far field operator F in polar coordinates in (3.4) and the definition of the extremal phases ϑ_* and ϑ^* in (3.5). Since the far field operator is normal and its eigenvalues have the same structure as in the previous chapter, the characterization of the smallest and largest phase from Lemma 2.4 and Lemma 2.13 still hold, i.e.

$$\cot\vartheta_* = \max_{g\in L^2(\mathbb{S}_1)} \frac{\mathrm{Re}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}, \qquad \cot\vartheta^* = \min_{g\in L^2(\mathbb{S}_1)} \frac{\mathrm{Re}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(Fg,\, g)_{L^2(\mathbb{S}_1)}}.$$

Remark 3.4. Using the factorization of $F = H^*TH$, we obtain that

$$(Fg,\, g)_{L^2(\mathbb{S}_1)} = (THg, Hg)_{L^2(D)} = (T\varphi, \varphi)_{L^2(D)},$$

where $\varphi = Hg \in \mathcal{R}(H)$ and rewrite the characterization for the smallest and the largest phase as

$$\cot\vartheta_* = \max_{\varphi\in X} \frac{\mathrm{Re}\,(T\varphi, \varphi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(T\varphi, \varphi)_{L^2(\mathbb{S}_1)}}, \qquad \cot\vartheta^* = \min_{\varphi\in X} \frac{\mathrm{Re}\,(T\varphi, \varphi)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(T\varphi, \varphi)_{L^2(\mathbb{S}_1)}}.$$

where we replaced the range of the Herglotz operator H by the space X from (3.8). This is necessary since the range of the Herglotz operator is not dense in $L^2(D)$.

After setting the framework we will now proceed to state the inside-outside duality for this scattering scenario. In the process we discuss how the presence of the space X in the characterization of the smallest and largest phase in Remark 3.4 changes the analysis and the results of the inside-outside duality.

We adopt the notation from the previous chapter and indicate the dependence of the relevant quantities on the wavenumber by writing $T = T_k, X = X_k, \vartheta_* = \vartheta_*(k)$. To point out the arising difficulty from the presence of the space X_k in the characterization of the extremal phases in Remark 3.4, assume that k_0^2 is a transmission eigenvalue such that there exists a function $w_0 \in X_{k_0}$ such that $(T_{k_0}w_0, w_0)_{L^2(D)} = 0$ according to Lemma 3.2. If we wanted to use the arguments of the proof of the first part of the inside-outside duality in Theorem 2.8, we would need to require that $w_0 \in X_k$ for all wavenumbers k in a neighborhood of k_0. But since this is clearly not the case, we need to work around this problem by eliminating the space X_k from the characterization of the extremal phases. We do this by following [KL13] and introduce a projection operator $P_k : X_k \to L^2(D)$ which we assume to be differentiable with respect to the wavenumber k. Then the characterization of the smallest and largest phase can be written as

$$\cot\vartheta_* = \max_{w\in L^2(D)} \frac{\mathrm{Re}\,(T_kP_kw, P_kw)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(T_kP_kw, P_kw)_{L^2(\mathbb{S}_1)}}, \qquad \cot\vartheta^* = \min_{w\in L^2(D)} \frac{\mathrm{Re}\,(T_kP_kw, P_kw)_{L^2(\mathbb{S}_1)}}{\mathrm{Im}\,(T_kP_kw, P_kw)_{L^2(\mathbb{S}_1)}}. \tag{3.9}$$

Note that although this eliminates the space X_k, we now have to include the projection operator into our considerations, which complicates the derivation of the inside-outside duality. To show that a projection operator exists, let W be the completion of $C_0^\infty(D)$ with respect to the norm $\|\psi\|_W := \|\Delta\psi + k^2\psi\|_{L^2(D)}$. Then a projection P_k is given by

$$P_k g = g - \Delta\hat{w} + k^2\hat{w}, \tag{3.10}$$

with $\hat{w} \in W$ as the solution to the W-coercive variational problem

$$\int_D (\Delta\hat{w} + k^2\hat{w})(\Delta\psi + k^2\psi)\,\mathrm{d}x = \int_D g(\Delta\psi + k^2\psi)\,\mathrm{d}x \qquad \forall\psi \in W.$$

The projection property is due to the coercivity of this formulation, since $g \in X_k$ implies that the right-side of the latter equation vanishes, implying that $\hat{w} = 0$ and therefore $P_k g = g$. On the other hand for an arbitrary $g \in L^2(D)$, $P_k g$ clearly solves the Helmholtz equation due to the construction of P such that $P_k g \in X_k$. Finally the operator is differentiable with respect to k since the mapping $k \to \hat{w}_k$ is differentiable with respect to k. Note that this particular definition of the projection operator is arbitrary and plays no further role in our analysis.

Recalling the proof of first part of the inside-outside duality in Theorem 2.8 we need to calculate the derivative

$$\alpha(k_0) := \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D)}\Big|_{k=k_0}. \tag{3.11}$$

In particular we need to show that this derivative is real-valued and does not vanish. First we will derive an explicit expression of $\alpha(k_0)$. As a first step, we calculate an auxiliary derivative.

Lemma 3.5. *Let $k_0^2 > 0$ be an interior transmission eigenvalue with transmission eigenpair (u_0, w_0). Then the map $k \to (T_k w_0, w_0)_{L^2(D)}$ is differentiable in k_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D)}\Big|_{k=k_0} = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x,$$

where $v_{k_0} \in H_0^2(D)$ is the radiating solution of (3.7) for $k = k_0$ and $f = w_0$, i.e.

$$\int_D \left(\nabla v_{k_0} \cdot \nabla\overline{\psi} - k_0^2(1+q)v_{k_0}\overline{\psi}\right)\,\mathrm{d}x = \int_D k_0^2 q w_0 \overline{\psi}\,\mathrm{d}x \qquad \forall\psi \in H^1(D). \tag{3.12}$$

Proof. Note that $v_{k_0} \in H_0^2(D)$ since its far field vanishes and therefore v_{k_0} vanishes in the exterior of D due to Rellich's lemma, see the proof of [KL13, Theorem 3.1] for details. Setting $v = v_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ as the radiating solution to (3.7) for variable wavenumber k and $f = w_0$, we find that differentiating that expression yields

$$\int_{\mathbb{R}^3} \left(\nabla v_k' \cdot \nabla\overline{\psi} - k^2(1+q)v_k'\overline{\psi}\right)\,\mathrm{d}x = 2k\left[\int_D q w_0\overline{\psi}\,\mathrm{d}x + \int_{\mathbb{R}^3}(1+q)v_k\overline{\psi}\,\mathrm{d}x\right] \qquad \forall\psi \in C_0^\infty(\mathbb{R}^3), \tag{3.13}$$

where $v_k' := \mathrm{d}/\mathrm{d}k\; v_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$. Note also that $(T_{k_0} w_0, w_0)_{L^2(D)} = 0$ by Theorem 3.2, i.e.

$$\int_D q\left(|w_0|^2 + v_{k_0}\overline{w_0}\right)\,\mathrm{d}x = 0.$$

Using this equation we get

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D)}\Big|_{k=k_0} = \frac{\mathrm{d}}{\mathrm{d}k}\int_D qk^2(w_0 + v_k)\overline{w_0}\,\mathrm{d}x\Big|_{k=k_0} = k_0^2\int_D q v_{k_0}'\overline{w_0}\,\mathrm{d}x.$$

Eliminating w_0 from this equation by using (3.12) for $\psi = v'_{k_0}$ and (3.13) for $\psi = v_{k_0}$, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D)}\Big|_{k=k_0} &= \int_D \left(\nabla\overline{v_{k_0}}\cdot\nabla v'_{k_0} - k_0^2(1+q)\overline{v_{k_0}}v'_{k_0}\right)\,\mathrm{d}x \\ &= 2k_0\int_D \left(qw_0\overline{v_{k_0}} + (1+q)v_{k_0}\overline{v_{k_0}}\right)\,\mathrm{d}x = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x,\end{aligned}$$

which concludes the proof. ■

We now use this auxiliary derivative to calculate the full derivative $\alpha(k_0)$. Note that the following proof in a simplified version of the proof of [KL13, Lemma 5.3].

Theorem 3.6. *Let $k_0^2 > 0$ be an interior transmission eigenvalue with transmission eigenpair (u_0, w_0) where $w_0 \in X_{k_0}$. Then the map $k \to (T_k P_k w_0, P_k w_0)_{L^2(D)}$ is differentiable in k_0 and*

$$\alpha(k_0) = \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D)}\Big|_{k=k_0} = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x + 4k_0 \,\mathrm{Re}\int_D \overline{w_0}v_{k_0}\,\mathrm{d}x, \tag{3.14}$$

where $v_{k_0} \in H_0^2(D)$ is again the radiating solution of (3.12).

Proof. By definition of P_k, we have that $P_k w_0 \in X_k$, so that $w_k := P_k w_0 \in L^2(D)$ solves the Helmholtz equation, i.e.

$$\int_D w_k(\Delta\psi + k^2\psi)\,\mathrm{d}x = 0 \qquad \forall\psi \in C_0^\infty(D).$$

Note also in this context that $w_{k_0} = w_0$. The projection P_k in (3.10) is differentiable and $w'_k := \mathrm{d}/\,\mathrm{d}k P_k w_0$ solves

$$\int_D w'_k(\Delta\overline{\psi} + k^2\overline{\psi})\,\mathrm{d}x = -2k\int_D w_k\overline{\psi}\,\mathrm{d}x \quad \forall\psi \in C_0^\infty(D). \tag{3.15}$$

By applying the chain rule, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, w_0)_{L^2(D)} &= (T'_k P_k w_0, w_0) + (T_k P'_k w_0, P_k w_0) + (T_k P_k w_0, P'_k w_0) \\ &= (T'_k P_k w_0, w_0) + \overline{(T_k^* P_k w_0, P'_k w_0)} + (T_k P_k w_0, P'_k w_0).\end{aligned}$$

To simplify this expression, we show that $T_{k_0} = T_{k_0}^*$ on the space X_{k_0}. To introduce the adjoint $T_{k_0}^*$, we note that since $v_{k_0} \in H_0^2(D)$ vanishes outside of D we can neglect the radiation condition and taking the complex conjugate of equation (3.12) yields that $\overline{v_{k_0}}$ solves

$$\int_D \left(\nabla\overline{v_{k_0}}\cdot\nabla\psi - k_0^2(1+q)\overline{v_{k_0}}\psi\right)\,\mathrm{d}x = \int_D k_0^2 q\overline{w_0}\psi\,\mathrm{d}x \qquad \forall\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3).$$

Furthermore, for $f \in L^2(D)$, let $v_f \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ be the radiating solution of (3.7), i.e.

$$\int_{\mathbb{R}^3} \left(\nabla v_f\cdot\nabla\overline{\psi} - k^2(1+q)v_f\overline{\psi}\right)\,\mathrm{d}x = \int_D k^2 qf\overline{\psi}\,\mathrm{d}x$$

for $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Then we calculate

$$
\begin{aligned}
(T_{k_0} f, w_0)_{L^2(D)} &= (k_0^2 q f, w_0)_{L^2(D)} + \int_D k_0^2 q v_f \overline{w_0}\,\mathrm{d}x \\
&= (f, k_0^2 q w_0)_{L^2(D)} - \int_D \left(\nabla \overline{v_{k_0}} \cdot \nabla v_f + k_0^2 (1+q) \overline{v_{k_0}} v_f\right) \mathrm{d}x \\
&= (f, k_0^2 q w_0)_{L^2(D)} + \int_D k_0^2 q f \overline{v_{k_0}}\,\mathrm{d}x = (f, k_0^2 q (w_0 + v_{k_0}))_{L^2(D)},
\end{aligned}
$$

such that $T^*_{k_0} w_0 = k_0^2 q(w_0 + v_{k_0})$ and therefore $T_{k_0} w_0 = T^*_{k_0} w_0$. Using this result and Lemma 3.5, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}k} (T_{k_0} w_0, w_0)_{L^2(D)} \Big|_{k=k_0} = \frac{2}{k_0} \int_D |\nabla v_{k_0}|^2 \,\mathrm{d}x + 2\mathrm{Re}\,(T_{k_0} w_0, P'_{k_0} w_0).
$$

Recall that $w'_k = \frac{\mathrm{d}}{\mathrm{d}k} P_k w_0 \in L^2(D)$, where w_k solves the Helmholtz equation. Since $v_{k_0} \in H^2_0(D)$, we can use (3.15) to obtain

$$
\begin{aligned}
2\mathrm{Re}\,(T_{k_0} w_0, P'_{k_0} w_0)_{L^2(D)} &= 2\ \mathrm{Re} \int_D q k_0^2 (v_{k_0} + w_0) \overline{w'_{k_0}}\,\mathrm{d}x = 2\ \mathrm{Re} \int_D (\Delta v_{k_0} + k_0^2 v_{k_0}) \overline{w'_{k_0}}\,\mathrm{d}x \\
&= 4 k_0\ \mathrm{Re} \int_D \overline{w_{k_0}} v_{k_0}\,\mathrm{d}x,
\end{aligned}
$$

where we used Green's identity and (3.12) for the first equality. This shows the assertion.

■

Now we can state the first part of the inside-outside duality. The proof is essentially a copy of the proof of Theorem 2.8. We include it anyway to show how the projection allows an adaption of the arguments.

Theorem 3.7 (Inside-Outside Duality - Part 1). *Let k_0^2 be an interior transmission eigenvalue and assume that $\alpha(k_0)$ in (3.14) does not vanish.*
(a) If $\sigma = -1$, then

$$
\lim_{k \nearrow k_0} \vartheta_*(k) = \pi \quad \text{if } \alpha > 0, \qquad \lim_{k \searrow k_0} \vartheta_*(k) = \pi \quad \text{if } \alpha < 0.
$$

(b) If $\sigma = 1$, then

$$
\lim_{k \nearrow k_0} \vartheta^*(k) = 0 \quad \text{if } \alpha > 0, \qquad \lim_{k \searrow k_0} \vartheta^*(k) = 0 \quad \text{if } \alpha < 0.
$$

Proof. (a) Due to Lemma 3.2, there is a function $w_0 \in X_{k_0}$ such that $(T_{k_0} w_0, w_0)_{L^2(\partial D)} = 0$. Assume that $I = (k_0 - \varepsilon, k_0 + \varepsilon)$ is an interval that does not contain other interior transmission eigenvalues. We have shown that

$$
\cot \vartheta_*(k) = \max_{w \in L^2(D)} \frac{\mathrm{Re}\,(T_k P_k w, P_k w)_{L^2(D}}{\mathrm{Im}\,(T_k P_k w, P_k w)_{L^2(D)}} \qquad \text{for } k \in I \setminus \{k_0\},
$$

Define $f(k) = (T_k P_k w_0, P_k w_0)_{L^2(D)}$ for $k \in I$ and note that the differentiability of P_k and T_k with respect to the wavenumber k implies that $f(k)$ is also differentiable with respect to k. Therefore we can apply Taylor's theorem again to obtain

$$
f(k) = f(k_0) + \alpha(k - k_0) + r(k),
$$

where

$$f(k_0) = (T_{k_0} P_{k_0} w_0, P_{k_0} w_0)_{L^2(D)} = (T_{k_0} w_0, w_0)_{L^2(D)} = 0$$

due to our choice of w_0. Furthermore the remainder $r(k)$ satisfies $r(k) = o(|k - k_0|)$ as $k \to k_0$. Since σ is negative, $\operatorname{Im}(r(k)) \le 0$ due to Lemma 3.2, because the derivative $\alpha = \mathrm{d}f/\mathrm{d}k\ f(k)$ at k_0 is real-valued and $\operatorname{Im} f(k) \le 0$. Hence,

$$\cot \vartheta_*(k) = \max_{w \in L^2(D)} \frac{\operatorname{Re}(T_k P_k w, P_k w)_{L^2(D)}}{\operatorname{Im}(T_k P_k w, P_k w)_{L^2(\mathbb{S}_1)}} \overset{w=w_0}{\ge} \frac{\alpha(k-k_0) + \operatorname{Re}(r(k))}{\operatorname{Im}(r(k))} \to \infty \quad \text{as } k \nearrow k_0. \tag{3.16}$$

If α is positive, $k \nearrow k_0$ implies that $\alpha(k-k_0) \le 0$ tends slower to zero than $0 > \operatorname{Im}(r(k)) = o(|k-k_0|)$, that is, $[\alpha(k-k_0) + \operatorname{Re}(r(k))]/\operatorname{Im}(r(k)) \to \infty$. Obviously, $\cot \vartheta_*(k) \to \infty$ for $\vartheta_*(k) \in (0, \pi)$ implies that $\vartheta_*(k) \to 0$. If α is negative, then $k \searrow k_0$ implies the same result.
(b) We use the characterization of the largest phase and adapt the arguments from the (a)-part suitably to arrive at our assertion. ■

Now we state the second part of the inside-outside duality. Note that unlike in the first part, where we assumed that the derivative $\alpha(k_0)$ does not vanish at an interior transmission eigenvalue, we don't need to make an additional assumptions for the second part. This pattern will be a recurrent theme in all the discussions of scattering by penetrable scattering objects.

Theorem 3.8 (Inside-Outside Duality - Part 2)**.** *Let k_0 be such that $I = (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ contains no wavenumber k such that k^2 is an interior transmission eigenvalue. If*

$$\sigma = -1 \qquad \textit{and} \qquad \lim_{I \ni k \to k_0} \vartheta_*(k) = 0 \tag{3.17}$$

or if

$$\sigma = 1 \qquad \textit{and} \qquad \lim_{I \ni k \to k_0} \vartheta^*(k) = \pi \tag{3.18}$$

then k_0^2 is an interior transmission eigenvalue.

Proof. For a proof we refer to either [KL13, Theorem 6.3] or to Theorem 4.16, where this statement will be proven for a more complicated scattering scenario, involving two different material parameters. The arguments can easily be simplified to this case. ■

3.3. The Influence of the Presence of Cavities

In this section, we will revisit the scattering scenario that we have already discussed in the last section but will additionally allow our scattering object to contain cavities, i.e. regions where the contrast vanishes. While we in principle follow the analysis from the last section, there are several points where we need to expand the arguments. This becomes particularly relevant when we discuss the factorization and try to characterize the range of the arising Herglotz wave operator. Tools we use in this context are extensions of the single layer and double layer potential, which will allow us to represent and extend solutions of the Helmholtz equation for the scattering object.

For this section, we adapt our model assumptions from the introduction to this chapter in the following way. We assume that the scattering object $D \subset \mathbb{R}^3$ is simply connected with boundary $\partial D \in C^2$. Note that due to the analytical tools we use later, we require more boundary regularity than we required in the previous sections. Inside of D we consider a region $D_0 \subset D$ that represents a cavity inside the scattering object. The cavity D_0 can be multiple connected, such that $D \setminus \overline{D_0}$ is connected and assume that its boundary ∂D_0 is also a C^2 curve. In the following ν denotes the outward normal to ∂D or ∂D_0. The scattering object is described by a real-valued function $n \ge 1$, where $n = q + 1$ for a contrast function $q \in L^\infty(D)$ such that $n = 1$ and $q = 0$ almost everywhere in D_0. Unlike in the previous section where the contrast was allowed to be either negative or positive, we will in this section focus only on positive contrasts, assuming there is a constant $c_0 > 0$ such that $q(x) \geq c_0$ for almost all $x \in D \setminus \overline{D_0}$. Note that the arguments can easily be adapted for negative contrasts. For this setting we consider the following scattering problem: For an incident wave $u^i = e^{ikx\cdot\theta}$ with direction $\theta \in \mathbb{S}_1$ we seek a total field u that solves

$$\Delta u + k^2 n u = 0 \qquad \text{in } \mathbb{R}^3, \tag{3.19}$$

so that the scattered field $u^s = u - u^i$ satisfies Sommerfeld's radiation condition (2.2). A variational formulation for the scattered field has already been stated in (3.1). Extending the contrast q by zero outside the scattering object and using the fact that the contrast q vanishes in the cavity D_0, we can restate it in the following way: We seek a function $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, such that

$$\int_{\mathbb{R}^3} \left(\nabla u^s \cdot \nabla\overline{\psi} - k^2(1+q)v\overline{\psi}\right) \mathrm{d}x = \int_{D\setminus\overline{D_0}} k^2 q u^i \overline{\psi}\, \mathrm{d}x \tag{3.20}$$

for all test functions $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Again, the existence and uniqueness of a solution has been established in [CK13]. The interior transmission eigenvalue problem that corresponds to this scattering problem has been formulated in (3.3).

Since the scattered field satisfies Sommerfeld's radiation condition, it can therefore be expanded in terms of its far field u^∞ as in (2.3). The far field gives rise to the far field operator $F : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1)$ from (3.2). In the absence of cavities, we have already discussed its properties in the introduction, i.e. it is compact and normal and its eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lie on a circle in the complex plane with radius $8\pi^2/k$ and center point $\mathrm{i}8\pi^2/k$. From the derivation of these properties in, e.g. [CK13], it is clear that they still hold in the presence of cavities.

From now on we proceed in the following way. First we will adapt the factorization that has been used in the last section and recall its properties in Lemma 3.9. These properties imply a specific convergence direction of the eigenvalues of the far field operator. Then we will show how we need to adapt the characterization of the range of the Herglotz wave operator in Lemma 3.10. In this context we introduce extension and restriction operators that can bridge the gap between functions that are defined on the whole domain D and functions that are only defined on the region $D \setminus \overline{D_0}$, where the contrast is supported. We use these operators in Lemma 3.11 to provide the link between the middle operator of the factorization and the interior transmission eigenvalues. Using the typical phase characterizations, we will then calculate the derivative $\alpha(k_0)$ in Lemma 3.15 in order to finally state the first part and the second part of the inside-outside duality in Theorem 3.16 and Theorem 3.17.

We start by adapting the factorization that has been shown in the last section. The adaption mainly consists of defining the relevant operators on the domain $D \setminus \overline{D_0}$ instead of D. The operators in this factorization will later provide us with the necessary link to the transmission eigenvalue problem. For that purpose we introduce the Herglotz wave operator $H : L^2(\mathbb{S}_1) \to L^2(D \setminus \overline{D_0})$ by

$$(H\psi)(x) = \int_{\mathbb{S}_1} \psi(\theta) e^{ikx\cdot\theta}\, \mathrm{d}s(\theta), \quad x \in D \setminus \overline{D_0}.$$

Its adjoint $H^* : L^2(D \setminus \overline{D_0}) \to L^2(\mathbb{S}_1)$ is then given by

$$(H^*\psi)(x) = \int_{D\setminus\overline{D_0}} \psi(\theta) e^{-ikx\cdot\theta}\, \mathrm{d}s(\theta), \quad x \in \mathbb{S}_1,$$

which is the far field w^∞ of the volume potential

$$w(x) = \int_{D\setminus\overline{D_0}} \psi(y)\Phi(x,y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

Due to the properties of the fundamental solution $\Phi(x,y) = e^{ikx\cdot y}/|x-y|, x \neq y$, it holds that w is a radiating solution to $\Delta w + k^2 w = -\psi$ in $\mathbb{R}^3$. Finally we introduce the operator $T : L^2(D\setminus\overline{D_0}) \to L^2(D\setminus\overline{D_0})$ by $Tf = k^2 q(f + v|_{D\setminus\overline{D_0}})$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the radiating weak solution to

$$\Delta v + k^2(1+q)v = -k^2 qf \quad \text{in } \mathbb{R}^3, \tag{3.21}$$

i.e.

$$\int_{\mathbb{R}^3} \nabla v\cdot\nabla\overline{\psi} - k^2(1+q)v\overline{\psi}\,\mathrm{d}x = \int_{D\setminus\overline{D_0}} k^2 qf\overline{\psi}\,\mathrm{d}x \tag{3.22}$$

for all $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Uniqueness and existence of the solution to (3.22) has already been established in [KG08]. We can now state the following factorization.

Lemma 3.9. *(a) The far field operator can be factorized as $F = H^*TH$.*
(b) It holds that $T = k^2q(I + C)$, where I is the identity operator and $C : L^2(D\setminus\overline{D_0}) \to L^2(D\setminus\overline{D_0})$ is a compact operator.
(c) $\mathrm{Im}\,(Tf,f)_{L^2(D\setminus\overline{D_0})} \geq 0$ *for all $f \in L^2(D\setminus\overline{D_0})$.*

For a proof we refer to the proof of [KL13, Theorem 2.5], where this assertion has been proven for scattering object without cavities. The arguments transfer one-to-one to this case.

Note that in the corresponding theorems in the previous sections, see e.g. Lemma 3.2, we added more properties of the middle operator T in order to link transmission eigenvalues to the eigenvalues of the far field operator. These properties usually involved functions in the range of the outer operator H of the factorization. Note that in this case the range of the Herglotz operator consists of functions whose domain is $D\setminus\overline{D_0}$, while interior transmission eigenfunctions have the scattering object D as a domain. To bridge this gap, we introduce an extension operator and some corresponding function spaces. In this context we will also characterize the image space of the Herglotz wave operator. It consists of those functions in $L^2(D\setminus\overline{D_0})$ which have an extension to D that solves the Helmholtz equation. We begin by defining

$$L^2_\Delta(D) := \{w \in L^2(D),\ \Delta w \in L^2(D)\},$$

where Δw is the weak Laplacian, i.e. there exists $\eta \in L^2(D)$, so that $\int_D \eta v\,\mathrm{d}x = \int_D w\Delta v\,\mathrm{d}x$ for all $v \in C_0^\infty(D)$ and $\Delta w = \eta$. This space is equipped with the graph norm

$$\|w\|_{L^2_\Delta(D)} := \|w\|_{L^2(D)} + \|\Delta w\|_{L^2(D)}. \tag{3.23}$$

Let now $u \in L^2(D)$ be a weak solution to the Helmholtz equation,

$$\int_D u(\Delta\psi + k^2\psi)\,\mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D). \tag{3.24}$$

Then it is obvious that $u \in L^2_\Delta(D)$. We follow [CH13b, Section 3] and use Green's second identity to define the Dirichlet trace $\gamma_D u := u|_{\partial D} \in H^{-1/2}(\partial D)$ by

$$\langle \gamma_D u, \phi\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} = \int_D (u\Delta w - w\Delta u)\,\mathrm{d}x,$$

where $w \in H^2(D)$ such that $w = 0$ and $\partial w/\partial\nu = \phi$ on ∂D. Continuity of the trace operator

$\gamma_D : L^2_\Delta(D) \to H^{-1/2}(\partial D)$ is due to

$$\|\gamma_D u\|_{H^{-1/2}(\partial D)} := \sup_{\|\phi\|_{H^{1/2}(\partial D)}=1} \langle \gamma_D u, \phi\rangle_{H^{-1/2}(\partial D)\times H^{1/2}(\partial D)} \leq C\|u\|_{L^2_\Delta(D)}.$$

In the same manner we can define the trace of the normal derivative $\gamma_N u := \partial u/\partial\nu\big|_{\partial D} \in H^{-3/2}(\partial D)$ by

$$\langle \gamma_N u, \phi\rangle_{H^{-3/2}(\partial D)\times H^{3/2}(\partial D)} = -\int_D (u\Delta w - w\Delta u)\,\mathrm{d}x,$$

where $w \in H^2(D)$ is such that $w = \phi$ and $\partial w/\partial\nu = 0$ on ∂D. The operator $\gamma_N : L^2_\Delta(D) \to H^{-3/2}(\partial D)$ is also continuous due to

$$\|\gamma_N u\|_{H^{-3/2}(\partial D)} := \sup_{\|\phi\|_{H^{3/2}(\partial D)=1}} \langle \gamma_N u, \phi\rangle_{H^{-3/2}(\partial D)\times H^{3/2}(\partial D)} \leq C\|u\|_{L^2_\Delta(D)}.$$

It is well known that H^1-solutions of the Helmholtz equation can be represented by Green's formula. In [CH13b, Section 3], this result was extended to L^2-solutions, showing that a solution $u \in L^2_\Delta(D)$ to (3.24) can be written as

$$u = \mathrm{SL}\,(\gamma_N u) - \mathrm{DL}(\gamma_D u), \tag{3.25}$$

where $\mathrm{SL} : H^{-3/2}(\partial D) \to L^2(D)$ and $\mathrm{DL} : H^{-1/2}(\partial D) \to L^2(D)$ are continuous extensions of the single layer potential and the double layer potential, given by

$$(\mathrm{SL}\phi)(x) := \int_{\partial D} \Phi(x,y)\phi(y)\,\mathrm{d}y, \qquad \text{in } \mathbb{R}^3 \setminus \partial D, \tag{3.26}$$

$$(\mathrm{DL}\,\psi)(x) := \int_{\partial D} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y)\,\mathrm{d}y, \qquad \text{in } \mathbb{R}^3 \setminus \partial D. \tag{3.27}$$

Now we introduce two different spaces $X_{D\setminus\overline{D_0}}$ and X_D, which contain those L^2-functions that are solutions to the Helmholtz equation on the domains $D \setminus \overline{D_0}$ and D:

$$X_{D\setminus\overline{D_0}} = \Big\{ w \in L^2(D\setminus\overline{D_0}) : \int_{D\setminus\overline{D_0}} w(\Delta\psi + k^2\psi)\,\mathrm{d}x = 0 \quad \forall\psi \in C_0^\infty(D\setminus\overline{D_0}) \Big\}$$

and

$$X_D = \Big\{ W \in L^2(D) : \int_D W(\Delta\psi + k^2\psi)\,\mathrm{d}x = 0 \quad \forall\psi \in C_0^\infty(D) \Big\}.$$

The image of the Herglotz wave operator can now be characterized by a space X, which contains those functions in $L^2(D\setminus\overline{D_0})$ that have an extension which solves the Helmholtz equation in D. Therefore this space can be seen as a kind of interpolation space between X_D and $X_{D\setminus\overline{D_0}}$. We define

$$X = \Big\{ w \in L^2(D\setminus\overline{D_0}) : \exists W \in X_D,\ w = W|_{D\setminus\overline{D_0}} \Big\}. \tag{3.28}$$

Motivated by the definition of the space X, we define an extension operator by $E : X \to X_D$ by $E(w) = W$, where $W \in X_D$ is the unique extension of w that solves the Helmholtz equation on D. Due to Green's representation theorem for L^2-solutions of the Helmholtz equation, the extension operator has the explicit representation

$$Ew(x) = \mathrm{SL}\,(\gamma_N w)\,(x) - \mathrm{DL}(\gamma_D w)(x), \qquad x \in D. \tag{3.29}$$

Obviously it holds that $X \subset X_{D\setminus\overline{D_0}}$. Again due to Green's representation theorem, we can write a function $w \in X_{D\setminus\overline{D_0}}$ as

$$w(x) = \mathrm{DL}(\gamma_D w)(x) - \mathrm{SL}\left(\gamma_N w\right)(x) + \mathrm{DL}(w|_{\partial D_0})(x) - \mathrm{SL}\left(\frac{\partial w}{\partial \nu}\Big|_{\partial D_0}\right)(x), \qquad x \in D \setminus \overline{D_0}.$$

Note that if $w \in X_{D\setminus\overline{D_0}} \cap X = X$, the second part of the equation is zero, since the jump of w and its normal derivative $\partial w/\partial\nu$ over ∂D_0 vanish. Therefore a map $A : X_{D\setminus\overline{D_0}} \to X$ is given by

$$Aw(x) = \mathrm{DL}(\gamma_D w)(x) - \mathrm{SL}\left(\gamma_N w\right)(x), \qquad x \in D \setminus \overline{D_0}, \tag{3.30}$$

where $Aw = w$ for $w \in X$. We will use this operator later to define a projection onto the space X. First we characterize the image of the Herglotz operator.

Lemma 3.10. *It holds that* $X = \mathrm{closure}_{L^2(D\setminus\overline{D_0})}\, \mathcal{R}(H)$.

Proof. We first define an extension $\tilde{H} : L^2(\mathbb{S}_1) \to L^2(D)$ of the Herglotz operator H by

$$\tilde{H}\psi(x) = \int_{\mathbb{S}_1} e^{ikx\cdot\theta}\psi(\theta)\,\mathrm{d}\theta \qquad x \in D,$$

so that $H\psi = \tilde{H}\psi|_{D\setminus\overline{D_0}}$. Let now $w = H\psi$ for an arbitrary function $\psi \in L^2(\mathbb{S}_1)$. Then the extension $W = \tilde{H}\psi$ solves the Helmholtz equation in D and $w = W|_{D\setminus\overline{D_0}}$ shows that $w \in X$. Next we show that the space X is closed to conclude that $\overline{\mathcal{R}(H)} \subset X$. To this end let $(w_j)_{j\in\mathbb{N}}$ be an arbitrary sequence in X, where $w_j \to w$ in $L^2(D \setminus \overline{D_0})$. We will show that $w \in X$. Due to the attributes of the space X, there is a corresponding sequence $(W_j)_{j\in\mathbb{N}} \subset X_D$ such that $W_j|_{D\setminus\overline{D_0}} = w_j$. Since each function W_j solves the Helmholtz equation in a weak sense, we know from standard regularity results, see e.g. [McL00], that W_j is analytic inside of D. We choose a function $\phi \in C_0^\infty(D)$, such that $\phi = 1$ in $\overline{D_0}$ and use Green's classical representation formula for ϕW_j and partial integration to obtain for $x \in D_0$:

$$\begin{aligned} W_j(x) &= -\int_D \left[\Delta\left(\phi(y)W_j(y)\right) + k^2\phi(y)W_j(y)\right]\Phi(x,y)\,\mathrm{d}y \\ &= -\int_{D\setminus\overline{D_0}} \left[2\nabla\phi(y)\cdot\nabla W_j(y) + W_j(y)\Delta\phi(y)\right]\,\mathrm{d}y \\ &= \int_{D\setminus\overline{D_0}} W_j(y)\left[2\,\mathrm{div}\left(\nabla\phi(y)\Phi(x,y)\right) - \Delta\phi(y)\Phi(x,y)\right]\,\mathrm{d}y. \end{aligned}$$

Since $(W_j|_{D\setminus\overline{D_0}})_{j\in\mathbb{N}}$ is a Cauchy-sequence in $L^2(D \setminus \overline{D_0})$, we conclude from the last calculation that $(W_j|_{\overline{D_0}})_{j\in\mathbb{N}}$ is a Cauchy-sequence in $L^2(\overline{D_0})$. This implies that W_j is a Cauchy-sequence in X_D and since this space is closed, there is a function $W \in X_D$ such that $W_j \to W$ and $W|_{D\setminus\overline{D_0}} = w$. Therefore $w \in X$, which shows the closedness of the space X.

To complete the proof we choose an arbitrary $w \in X$ and show, that $w \in \overline{\mathcal{R}(H)}$. Since $w \in X$, it follows that there exists $W \in L^2(D)$ with $Ew = W$ and W solves the Helmholtz equation in D. Then it follows that $W \in \overline{\mathcal{R}(\tilde{H})}$. Therefore there is a sequence $W_j \subset \mathcal{R}(\tilde{H})$, so that $\|W_j - W\|_{L^2(D)} \to 0$ as $j \to \infty$. It follows that $\|W_j|_{D\setminus\overline{D_0}} - w\|_{L^2(D)} \to 0$ and as $W_j|_{D\setminus\overline{D_0}} \in \mathcal{R}(H)$, we conclude that $w \in \overline{\mathcal{R}(H)}$, which shows the assertion. ■

Now we can add the missing properties of the operator T, which help us to link the interior transmission eigenvalue problem to the far field data. For a proof, we again refer to [KL13, Theorem 3.1], where this theorem has been proven for scattering objects without cavities.

Theorem 3.11. *(a) Let k^2 be an interior transmission eigenvalues with transmission pair $(U, W) \in L^2(D) \times L^2(D)$ and set $w := W|_{D\setminus\overline{D_0}}$. Then $w \in X$ and it holds that $(Tw, w)_{L^2(D\setminus\overline{D_0})} = 0$.*
(b) If $w \in X$ satisfies $(Tw, w)_{L^2(D\setminus\overline{D_0})} = 0$, then there exists a function $u \in L^2(D)$ such that k^2 is an interior transmission eigenvalue with corresponding eigenpair (u, Ew). Furthermore $u - Ew \in H_0^2(D)$.

As mentioned above the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of the far field operator F lie on a circle in the complex plane with radius $8\pi^2/k$ and center point $\mathrm{i}8\pi^2/k$ and converge to zero due to the compactness of the far field operator. Note that the properties of the operator T from Lemma 3.9 imply that the eigenvalues λ_n converge to zero from the right.

Lemma 3.12. *Let k^2 be no interior transmission eigenvalue. Then λ_n converges to zero from the right, i.e.* $\mathrm{Re}\,(\lambda_n) > 0$ *for $n \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues λ_j in polar coordinates in (3.4) and the definition of the largest phase ϑ^* in (3.5). We use the standard characterization for the largest phase from Remark 3.4 to write

$$\cot\vartheta^* = \min_{w\in X} \frac{\mathrm{Re}\,(Tw, w)_{L^2(D\setminus\overline{D_0})}}{\mathrm{Im}\,(Tw, w)_{L^2(D\setminus\overline{D_0})}}. \tag{3.31}$$

As usual, we indicate the dependence of all quantities on the wavenumber by writing $X = X_k$, $F = F_k$, $T = T_k$, $A = A_k$ and so on. For the first part of the inside-outside duality we need to replace the space X_k in (3.31) by using a projection onto this space. To define the projection, we introduce the space W as the completion of $C_0^\infty(D\setminus\overline{D_0})$ with respect to the norm $\|\psi\|_W = \|(\Delta\psi + k^2\psi)\|_{L^2(D\setminus\overline{D_0})}$. Now we define $P_k : L^2(D\setminus\overline{D_0}) \to X_k$ by

$$P_k g = A_k(g - (\Delta + k^2)\hat{w}_k) \tag{3.32}$$

where $\hat{w}_k \in W$ is the unique solution to the W-coercive problem

$$\int_{D\setminus\overline{D_0}} (\Delta\hat{w}_k + k^2\hat{w}_k)(\Delta\psi + k^2\psi)\,\mathrm{d}x = \int_{D\setminus\overline{D_0}} g(\Delta\psi + k^2\psi)\,\mathrm{d}x \qquad \forall\psi\in W \tag{3.33}$$

and $A_k : X^k_{D\setminus\overline{D_0}} \to X_k$ is the map defined in (3.30).

Lemma 3.13. *The map $P_k : L^2(D) \to X_k$ is a projection operator and the derivative* $\mathrm{d}/\mathrm{d}kP_k$ *exists and is well-defined.*

Proof. To show differentiability of P_k with respect to k, note that the operator A_k essentially consists of a sum of a single layer and a double layer potential. From the Taylor expansion of the fundamental solution Φ_k with respect to the variable k, the differentiability of the single layer and double layer potential follows. More precisely, the series expansion of the fundamental solution is

$$\Phi_k(x, y) = \frac{e^{ik|x-y|}}{|x-y|} = \sum_{n=0}^{\infty} \frac{(ik|x-y|)^n}{n!|x-y|} = \sum_{n=0}^{\infty} \frac{i^n k^n |x-y|^{n-1}}{n!}$$

and therefore the single layer potential from (3.26) can be written as

$$\mathrm{SL}_k\phi(x) = \int_{\partial D} \sum_{n=0}^{\infty} \frac{i^n k^n |x-y|^{n-1}}{n!} \phi(y)\,\mathrm{d}s(y) = \sum_{n=0}^{\infty} \frac{i^n k^n}{n!} \int_{\partial D} |x-y|^{n-1}\phi(y)\,\mathrm{d}s(y).$$

Therefore differentiating with respect to k yields

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{SL}_k\phi(x) = \sum_{n=0}^{\infty} \frac{n i^n k^{n-1}}{n!} \int_{\partial D} |x-y|^{n-1}\phi(y)\,\mathrm{d}s(y) = \sum_{n=1}^{\infty} \frac{i^n k^{n-1}}{(n-1)!} \int_{\partial D} |x-y|^{n-1}\phi(y)\,\mathrm{d}s(y)$$
$$\leq |\partial D| \|\phi\|_{L^\infty(D)} \sum_{n=1}^{\infty} \frac{i^n k^{n-1}}{(n-1)!} \mathrm{diam}(D)^{n-1} < \infty,$$

which shows the well-definedness of the derivative of the single layer potential. By the same arguments, the differentiability of the double layer potential is implied. Since the function $\hat{w}_k$ is also differentiable with respect to k, it follows that the derivative of $\mathrm{d}/\mathrm{d}kP_k$ exists and is well-defined.

To show that P_k is a projection, we choose an arbitrary function $g \in L^2(D \setminus \overline{D_0})$. Then

$$g - (\Delta + k^2)\hat{w} \in X^k_{D\setminus\overline{D_0}}$$

due the definition of $\hat{w}$. Consequently we have that $A_k[g - (\Delta + k^2)\hat{w}] \in X_k$. Finally if the function $g \in X_k$, it solves the Helmholtz equation in $D \setminus \overline{D_0}$, which implies that the right side of (3.33) vanishes. The coercivity of the sesquilinear form furthermore implies that $\hat{w}_k = 0$. Therefore $P_k g = A_k g = g$ due to the properties of the map A_k. This proves the assertion. ∎

Using this projection, we rewrite the expression (3.31) as

$$\cot\vartheta^*(k) = \min_{w\in L^2(D\setminus\overline{D_0})} \frac{\mathrm{Re}\,(T_k P_k w, P_k w)_{L^2(D\setminus\overline{D_0})}}{\mathrm{Im}\,(T_k P_k w, P_k w)_{L^2(D\setminus\overline{D_0})}}.$$

After this preliminary considerations, we can derive the first part of the inside-outside duality. For that purpose, we first calculate the auxiliary derivative α in Theorem 3.15, which allows us to give a conditional characterization of interior transmission eigenvalues. The following lemma is a preparation for this theorem.

Lemma 3.14. *Let $k_0^2 > 0$ be a transmission eigenvalue with transmission eigenpair (U_0, W_0) and set $w_0 := W_0|_{D\setminus\overline{D_0}} \in X_{k_0}$. Then the map $k \to (T_k w_0, w_0)_{L^2(D\setminus\overline{D_0})}$ is differentiable in k_0 and*

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D\setminus\overline{D_0})}\Big|_{k=k_0} = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x,$$

where $v_{k_0} \in H_0^2(D)$ is the radiating solution of (3.7) for $k = k_0$ and $f = w_0$, i.e.

$$\int_D \left(\nabla v_{k_0}\cdot\nabla\overline{\psi} - k_0^2(1+q)v_{k_0}\overline{\psi}\right)\,\mathrm{d}x = \int_{D\setminus\overline{D_0}} k_0^2 q w_0\overline{\psi}\,\mathrm{d}x \qquad \forall\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{3.34}$$

Proof. Due to Rellich's Identity v_{k_0} vanishes outside of D and therefore $v_{k_0} \in H_0^2(D)$. Furthermore setting $v = v_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ as the radiating solution to (3.34) for variable wavenumber k, we find that differentiating that expression yields

$$\int_{\mathbb{R}^3} \left(\nabla v_k'\cdot\nabla\overline{\psi} - k^2(1+q)v_k'\overline{\psi}\right)\,\mathrm{d}x = 2k\left[\int_{D\setminus\overline{D_0}} q w_0\overline{\psi}\,\mathrm{d}x + \int_{\mathbb{R}^3}(1+q)v_k\overline{\psi}\,\mathrm{d}x\right] \qquad \forall\psi \in C_0^\infty(\mathbb{R}^3) \tag{3.35}$$

Note also that $(T_{k_0}w_0, w_0)_{L^2(D\setminus\overline{D_0})} = 0$ by Theorem 3.11, i.e. $\int_{D\setminus\overline{D_0}} q\left(|w_0|^2 + v_{k_0}\overline{w_0}\right)\,\mathrm{d}x = 0$. Using this equation we get

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D\setminus\overline{D_0})}\big|_{k=k_0} = \frac{\mathrm{d}}{\mathrm{d}k}\int_{D\setminus\overline{D_0}} qk^2(w_0+v_k)\overline{w_0}\,\mathrm{d}x\Big|_{k=k_0} = k_0^2\int_{D\setminus\overline{D_0}} qv'_{k_0}\overline{w_0}\,\mathrm{d}x.$$

Eliminating w_0 from this equation by using (3.34) for $\psi = v'_{k_0}$ and (3.35) for $\psi = v_{k_0}$, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D\setminus\overline{D_0})}\big|_{k=k_0} &= \int_D \left(\nabla\overline{v_{k_0}}\cdot\nabla v'_{k_0} - k_0^2(1+q)\overline{v_{k_0}}v'_{k_0}\right)\,\mathrm{d}x\\ &= 2k_0\int_D \left(qw_0\overline{v_{k_0}} + (1+q)v_{k_0}\overline{v_{k_0}}\right)\,\mathrm{d}x = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x,\end{aligned}$$

which concludes the proof. ■

Theorem 3.15. *Let $k_0^2 > 0$ be a transmission eigenvalue with transmission eigenpair (U_0, W_0) and set $w_0 := W_0|_{D\setminus\overline{D_0}} \in X_{k_0}$. Then the map $k \to (T_k P_k w_0, P_k w_0)_{L^2(D\setminus\overline{D_0})}$ is differentiable in k_0 and*

$$\alpha(k_0) := \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D\setminus\overline{D_0})}\Big|_{k=k_0} = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x + 2k_0\ \mathrm{Re}\int_D \overline{W_0}v_{k_0}\,\mathrm{d}x, \tag{3.36}$$

where v_{k_0} is again the radiating solution of (3.34).

Proof. By definition of P_k, we have that $P_k w_0 \in X_k$, so that $w_k := E_k P_k w_0 \in L^2(D)$ solves the Helmholtz equation, i.e.

$$\int_D w_k(\Delta\psi + k^2\psi)\,\mathrm{d}x = 0 \qquad \forall\psi\in C_0^\infty(D).$$

Note also in this context that $w_{k_0} = W_0$, i.e. the extension of w_0 to D, since $w_{k_0} = E_{k_0}P_{k_0}w_0 = E_{k_0}w_0$. The projection P_k in (3.32) is differentiable and it is clear that $w'_k := \mathrm{d}/\mathrm{d}k(E_k P_k w_0)$ exists and solves

$$\int_D w'_k(\Delta\overline{\psi} + k^2\overline{\psi})\,\mathrm{d}x = -2k\int_D w_k\overline{\psi}\,\mathrm{d}x \quad \forall\psi\in C_0^\infty(D). \tag{3.37}$$

Also due to Green's representation theorem, we have that for an arbitrary $P_k w_0 \in X_k$ that

$$P_k w_0(x) = \mathrm{DL}(P_k w_0|_{\partial D})(x) - \mathrm{SL}\left(\frac{\partial P_k w_0}{\partial\nu}\Big|_{\partial D}\right)(x), \qquad x\in D\setminus\overline{D_0}$$

and by equation (3.29)

$$E_k P_k w_0(x) = \mathrm{DL}(P_k w_0|_{\partial D})(x) - \mathrm{SL}\left(\frac{\partial P_k w_0}{\partial\nu}\Big|_{\partial D}\right)(x), \qquad x\in D.$$

Therefore it is clear, that $\mathrm{d}/\mathrm{d}k w_k = \mathrm{d}/\mathrm{d}k(E_k P_k w_0)|_{D\setminus\overline{D_0}} = \mathrm{d}/\mathrm{d}k P_k w_0$. By applying the chain rule, we obtain that

$$\begin{aligned}\frac{d}{dk}(T_k P_k w_0, w_0) &= (T'_k P_k w_0, w_0) + (T_k P'_k w_0, P_k w_0) + (T_k P_k w_0, P'_k w_0)\\ &= (T'_k P_k w_0, w_0) + \overline{(T^*_k P_k w_0, P'_k w_0)} + (T_k P_k w_0, P'_k w_0).\end{aligned}$$

To simplify this expression, we note that $T_{k_0}w_0 = T^*_{k_0}w_0$ by the same arguments which we have already used in the proof of Lemma 3.6. This yields that

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_{k_0}w_0, w_0)_{L^2(D\setminus\overline{D_0})}\Big|_{k=k_0} = 2k_0\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x + 2\mathrm{Re}\,(T_{k_0}w_0, P'_{k_0}w_0).$$

Recall that $w'_k := \mathrm{d}/\,\mathrm{d}k(E_kP_kw_0) \in L^2(D)$, where w_k solves the Helmholtz equation. Furthermore from the discussion above, it is clear that $w'_{k_0}|_{D\setminus\overline{D_0}} = P'_{k_0}w_0$. Since $v_{k_0} \in H_0^2(D)$, we can use (3.37) to obtain

$$\begin{aligned}2\mathrm{Re}\,(T_{k_0}w_0, P'_{k_0}w_0)_{L^2(D\setminus\overline{D_0})} &= 2\ \mathrm{Re}\int_{D\setminus\overline{D_0}} qk_0^2(v_{k_0}+w_0)\overline{w'_{k_0}}\,\mathrm{d}x = 2\ \mathrm{Re}\int_D(\Delta v_{k_0}+k_0^2v_{k_0})\overline{w'_{k_0}}\,\mathrm{d}x\\ &= 2k_0\ \mathrm{Re}\int_D\overline{w_{k_0}}v_{k_0}\,\mathrm{d}x = 2k_0\ \mathrm{Re}\int_D E_{k_0}\overline{w_0}v_{k_0}\,\mathrm{d}x\end{aligned}$$

All in all, we get

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_kP_kw_0, P_kw_0)_{L^2(D\setminus\overline{D_0})}\Big|_{k=k_0} = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x + 2k_0\ \mathrm{Re}\int_D E_{k_0}\overline{w_0}v_{k_0}\,\mathrm{d}x.$$

Using $E_{k_0}\overline{w_0} = \overline{W_0}$ shows the assertion. ∎

Using the explicit expression we obtained for $\alpha(k_0)$ in the last lemma, we can state the first part of the inside-outside duality, where we refer to Theorem 3.7 for a proof.

Theorem 3.16 (Inside-Outside Duality - Part 1)**.** *Let k_0^2 be a transmission eigenvalue with transmission eigenpair (U_0, W_0) and $\alpha(k_0)$ the expression defined in* (3.36)*. Then it follows that $\lim_{k\nearrow k_0}\vartheta^*(k) = \pi$ or $\lim_{k\searrow k_0}\vartheta^*(k) = \pi$ if $\alpha(k_0) > 0$ or $\alpha(k_0) < 0$, respectively.*

Note that in all our numerical experiments, the phase curve approaches the value π from the left side, implying that $\alpha(k_0) > 0$ might hold for all transmission eigenvalues k_0. However, it remains an open problem to prove such a characteristic.

The second part of the inside-outside duality provides a sufficient condition for the squared wavenumber k_0^2 to be a transmission eigenvalue (see [KL13, Theorem 6.3(b)] for a proof).

Theorem 3.17 (Inside-Outside Duality - Part 2)**.** *Choose $k_0 > 0$ such that $I := (k_0-\varepsilon, k_0+\varepsilon)\setminus\{k_0\}$ contains no wavenumber k such that k^2 is an interior transmission eigenvalue. If it holds that $\lim_{I\ni k\to k_0}\vartheta^*(k) = \pi$, then k_0^2 is an interior transmission eigenvalue.*

3.4. Conditions for the Material Parameter

In this section, we want to further examine the properties of the derivative $\alpha(k_0)$ from (3.14) if there are no cavities in the scattering object. Let k_0^2 be an interior transmission eigenvalue with eigenpair $(u_0, w_0) \in L^2(D) \times X_{k_0}$, where the space X_{k_0} was defined in (3.8). Recall that the derivativewas given by

$$\alpha(k_0) = \frac{2}{k_0}\int_D |\nabla v_{k_0}|^2\,\mathrm{d}x + 4k_0\ \mathrm{Re}\int_D \overline{w_0}v_{k_0}\,\mathrm{d}x,$$

where k_0^2 is an interior transmission eigenvalue and $v_{k_0} \in H_0^2(D)$ is the radiating solution of (3.12), i.e.

$$\int_D \left(\nabla v_{k_0}\cdot\nabla\overline{\psi} - k_0^2(1+q)v_{k_0}\overline{\psi}\right)\,\mathrm{d}x = \int_D k_0^2qw_0\overline{\psi}\,\mathrm{d}x \qquad \forall\psi \in H^1(D). \tag{3.38}$$

For the first part of the inside-outside duality it is of particular importance to show that $\alpha(k_0)$ does not vanish. Therefore we want to derive conditions for the contrast q for which $\alpha(k_0)$ is either positive or negative for certain transmission eigenvalues k_0^2. We start by showing that the derivative $\alpha(k_0)$ for the smallest interior transmission eigenvalue k_0^2 has a distinct sign depending on the sign of the contrast q, if it does not vanish. To simplify notation we will in the following set $v_0 := v_{k_0}$.

Theorem 3.18. *Let k_0^2 be the smallest interior transmission eigenvalue. Then it holds that $\alpha(k_0) \geq 0$ if $q > 0$ and $\alpha(k_0) \leq 0$ if $q < 0$.*

Proof. We define $\lambda = k^2$ and $\lambda_0 = k_0^2$. Let a sesquilinear form $a_\lambda(v, \psi) : H_0^2(D) \times H_0^2(D) \to \mathbb{C}$ be given by

$$a_\lambda(v, \psi) := \int_D \frac{1}{q}[\Delta v + \lambda(1+q)v][\Delta\overline{\psi} + \lambda\overline{\psi}]\,\mathrm{d}x.$$

This sesquilinear form can be used to define interior transmission eigenvalues by a fourth-order equation that has been used to prove the existence of interior transmission eigenvalues. Indeed, we know from [Kir09] that λ is an interior transmission eigenvalue if, and only if, $a_\lambda(v, \psi) = 0$ for all test functions $\psi \in H_0^2(D)$. Note that the sesquilinear form $a_\lambda(\cdot, \cdot)$ defines an operator A_λ, such that $a(v, \psi) = (A_\lambda v, \psi)_{H_0^2(D)}$. In particular, λ is interior transmission eigenvalue if and only if zero is an eigenvalue of A_λ. The operator A_λ possesses a representation

$$A_\lambda = I + \lambda B_1 + \lambda^2 B_2,$$

where B_1, B_2 are self-adjoint, compact and B_2 is positive, see [Kir09, p.3]. Therefore we can conclude that the spectrum of A_λ is real and discrete with 1 as the only possible accumulation point. Furthermore the eigenvalues of A_λ depend continuously on the wavenumber and since $A_0 = I$, the spectrum of A_0 only contains $\{1\}$. Rewriting the definition of a_λ and substituting $v, \psi = v_0$, we obtain

$$a_\lambda(v_0, v_0) = \int_D \frac{1}{q}[|\Delta v_0|^2 + \lambda(1+q)v_0\Delta\overline{v_0} + \lambda\Delta v_0\overline{v_0} + \lambda^2(1+q)|v_0|^2]\,\mathrm{d}x.$$

Green's first identity implies that $a_\lambda(v_0, v_0)$ is a real-valued, quadratic polynomial in λ. This implies that the equation $a_\lambda(v_0, v_0) = 0$ has either exactly one solution $\lambda = \lambda_0$, in which case $\frac{d}{d\lambda}a_\lambda(v_0, v_0)\big|_{\lambda=\lambda_0} = 0$, or $a_\lambda(v_0, v_0) = 0$ has two solutions $\tilde{\lambda}, \hat{\lambda}$, of which at most one can be a transmission eigenvalue, since the eigenfunctions are linearly independent. Assume first that $\hat{\lambda} > \tilde{\lambda}$ and $\lambda \in (\tilde{\lambda}, \hat{\lambda})$. Then $a_\lambda(v_0, v_0) < 0$ and therefore

$$\inf_{f \in H_0^2(D)} a_\lambda(f, f) < 0.$$

Now the min-max principle implies that the smallest eigenvalue of A_λ is negative. Since the eigenvalue depends continuously on the wavenumber and since the first eigenvalue of A_0 is positive, it follows that the first interior transmission eigenvalues is between λ and 0. This implies that $\tilde{\lambda} = \lambda_0$ is the first interior transmission eigenvalue.

If the contrast q is negative, we apply similar arguments to $-a_\lambda(v_0, v_0)$ and the corresponding operator $-A_\lambda$ and obtain that the first interior transmission eigenvalue is at $\tilde{\lambda} = \lambda_0$. From now on we assume that $q > 0$. The following arguments can easily be adapted for negative contrasts. We now want to derive a different expression for $\alpha(k_0)$. Therefore we first calculate the term $\mathrm{d}/\,\mathrm{d}\lambda\; a_\lambda(v_0, v_0)$ explicitly. It is given by

$$\begin{aligned}\frac{d}{d\lambda}a_\lambda(v_0, v_0) &= \int_D \frac{1}{q}[(1+q)v_0\Delta\overline{v_0} + \Delta v_0\overline{v_0} + 2\lambda(1+q)|v_0|^2]\,\mathrm{d}x \\ &= \int_D \left[\frac{2}{q}\mathrm{Re}\,(\Delta v_0\overline{v_0}) + \Delta v_0\overline{v_0} + 2\lambda\frac{1+q}{q}|v_0|^2\right]\,\mathrm{d}x.\end{aligned}$$

Now we choose $\lambda = (\hat{\lambda} - \tilde{\lambda})/2$ such that $\frac{d}{d\lambda}a_\lambda(v_0, v_0) = 0$. Furthermore we use $\Delta v_0 = -\lambda_0 q w_0 -$

$\lambda_0(1+q)v_0$ to obtain

$$\begin{aligned}\frac{d}{d\lambda}a_\lambda(v_0,v_0) &= \int_D \left[\frac{2}{q}\mathrm{Re}\,(-\lambda_0 q w_0 - \lambda_0(1+q)v_0)\overline{v_0} + (-\lambda_0 q w_0 - \lambda_0(1+q)v_0)\overline{v_0} + 2\lambda\frac{1+q}{q}|v_0|^2\right]\,\mathrm{d}x \\ &= \int_D \left[-2\mathrm{Re}\,(\lambda_0 v_0\overline{w_0}) - q\lambda_0 v_0\overline{w_0} - |v_0|^2(1+q)\lambda_0\right]\,\mathrm{d}x + \int_D 2\frac{1+q}{q}|v_0|^2(\lambda-\lambda_0)\,\mathrm{d}x = 0.\end{aligned}$$

Now we rearrange terms and use (3.38) to obtain

$$\begin{aligned}2\lambda_0\int_D \mathrm{Re}\,(v_0\overline{w_0})\,\,\mathrm{d}x &= \int_D [-|v_0|^2(1+q)\lambda_0 - \lambda_0 q v_0\overline{w_0}]\,\mathrm{d}x + \int_D 2\frac{1+q}{q}|v_0|^2(\lambda-\lambda_0)\,\mathrm{d}x \\ &= -\|\nabla v_0\|^2 + \int_D 2\frac{1+q}{q}|v_0|^2(\lambda-\lambda_0)\,\mathrm{d}x.\end{aligned}$$

With this result we can finally show that the derivative is greater or equal to zero by calculating

$$\frac{\alpha k_0}{2} = \sigma\left[\int_D |\nabla v_0|^2\,\mathrm{d}x + 2\lambda_0\mathrm{Re}\int_D v_0\overline{w_0}\,\mathrm{d}x\right] = \sigma\int_D 2\frac{1+q}{q}|v_0|^2(\lambda-\lambda_0)\,\mathrm{d}x.$$

Due to our choice $\lambda = (\hat{\lambda}-\tilde{\lambda})/2$ and since $\lambda_0 = \tilde{\lambda}$ it follows that $\lambda \geq \lambda_0$ and since $\sigma > 0$, the assertion holds. ■

Remark 3.19. Note that the derivative $\alpha(k_0)$ only vanishes if the polynomial $a_\lambda(v_0,v_0)$ is equal to zero only at the interior transmission eigenvalue λ_0. However we cannot exclude this possibility so far. Note also that the arguments above are only valid for the first interior transmission eigenvalue, although all our numerical experiments indicate that this sign property might hold for all interior transmission eigenvalues.

For the remainder of this section we will focus on the case where $q > 0$. For $q < 0$ we quote the following result from [KL13, Theorem 6.2]:

Theorem 3.20. *Let k_0^2 be the smallest interior transmission eigenvalue and $q(x) = q_0$ constant for $q_0 \in (-1,0)$. Then there exists a $\hat{q} \in (-1,0)$ such that $-1 \leq q_0 \leq \hat{q}$ implies that $\alpha(k_0) < 0$ for all non-trivial $w_0 \in X_{k_0}$ such that $\mathrm{Im}\,(T_{k_0}w_0,w_0)_{L^2(D)} = 0$.*

From now on we assume the $q > 0$. If the contrast q is constant, i.e. $q = q_0$ for a number $q_0 > 0$, we can use the result in [KL13, Theorem 6.1]. Denote by ρ_0 the smallest Dirichlet eigenvalues of $-\Delta$ and by ρ_1 the smallest Dirichlet eigenvalues of Δ^2. Then the following holds.

Theorem 3.21. *Let k_0^2 be the smallest interior transmission eigenvalue and $q(x) = q_0 > 0$ such that*

$$q_0 > 2\left[\left(\frac{\rho_1}{\rho_0^2}-1\right) + \frac{\sqrt{\rho_1}}{\rho_0}\sqrt{\frac{\rho_1}{\rho_0^2}-1}\right].$$

Then it holds that $\alpha(k_0) > 0$.

As a conclusion to this section, we want to show that there are non-constant contrast for which the derivative does not vanish. First we prove the following auxiliary lemma.

Lemma 3.22. *Let $c \in (0,\frac{1}{2})$ and assume that an interior transmission eigenvalue k_0^2 fulfills $k_0^2 < c\rho_0$ for a contrast $q \in L^\infty(D)$, such that $0 < q_0 \leq q(x) \leq \frac{q_0}{2c}$. Then it holds that $\alpha(k_0) > 0$.*

Proof. To simplify notation we set $\|\cdot\| := \|\cdot\|_{L^2(D)}$. The derivative is given by $\alpha(k_0) = \frac{2}{k_0}A$, where

$$A = \|\nabla v_0\|^2 + 2k_0^2\mathrm{Re}\int_D v_0\overline{w_0}\,\mathrm{d}x.$$

We show that A is positive. To this end we first derive a lower bound for $\|v_0\|$. Since k_0^2 is interior transmission eigenvalue, there exists a function $w_0 \in X_{k_0}$ such that $(Tw_0, w_0) = 0$, i.e.

$$-\int_D q v_0 \overline{w_0}\,\mathrm{d}x = \int_D q|w_0|^2\,\mathrm{d}x.$$

This implies

$$\frac{q_0}{2c}\|v_0\|\|w_0\| \geq \left|\int_D q\overline{w_0}v_0\,\mathrm{d}x\right| = \int_D q|w_0|^2\,\mathrm{d}x \geq q_0\|w_0\|^2,$$

and therefore $\|v_0\| \geq 2c\|w_0\|$ and since $k_0^2 < c\rho_0$ by assumption, it follows that $\|v_0\| > 2k_0^2/\rho_0\|w_0\|$. We scale w_0 such that $k_0^2\|w_0\| = \frac{\rho_0}{2}$. This implies that $\|v_0\| > 1$ and in particular $\|v_0\| < \|v_0\|^2$. Under this conditions we obtain

$$A \geq \|\nabla v_0\|^2 - 2k_0^2\|v_0\|\|w_0\| > \|\nabla v_0\|^2 - 2k_0^2\|v_0\|^2\|w_0\| \geq \rho_0\|v_0\|^2 - 2\|v_0\|^2\frac{\rho_0}{2} = 0,$$

which shows our claim. ∎

If the contrast is large enough, there exist interior transmission eigenvalues that fulfill this condition. Indeed, we can show the following result for non-constant contrasts.

Theorem 3.23. *Let μ_p be the p-th eigenvalues of the bi-Laplacian Δ^2 and let $q_0 > \frac{4\mu_p}{\rho_0^2} - 3$. If*

$$q_0 < q(x) < \frac{\rho_0^2(3q_0 + q_0^2)}{4\mu_p}$$

then there are at least p interior transmission eigenvalues k_0^2 with $k_0^2 < \frac{2\mu_p}{\rho_0(3+q_0)}$ and for all interior transmission eigenvalues k_0^2 that fulfill this estimate, it holds that $\alpha(k_0) > 0$.

Proof. To show the existence of the p interior transmission eigenvalues that fulfill the bound of the theorem, it suffices to show that $\mu_p + k^4(1+q_0) - k^2\rho_0(2+q_0) < 0$ by [Kir09, p.4]. We set $k^2 = c\rho_0$ for a number $c \in (0, \frac{1}{2})$, so that the conditions can be written as

$$\mu_p + c^2\rho_0^2(1+q_0) - c\rho_0^2(2+q_0) < 0.$$

Since for $c \in (0, \frac{1}{2})$ it holds that $c^2 < \frac{1}{2}c$, it suffices to show that

$$\mu_p + \frac{1}{2}c\rho_0^2(1+q_0) - c\rho_0^2(2+q_0) \leq 0,$$

or rather

$$2\mu_p - 3c\rho_0^2 - cq_0\rho_0^2 < 0.$$

This can equivalently be written as

$$(3\rho_0^2 + q_0\rho_0^2)c \geq 2\mu_p.$$

Setting $c = 2\mu_p/(3\rho_0^2 + q_0\rho_0^2)$ and recalling that $c < \frac{1}{2}$ by assumption, we obtain the $q_0 > 4\mu_p/\rho_0^2 - 3$. Finally we calculate

$$\frac{q_0}{2c} = \frac{q_0(3\rho_0^2 + q_0\rho_0^2)}{4\mu_p} = \frac{\rho_0^2(3q_0 + q_0^2)}{4\mu_p} \qquad \text{and} \qquad c\rho_0 = \frac{2\mu_p}{\rho_0(3+q_0)}.$$

Now applying Lemma 3.22 yields the claim. ∎

Remark 3.24. The results of this section are certainly not conclusive and serve to show that there are indeed non-trivial derivatives α rather then trying to fully exhaust all possibilities in deriving conditions for the derivative. In general it would be very desirable to show that the derivative does not vanish for all contrasts and all interior transmission eigenvalues. However new ideas are certainly needed to advance the analysis in this direction.

3.5. Numerically Detecting Interior Transmission Eigenvalues from Far Field Data

In this section, we present numerical results of the inside-outside duality approach for the acoustic interior transmission problem for a variety of obstacles in three dimensions without and with inclusions. First, we describe the obstacles under consideration and then use the inside-outside duality algorithm to calculate interior transmission eigenvalues for these obstacles. When we consider the unit ball as a scattering object, we can calculate analytical reference values for the interior transmission eigenvalues. For the other scattering objects, we use reference values that have been calculated in [PK16] by an extension of the algorithm introduced in [Kle13].

We present five different obstacles which can be described via spherical coordinates. The spherical coordinates (ϱ, θ, ϕ) of a point in rectangular coordinates (x, y, z) are given by

$$x = \varrho \sin(\phi)\cos(\theta)\,, \qquad y = \varrho \sin(\phi)\sin(\theta)\,, \qquad z = \varrho\cos(\phi)\,,$$

where $\varrho \in [0, \infty)$ is the radial distance, $\phi \in [0, \pi]$ is the azimuthal angle, and $\theta \in [0, 2\pi]$ is the polar angle. The first surface under consideration is a unit sphere which can be obtained by picking $\varrho = 1$. The second surface is a prolate ellipsoid of revolution with semi-principle axes of length 1, 1, and 1.2; i.e., ϱ is chosen to be 1 for the x- and y-coordinates and 1.2 for the z-coordinate. The third surface is constructed by choosing $\varrho = 1.5\sqrt{0.25\sin^2(\phi) + \cos^2(\phi)}$ and it is peanut-shaped. The acorn-shaped obstacle is obtained by the choice $\varrho = 0.6\sqrt{4.25 + 2\cos(3\phi)}$ and is the fourth surface under consideration. The last surfaces is a round short cylinder. It is given by the choice $\varrho^{10} = 1/((2\sin(\phi)/3)^{10} + \cos^{10}(\phi))$. In the sequel, the five surfaces are abbreviated by **Sph**, **Eli**, **Pea**, **Aco**, and **SCyl**, respectively. In Figure 3.1 we show the five obstacles under consideration. Note that they have already been used before in [Kle13].

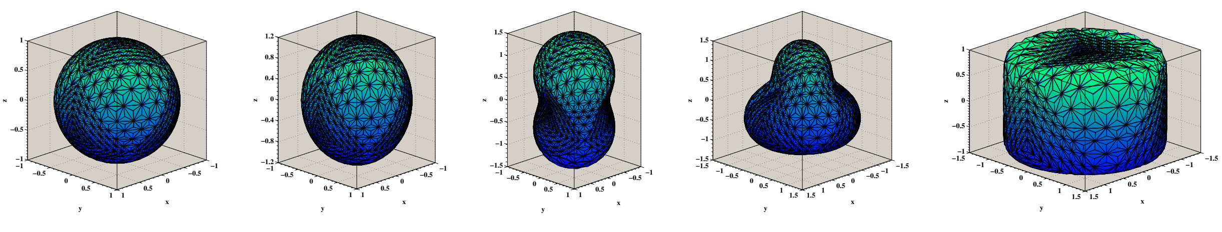


Figure 3.1.: Left to right: Scattering objects **Sph**, **Eli**, **Pea**, **Aco**, and **SCyl**.

For the inside-outside duality we need to approximate the far field operator in (3.2) numerically. To use the algorithm provided in [Kle13] by Andreas Kleefeld, we assume on this section either constant or piecewise constant index of refraction n in order to apply boundary integral methods. Then we obtain the discrete far field matrix

$$\mathbb{F}_N^\delta := u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(l)})_{j,l=1}^N \in \mathbb{C}^{N\times N},$$

from (2.58) as an approximation to the far field operator F, where we choose again $N = 120$. The construction of $\mathbb{F}_N^\delta$ has extensively been described in Section 2.4. In particular, it has been shown that is is sufficient to work with the phases of the eigenvalues λ_j^N of $\mathbb{F}_N^\delta$ in order to verify the inside-outside duality. To verify the inside-outside duality for acoustic scattering from Theorem 3.7 and Theorem 3.8 for scattering objects without cavities and Theorem 3.16 and Theorem 3.17 for objects with cavities, we therefore compute the eigenvalues $\lambda_j^N, j = 1, .., 120$ of $\mathbb{F}_N^\delta$ as an approximation to the eigenvalues λ_j of F for a sequence of wavenumbers, suitable for the scattering object under consideration. We then examine how the corresponding phases $\vartheta_{j,N}$ behave with varying wavenumber, in particular where the phase $\vartheta_N^* = \max_{j=1,.,120} \vartheta_{j,N}$ converges to π. Recall that small errors in eigenvalues close to zero lead to large errors in the corresponding phases. We therefore use the same regularization scheme that we have already used in Section 2.4. If required, we first neglect eigenvalues that are too close to zero, i.e. eigenvalues which lie in the ball $\{z \in \mathbb{C}, |z| < \varepsilon\}$, where ε is the noise level of F_N, given by $\|F_N - F\|$. In a second step we use the knowledge that the eigenvalues λ_j of F lie on a circle $\{z \in \mathbb{C}, |z - 8\pi^2\mathrm{i}/k|\}$ in the complex plane to project the numerically approximated eigenvalues λ_j^N orthogonally onto this circle, using the projection mapping

$$\mathcal{Q} : \lambda \mapsto \frac{8\pi^2\mathrm{i}}{k} + \frac{8\pi^2}{k} \frac{\lambda - 8\pi^2\mathrm{i}/k}{|\lambda - 8\pi^2\mathrm{i}/k|}. \tag{3.39}$$

We plot the phases $\vartheta_{j,N}^P$ of the projected eigenvalues $\mathcal{Q}[\lambda_j^N](k)$ for a sequence of wavenumbers k_n. Note that unlike in previous sections in which we also added noise to the far field data, our objective in this section is to test the inside-outside duality under "optimal circumstances" to evaluate its advantages and shortcomings as a method. That is also why we neglected to add artificial noise and calculated far field data as precisely as possible.

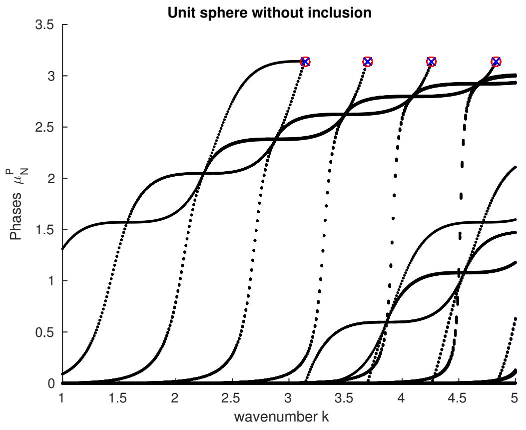
In this section a typical example for a phase plot is shown in Figure 3.2, where we used the inside-outside duality approach to detect interior transmission eigenvalues of a unit ball with constant index of refraction $n = 4$ with or without inclusion. As approximations for the transmission eigenvalues, we choose the wavenumbers that corresponds to the phases closest to π in the eigenvalue curve under consideration. This approach works particularly well if the eigenvalue curve shows a steep ascend close to π, which is the case in the example of scattering by a unit sphere, as we will discuss in the next subsection in more detail.

The unit sphere

In this subsection we present the numerical calculation for interior transmission eigenvalues for a unit sphere that may or may not contain an inclusion by using the inside-outside duality approach. We analytically calculate transmission eigenvalues for the unit sphere and then discuss the quality of the inside-outside duality approach to approximate these transmission eigenvalues. The interior transmission eigenvalues for a unit sphere without inclusion are given by the roots of the function

$$f(k) = \det \begin{bmatrix} j_p(k) & -j_p(k\sqrt{n}) \\ j_p'(k) & -\sqrt{n} j_p'(k\sqrt{n}) \end{bmatrix}, \tag{3.40}$$

for $p \geq 0$, where j_p are the spherical Bessel functions (see [Kle13, Section 6.1] for a derivation). For the index of refraction $n = 4$, we get the first four interior transmission values 3.141 59, 3.692 45, 4.261 68, and 4.831 86, which can also be seen in the first column of Table 3.1, which contains analytical values for all the cases we are going to discuss in this subsection. The values are also confirmed by the use of the inside-outside duality approach, where we used the interval $[1, 5]$ and the grid size 0.01. As one can see in Figure 3.2(a), we are able to detect the first four interior transmission eigenvalues. Precisely, we obtain the results 3.14, 3.69, 4.26, and 4.83 that are accurate within the chosen grid size. Note that for the first transmission eigenvalue, there are two phase curves that approach this value. Zooming into the curves shows that in one of the two curves are two eigenvalues contained that approaches the value 3.14. The first transmission eigenvalue has



(a) 401 points for the interval 1 to 5

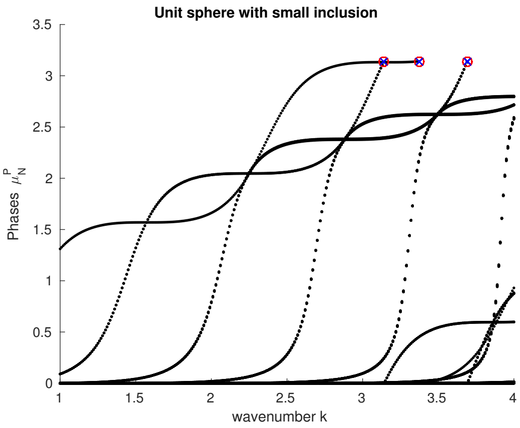
(b) 301 points for the interval 1 to 4

Figure 3.2.: (a) The detection of four interior transmission eigenvalues with the inside-outside duality approach for a unit sphere without inclusion. (b) The detection of three interior transmission eigenvalues for a unit sphere containing a spherical cavity of radius $R = 0.1$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.

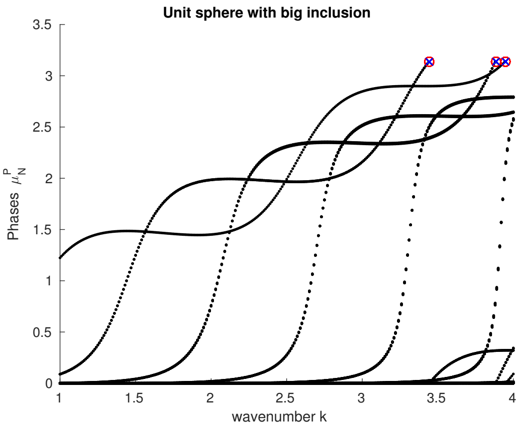
multiplicity three. This shows again that the inside-outside duality approach also takes multiplicity of transmission eigenvalues into account, see Remark 2.10. Note also that the slope of the first curve approaching the first transmission eigenvalue decreases rapidly in the end. In this particular example this in no problem due to the high accuracy in computation, but we will see later that the potential flatness of eigenvalue curves leads to a decrease in accuracy for the approximation of transmission eigenvalues for other scattering objects. This is also why we avoid using the extrapolation algorithm provided in [LP15a].
Next we use the same unit sphere with index of refraction $n_1 = 4$ but now include a cavity in form of a sphere of radius $R_1 = 0.1$ and index of refraction $n_1 = 1$. The results can be seen in Figure 3.2(b), from which we obtain the values 3.14, 3.49, and 3.69. Comparing this to the analytical values in the second column of Table 3.1 shows that we stay within the accuracy of the chosen grid size. For a formula for the analytic values, we refer to [PK16, Section 4.2]. The accuracy may seem remarkable for the second interior transmission eigenvalue because the corresponding phase curve seems rather flat but zooming into the graph shows a definite increase in slope towards the end of the curve, allowing for a precise estimation of the interior transmission eigenvalue.

As a conclusion to this subsection we want to show that we can also use spherical inclusions that have index of refraction different from one. We use one spherical inclusion of radius $R_2 = 0.1$ with index of refraction $n_2 = 3$ and one inclusion of radius $R_3 = 0.5$ and index of refraction $n_3 = 3$. The results can be seen in Figure 3.3. The graph in Figure 3.3(a) is similar to the case with the cavity in Figure 3.2(b). In particular the flatness of the second phase curve decreases towards the end of the curve, allowing for the precise estimation 3.37 of the second transmission eigenvalue within the gird size. The other two values 3.14 and 3.69 are also accurate within the chosen grid size. Hence, we are able to show that the inside-outside duality approach also works for an inclusion that has a different contrast that is not one. The same is true for the results shown in Figure 3.3(b). We obtain the values 3.44, 3.88. Later we will encounter obstacles for which the phase curve stays flat and an estimation of the transmission eigenvalues is less precise.

The parameters and the analytical reference values for the interior transmission eigenvalues are listed in Table 3.1 along with the results for a sphere without inclusion.



(a) 301 points for the interval 1 to 4



(b) 301 points for the interval 1 to 4

Figure 3.3.: (a) The detection of three interior transmission eigenvalues with the inside-outside duality approach for a unit sphere with a spherical inclusion of radius $R = 0.1$ and refractive index $n = 3$. (b) The detection of three interior transmission eigenvalues for a unit sphere containing a spherical inclusion of radius $R = 0.5$ and refractive index $n = 3$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.

ITE	no inclusion	$R_1 = 0.1$, $n_1 = 1$	$R_2 = 0.1$, $n_2 = 3$	$R_3 = 0.5$, $n_3 = 3$
1.	3.141 59	3.142 59	3.141 93	3.443 64
2.	3.692 45	3.490 66	3.373 33	3.883 18
3.	4.261 68	3.692 48	3.692 46	3.947 66
4.	4.831 86	4.261 68	4.261 68	4.382 33

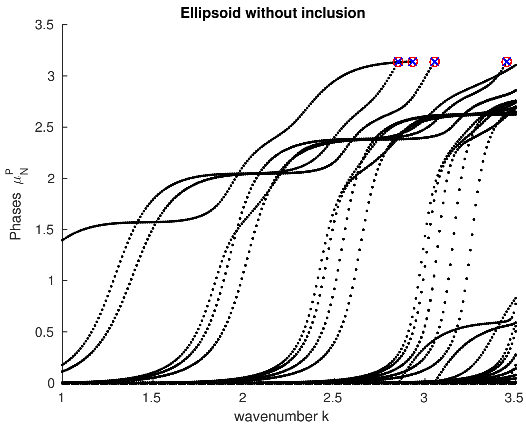
Table 3.1.: Different parameters for the unit sphere containing a sphere of different size and different index of refraction

The ellipsoid

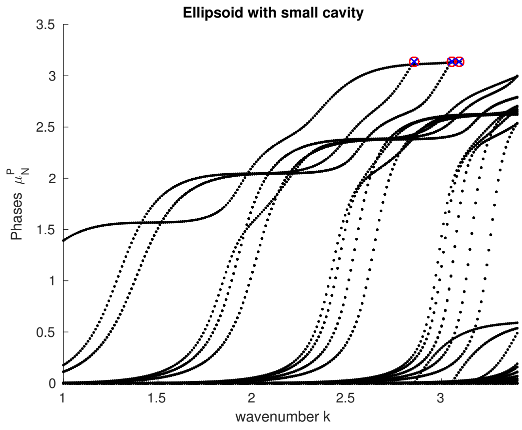
After taking a closer look at the detection of transmission eigenvalues for a sphere without inclusion and with inclusions that may or may not be cavities, we will from now on focus only on inclusions that are cavities, i.e. have refractive index of $n = 1$. We start by considering the ellipsoid as scattering object and consider the cases of an ellipsoid without cavities or with spherical cavities of size $R = 0.1$, $R = 0.2$ and $R = 0.3$. As one can see in Figure 3.4, the ellipsoid allows for a precise characterization of transmission eigenvalues due to the steep ascend of the eigenvalue curves. The approximations we obtain can be seen in Table 3.2 under the name “IO-value”. As in the case of the sphere, the approximation of the transmission eigenvalues is precise within the step size of the wavenumber grid, except for the last value in the second column, which shows a slight deviation. Note that the six-digit reference value in Table 3.2 are numerically computed by the integral equation method for transmission eigenvalues, introduced in [Kle13].

ITE ellipsoid	no inclusion	IO-value	small cavity	IO-value	bigger cavity	IO-value	biggest cavity	IO-value
1.	2.855721	2.85	2.855265	2.85	2.869239	2.86	2.937557	2.93
2.	2.931834	2.93	3.053040	3.05	3.073341	3.07	3.169967	3.16
3.	3.052080	3.05	3.095740	3.10	3.301488	3.30	3.379890	3.37

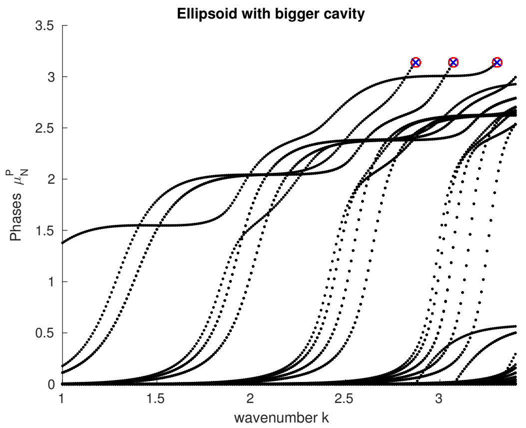
Table 3.2.: Approximations for the first three transmission eigenvalues for the ellipsoid with the inside-outside duality.



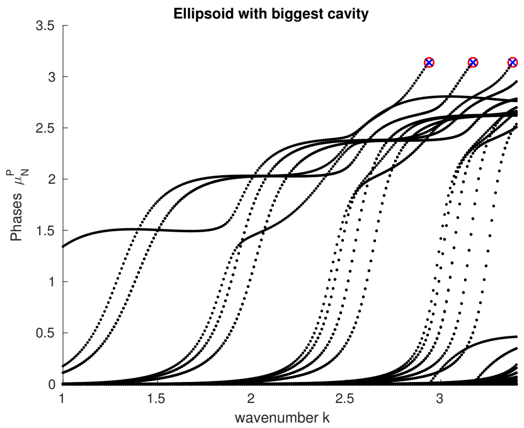
(a) 251 points for the interval 1 to 3.5



(b) 241 points for the interval 1 to 3.4



(c) 241 points for the interval 1 to 3.4

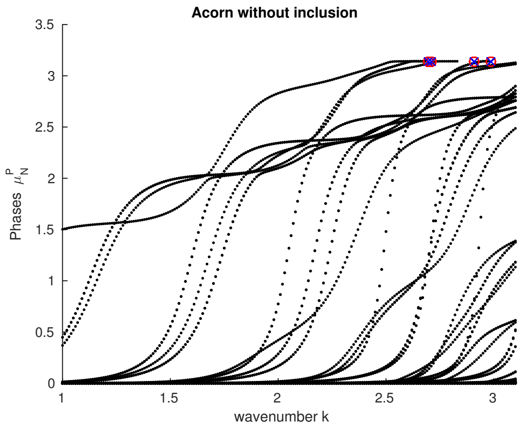


(d) 241 points for the interval 1 to 3.4

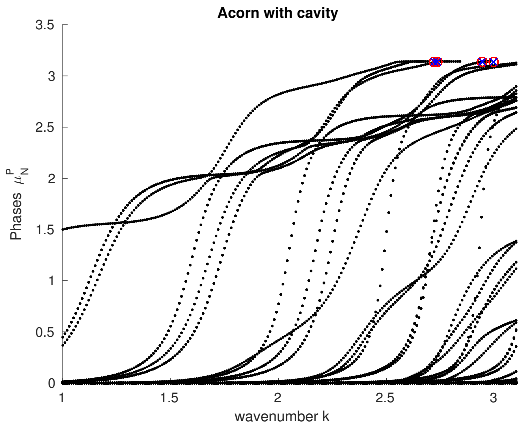
Figure 3.4.: Detection of interior transmission eigenvalues of the ellipsoid (a) without cavity (b) with spherical cavity of radius $R = 0.1$ (c) with spherical cavity of radius $R = 0.2$ (d) with spherical cavity of radius $R = 0.3$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.

The acorn

As an example for which the inside-outside duality fails in precisely detecting interior transmission eigenvalues is the scattering object acorn. As one can see in Figure 3.5(a) all phase curves, except for the last one, become very flat as they approach the critical value π. Zooming into the phase curve shows that in particular the third transmission eigenvalue is only approximated very roughly since the corresponding phase curve vanishes too early. The values for the first, second and fourth transmission eigenvalue are closer, but still not as precise as one would hope from the examples given above. The problem is worsened by including a cavity into the acorn, depicted in Figure 3.5(b). As one can see in Table 3.3, only the fourth transmission eigenvalue is approximated decently. It appears that by increasing the "geometric complexity" of a scattering object, the eigenvalue curves tend to become flatter, making the inside-outside duality procedure a less than optimal tool to accurately detect interior transmission eigenvalues.



(a) 211 points for the interval 1 to 3.1



(b) 211 points for the interval 1 to 3.1

Figure 3.5.: Detection of interior transmission eigenvalues of the acorn (a) without cavity (b) with spherical cavity of radius $R = 0.1$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.

ITE acorn	no inclusion	IO-value	one inclusion	IO-value
1.	2.694649	2.67	2.718420	2.64
2.	2.711716	2.69	2.733531	2.67
2.	2.910972	2.83	2.941369	2.84
2.	2.986754	2.98	2.994080	2.98

Table 3.3.: Approximations for the first four interior transmission eigenvalues for the acorn with the inside-outside duality.

The peanut

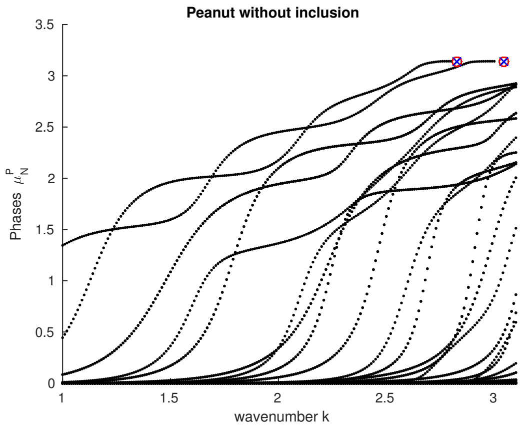
Next we consider the scattering object peanut. The results can be seen in Figure 3.6 and Table 3.4. Here a similar difficulty arises as in the previous case where we considered the acorn. Both eigenvalue curves become rather flat but unlike in the previous case, the curves still allow an approximation of the transmission eigenvalues that is at least precise for one place after the decimal point.

ITE peanut	no inclusion	IO-value	one inclusion	IO-value
1.	2.825465	2.80	2.825837	2.80
2.	3.044714	3.00	3.066903	3.02

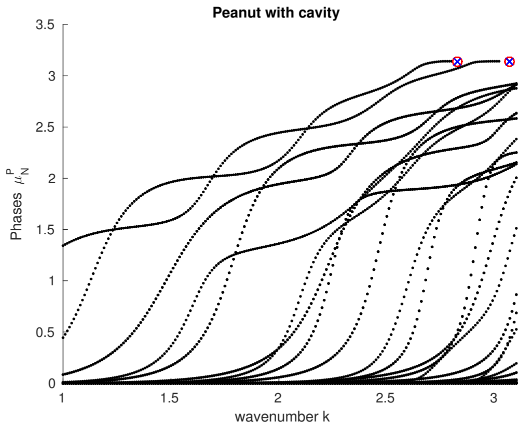
Table 3.4.: Approximations for the first two transmission eigenvalues for the peanut with the inside-outside duality.

The short cylinder

As a final scattering object we consider the short cylinder. Here the approximations of the transmission eigenvalues are again precise within the accuracy of the chosen grid size or show only very small derivations as one can see in Table 3.5 and Figure 3.7. As we noted above, this may again be due to the decrease in "geometric complexity" of the scattering object when compared to the peanut or the acorn. In this context it would be interesting to examine if geometric complexity is an actual quantity that can be measured in a way such that it corresponds certain behavioral patterns of the eigenvalue curves. For example one could take the surface-to-volume ratio of a scattering object as

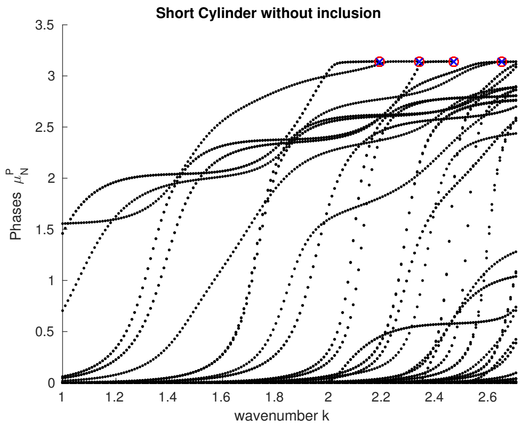


(a) 211 points for the interval 1 to 3.1

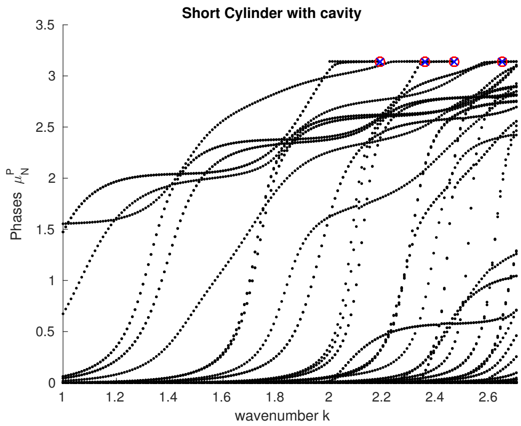


(b) 211 points for the interval 1 to 3.1

Figure 3.6.: Detection of interior transmission eigenvalues of the peanut (a) without cavity (b) with spherical cavity of radius $R = 0.1$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.



(a) 171 points for the interval 1 to 2.7



(b) 171 points for the interval 1 to 2.7

Figure 3.7.: Detection of interior transmission eigenvalues of the short cylinder (a) without cavity (b) with spherical cavity of radius $R = 0.1$. Blue crosses on the π axis mark the exact position of the transmission eigenvalues.

a measure for geometric complexity and conclude that since this ratio is smallest for the ball, the eigenvalue curve should have a steep ascend close to π. However it is far from obvious if such a link exists and how it could be established.

ITE short cylinder	no inclusion	IO-value	one inclusion	IO-value
1.	2.187215	2.18	2.187329	2.18
2.	2.337717	2.33	2.357965	2.34
3.	2.468408	2.46	2.468410	2.46
4.	2.645202	2.64	2.645487	2.65

Table 3.5.: Approximations for the first four transmission eigenvalues for the short cylinder with the inside-outside duality.

CHAPTER 4

SCATTERING FROM PENETRABLE OBJECTS WITH ANISOTROPIC DENSITY

4.1. Introduction

In this chapter we will consider a more general form of acoustic scattering by adding an anisotropic density to the scattering equation. Let the propagation of an acoustic wave in $\mathbb{R}^3$ be described by the time-harmonic wave equation

$$\operatorname{div}(A\nabla u) + k^2 n u = 0, \tag{4.1}$$

where the anisotropic density $A = I + Q$ is assumed to be real-valued, symmetric and positive definite in $\mathbb{R}^3$ and the matrix-valued contrast $Q : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is supported and sign-definite in the closure of the scatterer D. Furthermore the refractive index n is assumed to be bounded away from zero and has a sign-property we discuss below. Again denoting by ν and $[\cdot]_{\partial D}$ the exterior normal to D and the jump of a function across the boundary ∂D, we require our solution and its conormal derivative not to jump across the boundary of D, i.e.

$$[u]_{\partial D} = 0 \qquad \text{and} \qquad \left[\nu^T A\nabla u\right]_{\partial D} = 0.$$

This chapter can be understood as a natural continuation of the last chapter, where we discussed scattering equations that involved only the refractive index n. However the presence of both an anisotropic density and a refractive index leads to some difficulty. This is mainly due to the fact that the two parameter appear in the weak formulation of the scattering equations with different sign. Therefore we will at first, after introducing the relevant quantities, only consider the special case where the refractive index is set to one. In a second step we will then discuss which additional model assumptions we need to make to obtain the necessary estimates for both parameters.

As in the previous chapter, the total field $u = u^s + u^i$ decomposes into a scattered field u^s and an incoming field that we choose to be a plane wave $u^i(x, \theta) = e^{ikx\cdot\theta}$ with direction $\theta \in \mathbb{S}_1$. To guarantee uniqueness of solution, we require the scattered field to fulfill Sommerfeld's radiation condition (2.2). To write (4.1) for the scattered field, we use that the incident field solves the homogeneous Helmholtz equation in three-dimensional space to obtain

$$\operatorname{div}(A\nabla u^s) + k^2 n u^s = -\operatorname{div}(Q\nabla u^i) - k^2 q u^i \tag{4.2}$$

where $q = n - 1$ is the contrast functions corresponding to the refractive index n. To state a weak formulation of this problem, we define the space $\boldsymbol{L}^2(D) := L^2(D, \mathbb{C}) \times L^2(D, \mathbb{C}^3)$. This space is equipped with the standard scalar product $(\cdot, \cdot)$, which is for functions $f = (f_1, f_2)^T, g = (g_1, g_2)^T \in$

$\boldsymbol{L}^2(D)$ given by

$$(f,g) := (f_1,g_1)_{L^2(D,\mathbb{C})} + (f_2,g_2)_{L^2(D,\mathbb{C}^3)}.$$

Then we consider the following problem: For a source term $f = (f_1,f_2) \in \boldsymbol{L}^2(D)$, we seek a radiating scattered field $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, such that

$$\int_{\mathbb{R}^3} \left(A\nabla u^s \cdot \nabla\overline{\psi} - k^2 n u^s \overline{\psi}\right) \mathrm{d}x = -\int_D \left(f_2 \cdot \nabla\overline{\psi} - k^2 f_1 \overline{\psi}\right) \mathrm{d}x \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3). \tag{4.3}$$

Setting $f_2 = Q\nabla u^i$ and $f_1 = qu^i$ yields the weak formulation for equation (4.2). Using either an volume integral equation approach or a variational formulation on a bounded domain involving an exterior Dirichlet-to-Neumann operator [Kir08, Ned01] one shows that the latter problem is of Fredholm type, more precisely, that uniqueness of solution implies existence of solution for all source terms $f \in \boldsymbol{L}^2(D)$. As a standing assumption, we assume in this chapter that uniqueness of solution to (4.3) holds. This assumption is satisfied if, e.g., A is a sufficiently smooth function on $\mathbb{R}^3$, or if A is piecewise smooth with sufficiently regular jump discontinuities such that a unique continuation principle holds (for details see, e.g., [Pia98]). As in all acoustic scattering scenarios discussed in this thesis, the radiating solution $u^s = u^s(\cdot,\theta)$ to the Helmholtz equations (4.3) can be expressed in terms of its far field $u^\infty(\cdot,\theta)$, see (2.3), and the far field operator $F : L^2(\mathbb{S}_1) \to L^2(\mathbb{S}_1)$ is then defined in (2.5), i.e.

$$Fg(\hat{x}) = \int_{\mathbb{S}_1} u^\infty(\hat{x},\theta) g(\theta)\, \mathrm{d}S(\theta), \qquad \hat{x} \in \mathbb{S}_1. \tag{4.4}$$

The far field operator is compact due to the smoothness of its kernel. Since our material parameters Q and n are assumed to by real-valued, the far field operator is normal, see [KL14]. Additionally its eigenvalues lie on the circle $\{z \in \mathbb{C} : |z - 8\pi^2 \mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane. As in the previous cases, the injectivity of the far field operator is related to an interior transmission eigenvalue problem.

The squared wavenumber $k^2 > 0$ is called an interior transmission eigenvalue if there exists a non-trivial pair (u,w) of functions defined in D such that

$$\operatorname{div}(A\nabla u) + k^2 n u = 0 \quad \text{in } D, \qquad \Delta w + k^2 w = 0 \quad \text{in } D, \tag{4.5}$$

$$u = w \quad \text{on } \partial D, \qquad \nu^\top A \nabla u = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D. \tag{4.6}$$

This eigenvalue problem has to be understood in a weak sense: The squared wavenumber k^2 is an interior transmission eigenvalue if there exists a non-trivial eigenpair $(u,w) \in H^1(D) \times H^1(D)$ such that $u - w \in H^1_0(D)$ and

$$\int_D \left(A\nabla u \cdot \nabla\overline{\psi} - k^2 n u\overline{\psi}\right) \mathrm{d}x = 0, \qquad \int_D \left(\nabla w \cdot \nabla\overline{\psi} - k^2 w\overline{\psi}\right) \mathrm{d}x = 0 \qquad \forall \psi \in H^1_0(D),$$

$$\int_D \left(A\nabla u \cdot \nabla\overline{\psi} - k^2 n u\overline{\psi}\right) \mathrm{d}x = \int_D \left(\nabla w \cdot \nabla\overline{\psi} - k^2 w\overline{\psi}\right) \mathrm{d}x \qquad \forall \psi \in H^1(D). \tag{4.7}$$

Let us now indicate the main result of this chapter. Depending on the sign of the contrast function $Q = A - I$, the eigenvalues $\lambda_j = \lambda_j(k)$ converge to zero either from the left or from the right as $j \to \infty$ such that $\operatorname{Re}\lambda_j \lesseqgtr 0$ for $j \in \mathbb{N}$ large enough. We represent the eigenvalues in polar coordinates

$$\lambda_j = |\lambda_j| \exp(\mathrm{i}\vartheta_j), \quad \vartheta_j \in [0,\pi], \tag{4.8}$$

such that each eigenvalue λ_j corresponds to a phase ϑ_j. The convergence characteristic of the eigenvalues λ_j allows the definition of a smallest and a largest phase, i.e. if Q is positive definite or negative definite, then we define either

$$\vartheta_* := \min_{j\in\mathbb{N}} \vartheta_j \qquad \text{or} \qquad \vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j. \tag{4.9}$$

In this chapter we show that the inside-outside duality can be used to characterize interior transmission eigenvalues by the behavior of the smallest, or largest phase respectively. More precisely, we will show that interior transmission eigenvalues k_0^2, for which the derivative $\alpha(k_0)$ in (4.32) or (4.40) does not vanish, are characterized by the fact that the smallest phase $\vartheta_* = \vartheta_*(k)$ of $F = F_k$ tends to 0 as k tends to k_0 in case that Q is positive definite, see Theorem 4.15 and Theorem 4.23. Additionally, if $\vartheta_*(k)$ tends to zero as k tends to k_0, then $k_0^2 > 0$ is an interior transmission eigenvalue. A similar statement holds for the largest phase ϑ^* of Q is negative definite, see Theorem 4.16 and Theorem 4.24.

We proceed as in the previous chapter and start our derivation by providing a factorization of the far field operator and examine the properties of the arising operators in Section 4.2. Note that we do not exclude the case $n = 1$ from the derivations, such that the following results can easily adapted to the special case where $n = 1$ that we are going to consider in Section 4.3. In Section 4.4, we will then consider the general case where $n \neq 1$. The last two sections focus on the special case where $n = 1$. In Section 4.5 we will derive conditions for which the derivative α in (4.32) does not vanish. Finally in Section 4.6 we will show that the inside-outside duality can be used numerically to detect interior transmission eigenvalues for scattering models that include anisotropic densities.

4.2. A Factorization of the Far Field Operator

Before we introduce a factorization of the far field operator let us first state the model assumption more precisely. We assume that $D \subset \mathbb{R}^3$ is a bounded, simply connected Lipschitz domain and that $Q \in L^\infty(D, \mathbb{R}^{3\times 3})$ takes (almost everywhere) values in the space of symmetric 3×3 matrices. Moreover, denoting $z^* = \overline{z}^\top$, we assume for all $z \in \mathbb{C}^3$ and almost all $x \in D$ that either $z^* Q(x) z \geq q_0 |z|^2$ for some $q_0 > 0$, or that $z^* Q(x) z \leq q_0 |z|^2$ for $-1 < q_0 < 0$. In the first and second case Q is positive and negative definite, respectively, and extending Q by zero to all of $\mathbb{R}^3$, the material parameter $A = I + Q$ is positive definite everywhere. Furthermore let $q \in L^\infty(D)$ be a real-valued contrast such that, extending q also by zero outside of D, the refractive index $n = q + 1$ is positive. We consider in this chapter the two cases where either q vanishes or is bounded away from zero and has a sign property we discuss below. Since there are two material parameters in the wave equation, we need to account for both of them in our factorization by using multidimensional operators. In particular we need to consider functions in the space $\boldsymbol{L}^2(D) := L^2(D, \mathbb{C}) \times L^2(D, \mathbb{C}^3)$. In order to factorize the far field operator, we define an injective Herglotz wave operator $H : L^2(\mathbb{S}_1) \to \boldsymbol{L}^2(D)$ by

$$Hg = \begin{pmatrix} v_g \\ \nabla v_g \end{pmatrix} \qquad \text{where } v_g(x) = \int_{\mathbb{S}_1} g(\theta) e^{\mathrm{i}k\theta\cdot x}\, \mathrm{d}S(\theta), \qquad x \in D. \tag{4.10}$$

The adjoint $H^* : \boldsymbol{L}^2(D) \to L^2(\mathbb{S}_1)$ is then given by

$$H^* \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} (\hat{x}) = \int_D h_1(y) e^{-ik\hat{x}\cdot y}\, \mathrm{d}y + \int_D (\nabla_y e^{-\mathrm{i}k\hat{x}\cdot y}) \cdot h_2(y) \mathrm{d}y, \qquad \hat{x} \in \mathbb{S}_1.$$

This function is the far field of a combination of a volume potential for the Helmholtz equation and the divergence of such a potential. Recall that $\Phi(x, y) = \exp(\mathrm{i}k|x-y|)/(4\pi|x-y|)$, $x \neq y \in \mathbb{R}^3$, is the radiating fundamental solution to the Helmholtz equation. Then we have that

$$\begin{aligned} H^* \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= \left(\int_D \Phi(\cdot, y) h_1(y) \, \mathrm{d}y \right)^\infty - \left(\operatorname{div} \int_D \Phi(\cdot, y) h_2(y) \, \mathrm{d}y \right)^\infty \\ &= (V h_1)^\infty - (\operatorname{div} V h_2)^\infty = (V h_1 - \operatorname{div} V h_2)^\infty, \end{aligned}$$

where $V : L^2(D, \mathbb{C}) \to H^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ maps a source term to its volume potential,

$$Vh = \int_D \Phi(\cdot, y) h(y) \, \mathrm{d}y,$$

and V acts element-wise on h_2. Finally we define an operator $T : \boldsymbol{L}^2(D) \to \boldsymbol{L}^2(D)$ by

$$T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -k^2 q (f_1 - v) \\ Q[f_2 - \nabla v] \end{pmatrix}$$

where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak, radiating solution to

$$\operatorname{div}(A \nabla v) + k^2 n v = \operatorname{div}(Q f_2) + k^2 q f_1 \qquad \text{in } \mathbb{R}^3, \tag{4.11}$$

that is

$$\int_{\mathbb{R}^3} (A \nabla v \cdot \nabla \psi - k^2 n v \psi) \, \mathrm{d}x = \int_D \left(Q f_2 \cdot \nabla \psi - k^2 q f_1 \psi \right) \mathrm{d}x \qquad \forall \psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.12}$$

Before we state a factorization of the far field operator, we will show that the solution operator corresponding to the latter problem is closed. This is a property that we will use later when we prove some of the properties of the factorization.

Lemma 4.1. *The solution operator $L : f \to v$, corresponding to the problem in* (4.11)*, is closed from $\boldsymbol{L}^2(D)$ into $H^1(D)$.*

Proof. For this proof we assume that q does not vanish in D. If $q = 0$, the following arguments can easily be adapted, see [LP15a, Lemma A.1]. Choose a sequence $v_j := L f^{(j)} = L(f_1^{(j)}, f_2^{(j)})$ in the range of L with $\lim_{j\to\infty} v_j = v$ in $H^1(D)$. We have to show that there exists a function $f \in \boldsymbol{L}^2(D)$ such that $Lf = v$ and abbreviate the variational problem (4.12) as $a(v_j, \psi) = F_j(\psi)$ for all $\psi \in H^1(\mathbb{R}^3)$ with the continuous linear functional

$$F_j(\psi) := \int_D \left(\nabla \overline{\psi} \cdot Q f_2^{(j)} - k^2 q f_1^{(j)} \overline{\psi} \right) \mathrm{d}x \qquad \psi \in H^1(\mathbb{R}^3),$$

as right-hand side. The sequence v_j converges in $H^1(\mathbb{R}^3)$ and defines $F \in H^1(\mathbb{R}^3)^*$ by $F(\psi) := a(v, \psi)$ for $\psi \in H^1(\mathbb{R}^3)$. Continuity of a implies that $\|F_j - F\|_{H^1(\mathbb{R}^3)^*} \to 0$ as $j \to \infty$. Thus, it suffices to show that there is $f = (f_1, f_2)^T \in L^2(D, \mathbb{C}) \times L^2(D, \mathbb{C}^3)$ such that

$$F(\psi) = \int_D \left(\nabla \overline{\psi} \cdot Q f_2 - k^2 q f_1 \overline{\psi} \right) \mathrm{d}x.$$

From Riesz's representation theorem we obtain the existence of $\hat{v} \in H^1(D)$ such that

$$F(\psi) = \int_D \left(\nabla \hat{v} \cdot \nabla \overline{\psi} + \hat{v} \overline{\psi} \right) \mathrm{d}x \qquad \forall \psi \in H^1(D).$$

Setting $f := (-\frac{1}{k^2 q}\hat{v}, Q^{-1}\nabla\hat{v})^T$ finally yields

$$F(\psi) = \int_D \left(\nabla\hat{v}\cdot\nabla\overline{\psi} + \hat{v}\overline{\psi}\right)\,\mathrm{d}x = \int_D \left(\nabla\overline{\psi}\cdot Q f_2 - \overline{\psi}k^2 q f_1\right)\,\mathrm{d}x \qquad \forall\psi\in H^1(\mathbb{R}^3),$$

where the sign-definiteness of the matrix-valued contrast Q implies invertibility. ■

In the following two lemmas, we prove the standard factorization of the far field operator and examine the properties of the arising operators, which will help to establish a link between the far field operator and the interior transmission eigenvalues.

Lemma 4.2. *The far field operator F can be factorized as $F = -H^*TH$.*

Proof. As is the standard technique, we first define an auxiliary operator $G : \boldsymbol{L}^2(D) \to L^2(\mathbb{S}_1)$ by $G(f_1, f_2)^T = v^\infty$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak, radiating solution of (4.11). For $g \in L^2(\mathbb{S}_1)$ it then follows that $G(Hg) = v^\infty$, where v solves

$$\operatorname{div}(A\nabla v) + k^2 n v = \operatorname{div}(Q\nabla f) + k^2 q f \qquad \text{with } f(x) = \int_D f(\theta)e^{ik\hat{x}\cdot\theta}\mathrm{d}S(\theta), \quad x\in\mathbb{R}^3.$$

By the superposition principle, it follows that $F = -GH$. Taking a function $h = (h_1, h_2) \in \boldsymbol{L}^2(D)$ we note that if $w^\infty = H^*(h_1, h_2)^T = (Vh_1 - \operatorname{div} Vh_2)^\infty$ as above, then $w \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is a weak, radiating solution to

$$\Delta w + k^2 w = \operatorname{div} h_2 - h_1 \qquad \text{in } \mathbb{R}^3. \tag{4.13}$$

Since $A = Q + I$ and $n = q + 1$, equation (4.11) can equivalently be written as

$$\Delta v + k^2 v = \operatorname{div}[Q(f_2 - \nabla v)] + k^2 q(f_1 - v) \qquad \text{in } \mathbb{R}^3.$$

Substituting $h_2 = Q(f_2 - \nabla v)$ and $h_1 = -k^2 q(f_1 - v)$ in (4.13), we find that $G = H^*T$ and therefore $F = -H^*TH$ follows. ■

In the following lemma, we gather important properties of the operator $T = T_k$. For this purpose, we denote by $\overline{\mathcal{R}(H)}$ the closure of the range of H in $\boldsymbol{L}^2(D)$.

Lemma 4.3. *(a) For all $f \in \boldsymbol{L}^2(D)$ and $k > 0$ it holds that* $\operatorname{Im}(T_k f, f) \leq 0$.
(b) If $\operatorname{Im}(T_k f, f) = 0$ *for a non-trivial $f \in \overline{\mathcal{R}(H)}$ and $k > 0$, then there is a function $w \in H^1(D)$ with $(w, \nabla w)^T = f$ such that k^2 is an interior transmission eigenvalue with corresponding transmission eigenpair $(w - v, w)$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak solution to* (4.11).
(c) If $k^2 > 0$ is an interior transmission eigenvalue with corresponding transmission eigenpair (u, w), then $(T_k f, f) = 0$ for $f := (w, \nabla w)^T \in \overline{\mathcal{R}(H)}$.
(d) If Q is positive definite, q non-negative and $k = \mathrm{i}$, then T_i is coercive: There exists $c_0 > 0$ such that

$$(T_\mathrm{i} f, f) \geq c_0\|f\|^2 \qquad \forall f\in\boldsymbol{L}^2(D).$$

If Q is negative definite, q non-positive and $k = \mathrm{i}$, then $-T_\mathrm{i}$ is coercive: There exists $c_0 > 0$ such that

$$-(T_\mathrm{i} f, f) \geq c_0\|f\|^2 \qquad \forall f\in\boldsymbol{L}^2(D).$$

(e) For $k > 0$ the difference $T_k - T_\mathrm{i}$ is a compact operator from $\boldsymbol{L}^2(D)$ into itself.

Proof. (a) Let $f = (f_1, f_2) \in \boldsymbol{L}^2(D)$. We have by definition, that

$$(T_k f, f) = (Q[f_2 - \nabla v], f_2)_{L^2(D,\mathbb{C}^3)} - (k^2 q(f_1 - v), f_1)_{L^2(D,\mathbb{C})}.$$

Define now $g_2 := f_2 - \nabla v$ and $g_1 := f_1 - v$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the radiating weak solution to (4.11), i.e.

$$\int_{\mathbb{R}^3} (\nabla v \cdot \nabla\overline{\psi} - k^2 v\overline{\psi})\,\mathrm{d}x = \int_D \left(Q(f_2 - \nabla v)\cdot\nabla\overline{\psi} - k^2 q(f_1 - v)\overline{\psi}\right)\mathrm{d}x = \int_D \left(Qg_2\cdot\nabla\overline{\psi} - k^2 qg_1\overline{\psi}\right)\mathrm{d}x. \tag{4.14}$$

We get that

$$\begin{aligned}(T_k f, f) &= (Qg_2, g_2 + \nabla v)_{L^2(D,\mathbb{C}^3)} - k^2(qg_1, g_1 + v)_{L^2(D,\mathbb{C})}\\ &= (Qg_2, g_2)_{L^2(D,\mathbb{C}^3)} - k^2(qg_1, g_1)_{L^2(D,\mathbb{C})} + \int_D \left(Qg_2\cdot\nabla v - k^2 qg_1 v\right)\,\mathrm{d}x.\end{aligned}$$

Finally, equation (4.14) and standard arguments yield

$$(T_k f, f) = (Qg_2, g_2)_{L^2(D,\mathbb{C}^3)} - (k^2 qg_1, g_1)_{L^2(D,\mathbb{C})} + \int_{|x|<R} [|\nabla v|^2 - k^2|v|^2]\,\mathrm{d}x - \int_{|x|=R} \overline{v}\frac{\partial v}{\partial\nu}\,\mathrm{d}S. \tag{4.15}$$

Since $(Qg, g)_{L^2(D,\mathbb{C}^3)}$ and $(k^2 qg_1, g_1)_{L^2(D,\mathbb{C})}$ are real valued, taking the imaginary part of the last equation and using the radiation condition, we obtain that

$$\mathrm{Im}\,(T_k f, f) = -\frac{k}{4\pi^2}\int_{\mathbb{S}_1} |v^\infty|^2\,\mathrm{d}S \le 0. \tag{4.16}$$

(b) Let $\mathrm{Im}\,(T_k f, f) = 0$ for $f = (f_1, f_2)^T \in \overline{\mathcal{R}(H)}$ and define v as in the proof (a). Equation (4.16) implies that $v^\infty = 0$. Due to Rellich's lemma, this implies that v vanishes in $\mathbb{R}^3 \setminus \overline{D}$. Thus, the variational formulation (4.12) for v reduces to

$$\int_D [\nabla\overline{\psi}\cdot A\nabla v - k^2 n\overline{\psi}v]\,\mathrm{d}x = \int_D \left(\nabla\overline{\psi}\cdot Qf_2 - k^2 qf_1\overline{\psi}\right)\,\mathrm{d}x \qquad \forall\psi \in H^1(D). \tag{4.17}$$

Since $f \in \overline{\mathcal{R}(H)}$, there is a sequence of Herglotz wave functions

$$w_j(x) = \int_{\mathbb{S}_1} g_j(\theta)e^{\mathrm{i}k\,x\cdot\theta}\,\mathrm{d}S(\theta), \qquad x \in \mathbb{R}^3,\ j \in \mathbb{N},$$

such that $f_j = (w_j, \nabla w_j)^T$ converges to $f = (f_1, f_2)^T \in \boldsymbol{L}^2(D)$ as $j \to \infty$. We define v_j as the solution to (4.12) with f replaced by f_j. The continuity of the corresponding solution operator implies that $\|v_j - v\|_{H^1(D)} \le C\|f_j - f\|_{\boldsymbol{L}^2(D)} \to 0$ as $j \to \infty$. Convergence of the $f_j = (w_j, \nabla w_j)^T$ in $\boldsymbol{L}^2(D)$ moreover implies that the restrictions of $w_j \in C^\infty(\mathbb{R}^3)$ to D converge to some function $w \in H^1(D)$. Since w_j satisfies the homogeneous Helmholtz equation $\Delta w_j + k^2 w_j = 0$ in D, this property carries over to w. In particular, $(w, \nabla w) = (f_1, f_2)$ and $Qf_2 = Q\nabla w$. We rewrite (4.17) as

$$\int_D \left[\nabla\overline{\psi}\cdot A\nabla v - k^2 n\overline{\psi}v\right]\,\mathrm{d}x = \int_D \left(\nabla\overline{\psi}\cdot Q\nabla w - k^2 qw\overline{\psi}\right)\,\mathrm{d}x. \qquad \forall\psi \in H^1(D).$$

Using $Q = A - I$ and $q = n - 1$, the latter variational equation is equivalent to

$$\int_D \left[\nabla\overline{\psi}\cdot A\nabla(w - v) - k^2 n\overline{\psi}(w - v)\right]\,\mathrm{d}x = \int_D \left[\nabla\overline{\psi}\cdot\nabla w - k^2\overline{\psi}w\right]\,\mathrm{d}x = 0 \qquad \forall\psi \in H^1(D). \tag{4.18}$$

Choosing the test function ψ in $H^1_0(D)$ the last term on the right vanishes since $w \in H^1(D)$ is a weak solution to the Helmholtz equation in D, i.e. $\int_D \nabla w\cdot\nabla\overline{\psi}\,\mathrm{d}x = \int_D k^2 w\overline{\psi}\,\mathrm{d}x$ for all $\psi \in H^1_0(D)$. In consequence, (4.18) shows that $w - v$ is a weak solution to $\mathrm{div}(A\nabla(w - v)) + k^2 n(w - v) = 0$ in

D. If w vanishes then $f = (w, \nabla w)^T$ vanishes, which is excluded by assumption. Thus, the above equations show that $(w - v, w)$ is a transmission eigenpair to the eigenvalue k^2, compare (4.7).
(c) Let $k^2 > 0$ be an interior transmission eigenvalue with eigenpair $(u, w) \in H^1(D) \times H^1(D)$. Setting $f = (w, \nabla w)^T$ we will show that $(T_k f, f) = 0$. To this end, recall that the set of Herglotz wave functions for densities $g \in L^2(\mathbb{S}_1)$ is dense in the set of H^1-solutions to the Helmholtz equation in D, see [CK01]. Thus, there exists a sequence $g_j \in L^2(\mathbb{S}_1)$ such that the corresponding Herglotz wave functions w_j converge to w in $H^1(D)$. In consequence, $f = (w, \nabla w)^T \in \overline{\mathcal{R}(H)}$.

Since k^2 is an interior transmission eigenvalue, (4.7) implies that $v = u - w \in H^1_0(D)$ satisfies

$$\int_D \left[\nabla v \cdot \nabla \overline{\psi} - k^2 v \overline{\psi}\right] \mathrm{d}x = \int_D \left(\nabla \overline{\psi} \cdot Q \nabla (w - v) - k^2 q (w - v) \psi\right) \mathrm{d}x \qquad \forall \psi \in H^1(D).$$

Setting $\psi = w$ yields

$$\int_D \left[\nabla v \cdot \nabla \overline{w} - k^2 v \overline{w}\right] \mathrm{d}x = \int_D \left(Q(f_2 - \nabla v) \cdot \overline{f_2} - k^2 c (f_1 - v) \overline{f}_1\right) \mathrm{d}x = (T_k f, f).$$

As $f \in \overline{\mathcal{R}(H)}$ there is a sequence $(w_j)_{j \in \mathbb{N}}$ of Herglotz wave functions such that $(w_j, \nabla w_j)^T \to f$ as $j \to \infty$. By definition, $f = (w, \nabla w)$, which implies that $\|\nabla(w - w_j)\|_{L^2(D, \mathbb{C}^3)} \to 0$ and $\|w - w_j\|_{L^2(D, \mathbb{C})} \to 0$ as $j \to \infty$. Since w_j solves the Helmholtz equation and $v \in H^1_0(D)$, we get

$$\begin{aligned}\int_D \left[\nabla v \cdot \nabla \overline{w} - k^2 v \overline{w}\right] \mathrm{d}x &= \lim_{j \to \infty} \int_D \left[\nabla v \cdot \nabla \overline{w_j} - k^2 v \overline{w_j}\right] \mathrm{d}x \\ &= \lim_{j \to \infty} \int_D \left[\nabla v \cdot \nabla \overline{w_j} + v \operatorname{div} \nabla \overline{w_j}\right] \mathrm{d}x = 0\end{aligned}$$

by Green's first identity. In consequence, $(T_k f, f) = 0$.
(d) Relying on (4.15) for $k = \mathrm{i}$ and $f \in \boldsymbol{L}^2(D)$, we exploit the ellipticity of the sesquilinear form for $k = \mathrm{i}$ to conclude that $(T_{\mathrm{i}} f, f) \geq \|v\|^2_{H^1(\mathbb{R}^3)} \geq \|v\|^2_{H^1(D)}$, with v solving (4.12) for $k = \mathrm{i}$. Finally from the closedness of the solution operator $L : f \to v$ from $\boldsymbol{L}^2(D)$ into $H^1(D)$, it follows that $\|v\|_{H^1(D)} \geq C \|f\|_{\boldsymbol{L}^2(D)}$.
(e) This assertion follows from standard embedding arguments, see, e.g. [KL09]. ■

In the next Section, we will consider the special case $n = 1$, since we need less assumptions for this scattering model and explicit material bounds can be found for the first part of the inside-outside duality.

4.3. The case $n = 1$

Recall the model assumption from the beginning of the last section, i.e. $D \subset \mathbb{R}^3$ is a bounded, simply connected Lipschitz domain and the contrast Q is real-valued and sign-definite, such that the density A is positive-definite. In this section we consider the special case $n = 1$ and $q = 0$ in (4.3) and obtain the following scattering problem: We seek a radiating scattered field $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \left(A \nabla u^s \cdot \nabla \psi - k^2 u^s \psi\right) \mathrm{d}x = - \int_D Q \nabla u^i \cdot \nabla \psi \, \mathrm{d}x \qquad \forall \psi \in C^\infty_0(\mathbb{R}^3). \tag{4.19}$$

Recall the definition of the far field operator from the introduction to this chapter in (4.4), where $u^\infty(\cdot, \theta)$ is now the far field to the solutions of (4.19). Under this conditions, we will adapt the factorization of the far field operator from Lemma 4.2 for this case. For that purpose we first introduce a special version of the Herglotz wave operator from (4.10). In this section we define the injective Herglotz operator $H = H_k : L^2(\mathbb{S}_1) \to L^2(D, \mathbb{C}^3)$ as

$$Hg = \nabla v_g, \qquad v_g(x) = \int_{\mathbb{S}_1} g(\theta) e^{\mathrm{i}k\theta\cdot x} \, \mathrm{d}S(\theta), \qquad x \in D \tag{4.20}$$

with the adjoint $H^* : L^2(D, \mathbb{C}^3) \to L^2(\mathbb{S}_1)$ given by

$$(H^*h)(\hat{x}) = -\mathrm{i}k\hat{x} \cdot \int_D h(y) e^{-\mathrm{i}k\hat{x}\cdot y} \, \mathrm{d}y = \int_D (\nabla_y e^{-\mathrm{i}k\hat{x}\cdot y}) \cdot h(y) \, \mathrm{d}y, \qquad \hat{x} \in \mathbb{S}_1.$$

The middle operator T is now given by $T = T_k : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ by

$$Tf = Q(f - \nabla v), \tag{4.21}$$

where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak, radiating solution to

$$\mathrm{div}(A\nabla v) + k^2 v = \mathrm{div}(Qf) \qquad \text{in } \mathbb{R}^3, \tag{4.22}$$

that is, $\int_{\mathbb{R}^3} (A\nabla v \cdot \nabla\psi - k^2 v\psi) \, \mathrm{d}x = \int_D Qf \cdot \nabla\psi \, \mathrm{d}x$ holds for all $\psi \in H^1(\mathbb{R}^3)$ with compact support. Then as Corollaries to Lemma 4.2 and Lemma 4.3, we obtain a factorization of F and the following properties of the middle operator T. Note that in the following $\overline{\mathcal{R}(H)}$ denotes the closure of H in $L^2(D, \mathbb{C}^3)$.

Corollary 4.4. *The far field operator can be written as $F = -H^*TH$.*

Corollary 4.5. *(a) For all $f \in L^2(D, \mathbb{C}^3)$ and $k > 0$ it holds that $\mathrm{Im}\,(T_k f, f)_{L^2(D,\mathbb{C}^3)} \leq 0$.*
(b) If $\mathrm{Im}\,(T_k f, f)_{L^2(D,\mathbb{C}^3)} = 0$ for a non-trivial $f \in \overline{\mathcal{R}(H)}$ and $k > 0$, then there is a function $w \in H^1(D)$ with $\nabla w = f$ such that k^2 is an interior transmission eigenvalue with corresponding transmission eigenpair $(w - v, w)$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak solution to (4.22).
(c) If $k^2 > 0$ is an interior transmission eigenvalue with corresponding transmission eigenpair (u, w), then $(T_k f, f)_{L^2(D,\mathbb{C}^3)} = 0$ for $f := \nabla w \in \overline{\mathcal{R}(H)}$.
(d) If Q is positive definite and $k = \mathrm{i}$, then T_{i} is coercive: There exists $c > 0$ such that

$$(T_{\mathrm{i}} f, f)_{L^2(D,\mathbb{C}^3)} \geq c\|f\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall f \in L^2(D, \mathbb{C}^3).$$

If Q is negative definite, then the operator $-T_{\mathrm{i}}$ is coercive: There exists $c > 0$ such that

$$-(T_{\mathrm{i}} f, f)_{L^2(D,\mathbb{C}^3)} \geq c\|f\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall f \in L^2(D, \mathbb{C}^3).$$

(e) For $k > 0$ the difference $T_k - T_{\mathrm{i}}$ is a compact operator from $L^2(D, \mathbb{C}^3)$ into $L^2(D, \mathbb{C}^3)$.

The eigenvalues λ_j of the far field operator F lie on a circle with radius $8\pi^2/k$ and center at $8\pi^2\mathrm{i}/k$ in the complex plane. Since F is compact, these eigenvalues converge to zero as $j \to \infty$. If the contrast Q is sign-definite, they either approach the origin from the left or from the right. Using the results from Corollary 4.5, this can be shown as in Lemma 2.2.

Lemma 4.6. *Assume that k^2 is no interior transmission eigenvalue. If Q is positive definite or negative definite, then $\mathrm{Re}\,\lambda_j < 0$ or $\mathrm{Re}\,\lambda_j > 0$ for $j \in \mathbb{N}$ large enough, respectively.*

Recall the representation of the eigenvalues λ_j in polar coordinates in (4.8), which relates the eigenvalues λ_j to phases ϑ_j, and the definition of the smallest phase ϑ_* and ϑ^* in (4.9). If k^2 is no interior transmission eigenvalue, we can use the factorization $F = -H^*TH$ as in Remark 3.4 to obtain the representation

$$\cot \vartheta_* = \max_{w \in X} \frac{\operatorname{Re}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}} \qquad \text{and} \qquad \cot \vartheta^* = \min_{w \in X} \frac{\operatorname{Re}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}}. \tag{4.23}$$

where $X = X_k := \overline{\mathcal{R}(H)} \subset L^2(D, \mathbb{C}^3)$. At this point it becomes important to find a suitable characterization of the space X to define a projection operator which allows us to replace the space X in the characterization of the extremal phases by the space $L^2(D, \mathbb{C}^3)$. In the previous chapter, this characterization was well-known, i.e. the space X contained the L^2-solutions to the Helmholtz equation, see (3.8). Here, such a characterization is not so obvious. We will now prove that the space X contains those functions in $L^2(D, \mathbb{C}^3)$ that are curl-free and have potentials that solve the Helmholtz equation. First we need to introduce some technical details.

Remark 4.7. At this point we need the assumption that the Lipschitz domain is simply connected, since this allows us to express curl-free functions in terms of their potentials, see [Mon03, Theorem 3.37]. Note that it would also be sufficient to assume that the Lipschitz domain $D = \cup_{i=1}^{I} D_i$ can be decomposed into $I \in \mathbb{N}$ connected components D_i such that each D_i is a simply connected Lipschitz subdomain with connected boundary and $\overline{D_i} \cap \overline{D_j} = \emptyset$ if $1 \le i \ne j \le I$. This case has been examined in [LP15a]. For simplicity of presentation, we omit this decomposition.

Theorem 4.8 ([Mon03, Theorem 3.37]). *If $w \in L^2(D, \mathbb{C}^3)$ satisfies $\operatorname{curl}(w) = 0$ in the distributional sense, i.e.*

$$\int_D w \cdot \nabla \times \overline{\psi}\, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D, \mathbb{C}^3),$$

then there is a scalar potential $\phi_w \in H^1(D)$ such that $w = \nabla \phi_w$. The potential ϕ_w is unique up to adding a function that is constant on D.

To exclude additive constants, we use the space

$$H^1_\diamond(D) := \left\{ w \in H^1(D),\ \int_D w\, \mathrm{d}x = 0 \right\}. \tag{4.24}$$

This space is a Hilbert space for the inner product $(\phi, \psi) \mapsto \int_D \nabla\phi \cdot \nabla\overline{\psi}\, \mathrm{d}x$ due to a Poincaré inequality. Defining $L^2(D, \mathbb{C}^3, \operatorname{curl}0) := \{u \in L^2(D, \mathbb{C}^3),\ \operatorname{curl}(u) = 0\}$ as the space of curl-free functions in $L^2(D, \mathbb{C}^3)$, we can thus define an operator

$$E:\ L^2(D, \mathbb{C}^3, \operatorname{curl}0) \to H^1_\diamond(D), \qquad w \mapsto E(w) = \phi_w,$$

mapping a curl-free vector field w to its unique scalar potential ϕ_w in $H^1_\diamond(D)$, such that $\nabla E(w) = w$ in $L^2(D, \mathbb{C}^3)$. Obviously, E is continuous,

$$C\|\phi_w\|_{H^1(D)} \le \|\phi_w\|_{H^1_\diamond(D)} = \|\nabla\phi_w\|_{L^2(D,\mathbb{C}^3)} = \|w\|_{L^2(D,\mathbb{C}^3)} \qquad \forall w \in L^2(D, \mathbb{C}^3, \operatorname{curl}0).$$

Now we have introduced all technical tools necessary to characterize the closure of the range of the Herglotz operator, which is done in the following lemma.

Lemma 4.9. *It holds that*

$$X = \overline{\mathcal{R}(H)} = \Big\{ w \in L^2(D, \mathbb{C}^3),\ \int_D w \cdot \nabla \times \phi\, \mathrm{d}x = 0 \quad \forall \phi \in C_0^\infty(D, \mathbb{C}^3), \tag{4.25}$$

$$\exists d \in \mathbb{C} : \int_D \left[\nabla E(w) \cdot \nabla\psi - k^2 (E(w) + d)\psi \right] \mathrm{d}x = 0 \quad \forall \psi \in C_0^\infty(D) \Big\}. \tag{4.26}$$

Proof. Recall from the definition of the Herglotz wave function v_g in (4.20) that $Hg = \nabla v_g$. First we show that $\overline{\mathcal{R}(H)} \subset X$. Let $w \in \mathcal{R}(H)$ be such that $w = \nabla v_g$ for a function $g \in L^2(\mathbb{S}_1)$. Since w is a gradient field it follows immediately that w is curl-free, i.e., $\int_D w \cdot \nabla \times \phi \, \mathrm{d}x = 0$ for all $\phi \in C_0^\infty(D, \mathbb{C}^3)$. Moreover, $\nabla v_g = \nabla E(w) = w$, which implies that there exists $d \in \mathbb{C}$ such that $v_g = E(w) + d$. Since v_g solves the Helmholtz equation,

$$0 = \int_D \left[\nabla(E(w) + d) \cdot \nabla \psi - k^2 (E(w) + d)\psi \right] \mathrm{d}x \qquad \forall \psi \in C_0^\infty(D).$$

Thus, $w \in X$. If we additionally show that X is closed in the topology of $L^2(D, \mathbb{C}^3)$ it follows that $\overline{\mathcal{R}(H)} \subset X$. To this end, assume that $X \ni w_j \to w$ in $L^2(D, \mathbb{C}^3)$ as $j \to \infty$ and that $E(w_j) + d^{(j)} \mathbf{1}_D$ solves the Helmholtz equation. It is clear that the first condition in (4.25) for w_j implies, that $\int_D w \cdot \nabla \times \phi \, \mathrm{d}x = 0$ for all $\phi \in C_0^\infty(D, \mathbb{C}^3)$. Rewriting (4.26) as

$$\int_D \left[w_j \cdot \nabla \psi - k^2 \left(E(w_j) + d^{(j)} \right) \psi \right] \mathrm{d}x = 0 \quad \forall \psi \in C_0^\infty(D),$$

the continuity of E from $L^2(D, \mathbb{C}^3)$ into $H^1_\diamond(D)$ shows that merely the convergence of the vectors $d^{(j)} \in \mathbb{C}$ needs to be shown. This follows from the observation that, for arbitrary $\psi \in C_0^\infty(D)$,

$$(d^{(j)} - d^{(\ell)}) \int_D \psi \, \mathrm{d}x = \int_D \left[(w_j - w_\ell) \cdot \nabla \psi - k^2 E(w_j - w_\ell) \psi \right] \mathrm{d}x \to 0 \qquad (j, \ell \to \infty).$$

Now we consider the orthogonal decomposition $X = \overline{\mathcal{R}(H)} \oplus \overline{\mathcal{R}(H)}^\perp$ and show that the orthogonal complement of $\overline{\mathcal{R}(H)}$ is trivial. Assume that $w_0 \in \overline{\mathcal{R}(H)}^\perp \subset X$. Since $w_0 \in X$, condition (4.26) shows that there is $d \in \mathbb{C}^I$ such that $E(w_0) + d\mathbf{1}_D$ solves

$$\int_D \left[\nabla E(w_0) \cdot \nabla \psi - k^2 (E(w_0) + d) \psi \right] \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D).$$

According to [KG08, Theorem 7.3] the space of Herglotz wave functions v_g is dense in the H^1-solutions of the Helmholtz equation. Therefore there is a sequence $(g_j)_{j \in \mathbb{N}} \subset L^2(\mathbb{S}_1)$ such that $v_{g_j} \to E(w_0) + d\mathbf{1}_D$ in $H^1(D)$ as $j \to \infty$. In particular,

$$\left| \int_D (\nabla E(w_0) - \nabla v_{g_j}) \cdot \nabla \overline{v_{g_j}} \, \mathrm{d}x \right| \leq \| \nabla E(w_0) - \nabla v_{g_j} \|_{L^2(D, \mathbb{C}^3)} \| \nabla v_{g_j} \|_{L^2(D, \mathbb{C}^3)} \to 0,$$

since $\| \nabla v_{g_j} \|_{L^2(D, \mathbb{C}^3)}$ is bounded. In consequence,

$$\int_D |\nabla v_{g_j}|^2 \, \mathrm{d}x - \int_D \nabla E(w_0) \cdot \nabla \overline{v_{g_j}} \, \mathrm{d}x \to 0 \qquad (j \to \infty).$$

Since $w_0 = \nabla E(w_0) \in \overline{\mathcal{R}(H)}^\perp$, the second term on the left vanishes for all $j \in \mathbb{N}$. It follows that $w_0 = \lim_{j \to \infty} \nabla v_{g_j} = 0$, which concludes the proof. ■

Note that we needed to include a constant $d \in \mathbb{C}$ in the definition of the space X, since the operator E merely extracts the potential ϕ_w of a function $w \in X_k$ that has vanishing means but does not take the Helmholtz equation into consideration. To avoid the need to deal with these constants we next define $E_k : L^2(D, \mathbb{C}^3, \mathrm{curl}0) \to H^1(D)$ that again maps w to the unique scalar potential that solves the Helmholtz equation. To this end we define the function $\chi \in C_0^\infty(D)$, such that χ has support in D and $\int_D \chi \, \mathrm{d}x = 1$. Plugging in χ into (4.26) and solving for d shows that $d = - \int_D (k^{-2} \nabla E(w) \cdot \nabla \chi - E(w) \chi) \, \mathrm{d}x$.

Lemma 4.10. *Define* $E_k : L^2(D, \mathbb{C}^3, \mathrm{curl}0) \to H^1(D)$ *for* $k > 0$ *by*

$$E_k : w \to \phi_w = E(w) + \int_D \left[E(w)\chi - \frac{1}{k^2} \nabla E(w) \cdot \nabla \chi \right] \,\mathrm{d}x. \tag{4.27}$$

Then E_k is well-defined and bounded and for fixed w the function $k \mapsto E_k(w)$ is continuously differentiable taking values in $H^1(D)$. If $w \in X_k$, then $\phi_w = E_k(w)$ solves the Helmholtz equation,

$$\int_D [\nabla \phi_w \cdot \nabla \overline{\psi} - k^2 \phi_w \overline{\psi}] \,\mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D).$$

Proof. It remains to compute the derivative of $k \mapsto E_k(w)$. Considering (4.27), E_k is clearly differentiable and the derivative $\mathrm{d}E_k(w)/\mathrm{d}k$ takes the value $2k^{-3}\mathbf{1}_D \int_D w \cdot \nabla \chi \,\mathrm{d}x$ in the domain D. ∎

In the next step, we will introduce a projection operator onto the space X to rewrite the characterization of the extremal phases. Furthermore we will use the properties of the space X to calculate the derivative α from (3.14), which is essential for the first part of the inside-outside duality.

Since we will now investigate the behavior of the largest or the smallest phase on the wavenumber $k > 0$, the dependency of all introduced quantities on k becomes relevant. Therefore we denote this dependence whenever necessary, e.g., as $X_k, T_k, \vartheta_*(k)$ and $\vartheta^*(k)$. To account for the dependency of X_k on k, we follow the procedure from the last chapter and introduce a projection operator P_k from $L^2(D, \mathbb{C}^3)$ onto X_k. We will then use such a projection to rewrite (4.23) using the k-independent space $L^2(D, \mathbb{C}^3)$ instead of X_k,

$$\cot \vartheta_*(k) = \max_{w \in L^2(D,\mathbb{C}^3)} \frac{\mathrm{Re}\,(T_k P_k w, P_k w)}{\mathrm{Im}\,(T_k P_k w, P_k w)} \qquad \text{and} \qquad \cot \vartheta^*(k) = \min_{w \in L^2(D,\mathbb{C}^3)} \frac{\mathrm{Re}\,(T_k P_k w, P_k w)}{\mathrm{Im}\,(T_k P_k w, P_k w)}.$$

We will now show that such a projection operator exists, although we will not use its explicit form in the further analysis. For this purpose we introduce the spaces

$$\begin{aligned} H(\mathrm{div}\,0, D) &:= \{u \in H(\mathrm{div}, D) : \mathrm{div}\, u = 0 \quad \text{in } D\}, \\ H_0(\mathrm{curl}, D) &:= \{u \in H(\mathrm{curl}, D) : \; u \times \nu = 0 \quad \text{on } \partial D\}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} H(\mathrm{div}, D) &:= \{u \in L^2(D, \mathbb{C}^3) : \; \mathrm{div}\, u \in L^2(D, \mathbb{C})\}, \\ H(\mathrm{curl}, D) &:= \{u \in L^2(D, \mathbb{C}^3) : \; \mathrm{curl}\, u \in L^2(D, \mathbb{C}^3)\}. \end{aligned} \tag{4.29}$$

Now the following lemma will be helpful.

Lemma 4.11. *For $w \in L^2(D, \mathbb{C}^3)$ and $k > 0$ there exists a unique vector potential $A = A_w \in H_0(\mathrm{curl}, D) \cap H(\mathrm{div}0, D)$ such that*

$$w = \nabla \phi_w + \nabla \times A_w \qquad \text{where } \phi_w := E_k(w - \nabla \times A_w) \in H^1(D).$$

If $w \in X_k$ then $A_w = 0$ and ϕ_w is a weak solution to the Helmholtz equation,

$$\int_D (\nabla \phi_w \cdot \nabla \overline{\psi} - k^2 \phi_w \overline{\psi}) \,\mathrm{d}x = 0 \qquad \forall \phi \in C_0^\infty(D).$$

Proof. Due to [Mon03, Theorem 3.45, Remark 3.46] a function w in $L^2(D, \mathbb{C}^3)$ can be decomposed as

$$w = \nabla \phi_w + \nabla \times A_w$$

with a scalar potential $\phi_w \in H^1(D)$ and a vector potential $A_w \in H_0(\mathrm{curl}, D) \cap H(\mathrm{div}0, D)$, i.e., $\mathrm{div}\, A_w = 0$ in D. The potential A_w is unique since the difference $A = A_w^1 - A_w^2 \in H_0(\mathrm{curl}, D)$ of two vector potentials $A_w^{1,2}$ solves $\nabla \times \nabla \times A_w = 0$. Thus, $\|\nabla \times A_w\|_{L^2(D,\mathbb{C}^3)} = 0$ and Friedrich's inequality (see, e.g., [Mon03, Corollary 3.51]) implies that A_w vanishes. Moreover, $w - \nabla \times A_w \in L^2(D, \mathbb{C}^3, \mathrm{curl}0)$ is curl-free, such that $\phi_w := E_k(w - \nabla A_w) \in H^1(D)$ is well-defined. If $w \in X_k$, then w is a gradient field and $\nabla \times w = \nabla \times \nabla \times A_w = 0$ in D. Thus, A_w vanishes due to the same arguments as above and $\phi_w = E_k(w)$ solves the Helmholtz equation due to Theorem 4.10. ■

To define a projection operator P_k we exploit, as in the last lemma, the relation $\phi_w = E_k(w - \nabla \times A_w)$ for arbitrary $w \in L^2(D, \mathbb{C}^3)$. Assuming that $k^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D, we additionally define $\hat{w} = \hat{w}_{w,k} \in H_0^1(D)$ to be the unique weak solution to the boundary value problem $\Delta\hat{w} + k^2\hat{w} = \Delta\phi_w + k^2\phi_w$ in D and $\hat{w} = 0$ on ∂D. More precisely,

$$\int_D \left[\nabla\hat{w} \cdot \nabla\overline{\psi} - k^2\hat{w}\overline{\psi}\right] \mathrm{d}x = \int_D \left[\nabla\phi_w \cdot \nabla\overline{\psi} - k^2\phi_w\overline{\psi}\right] \mathrm{d}x \qquad \forall\psi \in H_0^1(D). \tag{4.30}$$

The latter problem is of Fredholm type. By the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D a unique solution $\hat{w} \in H_0^1(D)$ exists and depends continuously on ϕ_w.

Lemma 4.12. *If $k_0^2 > 0$ is not a Dirichlet eigenvalue of $-\Delta$ in D, then $P_{k_0} : L^2(D, \mathbb{C}^3) \to X_{k_0}$,*

$$P_{k_0} w = \nabla\phi_w - \nabla\hat{w} \quad \textit{for } w \in L^2(D, \mathbb{C}^3), \tag{4.31}$$

where $\phi_w = E_{k_0}(w - \nabla \times A_w)$ and $\hat{w} = \hat{w}_{w,k_0} \in H_0^1(D)$ solves (4.30), is a continuous projection onto X_{k_0}. There exists $\varepsilon = \varepsilon(k_0) > 0$ such that for each $w \in L^2(D, \mathbb{C}^3)$ the function $(k_0 - \varepsilon, k_0 + \varepsilon) \ni k \mapsto P_k w$ is continuously differentiable in k with values in $L^2(D, \mathbb{C}^3)$.

Proof. To check that P_{k_0} maps into X_{k_0} we note that $\nabla(\phi_w - \hat{w})$ is a vector field that possesses a scalar potential solving the Helmholtz equation weakly in D. Thus, (4.25) and (4.26) imply that $P_{k_0} w \in X_{k_0}$. Continuity of P_{k_0} from $L^2(D, \mathbb{C}^3)$ into $X_{k_0} \subset L^2(D, \mathbb{C}^3)$ with respect to the norm in $L^2(D, \mathbb{C}^3)$ is clear. To check that P_{k_0} is indeed a projection onto X_{k_0}, choose $w \in X_{k_0}$ and consider $\phi_w = E_{k_0}(w - \nabla \times A_w)$. Lemma 4.11 states that $A_w = 0$, i.e., $\phi_w = E_{k_0}(w) \in H^1(D)$ and ϕ_w solves the Helmholtz equation, that is, the right-hand side in (4.30) vanishes. The latter is by assumption uniquely solvable, which shows that $\hat{w} = 0$ and $P_{k_0} w = \nabla\phi_w = w$.

Concerning differentiability, recall from Lemma 4.10 that $k \mapsto \phi_w = E_k(w - \nabla \times A_w)$ is differentiable with values in $L^2(D, \mathbb{C}^3)$ and, moreover, that the derivative $k \mapsto \phi_w'$ is constant on each connected component of D. Thus, $k \mapsto \nabla\phi_w' = 0$, that is, $k \mapsto \nabla\phi_w$ is constant. Differentiability of $k \mapsto \nabla\hat{w}$ follows from differentiating (4.30) with respect to k. ■

Remark 4.13. If the boundary ∂D is sufficiently regular, i.e. $\partial D \in C^4$, then it is possible to avoid excluding Dirichlet eigenvalues in the definition of a projection, see [LP15a, Lemma 4.6].

We will now use the characterization of the space X and the properties of the projection to calculate the auxiliary derivative α from (3.11), which is essential to prove the first part of the inside-outside duality as we have already seen in the proof of Theorem 3.7. Recall now the operator E_k from (4.27) mapping curl-free vector fields to a scalar potential.

Theorem 4.14. *Let k_0^2 be an interior transmission eigenvalues such that there is $0 \neq w_0 \in X_{k_0}$ that satisfies $(T_{k_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$. We set $\phi_{w_0} = E_{k_0} w_0 \in H^1(D)$. Assume that $P_k : L^2(D, \mathbb{C}^3) \to X_k$ is a projection that is continuously differentiable in $k > 0$. Then*

$$\alpha(k_0) := \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = -2k_0 \int_D |v_{k_0}|^2 \,\mathrm{d}x + 4k_0 \mathrm{Re} \int_D v_{k_0} \overline{\phi_{w_0}} \,\mathrm{d}x. \tag{4.32}$$

Proof. For a proof we refer to either [LP15a, Theorem 4.3] or the proof of the corresponding statement when $n \neq 1$, see Theorem 4.22. Setting $n = 1$ and $q = 0$ in this proof yields the assertion. ■

Now we can state both parts of the inside-outside duality for this scattering model. For the first part, we use the derivative α, whose explicit form we have stated in the last theorem. For a proof of the first part, see the proof of Theorem 3.7 and for a proof of the second part, we refer to the proof of Theorem 4.24, where we consider the general case.

Theorem 4.15 (Inside-Outside Duality - Part 1). *Let k_0^2 be an interior transmission eigenvalue and α be the expression in (4.32). If k_0^2 is no Dirichlet eigenvalue of $-\Delta$ in D then the following statement holds: If Q is positive definite, then*

$$\lim_{k \nearrow k_0} \vartheta_*(k) = 0 \qquad \text{if } \alpha(k_0) > 0 \qquad \text{and} \qquad \lim_{k \searrow k_0} \vartheta_*(k) = 0 \qquad \text{if } \alpha(k_0) < 0.$$

If Q is negative definite, then

$$\lim_{k \searrow k_0} \vartheta^*(k) = \pi \qquad \text{if } \alpha(k_0) > 0 \qquad \text{and} \qquad \lim_{k \nearrow k_0} \vartheta^*(k) = \pi \qquad \text{if } \alpha(k_0) < 0.$$

Theorem 4.16 (Inside-Outside Duality - Part 2). *Assume that $k_0 > 0$ and that $I := (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ does not contain wavenumbers k such that k^2 is an interior transmission eigenvalue. If Q is positive definite and*

$$\lim_{I \ni k \to k_0} \vartheta_*(k) = 0$$

or if Q is negative definite and

$$\lim_{I \ni k \to k_0} \vartheta^*(k) = \pi$$

then k_0^2 is an interior transmission eigenvalue.

Remark 4.17. As in the previous chapter, our numerical experiments indicate that the derivative α may have a distinct sign. But it is as yet not clear how to prove such a property in general. See, however, [LV15] for partial results.

4.4. The case $n \neq 1$

In this section we consider the general scattering equation (4.3), i.e. we seek a radiating scattered field $u^s \in H^1_{\text{loc}}(\mathbb{R}^3)$ that solves

$$\int_{\mathbb{R}^3} \left(A\nabla u^s \cdot \nabla\psi - k^2 n u^s \psi\right) \mathrm{d}x = -\int_D \left(Q\nabla u^i \cdot \nabla\psi - k^2 q u^i \psi\right) \mathrm{d}x \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3). \tag{4.33}$$

Without the model assumption $n = 1$ from the beginning of the last section however there arises some difficulty. This is due to the fact that the two contrasts Q and q appear with different sign in the scattering equation above. For the proof of the second part of the inside-outside duality and in particular the estimate that leads to equation (4.45) we will therefore need to assume that $q = n - 1$ is negative. This however implies that the coercivity of the middle operator T of the factorization that has been proven in Lemma 4.3(d) is lost and therefore the standard proof that the eigenvalues λ_j converge to zero from specific side is no longer possible since it uses the coercivity of the middle operator T. Therefore we need to make some additional assumptions, which preserve this property. Let D be an open, bounded and simply-connected domain with connected complement $\mathbb{R}^3 \setminus \overline{D}$ such that $\partial D \in C^{1,1}$. In this section, we assume the matrix Q to be positive definite, i.e. $\overline{z}^T Q(x) z \geq q_0 |z|^2$ for $z \in \mathbb{C}^3$ for some $q_0 > 0$ and almost all $x \in D$ and the contrast function q to be negative, i.e. $-1 < q(x) \leq c_0 < 0$ for a constant $c_0 < 0$ and almost all $x \in D$. Further additional requirements in the section are that the entries in Q and the function q need to be smooth enough such that a solution u^s of (4.33) is in $H^{1+\varepsilon}(D)$ for an $\varepsilon > 0$. This assumption is fulfilled for example if $Q \in C^1(D, \mathbb{C}^{3\times 3})$ and $q \in C^1(D, \mathbb{C})$, see e.g. [McL00, Theorem 4.20].

This assumptions allow us to prove that the eigenvalues λ_j converge to zero from one specific side and therefore lay the necessary groundwork for the derivation of the inside-outside duality. The remainder of this Section is structured as follows. First we show in Lemma 4.18 which kind of additional consequences can be shown from our model assumptions for the properties of the middle operator T of the factorization. Then we will show in Lemma 4.19 how this properties can be used to adapt the proof for the convergence directions of the eigenvalues λ_j of the far field operator. After these preliminary considerations, we will derive both parts of the inside-outside duality in Theorem 4.23 and Theorem 4.24. Note that the first part of the inside-outside duality again only holds under the assumption that an auxiliary derivative α from equation (4.40) does not vanish.

Recall the factorization of the far field operator $F = -H^*TH$ from Lemma 4.2, where the properties of the middle operator T have been stated in Lemma 4.3. Due to our model assumptions, the middle operator obtains the following additional property.

Lemma 4.18. *The operator T can be written as*

$$T = \begin{pmatrix} -k^2 q(I - K_1) \\ Q(I_3 - K_2) \end{pmatrix}$$

where K_1 and K_2 are compact operators in $L^2(D, \mathbb{C})$ and $L^2(D, \mathbb{C}^3)$ and I and I_3 denote the identity operators for these spaces.

Proof. Due to our model assumptions, we obtain that the mappings $K_1 : f_1 \to v$ and $K_2 : f_2 \to \nabla v$ are compact mappings from $L^2(D, \mathbb{C}^d)$ into itself, where $v \in H^1_{\text{loc}}(\mathbb{R}^3)$ is the radiating, weak solution to (4.12). ∎

For this scattering scenario, the eigenvalues λ_j of the far field operator F also lie on a circle with radius $8\pi^2/k$ and center at $8\pi^2\mathrm{i}/k$ in the complex plane. Since the contrast Q is positive-definite and $n > 0$, we can prove that they approach the origin from the left side. Although the proof uses the principle technique that we have already seen in the proof of Lemma 2.2, we include it anyway to show how to deal with the two different material parameters and how to use the compact embeddings K_1, K_2 from Lemma 4.18.

Lemma 4.19. *Assume that $k^2 > 0$ is no interior transmission eigenvalue. Then $\operatorname{Re}\lambda_j < 0$ for $j \in \mathbb{N}$ large enough.*

Proof. Since the far field operator F is compact and normal, we find a orthonormal basis of eigenfunctions $\psi_n \in L^2(\mathbb{S}_1)$ of F corresponding to the eigenvalues λ_n. We define $\phi_n \in \mathcal{R}(H)$ by

$$\phi_n = \frac{1}{\sqrt{|\lambda_n|}} H\psi_n, \qquad n \in \mathbb{N},$$

where $\phi_n = (\phi_n^{(1)}, \phi_n^{(2)}) \in \boldsymbol{L}^2(D)$. Then we use the factorization of F to get

$$-(T\phi_n, \phi_n) = -\frac{1}{|\lambda_n|}(TH\psi_n, H\psi_n) = -\frac{1}{|\lambda_n|}(H^*TH\psi_n, \psi_n) = \frac{1}{|\lambda_n|}(F\psi_n, \psi_n) = \frac{\lambda_n}{|\lambda_n|} := s_n.$$

We know that $|s_n| = 1$ and $\operatorname{Im} s_n > 0$. Furthermore from the compactness of F it follows that λ_n converges to zero and therefore s_n will converge to -1 or 1. We will show that it converges to -1, proving the assertion. We can use Lemma 4.18 to obtain

$$
\begin{aligned}
(T\phi_n, \phi_n) = &-(k^2 q\phi_n^{(1)}, \phi_n^{(1)})_{L^2(D,\mathbb{C})} + (k^2 qK_1\phi_n^{(1)}, \phi_n^{(1)})_{L^2(D,\mathbb{C})} \\
&+ (Q\phi_n^{(2)}, \phi_n^{(2)})_{L^2(D,\mathbb{C}^3)} - (QK_2\phi_n^{(2)}, \phi_n^{(2)})_{L^2(D,\mathbb{C}^3)} = -s_n,
\end{aligned}
\tag{4.34}
$$

where K_1, K_2 are compact operators. We already know from the proof of [KL13, Lemma 4.1] that the sequence $\phi_n^{(1)}$ is bounded and using the same arguments, one can easily show that $\phi_n^{(2)}$ is also bounded. Therefore the sequence ϕ_n is bounded component-wise, such that the sequence ϕ_n itself is bounded. In particular we find a weakly convergent subsequence $\phi_n = (\phi_n^{(1)}, \phi_n^{(2)}) \rightharpoonup \phi = (\phi^{(1)}, \phi^{(2)})$. Note that since K_1, K_2 are compact, we have that

$$
\begin{aligned}
(k^2 qK_1\phi_n^{(1)}, \phi_n^{(1)})_{L^2(D,\mathbb{C})} - (QK_2\phi_n^{(2)}, \phi_n^{(2)})_{L^2(D,\mathbb{C}^3)} \longrightarrow &(k^2 qK_1\phi^{(1)}, \phi^{(1)})_{L^2(D,\mathbb{C})} \\
&-(QK_2\phi^{(2)}, \phi^{(2)})_{L^2(D,\mathbb{C}^3)}.
\end{aligned}
$$

Now we use that Q and q are real valued and by taking the imaginary part of equation (4.34) and using the fact that $\operatorname{Im}(s_n) \to 0$, we get that

$$
\operatorname{Im}\left[(k^2 qK_1\phi^{(1)}, \phi^{(1)})_{L^2(D,\mathbb{C})} - (QK_2\phi^{(2)}, \phi^{(2)})_{L^2(D,\mathbb{C}^3)}\right] = 0,
$$

which in particular implies that $\operatorname{Im}(T\phi, \phi) = 0$. Since k^2 is no interior transmission eigenvalue, we conclude from Lemma 4.3(b) that $\phi = 0$. Therefore we have that

$$
(k^2 qK_1\phi_n^{(1)}, \phi_n^{(1)})_{L^2(D,\mathbb{C})} - (QK_2\phi_n^{(2)}, \phi_n^{(2)})_{L^2(D,\mathbb{C}^3)} \to 0
$$

as $n \to \infty$. Using equation (4.34) again, this implies that

$$
-(k^2 q\phi_n^{(1)}, \phi_n^{(1)})_{L^2(D,\mathbb{C})} + (Q\phi_n^{(2)}, \phi_n^{(2)})_{L^2(D,\mathbb{C}^3)} + s_n \to 0. \tag{4.35}
$$

Since $-q > 0$, the first two terms in the latter equation are both positive, which implies that $s_n \to -1$, which proves the assertion. ■

Recall the definition of the smallest phase ϑ_* in (4.9) and the characterization

$$
\cot \vartheta_* = \max_{w \in X} \frac{\operatorname{Re}(Tw, w)}{\operatorname{Im}(Tw, w)}, \tag{4.36}
$$

where $X = \overline{\mathcal{R}(H)} \subset \boldsymbol{L}^2(D)$ with H being the Herglotz operator defined in (4.10). In a next step, we want to further characterize the space X. It contains those functions $w = (w_1, w_2) \in \boldsymbol{L}^2(D)$ for which the first component $w_1 \in L^2(D, \mathbb{C})$ solves the Helmholtz equation and is the potential of the second component $w_2 \in L^2(D, \mathbb{C}^3)$.

Lemma 4.20. *For $w = (w_1, w_2)^T \in \boldsymbol{L}^2(D)$, the space X can be characterized by*

$$
\begin{aligned}
X = \overline{\mathcal{R}(H)} = \Big\{w \in \boldsymbol{L}^2(D),\ &\int_D \left(\nabla w_1 \cdot \nabla \psi - k^2 w_1 \psi\right)\,\mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D, \mathbb{C}), \\
&\int_D w_2 \cdot \psi\ \mathrm{d}x = -\int_D w_1 \operatorname{div} \psi \,\mathrm{d}x \qquad \forall \psi \in C_0^\infty(D, \mathbb{C}^3)\Big\}.
\end{aligned}
\tag{4.37}
$$

Proof. Let $f \in \mathcal{R}(H)$, such that f can be written as $f = (v_g, \nabla v_g)$, where the Herglotz wave function v_g is given by

$$
v_g(x) = \int_{\mathbb{S}_1} g(\theta) e^{\mathrm{i}k\theta \cdot x}\,\mathrm{d}S(\theta)
$$

for a function $g \in L^2(\mathbb{S}_1)$. Then v_g fulfills the Helmholtz equation and therefore it instantly follows that $f \in X$. Let now $f = (f_1, f_2) \in \overline{\mathcal{R}(H)}$. Then there exists a sequence of functions $(f_j)_{j\in\mathbb{N}} \subset \mathcal{R}(H)$, such that $\|f_j - f\|_{\boldsymbol{L}^2(D)} \to 0$ for $j \to \infty$. In particular, there is a sequence $(v_g^{(j)})_{j\in\mathbb{N}}$ of Herglotz wave functions, such that $\|f_1 - v_g^{(j)}\|_{L^2(D,\mathbb{C})} \to 0$ and $\|f_2 - \nabla v_g^{(j)}\|_{L^2(D,\mathbb{C}^3)} \to 0$ for $j \to \infty$. It follows that

$$\int_D f_2 \cdot \phi \ \mathrm{d}x = \lim_{j\to\infty} \int_D \nabla v_g^{(j)} \cdot \phi \ \mathrm{d}x = \lim_{j\to\infty} \int_D v_g^{(j)} \operatorname{div}(\phi) \ \mathrm{d}x = \int_D f_1 \operatorname{div}(\phi) \ \mathrm{d}x$$

for all $\phi \in C_0^\infty(D, \mathbb{C}^3)$. Since each $v_g^{(j)}$ solves the Helmholtz equation, this property carries over to f_1, such that

$$\int_D \left(\nabla f_1 \cdot \nabla \psi - k^2 f_1 \psi\right) \ \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D),$$

such that $f \in X$.
Let now $f = (f_1, f_2) \in X$ be arbitrary. The density of the Herglotz wave functions in the H^1-solution of the Helmholtz equation, see [CK01], now implies there exists a sequence $v_g^{(j)} \in H^1(D)$, such that $\|v_g^{(j)} - f_1\|_{H^1(D)} \to 0$ for $j \to \infty$ and therefore $\|\nabla v_g^{(j)} - \nabla f_1\|_{L^2(D,\mathbb{C}^3)} = \|\nabla v_g^{(j)} - f_2\|_{L^2(D,\mathbb{C}^3)} \to 0$, which concludes the proof. ■

Since we will now investigate the behavior of the smallest phase with varying wavenumber $k > 0$, the dependency of all introduced quantities on k becomes relevant. As in the previous section, we denote this dependence by $X = X_k, T = T_k, \vartheta_* = \vartheta_*(k)$. We will now generalize the projection operator from (4.31) by introducing a projection operator $P_k : \boldsymbol{L}^2(D) \to X_k$ that is differentiable with respect to k. If k^2 is no interior transmission eigenvalue, we can use such a projection to obtain

$$\cot \vartheta_*(k) = \max_{w\in \boldsymbol{L}^2(D)} \frac{\operatorname{Re}\left(T_k P_k w, P_k w\right)}{\operatorname{Im}\left(T_k P_k w, P_k w\right)}.$$

In the following we want to show that such a projection exists, although we will not use its explicit form. Recall from the last section that functions $w \in L^2(D, \mathbb{C}^3)$ can be decomposed into a unique vector potential $A = A_w \in H_0(\operatorname{curl}, D) \cap H(\operatorname{div} 0, D)$ and a potential function $\phi_w \in H^1_\diamond(D)$, such that $w = \nabla \phi_w + \nabla \times A_w$, where $H^1_\diamond(D)$ was defined in (4.24). Recall also the operator $E_k : L^2(D, \mathbb{C}^3) \to H^1(D)$ from Lemma 4.10 in the previous section, which adds constants to a potential, such that, if possible, the modified potential solves the Helmholtz equation. Let now $w = (w_1, w_2)^T \in \boldsymbol{L}^2(D)$. If we exclude k^2 from being a Dirichlet eigenvalue of $-\Delta$, a projection P_k onto X_k is given by

$$P_k w = \begin{pmatrix} E_k(w_2) - \hat{w} \\ \nabla\left[E_k(w_2) - \hat{w}\right], \end{pmatrix}$$

where $\hat{w} \in H^1_0(D)$ solves

$$\int_D \left(\nabla \hat{w} \cdot \nabla \overline{\phi} - k^2 \hat{w} \overline{\phi}\right) \ \mathrm{d}x = \int_D \left(\nabla E_k(w_2) \cdot \nabla \overline{\phi} - k^2 E_k(w_2) \overline{\phi}\right) \ \mathrm{d}x \qquad \forall \phi \in H^1_0(D).$$

Obviously, P_k is a map onto X_k. Furthermore, $\nabla E_k w_2 = w_2$ and for $w = (w_1, w_2)^T \in X_k$, it follows that $E_k w_2 = w_1$. Since we have excluded k^2 from being an Dirichlet eigenvalue, $\hat{w}$ vanishes since w_1 solves the Helmholtz equation and therefore $P_k w = w$. In order to state the first part of the inside-outside duality, we generalize the derivative α from (4.32) for this scattering scenario. In a first step we will compute the derivative of $k \mapsto (T_k w,\, w)$.

Lemma 4.21. *Let $k_0^2 > 0$ be an interior transmission eigenvalue and let $0 \neq w = (w_1, w_2) \in X_{k_0}$ such that $(T_{k_0} w, w) = 0$. Then the weak radiating solution v_{k_0} to $\operatorname{div}(A \nabla v_{k_0}) + k_0^2 n v_{k_0} = \operatorname{div}(Q w_2) +$*

$k^2 q w_1$ in $\mathbb{R}^3$ belongs to $H_0^1(D)$ and

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w, w)\bigg|_{k=k_0} = -2k_0 \int_D \left(n|v_{k_0}|^2 + q\left(|w_1|^2 - 2\mathrm{Re}\left[\overline{w_1} v_{k_0}\right]\right)\right) \mathrm{d}x.$$

Proof. Due to $(T_{k_0} w, w) = 0$ it follows from (4.16) that the far field $v_{k_0}^\infty = 0$ and v_{k_0} vanishes in $\mathbb{R}^3 \setminus D$. Thus, $v_{k_0} \in H_0^1(D)$. By definition we have $T_{k_0} w = \left(Q(w_2 - \nabla v_{k_0}), -k^2 q(w_1 - v_{k_0})\right)^T$. For arbitrary $k > 0$ we define $v_k \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ as the radiating solution to

$$\int_{\mathbb{R}^3} (A\nabla v_k \cdot \nabla\overline{\psi} - k^2 n v_k \overline{\psi})\, \mathrm{d}x = \int_D \left[(Q w_2) \cdot \nabla\overline{\psi} - k^2 q w_1 \overline{\psi}\right] \mathrm{d}x \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3). \tag{4.38}$$

The mapping $k \mapsto v_k$ is Fréchet-differentiable and $v'_{k_0} := [\mathrm{d}v/\mathrm{d}k\ v_k]|_{k=k_0} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ solves

$$\int_{\mathbb{R}^3} (A\nabla v'_{k_0} \cdot \nabla\overline{\psi} - 2k_0 n v_{k_0} \overline{\psi} - k_0^2 n v'_{k_0} \overline{\psi})\, \mathrm{d}x = -2k_0 \int_D q w_1 \overline{\psi}\, \mathrm{d}x \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3). \tag{4.39}$$

By a density argument, both (4.38) and (4.39) also hold for all $\psi \in H^1(\mathbb{R}^3)$ with compact support. Moreover, for $k = k_0$ the solution $v_{k_0} \in H_0^1(D)$ has compact support and hence (4.38) holds in this case even for all $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}k}(T_k w, w)\bigg|_{k=k_0} &= -\int_D \left[Q\nabla v'_{k_0} \cdot \overline{w_2} - k_0^2 q v'_{k_0} \overline{w_1} + 2k_0 q(w_1 - v_{k_0})\overline{w_1}\right] \mathrm{d}x \\
&= -\int_D \left[(Q\overline{w_2}) \cdot \nabla v'_{k_0} - k^2 q v'_{k_0} \overline{w_1} + 2k_0 q(w_1 - v_{k_0})\overline{w_1}\right] \mathrm{d}x \\
&\overset{(4.38)}{=} -\int_D \left[A\nabla\overline{v_{k_0}} \cdot \nabla v'_{k_0} - k_0^2 n \overline{v_{k_0}} v'_{k_0} + 2k_0 q(w_1 - v_{k_0})\overline{w_1}\right] \mathrm{d}x.
\end{aligned}$$

Exploiting Green's identity, (4.39), and the symmetry of A yields

$$\int_D A\nabla\overline{v_{k_0}} \cdot \nabla v'_{k_0}\, \mathrm{d}x = \int_D \nabla\overline{v_{k_0}} \cdot A\nabla v'_{k_0}\, \mathrm{d}x = \int_D \left(2k_0 n v_{k_0} \overline{v_{k_0}} + k_0^2 n v'_{k_0} \overline{v_{k_0}} - 2k_0 q w_1 \overline{v_{k_0}}\right) \mathrm{d}x,$$

that is, $(\mathrm{d}/\mathrm{d}k)\,(T_k w, w)\big|_{k=k_0} = -2k_0 \int_D \left(n|v_{k_0}|^2 + q\left(|w_1|^2 - 2\mathrm{Re}\left[\overline{w_1} v_{k_0}\right]\right)\right) \mathrm{d}x.$ ■

Theorem 4.22. *Let $k_0^2 > 0$ be an interior transmission eigenvalue and let $0 \neq w = (w_1, w_2) \in X_{k_0}$ such that $(T_{k_0} w, w) = 0$. Then*

$$\alpha(k_0) := \frac{d}{dk}(T_k P_k w, P_k w)\bigg|_{k=k_0} = -2k_0 \int_D \left(n|v_{k_0}|^2 + q|w_1|^2\right) \mathrm{d}x + 4k_0 \mathrm{Re} \int_D n \overline{w_1} v_{k_0}\, \mathrm{d}x. \tag{4.40}$$

Proof. By definition of P_k we have that $P_k w = (w_1, w_2)^T \in X_k$, such that $w_2 = \nabla w_1$. Here w_1 solves the Helmholtz equation, i.e.

$$\int_D \left(\nabla w_1 \cdot \nabla\overline{\psi} - k^2 w_1 \overline{\psi}\right)\ \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D).$$

Then the derivative P'_k of P_k with respect to k is given by $\mathrm{d}/\mathrm{d}k\ P_k w = (w'_1, w'_2)^T = (w'_1, \nabla w'_1)^T$, where $w'_1 \in H^1(D)$ solves.

$$\int_D \left(\nabla w'_1 \cdot \nabla\overline{\psi} - k^2 w'_1 \overline{\psi}\right)\ \mathrm{d}x = 2k \int_D \overline{\psi} w_1\ \mathrm{d}x. \tag{4.41}$$

Applying the chain rule, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(T_kP_kw,P_kw) &= (T_k'P_kw,P_kw)+(T_kP_k'w,P_kw)+(T_kP_kw,P_k'w)\\ &= (T_k'P_kw,P_kw)+\overline{(T_k^*P_kw,P_k'w)}+(T_kP_kw,P_k'w).\end{aligned}$$

Next we show that $T_{k_0}w=T_{k_0}^*w$. We have

$$T_k\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}-k^2q(w_1-v_k)\\ Q[w_2-\nabla v_k]\end{pmatrix}$$

where $v_k\in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the weak, radiating solution to

$$\operatorname{div}(A\nabla v_k)+k^2nv_k=\operatorname{div}(Qw_2)+k^2qw_1 \qquad \text{in } \mathbb{R}^3, \tag{4.42}$$

that is

$$\int_{\mathbb{R}^3}(A\nabla v_k\cdot\nabla\overline{\psi}-k^2nv_k\overline{\psi})\,\mathrm{d}x=\int_D\left(Qw_2\cdot\nabla\overline{\psi}-k^2qw_1\overline{\psi}\right)\mathrm{d}x \qquad \forall\psi\in H^1_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.43}$$

Using the symmetry of Q we find that

$$\begin{aligned}(T_{k_0}w,w) &= (Qw_2,w_2)_{L^2(D,\mathbb{C}^3)}-(k_0^2qw_1,w_1)_{L^2(D,\mathbb{C})}-\int_D\left(Q\nabla v_{k_0}\cdot\overline{w_2}-k_0^2qv\overline{w_1}\right)\,\mathrm{d}x\\ &= (w_2,Qw_2)_{L^2(D,\mathbb{C}^3)}-(w_1,k_0^2qw_1)_{L^2(D,\mathbb{C})}-\int_D\left(\nabla v_{k_0}\cdot(Q\overline{w_2})-k_0^2qv\overline{w_1}\right)\,\mathrm{d}x\end{aligned}$$

such that, bearing (4.43) in mind, it follows that

$$\begin{aligned}(T_{k_0}w,w) &= (w_2,Qw_2)_{L^2(D,\mathbb{C}^3)}-(w_1,k_0^2qw_1)_{L^2(D,\mathbb{C})}-\int_D(A\nabla v_{k_0}\cdot\nabla\overline{v_{k_0}}-k_0^2nv_{k_0}\overline{v_{k_0}})\,\mathrm{d}x\\ &= (w_2,Qw_2)_{L^2(D,\mathbb{C}^3)}-(w_1,k_0^2qw_1)_{L^2(D,\mathbb{C})}-\int_D\left(Q\nabla\overline{v_{k_0}}\cdot w_2-k_0^2q\overline{v_{k_0}}w_1\right)\,\mathrm{d}x\\ &= (w,T_{k_0}w).\end{aligned}$$

This symmetry property implies $T_{k_0}^*w=T_{k_0}w$, see the proof of Theorem 3.6 for details. Using the result of the last lemma yields

$$\begin{aligned}\left[\frac{d}{dk}(T_kP_kw,P_kw)_{L^2(D,\mathbb{C}^3)}\right]\Bigg|_{k=k_0} &= -2k_0\int_D\left(n|v_{k_0}|^2+q\left(|w_1|^2-2\mathrm{Re}\left[\overline{w_1}v_{k_0}\right]\right)\right)\,\mathrm{d}x\\ &\quad+2\mathrm{Re}\,(T_{k_0}w,P_{k_0}'w).\end{aligned}$$

Since $Q=A-I$ we can use (4.41) and partial integration to get

$$\begin{aligned}2\mathrm{Re}\,(T_{k_0}w,P_{k_0}'w) &= 2\mathrm{Re}\int_D\left[(Qw_2-Q\nabla v_{k_0})\cdot\nabla\overline{w_1'}-k_0^2q(w_1-v_{k_0})\overline{w_1'}\right]\,\mathrm{d}x\\ &= 2\mathrm{Re}\int_D\left[(A\nabla v_{k_0}-Q\nabla v_{k_0})\cdot\nabla\overline{w_1'}-k_0^2v_{k_0}(n-q)\overline{w_1'}\right]\,\mathrm{d}x\\ &= 2\mathrm{Re}\int_D\left[\nabla v_{k_0}\cdot\nabla\overline{w_1'}-k^2v_{k_0}\overline{w_1'}\right]\,\mathrm{d}x\\ &= 4k_0\mathrm{Re}\int_D v_{k_0}\overline{w_1}\,\mathrm{d}x.\end{aligned}$$

Using $n = q + 1$ and rearranging terms then shows the assertion. ■

Now we state the first part and the second part of the inside-outside duality. We include the proof of the second part for this scattering scenario which involves the density A and the refractive index n. The arguments can easily be simplified to do the proof of the second part of the inside-outside duality in previous sections on scattering by penetrable scattering objects, which we have left out so far. In this proof we make use of our assumption that Q and $-q$ share the same sign. For a proof of the first part of the inside-outside duality, we again refer to the proof of Theorem 3.7.

Theorem 4.23 (Inside-Outside Duality - Part 1)**.** *Let k_0^2 be an interior transmission that is no Dirichlet eigenvalue and let α be the expression in* (4.40)*. Then the following statement holds:*

$$\lim_{k \nearrow k_0} \vartheta_*(k) = 0 \qquad \text{if } \alpha(k_0) > 0 \qquad \text{and} \qquad \lim_{k \searrow k_0} \vartheta_*(k) = 0 \qquad \text{if } \alpha(k_0) < 0.$$

Theorem 4.24 (Inside-Outside Duality - Part 2)**.** *Assume that $k_0 > 0$ and that the interval $I := (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ does not contain wavenumbers k such that k^2 is an interior transmission eigenvalues. If $\vartheta_*(k) \to 0$ for $I \ni k \to k_0$ then k_0^2 is an interior transmission eigenvalue.*

Proof. Assume that $\vartheta_*(k) \to 0$ for $I \ni k \to k_0$ and k_0^2 is no interior transmission eigenvalue. Due to equation (4.36), we have

$$\max_{w \in X_k} \frac{\mathrm{Re}\,(T_k w, w)}{\mathrm{Im}\,(T_k w, w)} \longrightarrow \infty \qquad \text{for } k \to k_0.$$

Thus, there is a sequence $\{k_j\}_{j \in \mathbb{N}} \subset I$ such that $k_j \to k_0$ and $w^{(j)} = (w_1^{(j)}, w_2^{(j)}) \in X_{k_j}$ with $\|w^{(j)}\|_{\boldsymbol{L}^2(D)} = 1$ such that

$$0 > \mathrm{Im}\,(T_{k_j} w^{(j)}, w^{(j)}) \to 0, \text{ for } j \to \infty$$

and $\mathrm{Re}\,(T_{k_j} w^{(j)}, w^{(j)}) \leq 0$ for j large enough. Let $v_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ be the corresponding weak radiating solution to

$$\mathrm{div}(A \nabla v_j) + k^2 n v_j = \mathrm{div}(Q w_2^{(j)}) + k^2 q w_1^{(j)} \qquad \text{in } \mathbb{R}^3, \tag{4.44}$$

see (4.12). Since the sequence $w^{(j)}$ is bounded in $L^2(D, \mathbb{C}^3)$ there exists a weakly convergent subsequence $w^{(j)} \rightharpoonup w^{(0)}$ in $L^2(D, \mathbb{C}^3)$ as $j \to \infty$. In particular $w^{(0)} \in X_{k_0}$ and $v_j \rightharpoonup v_{k_0}$ weakly in $H^1(B(0,R))$ for all radii $R > 0$, where $v_{k_0} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the corresponding weak radiating solution to (4.44) with right-hand side $\mathrm{div}(Q w_2^{(0)}) + k^2 q w_1^{(0)}$. In the proof of Lemma 4.3 we have already shown that

$$\mathrm{Im}\,(T_{k_j} w^{(j)}, w^{(j)}) = \frac{k_j}{4\pi^2} \|v_j^\infty\|^2_{L^2(\mathbb{S}_1)}.$$

The left hand side converges to zero and the right hand side to $k_0/(4\pi^2)\,\|v_{k_0}^\infty\|_{L^2(\mathbb{S}_1)}$. We conclude that $v_{k_0}^\infty = 0$ and v_{k_0} vanish in the exterior of D by Rellich's lemma. Due to our assumption that $k_0^2 > 0$ is no interior transmission eigenvalue, it follows that $w^{(0)}$ and v_{k_0} vanish everywhere, such that $w^{(j)}$ and v_j converge weakly to zero as $j \to \infty$. As in the proof of Lemma 4.3 we define $g_2^{(j)} := w_2^{(j)} - \nabla v_j$ and $g_1^{(j)} := w_1^{(j)} - v_j$ and find

$$\begin{aligned}(T_{k_j} w^{(j)}, w^{(j)}) &= (Q g_2^{(j)}, g_2^{(j)})_{L^2(D,\mathbb{C}^3)} - (k_j^2 q g_1^{(j)}, g_1^{(j)})_{L^2(D,\mathbb{C})} \\ &\quad + \int_{|x|<R} [|\nabla v_j|^2 - k_j^2 |v_j|^2]\,\mathrm{d}x - \int_{|x|=R} \overline{v_j} \frac{\partial v_j}{\partial \nu}\,\mathrm{d}S\end{aligned}$$

Due to our assumptions for the material parameters Q and q, we have that the first two terms of the right hand side of the last equation are positive, such that by taking the real part of this equation and using $\operatorname{Re}(T_{k_j} w^{(j)}, w^{(j)})_{L^2(D,\mathbb{C}^3)} \leq 0$, we have that

$$0 \geq \int_{|x|<R} [|\nabla v_j|^2 - k_j^2 |v_j|^2] \, \mathrm{d}x - \int_{|x|=R} \overline{v_j} \frac{\partial v_j}{\partial \nu} \, \mathrm{d}S, \tag{4.45}$$

or rather

$$\int_{|x|<R} |\nabla v_j|^2 \, \mathrm{d}x \leq \int_{|x|<R} k_j^2 |v_j|^2 \, \mathrm{d}x + \int_{|x|=R} \overline{v_j} \frac{\partial v_j}{\partial \nu} \, \mathrm{d}S.$$

From the weak convergence of v_j in $H^1(B(0,R))$ for all balls of arbitrary radius R, we conclude that v_j converges to zero strongly in $L^2(B(0,R))$. Note that the integrals $\int_{|x|=R} \overline{v_j} \frac{\partial v_j}{\partial \nu} \, \mathrm{d}S$ converge to zero as $j \to \infty$ since the far field v^∞ of v_{k_0} vanishes. Therefore the right side converges to zero and therefore also the left side, which implies that $v_j \to 0$ strongly in $H^1(B(0,R))$. This also implies that $w^{(j)} \to 0$, which contradicts our assumptions that $\|w^{(j)}\| = 1$. ■

4.5. Conditions for the Material Parameters

One of the reasons why we additionally consider the simple case where $n = 1$ is because we can then obtain explicit conditions for the contrast Q such that the derivative α does not vanish. Therefore we assume in this section that $n = 1$ and $q = 0$. To outline the subsequent estimates, we will first set up conditions for constant and isotropic contrast $Q = q_0 \, I_3$ and in a second step derive conditions for perturbations of such contrasts. To simplify notation we abbreviate the L^2-norm by $\|u\| := \|u\|_{L^2(D,\mathbb{C})}$ or $\|u\| := \|u\|_{L^2(D,\mathbb{C}^3)}$, depending on the space of u.

Assume for a moment that k_0^2 is an interior transmission eigenvalue with eigenpair $(u_0, w_0) \in H^1(D) \times H^1(D)$ for contrast Q and recall that $v_{k_0} = v_0 := u_0 - w_0 \in H_0^1$ solves

$$\int_D \left(A \nabla v_{k_0} \cdot \nabla \overline{\psi} - k_0^2 v_{k_0} \overline{\psi} \right) \mathrm{d}x = \int_D Q \nabla w_0 \cdot \nabla \overline{\psi} \, \mathrm{d}x \qquad \forall \psi \in H^1(D). \tag{4.46}$$

The choice $\psi = 1$ shows that $v_{k_0} \in \tilde{H}_0^1(D)$, where

$$\tilde{H}_0^1(D) := \left\{ \varphi \in H_0^1(D), \int_D \varphi \, \mathrm{d}x = 0 \right\}.$$

Before setting up conditions for Q we further note by the min-max principle that the smallest eigenvalue ρ_0 of the eigenvalue problem to find $(\rho, \varphi) \in \mathbb{R} \times \tilde{H}_0^1(D)$ such that

$$\int_D \nabla \varphi \cdot \nabla \overline{\psi} \, \mathrm{d}x = \rho \int_D \varphi \overline{\psi} \, \mathrm{d}x \qquad \forall \psi \in \tilde{H}_0^1(D) \tag{4.47}$$

is larger than the first Dirichlet eigenvalue of $-\Delta$ in D and given by $\rho_0 = \inf_{\varphi \in \tilde{H}_0^1(D)} \|\nabla \varphi\|^2 / \|\varphi\|^2$. Moreover, we denote by $1/\mu_0$ the smallest non-trivial Neumann eigenvalue of $-\Delta$ in D.

Theorem 4.25. *Choose $0 < k^2 < 2\rho_0$ and $q_0 > 0$ such that*

$$q_0 > \max\left\{ \frac{k^2 - \rho_0}{\rho_0 - k^2/2}, 0 \right\} \qquad \text{and} \qquad \gamma(q_0) := \frac{(q_0 + 2)\big((q_0 - 1)^2 - 5\big)}{(q_0 + 1)^2} > 8 \rho_0 \mu_0. \tag{4.48}$$

Setting $Q = q_0 \, I_3$ then guarantees the existence of at least one transmission eigenvalue $k_0^2 < k^2$ and for all interior transmission eigenvalues $k_0^2 < k^2$ the derivative $\alpha(k_0^2) > 0$ is positive.

Proof. If we assume that $(u_0, \phi_{w_0}) \in H^1(D) \times H^1(D)$ is a transmission eigenpair for the eigenvalue $k_0^2 > 0$ and contrast $Q = q_0 \, I_3$, Theorem 4.5 implies that $\nabla \phi_{w_0} =: w_0 \in X_{k_0}$. Thus, $\phi_{w_0} = E_{k_0}(w_0) \in$

$H^1(D)$ solves the Helmholtz equation and

$$4k_0 \mathrm{Re} \int_D \overline{\phi_{w_0}} v_{k_0} \, \mathrm{d}x = 4k_0 \mathrm{Re} \int_D \frac{1}{k_0^2} \nabla \overline{\phi_{w_0}} \cdot \nabla v_{k_0} \, \mathrm{d}x = \frac{4}{k_0} \mathrm{Re} \int_D \overline{w_0} \cdot \nabla v_{k_0} \, \mathrm{d}x.$$

Due to our assumption that Q can be written as $Q = q_0 \, I_3$ for a constant $q_0 > 0$, we have that $A = (1 + q_0) I_3$ and by substituting $\psi = v_{k_0}$ in the variational formulation (4.46) we obtain

$$\frac{4}{k_0} \int_D \overline{w_0} \cdot \nabla v_{k_0} \, \mathrm{d}x = \frac{4(q_0+1)}{q_0 k_0} \|\nabla v_{k_0}\|^2 - \frac{4k_0}{q_0} \|v_{k_0}\|^2.$$

Thus, $\alpha(k_0)$ is given by

$$\alpha(k_0) = -2k_0 \|v_{k_0}\|^2 + \frac{4}{k} \mathrm{Re} \int_D \overline{w_0} \cdot \nabla v_{k_0} \, \mathrm{d}x = \frac{4(q_0+1)}{q_0 k_0} \|\nabla v_{k_0}\|^2 - 2k_0 \left(\frac{2}{q_0} + 1 \right) \|v_{k_0}\|^2. \quad (4.49)$$

Furthermore, the definition of ρ_0 from (4.47) implies that $\rho_0 \|v\|^2 \leq \|\nabla v\|^2$ for all $v \in \tilde{H}_0^1(D)$, i.e.,

$$\alpha(k_0) \geq \left(\frac{4(q_0+1)}{q_0 k_0} \rho_0 - 2k_0 \left(\frac{2}{q_0} + 1 \right) \right) \|v_{k_0}\|^2 > 0 \qquad \text{if} \quad 4(q_0+1)\rho_0 - 2(q_0+2) k_0^2 > 0.$$

The derivative $\alpha(k_0)$ is hence positive whenever

$$k_0^2 < \frac{2(q_0+1)\rho_0}{q_0+2} =: C(q_0) \qquad \text{or, equivalently,} \qquad q_0 > \max\left\{ \frac{k_0^2 - \rho_0}{\rho_0 - k_0^2/2}, 0 \right\}. \quad (4.50)$$

The left inequality in particular implies that for transmission eigenvalues $k_0^2 < C(q_0) < 2\rho_0$ the derivative $\alpha(k_0)$ is positive. To show existence of transmission eigenvalues k_0^2 satisfying the latter bound we use a result from [Kir09]: There exists at least one transmission eigenvalue less than $C(q_0)$ if $(q_0+2)\rho_0 + 2C(q_0)^2 \mu_0 < C(q_0) q_0$. (We exploited the equation before (3.23)in [Kir09]; note that the definitions of ρ_0 and $\mu = \mu_0$ are exchanged.) Since $C(q_0) > \rho_0$ we write this condition equivalently as

$$q_0 > \frac{2\rho_0 + 2C(q_0)^2 \mu_0}{C(q_0) - \rho_0}, \qquad \text{i.e.,} \qquad 8\rho_0 \mu_0 < \frac{(q_0+2)\big((q_0-1)^2 - 5\big)}{(q_0+1)^2} =: \gamma(q_0). \quad (4.51)$$

■

Remark 4.26. When the function γ from (4.48) is restricted to $(1+\sqrt{5}, \infty)$, then it is monotonously increasing and thus invertible. The area in the (k_0, q_0)-plane where we showed that $\alpha(k_0)$ is positive is sketched in Figure 4.1.

Finally, we derive conditions for non-constant contrast by a perturbation argument. We assume $Q = q_0 \, I_3 + Q'$, or equivalently $A = (1 + q_0) I + Q'$, where $Q' \in L^\infty(D, \mathbb{R}^{3\times 3})$ is a function taking values in the symmetric matrices such that for $c_0 > 0$ constant

$$z^* Q(x) z = z^* [q_0 I + Q'(x)] z \geq c_0 |z|^2 \qquad \text{for almost all } x \in D \text{ and } z \in \mathbb{C}^3. \quad (4.52)$$

Theorem 4.27. *Let $Q = q_0 \, I_3 + Q'$ for a $q_0 > 0$ and $Q' \in L^\infty(D, \mathbb{R}^{3\times 3})$ be symmetric such that* (4.52) *holds and, additionally, $\|Q'\| [q_0 + \|Q'\|] < c_0(1 + c_0)$. Choose $0 < k < 2\rho_0$ such that*

$$k^2 < \frac{2\rho_0}{q_0+2} \left[1 + c_0 - \frac{\|Q'\|_\infty}{c_0} [1 + \|Q'\|_\infty] \right].$$

If $\|Q'\|_\infty$ is small enough such that (4.54) *holds, then there exists at least one transmission eigenvalue less than k^2 and for all such transmission eigenvalues it holds that $\alpha(k_0) > 0$.*

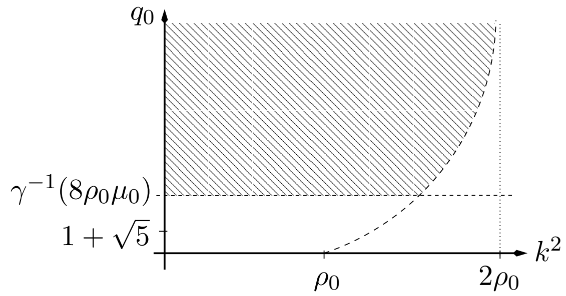


Figure 4.1.: If k_0^2 is an interior transmission eigenvalue for contrast $Q = q_0\, I_3$ and if (k_0, q_0) inside the dashed area in the (k^2, q_0)-plane then $\alpha(k_0) > 0$. Moreover, for each $q_0 > \gamma^{-1}(8\rho_0\mu_0)$ there exists an interior transmission eigenvalue k_0^2 such that (k_0, q_0) lies inside the dashed area.

Proof. Assume that $(u_0, w_0) \in H^1(D) \times H^1(D)$ is a transmission eigenpair for the eigenvalue $k_0^2 > 0$ and contrast Q. Choosing $\psi = v_{k_0} = u_0 - w_0$ in (4.46) and substituting the representations for Q and A we obtain that

$$(1+q_0)\|\nabla v_{k_0}\|^2 + \int_D Q' \nabla v_{k_0} \cdot \nabla \overline{v_{k_0}}\, \mathrm{d}x - k_0^2 \|v_{k_0}\|^2 = \int_D \left(q_0 w_0 \cdot \nabla \overline{v_{k_0}} + Q' w_0 \cdot \nabla \overline{v_{k_0}} \right) \mathrm{d}x.$$

Starting again as in (4.49), the derivative $\alpha(k_0)$ can hence be estimated by

$$\begin{aligned} \alpha(k_0) &= -2k_0 \|v_{k_0}\|^2 + \frac{4}{k_0} \mathrm{Re} \int_D \overline{w_0} \cdot \nabla v_{k_0}\, \mathrm{d}x \\ &= -2k_0 \|v_{k_0}\|^2 + \frac{4}{k_0} \left[\frac{1+q_0}{q_0} \|\nabla v_{k_0}\|^2 - \frac{k_0^2}{q_0} \|v_{k_0}\|^2 + \frac{1}{q_0} \int_D Q'(\nabla v_{k_0} - w_0) \cdot \nabla \overline{v_{k_0}}\, \mathrm{d}x \right] \\ &\geq \frac{4}{k_0 q_0}(1+c_0) \|\nabla v_{k_0}\|^2 - 2k_0 \left(1 + \frac{2}{q_0}\right) \|v_{k_0}\|^2 - \frac{4}{k_0 q_0} \|Q'\|_\infty \|w_0\| \|\nabla v_{k_0}\|. \end{aligned}$$

To substitute w_0 in the last expression, we exploit that $(T_{k_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$ due to Theorem 4.5 and estimate

$$c_0 \|w_0\|^2 \leq \int_D Q w_0 \cdot \overline{w_0}\, \mathrm{d}x = (T_{k_0} w_0, w_0) - \int_D Q \nabla v_{k_0} \cdot \overline{w_0}\, \mathrm{d}x \leq \|Q\|_\infty \|\nabla v_{k_0}\| \|w_0\|.$$

Thus, $\|w_0\| \leq (q_0 + \|Q'\|_\infty)/c_0\, \|\nabla v_{k_0}\|$ and

$$\alpha(k_0) \geq \frac{4}{k_0 q_0} \left[1 + c_0 - \|Q'\|_\infty (q_0 + \|Q'\|_\infty)/c_0 \right] \|\nabla v_{k_0}\|^2 - 2k_0 \left(1 + \frac{2}{q_0}\right) \|v_{k_0}\|^2.$$

Recall from the last proof that $\rho_0 \|v\|^2 \leq \|\nabla v\|^2$ for all $v \in \tilde{H}_0^1(D)$, to estimate

$$\alpha(k_0) \geq \frac{4}{k_0 q_0} \left[1 + c_0 - \frac{\|Q'\|_\infty}{c_0} [q_0 + \|Q'\|_\infty] - \frac{k_0^2 q_0}{2\rho_0} \left(1 + \frac{2}{q_0}\right) \right] \|\nabla v_{k_0}\|^2.$$

Since $v_{k_0} \in H_0^1(D)$ cannot be constant, multiplication with $2\rho_0/(q_0+2)$ yields the two conditions

$$k_0^2 < \frac{2\rho_0}{q_0+2} \left[1 + c_0 - \frac{\|Q'\|_\infty}{c_0} [q_0 + \|Q'\|_\infty] \right] := C(q_0, Q') \quad \text{and} \quad \|Q'\|_\infty [q_0 + \|Q'\|_\infty] < c_0(1+c_0). \tag{4.53}$$

We proceed as in the case of constant contrast. (Recall that $1/\mu_0^2$ is the smallest non-trivial Neumann eigenvalues of $-\Delta$ in D.) Exploiting the bound from [Kir09] for the existence of transmission eigenvalues as in the proof of Theorem 4.25 shows that there exists at least one transmission eigenvalue less than $C(q_0, Q')$ if $(c_0+2)\rho_0 + 2C(q_0, Q')^2\mu_0 < C(q_0, Q')c_0$. Plugging in $C(q_0, Q')$ explicitly shows that the last inequality can be rewritten as

$$\frac{(c_0+2)(q_0+2)}{2c_0} < \left[1 + c_0 - \frac{\|Q'\|_\infty}{c_0}[q_0 + \|Q'\|_\infty]\right]\left[1 - \frac{4\rho_0\mu_0}{(q_0+2)c_0}\left[1 + c_0 - \frac{\|Q'\|_\infty}{c_0}[q_0 + \|Q'\|_\infty]\right]\right]. \tag{4.54}$$

For a given $q_0 > 0$ this inequality holds true if the perturbation $\|Q'\|_\infty$ is small enough. ■

Remark 4.28. As in the previous chapter, these results are certainly not conclusive in showing when the derivative α is non-trivial, but rather serve to show that there exists scattering scenarios at all, for which the derivative does not vanish. As our numerical experiments indicate, it might be possible that α is non-zero for all interior transmission eigenvalues. However, it is currently not clear to us how to show such a feature.

4.6. Numerically Detecting Interior Transmission Eigenvalues from Far Field Data

In this section we present computations of interior transmission eigenvalues for the scattering scenario from Section 4.3 where it was assumed that $n = 1$. In order to apply integral equations as in the previous chapters, we assume the contrast function $Q = q_0 I_3$ for a constant $q_0 > 0$. We furthermore present numerical results both for positive and negative definite contrast and also for three different scatterers: the unit ball, the unit cube and a non-convex object consisting of a cylinder attached to a cube. First we need to obtain the discrete far field matrix

$$\mathbb{F}_N^\delta := u_\delta^\infty(\theta_N^{(j)}, \theta_N^{(l)})_{j,l=1}^N \in \mathbb{C}^{N\times N}$$

from (2.58) as an approximation to the far field operator F. Again we refer to Section 2.4 for the construction of directions $\{\theta_j^N\}_{j=1,..,N} \subset \mathbb{S}_1$ leading to surface quadrangulations of equal area, and the relation between the eigenvalues λ_j of F and the eigenvalues λ_j^N of its discrete representation $\mathbb{F}_N^\delta$ in the context of the inside-outside duality. Let us first describe how we obtain the far field data of the scattering equation under consideration. We rewrite the Helmholtz equation for the total field as

$$\Delta u + k_{\text{int}}^2 u = 0 \quad \text{in } D, \qquad \Delta u + k_{\text{ext}}^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

where $k_{\text{ext}}^2 = k^2$ and $k_{\text{int}}^2 = 1/(q_0+1)k^2$. Writing $[\cdot]|^+$ and $[\cdot]|^-$ for the exterior and interior trace operators, respectively, the jump conditions for u on ∂D are

$$u|^+ = u|^- \qquad \text{and} \qquad (q_0+1)\frac{\partial u}{\partial\nu}\Big|^- = \frac{\partial u}{\partial\nu}\Big|^+ \qquad \text{on } \partial D.$$

Recall that the total field $u = u^i + u^s$ decomposes into an incident plane wave u^i and the corresponding radiating scattered field u^s. To compute numerical approximations of the scattered and far field we use a boundary integral equation due to Kleinman and Martin, see [KM88, SBA+15],

$$\begin{bmatrix} N_{k_{\text{ext}}} + (1+q)N_{k_{\text{int}}} & K'_{k_{\text{ext}}} + K'_{k_{\text{int}}} \\ K_{k_{\text{ext}}} + K_{k_{\text{int}}} & -S_{k_{\text{ext}}} - 1/(q+1)S_{k_{\text{int}}} \end{bmatrix} \begin{bmatrix} u|^+ \\ \frac{\partial u}{\partial\nu}\Big|^+ \end{bmatrix} = \begin{bmatrix} \frac{\partial u^i}{\partial\nu}\Big|^+ \\ -u^i|^+ \end{bmatrix} \tag{4.55}$$

where S_k, K_k, K'_k and N_k are the single-layer potential, double-layer potential, adjoint double-layer potential and hypersingular boundary operators for wavenumber k. Using the software package

BEM++ (see [SBA+15]), we approximate the solution to this system of boundary integral equations using a Galerkin method. This yields the approximate far field data $\mathbb{F}_N^\delta = (u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$.

To verify that our numerical approximation of $F = F_k$ is sufficiently accurate we exploit that if the scatterer D is the unit ball with positive contrast $q_0 = 10$ one can compute the eigenvalues of the far field operator F analytically in terms of Bessel functions, relying on a series representation of the scattered field, compare [CK13]. In Figure 4.2(a) we plot the eigenvalues λ_j and λ_j^N of the far field operator F and its approximation $\mathbb{F}_N^\delta$ for a single wavenumber $k = 5$. Since analytic expressions for a cubic scatterers are not available we plot in Figure 4.2(b) the corresponding computed eigenvalues λ_j^N for the unit cube with contrast $q_0 = -0.9$ and wavenumber $k = 2.0$ together with the circle on which the exact eigenvalues lie.

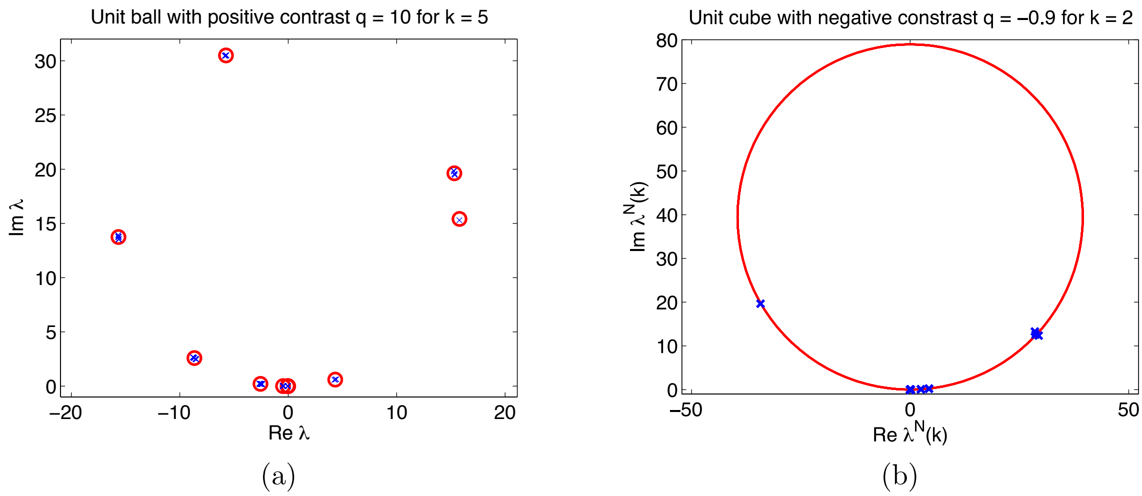
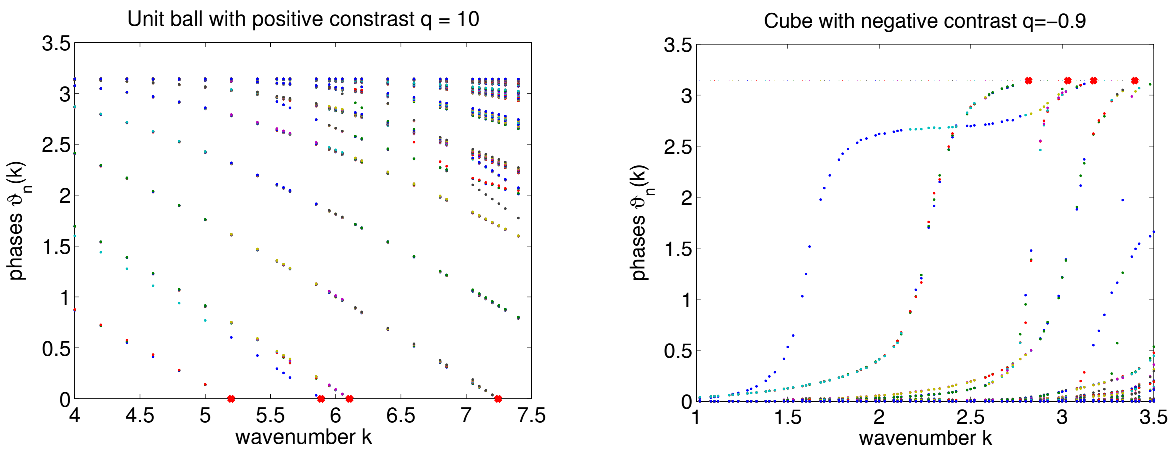


Figure 4.2.: (a) $D = B(0,1)$, $q_0 = 10$, $k = 5$. Red circles mark analytical eigenvalues of F and blue crosses mark numerically computed eigenvalues of $\mathbb{F}_N^\delta$. (b) $D = [-1,1]^3$, $q_0 = -0.9$, $k = 2.0$. Red crosses mark numerically computed eigenvalues of $\mathbb{F}_N^\delta$; the exact eigenvalues of D lie on the blue circle.

Next we compute the eigenvalues λ_j^N, $j = 1, \dots, 120$ of $\mathbb{F}_N^\delta(k)$ for a grid of wavenumbers and examine how their phases behave close to interior transmission eigenvalues k_0^2. Due to Theorem 4.24 we expect the eigenvalue λ^* with the largest phase ϑ^* or the eigenvalue λ_* with the smallest phase ϑ_* to converge to zero from either the left or the right side, implying that either the largest phase ϑ^* converges to π or the smallest phase ϑ_* converges to zero. Due to the polar coordinate representation of the eigenvalues, small errors in the approximated eigenvalues close to zero lead to large errors in the corresponding phases. Thus, we need to stabilize the computation of the phases of the approximate eigenvalues λ_j^N and proceed as in Section 2.4. Assuming that the noise level of $\mathbb{F}_N^\delta(k)$ is $\varepsilon(k) = \|\mathbb{F}_N^\delta(k) - \mathbb{F}(k)\|$ we omit all eigenvalues in the circle $\{|z| \leq \varepsilon(k)\}$ around zero. To further stabilize the phase computations, we afterwards exploit the a-priori knowledge that the exact eigenvalues $\lambda_j(k)$ lie on the circle $\{z \in \mathbb{C}, |z - 8\pi^2\mathrm{i}/k| = 8\pi^2/k\}$ in the complex plane and project the eigenvalues $\lambda_j^N(k)$ orthogonally onto this circle, using the mapping

$$\mathcal{Q} : \lambda \mapsto \frac{8\pi^2\mathrm{i}}{k} + \frac{8\pi^2}{k}\frac{\lambda - 8\pi^2\mathrm{i}/k}{|\lambda - 8\pi^2\mathrm{i}/k|}. \tag{4.56}$$

Then we finally compute the phases of the projected eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$. Figure 4.3 shows the dependence of these numerically computed phases on the wavenumber k, both for a the unit ball with positive contrast and the unit cube with negative contrast as scattering objects. To indicate the stability of these phase curves under random noise we have perturbed the numerically computed data $(u_\delta^\infty(\theta_j, \theta_\ell))_{j,\ell=1}^{120}$ by adding a random matrix of size 120×120 containing normally distributed entries with mean zero such that the relative noise level in the spectral matrix norm equals 5% before computing $\mathcal{Q}[\lambda_j^N(k)]$. Due to this artificial noise and unavoidable numerical inaccuracies, the phase of eigenvalues with a multiplicity $m > 1$ appears as a vertical cluster of m dots above the corresponding wavenumber k in Figure 4.3(a) and (b).



(a) (b)

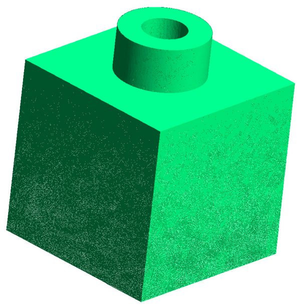
Figure 4.3.: Dots mark the phases of the projected numerical eigenvalues $\mathcal{Q}[\lambda_j^N]$ of $\mathbb{F}_N^\delta(k)$. (a) $D = B(0,1)$, $q_0 = 10$. (b) $D = [-1,1]^3$, $q_0 = -0.9$.

Finally, to obtain numerical approximations to interior transmission eigenvalues we suggest the following method: In a first step we neglect all those phases stemming from far field operator approximations $\mathbb{F}_N^\delta(k)$ with normality error $\|(\mathbb{F}_N^\delta)^*(k)\mathbb{F}_N^\delta(k) - \mathbb{F}_N^\delta(k)(\mathbb{F}_N^\delta)^*(k)\| / \|\mathbb{F}_N^\delta(k)(\mathbb{F}_N^\delta)^*(k)\|$ above a threshold that we consider as too high to provide accurate phase information errors. From the remaining phases we compute those wavenumbers where the discrete derivative of the smallest or largest phase changes sign, i.e., wavenumbers where the extremal phase jumps. Depending on whether the extremal phase approaches the eigenvalues from the left or the right we use the last two smallest or largest phases before the jump to linearly extrapolate the wavenumbers where the phase curve intersects the lines $\{\vartheta = 0\}$ or $\{\vartheta = \pi\}$. The squares of these wavenumbers are approximations of transmission eigenvalues. Table 4.1 indicates the round-about two-digit accuracy of these eigenvalue approximation scheme when the scatterer is a ball; the computed eigenvalues are marked in Figure 4.3 as red dots on $\{\vartheta = 0\}$ in (a) and $\{\vartheta = \pi\}$ in (b).

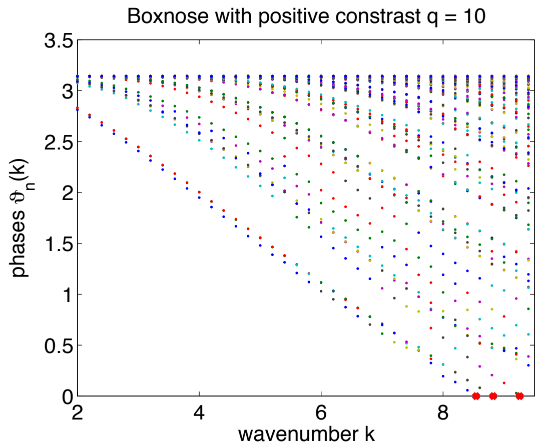
		$k_{0,1}$	$k_{0,2}$	$k_{0,3}$	$k_{0,4}$
$D = B(0,1)$, $q_0 = 10$	computed ITE	5.199	5.888	6.106	7.245
	exact ITE	5.204	5.886	6.104	7.244
$D = [-1,1]^3$, $q_0 = -0.9$	computed ITE	2.863	3.029	3.164	3.397

Table 4.1.: Numerical approximations to the square roots $k_{0,j}$, $j = 1, \ldots, 4$, of four interior transmission eigenvalues $k_{0,j}^2$ for the two settings introduced above.

To show that the numerical scheme also works for non-convex scattering objects, we repeat this procedure for a scatterer consisting of a cylinder placed on a cube with contrast $q_0 = 10$. Figure 4.4(a) shows the geometry of this object, called *boxnose* in the sequel. Precisely the same computational technique as indicated above yield the phase curves shown in Figure 4.4(b). Finally, the above extrapolation algorithm leads to the approximations $k_{0,1} = 8.54$, $k_{0,2} = 8.823$, and $k_{0,3} = 9.259$ for the three smallest transmission eigenvalues $k_{0,j}^2$, $j = 1, 2, 3$.



(a)



(b)

Figure 4.4.: (a) The *boxnose*. (b) Dots mark the phases of the projected numerical eigenvalues $\mathcal{Q}[\lambda_j^N(k)]$ of $\mathbb{F}_N^\delta(k)$. Red crosses on the k-axis mark the positions of the estimated square roots of three interior transmission eigenvalues.

Part II.

The Inside-Outside Duality for Elastic and Electromagnetic Scattering

CHAPTER 5

INTRODUCTION

In the second part of this thesis we will derive the inside-outside duality for elastic and electromagnetic scattering models. In Chapter 6 on elastic scattering we will first consider the propagation of time-harmonic elastic waves in an elastic background medium. In Section 6.2 we then add a rigid body to this medium and consider an exterior scattering problem with Dirichlet boundary conditions. In Section 6.3 we consider scattering from a penetrable, inhomogeneous object that shares the material parameters of the background medium and whose mass density is described by a scalar function. An essential part of the differential equations that describe the scattering process is the Navier-operator Δ^*, which we define in (6.2). This operator can be considered as the extension of the Laplacian for elastic scattering models. Therefore it also shares some properties with the Laplacian. For example, if we assume Dirichlet boundary conditions, the Navier operator has a set of infinitely many, discrete eigenvalues which tend to infinity. As for the Laplacian, this property is an immediate consequence of the coercivity of these operators, see e.g. [McL00].

A typical problem in inverse, elastic scattering theory is the determination of the shape of a rigid obstacle from far field measurements. An obvious attempt to approach this problem is to extend the available methods for acoustic scattering theory to the present case. Results of this approach are for example extensions of the linear sampling method and the factorization method to elastic scattering problems [Are01, AK02, HKS13]. However these methods can fail at interior eigenvalues of the Navier-operator. Therefore there is a natural interest in determining these eigenvalues from far field data without knowledge of the scattering object. The inside-outside duality for rigid obstacles, which we derive in Section 6.2, can potentially be used to determine interior eigenvalues of rigid obstacles or at least guarantee certain frequency bands that contain no interior eigenvalues. As in the case of acoustic scattering by impenetrable scattering objects, the inside-outside duality yields a full characterization of interior Dirichlet eigenvalues of $-\Delta^*$. This is again due to the fact that the far field operator under consideration has a factorization with outer operators that have dense range in their image space.

Next we consider penetrable, inhomogeneous scattering objects within the homogeneous background medium. This scattering problem corresponds to an interior transmission eigenvalue problem, which has been examined in [Cha02, CA08, BG10]. In these studies, the well-posedness of the interior transmission eigenvalue problem has been examined and the existence of at most a countable set of interior transmission eigenvalues has been shown under strict conditions for the material parameters. These results have been generalized in [BCG13], where the existence of an infinite, discrete set of interior transmission eigenvalues has been shown for general settings that include the setting we consider in this chapter. The study of the interior transmission eigenvalue problem is interesting in relation to the application of the linear sampling method [CGK02, BGCM06, GM07] and the factorization method [CKA$^+$07]. As in the case of rigid obstacles, these methods can fail at interior transmission eigenvalues. We will show that the inside-outside duality can be used to provide a sufficient condition to determine interior transmission eigenvalues. A full characterization is again only possible under certain conditions for the material parameters, which is a common problem when considering scattering by penetrable objects.

The final setting of this thesis in Chapter 7 is electromagnetic scattering from anisotropic, dielectric scattering objects that may contain cavities. We will use the inside-outside duality to determine electromagnetic transmission eigenvalues of the related interior transmission eigenvalue problem. This eigenvalue problem has been studied in [Kir09, CK10, CCM11, CH09, CHG10] among other. It has been shown that there is an infinite discrete set of interior transmission eigenvalues with infinity as the only possible accumulation point. Since the scattering object is inhomogeneous, we encounter the same problem of Chapter 3 and Chapter 4, being able to prove a full characterization of interior transmission eigenvalues only under certain conditions on the material parameters. In the second part of the discussion on electromagnetic scattering, we will also allow our scattering object to contain cavities. In this context we will adapt the analysis of acoustic scattering objects with cavities to the present case.

CHAPTER 6

ELASTIC SCATTERING FROM PENETRABLE AND IMPENETRABLE OBJECTS

6.1. Introduction

In this chapter we will show that it is possible to characterize Dirichlet eigenvalues of the Navier operator and interior transmission eigenvalues from far field data by using the inside-outside duality method. The far field data arises from a corresponding scattering problem. First we consider scattering from a rigid body, i.e. we consider an exterior Dirichlet scattering problem. Second, we consider scattering from a penetrable, inhomogeneous scattering object. For either case, we assume that the scattering object is embedded in an isotropic, homogeneous and elastic background medium that fills the three-dimensional space $\mathbb{R}^3$ and is described by the constant Lamé parameters μ and λ and the normalized mass density $\rho = 1$.

The propagation of time-harmonic elastic waves in this background medium is described by the Navier equation

$$\Delta^* u + \omega^2 u = 0, \tag{6.1}$$

where $\omega > 0$ is the frequency and the Navier operator Δ^* is given by

$$\Delta^* := \mu\Delta + (\lambda + \mu)\nabla \operatorname{div}. \tag{6.2}$$

Note that since the displacement field u is vector-valued, the Laplace operator Δ is applied componentwise and $\nabla u = (\nabla u_1, \nabla u_2, \nabla u_3)^T$ is the Jacobi matrix of u. To guarantee propagation of an elastic wave in this medium, we require the Lamé constants to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$. The displacement field u can be decomposed as $u = u_\mathrm{p} + u_\mathrm{s}$, where u_p describes its longitudinal (pressure) part and u_s describes its transversal (shear) part. Note that both of these parts solve the Helmholtz equations

$$\Delta u_\mathrm{p} + k_\mathrm{p}^2 u_\mathrm{p} = 0, \qquad \Delta u_\mathrm{s} + k_\mathrm{s}^2 u_\mathrm{s} = 0,$$

with positive wavenumbers

$$k_\mathrm{p}^2 = \frac{\omega^2}{\lambda + 2\mu}, \quad k_\mathrm{s}^2 = \frac{\omega^2}{\mu}. \tag{6.3}$$

Now we consider the exterior time-harmonic Dirichlet scattering problem. For an impenetrable scattering object $D \subset \mathbb{R}^3$ with Lipschitz boundary, we seek a solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ to

$$\Delta^* u + \omega^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D. \tag{6.4}$$

The total field $u = u^s + u^i$ is the sum of a scattered field u^s and an incident plane wave u^i. To define the incident plane wave more precisely, we introduce longitudinal and transversal plane waves as incoming waves with direction of propagation $\theta \in \mathbb{S}_1 := \{x \in \mathbb{R}^3 : |x| = 1\}$ by

$$u^i_{\mathrm{p}}(x,\theta) = q_{\mathrm{p}} e^{\mathrm{i}k_{\mathrm{p}} x\cdot\theta}, \qquad u^i_{\mathrm{s}}(x,\theta) = q_{\mathrm{s}} e^{\mathrm{i}k_{\mathrm{s}} x\cdot\theta}, \qquad x \in \mathbb{R}^3. \tag{6.5}$$

Here q_{p} and q_{s} are polarization vectors that are parallel, or orthogonal, to θ respectively. Both plane waves are entire solutions of the Navier equation and so is the linear combination

$$u^i(x,\theta) = u^i_{\mathrm{p}}(x,\theta) + u^i_{\mathrm{s}}(x,\theta). \tag{6.6}$$

We require the scattered field u^s to fulfill the Kupradze radiation condition

$$\lim_{r\to\infty}\left(\frac{\partial u^s_{\mathrm{p}}}{\partial r} - \mathrm{i}k_{\mathrm{p}} u^s_{\mathrm{p}}\right) = 0, \quad \lim_{r\to\infty}\left(\frac{\partial u^s_{\mathrm{s}}}{\partial r} - \mathrm{i}k_{\mathrm{s}} u^s_{\mathrm{s}}\right) = 0, \quad r = |x|, \tag{6.7}$$

uniformly in all directions. Here the radiation condition is defined in terms of the longitudinal wave $u^s_{\mathrm{p}} = -k_{\mathrm{p}}^{-2}\nabla \operatorname{div} u^s$ and the transversal wave $u^s_{\mathrm{s}} = u^s - u^s_{\mathrm{p}}$. Note that solutions that fulfill Kupradze's radiation condition are in this chapter called radiating solutions. We now introduce two function spaces of longitudinal and transversal vector fields on $\mathbb{S}_1$ by

$$\begin{aligned} L^2_{\mathrm{p}}(\mathbb{S}_1) &:= \{g_{\mathrm{p}} : \mathbb{S}_1 \to \mathbb{R}^3 : g_{\mathrm{p}}(\theta) \times \theta = 0, |g_{\mathrm{p}}| \in L^2(\mathbb{S}_1)\}, \\ L^2_{\mathrm{s}}(\mathbb{S}_1) &:= \{g_{\mathrm{s}} : \mathbb{S}_1 \to \mathbb{R}^3 : g_{\mathrm{s}}(\theta) \cdot \theta = 0, |g_{\mathrm{s}}| \in L^2(\mathbb{S}_1)\}. \end{aligned}$$

On the space $L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$ a scalar product is given by

$$(g,h) := \frac{\omega}{k_{\mathrm{p}}} \int_{\mathbb{S}_1} g_{\mathrm{p}}(\theta) \cdot \overline{h_{\mathrm{p}}(\theta)} \,\mathrm{d}s(\theta) + \frac{\omega}{k_{\mathrm{s}}} \int_{\mathbb{S}_1} g_{\mathrm{s}}(\theta) \cdot \overline{h_{\mathrm{s}}(\theta)} \,\mathrm{d}s(\theta)$$

for $g = (g_{\mathrm{p}}, g_{\mathrm{s}}), h = (h_{\mathrm{p}}, h_{\mathrm{s}}) \in L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$. Radiating solutions to the Navier equation have the asymptotic behavior

$$u^s(x) = \frac{e^{\mathrm{i}k_{\mathrm{p}}|x|}}{|x|} u_{\mathrm{p},\infty}(\hat{x}) + \frac{e^{\mathrm{i}k_{\mathrm{s}}|x|}}{|x|} u_{\mathrm{s},\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty, \tag{6.8}$$

uniformly in all directions $\hat{x} := x/|x|$. Here $u_{\mathrm{p},\infty}$ and $u_{\mathrm{s},\infty}$ are the longitudinal and transversal far fields and we will call the pair $u_\infty := (u_{\mathrm{p},\infty}, u_{\mathrm{s},\infty}) \in L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$ the far field of u. In order to introduce the far field operator we will first generalize the incident field and introduce the Herglotz wave field v^{in}_g for a function $g \in L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$ by

$$v^{\mathrm{in}}_g(x) := \int_{\mathbb{S}_1} \left(e^{\mathrm{i}k_{\mathrm{p}} x\cdot\theta} g_{\mathrm{p}}(\theta) + e^{ik_{\mathrm{s}} x\cdot\theta} g_{\mathrm{s}}(\theta)\right) \mathrm{d}s(\theta), \qquad x \in \mathbb{R}^3. \tag{6.9}$$

We now define the far field operator as the far field of the solution v_g to the exterior Dirichlet scattering problem, where the incident wave field is the the Herglotz wave function v^{in}_g, i.e. $F : L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1) \to L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$ is given by

$$Fg := v^\infty_g, \tag{6.10}$$

where $v^\infty_g = (v^\infty_{g,\mathrm{p}}, v^\infty_{g,\mathrm{s}}) \in L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$. This far field operator has some crucial properties which are necessary to derive the inside-outside duality. We know from [AK02, Theorem 3.3, Theorem 3.4] that the far field operator F is compact and normal and that its eigenvalues λ_j lie on a circle in the complex plane with center $2\pi\mathrm{i}/\omega$ and radius $2\pi/\omega$. As we will show in Theorem 6.2, the eigenvalues

$(\lambda_j)_{j\in\mathbb{N}}$ converge to zero from the left side. We represent the eigenvalues λ_j of the far field operator F in polar coordinates, i.e.

$$\lambda_j = |\lambda_j| e^{\mathrm{i}\vartheta_j}, \qquad \vartheta_j \in [0, \pi], \tag{6.11}$$

where we set $\vartheta_j = 0$ if $\lambda_j = 0$. By this representation, each eigenvalue λ_j corresponds to a phase ϑ_j and since the eigenvalues converge to zero from the left side, there is one distinct eigenvalue λ_* with a smallest phase

$$\vartheta_* := \min_{j\in\mathbb{N}} \vartheta_j. \tag{6.12}$$

Note that the eigenvalues $\lambda_j = \lambda_j(\omega)$ and their phases $\vartheta_j = \vartheta_j(\omega)$ depend on the frequency ω. The inside-outside duality now states that ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ if, and only if, $\vartheta_*(\omega) \to 0$ as ω approaches ω_0, see Theorem 6.4 and Theorem 6.5 for a precise statement.

The next scattering problem we will consider is scattering from penetrable, inhomogeneous bodies. For a real-valued mass density $\rho \in L^\infty(\mathbb{R}^3)$ such that $\rho = 1$ in the exterior of D, we seek a solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ to the equation

$$\Delta^* u + \omega^2 \rho u = 0 \qquad \text{in } \mathbb{R}^3, \tag{6.13}$$

such that

$$[u]_{\partial D} = 0 \qquad \text{and} \qquad [T_\nu u]_{\partial D} = 0,$$

where ν denotes the outward normal to ∂D and $[\cdot]_{\partial D}$ the jump of a vector-valued function over the boundary ∂D. Finally T_ν is the stress tensor, given by

$$T_\nu := 2\mu\nu\cdot\nabla + \lambda\nu \operatorname{div} + \mu\nu \times \operatorname{curl}.$$

The total field $u = u^s + u^i$ is the sum of a scattered field u^s and the incident field u^i that has been defined in (6.6). The scattered field u^s is assumed to satisfy Kupradze's radiation condition (6.7). Then the scattered field u^s has a representation in terms of its far field u^∞ as in (6.8). Choosing the incident field to be the Herglotz wave field v_g^{in}, defined in (6.9) for a function $g \in L^2_{\mathrm{p}}(\mathbb{S}_1) \times L^2_{\mathrm{s}}(\mathbb{S}_1)$, the far field operator F is defined in (6.10). The far field operator retains the properties that we have already mentioned for the exterior Dirichlet scattering problem, i.e. it is compact and normal and its eigenvalues lie on a circle in the complex plane with center $2\pi\mathrm{i}/\omega$ and radius $2\pi/\omega$, see [Sev05]. The scattering problem is related to an interior transmission eigenvalue problem for elastic scattering. To state this problem, we define

$$H^2_0(D, \mathbb{C}^3) := \{u \in H^2(D, \mathbb{C}^3) : u = 0 \text{ and } T_\nu u = 0 \text{ on } \partial D\}.$$

Then the squared frequency ω^2 is called an interior transmission eigenvalue if there are non-trivial functions $u, w \in L^2(D, \mathbb{C}^3)$ such that $u - w \in H^2_0(D, \mathbb{C}^3)$ and

$$\begin{aligned} \Delta^* u(x) + \omega^2 \rho u(x) &= 0 \qquad \text{in } D,\\ \Delta^* w(x) + \omega^2 w(x) &= 0 \qquad \text{in } D,\\ u(x) - w(x) &= 0 \qquad \text{on } \partial D,\\ T_\nu u(x) - T_\nu w(x) &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{6.14}$$

It has been shown that there exists an infinite number of discrete interior transmission eigenvalues with infinity as the only possible accumulation point, see [BCG13]. We want to determine these interior transmission eigenvalues by the inside-outside duality. Note that although we only consider positive mass densities $\rho \in L^\infty(D)$ with positive contrast $q = \rho - 1$ in this chapter, the arguments can easily be adapted for negative contrasts as in Chapter 3. To indicate our main result, note that for mass densities with positive contrast $q \in L^\infty(D)$ the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of the far field operator F converge to zero from the right side, see Lemma 6.8 below. Then there is one distinct eigenvalue λ^* with a largest phase

$$\vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j. \tag{6.15}$$

Again, we denote the dependence of the phases on the frequency by $\vartheta_j = \vartheta_j(\omega)$. The first part of our main result now states the following: If ω_0^2 is an interior transmission eigenvalue and the expression $\alpha(\omega_0)$ in (6.41) does not vanish, then $\vartheta^*(\omega) \to \pi$ as $\omega \to \omega_0$. On the other hand, if $\vartheta^*(\omega) \to \pi$ for $\omega \to \omega_0$, then ω_0^2 is an interior transmission eigenvalue, see Theorem 6.11 and Theorem 6.12 for a precise statement.

Before we proceed with the discussion of the exterior Dirichlet scattering problem, we introduce some technical details. For the elastic wave equations we will later seek solutions in the space of vectorial Sobolev functions $H^1(D, \mathbb{C}^3)$. For our purpose we equip the space with the norm

$$\|u\|^2_{H^1(D,\mathbb{C}^3)} := \|u\|^2_{L^2(D,\mathbb{C}^3)} + \|\operatorname{div} u\|^2_{L^2(D,\mathbb{C})} + \|\nabla u\|^2_{L^2(D,\mathbb{C}^{3\times 3})}.$$

Using now Green's first theorem and Gauss' integral theorem for the operator Δ^* from (6.2), we obtain Betti's first formula, i.e. for two functions $u, \varphi \in H^1(D, \mathbb{C}^3)$ such that $\Delta^* u \in L^2(D, \mathbb{C}^3)$, we get that

$$\int_D \Delta^* u \cdot \overline{\varphi} \,\mathrm{d}x = -\int_D \left(\mu \nabla u : \nabla\overline{\varphi} + (\mu+\lambda) \operatorname{div} u \operatorname{div}\overline{\varphi}\right) \mathrm{d}x + \int_{\partial D} T_\nu u \cdot \overline{\varphi} \,\mathrm{d}s. \tag{6.16}$$

Here, $A : B$ denotes the Frobenius scalar product of the matrices A, B, defined by $A : B = \sum_{i,j} a_{ij} b_{ij}$ and the boundary integral represents the dual product of $H^{\pm 1/2}(\partial D, \mathbb{C}^3)$. After this preliminary considerations, we will in the next section consider elastic scattering from an impenetrable scattering object with Dirichlet boundary conditions.

6.2. The Exterior Dirichlet Problem

In this section we assume the presence of an impenetrable scattering object $D \subset \mathbb{R}^3$ within the homogeneous background medium, such that the exterior of D is connected and the boundary ∂D is Lipschitz. Then ω^2 is a Dirichlet eigenvalue of $-\Delta^*$ if there exists a solution $v \in H^1_0(D, \mathbb{C}^3)$ such that

$$\Delta^* v + \omega^2 v = 0 \quad \text{in } D \qquad \text{and} \qquad v = 0 \quad \text{on } \partial D.$$

This eigenvalue problem is understood in a weak sense, i.e. $v \in H^1_0(D, \mathbb{C}^3)$ needs to satisfy

$$\int_D \left(\mu \nabla v : \nabla\overline{\varphi} + (\lambda+\mu) \operatorname{div} v \operatorname{div}\overline{\varphi} - \omega^2 v \cdot \overline{\varphi}\right) \mathrm{d}x = 0$$

for all $\varphi \in H^1(D, \mathbb{C}^3)$. Closely related to this problem is the exterior Dirichlet boundary value problem (6.4) which is also understood in a weak sense, i.e. in the formulation for the scattered field, we seek a radiating solution $u^s \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ to

$$\int_{\mathbb{R}^3\setminus\overline{D}} \left(\mu \nabla u^s : \nabla\overline{\varphi} + (\lambda+\mu) \operatorname{div} u^s \operatorname{div}\overline{\varphi} - \omega^2 u^s \cdot \overline{\varphi}\right) \mathrm{d}x = 0 \tag{6.17}$$

for all test functions $\varphi \in H^1(\mathbb{R}^3 \setminus \overline{D}, \mathbb{C}^3)$ with compact support in $\mathbb{R}^3 \setminus \overline{D}$ such that $u^s = -u^i$ on the boundary ∂D, where $u^i(x, \theta)$ is the incident plane wave with direction $\theta \in \mathbb{S}_1$, defined in (6.6). In this section we will proceed as follows: As a first step we will state a factorization of the far field operator F from (6.10) and examine the properties of the arising operators in Lemma 6.1. Then we will use these properties to show in Lemma 6.2 that the eigenvalues λ_j of the far field operator converge to zero only from the left side. Using a particular characterization of the smallest phase, we will then calculate the necessary auxiliary derivative in Lemma 6.3 in order to state the first part and the second part of the inside-outside duality in Theorem 6.4 and Theorem 6.5.

We start by discussing a factorization of the far field operator and introduce the elastic single layer potential

$$\mathrm{SL}\varphi(x) := \int_{\partial D} \Phi_N(x,y)\varphi(y)\,\mathrm{d}s(y), \qquad x \in \mathbb{R}^3 \setminus \partial D, \tag{6.18}$$

where Φ_N is the fundamental solution to the Navier equation,

$$\Phi_N(x,y) := \frac{k_\mathrm{s}^2}{4\pi\omega^2}\frac{e^{ik_\mathrm{s}|x-y|}}{|x-y|}I + \frac{1}{4\pi\omega^2}\nabla_x\nabla_x\left[\frac{e^{\mathrm{i}k_\mathrm{s}|x-y|}}{|x-y|} - \frac{e^{\mathrm{i}k_\mathrm{p}|x-y|}}{|x-y|}\right], \qquad x,y \in \mathbb{R}^3, \quad x \neq y, \tag{6.19}$$

and I denotes the identity matrix. This single layer potential is a linear, bounded operator from $H^{-1/2}(\partial D, \mathbb{C}^3)$ into $H^1(B_R, \mathbb{C}^3)$. Denoting by $[\cdot]|^\pm$ the trace of a function taken from the outside $(+)$ or the inside $(-)$, it holds that $\mathrm{SL}\varphi|^\pm = S\varphi$ in $H^{1/2}(\partial D, \mathbb{C}^3)$, where the elastic single layer operator $S : H^{-1/2}(\partial D, \mathbb{C}^3) \to H^{1/2}(\partial D, \mathbb{C}^3)$ is given by

$$(S\varphi)(x) := \int_{\partial D} \Phi_N(x,y)\varphi(y)\,\mathrm{d}s(y), \qquad x \in \partial D.$$

Furthermore for a function $\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)$, the jump relation

$$T_\nu \mathrm{SL}\varphi|^- - T_\nu \mathrm{SL}\varphi|^+ = \varphi \tag{6.20}$$

holds, see [Kup65, KGBB79] for the properties of these operators.

We denote the duality pairing $\langle H^{-1/2}(\partial D, \mathbb{C}^3), H^{1/2}(\partial D, \mathbb{C}^3)\rangle$ by $(\cdot,\cdot)$ and summarize the properties of S in the following lemma. For a proof, we refer to [AK02].

Lemma 6.1. *Let ω^2 be no Dirichlet eigenvalue of the Navier equation.*
(a) For all $\varphi \in H^{-1/2}(\partial D, \mathbb{C}^3)$ it holds that $(\varphi, S\varphi) \leq 0$.
(b) It holds that $(\varphi, S\varphi) = 0$ if and only if $\varphi = 0$.
(c) Denote by S_i the single-layer operator for the frequency $\omega = \mathrm{i}$. Then S_i is compact, self-adjoint and positive definite, i.e. for a constant $c > 0$

$$(\varphi, S_\mathrm{i}\varphi) \geq c\|\varphi\|_{H^{-1/2}(\partial D, \mathbb{C}^3)} \qquad \forall \varphi \in H^{-1/2}(\partial D, \mathbb{C}^3).$$

(d) The difference $S - S_\mathrm{i}$ is compact from $H^{-1/2}(\partial D, \mathbb{C}^3)$ into $H^{1/2}(\partial D, \mathbb{C}^3)$.

As a second ingredient for a factorization we introduce the injective, bounded operator
$A : H^{1/2}(\partial D, \mathbb{C}^3) \to L^2_\mathrm{p}(\mathbb{S}_1) \times L^2_\mathrm{s}(\mathbb{S}_1)$ by $Af = v^\infty$, where v^∞ is the far field of the radiating solution $v \in H^1_\mathrm{loc}(\mathbb{R}^3 \setminus \overline{D})$ to the problem

$$\Delta^* v + \omega^2 v = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \qquad v = f \quad \text{on } \partial D.$$

Using for example a boundary integral equation approach, see [Kup65, KGBB79], it can be shown that this problem is uniquely solvable. Before we state the factorization, note finally that the solution operator A has dense range in $L^2_\mathrm{p}(\mathbb{S}_1) \times L^2_\mathrm{s}(\mathbb{S}_1)$. Now we can state a factorization of the far field operator. It holds that

$$F = -4\pi A S^* A^*. \tag{6.21}$$

A proof for this factorization and the properties of these operators can be found in [AK02]. Using this factorization and the properties of the operator S from Lemma 6.1, one can easily adapt the arguments from the proof of Lemma 2.2 to show that the eigenvalues of the far field operator converge to zero only from the left side.

Theorem 6.2. *Assume that ω^2 is no Dirichlet eigenvalue of $-\Delta^*$. Then the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of F converge to zero from the left side, i.e.* $\operatorname{Re}\lambda_j < 0$ *for $j \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues λ_j in polar coordinates in (6.11) and the definition of the smallest phase ϑ_* in (6.12). Due to the compactness and normality of the far field operator and the distinct structure of the eigenvalues, the typical characterization of the cotangent of the smallest phase holds, i.e. if ω^2 is no Dirichlet eigenvalue of $-\Delta^*$, then

$$\cot\vartheta_* = \max_{g\in L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)} \frac{\operatorname{Re}(Fg,g)_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)}}{\operatorname{Im}(Fg,g)_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)}},$$

see Lemma 2.4 for a proof. Using the factorization $F = -4\pi A S^* A^*$ and the denseness of the range of A^* in $H^{-1/2}(\partial D, \mathbb{C}^3)$, this characterization can also be expressed using the single-layer operator S. Since $(Fg,g)_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)} = -4\pi(S^*A^*g,\, A^*g) = -4\pi(\varphi,\, S\varphi)$ for $\varphi = A^*g \in H^{-1/2}(\partial D, \mathbb{C}^3)$, it follows that

$$\cot\vartheta_* = \max_{\psi\in H^{-1/2}(\partial D,\mathbb{C}^3)} \frac{\operatorname{Re}(\psi, S\psi)}{\operatorname{Im}(\psi, S\psi)}. \tag{6.22}$$

From now on we will indicate the dependency of relevant quantities on the frequency ω by writing $S = S_\omega$, $\mathrm{SL} = \mathrm{SL}_\omega$, $\lambda_j = \lambda_j(\omega), \vartheta = \vartheta(\omega)$ and so on. In the next lemma we compute an auxiliary derivative that is important for our final result.

Lemma 6.3. *Assume that ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ in D. Then S_{ω_0} has a non-trivial kernel and for all elements φ_0 in this kernel it holds that $(\varphi_0, S_{\omega_0}\varphi_0) = 0$. Furthermore, the mapping $\omega \mapsto (\varphi_0, S_\omega\varphi_0)$ is differentiable in ω_0 and*

$$\alpha(\omega_0) := \frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0)\Big|_{\omega=\omega_0} = 2\int_D |v_{\omega_0}|^2\,\mathrm{d}x, \qquad \textit{where } v_{\omega_0} = \mathrm{SL}_{\omega_0}\varphi_0. \tag{6.23}$$

Proof. For arbitrary $\omega \in \mathbb{R}$, we have that $v_\omega := \mathrm{SL}_\omega\varphi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is a solution of $\Delta^* v_\omega + \omega^2 v_\omega = 0$ in $\mathbb{R}^3$. If $\omega = \omega_0$, the far field $v^\infty_{\omega_0}$ of v_{ω_0} vanishes as a consequence of the proof of [AK02, Lemma 6.1] and by Rellich's lemma, v_{ω_0} vanishes in the exterior of D such that $v_{\omega_0} \in H^2_0(D, \mathbb{C}^3)$ is a Dirichlet eigenfunction of $-\Delta^*$, i.e. $\Delta^* v_{\omega_0} + \omega_0^2 v_{\omega_0} = 0$ in D and $u = 0$ on ∂D. This implies that the kernel of S_{ω_0} is non-trivial and includes the function $\varphi_0 \in H^{-1/2}(\partial D, \mathbb{C}^3)$. By applying the chain rule, the derivative $v'_{\omega_*} := (\mathrm{d}/\mathrm{d}\omega\; v_\omega)|_{\omega=\omega_*} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ solves

$$\Delta^* v'_{\omega_*} + \omega_*^2 v'_{\omega_*} + 2\omega_* v_{\omega_*} = 0 \quad \text{in } \mathbb{R}^3. \tag{6.24}$$

Now we use the jump relation for the single layer potential from (6.20) to compute

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0) = \left(\varphi_0, \frac{\mathrm{d}}{\mathrm{d}\omega}S_\omega\varphi_0\right) = \left(\varphi_0, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right) = \left(T_\nu v_\omega|^- - T_\nu v_\omega|^+, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right).$$

Since v_{ω_0} vanishes in the exterior of D, the exterior surface traction also vanishes, such that

$T_\nu v_{\omega_0}|^+ = 0$. Using Betti's formula from (6.16) twice, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}(\varphi_0, S_\omega\varphi_0)\Big|_{\omega=\omega_0} &= \left(T_\nu v_\omega|^- - T_\nu v_\omega|^+, \frac{\mathrm{d}}{\mathrm{d}\omega}v_\omega\right)\Big|_{\omega=\omega_0} = \left(T_\nu v_{\omega_0}|^-, v'_{\omega_0}\right) \\ &= \int_D \left(\Delta^* v_{\omega_0}\cdot\overline{v'_{\omega_0}} + \mu\nabla v_{\omega_0} : \nabla\overline{v'_{\omega_0}} + (\mu+\lambda)\operatorname{div} v_{\omega_0}\operatorname{div}\overline{v'_{\omega_0}}\right)\,\mathrm{d}x \\ &= \int_D \left(-\omega_0^2 v_{\omega_0}\cdot\overline{v'_{\omega_0}} + v_{\omega_0}\cdot\Delta^*\overline{v'_{\omega_0}}\right)\,\mathrm{d}x \\ &= \int_D \left(-\omega_0^2 v_{\omega_0}\cdot\overline{v'_{\omega_0}} + \omega_0^2\overline{v'_{\omega_0}}\cdot v_{\omega_0} + 2\overline{v_{\omega_0}}v_{\omega_0}\right)\,\mathrm{d}x = 2\int_D |v_{\omega_0}|^2\,\mathrm{d}x,\end{aligned}$$

where we used (6.24) for the second to last equality. ■

We can now state the first and second part of the inside-outside duality. The first part makes use of the positivity of the derivative $\alpha(\omega)$, which we calculated in the last lemma, to set up a Taylor expansion for the characterization of the cotangent of the smallest phase. For a proof we refer to the proof of Theorem 2.8.

Theorem 6.4 (Inside-Outside Duality - Part 1). *Let $\omega_0^2 > 0$ be a Dirichlet eigenvalue of $-\Delta^*$. Then it holds that $\lim_{\omega\nearrow\omega_0}\vartheta_*(\omega) = 0$.*

Theorem 6.5 (Inside-Outside Duality - Part 2). *Assume that the interval $I = (\omega_0-\varepsilon, \omega_0)$ contains no ω such that ω^2 is a Dirichlet eigenvalue of $-\Delta^*$. If $\lim_{\omega\nearrow\omega_0}\vartheta_*(\omega)\to 0$, then ω_0^2 is a Dirichlet eigenvalue of $-\Delta^*$ in D.*

Proof. Arguing by contradiction, we assume that $\lim_{\omega\nearrow\omega_0}\vartheta_*(\omega) = 0$ but ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$. Using the characterization of the smallest phase from (6.22), this implies that

$$\max_{\varphi\in H^{-1/2}(\partial D)}\frac{\operatorname{Re}(\varphi,\, S_\omega\varphi)_{L^2(\mathbb{S}_1)}}{\operatorname{Im}(\varphi,\, S_\omega\varphi)_{L^2(\mathbb{S}_1)}} \longrightarrow \infty \qquad \text{as } \omega\nearrow\omega_0.$$

Then it follows that there is a sequence $\omega_j \nearrow \omega_0 \in I$ and a sequence $\varphi_j \in H^{-1/2}(\partial D, \mathbb{C}^3)$ with $\|\varphi_j\|_{H^{-1/2}(\partial D,\mathbb{C}^3)} = 1$ such that

$$0 > \operatorname{Im}(\varphi_j, S_{\omega_j}\varphi_j) \to 0 \qquad \text{and} \qquad \operatorname{Re}(\varphi_j, S_{\omega_j}\varphi_j) \le 0 \tag{6.25}$$

as j becomes large. Since the sequence $(\varphi_j)_{j\in\mathbb{N}}$ is bounded, we find a subsequence, also denoted by $(\varphi_j)_{j\in\mathbb{N}}$, which weakly converges to a $\varphi_0 \in H^{-1/2}(\partial D, \mathbb{C}^3)$. From [AK02] we know that

$$\operatorname{Im}(\varphi_j, S_{\omega_j}\varphi_j) = -\omega_j\|v_j^\infty\|^2_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)}, \tag{6.26}$$

where $v_j = \mathrm{SL}_{\omega_j}\varphi_j$. Since the mapping from φ_j to v_j^∞ is compact, it follows that v_j^∞ converges strongly to a function v_0^∞, which is the far field of the function $v_0 = \mathrm{SL}_{\omega_0}\varphi_0$. The far field vanishes due to equations (6.25) and (6.26). Since we assumed that ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$, we conclude that $v_0 = 0$ everywhere and therefore φ_0 also vanishes such that $\varphi_j \rightharpoonup 0$. Now we can apply Betti's first formula for a ball B_R which contains the scatterer D, to compute that

$$\begin{aligned}(\varphi_j, S_{\omega_j}\varphi_j) &= \int_{\partial D}\overline{v_j}\cdot(T_\nu v_j|_- - T_\nu v_j|_+)\,\mathrm{d}s \\ &= \int_{B_R}\left(\mu\nabla v_j : \nabla\overline{v_j} + (\lambda+\mu)\operatorname{div} v_j\operatorname{div}\overline{v_j} - \omega^2|v_j|^2\right)\,\mathrm{d}x + \int_{\partial B_R} T_\nu v_j\cdot\overline{v_j}\,\mathrm{d}S.\end{aligned}$$

Note that the last integral tends strongly to zero as v_j converges strongly to zero on ∂B_R by elliptic regularity and compact embedding results, see also the proof of Theorem 2.9 for the acoustic case. Note also that since v_j converges weakly to $v_0 = 0$ in $H^1(B_R, \mathbb{C}^3)$, it strongly converges to zero in $L^2(B_R, \mathbb{C}^3)$. Now we can use (6.25) and the real part of the last equation to obtain

$$0 \geq \mathrm{Re}\,(\varphi_j, S_{\omega_j}\varphi_j) = \int_{B_R} \left(\mu \nabla v_j : \nabla \overline{v_j} + (\mu+\lambda)\,\mathrm{div}\, v_j\, \mathrm{div}\, \overline{v_j} - \omega_j^2 v_j \cdot \overline{v_j}\right) \mathrm{d}x + \mathrm{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\, \mathrm{d}S$$

or equivalently

$$\int_{B_R} \left(\mu \nabla v_j : \nabla \overline{v_j} + (\lambda+\mu)\,\mathrm{div}\, v_j\, \mathrm{div}\, \overline{v_j}\right) \mathrm{d}x \leq \omega_j^2 \int_{B_R} |v_j|^2\, \mathrm{d}x + \mathrm{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j}\, \mathrm{d}S \to 0$$

as $j \to \infty$. Therefore v_j converges strongly to zero in $H^1(B_R, \mathbb{C}^3)$ by our definition of the H^1-norm and also the trace $v_j|_{\partial D} = S_{\omega_j}\varphi_j$ tends strongly to zero in $H^{1/2}(\partial D, \mathbb{C}^3)$. Since ω_0^2 is no Dirichlet eigenvalue of $-\Delta^*$, the single layer boundary operator S is an isomorphism and therefore we conclude that $\varphi_j \to 0$ as $j \to \infty$. But this is a contradiction to our assumption that $\|\varphi_j\| = 1$ for all $j \in \mathbb{N}$. ■

6.3. Scattering from Penetrable Inhomogeneous Media

As in the previous section, we assume the presence of an isotropic and homogeneous elastic background medium that is described by the Lamé constants λ, μ and has normalized constant mass density equal to one. Embedded in the medium is a penetrable, inhomogeneous scattering object $D \subset \mathbb{R}^3$ with Lipschitz boundary. The scattering object has the same Lamé parameters as the background medium and its mass density is given by a bounded function $\rho \in L^\infty(D)$ such that contrast $q = \rho - 1$ is positive and bounded away from zero, i.e. there exists a constant $q_0 > 0$ such that $q(x) \geq q_0$ almost everywhere in D. We consider a variational formulation of equation (6.13) and seek a radiating solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ to

$$\int_{\mathbb{R}^3} \left(\mu \nabla u : \nabla \overline{\varphi} + (\mu+\lambda)\,\mathrm{div}\, u\, \mathrm{div}\, \overline{\varphi} - \omega^2 \rho u \cdot \overline{\varphi}\right) \mathrm{d}x = 0 \tag{6.27}$$

for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support, where we extended $q = \rho - 1$ by zero outside of D. Recall from the introduction that the total field $u = u^i + u^s$ can be decomposed into the incoming plane wave u^i from (6.6) and a scattered field u^s that fulfills the radiation condition (6.7) and can therefore be represented in terms of its far field as in (6.8). Recall in this context also the definition of the far field operator F in (6.10). Let us consider the equation for the scattered field and slightly generalize the scattering problem by allowing any source term $f \in L^2(D, \mathbb{C}^3)$. We seek a radiating solution $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ to the problem

$$\int_{\mathbb{R}^3} \left(\mu \nabla v : \nabla \overline{\varphi} + (\mu+\lambda)\,\mathrm{div}\, v\, \mathrm{div}\, \overline{\varphi} - \omega^2 \rho v \cdot \overline{\varphi}\right) \mathrm{d}x = -\omega^2 \int_D q f \cdot \overline{\varphi}\, \mathrm{d}x \tag{6.28}$$

for all test functions $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support. Choosing $f = -u^i$ then yields the original scattering problem. Existence and uniqueness of a solution to this problem can for example be shown by an integral equation approach, see, e.g. [Pet93, Sev05]. Recall the definition of interior transmission eigenvalues in (6.14). This eigenvalue problem is understood in a variational sense, i.e. ω^2 is an interior transmission eigenvalue if there is a pair $(u, w) \in L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$, such that

$u - w \in H_0^2(D, \mathbb{C}^3)$ and

$$\int_D u \cdot \left(\Delta^* \varphi - \omega^2 \rho\varphi\right) \,\mathrm{d}x = 0, \qquad \int_D w \cdot \left(\Delta^* \varphi - \omega^2 \varphi\right) \,\mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(D, \mathbb{C}^3), \tag{6.29}$$

$$\int_D u \cdot \left(\Delta^* \varphi - \omega^2 \rho\varphi\right) \,\mathrm{d}x = \int_D w \cdot \left(\Delta^* \varphi - \omega^2 \varphi\right) \,\mathrm{d}x \qquad \varphi \in C^\infty(D, \mathbb{C}^3). \tag{6.30}$$

We know from [Cha02, BCG13] that there is only a discrete set of interior transmission eigenvalues. As in the case of acoustic scattering, interior transmission eigenvalues are related to the properties of the far field operator F. Whenever ω^2 is no interior transmission eigenvalue, then the far field operator F is injective or conversely, when F is not injective, then ω^2 must be an interior transmission eigenvalue. From now on we proceed as follows: As a first step, we will derive a factorization for the far field operator and examine the properties of the arising operators in Lemma 6.7. Then we will use these properties to show that the eigenvalues λ_j of F converge to zero from one specific side in Lemma 6.8. Using a characterization of the cotangent of the largest phase similar to the last section, we will then calculate a crucial auxiliary derivative in Lemma 6.10. Finally we will use this auxiliary derivative to prove the inside-outside duality in Theorem 6.11 and Theorem 6.12.

We will now show that the eigenvalues converge to zero from one specific side. To this end we derive a factorization of the far field operator and examine the properties of the arising operators. The definition of the Herglotz wave field in (6.9) implies the existence of a Herglotz wave operator $H : L^2_\mathrm{p}(\mathbb{S}_1) \times L^2_\mathrm{s}(\mathbb{S}_1) \to L^2(D, \mathbb{C}^3)$, which is given by

$$Hg = v_g \qquad \text{where } v_g(x) = \int_{\mathbb{S}_1} \left[e^{ik_\mathrm{p} x\cdot\theta} g_\mathrm{p}(\theta) + e^{ik_\mathrm{s} x\cdot\theta} g_\mathrm{s}(\theta) \right] \mathrm{d}s(\theta), \qquad x \in D.$$

The adjoint of the Herglotz operator $H^* : L^2(D, \mathbb{C}^3) \to L^2_\mathrm{p}(\mathbb{S}_1) \times L^2_\mathrm{s}(\mathbb{S}_1)$ is given by

$$H^*\varphi(\theta) = \int_D \left[e^{-ik_\mathrm{p} x\cdot\theta} \varphi_\mathrm{p}(\theta) + e^{-ik_\mathrm{s} x\cdot\theta} \varphi_\mathrm{s}(\theta) \right] \mathrm{d}x.$$

Let us define a volume potential $V : L^2(D, \mathbb{C}^3) \to H^2_\mathrm{loc}(\mathbb{R}^3, \mathbb{C}^3)$ by

$$Vh(x) = \int_D \Phi_N(x, y) h(y) \,\mathrm{d}y,$$

where Φ_N is the fundamental solution of the Navier equation from (6.19) such that Vh solves

$$(\Delta^* + \omega^2) Vh = -h, \qquad \text{in } \mathbb{R}^3,$$

see [McL00]. We also know from the proof of [HKS13, Lemma 3.1] that H^*h is the far field w^∞ of the function $w = Vh$. As the final ingredient for our factorization, we introduce the operator $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ by

$$Tf = \omega^2 q(f - v)$$

where v is the radiating solution of (6.28). Then we can prove the following factorization.

Theorem 6.6. *It holds that $F = H^*TH$.*

Proof. We follow the standard procedure and introduce an auxiliary operator $G : L^2(D, \mathbb{C}^3) \to L^2_\mathrm{p}(\mathbb{S}_1) \times L^2_\mathrm{s}(\mathbb{S}_1)$ that maps a function f onto the far field v^∞ of the solution of v to (6.28). Then $F = GH$ by the superposition principle. As we noted above, H^*h is the far field of the function

$w = Vh$. Now we write (6.28) equivalently as

$$\int_{\mathbb{R}^3} \left(\mu \nabla v : \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v \operatorname{div} \overline{\varphi} - \omega^2 v \cdot \overline{\varphi}\right) \, \mathrm{d}x = \int_D \omega^2 q(f - v) \cdot \overline{\varphi} \, \mathrm{d}x. \tag{6.31}$$

From the discussion above, it then follows that $G = H^*T$. Since $F = GH$, this implies the factorization of the far field operator. ■

Before we proceed we characterize the closure of the range of the Herglotz wave operator. If we denote by $\overline{\mathcal{R}(H)}$ this closure in $L^2(D, \mathbb{C}^3)$, then

$$X := \overline{\mathcal{R}(H)} = \left\{ w \in L^2(D, \mathbb{C}^3) : \int_D w \cdot (\Delta^* \phi + \omega^2 \phi) \, \mathrm{d}x = 0 \qquad \phi \in C_0^\infty(D, \mathbb{C}^3) \right\} \tag{6.32}$$

as a consequence of, e.g., [Are03, Theorem 4.2]. Now we summarize important properties of the middle operator T in the following lemma.

Lemma 6.7. *(a) For all $f \in L^2(D, \mathbb{C}^3)$ and $\omega > 0$ it holds that* $\operatorname{Im} (Tf, f)_{L^2(D,\mathbb{C}^3)} \geq 0$.
(b) If $\operatorname{Im} (Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$ *for a non-trivial $w \in X$ and $\omega > 0$, then ω^2 is an interior transmission eigenvalue with corresponding transmission eigenpair $(w - v, w)$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is the weak radiating solution to* (6.28).
(c) If $\omega^2 > 0$ is an interior transmission eigenvalue with corresponding transmission eigenpair (u, w), *then $w \in X$ and* $(Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$.
(d) The operator T can be written as $T = \omega^2 q(I + C)$ for a compact operator $C : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$.

Proof. (a) In order to simplify notation below, we introduce a sesquilinear form Ψ by

$$\Psi_{\Omega,\rho}(u, \varphi) := \int_\Omega \left(\mu \nabla u : \nabla \overline{\varphi} + (\lambda + \mu) \operatorname{div} u \;\operatorname{div} \overline{\varphi} - \omega^2 \tilde{\rho} u \cdot \overline{\varphi}\right) \, \mathrm{d}x \tag{6.33}$$

for an open set $\Omega \subset \mathbb{R}^3$ and functions $\tilde{\rho} \in L^\infty(\Omega)$, $u, \varphi \in H^1(\Omega, \mathbb{C}^3)$. Now we start with an auxiliary calculation. We choose a cut-off function $\phi \in C^\infty(\mathbb{R}^3)$ with compact support such that $\phi = 1$ in a ball $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$, where the radius of the ball in chosen large enough such that D is contained in B_R. Then we set the test function $\varphi = \phi v$ in (6.31), where v is the solution to this problem. Then we get that

$$\Psi_{B_R,1}(v, v) + \Psi_{\mathbb{R}^3 \setminus B_R,1}(v, v) = \omega^2 \int_D q(f - v) \cdot \overline{v} \, \mathrm{d}x.$$

Note that v is a smooth solution to the Navier equation outside the ball B_R, i.e. $\Delta^* v + \omega^2 v = 0$. We apply Betti's formula to obtain that

$$\Psi_{\mathbb{R}^3 \setminus B_R,1}(v, \varphi) = \int_{|x|=R} T_\nu v \cdot \overline{v} \, \mathrm{d}s$$

and therefore we have in total that

$$\Psi_{B_R,1}(v, v) + \int_{|x|=R} T_\nu v \cdot \overline{v} \, \mathrm{d}s = \omega^2 \int_D q(f - v) \cdot \overline{v} \, \mathrm{d}x. \tag{6.34}$$

After this preliminary considerations, we come to our main assertion. Choose an arbitrary $f \in L^2(D, \mathbb{C}^3)$. We have by definition that

$$(Tf, f)_{L^2(D,\mathbb{C}^3)} = -\omega^2 (q(f - v), f)_{L^2(D,\mathbb{C}^3)}.$$

Define now $g \in L^2(D, \mathbb{C}^3)$ by $g := f - v$, where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ solves (6.28). Then we get that

$$(Tf, f)_{L^2(D,\mathbb{C}^3)} = \omega^2 (qg, g + v)_{L^2(D,\mathbb{C}^3)} = \omega^2 (qg, g)_{L^2(D,\mathbb{C}^3)} + \omega^2 \int_D qg \cdot \overline{v} \;\mathrm{d}x.$$

Re-substituting g and then using equation (6.34) shows that

$$(Tf, f)_{L^2(D,\mathbb{C}^3)} = \omega^2 (qg, g)_{L^2(D,\mathbb{C}^3)} + \Psi_{B_R,1}(v, v) + \int_{|x|=R} T_\nu v \cdot \overline{v} \;\mathrm{d}s \tag{6.35}$$

which implies that

$$\mathrm{Im}\,(Tf, f)_{L^2(D,\mathbb{C}^3)} = \mathrm{Im} \int_{|x|=R} T_\nu v \cdot \overline{v} \;\mathrm{d}s$$

since q and μ, λ are all real-valued. Now we can apply [Sev05, Lemma 1] and get that

$$\mathrm{Im}\,(Tf, f)_{L^2(D,\mathbb{C}^3)} = 2\omega \|v^\infty\|_{L^2(D,\mathbb{C}^3)}. \tag{6.36}$$

(b) Assume there exists a non-trivial $w \in \overline{\mathcal{R}(H)}$ such that $\mathrm{Im}\,(Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$ and let v be the solution of (6.31) for $f = w$. Then we conclude from the (a)-part of this proof that the far field v^∞ vanishes and by Rellich's lemma v vanishes outside of D, which implies that $v \in H^2_0(D, \mathbb{C}^3)$. Setting $u = w + v$, we calculate for $\phi \in C^\infty_0(D, \mathbb{C}^3)$ that

$$\begin{aligned}\int_D u \cdot \left[\Delta^* \phi + \omega^2 (1+q)\phi\right] \;\mathrm{d}x &= \int_D v \cdot \left[\Delta^* \phi + \omega^2 (1+q)\phi\right] + \int_D w \cdot \left[\Delta^* \phi + \omega^2 (1+q)\phi\right] \\ &= \int_D \left(\mu \nabla v : \nabla \phi + (\mu + \lambda) \operatorname{div} v \operatorname{div} \phi - \omega^2 v \cdot \phi\right) \mathrm{d}x - \omega^2 \int_D qw \cdot \phi \;\mathrm{d}x = 0,\end{aligned}$$

where we used that $w \in X$ solves the Navier equation $\Delta^* w + \omega^2 w = 0$. From this calculation, we conclude that (u, w) fulfills (6.29) and substituting $u = w + v$ shows that (6.30) also holds, such that (u, w) is an transmission eigenvalue pair and ω^2 is the corresponding interior transmission eigenvalue.
(c) Let $\omega^2 > 0$ be an interior transmission eigenvalue with eigenpair $(u, w) \in L^2(D, \mathbb{C}^3) \times L^2(D, \mathbb{C}^3)$. We will show that $(T_\omega w, w)_{L^2(D,\mathbb{C}^3)} = 0$. Since $w \in X = \overline{\mathcal{R}(H)}$, there exists a sequence $g_j \in L^2(\mathbb{S}^2)$ such that the corresponding Herglotz wave functions w_j converge to w in $L^2(D, \mathbb{C}^3)$. Since ω^2 is an interior transmission eigenvalue, (6.30) implies that $v = u - w \in H^2_0(D, \mathbb{C}^3)$ satisfies

$$\int_D [\Delta^* v + \omega^2 v] \cdot \overline{\phi} \;\mathrm{d}x = \omega^2 \int_D q(w - v) \cdot \overline{\phi} \;\mathrm{d}x$$

for all $\phi \in L^2(D, \mathbb{C}^3)$. Choosing $\phi = w$ yields

$$\int_D [\Delta^* v + \omega^2 v] \cdot \overline{w} \;\mathrm{d}x = \omega^2 \int_D q(w - v) \cdot \overline{w} \;\mathrm{d}x = (Tw, w)_{L^2(D,\mathbb{C}^3)}.$$

Since w_j solves the Navier equation and $v \in H^2_0(D, \mathbb{C}^3)$, we get

$$\int_D [\Delta^* v + \omega^2 v] \cdot \overline{w} \;\mathrm{d}x = \lim_{j\to\infty} \int_D [\Delta^* v + \omega^2 v] \cdot \overline{w_j} \;\mathrm{d}x = 0$$

by Betti's first identity. In consequence, $(T_\omega w, w)_{L^2(D,\mathbb{C}^3)} = 0$.
(d) This is clear due to the compactness of the embedding of $H^1(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$. ■

The properties of the operator T and the specific structure of the eigenvalues λ_j of F imply that the eigenvalues converge to zero from the right side, see again Lemma 2.4 for a proof.

Theorem 6.8. *Assume that ω^2 is no interior transmission eigenvalue. Then the eigenvalues λ_j of F converge to zero from the right side, i.e. $\operatorname{Re}\lambda_j > 0$ for $j \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ in polar coordinates in (6.11) and the definition of the largest phase $\vartheta^* := \max_{j\in\mathbb{N}} \vartheta_j$ in (6.15). Since the far field operator retains normality and compactness and due to the distinct properties of its eigenvalues, we know that if ω^2 is no interior transmission eigenvalue, then

$$\cot\vartheta^* = \min_{g\in L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)} \frac{\operatorname{Re}(Fg,g)_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)}}{\operatorname{Im}(Fg,g)_{L^2_{\mathrm{p}}(\mathbb{S}_1)\times L^2_{\mathrm{s}}(\mathbb{S}_1)}}, \tag{6.37}$$

see Lemma 2.4 for a proof. As in the previous section, we use the factorization of $F = H^*TH$ and rewrite the characterization of the largest phase in (6.37) to obtain

$$\cot\vartheta^* = \min_{f\in L^2(D,\mathbb{C}^3)} \frac{\operatorname{Re}(THf,Hf)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(THf,Hf)_{L^2(D,\mathbb{C}^3)}} = \min_{\varphi\in X} \frac{\operatorname{Re}(T\varphi,\varphi)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T\varphi,\varphi)_{L^2(D,\mathbb{C}^3)}},$$

where $X = \overline{\mathcal{R}(H)}$ was defined in (6.32). From now on the dependency of all quantities on the frequency ω becomes important. We indicate it by writing $T = T_\omega, X = X_\omega, \lambda_j = \lambda_j(\omega)$, etc. Assume now that there is a projection $P_\omega : L^2(D,\mathbb{C}^3) \to X_\omega$ that is differentiable with respect to ω. We can use this projection to rewrite the characterization for the largest phase as

$$\cot\vartheta^*(\omega) = \min_{w\in L^2(D,\mathbb{C}^3)} \frac{\operatorname{Re}(T_\omega P_\omega w, P_\omega w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_\omega P_\omega w, P_\omega w)_{L^2(D,\mathbb{C}^3)}}.$$

To show the existence of the projection $P_\omega : L^2(D,\mathbb{C}^3) \to X_\omega$ we give an explicit representation of P_ω. First we denote by W the completion of $C_0^\infty(D,\mathbb{C}^3)$ with respect to the norm $\|\varphi\|_W := \|\Delta^*\varphi + \omega^2\varphi\|_{L^2(D,\mathbb{C}^3)}$. Note that this completion is well-defined, since if $\|\Delta^*\varphi + \omega^2\varphi\|_{L^2(D,\mathbb{C}^3)} = 0$ for $\varphi \in C_0^\infty$, the compact support of φ in D and representation formulas for solutions of the Navier equation as in [HW08] imply that $\varphi = 0$. Now we define P_ω by

$$P_\omega w = w - (\Delta^*\hat{w} + \omega^2\hat{w})$$

where $\hat{w} \in W$ solves the W-coercive variational problem

$$\int_D (\Delta^*\hat{w} + \omega^2\hat{w}) \cdot (\Delta^*\varphi + \omega^2\varphi)\,\mathrm{d}x = \int_D w \cdot (\Delta^*\varphi + \omega^2\varphi)\,\mathrm{d}x \qquad \forall\varphi \in W.$$

If $w \in X_\omega$, then the right side of the last equation vanishes and the coercivity of the sesquilinear form on W implies that $\hat{w} = 0$, which shows $P_\omega w = w$. On the other hand for an arbitrary $w \in L^2(D,\mathbb{C}^3)$ we have that $P_\omega w \in X_\omega$ due to the definition of $\hat{w}$. Hence P_ω is projection onto X_ω. The differentiability of this function is a consequence of the differentiability of the map $\omega \mapsto \hat{w} = \hat{w}(\omega)$. Assume now that ω_0^2 is an interior transmission eigenvalue. Then there exists a non-trivial function $w_0 \in X_{\omega_0}$ such that $(T_{\omega_0}w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$. To prove the first part of the inside-outside duality as in the proof of [KL13, Lemma 5.1] we need to calculate the derivative

$$\alpha(\omega_0) := \frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}\Big|_{\omega=\omega_0}. \tag{6.38}$$

We start by calculating an auxiliary derivative, which neglects the projection operator.

Lemma 6.9. *Let $\omega_0^2 > 0$ be an interior transmission eigenvalue with eigenpair $(u_0, w_0) \in L^2(D,\mathbb{C}^3)\times$*

X_{ω_0}. *Then* $v_0 = u_0 - w_0 \in H_0^2(D, \mathbb{C}^3)$ *is the radiating solution to*

$$\Delta^* v_0 + \omega_0^2 \rho v_0 = -\omega_0^2 q w_0 \tag{6.39}$$

and the mapping $\omega \to (T_\omega w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ *is differentiable at* ω_0 *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega w_0, w_0)\Big|_{\omega=\omega_0} = \frac{2}{\omega_0}\int_D (\mu \nabla v_0 : \nabla \overline{v_0} + (\lambda+\mu)\operatorname{div} v_0 \operatorname{div} \overline{v_0})\ \mathrm{d}x$$

Proof. Note that (6.39) holds due to the properties of the eigenpair (u_0, w_0), see also the proof of Lemma 6.7 for details. For arbitrary $\omega > 0$ we define $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ as the radiating solution to

$$\int_{\mathbb{R}^3} \left(\mu \nabla v_\omega : \nabla \overline{\varphi} + (\mu+\lambda)\operatorname{div} v_\omega \operatorname{div}\overline{\varphi} - \omega^2 \rho v_\omega \cdot \overline{\varphi}\right)\ \mathrm{d}x = \omega^2 \int_D q w_0 \cdot \overline{\varphi}\ \mathrm{d}x \tag{6.40}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Note that if $\omega = \omega_0$ then $v_{\omega_0} = v_0 \in H_0^2(D, \mathbb{C}^3)$ is the radiating solution to (6.39) by Betti's formula. The map $\omega \mapsto v_\omega$ is Fréchet-differentiable and $v'_{\omega_0} := [\mathrm{d}v/\mathrm{d}\omega\ v_\omega]|_{\omega=\omega_0} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ solves

$$\begin{aligned}\int_{\mathbb{R}^3} \left(\mu \nabla v'_{\omega_0} \nabla \overline{\varphi} + (\mu+\lambda)\operatorname{div} v'_{\omega_0} \operatorname{div}\overline{\varphi} - \omega_0^2 \rho v'_{\omega_0} \cdot \overline{\varphi}\right)\ \mathrm{d}x &= -\int_D 2\omega_0 q w_0 \cdot \overline{\varphi}\,\mathrm{d}x + \int_D 2\omega_0 \rho v_{\omega_0} \cdot \overline{\varphi}\,\mathrm{d}x \\ &= \frac{2}{\omega_0}\int_D (\mu \nabla v_{\omega_0} : \nabla\overline{\varphi} + (\lambda+\mu)\operatorname{div} v_{\omega_0} \operatorname{div}\overline{\varphi})\ \mathrm{d}x\end{aligned}$$

for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with compact support. Moreover, for $\omega = \omega_0$ the solution $v_{\omega_0} \in H_0^2(D)$ has compact support and hence (6.40) holds in this case even for all $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$. Using that $(T_{\omega_0} w_0, w_0)_{L^2(D,\mathbb{C}^3)} = 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega w_0, w_0)_{L^2(D,\mathbb{C}^3)}\Big|_{\omega=\omega_0} &= \int_D q\omega_0^2 v'_{\omega_0} \cdot \overline{w_0}\ \mathrm{d}x - \int_D 2\omega_0 q (w_0 - v_{\omega_0})\overline{w_0}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left(\mu \nabla v'_{\omega_0} \nabla \overline{v_{\omega_0}} + (\mu+\lambda)\operatorname{div} v'_{\omega_0} \operatorname{div}\overline{v_{\omega_0}} - \omega_0^2 \rho v'_{\omega_0} \cdot \overline{v_{\omega_0}}\right)\ \mathrm{d}x \\ &= \frac{2}{\omega_0}\int_D (\mu \nabla v_{\omega_0} : \nabla\overline{v_{\omega_0}} + (\lambda+\mu)\operatorname{div} v_{\omega_0} \operatorname{div}\overline{v_{\omega_0}})\ \mathrm{d}x\end{aligned}$$

which shows the assertion. ■

Lemma 6.10. *Let* ω_0^2 *be an interior transmission eigenvalue with eigenpair* $(u_0, w_0) \in L^2(D, \mathbb{C}^3) \times X_{\omega_0}$. *Then the map* $\omega \to (T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}$ *is differentiable in* ω_0 *such that*

$$\begin{aligned}\alpha(\omega_0) = \frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}\Big|_{\omega=\omega_0} &= \frac{2}{\omega_0}\int_D (\mu \nabla v_0 : \nabla \overline{v_0} + (\lambda+\mu)\operatorname{div} v_0 \operatorname{div}\overline{v_0})\ \mathrm{d}x \\ &\quad + 4\omega_0 \mathrm{Re} \int_D \overline{v_0} \cdot w_0\,\mathrm{d}x,\end{aligned} \tag{6.41}$$

where $v_0 \in H_0^2(D, \mathbb{C}^3)$ *is again the radiating solution to* (6.39).

Proof. Let $v_\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ be defined as in the proof of the last lemma, such that $v_0 = v_{\omega_0}$. By definition of the projection P_ω and the space X_ω, we have that $w_\omega := P_\omega w_0 \in X_\omega$ solves the Navier equation, i.e.

$$\int_D w_\omega \cdot [\Delta^* \varphi - \omega^2 \varphi]\ \mathrm{d}x = 0 \qquad \forall \varphi \in C_0^\infty(D, \mathbb{C}^3).$$

Using the differentiability of the projection operator P_ω, the derivative P'_ω of P_ω with respect to ω is given by $\mathrm{d}/\mathrm{d}\omega(P_\omega w_0) = w'_\omega$, where $w'_\omega \in L^2(D, \mathbb{C}^3)$ solves

$$\int_D w'_\omega \cdot [\Delta^* \varphi - \omega^2 \varphi] \ \mathrm{d}x = 2\omega \int_D \varphi \cdot w_\omega \, \mathrm{d}x \tag{6.42}$$

for all $\varphi \in C_0^\infty(D, \mathbb{C}^3)$. Applying the chain rule, we get

$$\begin{aligned}\frac{d}{\omega\omega}(T_\omega P_\omega w_0, P_\omega w_0) &= (T'_\omega P_\omega w_0, P_\omega w_0) + (T_\omega P'_\omega w_0, P_\omega w_0) + (T_\omega P_\omega w_0, P'_\omega w_0)\\ &= (T'_\omega P_\omega w_0, P_\omega w_0) + \overline{(T^*_\omega P_\omega w_0, P'_\omega w_0)} + (T_\omega P_\omega w_0, P'_\omega w_0).\end{aligned}$$

Furthermore the symmetry of the sesquilinear form in (6.40) for the choice $\varphi = v_\omega$ implies that T is self-adjoint on the kernel of $w_0 \to (Tw_0, w_0)_{L^2(D,\mathbb{C}^3)}$ such that $T_{\omega_0} w_0 = T^*_{\omega_0} w_0$, for details see the proof of Theorem 3.6 for acoustic scattering. Using the result of the last lemma, we obtain

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}\omega}(T_\omega P_\omega w_0, P_\omega w_0)_{L^2(D,\mathbb{C}^3)}\right]\Bigg|_{\omega=\omega_0} &= \int_D (\mu \nabla v_{\omega_0} : \nabla \overline{v_{\omega_0}} + (\lambda+\mu) \operatorname{div} v_{\omega_0} \operatorname{div} \overline{v_{\omega_0}}) \ \mathrm{d}x\\ &\qquad + 2\mathrm{Re}\,(T_{\omega_0} w_0, P'_{\omega_0} w_0)_{L^2(D,\mathbb{C}^3)}.\end{aligned}$$

Now we can use that $v_{\omega_0} \in H_0^2(D, \mathbb{C}^3)$ and partial integration to get

$$\begin{aligned}2\mathrm{Re}\,(T_{\omega_0} w_0, P'_{\omega_0} w_0)_{L^2(D,\mathbb{C}^3)} &= 2\mathrm{Re}\left[\int_D \omega^2 q w_0 \cdot \overline{w'_{\omega_0}} \, \mathrm{d}x - \int_D \omega^2 q v_{\omega_0} \cdot \overline{w'_{\omega_0}} \, \mathrm{d}x\right]\\ &= 2\mathrm{Re}\left[\int_D [\Delta^* v_{\omega_0} + \omega_0^2 (1+q) v_{\omega_0}] \cdot \overline{w'_{\omega_0}} \, \mathrm{d}x - \omega_0^2 \int_D q v_{\omega_0} \cdot \overline{w'_{\omega_0}} \ \mathrm{d}x\right]\\ &= 2\mathrm{Re} \int_D [\Delta^* v_{\omega_0} + \omega_0^2 v_{\omega_0}] \cdot w'_{\omega_0} \, \mathrm{d}x = 2\mathrm{Re} \int_D 2\omega_0 \overline{v_{\omega_0}} \cdot w_0 \, \mathrm{d}x,\end{aligned}$$

where we used (6.42). This shows our claim. ■

After this preliminary considerations, we now state the first and second part of the inside-outside duality. The proof of the first part of the inside-outside duality again makes use of the derivative α in (6.41) to set up a Taylor expansion of the characterization of the cotangent of the largest phase ϑ^*. For a proof which includes a projection P_ω, we refer to the proof of Theorem 3.7.

Theorem 6.11 (Inside-Outside Duality - Part 1)**.** *Let ω_0^2 be an interior transmission eigenvalue and $\alpha(\omega_0)$ be the expression in (6.41). Then*

$$\lim_{\omega \nearrow \omega_0} \vartheta^*(\omega) = \pi \qquad \textit{if } \alpha(\omega_0) > 0 \qquad \textit{and} \qquad \lim_{\omega \searrow \omega_0} \vartheta^*(\omega) = \pi \qquad \textit{if } \alpha(\omega_0) < 0.$$

Theorem 6.12 (Inside-outside duality - Part 2)**.** *Assume that $\omega_0 > 0$ and $I = (\omega_0 - \varepsilon, \omega_0 + \varepsilon) \setminus \{\omega_0\}$ does not contain frequencies ω such that ω^2 is an interior transmission eigenvalue. If $\vartheta^*(\omega) \to \pi$ for $I \ni \omega \to \omega_0$, then ω_0^2 is an interior transmission eigenvalue.*

Proof. Assume that $\vartheta^*(\omega) \to \pi$ for $I \ni \omega \to \omega_0$. We have that

$$\cot(\vartheta^*) = \min_{w \in X_\omega} \frac{\mathrm{Re}\,(T_\omega w, w)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(T_\omega w, w)_{L^2(D,\mathbb{C}^3)}} \to -\infty \qquad \text{for } I \ni \omega \to \omega_0.$$

Thus, there is a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subset I$ such that $\omega_j \to \omega_0$ and $w_j \in X_{\omega_j}$ with $\|w_j\|_{L^2(D,\mathbb{C}^3)} = 1$ such that $0 < \mathrm{Im}\,(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \to 0$ as $j \to \infty$ and $\mathrm{Re}\,(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} \le 0$ for j large enough.

Let $v_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ be the corresponding radiating solution to

$$\int_{\mathbb{R}^3} \left(\mu \nabla v_j : \nabla \overline{\varphi} + (\mu + \lambda) \operatorname{div} v_j \operatorname{div} \overline{\varphi} - \omega_j^2 \rho v_j \cdot \overline{\varphi}\right) \, \mathrm{d}x = \omega_j^2 \int_D q w_j \cdot \overline{\varphi} \, \mathrm{d}x \tag{6.43}$$

for test functions φ in $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ with compact support. Since the sequence w_j is bounded in $L^2(D, \mathbb{C}^3)$ there exists a weakly convergent subsequence $w_j \rightharpoonup w_0$ in $L^2(D, \mathbb{C}^3)$ as $j \to \infty$. In particular $w_0 \in X_{\omega_0}$ and $v_j \rightharpoonup v_0$ weakly in $H^1(B_R, \mathbb{C}^3)$ for all radii $R > 0$, where $v_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ is the corresponding weak radiating solution to (6.43) with ω_j, w_j replaced by ω_0, w_0. In the proof of Lemma 6.7 we have already shown that

$$\operatorname{Im} (T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} = \frac{\omega_j}{4\pi^2} \|v_j^\infty\|^2_{L^2(\mathbb{S}_1)}, \qquad j \in \mathbb{N}.$$

The left hand side converges to zero and the right hand side to $\omega_0/(4\pi^2) \, \|v_0^\infty\|_{L^2(\mathbb{S}_1)}$. We conclude that $v_0^\infty = 0$ and v_0 vanishes in the exterior of D by Rellich's Lemma.

Assume now that $\omega_0^2 > 0$ is not an interior transmission eigenvalue. Then it follows from Lemma 6.7(b) that w_0 and v_0 vanish everywhere, such that w_j and v_j converge weakly to zero as $j \to \infty$. We define $g_j = w_j - v_j$, recall the arguments of the proof of Lemma 6.7 and get that

$$(T_{\omega_j} w_j, w_j)_{L^2(D,\mathbb{C}^3)} = \omega_j^2 (q g_j, g_j)_{L^2(D,\mathbb{C}^3)} + \Psi_{B_R,1}(v_j, v_j) + \int_{|x|=R} T_\nu v \cdot \overline{v} \, \mathrm{d}s$$

Now we can use (6.25) and use the real part of the last equation to obtain

$$0 \geq \operatorname{Re} (T_{\omega_0} w_j, w_j) = \Psi_{B_R,1}(v_j, v_j) + \operatorname{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j} \, \mathrm{d}S$$

or equivalently

$$\int_{B_R} (\mu \nabla v_j : \nabla \overline{v_j} + (\lambda + \mu) \operatorname{div} v_j \operatorname{div} \overline{v_j}) \, \mathrm{d}x \leq \int_{B_R} \omega^2 |v_j|^2 \, \mathrm{d}x + \operatorname{Re} \int_{|x|=R} T_\nu v_j \cdot \overline{v_j} \, \mathrm{d}S, \qquad j \in \mathbb{N}.$$

As $\|v_j\|_{L^2(B_R,\mathbb{C}^3)} \to 0$ and $\|v_j\|_{H^{1/2}(\partial B_R,\mathbb{C}^3)} \to 0$ as $j \to \infty$ due to the compact embedding of $H^1(B_R, \mathbb{C}^3)$ in $L^2(B_R, \mathbb{C}^3)$ and the smoothness of v_j in a neighborhood of ∂B_R, the right-hand side of the latter inequality converges to zero as j tends to infinity. Therefore, v_j converges strongly to zero in $H^1(B_R, \mathbb{C}^3)$ due to the definition of the H^1-norm. Then it follows that $w_j \to 0$ in $L^2(D, \mathbb{C}^3)$. But this is a contradiction to our assumption that $\|w_j\| = 1$ for all $j \in \mathbb{N}$. ■

6.4. Conditions for the Material Parameter

In this section we show the existence of interior transmission eigenvalues ω_0^2 with positive derivative $\alpha(\omega_0)$, see (6.41). While the results in this section are certainly not conclusive and only hold under severe restrictions for the density ρ, they mainly serve to show that there exist interior transmission eigenvalues at all for which the derivative α does not vanish. In this section we proceed as follows: Following [Kir09, Section 2], we first prove an existence result for interior transmission eigenvalues if the contrast $q = \rho - 1 \in L^\infty(D, \mathbb{C})$ is large enough. Then we show under which conditions the derivative $\alpha(\omega_0)$ does not vanish and finally we bring these two results together to show the existence of interior transmission eigenvalues with non-trivial derivative α.

We will start by showing an existence result for interior transmission eigenvalues, given that the contrast q is large enough. To this end we equip the space $H_0^2(D, \mathbb{C}^3)$ with the inner product $(\phi, \psi)_{H_0^2(D,\mathbb{C}^3)} = (1/q \; \Delta^* \phi, \Delta^* \psi)_{L^2(D,\mathbb{C}^3)}$. To see that this is indeed an inner product, we need to show definiteness. Assume for any function $\phi \in H_0^2(D, \mathbb{C}^3)$ that

$$(\phi, \phi)_{H_0^2(D,\mathbb{C}^3)} = (1/q\ \Delta^* \phi, \Delta^* \phi)_{L^2(D,\mathbb{C}^3)} = 0.$$

Since $1/q > 0$ in D, we conclude that $\Delta^* \phi = 0$ almost everywhere in D. In particular it follows that $(\Delta^* \phi, \phi)_{L^2(D,\mathbb{C}^3)} = 0$, which by Betti's formula (6.16) implies that

$$\|\nabla \phi\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + \|\operatorname{div} \phi\|_{L^2(D,\mathbb{C}^3)} = 0.$$

Since ϕ has zero boundary conditions, this in turn implies $\phi = 0$ and therefore shows the definiteness of the inner product.

The interior transmission eigenvalue problem (6.14) can equivalently be written as a fourth-order differential equation for $v = u - w \in H_0^2(D, \mathbb{C}^3)$, which yields

$$(\Delta^* + \omega^2) \frac{1}{q} (\Delta^* + \omega^2 \rho) v = 0,$$

which in its weak formulation reads

$$a_\omega(v, \psi) := \int_D \frac{1}{q} \left[\Delta^* v + \omega^2 \rho v\right] \cdot \left[\Delta^* \psi + \omega^2 \psi\right] \mathrm{d}x = 0 \qquad \forall \psi \in H_0^2(D, \mathbb{C}^3). \tag{6.44}$$

Arguing as in [Kir09, Section 2], we have that ω^2 is an interior transmission eigenvalue if and only if there exists a non-trivial function $v \in H_0^2(D, \mathbb{C}^3)$ such that $a_\omega(v, \psi) = 0$ for all $\psi \in H_0^2(D, \mathbb{C}^3)$. To give an existence result, we define μ_1 as the smallest eigenvalue of the bi-Navier operator, i.e. $(\Delta^*)^2 \hat{v} = \mu_1 \hat{v}$ in D for an eigenfunction $\hat{v} \in H_0^2(D, \mathbb{C}^3)$. Furthermore let $\gamma = \gamma(\mu, \lambda)$ be a constant such that

$$\mu \|\nabla u\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda + \mu) \|\operatorname{div} u\|^2_{L^2(D,\mathbb{C})} \geq \gamma \|u\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall u \in H_0^2(D, \mathbb{C}^3).$$

It is clear that such a constant γ exists, since applying the Poincaré-inequality component-wise, we have that there is a constant γ_0 such that $\gamma_0 \|u\|_{L^2(D,\mathbb{C}^3)} \leq \|\nabla u\|_{L^2(D,\mathbb{C}^{3\times 3})}$. Then we can show that an interior transmission eigenvalue exists if the contrast q is large enough. Recall for this purpose that $q(x) \geq q_0$ for a constant $q_0 > 0$ for almost all $x \in D$.

Theorem 6.13. *If $q \in L^\infty(D, \mathbb{C}^3)$ is large enough such that*

$$\mu_1 < \frac{(1 + q_0/2)^2 \gamma^2}{1 + q_0}, \tag{6.45}$$

then there exists at least one transmission eigenvalue ω_0^2 in the interval $(0, (1 + q_0/2)\gamma/(1 + q_0))$.

Proof. We will follow [Kir09] to show existence of interior transmission eigenvalues. First we rewrite the bilinear form a_ω as

$$a_\omega(v, \psi) = \int_D \frac{1}{q} [\Delta^* v + \omega^2 v] \cdot [\Delta^* \psi + \omega^2 \psi] \, \mathrm{d}x + \omega^2 \int_D v \cdot [\Delta^* \psi + \omega^2 \psi] \, \mathrm{d}x \tag{6.46}$$

for all $\psi \in H_0^2(D, \mathbb{C}^3)$. We can rewrite a_ω as

$$a_\omega = a_0 + \omega^2 b_1 + \omega^4 b_2,$$

where b_1 and b_2 are bilinear forms, given by

$$b_1(v,\psi) = \int_D \frac{1}{q}[v\Delta^*\psi + \psi\Delta^* v]\,\mathrm{d}x + \int_D v\Delta^*\psi\,\mathrm{d}x,$$
$$b_2(v,\psi) = \int_D \frac{q+1}{q} v\psi\,\mathrm{d}x, \qquad v,\psi \in H_0^2(D,\mathbb{C}^3).$$

and a_0 is the inner product on H_0^2 that we introduced above. We use Riesz' representation theorem and find bounded operators B_1, B_2 from $H_0^2(D,\mathbb{C}^3)$ into itself such that

$$b_j(v,\psi) = (B_j v,\psi)_{H_0^2(D,\mathbb{C}^3)} \qquad \forall v,\psi \in H_0^2(D,\mathbb{C}^3), \quad j=1,2.$$

Therefore we can write the equation $a_\omega(v,\psi)=0$ for all $\psi \in H_0^2(D,\mathbb{C}^3)$ equivalently as

$$v + \omega^2 B_1 v + \omega^4 B_2 v = 0.$$

From the symmetry of b_j we conclude that B_1, B_2 are self-adjoint. Furthermore these operators are also compact, since they represent differential operators of order less than four, see [KG08] for the corresponding acoustic case. Finally the operator B_2 is positive. Now we define

$$A_\omega = I_3 + \omega^2 B_1 + \omega^4 B_2$$

and notice that this operator is self adjoint due to the self-adjointness of the operators that constitute the operator. Furthermore its spectrum is real and discrete and due to the compactness of B_1 and B_2, we know that the only possible accumulation point is 1. Moreover the eigenvalues depend continuously on the frequency ω. Notice that the spectrum of the operator $A_0 = I_3$ only consists of $\{1\}$. If we now find a function $\hat{v} \in H_0^2(D,\mathbb{C}^3)$ and a corresponding value $\hat{\omega}$, such that $a_{\hat{\omega}}(\hat{v},\hat{v}) < 0$, we know from the min-max principle that the smallest eigenvalue of $A_{\hat{\omega}}$ is negative. Since the smallest eigenvalue depends continuously on the frequency ω, it follows that there is a value ω between 0 and $\hat{\omega}$ such that the kernel of A_ω is non-trivial and therefore ω^2 is a transmission eigenvalue. We will now construct such a function $\hat{v}$. First we use (6.46) to estimate

$$\begin{aligned}
a_\omega(v,v) &\le \frac{1}{q_0}\int_D [\Delta^* v + \omega^2 v]^2\,\mathrm{d}x + \omega^2 \int_D v\cdot\Delta^* v\,\mathrm{d}x + \omega^4 \|v\|_{L^2(D,\mathbb{C}^3)} \\
&= \frac{1}{q_0}\int_D \left[(\Delta^* v)^2 + \omega^2(2+q_0) v\cdot\Delta^* v\right]\,\mathrm{d}x + \frac{(1+q_0)\omega^4}{q_0}\|v\|^2_{L^2(D,\mathbb{C}^3)} \\
&= \frac{1}{q_0}\int_D \left[(\Delta^* v)^2 - \omega^2(2+q_0)(\mu|\nabla v|^2 + (\lambda+\mu)|\operatorname{div} v|^2)\right]\,\mathrm{d}x + \frac{(1+q_0)\omega^4}{q_0}\|v\|^2_{L^2(D,\mathbb{C}^3)},
\end{aligned}$$

where we used Betti's formula. Let now $\hat{v}$ be an eigenfunction of the bi-Navier operator $(\Delta^*)^2$, corresponding to an eigenvalue μ_1, i.e. $(\Delta^*)^2\hat{v} = \mu_1\hat{v}$ in D. Therefore we obtain

$$a_\omega(\hat{v},\hat{v}) \le \frac{\mu_1 + \omega^4(1+q_0)}{q_0}\|\hat{v}\|^2_{L^2(D,\mathbb{C}^3)} - \frac{\omega^2(2+q_0)}{q_0}\left[\mu\|\nabla\hat{v}\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\operatorname{div}\hat{v}\|^2_{L^2(D,\mathbb{C}^3)}\right].$$

We can continue to estimate

$$a_\omega(\hat{v},\hat{v}) \le \frac{1}{q_0}\left[\mu_1 + \omega^4(1+q_0) - \omega^2(2+q_0)\gamma\right]\|\hat{v}\|^2_{L^2(D,\mathbb{C}^3)}.$$

Following [Kir09], we have

$$\mu_1 + \omega^4(1+q_0) - \omega^2(2+q_0)\gamma = \left(\omega^2\sqrt{1+q_0} - \frac{(1+q_0/2)}{\sqrt{1+q_0}}\right)^2 + \mu_1 - \frac{(1+q_0/2)^2\gamma^2}{1+q_0}.$$

Choosing $\omega^2 = (1+q_0/2)\gamma/(1+q_0)$, the first bracket vanishes such that if q_0 is big enough such that

$$\mu_1 < \frac{(1+q_0/2)^2\gamma^2}{1+q_0},$$

we can conclude that $a_k(\hat{v},\hat{v}) < 0$ and therefore there exists an interior transmission eigenvalue ω_0^2 in the interval $(0,(1+q_0/2)\gamma/(1+q_0))$. ■

For the remainder of this section we assume a constant contrast $q = q_0$ in D. Recall that for the eigenpair $(u_0, w_0) \in L^2(D,\mathbb{C}^3) \times X_{\omega_0}$, corresponding to the interior transmission eigenvalue ω_0^2, the derivative $\alpha(\omega_0)$ is given by

$$\alpha(\omega_0) = \frac{2}{\omega_0}\int_D \left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\,\mathrm{div}\, v_0\,\mathrm{div}\,\overline{v_0}\right)\,\mathrm{d}x + 4\omega_0\int_D \overline{v_0}\cdot w_0\,\mathrm{d}x,$$

where v_0 is the radiating solution to (6.39). Then $\tilde{\alpha}(\omega_0) := \frac{\omega_0}{2}\alpha(\omega_0)$ is given by

$$\begin{aligned}\tilde{\alpha}(\omega_0) &= \int_D \left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\,\mathrm{div}\, v_0\,\mathrm{div}\,\overline{v_0}\right)\,\mathrm{d}x + 2\omega_0^2\int_D \overline{v_0}\cdot w_0\,\mathrm{d}x\\ &= \mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\mathrm{div}\, v_0\|^2_{L^2(D,\mathbb{C})} + 2\omega_0^2\int_D \overline{v_0}\cdot w_0\,\mathrm{d}x.\end{aligned}$$

The following condition for the positivity of the derivative α holds.

Lemma 6.14. *Let ω_0^2 be an interior transmission eigenvalue and assume that*

$$\gamma\left(\frac{2}{q}+1\right) - 2\frac{q+1}{q}\omega_0^2 > 0. \tag{6.47}$$

Then $\alpha(\omega_0) > 0$.

Proof. We start by rewriting the integral

$$\begin{aligned}2\omega_0^2\int_D \overline{v_0}\cdot w_0\,\mathrm{d}x &= \frac{2}{q}\omega_0^2\int_D q\overline{v_0}\cdot w_0\,\mathrm{d}x\\ &= \frac{2}{q}\int_D \left(\mu\nabla v_0 : \nabla\overline{v_0} + (\lambda+\mu)\,\mathrm{div}\, v_0\,\mathrm{div}\,\overline{v_0} - \omega_0^2\rho v_0\cdot\overline{v_0}\right)\,\mathrm{d}x\\ &= \frac{2}{q}\left(\mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + (\lambda+\mu)\|\mathrm{div}\, v_0\|^2_{L^2(D,\mathbb{C})} - \rho\omega_0^2\|v_0\|^2_{L^2(D,\mathbb{C}^3)}\right).\end{aligned}$$

Using this expression, we obtain that

$$\tilde{\alpha}(\omega_0) = \left(\frac{2}{q}+1\right)\mu\|\nabla v_0\|^2_{L^2(D,\mathbb{C}^{3\times 3})} + \left(\frac{2}{q}+1\right)(\lambda+\mu)\|\mathrm{div}\, v_0\|^2_{L^2(D,\mathbb{C})} - \frac{2}{q}\rho\omega_0^2\|v_0\|^2_{L^2(D,\mathbb{C}^3)}$$

Using $\rho = q+1$, we get that

$$\tilde{\alpha}(\omega_0) \geq \left[\gamma\left(\frac{2}{q}+1\right) - \frac{2}{q}(q+1)\omega_0^2\right]\|v_0\|_{L^2(D,\mathbb{C}^3)}$$

which yields the condition

$$\gamma\left(\frac{2}{q}+1\right)-2\frac{q+1}{q}\omega_0^2>0$$

for the positivity of $\tilde{\alpha}(\omega)$ and $\alpha(\omega_0)$. ■

The condition in (6.47) shows that in our consideration transmission eigenvalues ω_0^2 must not be too large for the derivative $\alpha(\omega_0)$ to be positive. In the next corollary, we show that the derivative is positive for the interior transmission eigenvalue from Theorem 6.13.

Corollary 6.15. *Let the contrast q fulfill the condition* (6.45)*. Then there exists at least one interior transmission eigenvalue $\omega_0^2<(1+q/2)\gamma/(1+q)$ and for all interior transmission eigenvalues ω_0^2 that fulfill this bound, its holds that $\alpha(\omega_0)>0$.*

Proof. From Lemma 6.14 we know that $\alpha(\omega_0)>0$ if the condition

$$\gamma\left(\frac{2}{q}+1\right)-\frac{2}{q}(q+1)\omega_0^2>0$$

is fulfilled. Since $\omega_0^2\in(0,(1+q/2)\gamma/(1+q))$, it suffices to show that

$$\gamma\left(\frac{2}{q}+1\right)-\frac{2}{q}(q+1)(1+q/2)\gamma/(1+q)=\gamma\left(\frac{2}{q}+1\right)-\frac{2}{q}(1+q/2)\gamma\geq 0.$$

Dividing by γ and multiplying by q yields as a sufficient condition that

$$2+q-2(1+q/2)\geq 0,$$

which is obviously true. This shows that for the transmission eigenvalue ω_0^2 the derivative is indeed positive. ■

CHAPTER 7

ELECTROMAGNETIC SCATTERING FROM PENETRABLE SCATTERING OBJECTS

7.1. Introduction

Now we will derive the inside-outside duality for electromagnetic scattering from anisotropic, dielectric scattering objects which may contain cavities. In particular we will determine interior transmission eigenvalues from far field data. Structurally our procedure will be similar to the last chapters. However, since the setting for electromagnetic scattering is more complex and involves different function spaces, this also shows in the derivation of the inside-outside duality, where the consideration of correct function spaces plays an important role. This chapter is based on the work in [LR15] for scattering objects without cavities. In the presence of cavities, we adapt the arguments from Section 3.3 for the electromagnetic case. Before we state our main result, let us first introduce the electromagnetic scattering problem and the relevant quantities. Let $D \subset \mathbb{R}^3$ be a bounded, simply-connected Lipschitz domain with connected complement that represents the scattering object. In this introduction we assume that D contains no cavities. Later in Section 7.3 we will relax this assumption. We denote the circular frequency by $\omega > 0$, the electric permittivity of a given dielectric medium by $\varepsilon > 0$, the constant magnetic permittivity by $\mu_0 > 0$ and the vanishing conductivity by $\sigma > 0$. Then the propagation of time-harmonic electromagnetic waves in $\mathbb{R}^3$ is described by the following equations for the electric and magnetic field E and H,

$$\operatorname{curl} E - \mathrm{i}\omega\mu_0 H = 0, \qquad \operatorname{curl} H + \mathrm{i}\omega\varepsilon E = 0.$$

Let the wavenumber be given by $k = \omega\sqrt{\varepsilon_0\mu_0}$, where ε_0 is the constant background permittivity. Then the system above reduces to

$$\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} H\right) - k^2 H = 0 \quad \text{in } \mathbb{R}^3, \tag{7.1}$$

where $\varepsilon_r = \varepsilon/\varepsilon_0$ is the relative permittivity, which equals ε_0 outside the scatterer D. To formulate the scattering problem more precisely, we assume that the support of $I_3 - \varepsilon_r$ equals D. Furthermore the material parameter $\varepsilon_r^{-1} \in L^\infty(D, \mathbb{R}^{3\times3})$ is a real-valued, symmetric 3×3 matrix that is bounded away from zero, i.e. $0 < c \le \overline{\zeta}^T \varepsilon_r^{-1}(x)\zeta$ for almost all $x \in \mathbb{R}^3$ and $\zeta \in \mathbb{C}^3$. The contrast function $Q := I_3 - \varepsilon_r^{-1}$ is then supported in $\overline{D}$. For this chapter, we assume that the sign of Q is negative, i.e. $\overline{\zeta}^T Q(x)\zeta \le -c_0|\zeta|^2$ for $\zeta \in \mathbb{C}^3$, a constant $c_0 > 0$ and almost all $x \in D$. The tangential components of the magnetic field H and of $\varepsilon_r^{-1} \operatorname{curl} H$ are continuous across interfaces where ε_r^{-1} jumps, i.e., if

ε_r^{-1} jumps across the boundary ∂D of the scattering object, then

$$\nu \times [H]_{\partial D} = 0 \quad \text{and} \quad \nu \times [\varepsilon_r^{-1} \operatorname{curl} H]_{\partial D} = 0.$$

In our model we assume that electromagnetic scattering from the scatterer D is caused by an incident, time-harmonic electromagnetic plane wave

$$H^i(x, \theta; p) := p e^{ikx\cdot\theta}, \quad x \in \mathbb{R}^3, \quad \text{where } \theta \in \mathbb{S}_1, p \in \mathbb{C}^3, \text{ and } p \cdot \theta = 0,$$

with direction θ and polarization p. Since the incident field H^i solves $\operatorname{curl}^2 H^i - k^2 H^i = 0$ in $\mathbb{R}^3$, we can write equation (7.1) for the scattered field $H^s = H - H^i$ as

$$\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} H^s\right) - k^2 H^s = \operatorname{curl}(Q \operatorname{curl} H^i) \qquad \text{in } \mathbb{R}^3. \tag{7.2}$$

Furthermore, the scattered field H^s is assumed to satisfy the Silver-Müller radiation condition

$$\operatorname{curl} H^s(x) \times \hat{x} - \mathrm{i}k H^s(x) = \mathcal{O}(|x^{-2}|) \qquad \text{as } |x| \to \infty$$

uniformly with respect to $\hat{x} := x/|x| \in \mathbb{S}_1$. Solutions that satisfy this condition are in this chapter called radiating solutions. To generalize the scattering problem, we consider a source term $f \in L^2(D, \mathbb{C}^3)$ and seek a weak radiating solution $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ to

$$\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} v\right) - k^2 v = \operatorname{curl}(Qf) \qquad \text{in } \mathbb{R}^3,$$

where

$$H_{\text{loc}}(\text{curl}, \mathbb{R}^3) = \{v : \mathbb{R}^3 \to \mathbb{C}^3 : \ v|_{B_R} \in H(\text{curl}, B_R) \quad \text{for } R > 0\}$$

and

$$H(\text{curl}, B_R) = \{v \in L^2(B_R, \mathbb{C}^3) : \ \operatorname{curl} v \in L^2(B_R, \mathbb{C}^3)\}.$$

Note that setting $f = \operatorname{curl} H^i$ yields the original problem. In the variational formulation, the radiating solution $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ needs to solve

$$\int_{\mathbb{R}^3} \left(\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right) \mathrm{d}x = \int_D Qf \cdot \operatorname{curl} \overline{\psi} \,\mathrm{d}x \tag{7.3}$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with compact support. As a standing assumption in this chapter, we suppose that for all $f \in L^2(D, \mathbb{C}^3)$, equation (7.3) has a unique, radiating solution $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$. This is for example true if ε_r is globally Hölder continuous, see e.g. [Vog91]. Due to the radiation condition, the solution can be expressed in terms of far fields,

$$v(x) = \frac{\exp(\mathrm{i}k|x|)}{4\pi|x|} v^\infty(\hat{x}) + \mathcal{O}\left(|x|^{-2}\right), \qquad \text{as } |x| \to \infty, \tag{7.4}$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}_1$, where $v^\infty : \mathbb{S}_1 \to \mathbb{C}^3$ is the far field pattern of v. In order to introduce the far field operator corresponding to the scattering problem in (7.2), we first introduce the necessary function space. Since v^∞ is analytic and a tangential vector field on the unit sphere, i.e. $v^\infty(\hat{x}) \cdot \hat{x} = 0$ for all $\hat{x} \in \mathbb{S}_1$, it belongs to the space of tangential vector fields on the unit sphere,

$$L^2_t(\mathbb{S}_1) := \left\{v \in L^2(\mathbb{S}_1, \mathbb{C}^3),\ v(\hat{x}) \cdot \hat{x} = 0 \text{ for a.e. } \hat{x} \in \mathbb{S}_1\right\} \subset L^2(\mathbb{S}_1, \mathbb{C}^3),$$

see [CK13]. We can now introduce the far field operator $F : L^2_t(\mathbb{S}_1) \to L^2_t(\mathbb{S}_1)$ by

$$(Fp)(\hat{x}) := \int_{\mathbb{S}_1} H^\infty(\hat{x}, \theta; p(\theta))\, \mathrm{d}S(\theta) \qquad \text{for } \hat{x} \in \mathbb{S}_1, \tag{7.5}$$

where $H^\infty(\cdot, \theta; p)$ is the far field pattern corresponding to the incident plane wave $H^i(\cdot, \theta; p)$. Due to our assumption that ε_r^{-1} is real-valued, it is well known [CK13] that the far field operator is linear, compact and normal and its eigenvalues λ_j lie on the circle $\{\lambda \in \mathbb{C}, |8\pi^2\mathrm{i}/k - \lambda| = 8\pi^2/k\}$ in the complex plane. From this scattering problem arises an interior transmission eigenvalue problem. The squared wavenumber k^2 is an interior transmission eigenvalue if there is a pair of non-trivial functions $(u, w) \in H_0(\mathrm{curl}, D) \times H_0(\mathrm{curl}, D)$, such that

$$\mathrm{curl}\left(\varepsilon_r^{-1}\,\mathrm{curl}\, u\right) - k^2 u = 0 \quad \text{in } D \qquad \text{and} \qquad \mathrm{curl}^2\, w - k^2 w = 0 \quad \text{in } D, \tag{7.6}$$

$$\nu \times (u - w)|_{\partial D} = 0 \qquad \text{and} \qquad \nu \times \left(\varepsilon_r^{-1}\,\mathrm{curl}\, u - \mathrm{curl}\, w\right)\big|_{\partial D} = 0. \tag{7.7}$$

In order to indicate the main results of this chapter, we represent the eigenvalues λ_j of the the far field operator in polar coordinates

$$\lambda_j = |\lambda_j| e^{\mathrm{i}\vartheta_j}, \qquad \vartheta_j \in [0, \pi], \tag{7.8}$$

where we set the phase $\vartheta_j = 0$ if $\lambda_j = 0$. By this representation, each eigenvalues λ_j corresponds to a phase ϑ_j. Our assumption that the sign of contrast Q is negative implies that the eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ of the far field operator converge to zero from the left, see Lemma 7.3. Therefore there is one distinct eigenvalue λ_* with smallest phase

$$\vartheta_* = \min_{j \in \mathbb{N}} \vartheta_j, \tag{7.9}$$

where the phase $\vartheta_* = \vartheta_*(k)$ depends on the wavenumber k. Now one part of the inside-outside duality states that if $\vartheta_*(k) \to 0$ as k approaches a wavenumber k_0, then k_0^2 is an interior transmission eigenvalue, see Theorem 7.7. The converse direction only holds under the condition that the expression $\alpha(k_0)$ in (7.19) does not vanish, see Theorem 7.6. Then interior transmission eigenvalues k_0^2 are fully characterized by the behavior of the phase $\vartheta_*(k)$ as k approaches k_0. In Section 7.3 we will show that in the presence of cavities in the scattering object D these main results still hold, see Theorem 7.19 and Theorem 7.20. Note that in the absence of cavities, it is possible to derive bounds for the material parameters that imply the existence of interior transmission eigenvalues k_0^2, for which the derivative $\alpha(k_0)$ does not vanish. This is no longer possible in the presence of cavities, which is the only major drawback in allowing for cavities within the scattering object. The remainder of this chapter is structured as follows. In Section 7.2, we derive the inside-outside duality for objects that contain no cavities. We will also show for which material parameters the auxiliary derivative α does not vanish. In Section 7.3, we will then consider the influence of the presence of cavities on our derivation.

7.2. Characterizing Interior Transmission Eigenvalues From Far Field Data

In this section we will follow [LR15] and derive the inside-outside duality for scattering objects that contain no cavities. We proceed in the same manner as in the previous chapters. At first we derive a factorization for the far field operator and then link interior transmission eigenvalues to the middle operator of this factorization in Lemma 7.2. Then we give a characterization of the smallest phase of all the phases of the eigenvalues λ_j of F in Theorem 7.4. Next we introduce the auxiliary derivative

α in Lemma 7.5 and use it to finally state both parts of the inside-outside duality in Theorem 7.6 and Theorem 7.7. We start by deriving a factorization for the far field operator.

At first we introduce the linear, compact Herglotz operator $H : L^2_t(\mathbb{S}^2) \to L^2(D, \mathbb{C}^3)$, defined by

$$Hg = \operatorname{curl}_x v_g, \qquad v_g(x) := \int_{\mathbb{S}_1} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathrm{d}S(\theta) \qquad \text{for } x \in D. \tag{7.10}$$

The Herglotz wave function v_g is smooth and solves Maxwell's equations $\operatorname{curl}^2 v_g - k^2 v_g = 0$ and the vectorial Helmholtz equation $\Delta v_g + k^2 v_g = 0$ in $\mathbb{R}^3$ in the classical sense. The Herglotz operator is injective and from [LR15, Proposition 2], we know that its adjoint $H^* : L^2(D, \mathbb{C}^3) \to L^2_t(\mathbb{S}^2)$ is given by

$$(H^*\psi)(\theta) = \mathrm{i}k\, \theta \times \int_D \psi(x) e^{-\mathrm{i}k\, x\cdot\theta}\, \mathrm{d}x \qquad \text{for } \theta \in \mathbb{S}_1.$$

From this expression it is clear, that for a function $\psi \in L^2(D, \mathbb{C}^3)$ the function $H^*\psi \in L^2_t(\mathbb{S}_1)$ is the far field pattern w^∞ to

$$w(x) = \operatorname{curl} \int_D \Phi(x, y)\psi(y)\ \mathrm{d}y, \qquad x \in \mathbb{R}^3,$$

where $\Phi(x, y)$ is the fundamental solution to the Helmholtz equation,

$$\Phi(x, y) := \frac{\exp(\mathrm{i}k|x - y|)}{4\pi|x - y|}, \quad x \neq y.$$

The last component for the factorization is the operator

$$T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3), \qquad Tf := Q(f + \operatorname{curl} v|_D),$$

where $v \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ is the unique radiating weak solution to $\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} v\right) - k^2 v = \operatorname{curl}(Qf)$ in $\mathbb{R}^3$, that is, for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support, v satisfies the Silver-Müller radiation condition and

$$\int_{\mathbb{R}^3} \left[\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right]\ \mathrm{d}x = \int_D Qf \cdot \operatorname{curl} \overline{\psi}\, \mathrm{d}x. \tag{7.11}$$

These operators can now be used to state a factorization for the far field operator, see [LR15, Theorem 10].

Theorem 7.1. *For $k > 0$ the factorization $F = H^*TH$ holds.*

Furthermore the middle operator T provides the link between interior transmission eigenvalues and the far field data. Before we provide this link, we want to characterize the closure of the image of the Herglotz operator H, which contains those L^2-functions that solve Maxwell's equation in a weak sense. To be more precise, we define

$$X = \left\{ w \in L^2(D, \mathbb{C}^3), \int_D w \cdot (\operatorname{curl}^2 \psi - k^2 \psi)\, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D, \mathbb{C}^3) \right\}, \tag{7.12}$$

and note that by [LR15, Lemma 4] it holds that $X = \operatorname{closure}_{L^2(D,\mathbb{C}^3)} \mathcal{R}(H)$. Before we provide the link between interior transmission eigenvalues and the far field operator via its factorization, we will use the space X to define interior transmission eigenvalues in a way that is suitable for our derivation. Note that the following definition is equivalent to the definition provided in the introduction to this chapter.

The squared wavenumber k^2 is an interior transmission eigenvalue if there exists a non-trivial pair

$(u, w) \in H_0(\mathrm{curl}, D) \times X$ that satisfies

$$\begin{aligned} &\mathrm{curl}\left(\varepsilon_r^{-1}\,\mathrm{curl}\,u\right) - k^2 u = \mathrm{curl}\,(Qw) \quad \text{in } D, \qquad \mathrm{curl}^2\, w - k^2 w = 0 \quad \text{in } D, \quad \text{and} \\ &\qquad \nu \times \varepsilon_r^{-1}\,\mathrm{curl}\,u = \nu \times Qw \quad \text{on } \partial D. \end{aligned} \tag{7.13}$$

The differential equations and the boundary conditions are understood in a variational sense, i.e.,

$$\int_D \left[\varepsilon_r^{-1}\,\mathrm{curl}\,u \cdot \mathrm{curl}\,\overline{\psi} - k^2 u \cdot \overline{\psi}\right] \mathrm{d}x = \int_D Qw \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x \quad \forall \psi \in H(\mathrm{curl}, D). \tag{7.14}$$

In the following lemma we will examine the properties of the middle operator T of the factorization and in this context provide the link between interior transmission eigenvalues and the far field operator F. For a proof we refer to [LR15, Theorem 10, Theorem 11, Corollary 12].

Lemma 7.2. *(a) For weak solutions $v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ to (7.11), the mapping $f \mapsto \mathrm{curl}\,v|_D$ is compact from $L^2(D, \mathbb{C}^3)$ into $L^2(D, \mathbb{C}^3)$.*
(b) For $k > 0$ and $f \in L^2(D, \mathbb{C}^3)$ it holds that $\mathrm{Im}\,(Tf, f)_{L^2(D,\mathbb{C}^3)} \geq 0$.
(c) If k^2 is an interior transmission eigenvalue with eigenpair $(u, w) \in H_0(\mathrm{curl}, D) \times X$, then $(Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$.
(d) If there is a function $w \in X \setminus \{0\}$ such that $\mathrm{Im}\,(Tw, w)_{L^2(D,\mathbb{C}^3)} = 0$, then k^2 is an interior transmission eigenvalue and there is a function $u \in H_0(\mathrm{curl}, D)$ such that (u, w) is the corresponding eigenpair.

For the electromagnetic case, the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of the far field operator F also lie on a circle with radius $8\pi^2/k$ with center at $8\pi^2\mathrm{i}/k$. The first two properties of the last lemma provide us with the necessary information to show that the eigenvalues of the far field operator converge to zero from the left side, see e.g. Remark 2.3.

Lemma 7.3. *Assume that k^2 is no interior transmission eigenvalue. Then the eigenvalues λ_j of the far field operator F converge to zero from the left, i.e. $\mathrm{Re}\,(\lambda_j) < 0$ for $j \in \mathbb{N}$ large enough.*

Recall the representation of the eigenvalues λ_j in polar coordinates in (7.8) and the definition of the smallest phase ϑ_* in (7.9). We give the standard characterization for this phase via Remark 3.4.

Lemma 7.4. *Assume that k^2 is no interior transmission eigenvalue. Then it holds that*

$$\cot\vartheta_* = \max_{w\in X} \frac{\mathrm{Re}\,(Tw, w)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(Tw, w)_{L^2(D,\mathbb{C}^3)}}. \tag{7.15}$$

From now on we indicate the dependence of all the quantities on the wavenumber k by writing $T = T_k$, $X = X_k$, $\lambda_j = \lambda_j(k)$, $\vartheta_j = \vartheta_j(k)$ and so on. To remove the space X_k from the characterization we use a projection operator $P_k : L^2(D, \mathbb{C}^3) \to X_k$ that is differentiable with respect to the wavenumber k. When we introduced the projection operator in Chapter 4, we used a decomposition of functions in $L^2(D, \mathbb{C}^3)$ into a scalar and a vector potential. The same idea can be applied here. We introduce the space

$$W = \{\psi \in H_0^1(D, \mathbb{C}^3), \mathrm{curl}\,\psi \in H_0^1(D, \mathbb{C}^3)\}$$

with the norm $\|\psi\|_W^2 := \|\psi\|_{H(\mathrm{curl},D)}^2 + \|\mathrm{div}\,\psi\|_{L^2(D,\mathbb{C})}^2 + \|\mathrm{curl}^2\,\psi\|_{L^2(D,\mathbb{C}^3)}^2$. Recall from Lemma 4.11 that a function $g \in L^2(D, \mathbb{C}^3)$ can be decomposed as $g = \mathrm{curl}\,A_g + \nabla p_g$, where $p_g \in H_0^1(D)$ and $A_g \in H(\mathrm{curl}, D) \cap H(\mathrm{div}\,0, D)$ is a uniquely determined vector potential such that $A_g \cdot \nu = 0$ on ∂D. Here $H(\mathrm{div}\,0, D)$ is the space of functions with vanishing divergence in D, i.e.

$$H(\mathrm{div}\,0, D) := \{u \in L^2(D, \mathbb{C}^3) : \mathrm{div}\,u \in L^2(D, \mathbb{C}),\ \mathrm{div}\,u = 0 \text{ in } D\}.$$

For $k > 0$ we can now define the operator $P_k : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ by

$$P_k g := g - (\operatorname{curl}^2 - k^2)\hat{A}_g - \nabla p_g,$$

where $\hat{A}_g \in W$ solves the following variational problem

$$\int_D (\operatorname{curl}^2 - k^2)\hat{A}_g \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x + \int_D \operatorname{div}\hat{A}_g \cdot \operatorname{div}\overline{\psi}\,\mathrm{d}x \tag{7.16}$$

$$= \int_D \operatorname{curl} A_g \cdot (\operatorname{curl}^2 - k^2)\overline{\psi}\,\mathrm{d}x \qquad \forall \psi \in W. \tag{7.17}$$

Note in this context that $H_0^1(D, \mathbb{C}^3) = H_0(\operatorname{div}, D) \cap H_0(\operatorname{curl}, D)$ by [GR86b, Lemma 2.5], where $H_0(\operatorname{curl}, D)$ has been defined in (4.28) and $H_0(\operatorname{div}, D) := \{f \in H(\operatorname{div}, D) : \nu \cdot f = 0 \quad \text{on } \partial D\}$, where $H(\operatorname{div}, D)$ has been defined in (4.29). This shows that functions in the space W have the necessary regularity for the variational problem to be well-defined. Furthermore, the sesquilinear form that arises from this problem is coercive as a consequence of [LR15, Lemma 16] and therefore the variational problem is well-posed and has a unique solution in W. From [LR15, Lemma 18] we furthermore know that the map P_k has the desired properties, i.e. $P_k : L^2(D, \mathbb{C}^3) \to X_k$ is the orthogonal projection from $L^2(D, \mathbb{C}^3)$ onto X_k and for $g \in L^2(D, \mathbb{C}^3)$ the map $k \mapsto P_k g$ from $\mathbb{R}_+$ into $L^2(D, \mathbb{C}^3)$ is continuously differentiable. Now we can use the projection operator to rewrite the expression for the smallest phase in (7.15) as

$$\cot \vartheta_* = \max_{w \in X_k} \frac{\operatorname{Re}(T_k P_k, P_k w)_{L^2(D,\mathbb{C}^3)}}{\operatorname{Im}(T_k P_k w, w)_{L^2(D,\mathbb{C}^3)}}.$$

Assume now that k_0^2 is an interior transmission eigenvalue with eigenpair $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$. For the first part of the inside-outside duality we will calculate the auxiliary derivative

$$\alpha(k_0) = \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0}. \tag{7.18}$$

The following lemma calculates an explicit expression for $\alpha(k_0)$. For a proof see either [LR15, Lemma 22] or the proof of Lemma 7.18, where we consider the influence of the presence of cavities.

Lemma 7.5. *Assume that k_0^2 is an interior transmission eigenvalue with eigenpair $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$. Then the map $k \mapsto \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}$ is differentiable at $k = k_0$ and*

$$\alpha(k_0) = \frac{\mathrm{d}}{\mathrm{d}k}(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = 2k_0 \int_D |v_0|^2\,\mathrm{d}x + \frac{4}{k_0}\operatorname{Re}\int_D \operatorname{curl} v_0 \cdot \overline{w_0}\,\mathrm{d}x. \tag{7.19}$$

Now we can state the first part and the second of the inside-outside duality. For the proof of the first part, we refer to the proof of Theorem 3.7, where this theorem has been proven for acoustic scattering scenarios. The arguments transfer one-to-one to this case. For a proof of the second part, see [LR15, Theorem 15].

Theorem 7.6 (Inside-outside duality - Part 1)**.** *Let k_0^2 be an interior transmission eigenvalue with corresponding interior transmission eigenpair $(v_0, w_0) \in H_0(\operatorname{curl}, D) \times X_{k_0}$ and let $\alpha(k_0)$ be the expression in equation* (7.19)*. Then it holds that*

$$\lim_{k \searrow k_0} \vartheta_*(k) = 0 \quad \textit{if } \alpha(k_0) > 0 \qquad \textit{and} \qquad \lim_{k \nearrow k_0} \vartheta_*(k) = 0 \quad \textit{if } \alpha(k_0) < 0.$$

Theorem 7.7 (Inside-Outside Duality - Part 2)**.** *Let $k_0 > 0$ be such that $I := (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$*

contains no wavenumber k such that k^2 is an interior transmission eigenvalue. If $\lim_{I\ni k\to k_0} \vartheta_*(k) = 0$, then k_0^2 is an interior transmission eigenvalue.

In the last part of this section we want to give conditions for the contrast Q for which there are interior transmission eigenvalues k_0^2 such that the corresponding derivative $\alpha(k_0)$ does not vanish. For that purpose we define a space of divergence-free functions by

$$V = \left\{ v \in H_0(\mathrm{curl}, D), \quad \int_D v \cdot \nabla\phi \,\mathrm{d}x = 0 \qquad \forall \phi \in H^1(D) \right\}.$$

Now denote by $\mu_0 > 0$ the smallest eigenvalue of the eigenvalue problem to find $(\mu, v) \in \mathbb{R} \times V$ such that

$$\int_D \mathrm{curl}\, v \cdot \overline{\psi} \,\mathrm{d}x = \mu \int_D v \cdot \overline{\psi} \,\mathrm{d}x \qquad \forall \psi \in V.$$

Furthermore [Mon03, Corollary 3.51] implies that there is a number $\rho_0 > 0$ such that

$$\|v\|_{L^2(D,\mathbb{C}^3)} \leq \rho_0 \|\mathrm{curl}\, \psi\|^2_{L^2(D,\mathbb{C}^3)} \qquad \forall v \in H(\mathrm{curl}, D) \cap H_0(\mathrm{div}\, 0, D).$$

Let us at first assume that the contrast is constant, i.e. there exists a $q_0 \in (-\infty, 0)$ such that $Q = q_0 I_3$. Then we obtain the following estimate, which implies the existence of interior transmission eigenvalues for which the derivate does not vanish, see [LR15, Theorem 23].

Theorem 7.8. *If $Q = q_0 I_3$ and $q_0 < -(1 + \sqrt{5})$ satisfies*

$$8\rho_0^2 \mu \leq \frac{(2 - q_0)(1 + q_0)^2 - 5}{1 - q_0^2},$$

then there exists at least one interior transmission eigenvalue k_0^2 such that

$$k_0^2 < \frac{2\mu_0(1 - q_0)}{2 - q_0}$$

and for any interior transmission eigenvalue below this bound, it holds that the derivative $\alpha(k_0)$ is less than zero.

It is possible to expand on this bound by allowing perturbations of constant material parameters. We set

$$Q := q_0 I_3 + Q', \qquad 0 \geq Q' \in L^\infty(D, \mathrm{Sym}(3))$$

and denote by $\||Q'|_2\|_{L^\infty(D)}$ the essential supreme of the spectral matrix norm of Q'. Then we obtain the following bounds, see [LR15, Lemma 24].

Lemma 7.9. *If $q_0 < -(1 + \sqrt{5})$ and if Q' satisfies*

$$\frac{(2 - q_0)^2}{2|q_0|} \leq \left[1 - q_0 - \frac{\||Q'|_2\|_{L^\infty(D)}}{\sqrt{\mu_0}} \right] \left[1 - \frac{4\rho_0^2 \mu_0}{(2 - q_0)|q_0|} \left[1 - q_0 - \frac{\||Q'|_2\|_{L^\infty(D)}}{\sqrt{\mu_0}} \right] \right],$$

then there exists an interior transmission eigenvalue k_0^2 such that

$$k_0^2 < \frac{2\mu_0}{2 - q_0} \left[1 - q_0 - \||Q'|_2\|_{L^\infty(D)} / \sqrt{\mu_0} \right]$$

and for any interior transmission eigenvalue k_0^2 that satisfies this condition, the derivative $\alpha(k_0)$ is strictly negative.

Remark 7.10. Note the similarity between the bounds in Theorem 7.8 and Theorem 4.25 in the last chapter. This is on the one hand due to the structural similarity of both problems, which shows also in the explicit expression for the derivative α in (4.32) and (7.19), and on the other hand due to the existence proofs of the interior transmission eigenvalues for acoustic and electromagnetic scattering problems in [Kir09], which rely on the same principle technique.

7.3. The Influence of the Presence of Cavities

In this section we want to extend the inside-outside duality for electromagnetic scattering problems by allowing cavities D_0 in the scatterer D. We proceed by adapting the structure we used when we examined acoustic scatterers that contain cavities in Section 3.3 to the case of electromagnetic scattering. In particular we use the functional analytical framework that was provided in [CHM15] for the electromagnetic case. As in Section 3.3, we assume that the scattering object $D \subset \mathbb{R}^3$ is simply connected with boundary $\partial D \in C^2$. Inside of D we consider a region $D_0 \subset D$, that represents a cavity inside the scattering object. The cavity D_0 can be multiply connected, such that $D \setminus \overline{D_0}$ is simply-connected and assume that its boundary ∂D_0 is also C^2 smooth.

The circular frequency $\omega > 0$, the electric permittivity of a given dielectric medium $\varepsilon > 0$, the constant magnetic permittivity $\mu_0 > 0$ and the vanishing conductivity $\sigma > 0$ have already been introduced in the beginning of this chapter. Furthermore the relative permittivity ε_r now equals ε_0 outside the scatterer D and in the cavity D_0. As in the previous section the material parameter $\varepsilon_r^{-1} \in L^\infty(D, \mathbb{R}^{3\times 3})$ is a real-valued, symmetric 3×3 matrix, $0 < c \le \overline{\zeta}^T \varepsilon_r^{-1}(x)\zeta$ for almost all $x \in \mathbb{R}^3$ and $\zeta \in \mathbb{C}^3$. The contrast function $Q := I_3 - \varepsilon_r^{-1}$ is then supported in $D \setminus \overline{D_0}$.

We use the presence of the cavity to rewrite the scattering problem (7.3) in the following way. We seek a radiating solution $v \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ to the problem

$$\int_{\mathbb{R}^3} \left(\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \overline{\psi} - k^2 v \cdot \overline{\psi}\right) \mathrm{d}x = \int_{D\setminus\overline{D_0}} Qf \cdot \operatorname{curl} \overline{\psi} \, \mathrm{d}x \tag{7.20}$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with compact support. Since the solution satisfies the Silver-Müller radiation condition, the solution can be represented in terms of its far field v^∞ as in (7.4). Recall the definition of the far field operator in (7.5). The presence of cavities does not change the properties of this operator, which we have already discussed in the introduction to this chapter. It is still compact and normal and its eigenvalues lie on the circle $\{\lambda \in \mathbb{C}, \, |8\pi^2 \mathrm{i}/k - \lambda| = 8\pi^2/k\}$ in the complex plane. From now on we proceed in the following way. In a first step we will adapt the factorization that has been derived in the last section and state its properties in Theorem 7.11. These properties can be used to show that the eigenvalues of the far field operator converge to zero from the left side, due to our assumption that the contrast Q is negative in the introduction to this chapter. Then we will show how we need to adapt the characterization of the range of the Helglotz wave operator in Lemma 7.13. In this context we will introduce extension and restriction operators that will help us to relate functions that act on the whole domain D to functions that are defined only on $D \setminus \overline{D_0}$. These operators can then be used to link interior transmission eigenvalues to the middle operator T of the factorization in Lemma 7.14. Using the typical phase characterizations, we will then calculate the derivative $\alpha(k_0)$ in Lemma 7.18 in order to finally state the first part and the second part of the inside-outside duality in Theorem 7.19 and Theorem 7.20.

We start by adapting the factorization of the far field operator in order to link interior transmission eigenvalues to far field data. To this end we introduce the linear, compact Herglotz operator $H : L^2_t(\mathbb{S}^2) \to L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$, defined by

$$Hg = \operatorname{curl} v_g, \qquad v_g(x) := \int_{\mathbb{S}_1} e^{\mathrm{i}k\, x\cdot\theta} g(\theta) \, \mathrm{d}S(\theta) \qquad \text{for } x \in D \setminus \overline{D_0}. \tag{7.21}$$

As we noted in the previous section, the Herglotz wave function v_g is smooth and solves Maxwell's equations $\operatorname{curl}^2 v_g - k^2 v_g = 0$ and the vectorial Helmholtz equation $\Delta v_g + k^2 v_g = 0$ in $\mathbb{R}^3$ in the classical sense. The Herglotz operator is injective and due to [LR15, Proposition 2], we know that its adjoint $H^* : L^2(D \setminus \overline{D_0}, \mathbb{C}^3) \to L^2_t(\mathbb{S}^2)$ is given by

$$(H^*\psi)(\theta) = \mathrm{i}k\,\theta \times \int_{D\setminus\overline{D_0}} \psi(x) e^{-\mathrm{i}k\,x\cdot\theta}\,\mathrm{d}x \qquad \text{for } \theta \in \mathbb{S}_1.$$

It follows that for $\psi \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$, the function $H^*\psi \in L^2_t(\mathbb{S}_1)$ is the far field pattern v^∞ to

$$v(x) = \operatorname{curl}_x \int_{D\setminus\overline{D_0}} \Phi(x,y)\psi(y)\ \mathrm{d}y, \qquad x \in \mathbb{R}^3.$$

The last component for the factorization is the operator

$$T : L^2(D \setminus \overline{D_0}, \mathbb{C}^3) \to L^2(D \setminus \overline{D_0}, \mathbb{C}^3), \qquad Tf := Q(f + \operatorname{curl} v|_{D\setminus\overline{D_0}}),$$

where $v \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ is the unique radiating weak solution to $\operatorname{curl}\left(\varepsilon_r^{-1} \operatorname{curl} v\right) - k^2 v = \operatorname{curl}(Qf)$ in $\mathbb{R}^3$, that is, for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support, v satisfies

$$\int_{\mathbb{R}^3} \left[\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl}\overline{\psi} - k^2 v \cdot \overline{\psi}\right]\ \mathrm{d}x = \int_{D\setminus\overline{D_0}} Qf \cdot \operatorname{curl}\overline{\psi}\,\mathrm{d}x \tag{7.22}$$

together with the Silver-Müller radiation condition. Then a factorization of the far field operator is given in the following theorem, which also includes some properties of the middle operator of the factorization. It can be proven in the same manner as Theorem 7.2 in the previous section.

Theorem 7.11. *(a) For $k > 0$ the factorization $F = H^*TH$ holds.*
(b) If $v \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ is the radiating weak solution to (7.22) *then the mapping $f \mapsto \operatorname{curl} v|_{D\setminus\overline{D_0}}$ is compact from $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ into $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$.*
(c) For $k > 0$ and $f \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ it holds that $\operatorname{Im}\,(Tf,\, f)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} \geq 0$.

Before we show how the middle operator T links transmission eigenvalues to the scattering problem, we will state the transmission eigenvalue problem more precisely. For that purpose we introduce a space X_D which contains those functions in $L^2(D, \mathbb{C}^3)$ that solve Maxwell's equation:

$$X_D := \Big\{W \in L^2(D, \mathbb{C}^3), \quad \int_D W \cdot (\operatorname{curl}^2 \psi - k^2\psi)\,\mathrm{d}x = 0 \quad \forall \psi \in C_0^\infty(D, \mathbb{C}^3)\Big\}. \tag{7.23}$$

Note that in contrast to the previous section, where this space was denoted by X in (7.12), we changed notation to keep the principle notation that we have used so far. Recall the definition of interior transmission eigenvalues in (7.13), which we will also use in this section. To establish a link between transmission eigenvalues and far field data, we will use the factorization of the far field operator F and in particular the properties of the middle operator T. Therefore we would like to neglect the operator H from the factorization and therefore introduce a function space that characterizes its image. Before we do that however, we need to introduce some technical details.

First we will adapt the arguments from Section 3.3 for the representations of L^2-solutions of the Helmholtz equation to the Maxwell case. To this end we define the space

$$H(\operatorname{curl}^2, D) := \{w \in L^2(D, \mathbb{C}^3), \operatorname{curl}^2 w \in L^2(D, \mathbb{C}^3)\},$$

where $\operatorname{curl}^2 w$ is understood in a weak sense, i.e. for $w \in H(\operatorname{curl}^2, D)$ there exists $\eta \in L^2(D, \mathbb{C}^3)$, such that $\int_D \eta \cdot \psi\,\mathrm{d}x = \int_D w \cdot \operatorname{curl}^2 \psi\,\mathrm{d}x$ for all $\psi \in C_0^\infty(D, \mathbb{C}^3)$ and $\operatorname{curl}^2 w = \eta$. The space is equipped with the norm

$$\|u\|_{H(\mathrm{curl}^2,D)} := \|\mathrm{curl}^2 u\|_{L^2(D,\mathbb{C}^3)} + \|u\|_{L^2(D,\mathbb{C}^3)}.$$

Let now $u \in L^2(D,\mathbb{C}^3)$ be a weak solution to Maxwell's equation,

$$\int_D u \cdot (\mathrm{curl}^2 \psi - k^2\psi)\, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D,\mathbb{C}^3). \tag{7.24}$$

Then it is obvious that $u \in H(\mathrm{curl}^2, D)$. We will now consider the traces of such functions on the boundary of D. For that purpose we introduce the Sobolev spaces of tangential traces by

$$H_t^s(\partial D) := \{v \in H^s(\partial D, \mathbb{C}^3), v \times \nu = 0 \quad \text{a.e. on } \partial D\}.$$

Now we follow the arguments in the proof of [CHM15, Lemma 3.2] and define the first trace operator $\gamma_D u := \nu \times u|_{\partial D} \in H_t^{-1/2}(\partial D)$ in the following way: For $\alpha \in H_t^{1/2}(\partial D), \|\alpha\|_{H_t^{1/2}(\partial D)} = 1$, there exists a function $w \in H^2(D,\mathbb{C}^3)$ such that $\nu \times \mathrm{curl}\, w = \alpha$ and $\nu \times w = 0$ on ∂D by [Had04, Lemma 3.1]. Then

$$\langle \alpha, \gamma_D u\rangle_{H_t^{1/2}(\partial D), H_t^{-1/2}(\partial D)} = -\int_D \left(u \cdot \mathrm{curl}^2 w - w \cdot \mathrm{curl}^2 u\right) \mathrm{d}x.$$

The norm of $\gamma_D u$ is given by

$$\|\gamma_D u\|_{H_t^{-1/2}(\partial D)} := \sup_{\|\alpha\|=1} \langle \alpha, \gamma_D u\rangle_{H_t^{1/2}(\partial D), H_t^{-1/2}(\partial D)}$$

and therefore $\gamma_D : H(\mathrm{curl}^2, D) \to H_t^{-1/2}(\partial D)$ is a continuous map since

$$\begin{aligned}\langle \alpha, \gamma_D u\rangle_{H_t^{1/2}, H_t^{-1/2}} &\le \left(\|u\|_{L^2(D,\mathbb{C}^3)} + \|\mathrm{curl}^2 u\|_{L^2(D,\mathbb{C}^3)}\right) \|w\|_{H^2(D,\mathbb{C}^3)} \\ &\le c_1 \left(\|u\|_{L^2(D,\mathbb{C}^3)} + \|\mathrm{curl}^2 u\|_{L^2(D,\mathbb{C}^3)}\right) = c_1 \|u\|_{H(\mathrm{curl}^2,D)}\end{aligned}$$

for a constant c_1 independent of α by [Had04, Lemma 3.1]. In similar manner, we introduce a second trace operator $\gamma_N u := (\nu \times \mathrm{curl}\, u)\,|_{\partial D} \in H_t^{-3/2}(\partial D)$. For $\beta \in H_t^{3/2}(\partial D), \|\beta\|_{H_t^{3/2}(\partial D)} = 1$, we choose $w \in H^2(D,\mathbb{C}^3)$ such that $\nu \times \mathrm{curl}\, w = 0$ and $\nu \times w = \beta$ on ∂D. Then we have

$$\langle \gamma_N, \beta\rangle_{H^{-3/2}(\partial D), H^{3/2}(\partial D)} = \int_D \left(\mathrm{curl}^2 u \cdot w - u \cdot \mathrm{curl}^2 w\right) \mathrm{d}x,$$

By the same arguments as above it follows that γ_N maps from $H(\mathrm{curl}^2, D)$ continuously into $H_t^{-3/2}(\partial D)$. By means of these traces it is possible to derive a representation formula for L^2-solutions of Maxwell's equation. Important ingredients for such a representation are two potentials $\mathrm{SL} : H_t^{-3/2}(\partial D, \mathbb{C}^3) \to L^2(D,\mathbb{C}^3)$ and $\mathrm{DL} : H_t^{-1/2}(\partial D, \mathbb{C}^3) \to L^2(D,\mathbb{C}^3)$, defined by

$$\mathrm{SL}(\psi) = \int_{\partial D} \psi(y)\Phi(\cdot, y)\, \mathrm{d}S(y) \qquad \text{in } \mathbb{R}^3 \setminus \partial D, \tag{7.25}$$

$$\mathrm{DL}(\phi) = \mathrm{curl}_x \int_{\partial D} \phi(y)\Phi(\cdot, y)\, \mathrm{d}S(y) \qquad \text{in } \mathbb{R}^3 \setminus \partial D. \tag{7.26}$$

Note that both potentials are continuous maps by [CHM15, Lemma 2.3]. The following lemma shows how these potentials can be used to derive a representation formula for functions that solve Maxwell's equation.

Lemma 7.12. *A function $\psi \in X_D$ can be represented by the following Stratton-Chu formula,*

$$\psi = -\mathrm{DL}(\gamma_D(\psi)) - \mathrm{SL}(\gamma_N(\psi)).$$

Proof. Since $\psi \in X_D$ solves the Maxwell's equation $\operatorname{curl}^2 \psi - k^2\psi = 0$ in D in a distributional sense, it is clear that $\psi \in H(\operatorname{curl}^2, D)$. Assume for a moment that $\psi \in H(\operatorname{curl}^2, D) \cap H(\operatorname{curl}, D)$. Setting $\phi = -1/(\mathrm{i}k) \operatorname{curl} \psi$, we have that $\phi \in H(\operatorname{curl}, D)$ and the following Maxwell system

$$\operatorname{curl} \phi - \mathrm{i}k\psi = 0, \qquad \operatorname{curl} \psi + \mathrm{i}k\phi = 0,$$

holds in a distributional sense in D. Since both $\phi, \psi \in H(\operatorname{curl}, D)$, [Mon03, Theorem 9.2] states that

$$\psi(x) = -\operatorname{curl} \int_{\partial D} (\nu \times \psi)(y)\Phi(x, y)\, \mathrm{d}y + \frac{1}{\mathrm{i}k} \operatorname{curl}^2 \int_{\partial D} \nu(y) \times \phi(y)\Phi(x, y)\ \mathrm{d}y$$

for all $x \in D$. Substituting $\phi = 1/(\mathrm{i}k) \operatorname{curl} \psi$ yields

$$\psi(x) = -\operatorname{curl} \int_{\partial D} (\nu \times \psi)(y)\Phi(x, y)\, \mathrm{d}y - \frac{1}{k^2} \operatorname{curl}^2 \int_{\partial D} \nu(y) \times \operatorname{curl} \psi(y)\Phi(x, y)\ \mathrm{d}y. \tag{7.27}$$

Note that the second integral in the last equation solves Maxwell's equation, i.e.

$$(\operatorname{curl}^2 - k^2) \int_{\partial D} \nu(y) \times \operatorname{curl} \psi(y)\Phi(x, y)\ \mathrm{d}y = 0.$$

Substituting into (7.27) yields

$$\psi(x) = -\operatorname{curl} \int_{\partial D} (\nu \times \psi)(y)\Phi(x, y)\, \mathrm{d}y - \int_{\partial D} \nu(y) \times \operatorname{curl} \psi(y)\Phi(x, y)\ \mathrm{d}y.$$

Since $H(\operatorname{curl}^2, D) \cap H(\operatorname{curl}, D)$ is dense in $H(\operatorname{curl}^2, D)$, see, e.g. [Had04, Lemma A.1], the assertion follows from a density argument . ■

Let us finally introduce two function spaces that are necessary to define the necessary operators. The first space $X_{D\setminus\overline{D_0}}$ contains those functions in $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ that solve the Maxwell equation on this domain. The second space X contains those functions in $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ that can be extended to D such that the extension solves Maxwell's equation on this domain. The precise definition of the spaces is given by

$$X_{D\setminus\overline{D_0}} := \left\{ w \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3),\ \int_{D\setminus\overline{D_0}} w \cdot (\operatorname{curl}^2 \psi - k^2\psi)\, \mathrm{d}x = 0 \quad \forall \psi \in C_0^\infty(D \setminus \overline{D_0}, \mathbb{C}^3) \right\}$$

and

$$X = \left\{ w \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3) |\ \exists W \in X_D, w := W|_{D\setminus\overline{D_0}} \right\}$$

where X_D was the space of solution of Maxwell's equation on D, defined at the beginning of this section in (7.23). As in the case of acoustic scattering, we can define an extension operator E : $X \to X_D$ by $E(w) = W$, where W is the extension of w to D that solves Maxwell's equation. This extension operator can be written explicitly by using the Stratton-Chu formula from Lemma 7.12,

$$Ew(x) = -\mathrm{DL}(\gamma_D w)(x) - \mathrm{SL}(\gamma_N w).$$

Using this formula again, we can represent functions $w \in X_{D\setminus\overline{D_0}}$ by

$$\begin{aligned} w(x) = &-\mathrm{DL}(\gamma_D w)(x) - \mathrm{SL}\,(\gamma_N w)\,(x) \\ &- \mathrm{DL}((\nu \times w)\,|_{\partial D_0})(x) - \mathrm{SL}((\nu \times \operatorname{curl} w)|_{\partial D_0})(x), \qquad x \in D \setminus \overline{D_0}. \end{aligned}$$

As in the case of acoustic scattering we can also introduce an operator $A : X_{D\setminus\overline{D_0}} \to X$ by means of the representation formula for Maxwell's equations. We define

$$Aw(x) = -\mathrm{DL}(\gamma_D(w))(x) - \mathrm{SL}(\gamma_N(w))(x), \qquad x \in D \setminus \overline{D_0}. \tag{7.28}$$

This operator can later be used to define a projection onto the image space of the Herglotz wave operator. In the next lemma, we will characterize this space.

Lemma 7.13. *It holds that* $X = \mathrm{closure}_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}\mathcal{R}(H)$.

Proof. At first we define an extension $\tilde{H} : L^2_t(\mathbb{S}_1) \to L^2(D, \mathbb{C}^3)$ of the Herglotz operator H by

$$\tilde{H}\psi(x) = \operatorname{curl} \int_D e^{ikx\cdot\theta}\psi(\theta)\,\mathrm{d}x \qquad x \in D,$$

such that $Hg = \tilde{H}g|_{D\setminus\overline{D_0}}$. Let now $w = H\psi$ for an arbitrary function $\psi \in L^2(\mathbb{S}_1)$. Then the extension $W = \tilde{H}\psi$ solves Maxwell's equation in D and $w = W|_{D\setminus\overline{D_0}}$ shows that $w \in X$. Next we show that the space X is closed to conclude that $\overline{\mathcal{R}(H)} \subset X$. To this end let $(w_j)_{j\in\mathbb{N}}$ be an arbitrary sequence in X, where $w_j \to w$ in $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$. We will show that $w \in X$. Note that $w_j \in X_{D\setminus\overline{D_0}}$ and since this space is closed, it follows that $w \in X_{D\setminus\overline{D_0}}$. Therefore it follows that $w_j \to w$ in $H(\operatorname{curl}^2, D \setminus \overline{D_0})$, since both functions solve Maxwell's equation on $D \setminus \overline{D_0}$, such that

$$\|w_j - w\|_{H(\operatorname{curl}^2,D\setminus\overline{D_0})} = \|w_j - w\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} + \|\operatorname{curl}^2[w_j - w]\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} \tag{7.29}$$

$$= (1 + k^2)\|w_j - w\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} \to 0. \tag{7.30}$$

Furthermore by the Stratton-Chu formula

$$w = -\,\mathrm{DL}(\gamma_D w) - \mathrm{SL}\,(\gamma_N w) - \mathrm{DL}((\nu \times w)|_{\partial D_0}) - \mathrm{SL}\left((\nu \times \operatorname{curl} w)\Big|_{\partial D_0}\right) \qquad \text{in } D \setminus \overline{D_0}.$$

In particular, each term in the expression above is well-defined. Hence we can set

$$w(x) = -\,\mathrm{DL}\,(\gamma_{\mathrm{D}}(w))\,(x) - \mathrm{SL}\,(\gamma_{\mathrm{N}}(w))\,(x), \qquad x \in D.$$

Then w solves Maxwell's equation in D. Since $w_j \in X$, each w_j can also be represented by

$$w_j = \mathrm{DL}(w_j|_{\partial D})(x) - \mathrm{SL}\left(\frac{\partial w_j}{\partial \nu}\Big|_{\partial D}\right)(x), \qquad x \in D \setminus \overline{D_0}.$$

By the triangle inequality we have that

$$\begin{aligned} \left\| w_j - w|_{D\setminus\overline{D_0}} \right\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} &= \|\mathrm{DL}(\gamma_D[w_j - w]) - \mathrm{SL}\,(\gamma_N[w_j - w])\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} \\ &\le \|\mathrm{DL}(\gamma_D[w_j - w])\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} + \|\mathrm{SL}\,(\gamma_N[w_j - w])\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}\,. \end{aligned}$$

Since SL and DL are continuous, there are constants $c_1, c_2 > 0$ such that for $j \to \infty$ it follows that

$$\left\| w_j - w|_{D\setminus\overline{D_0}} \right\|_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} \leq c_1 \left(\|\gamma_D[w_j - w]\|_{H_t^{-1/2}(\partial D)} + \|\gamma_N[w_j - w]\|_{H_t^{-3/2}(\partial D)} \right)$$
$$\leq c_2 \left[\|w_j - w\|_{H(\mathrm{curl}^2, D\setminus\overline{D_0})} \right] \to 0,$$

where we used the continuity of the trace operators γ_D and γ_N. In particular it follows that $w_{D\setminus\overline{D_0}} = w$ and since w solves Maxwell's equation in D, it follows that $w \in X$ and therefore X is closed.

To complete the proof we choose an arbitrary $w \in X$ and show that $w \in \overline{\mathcal{R}(H)}$. Since $w \in X$, it follows that there exists $W \in L^2(D, \mathbb{C}^3)$ with $Ew = W$ and W solves Maxwell's equation in D. Then it follows that $W \in \overline{\mathcal{R}(\tilde{H})}$ by [LR15, Lemma 4]. Therefore there is a sequence $W_j \subset \mathcal{R}(\tilde{H})$, such that $\|W_j - W\|_{L^2(D)} \to 0$ as $j \to \infty$. It follows that $\|W_j|_{D\setminus\overline{D_0}} - w\|_{L^2(D\setminus\overline{D_0})} \to 0$ and as $W_j|_{D\setminus\overline{D_0}} \in \mathcal{R}(H)$, we conclude that $w \in \overline{\mathcal{R}(H)}$, which shows the assertion. ■

The following theorem shows the connection between the far field operator and interior transmission eigenvalues, which is due to the properties of the middle operator of the factorization of the far field operator. Slightly adapting the arguments of the proof of [LR15, Theorem 11] yields the following properties of the middle operator.

Theorem 7.14. *(a) Let k^2 be an interior transmission eigenvalue with corresponding eigenpair $(v, W) \in H_0(\mathrm{curl}, D) \times X_D$ and set $w := W|_{D\setminus\overline{D_0}} \in X$. Then $\mathrm{Im}\,(T_k w, w)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} = 0$.*
(b) Let $w \in X \setminus \{0\}$ such that $\mathrm{Im}\,(T_k w, w)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} = 0$. Then k^2 is an interior transmission eigenvalue and $(v, Ew) \in H_0(\mathrm{curl}, D) \times X_D$ is the corresponding eigenpair.

As we previously mentioned, the eigenvalues of the far field operator lie on the circle $\{\lambda \in \mathbb{C}, |8\pi^2 \mathrm{i}/k - \lambda| = 8\pi^2/k\}$ in the complex plane. Using the properties of the operator T and the factorization of the far field operator F, we can show in the usual way that the eigenvalues converge to zero from the left.

Lemma 7.15. *Let k^2 be no interior transmission eigenvalue. Then λ_j converges to zero from the left, i.e. $\mathrm{Re}\,(\lambda_j) < 0$ for $j \in \mathbb{N}$ large enough.*

We again use the representation of the eigenvalues λ_j in polar coordinates in (7.8) and the definition of the smallest phase ϑ_* in (7.9). Then the standard phase characterization holds,

$$\cot \vartheta_* = \max_{w \in X} \frac{\mathrm{Re}\,(Tw, w)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}}{\mathrm{Im}\,(Tw, w)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}}. \tag{7.31}$$

We want to vary the wavenumber and therefore indicate the dependence of relevant quantities on k by writing, e.g., $X = X_k, F = F_k, T = T_k, \lambda_j = \lambda_j(k)$ etc. For the first part of the inside-outside duality, we need to replace the space X_k. To this end we define a projection from X_k into $L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ in the following way. First we define the space

$$W := \left\{ \psi \in H_0^1(D \setminus \overline{D_0}, \mathbb{C}^3), \mathrm{curl}\, \psi \in H_0^1(D \setminus \overline{D_0}, \mathbb{C}^3) \right\}$$

with norm

$$\|\psi\|_W^2 := \|\psi\|^2_{H(\mathrm{curl}, D\setminus\overline{D_0})} + \|\,\mathrm{div}\,\psi\|^2_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} + \|\,\mathrm{curl}^2\,\psi\|^2_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}.$$

Recall the decomposition of a function $g \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ as $g = \mathrm{curl}\, B_g + \nabla p_g$ with a uniquely determined vector potential $B_g \in H(\mathrm{curl}, D\setminus\overline{D_0}) \cap H(\mathrm{div}\, 0, D\setminus\overline{D_0})$ such that $B_g \cdot \nu = 0$ on ∂D and a unique scalar potential $p_g \in H_0^1(D \setminus \overline{D_0})$. Now we can define a projection $P_k : L^2(D \setminus \overline{D_0}, \mathbb{C}^3) \to X_k$ by

$$P_k g := A_k P_k^{\text{aux}} g, \tag{7.32}$$

where $A_k : X^k_{D\setminus\overline{D_0}} \to X_k$ is the operator defined in (7.28) and the auxiliary projection P_k^{aux} : $L^2(D \setminus \overline{D_0}, \mathbb{C}^3) \to X^k_{D\setminus\overline{D_0}}$ is given by

$$P_k^{\text{aux}} g = g - (\operatorname{curl}^2 - k^2)\hat{B}_g - \nabla p_g,$$

with $\hat{B}_g \in W$ as solution to the variational problem,

$$\int_{D\setminus\overline{D_0}} \left(\operatorname{curl}^2 - k^2\right) \hat{B}_g \cdot \left(\operatorname{curl}^2 - k^2\right) \overline{\psi} \; \mathrm{d}x + \int_{D\setminus\overline{D_0}} \operatorname{div} \hat{B}_g \cdot \operatorname{div} \overline{\psi} \; \mathrm{d}x \tag{7.33}$$

$$= \int_{D\setminus\overline{D_0}} \operatorname{curl} \hat{B}_g \cdot \left(\operatorname{curl}^2 - k^2\right) \overline{\psi} \, \mathrm{d}x, \qquad \forall\, \psi \in W. \tag{7.34}$$

Recall from the previous section that $H_0^1(D \setminus \overline{D_0}, \mathbb{C}^3) = H_0(\operatorname{div}, D \setminus \overline{D_0}) \cap H_0(\operatorname{curl}, D \setminus \overline{D_0})$, such that functions in W have the necessary regularity for the variational problem to be well-defined. Also, the sesquilinear form that arises from this problem is coercive as a consequence of [LR15, Lemma 16] and therefore the variational problem is well-posed and has a unique solution in W. We will now show that the map P_k is a projection and fulfills additional requirements for the inside-outside duality.

Lemma 7.16. *The map $P_k : L^2(D \setminus \overline{D_0}) \to X_k$ in (7.32) is a projection that is differentiable with respect to k. Furthermore, the derivative $\mathrm{d}P_k/\,\mathrm{d}k$ is divergence-free, i.e. it holds that*

$$\int_{D\setminus\overline{D_0}} \frac{\mathrm{d}}{\mathrm{d}k} P_k w \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D \setminus \overline{D_0}).$$

Proof. For an arbitrary function $g \in L^2(D \setminus \overline{D_0})$, it has already been shown in [LR15, Theorem 18] that P_k^{aux} is a projection onto $X^k_{D\setminus\overline{D_0}}$, i.e. it holds that $P_k^{\text{aux}} g \in X^k_{D\setminus\overline{D_0}}$ and if $g \in X^k_{D\setminus\overline{D_0}}$, then $P_k^{\text{aux}} g = g$. By definition of the map A_k it then follows that $P_k g = A_k P_k^{\text{aux}} g \in X_k$ and if $g \in X^k_{D\setminus\overline{D_0}}$, then $P_k g = A_k g = g$. Note that the map A_k is differentiable with respect to k due to the differentiability of Maxwell's double layer and single layer potential from (7.25) and (7.26), see the proof of [CHM15, Lemma 2.2]. Since the function $k \to \hat{B}_g(k)$ is differentiable with respect to k, it follows that the auxiliary projection P_k^{aux} is also differentiable with respect to k and therefore also the projection P_k. Finally we show that the derivative $\mathrm{d}P_k/\,\mathrm{d}k$ is divergence-free. Note first that A_k is represented as a sum of Maxwell's single layer and double layer potential. This implies that $P_k w$ is divergence-free for all $w \in L^2(D \setminus \overline{D_0}, \mathbb{C}^3)$ and $k > 0$, i.e.

$$\int\limits_{D\setminus\overline{D_0}} P_k w \cdot \nabla \overline{\psi} \, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D \setminus \overline{D_0}).$$

Then it follows that

$$\int\limits_{D\setminus\overline{D_0}} \frac{\mathrm{d}}{\mathrm{d}k} P_k w \cdot \nabla \overline{\psi} \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}k} \int\limits_{D\setminus\overline{D_0}} P_k w \cdot \nabla \overline{\psi} \, \mathrm{d}x = 0 \qquad \forall \psi \in C_0^\infty(D \setminus \overline{D_0}),$$

which concludes the proof. ∎

Using this projection, we can write the characterization for the smallest phase equivalently as

$$\cot \vartheta_*(k) = \max_{w \in X_k \setminus \{0\}} \frac{\mathrm{Re}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(T_k w, w)_{L^2(D,\mathbb{C}^3)}} = \max_{g \in L^2(D,\mathbb{C}^3)\setminus\{0\}} \frac{\mathrm{Re}\,(T_k P_k g, P_k g)_{L^2(D,\mathbb{C}^3)}}{\mathrm{Im}\,(T_k P_k g, P_k g)_{L^2(D,\mathbb{C}^3)}}.$$

Now we calculate the derivative α for the first part of the inside-outside duality. We start by calculating an auxiliary derivative.

Lemma 7.17. *Assume that k_0^2 is an interior transmission eigenvalue with eigenfunctions (v_0, W_0) in $H_0(\mathrm{curl}, D) \times H_0(\mathrm{curl}, D)$. Setting $w_0 = W_0|_{D\setminus\overline{D_0}} \in X_{k_0}$, the mapping $k \mapsto (T_k w_0, w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}$ is differentiable in k at $k = k_0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}\bigg|_{k=k_0} = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x. \tag{7.35}$$

Proof. Define v_k for $k > 0$ as the unique radiating solution to the variational formulation

$$\int_{\mathbb{R}^3} \left[(\mathrm{Id} - Q)\,\mathrm{curl}\, v_k \cdot \mathrm{curl}\,\overline{\psi} - k^2 v_k \cdot \overline{\psi}\right] \mathrm{d}x = \int_{D\setminus\overline{D_0}} Q\, w_0 \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x \quad \forall \psi \in H(\mathrm{curl}, \mathbb{R}^3) \tag{7.36}$$

with compact support and note that $v_0 = v_{k_0} \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3) \cap H_0(\mathrm{curl}, D)$. Since this variational problem depends polynomially on k and since $v_{k_0} \in H_0(\mathrm{curl}, D)$ we note that the derivative $v_0' := \mathrm{d}v_k/\mathrm{d}k|_{k=k_0}$ of v_k with respect to $k > 0$ at $k = k_0$ satisfies

$$\int_D \left[(\mathrm{Id} - Q)\,\mathrm{curl}\, v_0' \cdot \mathrm{curl}\,\overline{\psi} - k_0^2 v_0' \cdot \overline{\psi}\right] \mathrm{d}x = 2k_0 \int_D v_0 \cdot \overline{\psi}\,\mathrm{d}x \quad \forall \psi \in H(\mathrm{curl}, D). \tag{7.37}$$

Now we compute the derivative of $k \mapsto (T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}$ with respect to k at $k = k_0$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} &= \frac{\mathrm{d}}{\mathrm{d}k}(Q(w_0 + v_k|_{D\setminus\overline{D_0}}), w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}\bigg|_{k=k_0} \\ &= \int_{D\setminus\overline{D_0}} Q\,\mathrm{curl}(v_0')\overline{w_0}\,\mathrm{d}x.\end{aligned}$$

Choosing $\psi = v_0'$ in (7.36) and taking the complex conjugate of this equation shows that

$$\frac{\mathrm{d}}{\mathrm{d}k}(T_k w_0, w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}\bigg|_{k=k_0} = \int_D \left[(\mathrm{Id} - Q)\,\mathrm{curl}(v_0') \cdot \mathrm{curl}(\overline{v_0}) - k_0^2 v_0' \cdot \overline{v_0}\right] \mathrm{d}x = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x,$$

which concludes the proof. ■

Lemma 7.18. *Assume that k_0^2 is an interior transmission eigenvalue with eigenpair $(v_0, W_0) \in H_0(\mathrm{curl}, D) \times H_0(\mathrm{curl}, D)$. Setting $w_0 := W_0|_{D\setminus\overline{D_0}}$, the mapping $k \mapsto (T_k P_k w_0, P_k w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)}$ is differentiable in k at k_0 and*

$$\alpha(k_0) := \frac{\mathrm{d}}{\mathrm{d}k}\,(T_k P_k w_0, P_k w_0)_{L^2(D,\mathbb{C}^3)}\bigg|_{k=k_0} = 2k_0 \int_D |v_0|^2 \,\mathrm{d}x + \frac{4}{k_0}\mathrm{Re} \int_D \mathrm{curl}\, v_0 \cdot \overline{W_0}\,\mathrm{d}x. \tag{7.38}$$

Proof. Recall that $k \mapsto P_k w_0$ is continuously differentiable and since P_k maps into the space X_k, we know that $w_k := E_k P_k w_0 \in L^2(D, \mathbb{C}^3)$ solves Maxwell's equation, i.e.

$$\int_D w_k(\mathrm{curl}^2\,\psi - k^2\psi)\,\mathrm{d}x = 0 \quad \forall \psi \in C_0^\infty(D, \mathbb{C}^3).$$

Since both E_k and P_k are differentiable with respect to k, the function w_k is also differentiable and its derivative $w_k' := \mathrm{d}w_k/\,\mathrm{d}k \in L^2(D,\mathbb{C}^3)$ solves

$$\int_D w_k'(\operatorname{curl}^2\psi - k^2\psi)\,\mathrm{d}x = 2k\int_D w_k\psi\,\mathrm{d}x \qquad \forall\psi \in C_0^\infty(D,\mathbb{C}^3). \tag{7.39}$$

Note that by the same arguments as in proof of Lemma 3.6, it is clear that the derivative in k is not influenced by the presence of the operator E_k, i.e. it holds that $w_k'|_{D\setminus\overline{D_0}} = P_k' w_0$.

We compute the derivative $\alpha(k_0)$ by the chain rule,

$$\begin{aligned}\alpha(k_0) &= \left[\frac{\mathrm{d}}{\mathrm{d}k}(T_kP_kw_0, P_kw_0)_{L^2(D,\mathbb{C}^3)}\right]\Big|_{k=k_0}\\ &= \left[(T_k'P_kw_0,P_kw_0)_{L^2(D,\mathbb{C}^3)} + (T_kP_k'w_0,P_kw_0)_{L^2(D,\mathbb{C}^3)} + (T_kP_kw_0,P_k'w_0)_{L^2(D,\mathbb{C}^3)}\right]\big|_{k=k_0}\\ &= 2k_0\int_D |v_0|^2\,\mathrm{d}x + \overline{(T_{k_0}^*w_0,P_{k_0}'w_0)}_{L^2(D,\mathbb{C}^3)} + (T_{k_0}w_0,P_{k_0}'w_0)_{L^2(D,\mathbb{C}^3)},\end{aligned}$$

where we used the result of the previous lemma. Next we show that $T_{k_0}^*w_0 = T_{k_0}w_0$ on the space X_{k_0}. To this end, recall that $T_{k_0}w_0 = Q(w_0 + v_0|_{D\setminus\overline{D_0}})$ where the first component $v_0 \in H_0(\operatorname{curl},D)$ of the eigenpair (v_0,W_0) to the transmission eigenvalue k_0 solves

$$\int_D \left[(\operatorname{Id}-Q)\operatorname{curl}v_0\cdot\operatorname{curl}\overline{\psi} - k_0^2 v_0\cdot\overline{\psi}\right]\,\mathrm{d}x = \int_{D\setminus\overline{D_0}} Q\,w_0\cdot\operatorname{curl}\overline{\psi}\,\mathrm{d}x \qquad \forall\psi\in H(\operatorname{curl},D).$$

Obviously, extending v_0 by zero outside D yields a radiating solution to (7.36). Moreover,

$$\begin{aligned}(T_{k_0}w_0,w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} &= (Qw_0,w_0)_{L^2(D\setminus\overline{D_0},\mathbb{C}^3)} + \int_{D\setminus\overline{D_0}}\operatorname{curl}v_0\cdot(Q\overline{w_0})\,\mathrm{d}x\\ &= \int_D \overline{w_0}^\top Qw_0\,\mathrm{d}x + \int_D\left[(\operatorname{curl}\overline{v_0})^\top(\operatorname{Id}-Q)\operatorname{curl}v_0 - k_0^2|v_0|^2\right]\,\mathrm{d}x.\end{aligned}$$

Since the latter expression is real-valued, T_{k_0} is self-adjoint on the kernel of $w_0 \mapsto (T_{k_0}w_0,w_0)$, i.e., $T_{k_0}w_0 = T_{k_0}^*w_0$, and

$$\frac{\mathrm{d}}{\mathrm{d}k}\,(T_kP_kw_0,P_kw_0)_{L^2(D,\mathbb{C}^3)}\Big|_{k=k_0} = 2k_0\int_D|v_0|^2\,\mathrm{d}x + 2\mathrm{Re}\,\left(T_{k_0}w_0,P_{k_0}'w_0\right)_{L^2(D,\mathbb{C}^3)}.$$

To compute the last term on the right we recall that $w_0 \in X_{k_0}$. Using the arguments of Lemma 7.16, we conclude that w_{k_0}' is divergence-free. Due to [GR86a, Theorem 3.6] it follows that there exists a unique vector potential $A_0 \in H_0(\operatorname{curl},D)\cap H(\operatorname{div}0,D)$ such that $\operatorname{curl}A_0 = w_{k_0}'$. Therefore we can apply equation (7.36) for $\psi = A_0$ to obtain

$$\begin{aligned}(T_{k_0}w_0,P_{k_0}'\overline{w_0}) &= -\int_{D\setminus\overline{D_0}} Q(w_0+\operatorname{curl}v_0)\cdot w_{k_0}'\,\mathrm{d}x = -\int_{D\setminus\overline{D_0}} Q(w_0+\operatorname{curl}v_0)\cdot\operatorname{curl}A_0\,\mathrm{d}x\\ &= -\int_D\left[\operatorname{curl}v_0\cdot\operatorname{curl}A_0 - k_0^2v_0\cdot\overline{A_0}\right]\,\mathrm{d}x.\end{aligned}$$

Since $v_0 \in H_0^1(D,\mathbb{C}^3)\cap H_0(\operatorname{div}0,D)$, we can exploit Theorem 3.6 in [GR86a] another time to obtain the existence of a unique vector potential $V_0 \in H_0(\operatorname{curl},D)\cap H(\operatorname{div}0,D)$ such that $\operatorname{curl}V_0 = v_0$. Obviously, $\operatorname{curl}V_0 \in H_0^1(D,\mathbb{C}^3)$, which allows to continue the last computation by a partial integration,

$$(T_{k_0}w_0, P'_{k_0}w_0) = -\int_D \left[\operatorname{curl} v_0 \cdot \operatorname{curl}\overline{A_0} - k_0^2 \operatorname{curl} v_0 \cdot \overline{A_0}\right] \,\mathrm{d}x$$
$$= -\int_D \left[\operatorname{curl}^2 v_0 \cdot \operatorname{curl}\overline{A_0} - k_0^2 V_0 \cdot \operatorname{curl}\overline{A_0}\right] \,\mathrm{d}x = -\int_D \left[\operatorname{curl}^2 - k^2\right] V_0 \cdot \overline{w'_{k_0}} \,\mathrm{d}x.$$

Now we can use (7.39) and $P_{k_0}w_0 = w_0$ to obtain

$$(T_{k_0}w_0, P'_{k_0}w_0) = 2k_0 \int_D E_{k_0}\overline{w_0} \cdot V_0 \,\mathrm{d}x.$$

Since $V_0 \in W$ satisfies $V_0 \in H_0(\operatorname{curl}, D)$ and $\operatorname{curl} V_0 = v_0 \in H_0(\operatorname{curl}, D)$ and since $E_{k_0}\overline{w_0} \in X_D$ it holds that $\int_D \overline{E_{k_0}w_0} \cdot [\operatorname{curl}^2 V_0 - k_0^2 V_0] \,\mathrm{d}x = 0$. In particular,

$$(T_{k_0}w_0, P'_{k_0}w_0) = 2k_0 \int_D E_{k_0}\overline{w_0} \cdot V_0 \,\mathrm{d}x = \frac{2}{k_0}\int_D \operatorname{curl}^2 V_0 \cdot E_{k_0}\overline{w_0} \,\mathrm{d}x = \frac{2}{k_0}\int_D \operatorname{curl} v_0 \cdot \overline{W_0} \,\mathrm{d}x,$$

which proves the assertion. ■

Now we state the first part and the second part of the inside-outside duality. In this context we use the auxiliary derivative α from (7.38). The proof of first part of the inside-outside duality is analogous to the proof Theorem 3.7 while the proof of the second part is analogous to the proof of Theorem [LR15, Theorem 15].

Theorem 7.19 (Inside-outside duality - Part 1)**.** *Let k_0^2 be an interior transmission eigenvalue with corresponding transmission eigenpair (v_0, W_0) and set $w_0 = W_0|_{D\setminus\overline{D_0}} \in X_{k_0}$. Let $\alpha(k_0)$ be the expression in (7.38). Then it holds that*

$$\lim_{k\nearrow k_0} \vartheta_*(k) = 0 \text{ if } \alpha(k_0) > 0 \qquad \text{and} \qquad \lim_{k\searrow k_0} \vartheta_*(k) = 0 \text{ if } \alpha(k_0) < 0.$$

Theorem 7.20 (Inside-outside duality - Part 2)**.** *Choose $k_0 > 0$ such that $I := (k_0 - \varepsilon, k_0 + \varepsilon) \setminus \{k_0\}$ contains no wavenumber k such that k^2 is an interior transmission eigenvalue. If it holds that $\lim_{I\ni k\to k_0} \vartheta_*(k) = 0$, then k_0^2 is an interior transmission eigenvalue.*

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D	scattering object
D_0	cavity inside of D
$[\cdot]_{\partial D}$	jump of a function over the boundary ∂D
ν	outward normal vector
∂D	boundary of the scattering object D
A	Anisotropic density
Q	Contrast corresponding to density A or relative permittivity ε_r^{-1}
T_ν	stress tensor
ε	electric permittivity
μ_0	constant magnetic permittivity
ω	circular frequency
σ	Chapter 2: Spectrum of a matrix, Chapter 3: sign of contrast q, Chapter 7: Conductivity
τ	function as part of the Robin boundary condition
k	wavenumber
n	index of refraction
p	polarization vector
q	contrast corresponding to the refractive index n.
$H(\mathrm{curl}, D)$	space of functions $u \in L^2(D, \mathbb{C}^3)$ such that $\mathrm{curl}\, u \in L^2(D, \mathbb{C}^3)$
$H(\mathrm{curl}^2, D)$	space of functions $u \in L^2(D, \mathbb{C}^3)$ such that $\mathrm{curl}^2\, u \in L^2(D, \mathbb{C}^3)$
$H(\mathrm{div}\, 0, D)$	space of functions $u \in H(\mathrm{div}, D)$ such that $\mathrm{div}\, u = 0$ in D
$H(\mathrm{div}, D)$	space of functions $u \in L^2(D, \mathbb{C}^3)$ such that $\mathrm{div}\, u \in L^2(D, \mathbb{C}^3)$
H^1	Sobolev space of order one
H^1_{loc}	space functions belonging locally to H^1
H^s	Sobolev space of order $s \in \mathbb{R}$
$H_0(\mathrm{curl}, D)$	space of functions $u \in H(\mathrm{curl}, D)$ such that $u \times \nu = 0$ on ∂D
$H_0(\mathrm{div}, D)$	space of functions $u \in H(\mathrm{div}, D)$ such that $\nu \cdot u = 0$ on ∂D
H^1_0	functions belonging to H^1 with zero boundary values
$H_{\mathrm{loc}}(\mathrm{curl}, D)$	functions belonging locally to $H(\mathrm{curl}, D)$
$L^2(D)$ or $L^2(D, \mathbb{C})$	space of square-integrable, complex functions
$L^2_\Delta(D)$	space of functions $u \in L^2(D)$ such that $\Delta u \in L^2(D)$
$L^2_t(\mathbb{S}_1)$	space of tangential vector fields on the unit sphere
$L^2_p(\mathbb{S}_1)$	space of longitudinal vector fields on the unit sphere

$L^2_s(\mathbb{S}_1)$	space transversal vector fields on the unit sphere
W	auxiliary space for the introduction of projection operators
$W(B)$	numerical range of a bounded, linear operator B
X	characterization of closure of the image of the Herglotz operator H
$\boldsymbol{L}^2(D)$	space of functions $u \in L^2(D,\mathbb{C}) \times L^2(D,\mathbb{C}^3)$
E	extension operator from D_0 to D
F	far field operator for the scattering problem under consideration
H	(modified) Herglotz wave operator
I	identity operator for scalar functions
I_n	identity operator of dimension $n \in \mathbb{N}$
K	double layer boundary operator
K'	adjoint double layer boundary operator
N	hypersingular boundary operator
N_R	near field operator on a sphere S_R
P_k	projection operator onto the space X
S	single layer boundary operator
TN	modified near field operator
V	volume potential operator
DL	(extension of the) double layer potential
$\mathbb{F}_N$	discrete far field operator of dimension N
$\mathbb{F}^\delta_N$	discrete, noisy far field operator of dimension N and noise level δ
$\mathbb{N}_N$	discrete, noisy near field operator of dimension N
SL	(extension of the) single layer potential
$\mathbb{TN}$	noisy discretization of the modified near field operator
γ_D	Dirichlet trace for functions in L^2_Δ or functions in $H(\mathrm{curl}^2, D)$
γ_N	Neumann trace for functions in L^2_Δ or functions in $H(\mathrm{curl}^2, D)$
$\mathcal{Q}$	operator that projects eigenvalues onto a circle in the complex plane
$\mathcal{R}$	range of an operator
u^∞, H^∞	far fields for acoustic, elastic and electromagnetic scattering
u^i, H^i	incoming wave field for acoustic, elastic and electromagnetic scattering
u^s, H^s	scattered wave field for acoustic, elastic and electromagnetic scattering
u_p	longitudinal (pressure) part of a wave field
u_s	transversal (shear) part of a wave field
δ_j	phases of the eigenvalues of μ_j
λ_j	eigenvalues of the far field operator F
μ_j	eigenvalues of the near field operator N_R
ϑ^*	largest phase among all phases $(\vartheta_j)_{j\in\mathbb{N}}$
ϑ_*	smallest phase among all phases $(\vartheta_j)_{j\in\mathbb{N}}$
ϑ_j	phases of the eigenvalues λ_j
B_R	ball of radius R
$\mathbb{S}_1$	unit sphere
$\mathbb{S}_R$	sphere of radius R
Δ	Laplace operator
Δ^*	Navier operator
Φ	fundamental solution of the Helmholtz equation for acoustic scattering

Φ_N	fundamental solution of the Navier equation for elastic scattering
Ψ	sesquilinear form for elastic scattering
α	auxiliary derivative for the first part of the inside-outside duality
$k \nearrow k_0$	k approaches k_0 from below
$k \searrow k_0$	k approaches k_0 from above

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